**Error bounds on the approximation of functions and partial derivatives by quadratic spline quasi-interpolants on non-uniform criss-cross triangulations of a rectangular domain**



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# Error bounds on the approximation of functions and partial derivatives by quadratic spline quasi-interpolants on non-uniform criss-cross triangulations of a rectangular domain

**Catterina Dagnino** · **Sara Remogna** · **Paul Sablonnière**

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**Abstract** Given a non-uniform criss-cross triangulation of a rectangular domain $\Omega$, we consider the approximation of a function $f$ and its partial derivatives, by general $C^1$ quadratic spline quasi-interpolants and their derivatives. We give error bounds in terms of the smoothness of $f$ and the characteristics of the triangulation. Then, the preceding theoretical results are compared with similar results in the literature. Finally, several examples are proposed for illustrating various applications of the quasi-interpolants studied in the paper.

**Keywords** Bivariate splines · Quasi-interpolation · Derivative approximation

**Mathematics Subject Classification (2000)** 65D07 · 65D10 · 41A25

## 1 Introduction

Spline quasi-interpolation is well known to be a good method for the approximation of bivariate functions. A nice property of spline quasi-interpolants (abbr. QIs) is that their construction does not need the solution of any system of equations. This property is particularly attractive in the bivariate case, where the number of data sites can be huge in practice.

In the literature, quadratic spline QIs on criss-cross triangulations $\mathscr{T}_{mn}$ of a rectangular domain $\Omega$ are proposed and studied by many authors (see e.g. [31, Chap. 2], [8, Chap. 8], [18, Chap. 12], [2,5,6,9,14,15,28] and the references therein). In

C. Dagnino · S. Remogna
Dipartimento di Matematica, Università di Torino, via C. Alberto, 10 – 10123 Torino, Italy
E-mail: catterina.dagnino@unito.it
E-mail: sara.remogna@unito.it

P. Sablonnière
Centre de mathématiques, INSA de Rennes, 20 avenue des Buttes de Coësmes, CS 70839, F-35708-Rennes Cédex 7, France
E-mail: Paul.Sablonniere@insa-rennes.fr

general, they are based on B-splines with supports not completely included in $\Omega$ and with some data points lying outside $\Omega$. Therefore, $f$ has to be defined in an open set containing $\Omega$. In particular, in [2] and [15], some quadratic $C^1$ QIs defined on uniform meshes are proposed to approximate $f$ and its partial derivatives. Moreover, error estimates are given in the case $f \in C^3(\Omega)$.

In [20,21] spline QIs based on B-splines with supports not completely included in $\Omega$ and all data sites inside or on the boundary of $\Omega$ are proposed.

In [10,25,27] the authors define $C^1$ quadratic spline QIs on $\mathscr{T}_{mn}$, as linear combinations of B-splines whose supports are contained in $\Omega$ and with functionals based on data sites lying inside or on the boundary of $\Omega$, for which extra values outside $\Omega$ are not necessary. This property can be very useful, for instance in numerical integration [19] and in the approximation of functions with boundary conditions [4,11]. In [12], the error for two special QIs is introduced and partially studied.

In [29], quadratic $C^1$ quasi-interpolating splines on uniform criss-cross triangulations of $\Omega$ are directly determined by setting their Bernstein-Bézier coefficients to appropriate combinations of the given data values, without using locally supported splines spanning the spaces.

We notice that $C^1$ quadratic splines are those of the lowest degree having continuous first partial derivatives. This property could be interesting and useful in several applications, such as in numerical methods for PDEs (see e.g. [16]), where gradients of the basis functions have to be computed in the space of variational approximants. They should also provide excellent approximants of the solutions of integral equations (see e.g. [1,30]).

In this paper we investigate the approximation of a function $f$, defined in $\Omega$, by general $C^1$ quadratic spline QIs on non-uniform criss-cross triangulations, based on B-splines with supports in $\Omega$ and data sites inside or on the boundary of $\Omega$. Denoting them by $Qf$, we take their partial derivatives as approximation to those of $f$. We propose an error analysis for $f$ and its derivatives, making a particular effort to give error bounds in terms of the smoothness of $f$ and the characteristics of the triangulation, considering also the case of functions that are not regular enough.

Furthermore, in order to test our results, we provide several applications.

Here is an outline of the paper. In Section 2, general notations and results on $C^1$ quadratic spline QIs and their first and second order partial derivatives are introduced and three operators are considered: the Schoenberg-Marsden operator $S_1$ (see e.g. [5–7,9,10,25,27]) and the two optimal operators $S_2$ (see [25,27]) and $W_2$ (which is a modified version of the one introduced in [6,7]). Here and in the following, the expression “an optimal quadratic spline QI” has to be understood in the sense that such a QI is exact on the space of bivariate quadratic polynomials. In Section 3, local and global estimates on the infinity norm of approximation errors on functions and first order derivatives are given. Local estimates are provided for second order derivatives in the interior of each triangular cell of the given triangulation of the domain. More specific results are given in Section 4 for uniform meshes. In Section 5, for the QIs $S_1$, $S_2$, $W_2$ and their derivatives, specific error bounds are derived from the general results given in Sections 3 and 4. Finally, in Section 6, some examples and applications are proposed where the above QIs are compared with other existing QIs of the literature.

## 2 Quadratic spline quasi-interpolants and their partial derivatives

Let $\Omega = [a,b] \times [c,d]$ be a rectangle decomposed into $mn$ subrectangles by the two partitions $X_m = \{x_i, 0 \le i \le m\}$, $Y_n = \{y_j, 0 \le j \le n\}$ of the segments $[a,b] = [x_0, x_m]$ and $[c,d] = [y_0, y_n]$, respectively. Let $\mathscr{T}_{mn}$ be the criss-cross triangulation of $\Omega$, defined by drawing the two diagonals in each subrectangle (see Fig. 2.2). We define the space

$$\mathscr{S}_2^1(\mathscr{T}_{mn}) = \{s \in C^1(\Omega) : \ s|_T \in \mathbb{P}_2, \text{ for each triangular cell } T \text{ of } \mathscr{T}_{mn}\},$$

whose dimension is $(m+2)(n+2)-1$, where $\mathbb{P}_\ell$ is the space of polynomials in two variables of total degree less than or equal to $\ell$ [5,18,31]. Setting $\mathscr{K}_{mn} := \{(i,j) : 0 \le i \le m+1, 0 \le j \le n+1\}$ and $\widehat{\mathscr{K}_{mn}} := \{(i,j), 1 \le i \le m, 1 \le j \le n\}$, let $\mathscr{B}_{mn} := \{B_{ij}, (i,j) \in \mathscr{K}_{mn}\}$ be the collection of $(m+2)(n+2)$ B-splines spanning the space $\mathscr{S}_2^1(\mathscr{T}_{mn})$ [27], with knots

$$\begin{aligned} x_{-2} &= x_{-1} = a = x_0 < x_1 < \ldots < x_m = b = x_{m+1} = x_{m+2}, \\ y_{-2} &= y_{-1} = c = y_0 < y_1 < \ldots < y_n = d = y_{n+1} = y_{n+2}. \end{aligned}$$

In $\mathscr{B}_{mn}$, we consider the $mn$ B-splines associated with the set of indices $\widehat{\mathscr{K}_{mn}}$, whose restrictions to the boundary $\Gamma$ of $\Omega$ are equal to zero. To the latter, we add $2m+2n+4$ boundary B-splines whose restrictions to $\Gamma$ are univariate quadratic B-splines. Their set of indices is

$$\widetilde{\mathscr{K}_{mn}} := \{(i,0), (i,n+1), 0 \le i \le m+1; (0,j), (m+1,j), 1 \le j \le n\}.$$

The Bernstein-Bézier coefficients (BB-coefficients) and the supports of the inner B-splines $\{B_{ij}, 2 \le i \le m-1, 2 \le j \le n-1\}$ are given in [22], the other ones can be found in [24,26]. Some examples of B-spline supports are shown in Fig. 2.2$(a)$. The B-splines are positive and form a partition of unity.

In $\mathscr{S}_2^1(\mathscr{T}_{mn})$, we consider QI operators $Q : C(\Omega) \to \mathscr{S}_2^1(\mathscr{T}_{mn})$, defined by

$$Qf(x,y) = \sum_{(i,j) \in \mathscr{K}_{mn}} \lambda_{ij}(f) B_{ij}(x,y), \tag{2.1}$$

where $B_{ij} \in \mathscr{B}_{mn}$ and $\lambda_{ij} : C(\Omega) \to \mathbb{R}$ are linear functionals of the following form:

$$\lambda_{ij}(f) = \sum_{\mu=1}^{p} w_\mu^{(i,j)} f(x_\mu^{(i)}, y_\mu^{(j)}), \tag{2.2}$$

involving only a finite fixed number, $p \ge 1$, of mesh-points $(x_\mu^{(i)}, y_\mu^{(j)})$ in the support $\Sigma_{ij}$ of $B_{ij}$, and of real non-zero weights $w_\mu^{(i,j)}$. Moreover, we assume that the $\lambda_{ij}$'s are such that $Qf = f$ for all $f \in \mathbb{P}_\ell, \ 0 \le \ell \le 2$.

We remark that if $(\bar{x},\bar{y}) \in \Omega$ is such that $x_{r-1} \le \bar{x} \le x_r$, $y_{s-1} \le \bar{y} \le y_s$, $1 \le r \le m$, $1 \le s \le n$, then it lies in the interior of one of the four triangular cells $T_{rs}^{(k)}$ of $\mathscr{T}_{mn}$, $k = 1,2,3,4$ or in a common edge of two $T_{rs}^{(k)}$ or in in an external edge or it is the common vertex of the four triangles (see Fig. 2.1). Moreover, every triangle $T_{rs}^{(k)}$ is covered by the supports of exactly seven B-splines $B_{ij}$, whose indices belong to the set $I(T_{rs}^{(k)}) := \{(i,j) : \Sigma_{ij} \cap int(T_{rs}^{(k)}) \ne \emptyset\}$, where $int(T_{rs}^{(k)})$ denotes the interior of $T_{rs}^{(k)}$.

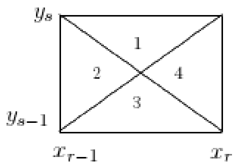



**Fig. 2.1** Triangular cells $T_{rs}^{(k)}$, $k = 1,2,3,4$.

Now, we consider $Qf$ and its partial derivatives

$$Q^\alpha f(\bar{x},\bar{y}) := D^\alpha Qf(\bar{x},\bar{y}) = \sum_{(i,j)\in I(T_{rs}^{(k)})} \lambda_{ij}(f) D^\alpha B_{ij}(\bar{x},\bar{y}). \tag{2.3}$$

with $0 \le |\alpha| \le 2$, $(\bar{x},\bar{y}) \in T_{rs}^{(k)}$ for $|\alpha| = 0,1$ and $(\bar{x},\bar{y}) \in int(T_{rs}^{(k)})$ for $|\alpha| = 2$, where $\alpha = (\alpha_1,\alpha_2)$ and $|\alpha| = \alpha_1 + \alpha_2$.

In (2.3), we compute the values of the B-splines and their derivatives by means of their BB-coefficients [26] and the de Casteljau algorithm for triangular surfaces [13, 18]. Since $B_{ij}$ is a polynomial of total degree two in $T_{rs}^{(k)}$, it is described by six BB-coefficients, ensuring the $C^1$ smoothness. Consequently, its first partial derivatives are polynomials of total degree one in such triangle, where they are described by three BB-coefficients ensuring the $C^0$ smoothness, while the second partial derivatives are constant polynomials inside $T_{rs}^{(k)}$.

