Asymptotic behavior for a class of multibump solutions to Duffing-like systems

ASYMPTOTIC BEHAVIOR FOR A CLASS OF MULTIBUMP SOLUTIONS TO DUFFING–LIKE SYSTEMS

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ABSTRACT

We consider a class of second order Hamiltonian systems $\ddot{q} = q - V'(t,q)$ where $V(t,q)$ is asymptotic at infinity to a time periodic and superquadratic function $V_+(t,q)$. We prove the existence of a class of multibump solutions whose ω-limit is a suitable homoclinic orbit of the system at infinity $\ddot{q} = q - V_+'(t,q)$.

1. Statement of the results

In this paper we study a class of second order Hamiltonian systems of the type:

$$\ddot{q} = -U'(t,q) \tag{HS}$$

where $U'(t,q)$ denotes the gradient with respect to q of a smooth potential $U : \mathbf{R} \times \mathbf{R}^N \to \mathbf{R}$, having a strict local maximum at the origin. Precisely, we assume:

$(h1)$ $U \in C^1(\mathbf{R} \times \mathbf{R}^N, \mathbf{R})$ with $U'(t,\cdot)$ locally Lipschitz continuous uniformly with respect to $t \in \mathbf{R}$;

$(h2)$ $U(t,0) = 0$ and $U(t,q) = -\frac{1}{2} q \cdot L(t) q + V(t,q)$ with $V'(t,q) = o(|q|)$, as $q \to 0$, uniformly with respect to $t \in \mathbf{R}$ and $L(t)$ is a symmetric matrix such that $c_1 |q|^2 \le q \cdot L(t) q \le c_2 |q|^2$ for any $(t,q) \in \mathbf{R} \times \mathbf{R}^N$ with c_1, c_2 positive constants.

Moreover, we ask the potential U to be asymptotic to a time periodic potential U_+ in the limit $t \to +\infty$. In fact we assume that there exists $U_+ : \mathbf{R} \times \mathbf{R}^N \to \mathbf{R}$ of the form $U_+(t,q) = -\frac{1}{2}\, q \cdot L_+(t)\, q + V_+(t,q)$, satisfying $(h1)$, $(h2)$ and

$(h3)$ there is $T > 0$ such that $U_+(t,q) = U_+(t+T,q)$ for any $(t,q) \in \mathbf{R} \times \mathbf{R}^N$;

$(h4)$ *(i)* there is $(\bar{t},\bar{q}) \in \mathbf{R} \times \mathbf{R}^N$ such that $U_+(\bar{t},\bar{q}) > 0$;

(ii) there are two constants $\beta > 2$ and $\alpha < \frac{\beta}{2} - 1$ such that:

$$\beta V_+(t,q) - V_+'(t,q) \cdot q \le \alpha q \cdot L_+(t) q \text{ for all } (t,q) \in \mathbf{R} \times \mathbf{R}^N;$$

$(h5)$ $U'(t,q) - U_+'(t,q) \to 0$ as $t \to +\infty$ uniformly on the compact sets of $\mathbf{R}^N$.

The problem of existence and multiplicity of homoclinic orbits (i.e., solutions to (HS) satisfying $q(t) \to 0$ and $\dot{q}(t) \to 0$ as $t \to \pm\infty$) has been deeply investigated by variational methods in several papers [1–6]. We also mention [7–11] for the first order systems.

In particular we refer to [12] for the case of asymptotically time periodic potential U satisfying $(h1)$–$(h5)$ (see also [13]). In [12] it is proved that if, in addition,

$(*)$ the set of homoclinics of the system at infinity

$$\ddot{q} = -U'_+(t,q) \qquad \text{(HS)}_+$$

is countable,

then (HS) admits an uncountable set of bounded motions and countably many homoclinics of multibump type. These solutions leave the origin and come back in a neighborhood of it finitely or infinitely many times staying near translations of a particular homoclinic solution v_+ of (HS)_+. This dynamics was firstly shown in [10] for first order convex Hamiltonian systems periodic in time.

In the present work we prove the following theorem.

