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## ASYMPTOTIC BEHAVIOR FOR A CLASS OF MULTIBUMP SOLUTIONS TO DUFFING-LIKE SYSTEMS

Paolo CALDIROLI, Piero MONTECCHIARI and Margherita NOLASCO SISSA, International School for Advanced Studies, via Beirut 4 34013 Trieste, Italy

#### ABSTRACT

We consider a class of second order Hamiltonian systems  $\ddot{q} = q - V'(t,q)$  where V(t,q) is asymptotic at infinity to a time periodic and superquadratic function  $V_{+}(t,q)$ . We prove the existence of a class of multibump solutions whose  $\omega$ -limit is a suitable homoclinic orbit of the system at infinity  $\ddot{q} = q - V'_{+}(t, q)$ .

### 1. Statement of the results

In this paper we study a class of second order Hamiltonian systems of the type:

$$= -U'(t,q)$$
 (HS)

where U'(t,q) denotes the gradient with respect to q of a smooth potential  $U: \mathbf{R} \times$  $\mathbf{R}^N \to \mathbf{R}$ , having a strict local maximum at the origin. Precisely, we assume:

 $\ddot{q} =$ 

- (h1)  $U \in C^1(\mathbf{R} \times \mathbf{R}^N, \mathbf{R})$  with  $U'(t, \cdot)$  locally Lipschitz continuous uniformly with respect to  $t \in \mathbf{R}$ ;
- (h2) U(t,0) = 0 and  $U(t,q) = -\frac{1}{2}q \cdot L(t)q + V(t,q)$  with V'(t,q) = o(|q|), as  $q \to 0$ , uniformly with respect to  $t \in \mathbf{R}$  and L(t) is a symmetric matrix such that  $c_1|q|^2 \leq$  $q \cdot L(t)q \leq c_2 |q|^2$  for any  $(t,q) \in \mathbf{R} \times \mathbf{R}^N$  with  $c_1, c_2$  positive constants.

Moreover, we ask the potential U to be asymptotic to a time periodic potential  $U_{+}$ in the limit  $t \to +\infty$ . In fact we assume that there exists  $U_+ : \mathbf{R} \times \mathbf{R}^N \to \mathbf{R}$  of the form  $U_{+}(t,q) = -\frac{1}{2} q \cdot L_{+}(t) q + V_{+}(t,q)$ , satisfying (h1), (h2) and

- (h3) there is T > 0 such that  $U_+(t,q) = U_+(t+T,q)$  for any  $(t,q) \in \mathbf{R} \times \mathbf{R}^N$ ;
- (h4) (i) there is  $(\bar{t}, \bar{q}) \in \mathbf{R} \times \mathbf{R}^N$  such that  $U_+(\bar{t}, \bar{q}) > 0$ ;

  - (ii) there are two constants  $\beta > 2$  and  $\alpha < \frac{\bar{\beta}}{2} 1$  such that:  $\beta V_+(t,q) V'_+(t,q) \cdot q \leq \alpha q \cdot L_+(t)q$  for all  $(t,q) \in \mathbf{R} \times \mathbf{R}^N$ ;

(h5)  $U'(t,q) - U'_+(t,q) \to 0$  as  $t \to +\infty$  uniformly on the compact sets of  $\mathbf{R}^N$ .

The problem of existence and multiplicity of homoclinic orbits (i.e., solutions to (HS) satisfying  $q(t) \to 0$  and  $\dot{q}(t) \to 0$  as  $t \to \pm \infty$ ) has been deeply investigated by variational methods in several papers [1-6]. We also mention [7-11] for the first order systems.

In particular we refer to [12] for the case of asymptotically time periodic potential U satisfying (h1)-(h5) (see also [13]). In [12] it is proved that if, in addition,

(\*) the set of homoclinics of the system at infinity

$$\ddot{q} = -U'_+(t,q) \tag{HS}_+$$

is countable,

then (HS) admits an uncountable set of bounded motions and countably many homoclinics of multibump type. These solutions leave the origin and come back in a neighborhood of it finitely or infinitely many times staying near translations of a particular homoclinic solution  $v_+$  of (HS)<sub>+</sub>. This dynamics was firstly shown in [10] for first order convex Hamiltonian systems periodic in time.

In the present work we prove the following theorem.

