Tacit vs Explicit Images of Mathematical Logic: the reflexions of the School of Peano

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Explicit Versus Tacit Knowledge in Mathematics

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ABSTRACT. This workshop aimed to bring together an international group of historians of mathematics to reflect upon the role played by tacit knowledge in doing mathematics at various times and places. The existence of tacit knowledge in contemporary mathematics is familiar to anyone who has ever been given an idea of how a particular proof or theory "works" by a verbal analogy or diagrammatic explanation that one would never consider publishing. Something of it is felt by every student of mathematics, when the process of learning mathematics often amounts to training the right reflexes. In more advanced contexts, the tacit understanding that a particular technique, instrument or approach is "the one to use" in a given circumstance gives another familiar instance. Tacit knowledge, a term introduced by the philosopher M. Polanyi, contrasts with the explicit knowledge that in almost all historical mathematical cultures is associated with mathematical text. The workshop invited a use of the categories of tacit and explicit knowledge to achieve a better knowledge of how mathematical creation proceeds, and also of how cultural habits play a tacit role in mathematical production. The meeting intended to offer the possibility of significant innovation and enrichment of historical method, as well as new and compelling insight into the process of creating mathematics in different times and places. The meeting was intended to afford the opportunity for a presentation of selected case studies by leading experts and new scholars. In retrospect, as we hope these abstracts show, the results promise to be of significant interest not only to historians, but to the mathematical community more broadly.

Introduction by the Organisers

The aim of this workshop was to bring together an international group of historians of mathematics to reflect upon the role played by *tacit, as opposed to explicit knowledge* in doing mathematics at various times and places. Methodological discussions on the use of this concept alternated with specific case studies from the history of mathematics. The aim was to allow a better understanding of mathematical practices in given contexts. The theme impinges on the transmission of existing mathematics as well on the creation of new theories and results.

The existence of tacit knowledge in contemporary mathematics is familiar to anyone who has ever been given an idea of how a particular proof or theory "works" by a verbal analogy or diagrammatic explanation that one would never consider publishing. Something of it is felt by every student of mathematics, when the process of learning mathematics often amounts to training the right reflexes. In more advanced contexts, the tacit understanding that a particular book or paper or approach is "the one to use" in a given circumstance gives another familiar instance. The theme was specifically chosen for this meeting on the history of mathematics in view of its inspirational and unifying potential, and in the ways that it promised to shed light on methods for understanding mathematical texts and practices of the past. Originally, our plan was to look at cases that range from the most ancient history of mathematics to current developments. We include here the original list of examples, and the reader can compare this to the actual papers, which achieved a comparable breadth while highlighting rather different features:

- The difference between algorithmic mathematics (like in ancient Mesopotamia or medieval China) and proof-oriented mathematics in the Euclidean tradition and the intermediate stages, like Chinese two-column algorithmic texts which are proof-driven but not in the Euclidean style are all too often analyzed without taking into account the parts of the practice that remain tacit and are not spelled out in the text, contributing thus to give a biased image of that difference.
- Tacit knowledge is present in various ways throughout the mathematical exchanges of the seventeenth century. Correspondence by letters included knowledge on how to write a letter, without spelling out the rules of letter writing. In cases where these tacit codes were not applied, it is interesting to give an interpretation of this step aside. More generally, tacit rules of scientific exchange dictated what was to be made explicit or public in a mathematical proof, and which parts were not. On the mathematical level, curves were identified by a catalogue of properties, which was never explicitly listed in its entirety. For instance, as soon as a curve was found to have the property that its subtangent is the double of the abscissa, it was identified with a parabola.
- A good deal of the development of mathematics in the nineteenth and twentieth centuries can be viewed as a process of making the practice of mathematics increasingly explicit, thereby reducing the amount of tacit knowledge and thus opening up a wide space of rational discussion and achievements. However, this tendency to greater technical explicitness,

which is evident in the typical manuscripts posted by mathematicians on ArXiv every day, may induce historians of mathematics to neglect the persistence of tacit knowledge in the most recent mathematics. The identification of such tacit elements seems capable of affording significant insights into the development of mathematics today.

- Similarly, several large scale mathematical enterprises of the last 100 years like Bourbaki's *Éléments de mathématique* or – in a different manner – computer-based mathematical research, like the more recent projects towards automated theorem proving (ATP), appear at first as signposts of a massive pushing back of tacit knowledge. Looking more closely, however, at details like the occasional warning signs in the margins of Bourbaki's volumes, or at problems related with the user interface, one sees that these undertakings carry in fact their own heavy collection of tacit mathematical practice.