Now we consider some quadratic spline QI operators. In order to do it we define the mesh-points:

$$s_i = \frac{x_{i-1}+x_i}{2}, \qquad t_j = \frac{y_{j-1}+y_j}{2}, \quad (i,j) \in \mathscr{K}_{mn}, \tag{2.4}$$

that are the $mn$ intersection points of diagonals in each subrectangle, the $2(m+n)$ midpoints of the subintervals on the four edges, and the four vertices of $\Omega$, see Fig. 2.2$(b)$.

## 2.1 The quasi-interpolant $S_1$

The first operator is the Schoenberg-Marsden near optimal operator (see e.g. [5–7,9, 10,25,27]) and it is obtained by assuming, in (2.2), $p = 1$, $w_1^{(i,j)} = 1$ and $(x_1^{(i)}, y_1^{(j)}) =$

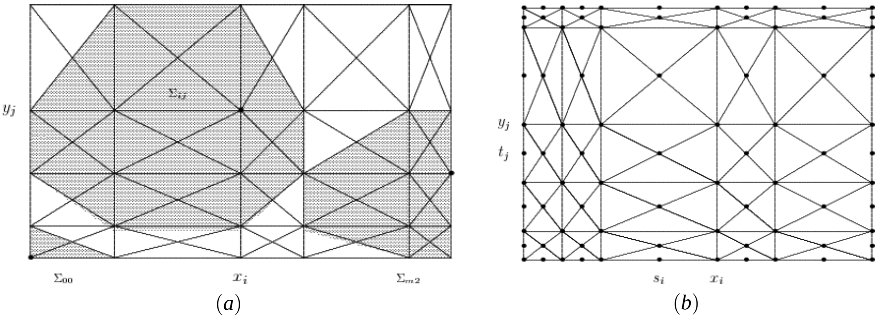



**Fig. 2.2** (*a*) Some supports of inner and boundary B-splines. (*b*) Grid points $(x_i, y_j)$ and mesh-points $(s_i, t_j)$ defined in (2.4).

$(s_i, t_j)$, $(i, j) \in \mathscr{K}_{mn}$, given in (2.4), i.e.

$$S_1 f(x,y) = \sum_{(i,j)\in\mathscr{K}_{mn}} f(s_i, t_j) B_{ij}(x,y).$$

It is exact for bilinear polynomials and $\|S_1\|_\infty = 1$, where $\|\cdot\|_\infty$ is the infinite norm. We notice that the number of data sites required by $S_1$ is

$$N_S = mn + 2m + 2n + 4. \tag{2.5}$$

## 2.2 The quasi-interpolant $S_2$

In order to define $S_2$, for $0 \le i \le m+1$, $0 \le j \le n+1$, we set:

$$a_i = -\frac{\pi_i^2 \pi'_{i+1}}{\pi_i + \pi'_{i+1}},\ c_i = -\frac{\pi_i (\pi'_{i+1})^2}{\pi_i + \pi'_{i+1}},\ \overline{a}_j = -\frac{\zeta_j^2 \zeta'_{j+1}}{\zeta_j + \zeta'_{j+1}},\ \overline{c}_j = -\frac{\zeta_j (\zeta'_{j+1})^2}{\zeta_j + \zeta'_{j+1}},$$
$$b_{ij} = 1 - (a_i + c_i + \overline{a}_j + \overline{c}_j),$$

with $a_0 = c_0 = a_{m+1} = c_{m+1} = \overline{a}_0 = \overline{c}_0 = \overline{a}_{n+1} = \overline{c}_{n+1} = 0$, $b_{00} = b_{m+1,0} = b_{0,n+1} = b_{m+1,n+1} = 1$ and $\pi_i = \frac{h_i}{h_{i-1}+h_i}$, $\pi'_i = \frac{h_{i-1}}{h_{i-1}+h_i} = 1 - \pi_i$, $\zeta_j = \frac{k_j}{k_{j-1}+k_j}$, $\zeta'_j = \frac{k_{j-1}}{k_{j-1}+k_j} = 1 - \zeta_j$, for $1 \le i \le m+1$ and $1 \le j \le n+1$, with $h_i = x_i - x_{i-1}$, $k_j = y_j - y_{j-1}$.

The quadratic spline QI $S_2$ [25, 27] is defined as follows:

$$S_2 f(x,y) = \sum_{(i,j)\in\mathscr{K}_{mn}} \lambda_{ij}(f) B_{ij}(x,y),$$

with coefficient functionals obtained assuming, in (2.2), $p = 5$, $w_1^{(i,j)} = b_{ij}$, $w_2^{(i,j)} = a_i$, $w_3^{(i,j)} = c_i$, $w_4^{(i,j)} = \overline{a}_j$, $w_5^{(i,j)} = \overline{c}_j$ and

$$(x_1^{(i)}, y_1^{(j)}) = (s_i, t_j), \quad (x_2^{(i)}, y_2^{(j)}) = (s_{i-1}, t_j),\ (x_3^{(i)}, y_3^{(j)}) = (s_{i+1}, t_j),$$
$$(x_4^{(i)}, y_4^{(j)}) = (s_i, t_{j-1}),\ (x_5^{(i)}, y_5^{(j)}) = (s_i, t_{j+1}),$$

where $s_i$ and $t_j$ are given in (2.4). The operator $S_2$ is optimal and, since

$$|a_i|, |c_i|, |\overline{a}_j|, |\overline{c}_j| \le 1/2 \ \text{ and } \ |b_{ij}| \le 3, \tag{2.6}$$

then $\|S_2\|_\infty \le 5$, see [25]. Moreover, we can notice that the number of data sites requested by $S_2$ is equal to $N_S$ given in (2.5).

### 2.3 The quasi-interpolant $W_2$

The third quasi-interpolant $W_2$ is defined as follows:

$$W_2 f(x,y) = \sum_{(i,j)\in\mathscr{K}_{mn}} \lambda_{ij}(f) B_{ij}(x,y),$$

where the coefficient functionals are obtained assuming, in (2.2), $p = 5$, $w_1^{(i,j)} = 2$, $w_2^{(i,j)} = w_3^{(i,j)} = w_4^{(i,j)} = w_5^{(i,j)} = -\frac{1}{4}$ and

$$(x_1^{(i)}, y_1^{(j)}) = (s_i, t_j), \ (x_2^{(i)}, y_2^{(j)}) = (x_{i-1}, y_{j-1}), \ (x_3^{(i)}, y_3^{(j)}) = (x_{i-1}, y_j),$$
$$(x_4^{(i)}, y_4^{(j)}) = (x_i, y_{j-1}), \ (x_5^{(i)}, y_5^{(j)}) = (x_i, y_j).$$

The definition of $W_2$ involves two kinds of data sites: the points $(s_i, t_j)$ given in (2.4) and the grid points $(x_i, y_j)$, see Fig. 2.2$(b)$. Moreover the number of data sites requested by $W_2$ is

$$N_W = 2mn + 3m + 3n + 1. \tag{2.7}$$

The operator $W_2$ is optimal and $\|W_2\|_\infty \leq 3$. We remark that in [6,7] an operator similar to $W_2$, but based on B-splines with octagonal support not completely included in $\Omega$, has been introduced.

## 3 Local and global error bounds for functions and derivatives

In this section general techniques to bound the errors on functions and partial derivatives (of order at most 2) are presented, which are valid for all the operators defined in Section 2. The results will be specified in Section 5 for each type of operator.

Let $T_{rs}^{(k)}$ be a triangle of $\mathscr{T}_{mn}$ and

$$E_{\alpha,\nu}(\bar{x},\bar{y}) := D^\alpha \left(f - Qf\right)(\bar{x},\bar{y}), \tag{3.1}$$

where $0 \leq |\alpha| \leq 2$, $(\bar{x},\bar{y}) \in T_{rs}^{(k)}$ for $|\alpha| = 0,1$, $(\bar{x},\bar{y}) \in int(T_{rs}^{(k)})$ for $|\alpha| = 2$ and $\nu \geq |\alpha|$ is an integer related to the smoothness of $f$.

Since $Q$ reproduces polynomials belonging to $\mathbb{P}_\ell$, there results that (3.1) is equivalent to $E_{\alpha,\nu}(\bar{x},\bar{y}) = D^\alpha R_\nu(\bar{x},\bar{y}) - D^\alpha Q R_\nu(\bar{x},\bar{y})$, where

$$R_\nu(x,y) := f(x,y) - q_\nu(x,y), \tag{3.2}$$

for any $q_\nu \in \mathbb{P}_\nu$ and any $f$ such that $D^\alpha f(\bar{x},\bar{y})$ exists, with $0 \leq |\alpha| \leq \nu \leq \ell \leq 2$. Since

$$|E_{\alpha,\nu}(\bar{x},\bar{y})| \leq |D^\alpha R_\nu(\bar{x},\bar{y})| + |D^\alpha Q R_\nu(\bar{x},\bar{y})|, \tag{3.3}$$

in Theorem 3.1 and Theorem 3.2, we give upper bounds both for $|D^\alpha R_\nu(\bar{x},\bar{y})|$ and $|D^\alpha Q R_\nu(\bar{x},\bar{y})|$ and finally, in Theorem 3.3, for $|E_{\alpha,\nu}(\bar{x},\bar{y})|$. In order to do it, we need the following two lemmas, that give upper bounds for $\sum_{(i,j)\in I(T_{rs}^{(k)})} \left|D^\alpha B_{ij}(\bar{x},\bar{y})\right|$ and $\left|\lambda_{ij}(R_\nu)\right|$.

First of all, we need to introduce the following notations:

- $h_r = x_r - x_{r-1}$, $k_s = y_s - y_{s-1}$, for $0 \le r \le m+1$, $0 \le s \le n+1$;
- $\widehat{\Delta}_{rs} = \max\{h_r, k_s\}$, $0 \le r \le m+1$, $0 \le s \le n+1$;
- $\overline{h}_r = \max_{r-1\le i\le r+1}\{h_i\}$, $\underline{h}_r = \min_{r-1\le i\le r+1}\{h_i, h_i \ne 0\}$, $2 \le r \le m-1$,
  $\overline{h} = \max\limits_{1\le i\le m}\{h_i\}$, $\underline{h} = \min\limits_{1\le i\le m}\{h_i\}$;
- $\overline{k}_s = \max_{s-1\le j\le s+1}\{k_j\}$, $\underline{k}_s = \min_{s-1\le j\le s+1}\{k_j, k_j \ne 0\}$, $2 \le s \le n-1$,
  $\overline{k} = \max\limits_{1\le j\le n}\{k_j\}$, $\underline{k} = \min\limits_{1\le j\le n}\{k_j\}$;
- $\Delta_{rs} = \max\{\overline{h}_r, \overline{k}_s\}$, $\overline{\Delta}_{rs} = \max\limits_{(i,j)\in I(T_{rs}^{(k)})}\{\Delta_{ij}\}$, $\delta_{rs} = \min\{\underline{h}_r, \underline{k}_s\}$;
- $\Delta = \max\{\overline{h}, \overline{k}\}$, $\delta = \min\{\underline{h}, \underline{k}\}$;
- $\Sigma_{rs}^{(k)} = \bigcup_{(i,j)\in I(T_{rs}^{(k)})} \Sigma_{ij}$;
- $\|\cdot\|_{\infty,B} = \|\cdot\|_B =$ supremum norm over $B$, with $B$ compact set in $\mathbb{R}^2$;
- $\omega(D^\nu f, t, B) = \max\{\omega(D^\alpha f, t, B), |\alpha| = \nu\}$, where

$$\omega(\varphi, t, B) = \max\{|\varphi(P_1) - \varphi(P_2)|; P_1, P_2 \in B, \|P_1 - P_2\| \le t\}$$

  is the modulus of continuity of $\varphi \in C(B)$, and $\|\cdot\|$ is the Euclidean norm;
- $\|D^\nu f\|_B = \max\limits_{|\beta|=\nu}\left\|D^\beta f\right\|_B$.