Theorem 1.1. *If U satisfies $(h1), (h2)$ and there exists U_+ for which $(h1)$–$(h5)$ and $(*)$ hold then there is a homoclinic solution v_+ of (HS)_+ such that for any sequence $(r_n) \subset \mathbf{R}_+$ there are $N \in \mathbf{R}$ and a sequence $(d_n) \subset \mathbf{N}$ for which if $(p_n) \subset \mathbf{Z}$ satisfies $p_1 \geq N$ and $p_{n+1} - p_n \geq d_n \quad (n \in \mathbf{N})$, and if $\sigma = (\sigma_n) \in \{0,1\}^{\mathbf{N}}$, then there is a solution v_σ of (HS) such that*

$$|v_\sigma(t) - \sigma_n v_+(t - p_n T)| < r_n \quad \text{and} \quad |\dot{v}_\sigma(t) - \sigma_n \dot{v}_+(t - p_n T)| < r_n$$

for any $t \in [\frac{1}{2}(p_{n-1} + p_n)T, \frac{1}{2}(p_n + p_{n+1})T]$ and $n \in \mathbf{N}$, whit the agreement $p_0 = -\infty$. In addition, any v_σ satisfies $v_\sigma(t) \to 0$ and $\dot{v}_\sigma(t) \to 0$, as $t \to -\infty$, and, if $\sigma_n = 0$ definitively, then v_σ is a homoclinic orbit.

We remark that for a constant sequence $r_n = r$ $(n \in \mathbf{N})$, theorem 1.1 gives the main result contained in [12]. By theorem 1.1, choosing $r_n \to 0$, we obtain the following result, which we think interesting in its own.

Corollary 1.2. *Under the same assumptions of theorem 1.1, (HS) admits an uncountable set of multibump solutions whose α-limit is $\{0\}$ and whose ω-limit is Γ_+, where $\Gamma_+ = \{(v_+(t), \dot{v}_+(t)) \;:\; t \in \mathbf{R}\} \cup \{0\}$.*

We recall that the α-limit and the ω-limit of a solution q are respectively the sets $\alpha(q) = \{\, (\bar{q}, \bar{p}) \in \mathbf{R}^{2N} \;:\; \exists\, t_n \to -\infty \text{ s.t. } (q(t_n), \dot{q}(t_n)) \to (\bar{q}, \bar{p}) \,\}$ and $\omega(q) = \{\, (\bar{q}, \bar{p}) \in \mathbf{R}^{2N} \;:\; \exists\, t_n \to +\infty \text{ s.t. } (q(t_n), \dot{q}(t_n)) \to (\bar{q}, \bar{p}) \,\}$.

If the potential U is doubly asymptotic to two, possibly distinct periodic potentials U_+ as $t \to +\infty$ and U_-, as $t \to -\infty$, we can prove the existence of multibump solutions of (HS) of mixed type.

Theorem 1.3. *If U satisfies $(h1)$, $(h2)$ and there exist $U_\pm$ for which $(h1)$–$(h5)$ and $(*)$ hold, then there are homoclinic orbits $v_\pm$ of $\text{(HS)}_\pm$ such that (HS) admits an uncountable set of multibump solutions whose α-limit is 0 or Γ_- and whose ω-limit is 0 or Γ_+.*

Remark 1.4. If we specialize theorem 1.3 to the case U periodic in time, we get the existence of a homoclinic v of (HS) and an uncountable set of connecting orbits between 0 and v and between v and itself.

We conclude by noting that, as shown in [12], the hypotheses $(h1)$–$(h5)$ and $(*)$ are verified in the case of the perturbed Duffing–like equation

$$\ddot{q} = q - a(t)(1 + \epsilon \cos(\omega(t)t))\, q^3$$

where $a, \omega \in C^1(\mathbf{R})$, $a(t) \to a_+ > 0$, $\omega(t) \to \omega_+ \neq 0$ as $t \to +\infty$, a is bounded and $\epsilon \neq 0$ is sufficiently small.

2. Outline of the proof of Theorem 1.1

For simplicity we consider the case $L(t) = L_+(t) = I$ and $T = 1$. The general case can be studied by similar arguments.

Variational setting and notation

It is well known that the system (HS) defines a variational problem in a natural way. In fact, the homoclinic solutions to (HS) are the critical points of the action functional $\varphi : X = H^1(\mathbf{R}, \mathbf{R}^N) \to \mathbf{R}$ defined by

$$\varphi(u) = \tfrac{1}{2}\|u\|^2 - \textstyle\int_{\mathbf{R}} V(t, u)\, dt$$

where $\|u\|$ is the standard norm of $H^1(\mathbf{R}, \mathbf{R}^N)$ induced by the inner product $\langle u, v\rangle = \int_{\mathbf{R}}(\dot u \cdot \dot v + u \cdot v)\, dt$. Analogously we define the functional φ_+ associated to V_+.

It turns out that φ and φ_+ are of class C^1 and $\varphi'(u)v = \langle u, v\rangle - \int_{\mathbf{R}} V'(t, u) \cdot v\, dt$ for any $u, v \in X$ (the corresponding expression holds for φ'_+).