**Theorem 1.1.** If U satisfies (h1), (h2) and there exists  $U_+$  for which (h1)-(h5) and (\*) hold then there is a homoclinic solution  $v_+$  of  $(HS)_+$  such that for any sequence  $(r_n) \subset \mathbf{R}_+$  there are  $N \in \mathbf{R}$  and a sequence  $(d_n) \subset \mathbf{N}$  for which if  $(p_n) \subset \mathbf{Z}$  satisfies  $p_1 \geq N$  and  $p_{n+1} - p_n \geq d_n$   $(n \in \mathbf{N})$ , and if  $\sigma = (\sigma_n) \in \{0, 1\}^{\mathbf{N}}$ , then there is a solution  $v_{\sigma}$  of (HS) such that

 $|v_{\sigma}(t) - \sigma_n v_+(t - p_n T)| < r_n$  and  $|\dot{v}_{\sigma}(t) - \sigma_n \dot{v}_+(t - p_n T)| < r_n$ 

for any  $t \in [\frac{1}{2}(p_{n-1}+p_n)T, \frac{1}{2}(p_n+p_{n+1})T]$  and  $n \in \mathbf{N}$ , whit the agreement  $p_0 = -\infty$ . In addition, any  $v_{\sigma}$  satisfies  $v_{\sigma}(t) \to 0$  and  $\dot{v}_{\sigma}(t) \to 0$ , as  $t \to -\infty$ , and, if  $\sigma_n = 0$  definitively, then  $v_{\sigma}$  is a homoclinic orbit.

We remark that for a constant sequence  $r_n = r$   $(n \in \mathbf{N})$ , theorem 1.1 gives the main result contained in [12]. By theorem 1.1, choosing  $r_n \to 0$ , we obtain the following result, which we think interesting in its own.

**Corollary 1.2.** Under the same assumptions of theorem 1.1, (HS) admits an uncountable set of multibump solutions whose  $\alpha$ -limit is  $\{0\}$  and whose  $\omega$ -limit is  $\Gamma_+$ , where  $\Gamma_+ = \{(v_+(t), \dot{v}_+(t)) : t \in \mathbf{R}\} \cup \{0\}.$ 

We recall that the  $\alpha$ -limit and the  $\omega$ -limit of a solution q are respectively the sets  $\alpha(q) = \{ (\bar{q}, \bar{p}) \in \mathbf{R}^{2N} : \exists t_n \to -\infty \text{ s.t. } (q(t_n), \dot{q}(t_n)) \to (\bar{q}, \bar{p}) \}$  and  $\omega(q) = \{ (\bar{q}, \bar{p}) \in \mathbf{R}^{2N} : \exists t_n \to +\infty \text{ s.t. } (q(t_n), \dot{q}(t_n)) \to (\bar{q}, \bar{p}) \}.$ 

If the potential U is doubly asymptotic to two, possibly distinct periodic potentials  $U_+$  as  $t \to +\infty$  and  $U_-$ , as  $t \to -\infty$ , we can prove the existence of multibump solutions of (HS) of mixed type.

**Theorem 1.3.** If U satisfies (h1), (h2) and there exist  $U_{\pm}$  for which (h1)-(h5) and (\*) hold, then there are homoclinic orbits  $v_{\pm}$  of  $(HS)_{\pm}$  such that (HS) admits an uncountable set of multibump solutions whose  $\alpha$ -limit is 0 or  $\Gamma_{-}$  and whose  $\omega$ -limit is 0 or  $\Gamma_{+}$ .

**Remark 1.4.** If we specialize theorem 1.3 to the case U periodic in time, we get the existence of a homoclinic v of (HS) and an uncountable set of connecting orbits between 0 and v and between v and itself.

We conclude by noting that, as shown in [12], the hypotheses (h1)-(h5) and (\*) are verified in the case of the perturbed Duffing–like equation

$$\ddot{q} = q - a(t)(1 + \epsilon \cos(\omega(t)t)) q^3$$

where  $a, \omega \in C^1(\mathbf{R}), a(t) \to a_+ > 0, \omega(t) \to \omega_+ \neq 0$  as  $t \to +\infty$ , a is bounded and  $\epsilon \neq 0$  is sufficiently small.

#### 2. Outline of the proof of Theorem 1.1

For simplicity we consider the case  $L(t) = L_+(t) = I$  and T = 1. The general case can be studied by similar arguments.

### Variational setting and notation

It is well known that the system (HS) defines a variational problem in a natural way. In fact, the homoclinic solutions to (HS) are the critical points of the action functional  $\varphi : X = H^1(\mathbf{R}, \mathbf{R}^N) \to \mathbf{R}$  defined by

$$\varphi(u) = \frac{1}{2} \|u\|^2 - \int_{\mathbf{R}} V(t, u) \, dt$$

where ||u|| is the standard norm of  $H^1(\mathbf{R}, \mathbf{R}^N)$  induced by the inner product  $\langle u, v \rangle = \int_{\mathbf{R}} (\dot{u} \cdot \dot{v} + u \cdot v) dt$ . Analogously we define the functional  $\varphi_+$  associated to  $V_+$ .

It turns out that  $\varphi$  and  $\varphi_+$  are of class  $C^1$  and  $\varphi'(u)v = \langle u, v \rangle - \int_{\mathbf{R}} V'(t, u) \cdot v \, dt$ for any  $u, v \in X$  (the corresponding expression holds for  $\varphi'_+$ ).