**Lemma 3.1** *Let $T_{rs}^{(k)}$ be a triangular cell of $\mathscr{T}_{mn}$, $(\bar{x}, \bar{y}) \in T_{rs}^{(k)}$ for $|\alpha| = 0, 1$ and $(\bar{x}, \bar{y}) \in int(T_{rs}^{(k)})$ for $|\alpha| = 2$. Then*

$$\sum_{(i,j)\in I(T_{rs}^{(k)})} \left|D^\alpha B_{ij}(\bar{x}, \bar{y})\right| \le K_{|\alpha|}(\underline{h}_r)^{-\alpha_1}(\underline{k}_s)^{-\alpha_2}, \tag{3.4}$$

*with $K_0 = 1$;*

$$K_1 = \begin{cases} 4, & \text{if } r = 1, m \text{ and/or } s = 1, n \\ 2, & \text{otherwise;} \end{cases} \tag{3.5}$$

$$K_2 = \begin{cases} 12, & \text{if } r = 1, m \text{ and/or } s = 1, n \\ 6, & \text{otherwise.} \end{cases} \tag{3.6}$$

*Proof* For $|\alpha| = 0$, due to the B-spline partition of unity, (3.4) is an equality, with $K_0 = 1$. In the case $|\alpha| = 1$, since $D^\alpha B_{ij}$ is a linear polynomial in the triangle $T_{rs}^{(k)}$ with vertices $A_1$, $A_2$, $A_3$, we have

$$\left|D^\alpha B_{ij}(\bar{x}, \bar{y})\right| \le \max\left\{\left|D^\alpha B_{ij}(A_1)\right|, \left|D^\alpha B_{ij}(A_2)\right|, \left|D^\alpha B_{ij}(A_3)\right|\right\}.$$

If $|\alpha| = 2$, then $D^\alpha B_{ij}$ is a constant inside $T_{rs}^{(k)}$. Using the values of the $B_{ij}$'s BB-coefficients, we can easily deduce the inequality (3.4) and the constants $K_1$ and $K_2$. We remark that the constants $K_1$ and $K_2$ are bigger for the triangles near the boundary of $\Omega$. □

**Lemma 3.2** *Let $f \in C^\nu(\Sigma_{rs}^{(k)})$, with $0 \le \nu \le \ell \le 2$. For any $(i, j) \in I(T_{rs}^{(k)})$*

$$\left|\lambda_{ij}(R_\nu)\right| \le C_\nu^{(i,j)} \Delta_{ij}^\nu \omega\left(D^\nu f, \frac{\Delta_{ij}}{2}, \Sigma_{rs}^{(k)}\right), \tag{3.7}$$

*where $C_\nu^{(i,j)}$ is a constant dependent on $\nu$, $i$ and $j$. If, in addition, $f \in C^{\ell+1}(\Sigma_{rs}^{(k)})$, then*

$$\left|\lambda_{ij}(R_\ell)\right| \leq C_{\ell+1}^{(i,j)} \Delta_{ij}^{\ell+1} \left\|D^{\ell+1} f\right\|_{\Sigma_{rs}^{(k)}}, \tag{3.8}$$

*where $C_{\ell+1}^{(i,j)}$ is a constant dependent on $\ell$, $i$ and $j$.*

*Proof* We set $\binom{\nu}{\beta} = \frac{\nu!}{\beta_1!\beta_2!}$ and, in (3.2), we choose as $q_\nu$ the Taylor polynomial in the expansion of $f$ at the midpoint $(\xi_0, \eta_0)$ of the external edge of $T_{rs}^{(k)}$, i.e.

$$q_\nu(x,y) = \frac{1}{\nu!} \sum_{|\beta| \leq \nu} \binom{\nu}{\beta} D^\beta f(\xi_0, \eta_0)(x-\xi_0)^{\beta_1}(y-\eta_0)^{\beta_2}. \tag{3.9}$$

Then, for $0 \leq \nu \leq \ell$ and $(u,v)$ in the segment joining $(x,y)$ to $(\xi_0, \eta_0)$, from (3.2) and (3.9) (see e.g. [6]),

$$R_\nu(x,y) = \frac{1}{\nu!} \sum_{|\beta|=\nu} \binom{\nu}{\beta} \left[D^\beta f(u,v) - D^\beta f(\xi_0, \eta_0)\right] (x-\xi_0)^{\beta_1}(y-\eta_0)^{\beta_2}. \tag{3.10}$$

Moreover, from (2.2), we have

$$\left|\lambda_{ij}(R_\nu)\right| \leq \sum_{\mu=1}^{p} \left|w_\mu^{(i,j)}\right| \left|R_\nu(x_\mu^{(i)}, y_\mu^{(j)})\right|. \tag{3.11}$$

Without loss of generality, we consider $(\xi_0, \eta_0) = (0,0)$. Therefore, from (3.10), for $\mu = 1, \ldots, p$, we get

$$R_\nu(x_\mu^{(i)}, y_\mu^{(j)}) = \frac{1}{\nu!} \sum_{|\beta|=\nu} \binom{\nu}{\beta} \left[D^\beta f(u,v) - D^\beta f(0,0)\right] (x_\mu^{(i)})^{\beta_1}(y_\mu^{(j)})^{\beta_2}. \tag{3.12}$$

Since $(u,v)$ lies in the segment joining $(x_\mu^{(i)}, y_\mu^{(j)})$ to $(0,0)$, it is possible to find a real constant $\sigma_\mu^{(i,j)}$, depending on $\mu$, $i$ and $j$, such that

$$\left\|(x_\mu^{(i)}, y_\mu^{(j)}) - (0,0)\right\| \leq \sigma_\mu^{(i,j)} \frac{\Delta_{ij}}{2}. \tag{3.13}$$

Then, in (3.12),

$$\left|D^\beta f(u,v) - D^\beta f(0,0)\right| \leq \omega\left(D^\beta f, \sigma_\mu^{(i,j)} \frac{\Delta_{ij}}{2}, \Sigma_{rs}^{(k)}\right) \leq \lceil \sigma_\mu^{(i,j)} \rceil \omega\left(D^\beta f, \frac{\Delta_{ij}}{2}, \Sigma_{rs}^{(k)}\right), \tag{3.14}$$

where $\lceil z \rceil = \min\{\text{integers } i : \ i \geq z\}$, for all $z > 0$. Similarly, it is possible to find a real constant $\rho_\mu^{(i,j)}$, depending on $\mu$, $i$ and $j$, such that

$$\left(|x_\mu^{(i)}| + |y_\mu^{(j)}|\right) \leq \rho_\mu^{(i,j)} \Delta_{ij}. \tag{3.15}$$

Therefore, from (3.12), (3.14) and (3.15)

$$\begin{aligned}\left|R_\nu(x_\mu^{(i)},y_\mu^{(j)})\right| &\leq \tfrac{1}{\nu!}\left[|x_\mu^{(i)}|+|y_\mu^{(j)}|\right]^\nu \lceil\sigma_\mu^{(i,j)}\rceil\,\omega\left(D^\nu f,\tfrac{\Delta_{ij}}{2},\Sigma_{rs}^{(k)}\right)\\ &\leq \tfrac{1}{\nu!}\left(\rho_\mu^{(i,j)}\Delta_{ij}\right)^\nu \lceil\sigma_\mu^{(i,j)}\rceil\,\omega\left(D^\nu f,\tfrac{\Delta_{ij}}{2},\Sigma_{rs}^{(k)}\right).\end{aligned} \tag{3.16}$$

So, from (3.11) and (3.16), we get (3.7), with

$$C_\nu^{(i,j)} = \frac{1}{\nu!}\sum_{\mu=1}^{p}\left|w_\mu^{(i,j)}\right|\lceil\sigma_\mu^{(i,j)}\rceil\left(\rho_\mu^{(i,j)}\right)^\nu. \tag{3.17}$$

Finally, if, in addition, $f\in C^{\ell+1}(\Sigma_{rs}^{(k)})$, then, from (3.10)

$$R_\ell(x,y) = \frac{1}{(\ell+1)!}\sum_{|\beta|=\ell+1}\binom{\ell+1}{\beta}D^\beta f(\overline{u},\overline{v})(x-\xi_0)^{\beta_1}(y-\eta_0)^{\beta_2}, \tag{3.18}$$

with $(\overline{u},\overline{v})$ lying in the segment joining $(x,y)$ to $(\xi_0,\eta_0)$. Following the same logical scheme used in the first part of the proof and assuming $(\xi_0,\eta_0)=(0,0)$, from (3.18), for $\mu=1,\dots,p$

$$\begin{aligned}\left|R_\ell(x_\mu^{(i)},y_\mu^{(j)})\right| &\leq \tfrac{1}{(\ell+1)!}\left\|D^{\ell+1}f\right\|_{\Sigma_{rs}^{(k)}}\left[|x_\mu^{(i)}|+|y_\mu^{(j)}|\right]^{\ell+1}\\ &\leq \tfrac{1}{(\ell+1)!}\left\|D^{\ell+1}f\right\|_{\Sigma_{rs}^{(k)}}\left(\rho_\mu^{(i,j)}\Delta_{ij}\right)^{\ell+1}.\end{aligned} \tag{3.19}$$

Therefore, from (3.11) and (3.19), we obtain (3.8), with

$$C_{\ell+1}^{(i,j)} = \frac{1}{(\ell+1)!}\sum_{\mu=1}^{p}\left|w_\mu^{(i,j)}\right|\left(\rho_\mu^{(i,j)}\right)^{\ell+1}. \tag{3.20}$$

□

*Remark 3.1* More details about the computation of the constants $\sigma_\mu^{(i,j)}$ and $\rho_\mu^{(i,j)}$ will be given in Section 5, with reference to the three operators there considered.