For $a, b \in \mathbf{R}$ we denote $\{a \le \varphi \le b\} = \{\, u \in X \ : \ a \le \varphi(u) \le b\,\}$, $K = \{\, u \in X \ : \ u \neq 0, \varphi'(u) = 0\,\}$, $K^b = K \cap \{\varphi \le b\}$ and $K(a) = K \cap \{\varphi = a\}$, and similarly for $\{a \le \varphi_+ \le b\}$, K_+, K_+^b and $K_+(a)$.

We denote $B_r(v)$ the open ball in X of radius r centered in $v \in X$ and for any interval $I \subset \mathbf{R}$, $B_r(v; I) = \{u \in X \, : \, \|u - v\|_I < r\}$, where $\|u\|_I^2 = \int_I (|\dot u|^2 + |u|^2)\, dt$. Moreover, for $S \subseteq X$ and $0 \le r_1 < r_2$ we denote $A_{r_1, r_2}(S) = \bigcup_{v \in S} B_{r_2}(v) \setminus \bar B_{r_1}(v)$.

Palais Smale sequences

First of all we note that thanks to $(h1)$ and $(h2)$ the origin is a strict local minimum for φ (and φ_+).

Lemma 2.1. *For any $\epsilon > 0$ there exists $\delta > 0$ such that for any given interval $I \subseteq \mathbf{R}$, with $|I| \ge 1$ and for any $u \in X$ with $\|u\|_I \le \delta$ we have*

$$\textstyle\int_I V(t, u)\, dt \le \epsilon \|u\|_I^2 \text{ and } \int_I V'(t, u) \cdot v\, dt \le \epsilon \|u\|_I \|v\|_I, \ \forall v \in X.$$

In particular we have that

$$\varphi(u) = \tfrac{1}{2}\|u\|^2 + o(\|u\|^2) \quad \text{and} \quad \varphi'(u) = \langle u, \cdot\rangle + o(\|u\|) \quad \text{as} \quad u \to 0\,.$$

Now, we study the bounded Palais Smale (PS) sequences for φ and φ_+. We point out that the results stated in the next two lemmas follow assuming only $(h1)$ and $(h2)$, and they are inspired to concentration–compactness arguments [14]. We refer to [12] for the proofs.

Lemma 2.2. *If $(u_n) \subset X$ is a PS sequence at the level b (namely $\varphi(u_n) \to b$ and $\|\varphi'(u_n)\| \to 0$) weakly convergent to some $u \in X$, then $\varphi'(u) = 0$ and $(u_n - u)$ is a PS sequence at the level $b - \varphi(u)$. Moreover, $u_n \to u$ strongly in $H^1_{\mathrm{loc}}(\mathbf{R}, \mathbf{R}^N)$ and the following alternative holds: either*

(i) $u_n \to u$ strongly in X, or (ii) $\exists\, |t_{n_k}| \to \infty$ s.t. $\inf_k |u_{n_k}(t_{n_k})| > 0$.

Therefore, if $(u_n) \subset X$ is a PS sequence which converges weakly but not strongly to some $u \in X$, then there exists a positive number r such that for any $T > 0$ we have

$\limsup \|u_n\|_{|t|>T} > r$. Thanks to lemma 2.1 this value r can be taken independent of the sequence (u_n). In fact,

$$\exists\, \rho > 0 \text{ such that if } \limsup \|u_n\| \leq 2\rho \text{ and } \varphi'(u_n) \to 0 \text{ then } u_n \to 0. \tag{2.3}$$

By (2.3) and lemma 2.2 we obtain the following local compactness property.

Lemma 2.4. *Let $u_n \to u$ weakly in X and $\varphi'(u_n) \to 0$. If there exists $T > 0$ for which $\limsup \|u_n\|_{|t|>T} < \rho$ or if $\operatorname{diam}\{u_n\} < \rho$, then $u_n \to u$ strongly in X.*

By assuming also the hypotheses $(h3)$ and $(h4)$ we can state further properties concerning the PS sequences for φ_+.

First of all we point out that the hypothesis $(h4.ii)$ implies that

$$(\tfrac{1}{2} - \tfrac{1}{\beta} - \tfrac{\alpha}{\beta})\|u\|^2 \leq \tfrac{1}{\beta}\|\varphi_+'(u)\|\,\|u\| + \varphi_+(u) \quad \forall\, u \in X.$$

Therefore, any PS sequence (u_n) for φ_+ is bounded in X and $\liminf \varphi_+(u_n) \geq 0$.

Thanks to $(h3)$ the functionals φ_+ and $\|\varphi_+'(\cdot)\|$ are invariant under $\mathbf{Z}$–translations. By these facts and lemma 2.2, it is possible to characterize in a sharp way the PS sequences for φ_+, as already done in [4] and [7].