For  $a, b \in \mathbf{R}$  we denote  $\{a \leq \varphi \leq b\} = \{u \in X : a \leq \varphi(u) \leq b\}, K = \{u \in X : u \neq 0, \varphi'(u) = 0\}, K^b = K \cap \{\varphi \leq b\}$  and  $K(a) = K \cap \{\varphi = a\}$ , and similarly for  $\{a \leq \varphi_+ \leq b\}, K_+, K_+^b$  and  $K_+(a)$ .

We denote  $B_r(v)$  the open ball in X of radius r centered in  $v \in X$  and for any interval  $I \subset \mathbf{R}$ ,  $B_r(v; I) = \{u \in X : ||u - v||_I < r\}$ , where  $||u||_I^2 = \int_I (|\dot{u}|^2 + |u|^2) dt$ . Moreover, for  $S \subseteq X$  and  $0 \le r_1 < r_2$  we denote  $A_{r_1,r_2}(S) = \bigcup_{v \in S} B_{r_2}(v) \setminus \overline{B}_{r_1}(v)$ .

### Palais Smale sequences

First of all we note that thanks to (h1) and (h2) the origin is a strict local minimum for  $\varphi$  (and  $\varphi_+$ ).

**Lemma 2.1.** For any  $\epsilon > 0$  there exists  $\delta > 0$  such that for any given interval  $I \subseteq \mathbf{R}$ , with  $|I| \ge 1$  and for any  $u \in X$  with  $||u||_I \le \delta$  we have

 $\int_{I} V(t,u) \, dt \leq \epsilon \|u\|_{I}^{2} \text{ and } \int_{I} V'(t,u) \cdot v \, dt \leq \epsilon \|u\|_{I} \|v\|_{I}, \, \forall v \in X.$ 

In particular we have that

 $\varphi(u) = \frac{1}{2} ||u||^2 + o(||u||^2) \text{ and } \varphi'(u) = \langle u, \cdot \rangle + o(||u||) \text{ as } u \to 0.$ 

Now, we study the bounded Palais Smale (PS) sequences for  $\varphi$  and  $\varphi_+$ . We point out that the results stated in the next two lemmas follow assuming only (h1) and (h2), and they are inspired to concentration-compactness arguments [14]. We refer to [12] for the proofs.

**Lemma 2.2.** If  $(u_n) \subset X$  is a PS sequence at the level b (namely  $\varphi(u_n) \to b$  and  $\|\varphi'(u_n)\| \to 0$ ) weakly convergent to some  $u \in X$ , then  $\varphi'(u) = 0$  and  $(u_n - u)$  is a PS sequence at the level  $b - \varphi(u)$ . Moreover,  $u_n \to u$  strongly in  $H^1_{\text{loc}}(\mathbf{R}, \mathbf{R}^N)$  and the following alternative holds: either

(i)  $u_n \to u$  strongly in X, or (ii)  $\exists |t_{n_k}| \to \infty$  s.t.  $\inf_k |u_{n_k}(t_{n_k})| > 0$ .

Therefore, if  $(u_n) \subset X$  is a PS sequence which converges weakly but not strongly to some  $u \in X$ , then there exists a positive number r such that for any T > 0 we have

 $\limsup \|u_n\|_{|t|>T} > r$ . Thanks to lemma 2.1 this value r can be taken independent of the sequence  $(u_n)$ . In fact,

 $\exists \rho > 0$  such that if  $\limsup \|u_n\| \le 2\rho$  and  $\varphi'(u_n) \to 0$  then  $u_n \to 0$ . (2.3) By (2.3) and lemma 2.2 we obtain the following local compactness property.

**Lemma 2.4.** Let  $u_n \to u$  weakly in X and  $\varphi'(u_n) \to 0$ . If there exists T > 0 for which  $\limsup \|u_n\|_{|t|>T} < \rho$  or if diam  $\{u_n\} < \rho$ , then  $u_n \to u$  strongly in X.

By assuming also the hypotheses (h3) and (h4) we can state further properties concerning the PS sequences for  $\varphi_+$ .

First of all we point out that the hypothesis (h4.ii) implies that

$$\frac{1}{2} - \frac{1}{\beta} - \frac{\alpha}{\beta} \| u \|^2 \le \frac{1}{\beta} \| \varphi'_+(u) \| \| u \| + \varphi_+(u) \quad \forall u \in X.$$

Therefore, any PS sequence  $(u_n)$  for  $\varphi_+$  is bounded in X and  $\liminf \varphi_+(u_n) \ge 0$ .

Thanks to (h3) the functionals  $\varphi_+$  and  $\|\varphi'_+(\cdot)\|$  are invariant under **Z**-translations. By these facts and lemma 2.2, it is possible to characterize in a sharp way the PS sequences for  $\varphi_+$ , as already done in [4] and [7].