**Theorem 3.1** *Let $f\in C^\nu(\Sigma_{rs}^{(k)})$, with $0\leq|\alpha|\leq\nu\leq\ell\leq 2$, $(\overline{x},\overline{y})\in T_{rs}^{(k)}$ for $|\alpha|=0,1$ and $(\overline{x},\overline{y})\in int(T_{rs}^{(k)})$ for $|\alpha|=2$. Then*

$$|D^\alpha R_\nu(\overline{x},\overline{y})|\leq\frac{1}{(\nu-|\alpha|)!}\left(\frac{\widehat{\Delta}_{rs}}{2}\right)^{\nu-|\alpha|}\omega\left(D^\nu f,\frac{\widehat{\Delta}_{rs}}{2},\Sigma_{rs}^{(k)}\right). \tag{3.21}$$

*If, in addition, $f\in C^{\ell+1}(\Sigma_{rs}^{(k)})$, then*

$$|D^\alpha R_\ell(\overline{x},\overline{y})|\leq\frac{1}{(\ell+1-|\alpha|)!}\left(\frac{\widehat{\Delta}_{rs}}{2}\right)^{\ell+1-|\alpha|}\left\|D^{\ell+1}f\right\|_{\Sigma_{rs}^{(k)}}. \tag{3.22}$$

*Proof* From (3.10), assuming $(\xi_0,\eta_0)=(0,0)$ and $(x,y)=(\bar{x},\bar{y})$, since $(u,v)$ lies in the segment joining $(0,0)$ to $(\bar{x},\bar{y})$ and $\|(\bar{x},\bar{y})-(0,0)\| \le \frac{\widehat{\Delta}_{rs}}{2}$, then

$$\begin{aligned}|D^\alpha R_\nu(\bar{x},\bar{y})| &\le \omega\left(D^\nu f, \tfrac{\widehat{\Delta}_{rs}}{2}, \Sigma_{rs}^{(k)}\right)\tfrac{1}{(\nu-|\alpha|)!}\left(|\bar{x}|+|\bar{y}|\right)^{\nu-|\alpha|}\\ &\le \omega\left(D^\nu f, \tfrac{\widehat{\Delta}_{rs}}{2}, \Sigma_{rs}^{(k)}\right)\tfrac{1}{(\nu-|\alpha|)!}\left(\tfrac{\widehat{\Delta}_{rs}}{2}\right)^{\nu-|\alpha|},\end{aligned}$$

that is (3.21). If, in addition, $f\in C^{\ell+1}(\Sigma_{rs}^{(k)})$, then, from (3.18), we get (3.22). □

**Theorem 3.2** *Let $f\in C^\nu(\Sigma_{rs}^{(k)})$, with $0\le|\alpha|\le\nu\le\ell\le 2$, $(\bar{x},\bar{y})\in T_{rs}^{(k)}$ for $|\alpha|=0,1$ and $(\bar{x},\bar{y})\in int(T_{rs}^{(k)})$ for $|\alpha|=2$. Then*

$$|D^\alpha QR_\nu(\bar{x},\bar{y})| \le \overline{C}_{|\alpha|,\nu}\overline{\Delta}_{rs}^{\nu-|\alpha|}\omega\left(D^\nu f, \frac{\overline{\Delta}_{rs}}{2}, \Sigma_{rs}^{(k)}\right), \tag{3.23}$$

*where*

$$\overline{C}_{|\alpha|,\nu} = K_{|\alpha|}C_\nu\left(\frac{\overline{\Delta}_{rs}}{\delta_{rs}}\right)^{|\alpha|}, \quad C_\nu = \max_{(i,j)\in I(T_{rs}^{(k)})}\left\{C_\nu^{(i,j)}\right\}, \tag{3.24}$$

*$C_\nu^{(i,j)}$ is given by (3.17) and $K_{|\alpha|}$ is defined as in Lemma 3.1. If, in addition, $f\in C^{\ell+1}(\Sigma_{rs}^{(k)})$, then*

$$|D^\alpha QR_\ell(\bar{x},\bar{y})| \le \overline{C}_{|\alpha|,\ell+1}\overline{\Delta}_{rs}^{\ell+1-|\alpha|}\left\|D^{\ell+1}f\right\|_{\Sigma_{rs}^{(k)}}, \tag{3.25}$$

*where*

$$\overline{C}_{|\alpha|,\ell+1} = K_{|\alpha|}C_{\ell+1}\left(\frac{\overline{\Delta}_{rs}}{\delta_{rs}}\right)^{|\alpha|}, \quad C_{\ell+1} = \max_{(i,j)\in I(T_{rs}^{(k)})}\left\{C_{\ell+1}^{(i,j)}\right\} \tag{3.26}$$

*and $C_{\ell+1}^{(i,j)}$ is given by (3.20).*

*Proof* The proof is a consequence of Lemmas 3.1 and 3.2. Indeed, from (2.3), (3.4) and (3.7), we get (3.23) and (3.24). If, in addition, $f\in C^{\ell+1}(\Sigma_{rs}^{(k)})$, then, from (2.3), (3.4) and (3.8), we get (3.25) and (3.26). □

**Theorem 3.3** *Let $f\in C^\nu(\Sigma_{rs}^{(k)})$ and $\dot{T}_{rs}^{(k)}=T_{rs}^{(k)}$ if $|\alpha|=0,1$ and $\dot{T}_{rs}^{(k)}=int(T_{rs}^{(k)})$ if $|\alpha|=2$, with $0\le|\alpha|\le\nu\le\ell\le 2$, then*

$$\|E_{\alpha,\nu}\|_{\dot{T}_{rs}^{(k)}} \le M_{|\alpha|,\nu}^Q\overline{\Delta}_{rs}^{\nu-|\alpha|}\omega\left(D^\nu f, \frac{\overline{\Delta}_{rs}}{2}, \Sigma_{rs}^{(k)}\right), \tag{3.27}$$

*with error constant*

$$M_{|\alpha|,\nu}^Q = \frac{1}{2^{\nu-|\alpha|}(\nu-|\alpha|)!}+\overline{C}_{|\alpha|,\nu} \tag{3.28}$$

*and $\overline{C}_{|\alpha|,\nu}$ given in (3.24). If, in addition, $f\in C^{\ell+1}(\Sigma_{rs}^{(k)})$, then*

$$\left\|E_{\alpha,\ell+1}\right\|_{\dot{T}_{rs}^{(k)}} \le M_{|\alpha|,\ell+1}^Q\overline{\Delta}_{rs}^{\ell+1-|\alpha|}\left\|D^{\ell+1}f\right\|_{\Sigma_{rs}^{(k)}}, \tag{3.29}$$

*with*

$$M^Q_{|\alpha|,\ell+1} = \frac{1}{2^{\ell+1-|\alpha|}(\ell+1-|\alpha|)!} + \overline{C}_{|\alpha|,\ell+1} \tag{3.30}$$

*and* $\overline{C}_{|\alpha|,\ell+1}$ *given in (3.26).*

*Proof* From (3.3), Theorem 3.1 and Theorem 3.2, since $\widehat{\Delta}_{rs} \le \overline{\Delta}_{rs}$, we get (3.27) and (3.29). □

*Remark 3.2* If we consider a triangular cell sufficiently far from the boundary of $\Omega$, the error constants $M^Q_{|\alpha|,\nu}$, $M^Q_{|\alpha|,\ell+1}$, $|\alpha| = 1,2$, are smaller. Indeed, in those cases, $K_1$ and $K_2$, given by (3.5) and (3.6), are equal to 2 and 6, instead of 4 and 12, respectively.

The local estimates lead immediately to the following global results for $|\alpha| = 0,1$.

**Theorem 3.4** *Let* $f \in C^\nu(\Omega)$, *with* $0 \le |\alpha| \le \nu \le \ell \le 2$, $|\alpha| = 0,1$, *then*

$$\|E_{\alpha,\nu}\|_\Omega \le \overline{M}^Q_{|\alpha|,\nu}\Delta^{\nu-|\alpha|}\omega\left(D^\nu f, \frac{\Delta}{2}, \Omega\right),$$

*with error constant* $\overline{M}^Q_{|\alpha|,\nu} = \frac{1}{2^{\nu-|\alpha|}(\nu-|\alpha|)!} + K_{|\alpha|}C_\nu\left(\frac{\Delta}{\delta}\right)^{|\alpha|}$, $K_{|\alpha|}$ *defined as in Lemma 3.1 and* $C_\nu$ *given in (3.24). If, in addition,* $f \in C^{\ell+1}(\Omega)$, *then*

$$\left\|E_{\alpha,\ell+1}\right\|_\Omega \le \overline{M}^Q_{|\alpha|,\ell+1}\Delta^{\ell+1-|\alpha|}\left\|D^{\ell+1} f\right\|_\Omega,$$

*with* $\overline{M}^Q_{|\alpha|,\ell+1} = \frac{1}{2^{\ell+1-|\alpha|}(\ell+1-|\alpha|)!} + K_{|\alpha|}C_{\ell+1}\left(\frac{\Delta}{\delta}\right)^{|\alpha|}$ *and* $C_{\ell+1}$ *given in (3.26).*

*Remark 3.3* In (3.27) and (3.29), the error constants for $|\alpha| = 0$ are independent of the mesh ratios $\left(\frac{\overline{\Delta}_{rs}}{\delta_{rs}}\right)$ and therefore, in $\dot{T}^{(k)}_{rs}$ we get $Qf \to f$ as $\overline{\Delta}_{rs} \to 0$.

In case $|\alpha| = 1,2$, the error constants depend on the above mesh ratios. When such ratios are bounded, from (3.27) and (3.29), we can conclude that in $\dot{T}^{(k)}_{rs}$

$$D^\alpha Qf \to D^\alpha f \quad \text{as} \quad \overline{\Delta}_{rs} \to 0. \tag{3.31}$$

For example, this condition occurs in case of uniform triangulation $\mathscr{T}_{mn}$. Moreover, if we assume that the sequence of partitions $\{X_m \times Y_n\}$ of $\Omega$ is $\gamma$-quasi uniform i.e. there exists a constant $\gamma > 1$ such that $0 < \Delta/\delta \le \gamma$, then (3.31) holds. From the local convergence properties, we immediately get global convergence results for $|\alpha| = 0,1$.

## 4 The case of uniform triangulations

If we consider the specific case of a uniform triangulation, for which $h_i = k_j = \Delta$, $1 \le i \le m$, $1 \le j \le n$, using a technique similar to the one proposed in [15], we get error constants that we expect to be substantially reduced, as shown in Section 5 for the QIs $S_1$, $S_2$ and $W_2$, considered in Section 2. Moreover their computation is easier, because, in this case, the BB-coefficients of the B-splines are independent of the triangulation.

First of all, we write $Qf$ in the "quasi-Lagrange" form. From (2.1), (2.2) and (2.3), if $\mathscr{D}_{mn} := \{(x^{(i)}, y^{(j)}),\ (i,j) \in \overline{\mathscr{K}}_{mn}\}$ is the set of all mesh-points used in the definition of $Qf$, for a proper set of indices $\overline{\mathscr{K}}_{mn}$, then

$$Q^{\alpha} f(x,y) = \sum_{(i,j)\in\overline{\mathscr{K}}_{mn}} f(x^{(i)}, y^{(j)}) D^{\alpha} L_{ij}(x,y),$$

where $\{L_{ij},\ (i,j) \in \overline{\mathscr{K}}_{mn}\}$ are the fundamental splines obtained, from (2.1) and (2.2), as linear combinations of B-splines $\{B_{ij},\ (i,j) \in \mathscr{K}_{mn}\}$ (see e.g. [15]). Also the fundamental splines have local support $S_{ij}$, obtained by the union of the B-spline supports involved in their definition. Given a point $(\bar{x},\bar{y}) \in T_{rs}^{(k)}$ only a finite number of fundamental splines are non zero at this point, whose indices belong to the set $J(T_{rs}^{(k)}) := \{(i,j) :\ S_{ij} \cap int(T_{rs}^{(k)}) \neq \emptyset\}$. Moreover, we define $S_{rs}^{(k)} = \bigcup_{(i,j)\in J(T_{rs}^{(k)})} S_{ij}$.