Lemma 2.5. *Let $(u_n) \subset X$ be a PS sequence for φ_+ at the level b. Then there are $v_0 \in K_+ \cup \{0\}$, $v_1, \dots, v_k \in K_+$, a subsequence of (u_n), denoted again (u_n), and corresponding sequences $(t_n^1), \dots, (t_n^k) \subseteq \mathbf{Z}$, with $|t_n^j| \to +\infty \quad (j = 1, \dots, k)$ and $t_n^{j+1} - t_n^j \to +\infty \quad (j = 1, \dots, k-1)$, as $n \to \infty$, and such that:*

$$\begin{aligned} &\|u_n - (v_0 + v_1(\cdot - t_n^1) + \ldots + v_k(\cdot - t_n^k))\| \to 0 \\ &b = \varphi_+(v_0) + \cdots + \varphi_+(v_k). \end{aligned}$$

By lemma 2.1 and the assumption $(h4)$, we infer (see [15]) that the functional φ_+ verifies the geometrical hypotheses of the mountain pass theorem.

Then, if we define $\Gamma = \{\,\gamma \in C([0,1], X) \ : \ \gamma(0) = 0\,,\ \varphi_+(\gamma(1)) < 0\,\}$ and $c = \inf_{\gamma \in \Gamma} \max_{s \in [0,1]} \varphi_+(\gamma(s))$, we have that $c > 0$ and that there is a PS sequence at the level c. This fact and lemma 2.5 imply that $K_+ \neq \emptyset$.

Consequences of the assumption $(*)$

To get further compactness properties of the functional φ_+, it is convenient to introduce, following [10], two suitable sets of real numbers. Let us fix a level $b > c$. Setting $\mathcal{S}_{PS}^b = \{(u_n) \subset X \ : \ \varphi_+'(u_n) \to 0\,,\ \limsup \varphi_+(u_n) \leq b\}$, we define

$$\begin{aligned} D &= \{r \in \mathbf{R} \ : \ \exists\ (u_n), (\bar{u}_n) \in \mathcal{S}_{PS}^b \ s.t.\ \|u_n - \bar{u}_n\| \to r\}. \\ \Phi &= \{l \in \mathbf{R} \ : \ \exists\ (u_n) \in \mathcal{S}_{PS}^b \ s.t.\ \varphi_+(u_n) \to l\}. \end{aligned}$$

By lemma 2.5, D and Φ can be characterized by means of the set K_+. As proved in [15, lemma 3.10] we get

$D = \{(\sum_{j=1}^k \|v_j - \bar{v}_j\|^2)^{\frac{1}{2}} : k \in \mathbf{N},\ v_j, \bar{v}_j \in K_+ \cup \{0\},\ \sum \varphi_+(v_j) \leq b, \sum \varphi_+(\bar{v}_j) \leq b\}$,

$\Phi = \{\,\sum_{j=1}^k \varphi_+(v_j) \ : \ k \in \mathbf{N}\,,\ v_j \in K_+\} \cap [0, b]$.

By the assumption $(*)$, the sets D and Φ are countable and, since they are closed ([15, lemma 3.7]), we obtain the following lemma.

Lemma 2.6. *(i) For any $r \subset (0, \frac{\rho}{2}) \setminus D$, there exists $\delta = \delta(r) > 0$ such that*

$$\inf\{\|\varphi_+'(u)\| : u \in A_{r-\delta,r+\delta}(K_+^b) \cap \{\varphi_+ \le b\}\} > 0.$$

(ii) *For any interval* $[a_1, a_2] \subset \mathbf{R}_+ \setminus \Phi$ *it holds that*

$$\inf\{\|\varphi_+'(u)\| : a_1 \le \varphi_+(u) \le a_2\} > 0 .$$

From lemmas 2.4 and 2.6*(i)*, using a deformation argument, it is possible to show that the functional φ_+ admits a critical point of local mountain pass type (see [15; section 4] for the proof).

Lemma 2.7. *There exist* $\bar c \in [c, b)$, $\bar r \in (0, \frac{\varrho}{2})$, *a sequence* $(r_n) \subset (0, \bar r) \setminus D$ *with* $r_n \to 0$ *and a sequence* $(v_n) \subset K_+(\bar c)$ *with* $v_n \to v_+ \in K_+(\bar c)$, *such that for any* $h > 0$ *there is a sequence of paths* $(\gamma_{n,h}) \subset C([0,1], X)$ *satisfying:*

(i) $\gamma_{n,h}(0), \gamma_{n,h}(1) \in \partial B_{r_n}(v_n)$;

(ii) $\gamma_{n,h}(0)$ *and* $\gamma_{n,h}(1)$ *are not connectible in* $B_{\bar r}(v_+) \cap \{\varphi_+ < \bar c\}$;