**Lemma 2.5.** Let  $(u_n) \subset X$  be a PS sequence for  $\varphi_+$  at the level b. Then there are  $v_0 \in K_+ \cup \{0\}$ ,  $v_1, \ldots, v_k \in K_+$ , a subsequence of  $(u_n)$ , denoted again  $(u_n)$ , and corresponding sequences  $(t_n^1), \ldots, (t_n^k) \subseteq \mathbf{Z}$ , with  $|t_n^j| \to +\infty$   $(j = 1, \ldots, k)$  and  $t_n^{j+1} - t_n^j \to +\infty$   $(j = 1, \ldots, k-1)$ , as  $n \to \infty$ , and such that:

$$||u_n - (v_0 + v_1(\cdot - t_n^1) + \ldots + v_k(\cdot - t_n^k))|| \to 0$$
  
$$b = \varphi_+(v_0) + \cdots + \varphi_+(v_k).$$

By lemma 2.1 and the assumption (h4), we infer (see [15]) that the functional  $\varphi_+$  verifies the geometrical hypotheses of the mountain pass theorem.

Then, if we define  $\Gamma = \{ \gamma \in C([0,1], X) : \gamma(0) = 0, \varphi_+(\gamma(1)) < 0 \}$  and  $c = \inf_{\gamma \in \Gamma} \max_{s \in [0,1]} \varphi_+(\gamma(s))$ , we have that c > 0 and that there is a PS sequence at the level c. This fact and lemma 2.5 imply that  $K_+ \neq \emptyset$ .

### Consequences of the assumption (\*)

To get further compactness properties of the functional  $\varphi_+$ , it is convenient to introduce, following [10], two suitable sets of real numbers. Let us fix a level b > c. Setting  $\mathcal{S}_{PS}^b = \{(u_n) \subset X : \varphi'_+(u_n) \to 0, \limsup \varphi_+(u_n) \leq b\}$ , we define

$$D = \{ r \in \mathbf{R} : \exists (u_n), (\bar{u}_n) \in \mathcal{S}_{PS}^b \ s.t. \ \|u_n - \bar{u}_n\| \to r \}.$$
  
$$\Phi = \{ l \in \mathbf{R} : \exists (u_n) \in \mathcal{S}_{PS}^b \ s.t. \ \varphi_+(u_n) \to l \}.$$

By lemma 2.5, D and  $\Phi$  can be characterized by means of the set  $K_+$ . As proved in [15, lemma 3.10] we get

$$D = \{ (\sum_{j=1}^{k} \|v_j - \bar{v}_j\|^2)^{\frac{1}{2}} : k \in \mathbf{N}, \, v_j, \bar{v}_j \in K_+ \cup \{0\}, \, \sum \varphi_+(v_j) \le b, \, \sum \varphi_+(\bar{v}_j) \le b \}, \\ \Phi = \{ \sum_{j=1}^{k} \varphi_+(v_j) : k \in \mathbf{N}, \, v_j \in K_+ \} \cap [0, b].$$

By the assumption (\*), the sets D and  $\Phi$  are countable and, since they are closed ([15, lemma 3.7]), we obtain the following lemma.

**Lemma 2.6.** (i) For any  $r \subset (0, \frac{\rho}{2}) \setminus D$ , there exists  $\delta = \delta(r) > 0$  such that

$$\inf\{\|\varphi'_{+}(u)\| : u \in A_{r-\delta, r+\delta}(K^{b}_{+}) \cap \{\varphi_{+} \le b\}\} > 0.$$

(*ii*) For any interval  $[a_1, a_2] \subset \mathbf{R}_+ \setminus \Phi$  it holds that

 $\inf \{ \|\varphi'_+(u)\| : a_1 \le \varphi_+(u) \le a_2 \} > 0 .$ 

From lemmas 2.4 and 2.6(*i*), using a deformation argument, it is possible to show that the functional  $\varphi_+$  admits a critical point of local mountain pass type (see [15; section 4] for the proof).

**Lemma 2.7.** There exist  $\bar{c} \in [c, b)$ ,  $\bar{r} \in (0, \frac{\rho}{2})$ , a sequence  $(r_n) \subset (0, \bar{r}) \setminus D$  with  $r_n \to 0$  and a sequence  $(v_n) \subset K_+(\bar{c})$  with  $v_n \to v_+ \in K_+(\bar{c})$ , such that for any h > 0 there is a sequence of paths  $(\gamma_{n,h}) \subset C([0,1], X)$  satisfying:

(i) 
$$\gamma_{n,h}(0), \gamma_{n,h}(1) \in \partial B_{r_n}(v_n);$$

(ii)  $\gamma_{n,h}(0)$  and  $\gamma_{n,h}(1)$  are not connectible in  $B_{\bar{r}}(v_+) \cap \{\varphi_+ < \bar{c}\};$ 