Now, if $(\bar{x},\bar{y}) \in \dot{T}_{rs}^{(k)}$ and we consider the Taylor expansion of $f$ at $(\bar{x},\bar{y})$, then $f(x,y) = \hat{q}_{\nu}(x,y) + \hat{R}_{\nu}(x,y)$, with remainder term

$$\hat{R}_{\nu}(x,y) = \frac{1}{\nu!} \sum_{|\beta|=\nu} \binom{\nu}{\beta} \left[ D^{\beta} f(u,v) - D^{\beta} f(\bar{x},\bar{y}) \right] (x-\bar{x})^{\beta_1} (y-\bar{y})^{\beta_2},$$

for $0 \leq \nu \leq \ell$ and $(u,v)$ lying in the segment joining $(x,y)$ to $(\bar{x},\bar{y})$. It is easy to verify that $\hat{R}_{\nu}$ and its derivatives are 0 at $(\bar{x},\bar{y})$. Since $Q$ is exact on $\mathbb{P}_l$, from (3.3) we obtain

$$|E_{\alpha,\nu}(\bar{x},\bar{y})| \leq \sum_{(i,j)\in J(T_{rs}^{(k)})} \left| \hat{R}_{\nu}(x^{(i)}, y^{(j)}) \right| \left| D^{\alpha} L_{ij}(\bar{x},\bar{y}) \right|. \tag{4.1}$$

Denoting $|x^{(i)} - \bar{x}| + |y^{(j)} - \bar{y}| = \Delta \tau_{ij}(\bar{x},\bar{y})$ and being $\theta_{i,j}$ a constant dependent on $i$ and $j$, such that $\left\| (x^{(i)}, y^{(j)}) - (\bar{x},\bar{y}) \right\| \leq \theta_{i,j} \frac{\Delta}{2}$, then, in (4.1)

$$\begin{aligned} \left| \hat{R}_{\nu}(x^{(i)}, y^{(j)}) \right| &\leq \tfrac{1}{\nu!}\, \omega\left( D^{\nu} f, \left\| (x^{(i)}, y^{(j)}) - (\bar{x},\bar{y}) \right\|, S_{rs}^{(k)} \right) \left( |x^{(i)} - \bar{x}| + |y^{(j)} - \bar{y}| \right)^{\nu} \\ &\leq \tfrac{1}{\nu!} \lceil \theta_{i,j} \rceil\, \omega\left( D^{\nu} f, \tfrac{\Delta}{2}, S_{rs}^{(k)} \right) \left( \Delta \tau_{ij}(\bar{x},\bar{y}) \right)^{\nu}. \end{aligned} \tag{4.2}$$

Consequently, from (4.1) and (4.2),

$$|E_{\alpha,\nu}(\bar{x},\bar{y})| \leq \omega\left( D^{\nu} f, \frac{\Delta}{2}, S_{rs}^{(k)} \right) \Delta^{\nu} \frac{1}{\nu!} \sum_{(i,j)\in J(T_{rs}^{(k)})} \lceil \theta_{i,j} \rceil \left( \tau_{ij}(\bar{x},\bar{y}) \right)^{\nu} \left| D^{\alpha} L_{ij}(\bar{x},\bar{y}) \right|. \tag{4.3}$$

We can bound the last term in (4.3) as follows

$$\frac{1}{\nu!} \sum_{(i,j)\in J(T_{rs}^{(k)})} \lceil \theta_{i,j} \rceil \left( \tau_{ij}(\bar{x},\bar{y}) \right)^{\nu} \left| D^{\alpha} L_{ij}(\bar{x},\bar{y}) \right| \leq \hat{M}^{Q}_{|\alpha|,\nu} \Delta^{-|\alpha|}, \tag{4.4}$$

where the constant $\hat{M}^{Q}_{|\alpha|,\nu}$ is dependent on $\alpha$ and $\nu$ and can be evaluated, for any $Q$, by a technique similar to the one used in [15, Theorem 1]. From (4.3) and (4.4), we obtain

$$\|E_{\alpha,\nu}\|_{\dot{T}_{rs}^{(k)}} \leq \hat{M}^{Q}_{|\alpha|,\nu} \Delta^{\nu-|\alpha|}\, \omega\left( D^{\nu} f, \frac{\Delta}{2}, S_{rs}^{(k)} \right). \tag{4.5}$$

Finally, if, in addition, $f \in C^{\ell+1}(S_{rs}^{(k)})$, then it is easy to deduce

$$\left\|E_{\alpha,\ell+1}\right\|_{\dot{T}_{rs}^{(k)}} \leq \hat{M}^{Q}_{|\alpha|,\ell+1} \Delta^{\ell+1-|\alpha|} \left\|D^{\ell+1} f\right\|_{S_{rs}^{(k)}}. \tag{4.6}$$

## 5 Error bounds for specific quasi-interpolants

In this section, we detail the general error constants of Sections 3 and 4 for the three specific operators introduced in Section 2.

### 5.1 The quasi-interpolant $S_1$

By Theorem 3.3, we obtain the following error constants for the operator $S_1$:

$$M_{0,0}^{S_1} = 4,\ M_{0,1}^{S_1} = 5,\ M_{1,1}^{S_1} = \left[1 + 18\left(\frac{\overline{\Delta}_{rs}}{\delta_{rs}}\right)\right], \tag{5.1}$$

$$M_{0,2}^{S_1} = \frac{5}{4},\ M_{1,2}^{S_1} = \left[\frac{1}{2} + \frac{9}{2}\left(\frac{\overline{\Delta}_{rs}}{\delta_{rs}}\right)\right],\ M_{2,2}^{S_1} = \left[1 + \frac{27}{2}\left(\frac{\overline{\Delta}_{rs}}{\delta_{rs}}\right)^2\right]. \tag{5.2}$$

Indeed, if for the sake of simplicity, we consider $k = 3$, i.e. the triangle $T_{rs}^{(3)}$, from (3.28), (3.24) and (3.17), we notice that the only value related to the particular choice of the QI operator is $C_\nu$. Moreover, it is easy to verify that

$$C_\nu = \max_{(i,j)\in I(T_{rs}^{(3)})} \left\{C_\nu^{(i,j)}\right\} = C_\nu^{(r,s+1)} = \frac{1}{\nu!}\lceil \sigma_1^{(r,s+1)} \rceil \left(\rho_1^{(r,s+1)}\right)^\nu.$$

Therefore, we have to compute $\sigma_1^{(r,s+1)}$ and $\rho_1^{(r,s+1)}$.

Since, from (3.13), the value $\sigma_1^{(r,s+1)}$ is such that $\left\|(x_1^{(r)}, y_1^{(s+1)}) - (0,0)\right\| \leq \sigma_1^{(r,s+1)} \frac{\Delta_{r,s+1}}{2}$, after some algebra, we get $\sigma_1^{(r,s+1)} = 3$. Similarly, from (3.15), since the value $\rho_1^{(r,s+1)}$ is such that $\left(|x_1^{(r)}| + |y_1^{(s+1)}|\right) \leq \rho_1^{(r,s+1)} \Delta_{r,s+1}$, after some algebra, we get $\rho_1^{(r,s+1)} = \frac{3}{2}$.

Therefore $C_\nu = \frac{3}{\nu!}\left(\frac{3}{2}\right)^\nu$ and, from (3.28),

$$M_{|\alpha|,\nu}^{S_1} = \frac{1}{2^{\nu-|\alpha|}(\nu-|\alpha|)!} + K_{|\alpha|}\frac{3}{\nu!}\left(\frac{3}{2}\right)^\nu \left(\frac{\overline{\Delta}_{rs}}{\delta_{rs}}\right)^{|\alpha|}, \tag{5.3}$$

where $K_{|\alpha|}$ is defined in Lemma 3.1. From (5.3), immediately we obtain (5.1).

If in addition $f \in C^2(\Sigma_{rs}^{(k)})$, then, following the same logical scheme, from (3.30), (3.26) and (3.20), we get $C_2 = C_2^{(r,s+1)} = \frac{1}{2}\left(\frac{3}{2}\right)^2$ and

$$M_{|\alpha|,2}^{S_1} = \frac{1}{2^{2-|\alpha|}(2-|\alpha|)!} + K_{|\alpha|}\frac{1}{2}\left(\frac{3}{2}\right)^2 \left(\frac{\overline{\Delta}_{rs}}{\delta_{rs}}\right)^{|\alpha|}. \tag{5.4}$$

From (5.4), immediately we obtain (5.2).

We reach the same results if we consider another triangle $T_{rs}^{(k)}$.

As noticed in Remark 3.2, in triangles $T_{rs}^{(k)}$ lying sufficiently far from the boundary of $\Omega$, the error constants for the first and second derivatives, in (5.1) and (5.2), are smaller and become

$$M_{1,1}^{S_1} = \left[1+9\left(\frac{\overline{\Delta}_{rs}}{\delta_{rs}}\right)\right],\ M_{1,2}^{S_1} = \left[\frac{1}{2}+\frac{9}{4}\left(\frac{\overline{\Delta}_{rs}}{\delta_{rs}}\right)\right],\ M_{2,2}^{S_1} = \left[1+\frac{27}{4}\left(\frac{\overline{\Delta}_{rs}}{\delta_{rs}}\right)^2\right]. \qquad (5.5)$$

For uniform meshes and $T_{rs}^{(k)}$ sufficiently far from the boundary of $\Omega$, from (4.5) and (4.6), we get:

$$\hat{M}_{0,0}^{S_1} = 2.7,\ \hat{M}_{0,1}^{S_1} = 2.5,\ \hat{M}_{0,2}^{S_1} = 0.5,\ \hat{M}_{1,1}^{S_1} = 6.3,\ \hat{M}_{1,2}^{S_1} = 1.3,\ \hat{M}_{2,2}^{S_1} = 5.$$

We remark that these constants are smaller than those we could get by (5.1), (5.2) and (5.5), assuming uniform meshes.

### 5.2 The quasi-interpolant $S_2$

By Theorem 3.3, we provide the following error constants for $S_2$:

$$\begin{aligned} &M_{0,0}^{S_2} = 17,\ M_{0,1}^{S_2} = \frac{61}{2},\ M_{0,2}^{S_2} = \frac{245}{8},\\ &M_{1,1}^{S_2} = \left[1+120\left(\frac{\overline{\Delta}_{rs}}{\delta_{rs}}\right)\right],\ M_{1,2}^{S_2} = \left[\frac{1}{2}+122\left(\frac{\overline{\Delta}_{rs}}{\delta_{rs}}\right)\right],\ M_{2,2}^{S_2} = \left[1+366\left(\frac{\overline{\Delta}_{rs}}{\delta_{rs}}\right)^2\right], \end{aligned} \qquad (5.6)$$

$$M_{0,3}^{S_2} = \frac{45}{8},\ M_{1,3}^{S_2} = \left[\frac{1}{8}+\frac{269}{12}\left(\frac{\overline{\Delta}_{rs}}{\delta_{rs}}\right)\right],\ M_{2,3}^{S_2} = \left[\frac{1}{2}+\frac{269}{4}\left(\frac{\overline{\Delta}_{rs}}{\delta_{rs}}\right)^2\right]. \qquad (5.7)$$

Indeed, by the same technique used for the operator $S_1$, we consider the triangle $T_{rs}^{(3)}$, but in this case, we have

$$C_\nu = \max_{(i,j)\in I(T_{rs}^{(3)})} \left\{C_\nu^{(i,j)}\right\} = C_\nu^{(r,s+1)} = \frac{1}{\nu!}\sum_{\mu=1}^{5} |w_\mu^{(i,j)}| \lceil \sigma_\mu^{(r,s+1)} \rceil \left(\rho_\mu^{(r,s+1)}\right)^\nu.$$

After some algebra, taking also into account (2.6), we compute the values $\sigma_\mu^{(r,s+1)}$, $\rho_\mu^{(r,s+1)}$, $\mu = 1,\dots 5$, obtaining

$$C_\nu \le \frac{1}{\nu!}\left[3\cdot 3\left(\frac{3}{2}\right)^\nu + \frac{1}{2}\left(5\left(\frac{5}{2}\right)^\nu + 4\left(\frac{5}{2}\right)^\nu + \left(\frac{1}{2}\right)^\nu + 4\left(\frac{5}{2}\right)^\nu\right)\right]$$

and

$$M_{|\alpha|,\nu}^{S_2} = \frac{1}{2^{\nu-|\alpha|}(\nu-|\alpha|)!} + K_{|\alpha|} C_\nu \left(\frac{\overline{\Delta}_{rs}}{\delta_{rs}}\right)^{|\alpha|}, \qquad (5.8)$$

where $K_{|\alpha|}$ is defined in Lemma 3.1. From (5.8), immediately we get (5.6).