(iii) $\mathrm{range}\,\gamma_{n,h} \subseteq \bar B_{r_n}(v_n) \cap \{\varphi_+ \le \bar c + h\}$;

(iv) $\mathrm{range}\,\gamma_{n,h} \cap A_{r_n - \frac12\delta_n, r_n}(v_n) \subseteq \{\varphi_+ \le \bar c - h\}$;

(v) $\mathrm{supp}\,\gamma_{n,h}(s) \subset [-R_{n,h}, R_{n,h}]$ *for any* $s \in [0,1]$,

where $R_{n,h} > 0$ *is independent of* s, *and* $\delta_n = \delta(r_n)$ *is given by lemma 2.6(i).*

We recall that two points $u_0, u_1 \in X$ are not connectible in $a \subset X$ if there is no path joining u_0 and u_1 with range contained in A.

We point out that, by the $\mathbf{Z}$–invariance of φ_+, any translation by $p \in \mathbf{Z}$ of a path $\gamma_{n,h}$ satisfies properties *(i)–(v)* with respect to $v_n(\cdot - p)$.

Multibump functions

We introduce some notation. For $k \in \mathbf{N}$ and $d = (d_1, \dots, d_{k-1}) \in \mathbf{N}^{k-1}$ we set $P(k,d) = \{(p_1, \dots, p_k) \in \mathbf{Z}^k \ : \ p_{i+1} - p_i \ge 2d_i^2 + 3d_i \ \forall\, i = 1, \dots, k-1\}$, and, for $p \in P(k,d)$ we define the intervals:

$$I_i = \left(\tfrac12(p_{i-1} + p_i)\,,\, \tfrac12(p_i + p_{i+1})\right) \qquad (i = 1, \dots, k)$$
$$M_i = (p_i + d_i(d_i+1), p_{i+1} - d_i(d_i+1)) \quad (i = 1, \dots, k-1)$$

$M_0 = (-\infty, p_1 - d_1(d_1+1))$, $M_k = (p_k + d_{k-1}(d_{k-1}+1), +\infty)$ and $M = \bigcup_{i=0}^k M_i$, with the agreement that $p_0 = -\infty$ and $p_{k+1} = +\infty$.

In addition we introduce the functionals $\varphi_i : X \to \mathbf{R}$ $(i = 1, \dots, k)$ defined by $\varphi_i(u) = \frac12\|u\|_{I_i}^2 - \int_{I_i} V_+(t,u)\,dt$. We notice that $\varphi_+ = \sum_{i=1}^k \varphi_i$ and any φ_i is of class C^1 on X with $\varphi_i'(u)v = \langle u, v\rangle_{I_i} - \int_{I_i} V_+'(t,u)\cdot v\,dt$ for any $u, v \in X$.

Thanks to lemma 2.7 we obtain the following result. Let $(r_n) \subset (0, \bar r)\setminus D$, $r_n \to 0$, and $(v_n) \subset K_+(\bar c)$ be given by lemma 2.7 and let $(\delta_n) \subset \mathbf{R}_+$ be assigned by lemma 2.6*(i)*. Let us denote $r_{1,n} = r_n - \frac13\delta_n$, $r_{2,n} = r_n - \frac14\delta_n$ and $r_{3,n} = r_n - \frac15\delta_n$. Then we have:

Corollary 2.8. *Taking a sequence* $(h_n) \subset \mathbf{R}_+$ *and setting for* $n \in \mathbf{N}$ $\tilde d_n = \max\{R_{n,h_n},\, R_{n+1,h_{n+1}}\}$, *then for any* $k \in \mathbf{N}$ *and* $p \in P(k, \tilde d)$, *the surface* $G : Q = [0,1]^k \to X$ *defined by* $G(\theta_1, \dots, \theta_k) = \sum_{i=1}^k \gamma_{i,h_i}(\theta_i)(\cdot - p_i)$. *satisfies the following properties:*

(i) $G(\partial Q) \subseteq X \setminus \bigcap_{i=1}^k B_{r_{3,i}}(v_i(\cdot - p_i); I_i)$;

(ii) $G(\theta)|_{M_i} = 0$ *for any* $\theta \in Q$ *and* $i \in \{1, \dots, k\}$;

(iii) *for any* $\theta \in Q$ *such that* $G(\theta) \in X \setminus \bigcap_{i=1}^k B_{r_{1,i}}(v_i(\cdot - p_i); I_i)$ *there exists* $i = i(\theta)$ *for which* $G(\theta) \in \{\varphi_i \leq \bar{c} - h_i\}$*;*

(iv) $\operatorname{range} G \subset \bigcap_{i=1}^k \{\varphi_i \leq \bar{c} + h_i\}$;

(v) $\varphi_i(G(\theta)) = \varphi_+(\gamma_{i,h_i}(\theta_i)(\cdot - p_i))$.