(*iii*) range  $\gamma_{n,h} \subseteq \overline{B}_{r_n}(v_n) \cap \{\varphi_+ \le \overline{c} + h\};$ 

(*iv*) range  $\gamma_{n,h} \cap A_{r_n - \frac{1}{2}\delta_n, r_n}(v_n) \subseteq \{\varphi_+ \leq \bar{c} - h\};$ 

(v) supp  $\gamma_{n,h}(s) \subset [-\tilde{R}_{n,h}, R_{n,h}]$  for any  $s \in [0, 1]$ ,

where  $R_{n,h} > 0$  is independent of s, and  $\delta_n = \delta(r_n)$  is given by lemma 2.6(i).

We recall that two points  $u_0, u_1 \in X$  are not connectible in  $a \subset X$  if there is no path joining  $u_0$  and  $u_1$  with range contained in A.

We point out that, by the **Z**-invariance of  $\varphi_+$ , any translation by  $p \in \mathbf{Z}$  of a path  $\gamma_{n,h}$  satisfies properties (i)-(v) with respect to  $v_n(\cdot - p)$ .

### Multibump functions

We introduce some notation. For  $k \in \mathbf{N}$  and  $d = (d_1, \ldots, d_{k-1}) \in \mathbf{N}^{k-1}$  we set  $P(k,d) = \{(p_1, \ldots, p_k) \in \mathbf{Z}^k : p_{i+1} - p_i \geq 2d_i^2 + 3d_i \ \forall i = 1, \ldots, k-1\}$ , and, for  $p \in P(k,d)$  we define the intervals:

$$I_i = \left(\frac{1}{2}(p_{i-1} + p_i), \frac{1}{2}(p_i + p_{i+1})\right) \qquad (i = 1, \dots, k)$$
  
$$M_i = \left(p_i + d_i(d_i + 1), p_{i+1} - d_i(d_i + 1)\right) \quad (i = 1, \dots, k - 1)$$

 $M_0 = (-\infty, p_1 - d_1(d_1 + 1)), M_k = (p_k + d_{k-1}(d_{k-1} + 1), +\infty) \text{ and } M = \bigcup_{i=0}^k M_i,$ with the agreement that  $p_0 = -\infty$  and  $p_{k+1} = +\infty$ .

In addition we introduce the functionals  $\varphi_i : X \to \mathbf{R}$  (i = 1, ..., k) defined by  $\varphi_i(u) = \frac{1}{2} \|u\|_{I_i}^2 - \int_{I_i} V_+(t, u) dt$ . We notice that  $\varphi_+ = \sum_{i=1}^k \varphi_i$  and any  $\varphi_i$  is of class  $C^1$  on X with  $\varphi'_i(u)v = \langle u, v \rangle_{I_i} - \int_{I_i} V'_+(t, u) \cdot v dt$  for any  $u, v \in X$ .

Thanks to lemma 2.7 we obtain the following result. Let  $(r_n) \subset (0, \bar{r}) \setminus D$ ,  $r_n \to 0$ , and  $(v_n) \subset K_+(\bar{c})$  be given by lemma 2.7 and let  $(\delta_n) \subset \mathbf{R}_+$  be assigned by lemma 2.6(*i*). Let us denote  $r_{1,n} = r_n - \frac{1}{3}\delta_n$ ,  $r_{2,n} = r_n - \frac{1}{4}\delta_n$  and  $r_{3,n} = r_n - \frac{1}{5}\delta_n$ . Then we have:

**Corollary 2.8.** Taking a sequence  $(h_n) \subset \mathbf{R}_+$  and setting for  $n \in \mathbf{N}$   $d_n = \max\{R_{n,h_n}, R_{n+1,h_{n+1}}\}$ , then for any  $k \in \mathbf{N}$  and  $p \in P(k, \tilde{d})$ , the surface  $G : Q = [0,1]^k \to X$  defined by  $G(\theta_1, \ldots, \theta_k) = \sum_{i=1}^k \gamma_{i,h_i}(\theta_i)(\cdot - p_i)$ . satisfies the following properties:

(i)  $G(\partial Q) \subseteq X \setminus \bigcap_{i=1}^{k} B_{r_{3,i}}(v_i(\cdot - p_i); I_i);$ 

(ii)  $G(\theta)|_{M_i} = 0$  for any  $\theta \in Q$  and  $i \in \{1, \ldots, k\}$ ;

- (iii) for any  $\theta \in Q$  such that  $G(\theta) \in X \setminus \bigcap_{i=1}^k B_{r_{1,i}}(v_i(\cdot p_i); I_i)$  there exists  $i = i(\theta)$ for which  $G(\theta) \in \{\varphi_i \leq \bar{c} - h_i\};$
- (*iv*) range  $G \subset \bigcap_{i=1}^{k} \{ \varphi_i \leq \bar{c} + h_i \};$
- (v)  $\varphi_i(G(\theta)) = \varphi_+(\gamma_{i,h_i}(\theta_i)(\cdot p_i)).$

## A common pseudogradient vector field for $\varphi$ and $\varphi_i$

We point out that by (h5), the operator  $\varphi'(u)$  is close to  $\varphi'_{+}(u)$  for those elements  $u \in X$  with support at infinity, as stated in the next lemma (see [12; lemma 4.2] for a proof).