If in addition $f \in C^3(\Sigma_{rs}^{(k)})$, then we get

$$C_3 = C_3^{(r,s+1)} = \frac{1}{3!}\left[3\left(\frac{3}{2}\right)^3 + \frac{1}{2}\left(\left(\frac{5}{2}\right)^3 + \left(\frac{5}{2}\right)^3 + \left(\frac{1}{2}\right)^3 + \left(\frac{5}{2}\right)^3\right)\right]$$

and

$$M_{|\alpha|,3}^{S_2} = \frac{1}{2^{3-|\alpha|}(3-|\alpha|)!} + K_{|\alpha|} C_3 \left(\frac{\overline{\Delta}_{rs}}{\delta_{rs}}\right)^{|\alpha|}. \tag{5.9}$$

From (5.9), immediately we obtain (5.7).

As noticed in Remark 3.2, in triangles $T_{rs}^{(k)}$ lying sufficiently far from the boundary of $\Omega$, the error constants for the first and second derivatives in (5.6) and (5.7) are smaller and become

$$\begin{aligned} &M_{1,1}^{S_2} = \left[1 + 60\left(\tfrac{\overline{\Delta}_{rs}}{\delta_{rs}}\right)\right],\ M_{1,2}^{S_2} = \left[\tfrac{1}{2} + 61\left(\tfrac{\overline{\Delta}_{rs}}{\delta_{rs}}\right)\right],\ M_{1,3}^{S_2} = \left[\tfrac{1}{8} + \tfrac{269}{24}\left(\tfrac{\overline{\Delta}_{rs}}{\delta_{rs}}\right)\right], \\ &M_{2,2}^{S_2} = \left[1 + 183\left(\tfrac{\overline{\Delta}_{rs}}{\delta_{rs}}\right)^2\right],\ M_{2,3}^{S_2} = \left[\tfrac{1}{2} + \tfrac{269}{8}\left(\tfrac{\overline{\Delta}_{rs}}{\delta_{rs}}\right)^2\right]. \end{aligned} \tag{5.10}$$

For uniform meshes and $T_{rs}^{(k)}$ sufficiently far from the boundary of $\Omega$, from (4.5) and (4.6), we get:

$$\begin{aligned} &\hat{M}_{0,0}^{S_2} = 4.4,\ \hat{M}_{0,1}^{S_2} = 5.4,\ \hat{M}_{0,2}^{S_2} = 3.8,\ \hat{M}_{0,3}^{S_2} = 0.55, \\ &\hat{M}_{1,1}^{S_2} = 14.1,\ \hat{M}_{1,2}^{S_2} = 11.4,\ \hat{M}_{1,3}^{S_2} = 1.7,\ \hat{M}_{2,2}^{S_2} = 40.7,\ \hat{M}_{2,3}^{S_2} = 6.7. \end{aligned} \tag{5.11}$$

We notice that, also in this case, the constants in (5.11) are smaller than those we could get by (5.6), (5.7) and (5.10), assuming uniform meshes. Moreover, in the case $f \in C^3(\Omega)$, we get the same error constants given in [15].

## 5.3 The quasi-interpolant $W_2$

By Theorem 3.3, we provide the following error constants for $W_2$:

$$\begin{aligned} &M_{0,0}^{W_2} = 11,\ M_{0,1}^{W_2} = 18,\ M_{0,2}^{W_2} = \tfrac{131}{8}, \\ &M_{1,1}^{W_2} = \left[1 + 70\left(\tfrac{\overline{\Delta}_{rs}}{\delta_{rs}}\right)\right],\ M_{1,2}^{W_2} = \left[\tfrac{1}{2} + 65\left(\tfrac{\overline{\Delta}_{rs}}{\delta_{rs}}\right)\right],\ M_{2,2}^{W_2} = \left[1 + 195\left(\tfrac{\overline{\Delta}_{rs}}{\delta_{rs}}\right)^2\right], \end{aligned} \tag{5.12}$$

$$M_{0,3}^{W_2} = \frac{131}{48},\ M_{1,3}^{W_2} = \left[\frac{1}{8} + \frac{65}{6}\left(\frac{\overline{\Delta}_{rs}}{\delta_{rs}}\right)\right],\ M_{2,3}^{W_2} = \left[\frac{1}{2} + \frac{65}{2}\left(\frac{\overline{\Delta}_{rs}}{\delta_{rs}}\right)^2\right]. \tag{5.13}$$

Indeed, by the same method used for $S_1$ and $S_2$, we consider the triangle $T_{rs}^{(3)}$ and, in this case, we have

$$C_\nu = \max_{(i,j)\in I(T_{rs}^{(3)})} \left\{C_\nu^{(i,j)}\right\} = C_\nu^{(r,s+1)} = \frac{1}{\nu!}\sum_{\mu=1}^{5} |w_\mu^{(i,j)}| \lceil \sigma_\mu^{(r,s+1)} \rceil \left(\rho_\mu^{(r,s+1)}\right)^\nu.$$

After some algebra, we compute the values $\sigma_\mu^{(r,s+1)}$, $\rho_\mu^{(r,s+1)}$, $\mu = 1,\dots 5$, obtaining

$$C_\nu = \frac{1}{\nu!}\left[2\cdot 3\left(\frac{3}{2}\right)^\nu + \frac{1}{4}\left(5\left(\frac{5}{2}\right)^\nu + 5\left(\frac{5}{2}\right)^\nu + 3\left(\frac{3}{2}\right)^\nu + 3\left(\frac{3}{2}\right)^\nu\right)\right]$$

and

$$M_{|\alpha|,\nu}^{W_2} = \frac{1}{2^{\nu-|\alpha|}(\nu-|\alpha|)!} + K_{|\alpha|}C_\nu\left(\frac{\overline{\Delta}_{rs}}{\delta_{rs}}\right)^{|\alpha|}, \tag{5.14}$$

where $K_{|\alpha|}$ is defined in Lemma 3.1. From (5.14), immediately we get (5.12).

If in addition $f \in C^3(\Sigma_{rs}^{(k)})$, then we get

$$C_3 = C_3^{(r,s+1)} = \frac{1}{3!}\left[2\left(\frac{3}{2}\right)^3 + \frac{1}{4}\left(\left(\frac{5}{2}\right)^3 + \left(\frac{5}{2}\right)^3 + \left(\frac{3}{2}\right)^3 + \left(\frac{3}{2}\right)^3\right)\right]$$

and

$$M_{|\alpha|,3}^{W_2} = \frac{1}{2^{3-|\alpha|}(3-|\alpha|)!} + K_{|\alpha|}C_3\left(\frac{\overline{\Delta}_{rs}}{\delta_{rs}}\right)^{|\alpha|}. \tag{5.15}$$

From (5.15), immediately we obtain (5.13).

As noticed in Remark 3.2, in triangles $T_{rs}^{(k)}$ lying sufficiently far from the boundary of $\Omega$, the error constants for the first and second derivatives in (5.12) and (5.13) are smaller and become

$$\begin{aligned}
&M_{1,1}^{W_2} = \left[1+35\left(\tfrac{\overline{\Delta}_{rs}}{\delta_{rs}}\right)\right],\ M_{1,2}^{W_2} = \left[\tfrac{1}{2}+\tfrac{65}{2}\left(\tfrac{\overline{\Delta}_{rs}}{\delta_{rs}}\right)\right],\ M_{1,3}^{W_2} = \left[\tfrac{1}{8}+\tfrac{65}{12}\left(\tfrac{\overline{\Delta}_{rs}}{\delta_{rs}}\right)\right],\\
&M_{2,2}^{W_2} = \left[1+\tfrac{195}{2}\left(\tfrac{\overline{\Delta}_{rs}}{\delta_{rs}}\right)^2\right],\ M_{2,3}^{W_2} = \left[\tfrac{1}{2}+\tfrac{65}{4}\left(\tfrac{\overline{\Delta}_{rs}}{\delta_{rs}}\right)^2\right].
\end{aligned} \tag{5.16}$$

For uniform meshes and $T_{rs}^{(k)}$ sufficiently far from the boundary of $\Omega$, from (4.5) and (4.6), we get:

$$\begin{aligned}
&\hat{M}_{0,0}^{W_2} = 8.3,\ \hat{M}_{0,1}^{W_2} = 8.7,\ \hat{M}_{0,2}^{W_2} = 5.5,\ \hat{M}_{0,3}^{W_2} = 0.79,\\
&\hat{M}_{1,1}^{W_2} = 19.5,\ \hat{M}_{1,2}^{W_2} = 15.1,\ \hat{M}_{1,3}^{W_2} = 2.3,\ \hat{M}_{2,2}^{W_2} = 55.5,\ \hat{M}_{2,3}^{W_2} = 9.4.
\end{aligned} \tag{5.17}$$

We notice that the constants in (5.17) are smaller than those we could get by (5.12), (5.13) and (5.16), assuming uniform meshes.

## 6 Some examples and applications

In this section we present some examples and applications, developed in Matlab, related to the QIs $S_1$, $S_2$, $W_2$ and comparisons with other quadratic $C^1$ spline QIs on criss-cross triangulations, proposed in the literature.

### 6.1 Quasi-interpolant error constants for functions and partial derivatives

We compare the optimal QIs $S_2$ and $W_2$ with other optimal ones defined in [29] and [2], on uniform criss-cross triangulations with $h_i = k_j = \Delta$, $\forall i, j$.