- Developments in the history of mathematics are often loosely described as moving from approximate, incompletely understood treatments, to fully explicit, formal statements and their rigorous proofs. (See for instance Breger's contribution to [1].) Paying attention to the kind of tacit knowledge which is mobilized before and after such a development often provides a much more satisfactory analysis of the historical process than the mere confrontation of precise *versus* imprecise methods. A case in point is the rewriting of Algebraic Geometry in the first half of the twentieth century. In a 1926 letter to Hermann Weyl, Salomon Lefschetz significantly described the Italian school of Algebraic Geometry, not as lacking rigor, but as requiring "a terrible entraînement". Later attempts, by Francesco Severi and others, to defend their classical Algebraic Geometry against growing criticism would invariably insist on the fact that all those technical assumptions or arguments which the modern algebraists could not find in the Italian papers where indeed tacitly assumed, and well-known to all geometers raised in the Italian school. The question whether the category of tacit knowledge may render such arguments historically convincing appears quite difficult, and can only be decided by very detailed case studies.

- In contemporary mathematics, blogs and Wikis – the most famous probably being Terence Tao's – currently provide an extended form of oral culture in which less formal, formerly tacit approaches are written down and opened to a broad mathematical public according to shifting and variable rules.

The term "tacit knowing" or "tacit knowledge" which we explored here in its bearing on the history of mathematics, comes from a philosophical context, but has been mobilized before for the history of science. Michael Polanyi introduced "tacit knowing", or "tacit knowledge" in order to describe abilities which cannot be fully described or explained (see [4]). In the history of science, the concept has been mobilized in the study of the craft aspects of experimental science from

the seventeenth century to the present day. The philosophical theory of tacit knowledge has been much discussed over the years for instance also in the context of mathematical education and curricula, which is not the purpose of the workshop proposed here. More recently, the sociologist Harry Collins reassessed this notion in [2], in particular distinguishing several types of tacit knowledge.

The theory of tacit knowledge marks a counterpoint to the "ideal of wholly explicit knowledge" which took shape through the scientific revolution of the seventeenth century. Among the different interpretations which have been given of the concept of "tacit knowledge", from a conscious under-articulation or deliberate secrecy to the strong thesis that there are specific kinds of knowledge that cannot in principle be fully articulated – the standard example being here riding a bike – the application to the history of mathematics will focus on the weak sense: tacit knowledge is what mathematicians selectively conceal, avoid articulating or under-articulate, consciously or not. This does include the possible concealment of information by mathematicians competing with others, as well as the case of descriptions which are left incomplete because their authors assume, or know by experience, that their readers share a certain knowledge with them. Tacit knowledge is then built on experience or action, and cannot be fully described by rules or words. It concerns any type of knowledge or skill used as subsidiary to the performance and control of a mathematical task. The notion of tacit knowledge could be applied to the history of mathematics, as suggested by Breger ten years ago who used the greater level of abstraction created by the ongoing development of mathematics to detect tacit elements in earlier texts. This is a challenging thesis but obviously history of mathematics should not be reduced to just re-reading old texts through the spectacles of more modern mathematical achievements.

At this point, more recent methods in the history of mathematics come to the rescue: following a tradition that can be traced back to Ludwig Wittgenstein and other authors of the 1930s and 1940s, the second half of the twentieth century has seen authors such as Imre Lakatos, Paul Feyerabend, and Hans-Jörg Rheinberger placing the detailed analysis of scientific practice at the heart of history of science. This goes hand in hand with the realization that tacit scientific knowing is acquired by the individual scientist through a social context or network whose members share a common know-how. Although unstated know-how tends to be difficult to identify in a single mathematical text, shared tacit knowledge or know-how is more accessible, often by way of comparison with other local mathematical cultures or broader networks. It also tells a lot about mathematical (and strategic) practices in a specified time period.

In the case of mathematics, Epple has adapted Rheinberger's approach to the history of mathematics in his book [3] on the history of knot theory. The notions of *epistemic objects* and *epistemic techniques* are his key concepts to describe the ways of the active researchers to handle the complex web of established theories ready for use, formal and informal operational skills to deal with new phenomena, and often vague general ideas about the kind of mathematical object under focus.

Furthermore, the mathematical tools made use of in specific contexts or sites are in most cases abstract techniques or objects, but may also be material devices, from the measuring rod and compass to the analog integrator and computer. In the Renaissance and early modern periods, the design and use of such instruments was a core feature of mathematical practice, and the tacit knowledge involved in acquiring the techniques of use or design was considerable. Yet such knowledge has left historical evidence: Albrecht Dürer, most famously, tried to describe explicitly what perpective artists were actually doing, including the gestures transmitted through long workshop traditions. One aim of the conference will be to assess the degree of continuity between these older traditions and those in evidence in more recent mathematical practice.