**Lemma 2.9.** For any  $\epsilon > 0$  and for any C > 0 there exists  $N \in \mathbf{R}$  such that  $\|\varphi'(u) - \varphi'_{+}(u)\| \le \epsilon,$ for any  $u \in X$  with  $\|u\| \le C$  and supp  $u \subseteq [N, +\infty)$ .

Next lemma states the existence of a common pseudogradient vector field for  $\varphi$ and  $\varphi_i$ .

Let  $(r_n) \subset (0, \bar{r}) \setminus D$ ,  $r_n \to 0$ , and  $(v_n) \subset K(\bar{c})$  be given by lemma 2.7. Let us fix  $r_{1,n}, r_{2,n}, r_{3,n}$  as above. Moreover, let us fix sequences  $(a_n), (b_n)$  and  $(\lambda_n) \subset \mathbf{R}_+$  such that  $[a_n - \lambda_n, a_n + 2\lambda_n] \subset (\bar{c} - h_n, \bar{c}) \setminus \Phi$  and  $[b_n - \lambda_n, b_n + 2\lambda_n] \subset (\bar{c} + h_n, \bar{c} + \frac{3}{2}h_n) \setminus \Phi$ . **Lemma 2.10.** There exist  $\mu_n = \mu_n(r_n) > 0$ ,  $N \in \mathbf{R}$  and  $\overline{\epsilon}_n = \overline{\epsilon}_n(r_n, a_n, b_n, \lambda_n) > 0$ such that: for any  $\epsilon_n \in (0, \bar{\epsilon}_n)$  there exists  $\bar{d}_n \in \mathbf{N}$   $(n \in \mathbf{N})$  for which for any  $k \in \mathbf{N}$ and  $p \in P(k,d)$ , with  $p_1 > N$  there exists a locally Lipschitz continuous function  $\mathcal{W}: X \to X$  which verifies

 $(\mathcal{W}_0) \max_{1 \le i \le k} \|\mathcal{W}(u)\|_{I_i} \le 1, \, \varphi'(u)\mathcal{W}(u) \ge 0, \, \forall \, u \in X, \, \mathcal{W}(u) = 0 \, \forall \, u \in X \setminus B_3;$ 

- $(\mathcal{W}_1) \ \varphi'_i(u)\mathcal{W}(u) \ge \mu_i \text{ if } r_{1,i} \le \|u v_i(\cdot p_i)\|_{I_i} \le r_{2,i}, \ u \in B_2 \cap \{\varphi_i \le b_i\};$
- $(\mathcal{W}_2) \ \varphi_i'(u)\mathcal{W}(u) \ge 0 \ \forall u \in \{b_i \le \varphi_i \le b_i + \lambda_i\} \cup \{a_i \le \varphi_i \le a_i + \lambda_i\};$

 $(\mathcal{W}_3) \ \langle u, \mathcal{W}(u) \rangle_{M_i} \ge 0 \ \forall i \in \{0, \dots, k\} \ \text{if } u \in X \setminus \mathcal{M},$ 

where  $\mathcal{M} \equiv \bigcap_{i=0}^{k} B_{\sqrt{\epsilon}}(0; M_i)$  and  $B_j \equiv \bigcap_{i=1}^{k} B_{r_{i,i}}(v_i(\cdot - p_i); I_i)$  for j = 1, 2, 3. Moreover if  $K \cap B_2 = \emptyset$  then there exists  $\mu_p > 0$  such that  $(\mathcal{W}_4) \ \varphi'(u)\mathcal{W}(u) \ge \mu_p, \ \forall u \in B_2.$ 

### Approximating k-bump solutions

**Theorem 2.11.** If U satisfies (h1), (h2) and there exists  $U_+$  for which (h1)-(h5)and (\*) hold then for any given sequence  $(\rho_n) \in \mathbf{R}_+$  there exist  $N \in \mathbf{R}$  and a sequence  $(d_n) \subset \mathbf{N}$  such that for every  $k \in \mathbf{N}$  and  $p \in P(k, d)$ , with  $p_1 > N$ , we have  $K \cap \bigcap_{i=1}^{k} B_{\rho_i}(v_+(\cdot - p_i); I_i) \neq \emptyset$ , where  $v_+$  is given by lemma 2.7.