In [29], the authors introduce an optimal quadratic $C^1$ quasi-interpolating spline $\mathcal{Q}f$, on a uniform criss-cross triangulation of a rectangular domain $\Omega$, directly determined by setting the BB-coefficients of the spline to appropriate combinations of the given data values. In case of $f \in C^3(\Omega)$, they provide error bounds given by

$$\|D^\alpha(f - \mathcal{Q}f)\|_{T_{rs}^{(k)}} \le M_{|\alpha|,3}^{\mathcal{Q}} \left\|D^3 f\right\|_\Omega \Delta^{3-|\alpha|}, \quad |\alpha| = 0,1,2.$$

In the first four columns of Table 6.1 we compare the constants $M_{|\alpha|,3}^{\mathcal{Q}}$, $|\alpha| = 0,1,2$ with the corresponding ones for $S_2$ and $W_2$, obtained in (5.7) and (5.13), assuming uniform triangulations. In [29], the authors also remark that if $T_{rs}^{(k)}$ is sufficiently far from the boundary of $\Omega$, the constants $M_{|\alpha|,3}^{\mathcal{Q}}$ are smaller. In such a case, the comparisons with the error constants of $S_2$ and $W_2$, given in (5.11) and (5.17), respectively, are reported in the last four columns of Table 6.1. We can notice that $M_{|\alpha|,3}^{\mathcal{Q}}$ are always bigger than our constants $\hat{M}_{|\alpha|,3}^{S_2}$ and $\hat{M}_{|\alpha|,3}^{W_2}$, $|\alpha| = 0,1,2$. However, in [29], the authors were not interested in obtaining good constants in the estimates of the error but in defining directly a quasi-interpolant by setting its BB-coefficients.

| Case of an arbitrary triangular cell $T_{rs}^{(k)}$ | | | | Case of a triangular cell $T_{rs}^{(k)}$ sufficiently far from the boundary of $\Omega$ | | | |
|---|---|---|---|---|---|---|---|
| | $\|\alpha\| = 0$ | $\|\alpha\| = 1$ | $\|\alpha\| = 2$ | | $\|\alpha\| = 0$ | $\|\alpha\| = 1$ | $\|\alpha\| = 2$ |
| $M_{\|\alpha\|,3}^{S_2}$ | 5.625 | 22.54 | 67.75 | $\hat{M}_{\|\alpha\|,3}^{S_2}$ | 0.55 | 1.7 | 6.7 |
| $M_{\|\alpha\|,3}^{W_2}$ | 2.73 | 10.96 | 33 | $\hat{M}_{\|\alpha\|,3}^{W_2}$ | 0.79 | 2.3 | 9.4 |
| $M_{\|\alpha\|,3}^{\mathcal{Q}}$ [29] | 18 | 274.5 | 867 | $M_{\|\alpha\|,3}^{\mathcal{Q}}$ [29] | 5.33 | 82 | 258 |
| | | | | $M_{\|\alpha\|,3}^{Q_0}$ [2] | 0.40 | 0.96 | 2.8 |
| | | | | $M_{\|\alpha\|,3}^{Q_{-1/16}}$ [2] | 0.40 | 0.93 | 2.7 |

**Table 6.1** Error constants for some optimal QIs.

In [2] the authors construct a class of optimal QIs based on bivariate quadratic $C^1$ B-splines on uniform criss-cross triangulations of the plane $\mathbb{R}^2$. The coefficient functionals are obtained by imposing the exactness on $\mathbb{P}_2$ and minimizing a constant appearing in the leading term of an appropriate error estimate. They also propose error bounds for two particular operators, denoted by $Q_0$ and $Q_{-1/16}$, in case of functions $f \in C^3(\mathbb{R}^2)$,

$$\left\|D^\alpha(f - Q_\gamma f)\right\|_{T_{rs}^{(k)}} \le M_{|\alpha|,3}^{Q_\gamma} \left\|D^3 f\right\|_{\Omega_{T_{rs}^{(k)},\gamma}} \Delta^{3-|\alpha|}, \quad |\alpha| = 0,1,2, \quad \gamma = 0, -\frac{1}{16},$$

where $\Omega_{T_{rs}^{(k)},\gamma}$ is an appropriate neighbourhood of $T_{rs}^{(k)}$. We remark that $Q_0$ is the same operator given in [15,23] and [17, Chap. 3]. The error constants for $Q_0$ and $Q_{-1/16}$

are reported in the last four columns of Table 6.1 and they are comparable with those we have obtained for $S_2$ and $W_2$. However, as noticed in [2], since $Q_0$ and $Q_{-1/16}$ are defined on $\mathbb{R}^2$, then, if we consider a bounded region $\Omega$, they require function values outside $\Omega$.

### 6.2 Estimate for the maximum steplength $\Delta$ such that $\|f-Qf\|_\Omega$ is less than a fixed tolerance $\varepsilon$

From the error estimates given in Theorem 3.4 and from (5.2), (5.7), (5.13), one can compute $\Delta$ in order to have $\|f-Qf\|_\Omega$ less than a given tolerance $\varepsilon$. Indeed, if we consider the function error estimates, from Theorem 3.4, with $\alpha=(0,0)$, we have

$$\|f-S_1 f\|_\Omega \le \tfrac{5}{4}\Delta^2 \left\|D^2 f\right\|_\Omega, \quad \|f-S_2 f\|_\Omega \le \tfrac{45}{8}\Delta^3 \left\|D^3 f\right\|_\Omega,$$
$$\|f-W_2 f\|_\Omega \le \tfrac{131}{48}\Delta^3 \left\|D^3 f\right\|_\Omega.$$

Then, if we choose

$$\Delta < \sqrt{\frac{4\varepsilon}{5\,\|D^2 f\|_\Omega}} \text{ for } S_1,\ \Delta < \sqrt[3]{\frac{8\varepsilon}{45\,\|D^3 f\|_\Omega}} \text{ for } S_2,\ \Delta < \sqrt[3]{\frac{48\varepsilon}{131\,\|D^3 f\|_\Omega}} \text{ for } W_2, \tag{6.1}$$

we are sure that $\|f-Qf\|_\Omega \le \varepsilon$.

For example, if we consider the test function (see Fig. 6.1$(a)$)

$$f_1 = \tfrac{1}{3}\exp\left(-\tfrac{81}{16}\left(\left(x-\tfrac{1}{2}\right)^2+\left(y-\tfrac{1}{2}\right)^2\right)\right)$$

on $\Omega=[0,1]^2$ and we assume $\varepsilon = 5\cdot 10^{-3}$, then, from (6.1), we get $\Delta < 5.2\cdot 10^{-2}$ for $S_1$, $\Delta < 3.9\cdot 10^{-2}$ for $S_2$ and $\Delta < 5.0\cdot 10^{-2}$ for $W_2$.

In Fig. 6.1 we report the graphs of $f_1$, $W_2 f_1$ and $|f_1 - W_2 f_1|$, computed on a $300\times 300$ uniform rectangular grid $G$ of evaluation points in $\Omega$, considering a uniform triangulation with $m=n=21$ (this choice ensures $\Delta < 5.0\cdot 10^{-2}$). We remark that $\max_{(u,v)\in G}|(f_1-W_2 f_1)(u,v)| \le 2.7\cdot 10^{-5}$.

The above procedure can be usefully applied to get error bounds in numerical evaluation of 2D integrals by quadrature rules based on bivariate spline quasi-interpolation.

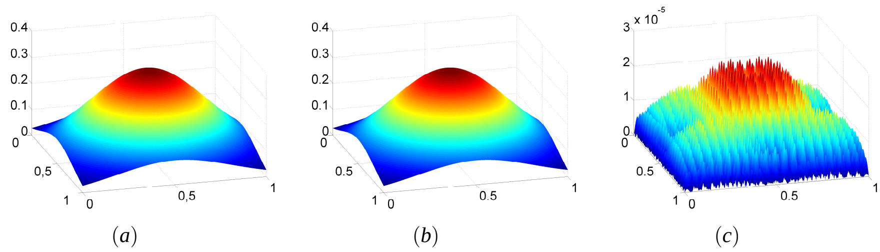


**Fig. 6.1** The graphs of $(a)$ $f_1$, $(b)$ $W_2 f_1$ and $(c)$ $|f_1 - W_2 f_1|$.

### 6.3 Approximation of functions and first partial derivatives

In this section we propose some numerical examples, where we take the first partial derivatives of $S_1 f$, $S_2 f$ and $W_2 f$ as approximations to those of the function $f$. We consider the two test functions

$$f_2(x,y) = 3(1-x)^2 \exp(-x^2-(y+1)^2) - 10\left(\frac{x}{5} - x^3 - y^5\right)\exp(-x^2-y^2) \\ -\frac{1}{3}\exp(-(x+1)^2-y^2)$$

and $f_3(x,y) = (xy)^{\frac{5}{3}} + \sin(xy)$, defined on the square domains $\Omega = [-4,4]^2$ and $\Omega = [-1,1]^2$, respectively. We notice that $f_3 \in C^1(\Omega)$. For a given function $f$ and for $Q = S_1, S_2, W_2$, we define

$$f\_\text{error} = \max_{(u,v)\in G} |(f-Qf)(u,v)|, \quad D^1 f\_\text{error} = \max_{|\alpha|=1}\{D^{(\alpha_1,\alpha_2)} f\_\text{error}\}, \tag{6.2}$$

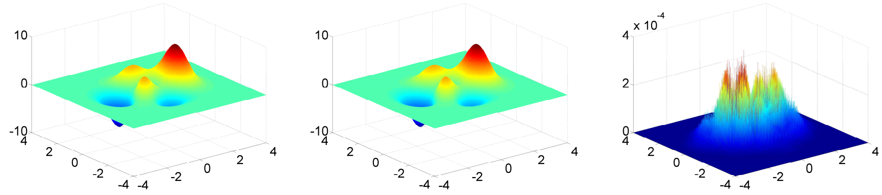
with $D^{(\alpha_1,\alpha_2)} f\_\text{error}=\max_{(u,v)\in G} |D^{(\alpha_1,\alpha_2)}(f-Qf)(u,v)|$, $|\alpha|=1$, where $G$ is a $300\times 300$ uniform rectangular grid of evaluation points in $\Omega$. We remark that the QIs $S_1$ and $S_2$ are based on $N_S$, given in (2.5), data sites, while $W_2$ is based on $N_W > N_S$, given in (2.7), data sites.

In Table 6.2, for increasing $m$ and $n$, we report the values (6.2), for $f = f_2$, considering uniform criss-cross triangulations and comparing the performances of the operators $S_1$, $S_2$ and $W_2$ with those of $Q_0$ and $Q_{-1/16}$, proposed in [2]. We can notice that the results related to the optimal operators $S_2$, $W_2$, $Q_0$ and $Q_{-1/16}$ are comparable. However, we remark that $Q_0$ and $Q_{-1/16}$ need evaluation points outside $\Omega$. In Fig. 6.2 we report the graphs of $f_2$, $S_2 f_2$ and $|f_2 - S_2 f_2|$ computed on the grid $G$, considering $m = n = 128$. In Figs. 6.3-6.4 we report the graphs of $D^{(1,0)} f_2$, $D^{(1,0)} S_2 f_2$, $\left|D^{(1,0)}(f_2 - S_2 f_2)\right|$ and $D^{(0,1)} f_2$, $D^{(0,1)} S_2 f_2$, $\left|D^{(0,1)}(f_2 - S_2 f_2)\right|$, respectively, computed on $G$, considering $m = n = 128$.

| | $S_1$ | $S_2$ | $W_2$ | $Q_0$ [2] | $Q_{-1/16}$ [2] |
|---|---|---|---|---|---|
| $m=n$ | $f_2$_error | | | | |
| 32 | 3.8(-1) | 5.6(-2) | 4.5(-2) | - | - |
| 64 | 9.7(-2) | 4.4(-3) | 3.7(-3) | 4.1(-3) | 4.8(-3) |
| 128 | 2.5(-2) | 3.8(-4) | 3.5(-4) | 3.8(-4) | 4.2(-4) |
| 256 | 6.1(-3) | 3.9(-5) | 3.8(-5) | 3.9(-5) | 4.2(-5) |
| $m=n$ | $D^1 f_2$_error | | | | |
| 32 | 1.3(0) | 4.5(-1) | 4.2(-1) | - | - |
| 64 | 3.4(-1) | 9.8(-2) | 1.0(-1) | 1.2(-1) | 1.2(-1) |
| 128 | 8.3(-2) | 2.6(-2) | 2.6(-2) | 3.0(-2) | 3.0(-2) |
| 256 | 2.2(-2) | 6.6(-3) | 6.7(-3) | 8.4(-3) | 8.4(-3) |

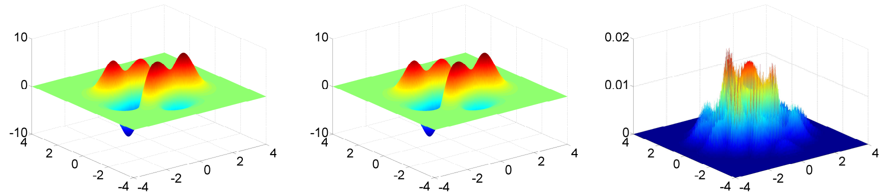
**Table 6.2** $f_2$_error and $D^1 f_2$_error.