A common pseudogradient vector field for φ *and* φ_i

We point out that by $(h5)$, the operator $\varphi'(u)$ is close to $\varphi'_+(u)$ for those elements $u \in X$ with support at infinity, as stated in the next lemma (see [12; lemma 4.2] for a proof).

Lemma 2.9. *For any* $\epsilon > 0$ *and for any* $C > 0$ *there exists* $N \in \mathbf{R}$ *such that*

$$\|\varphi'(u) - \varphi'_+(u)\| \leq \epsilon\,,$$

for any $u \in X$ *with* $\|u\| \leq C$ *and* $\operatorname{supp} u \subseteq [N, +\infty)$.

Next lemma states the existence of a common pseudogradient vector field for φ and φ_i.

Let $(r_n) \subset (0, \bar{r}) \setminus D$, $r_n \to 0$, and $(v_n) \subset K(\bar{c})$ be given by lemma 2.7. Let us fix $r_{1,n}, r_{2,n}, r_{3,n}$ as above. Moreover, let us fix sequences $(a_n), (b_n)$ and $(\lambda_n) \subset \mathbf{R}_+$ such that $[a_n - \lambda_n, a_n + 2\lambda_n] \subset (\bar{c} - h_n, \bar{c}) \setminus \Phi$ and $[b_n - \lambda_n, b_n + 2\lambda_n] \subset (\bar{c} + h_n, \bar{c} + \frac{3}{2}h_n) \setminus \Phi$.

Lemma 2.10. *There exist* $\mu_n = \mu_n(r_n) > 0$, $N \in \mathbf{R}$ *and* $\bar{\epsilon}_n = \bar{\epsilon}_n(r_n, a_n, b_n, \lambda_n) > 0$ *such that: for any* $\epsilon_n \in (0\,, \bar{\epsilon}_n)$ *there exists* $\bar{d}_n \in \mathbf{N}$ $(n \in \mathbf{N})$ *for which for any* $k \in \mathbf{N}$ *and* $p \in P(k, d)$*, with* $p_1 > N$ *there exists a locally Lipschitz continuous function* $\mathcal{W} : X \to X$ *which verifies*

$(\mathcal{W}_0)$ $\max_{1 \leq i \leq k} \|\mathcal{W}(u)\|_{I_i} \leq 1$, $\varphi'(u)\mathcal{W}(u) \geq 0$, $\forall\, u \in X$, $\mathcal{W}(u) = 0$ $\forall\, u \in X \setminus B_3$;

$(\mathcal{W}_1)$ $\varphi'_i(u)\mathcal{W}(u) \geq \mu_i$ *if* $r_{1,i} \leq \|u - v_i(\cdot - p_i)\|_{I_i} \leq r_{2,i}$, $u \in B_2 \cap \{\varphi_i \leq b_i\}$*;*

$(\mathcal{W}_2)$ $\varphi'_i(u)\mathcal{W}(u) \geq 0$ $\forall\, u \in \{b_i \leq \varphi_i \leq b_i + \lambda_i\} \cup \{a_i \leq \varphi_i \leq a_i + \lambda_i\}$*;*

$(\mathcal{W}_3)$ $\langle u, \mathcal{W}(u)\rangle_{M_i} \geq 0$ $\forall\, i \in \{0, \ldots, k\}$ *if* $u \in X \setminus \mathcal{M}$,

where $\mathcal{M} \equiv \bigcap_{i=0}^k B_{\sqrt{\epsilon}}(0; M_i)$ *and* $B_j \equiv \bigcap_{i=1}^k B_{r_{j,i}}(v_i(\cdot - p_i); I_i)$ *for* $j = 1, 2, 3$.

Moreover if $K \cap B_2 = \emptyset$ *then there exists* $\mu_p > 0$ *such that*

$(\mathcal{W}_4)$ $\varphi'(u)\mathcal{W}(u) \geq \mu_p$, $\forall\, u \in B_2$.