*Proof.* Arguing by contradiction there exists a sequence  $(\rho_n) \subset \mathbf{R}_+$  such that for any  $N \in \mathbf{R}$  and for any sequence  $(d_n) \subset \mathbf{N}$  there exist  $k \in \mathbf{N}$  and  $p \in P(k, d)$ , with  $p_1 > N$ , for which  $K \cap \bigcap_{i=1}^k B_{\rho_i}(v_+(\cdot - p_i); I_i) = \emptyset$ . Let  $(r_n) \subset (0, \overline{r}) \setminus D, r_n \to 0$ and  $(v_n) \subset K(\bar{c})$  be given by lemma 2.7. Without loss of generality, passing to a subsequence if necessary, we can assume  $B_{2r_n}(v_n) \subset B_{\rho_n}(v_+)$  for any  $n \in \mathbb{N}$ .

Let  $\mu_n$  and  $\overline{\epsilon}_n$  be given by lemma 2.10. Let us define  $\Delta_n = \frac{1}{4}\mu_n(r_{2,n} - r_{1,n})$ . Then, we fix  $h_n < \frac{1}{8}\Delta_n$ ,  $a_n$  and  $b_n$  as above, with  $b_n - a_n < \frac{1}{4}\Delta_n$  and  $0 < \epsilon_n < \frac{1}{4}\Delta_n$  $\min\{\bar{\epsilon}_n, \frac{1}{4}\delta_{r_{n-1}}^2, \frac{1}{4}\delta_{r_n}^2, \frac{1}{8}(\bar{c}-a_n)\} \text{ such that } \int_I |V_+(t,u) \, dt \leq \frac{1}{8} ||u||_I^2 \text{ for } |I| \geq 1.$ 

Now, we fix  $d_n > \max{\{\tilde{d}_n, \tilde{d}_n, 2\}}$  and such that  $\max{\{\|v_{n+1}\|_{t<-d_n}^2, \|v_n\|_{t>d_n}^2\}} < \epsilon_n$ , where  $\tilde{d}_n$  is given by corollary 2.8 and  $\bar{d}_n$  by lemma 2.10.

For these values of  $d_n$  and for N given by lemma 2.10 there exist  $k \in \mathbf{N}$  and  $p \in P(k, d)$ , with  $p_1 > N$ , for which  $\bigcap_{i=1}^k B_{r_i}(v_i(\cdot - p_i); I_i) \subseteq \bigcap_{i=1}^k B_{\rho_i}(v_+(\cdot - p_i); I_i)$ . So that, by the contrary assumptions, there exists a vector field  $\mathcal{W}$  satisfying properties  $(\mathcal{W}_0)$ - $(\mathcal{W}_4)$ .

Now, we consider the flow associated to the Cauchy problem

$$\frac{d\eta}{ds} = -\mathcal{W}(\eta) \,, \quad \eta(0, u) = u$$

to deform the surface G given by corollary 2.8 for this values of  $(h_i)_{i=1,\dots,k}$  and  $(p_i)_{i=1,\dots,k}$ .

Since  $\mathcal{W}$  is a bounded locally Lipschitz continuous vector field, the flow  $\eta$  is globally defined. Moreover, by  $(\mathcal{W}_0)$  the flow does not move the points outside  $B_3$ . This implies, by corollary 2.8(i),

$$\eta(s, G(\theta)) = G(\theta) \quad \forall \theta \in \partial Q, \quad \forall s \in \mathbf{R}.$$
(2.12)

Since  $\varphi(B_2)$  is a bounded set, by  $(\mathcal{W}_4)$  there exists  $\tau > 0$  such that for any  $\theta \in Q$  for which  $G(\theta) \in B_1$  there is  $[s_1, s_2] \subset (0, \tau]$  with  $\eta(s_1, G(\theta)) \in \partial B_1$ ,  $\eta(s_2, G(\theta)) \in \partial B_2$  and  $\eta(s, G(\theta)) \in B_2 \setminus \overline{B}_1$  for any  $s \in (s_1, s_2)$ . Therefore for any  $\theta \in Q$  there is an index  $i = i(\theta) \in \{1, \ldots, k\}$  such that, by  $(\mathcal{W}_1), \varphi_i(\eta(s_2, G(\theta))) \leq \varphi_i(\eta(s_1, G(\theta))) - 2\Delta_i$ . By corollary 2.8(*iv*) and since, by  $(\mathcal{W}_2)$ , the sets  $\{\varphi_i \leq b_i\}$  and  $\{\varphi_i \leq a_i\}$  are positively invariant, we obtain  $\varphi_i(\eta(s_2, G(\theta))) \leq b_i - 2\Delta_i < a_i$  and hence  $\varphi_i(\eta(\tau, G(\theta))) \leq a_i$ . Moreover, by (*iii*) of corollary 2.8, for any  $\theta$  for which  $G(\theta) \in X \setminus B_1$  there exists  $i = i(\theta)$  such that  $\eta(s, G(\theta)) \in \{\varphi_i \leq a_i\}$  for any  $s \in \mathbf{R}_+$ . Hence, setting  $\overline{G}(\theta) = \eta(\tau, G(\theta))$ , we get

$$\forall \theta \in Q, \ \exists i \in \{1, \dots, k\} \text{ such that } \varphi_i(\bar{G}(\theta)) < a_i.$$
(2.13)