Now, we consider two kinds of non-uniform triangulations. In order to construct them, we consider the following univariate non-uniform partitions of an arbitrary interval $[a,b]$, $X_m = \{x_i,\ i = 0,\dots,m\}$ and $\overline{X}_m = \{\overline{x}_i,\ i = 0,\dots,m\}$ (see e.g. [3]), where,



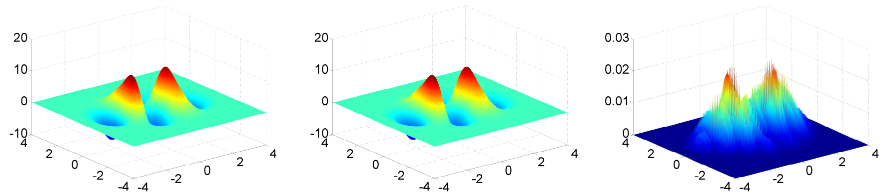
(a) (b) (c)

**Fig. 6.2** The graphs of $(a)$ $f_2$, $(b)$ $S_2 f_2$ and $(c)$ $|f_2 - S_2 f_2|$.



(a) (b) (c)

**Fig. 6.3** The graphs of $(a)$ $D^{(1,0)} f_2$, $(b)$ $D^{(1,0)} S_2 f_2$ and $(c)$ $\left|D^{(1,0)}(f_2 - S_2 f_2)\right|$.



(a) (b) (c)

**Fig. 6.4** The graphs of $(a)$ $D^{(0,1)} f_2$, $(b)$ $D^{(0,1)} S_2 f_2$ and $(c)$ $\left|D^{(0,1)}(f_2 - S_2 f_2)\right|$.

for $m$ even

$$\begin{aligned} & x_0 = a, \quad x_i = a + \left(\frac{\ln\left(1+\frac{i}{q}\right)}{\ln(2)}\right)\frac{b-a}{2}, \; i = 1,\ldots,q-1, \quad x_q = \frac{a+b}{2}, \\ & x_{i+q} = b - \left(\frac{\ln\left(1+\frac{q-i}{q}\right)}{\ln(2)}\right)\frac{b-a}{2}, \quad i = 1,\ldots,q-1, \; x_m = b, \end{aligned} \tag{6.3}$$

and

$$\begin{aligned} & \bar{x}_0 = a, \quad \bar{x}_i = \frac{a+b}{2} - \left(\frac{q-i}{q}\right)^2 \frac{b-a}{2}, \; i = 1,\ldots,q-1, \quad \bar{x}_q = \frac{a+b}{2}, \\ & \bar{x}_{i+q} = \frac{a+b}{2} + \left(\frac{i}{q}\right)^2 \frac{b-a}{2}, \; i = 1,\ldots,q-1, \quad \bar{x}_m = b, \end{aligned} \tag{6.4}$$

with $q = \frac{m}{2}$ and knots thickening around the midpoint $\frac{a+b}{2}$ (similarly for $m$ odd). It is easy to show that the sequence of partitions $\{X_m\}$ is $\gamma$-quasi uniform, with $\gamma = 2$ and the sequence $\{\bar{X}_m\}$ is locally uniform with constant $A = 3$. We recall that a sequence of univariate partitions $\{\bar{X}_m\}$ is locally uniform if there exists a constant $A \geq 1$ such that $\frac{1}{A} \leq \frac{\bar{x}_{i+1}-\bar{x}_i}{\bar{x}_{j+1}-\bar{x}_j} \leq A$, for all $i$ and $j = i \pm 1$. Similarly, we construct the partitions $Y_n$ and $\bar{Y}_n$ of $[c,d]$, by using the same scheme given in (6.3) and (6.4), respectively. We consider the corresponding criss-cross triangulations $\mathscr{T}_{mn}$ and $\bar{\mathscr{T}}_{mn}$, based on $X_m$, $Y_n$ and $\bar{X}_m$, $\bar{Y}_n$, respectively.

In Table 6.3 we compute (6.2), for $f = f_3$ and $Q = S_1$, $S_2$, $W_2$, defined on $\mathscr{T}_{mn}$ and $\bar{\mathscr{T}}_{mn}$, for increasing values of $m$ and $n$, with $m = n$. For the above operators, in case of a sequence of partitions $\{X_m \times Y_n\}$, thanks to Remark 3.3, the convergence of $\{D^\alpha Qf\}$ to $D^\alpha f$, for $|\alpha| = 1$ is guaranteed when $m,n \to \infty$. For the second sequence of partitions $\{\bar{X}_m \times \bar{Y}_n\}$ we have only numerical evidence for the convergence of $\{D^\alpha Qf\}$ to $D^\alpha f$, for $|\alpha| = 1$, when $m,n \to \infty$. We can notice that the use of the non-uniform triangulation $\bar{\mathscr{T}}_{mn}$ allows to get better results.

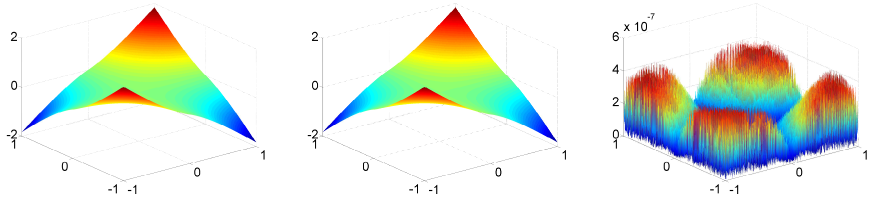
| | $S_1$ | $S_2$ | $W_2$ | $S_1$ | $S_2$ | $W_2$ |
|---|---|---|---|---|---|---|
| | non-uniform triangulations $\mathscr{T}_{mn}$ | | | non-uniform triangulations $\bar{\mathscr{T}}_{mn}$ | | |
| $m = n$ | $f_3$_error | | | | | |
| 4 | 3.3(-2) | 9.0(-3) | 6.6(-3) | 4.2(-2) | 1.4(-2) | 1.1(-2) |
| 8 | 1.0(-2) | 2.4(-3) | 1.4(-3) | 1.8(-2) | 2.4(-3) | 2.6(-3) |
| 16 | 2.8(-3) | 6.9(-4) | 3.6(-4) | 5.5(-3) | 3.4(-4) | 3.4(-4) |
| 32 | 8.4(-4) | 2.1(-4) | 1.0(-4) | 1.4(-3) | 4.4(-5) | 3.5(-5) |
| 64 | 2.6(-4) | 6.2(-5) | 2.9(-5) | 3.5(-4) | 5.6(-6) | 3.7(-6) |
| 128 | 8.1(-5) | 2.0(-5) | 9.3(-6) | 8.8(-5) | 7.2(-7) | 4.3(-7) |
| $m = n$ | $D^1 f_3$_error | | | | | |
| 4 | 3.1(-1) | 1.9(-1) | 1.3(-1) | 2.2(-1) | 1.3(-1) | 9.5(-2) |
| 8 | 1.8(-1) | 9.8(-2) | 6.1(-2) | 7.4(-2) | 3.5(-2) | 2.9(-2) |
| 16 | 9.9(-2) | 5.0(-2) | 2.7(-2) | 2.8(-2) | 7.9(-3) | 7.9(-3) |
| 32 | 5.3(-2) | 2.3(-2) | 9.7(-3) | 1.1(-2) | 2.0(-3) | 2.0(-3) |
| 64 | 2.5(-2) | 7.8(-3) | 5.2(-3) | 4.9(-3) | 4.9(-4) | 4.9(-4) |
| 128 | 9.7(-3) | 1.5(-3) | 2.7(-3) | 2.4(-3) | 1.3(-4) | 1.3(-4) |

**Table 6.3** $f_3$_error and $D^1 f_3$_error, in case of non-uniform triangulations $\mathscr{T}_{mn}$ and $\bar{\mathscr{T}}_{mn}$.

In Figs. 6.5-6.6 we report the graphs of $f_3, W_2 f_3, |f_3 - W_2 f_3|$ and $D^{(0,1)} f_3, D^{(0,1)} W_2 f_3$, $\left|D^{(0,1)}(f_3 - W_2 f_3)\right|$, respectively, computed on $G$, considering $m = n = 128$ and the triangulation $\bar{\mathscr{T}}_{128,128}$. The other first derivative $D^{(1,0)}$ is symmetrical and we do not report it.

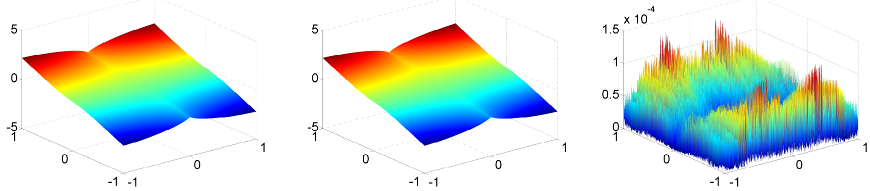
## 7 Conclusions and final remarks

In this paper we have analysed the error between a function $f$ and a general $C^1$ quadratic spline quasi-interpolant, $Qf$, defined on a non-uniform criss-cross triangulation of a rectangular domain $\Omega$. We have given error estimates for the infinity norms of $f - Qf$, of the first derivatives $D^\alpha(f - Qf)$, $|\alpha| = 1$, and of the second derivatives $D^\alpha(f - Qf)$, $|\alpha| = 2$ (in this case in the interior of each triangle of $\mathscr{T}_{mn}$). We have also considered the specific case of a uniform triangulation and, by a different technique, we have reduced the constants in the error bounds.




(a) (b) (c)

**Fig. 6.5** The graphs of $(a)$ $f_3$, $(b)$ $W_2 f_3$ and $(c)$ $|f_3 - W_2 f_3|$, with $\bar{\mathscr{T}}_{128,128}$.




(a) (b) (c)

**Fig. 6.6** The graphs of $(a)$ $D^{(0,1)} f_3$, $(b)$ $D^{(0,1)} W_2 f_3$ and $(c)$ $\left|D^{(0,1)}(f_3 - W_2 f_3)\right|$, with $\bar{\mathscr{T}}_{128,128}$.

Then, we have considered three local QI operators, we have computed their partial derivatives and bounded their errors.

Finally, we have proposed some applications concerning the approximation of functions and their partial derivatives by using the above QIs and we have compared the obtained results with those obtained from other $C^1$ quadratic spline QIs proposed in the literature.

## References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