Approximating k-bump solutions

Theorem 2.11. *If* U *satisfies* $(h1), (h2)$ *and there exists* U_+ *for which* $(h1)$*–*$(h5)$ *and* $(*)$ *hold then for any given sequence* $(\rho_n) \in \mathbf{R}_+$ *there exist* $N \in \mathbf{R}$ *and a sequence* $(d_n) \subset \mathbf{N}$ *such that for every* $k \in \mathbf{N}$ *and* $p \in P(k, d)$*, with* $p_1 > N$*, we have* $K \cap \bigcap_{i=1}^k B_{\rho_i}(v_+(\cdot - p_i); I_i) \neq \emptyset$*, where* v_+ *is given by lemma 2.7.*

Proof. Arguing by contradiction there exists a sequence $(\rho_n) \subset \mathbf{R}_+$ such that for any $N \in \mathbf{R}$ and for any sequence $(d_n) \subset \mathbf{N}$ there exist $k \in \mathbf{N}$ and $p \in P(k, d)$, with $p_1 > N$, for which $K \cap \bigcap_{i=1}^k B_{\rho_i}(v_+(\cdot - p_i); I_i) = \emptyset$. Let $(r_n) \subset (0, \bar{r}) \setminus D$, $r_n \to 0$ and $(v_n) \subset K(\bar{c})$ be given by lemma 2.7. Without loss of generality, passing to a subsequence if necessary, we can assume $B_{2r_n}(v_n) \subset B_{\rho_n}(v_+)$ for any $n \in \mathbf{N}$.

Let μ_n and $\bar{\epsilon}_n$ be given by lemma 2.10. Let us define $\Delta_n = \frac{1}{4}\mu_n(r_{2,n} - r_{1,n})$. Then, we fix $h_n < \frac{1}{8}\Delta_n$, a_n and b_n as above, with $b_n - a_n < \frac{1}{4}\Delta_n$ and $0 < \epsilon_n < \min\{\bar{\epsilon}_n, \frac{1}{4}\delta^2_{r_{n-1}}, \frac{1}{4}\delta^2_{r_n}, \frac{1}{8}(\bar{c} - a_n)\}$ such that $\int_I |V_+(t, u)\, dt \leq \frac{1}{8}\|u\|^2_I$ for $|I| \geq 1$.

Now, we fix $d_n > \max\{\tilde{d}_n, \bar{d}_n, 2\}$ and such that $\max\{\|v_{n+1}\|^2_{t<-d_n}, \|v_n\|^2_{t>d_n}\} < \epsilon_n$, where $\tilde{d}_n$ is given by corollary 2.8 and $\bar{d}_n$ by lemma 2.10.

For these values of d_n and for N given by lemma 2.10 there exist $k \in \mathbf{N}$ and $p \in P(k,d)$, with $p_1 > N$, for which $\bigcap_{i=1}^k B_{r_i}(v_i(\cdot - p_i); I_i) \subseteq \bigcap_{i=1}^k B_{\rho_i}(v_+(\cdot - p_i); I_i)$. So that, by the contrary assumptions, there exists a vector field $\mathcal{W}$ satisfying properties $(\mathcal{W}_0)$-$(\mathcal{W}_4)$.

Now, we consider the flow associated to the Cauchy problem

$$\frac{d\eta}{ds} = -\mathcal{W}(\eta)\,, \quad \eta(0,u) = u$$

to deform the surface G given by corollary 2.8 for this values of $(h_i)_{i=1,\dots,k}$ and $(p_i)_{i=1,\dots,k}$.

Since $\mathcal{W}$ is a bounded locally Lipschitz continuous vector field, the flow η is globally defined. Moreover, by $(\mathcal{W}_0)$ the flow does not move the points outside B_3. This implies, by corollary 2.8*(i)*,

$$\eta(s, G(\theta)) = G(\theta) \quad \forall\, \theta \in \partial Q\,, \quad \forall\, s \in \mathbf{R}. \tag{2.12}$$

Since $\varphi(B_2)$ is a bounded set, by $(\mathcal{W}_4)$ there exists $\tau > 0$ such that for any $\theta \in Q$ for which $G(\theta) \in B_1$ there is $[s_1, s_2] \subset (0,\tau]$ with $\eta(s_1, G(\theta)) \in \partial B_1$, $\eta(s_2, G(\theta)) \in \partial B_2$ and $\eta(s, G(\theta)) \in B_2 \setminus \bar{B}_1$ for any $s \in (s_1, s_2)$. Therefore for any $\theta \in Q$ there is an index $i = i(\theta) \in \{1,\dots,k\}$ such that, by $(\mathcal{W}_1)$, $\varphi_i(\eta(s_2, G(\theta))) \le \varphi_i(\eta(s_1, G(\theta))) - 2\Delta_i$. By corollary 2.8*(iv)* and since, by $(\mathcal{W}_2)$, the sets $\{\varphi_i \le b_i\}$ and $\{\varphi_i \le a_i\}$ are positively invariant, we obtain $\varphi_i(\eta(s_2, G(\theta))) \le b_i - 2\Delta_i < a_i$ and hence $\varphi_i(\eta(\tau, G(\theta))) \le a_i$. Moreover, by *(iii)* of corollary 2.8, for any θ for which $G(\theta) \in X \setminus B_1$ there exists $i = i(\theta)$ such that $\eta(s, G(\theta)) \in \{\varphi_i \le a_i\}$ for any $s \in \mathbf{R}_+$. Hence, setting $\bar{G}(\theta) = \eta(\tau, G(\theta))$, we get

$$\forall\, \theta \in Q,\ \exists i \in \{1,\dots,k\} \text{ such that } \varphi_i(\bar{G}(\theta)) < a_i. \tag{2.13}$$