By (2.13) we have that

(2.14) there exists  $i \in \{1, \ldots, k\}$  and  $\xi \in C([0, 1], Q)$  such that  $\xi(0) \in \{\theta_i = 0\}, \xi(1) \in \{\theta_i = 1\}$  and  $\varphi_i(\bar{G}(\theta)) < a_i$ , for any  $\theta \in \operatorname{range} \xi$ .

Indeed, assuming the contrary, the set  $D_i = \{\theta \in Q : \varphi_i(\overline{G}(\theta)) \ge a_i\}$  separates in Q the faces  $\{\theta_i = 0\}$  and  $\{\theta_i = 1\}$ , for any  $i \in \{1, \ldots, k\}$ . Then, using a Miranda fixed point theorem, it follows that  $\bigcap_i D_i \neq \emptyset$ , in contradiction with the property (2.13) (see [15]).

By  $(\mathcal{W}_3)$ , the set  $\mathcal{M}$  is positively invariant under the flow. Then, by corollary 2.8(ii),

$$\eta(s, G(Q)) \in \mathcal{M} \quad \forall s \in \mathbf{R}_+.$$
(2.15)

Now, let us take a cut-off function  $\chi \in C^{\infty}(\mathbf{R}, [0, 1])$  with  $\sup_{t \in \mathbf{R}} |\dot{\chi}(t)| < \frac{1}{2}$ (this can be done since  $\inf_{i=1...k} d_i > 2$ ) such that  $\chi(t) = 1$  if  $t \in I_i \setminus M$  and  $\chi(t) = 0$  if  $t \in \mathbf{R} \setminus I_i$ , where the index *i* is given by (2.14). Then  $\|\chi u\|_{I_i \cap M}^2 \leq 2\|u\|_{I_i \cap M}^2$  and  $\|(1-\chi)u\|_{I_i \cap M}^2 \leq 2\|u\|_{I_i \cap M}^2$  for any  $u \in X$ . We define a path  $g : [0,1] \to X$  by setting  $g(s) = \chi \overline{G}(\xi(s)), s \in [0,1]$ . By (2.12) and lemma 2.7(v), we have that  $g(0) = \gamma_{i,h_i}(0)(\cdot - p_i)$  and  $g(1) = \gamma_{i,h_i}(1)(\cdot - p_i)$ . We finally show that  $g(s) \in B_{\bar{r}}(v_+(\cdot - p_i)) \cap \{\varphi_+ < \bar{c}\}$ , for any  $s \in [0, 1]$  contradicting lemma 2.7(*ii*). In fact one easily get that range  $g \subset B_{2r_i}(v_i(\cdot - p_i))$ .

Then, setting  $u = \bar{G}(\xi(s))$ , we have  $\varphi_+(g(s)) = \varphi_i(g(s)) \le \varphi_i(u) + \frac{1}{2} \|\chi u\|_{I_i \cap M}^2 + \int_{I_i \cap M} (V_+(t, u) - V_+(t, \chi u)) dt \le a_i + 4 \|\chi u\|_{I_i \cap M}^2$ . Since, by (2.15)  $\|u\|_{I_i \cap M} \le \epsilon_i + \epsilon_{i-1}$ , we get  $\varphi_+(g(s)) < \bar{c}$ .

**Remarks.** (i) The multibump homoclinic solutions of (HS) given by theorem 2.11 are close to translations of  $v_+$  in the  $H^1$  norm on suitable intervals. Hence they are close in the  $C^0$  norm and, since they verify (HS), in the  $C^1$  norm, too.

(*ii*) Taking the  $C_{loc}^1$  closure of the set of the multibump homoclinic solutions of (HS), using the Ascoli–Arzelà theorem, we get solutions with infinitely many bumps, as stated in theorem 1.1.

(*iii*) Corollary 1.2 follows from theorem 1.1, taking a sequence  $r_n \to 0$  and any sequence  $(\sigma_n)$  with infinitely many 1's. Thus we have multiplicity both for the arbitrariness of  $(r_n)$  and for the arbitrariness of  $(\sigma_n)$ .

(iv) Similar arguments apply to prove theorem 1.3. We refer to [12] for the construction of multibump solutions of mixed type.

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