By (2.13) we have that

(2.14) there exists $i \in \{1,\dots,k\}$ and $\xi \in C([0,1], Q)$ such that $\xi(0) \in \{\theta_i = 0\}$, $\xi(1) \in \{\theta_i = 1\}$ and $\varphi_i(\bar{G}(\theta)) < a_i$, for any $\theta \in \operatorname{range}\xi$.

Indeed, assuming the contrary, the set $D_i = \{\theta \in Q \,:\, \varphi_i(\bar{G}(\theta)) \ge a_i\}$ separates in Q the faces $\{\theta_i = 0\}$ and $\{\theta_i = 1\}$, for any $i \in \{1,\dots,k\}$. Then, using a Miranda fixed point theorem, it follows that $\bigcap_i D_i \ne \emptyset$, in contradiction with the property (2.13) (see [15]).

By $(\mathcal{W}_3)$, the set $\mathcal{M}$ is positively invariant under the flow. Then, by corollary 2.8*(ii)*,

$$\eta(s, G(Q)) \in \mathcal{M} \quad \forall\, s \in \mathbf{R}_+\,. \tag{2.15}$$

Now, let us take a cut-off function $\chi \in C^\infty(\mathbf{R}, [0,1])$ with $\sup_{t\in\mathbf{R}} |\dot{\chi}(t)| < \frac{1}{2}$ (this can be done since $\inf_{i=1\dots k} d_i > 2$) such that $\chi(t) = 1$ if $t \in I_i \setminus M$ and $\chi(t) = 0$ if $t \in \mathbf{R} \setminus I_i$, where the index i is given by (2.14). Then $\|\chi\, u\|^2_{I_i \cap M} \le 2\|u\|^2_{I_i\cap M}$ and $\|(1-\chi)\, u\|^2_{I_i\cap M} \le 2\|u\|^2_{I_i\cap M}$ for any $u \in X$. We define a path $g : [0,1] \to X$ by setting $g(s) = \chi\bar{G}(\xi(s))$, $s \in [0,1]$. By (2.12) and lemma 2.7*(v)*, we have that $g(0) = \gamma_{i,h_i}(0)(\cdot - p_i)$ and $g(1) = \gamma_{i,h_i}(1)(\cdot - p_i)$. We finally show that $g(s) \in B_{\bar{r}}(v_+(\cdot - p_i)) \cap \{\varphi_+ < \bar{c}\}$, for any $s \in [0,1]$ contradicting lemma 2.7*(ii)*. In fact one easily get that range $g \subset B_{2r_i}(v_i(\cdot - p_i))$.

Then, setting $u = \bar{G}(\xi(s))$, we have $\varphi_+(g(s)) = \varphi_i(g(s)) \le \varphi_i(u) + \frac{1}{2}\|\chi\, u\|^2_{I_i\cap M} + \int_{I_i\cap M}(V_+(t,u) - V_+(t,\chi\, u))\, dt \le a_i + 4\, \|\chi\, u\|^2_{I_i\cap M}$. Since, by (2.15) $\|u\|_{I_i\cap M} \le \epsilon_i + \epsilon_{i-1}$, we get $\varphi_+(g(s)) < \bar{c}$.

□

Remarks. *(i)* The multibump homoclinic solutions of (HS) given by theorem 2.11 are close to translations of v_+ in the H^1 norm on suitable intervals. Hence they are close in the C^0 norm and, since they verify (HS), in the C^1 norm, too.
(ii) Taking the C^1_{loc} closure of the set of the multibump homoclinic solutions of (HS), using the Ascoli–Arzelà theorem, we get solutions with infinitely many bumps, as stated in theorem 1.1.
(iii) Corollary 1.2 follows from theorem 1.1, taking a sequence $r_n \to 0$ and any sequence (σ_n) with infinitely many 1's. Thus we have multiplicity both for the arbitrariness of (r_n) and for the arbitrariness of (σ_n).
(iv) Similar arguments apply to prove theorem 1.3. We refer to [12] for the construction of multibump solutions of mixed type.

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