Our main objective for the conference proposed here was thus to use the peculiar bias of the distinction between tacit and explicit knowledge in order to re-invigorate discussions about how the analysis of social networks on the one hand and of the research practice of mathematicians on the other come together to afford a close-up understanding of the historical process which we call mathematics. Last but not least we hoped it would allow a better understanding of how mathematical practices depend on larger cultural habits, or are embedded in larger cultural contexts, including language, writing cultures, literary and rhetorical devices, and craft knowledge.

The abstracts below show that the chosen theme has proved inspiring for most of the speakers, whom it enabled to highlight aspects of the production and transmission of mathematics which have often been neglected. The use of instruments, which may imply a lot of bodily skills which can scarcely be transmitted through words, is typically an example of such an understudied aspect of mathematics. The practice of skills was also one of the starting points of Michael Polanyi in his 1958 *Personal Knowledge*, as Jeanne Peiffer recalled in her short introduction to the workshop. Before describing the tacit component of The Art of Knowing, Polanyi suggests to grasp "the nature of the scientist's personal participation by examining the structure of skills" ([4, 49]and as his clue for this investigation, he takes what he calls the well-known fact "that the aim of a skilful performance is achieved by the observance of a set of rules which are not known as such to the person following them" (ibid.). For Polanyi, an art, a skill, which cannot be specified in detail – think at the famous example of riding a bike – cannot be transmitted by prescription, since no such prescription exists. It can be passed on only by example from master to apprentice. It follows that an art which has fallen into disuse for the period of a generation is altogether lost. And here questions for the historian come in, mostly methodological questions. How can we, as historians, recover not specified, not explicated skills, arts or knowledge ? Besides methodological reflections, a whole range of case studies have been presented by the participants of the workshop which have shown the various forms of tacitness.

Norbert Schappacher in his introduction briefly reminded the audience of Michael Polanyi's record:

(a) As a researcher, in particular as director of the chemical-kinetics research group in Fritz Haber's *Kaiser-Wilhelm-Institute* for physical chemistry and electrochemistry in Berlin-Dahlem starting in 1923; see [5], esp. chap. 2 and 3; cf. Polanyi's ranking among the leading scientists of the Kaiser-Wilhelm-Gesellschaft at the time in [6], vol. 2, p. 1254.

(b) As a thinker on economic theory, fighting the rather marxist tendencies of his brother Karl. Hachtmann in [5], vol. 1, pp. 31-32, points to text of Polanyi's from as early as 1930, on the return of investment into the sciences (*Rentabilität der Wissenschaften*) which kind of anticipated, in the concrete context of the *Kaiser-Wilhelm-Institutes* [KWIs] menaced by spending cuts after the big economic crisis, Pierre Bourdieu's later theory of the exchangeability of *actual, cultural, symbolic*, and *social* capital.

(c) Of the later unfolding of Polanyi's ideas of personal later tacit knowledge, driven by a desire (probably partly inspired by Ludwik Fleck) to balance Popper's so-called *Logic of scientific discovery* by more genuine descriptions of scientific practice, and by a more Gestalt-theoretic approach of scientific work, and its part in human culture at large.

More to the point of the subject of this meeting, i.e., the history of mathematics, Polanyi's letter to Lakatos of August 14, 1961 (from the Archives of the London School of Economics; thanks to H.J. Dahms for sharing it with us) was quoted, written in response to reading a draft of Lakatos' *Proofs and Refutations.*There one reads in particular : "If you are interested to find out as I am, how it can be that these procedures of acquiring what we call knowledge, do in fact lead to something that is knowledge, though it is, and must remain, impossible to define these procedures, or set up criteria of their success, without appealing to powers which are defined by no rules, then one feels that to speak of conjectures and refutations etc. as answering my question, is to beg it."

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Workshop: Explicit Versus Tacit Knowledge in Mathematics

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approaches have their well-known problems, namely to find a systematic basis and the fit with actual phenomena. Pasch focused on mathematical practice and contentful reasoning, which left hims struggling with what we now call the surface structure of language and ultimately led to unsurmountable problems, like those of identity of propositions and sameness of content. Because of this, and despite the fact that Pasch knew some of Frege's, Peano's, and Hilbert's work, he failed (or refused) to latch on to the development of modern logic. By looking sympathetically at such failed attempts we get a better understanding of the conceptual difficulties that were involved in arriving at the modern conception of logic.

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Tacit vs Explicit Images of Mathematical Logic: the reflexions of the School of Peano

Erika Luciano

Since the end of the 19th century, the Peano School presents itself and is presented as a group of scholars whose aim was to make their own scientific works as explicit as possible. In fact, according to Peano, all the hypotheses and deductive steps, even the most banal, must always be made explicit both in publications and in teaching practices. Moreover, the ideography was constructed in such a way that the meaning attached to the symbols was completely and unambiguously clarified. At the level of history as well, each definition and proposition was to be accompanied by an explicit account of its origin and development.

In actual fact, however, this historiographical depiction is less precise than might be supposed. In fact, in spite of the intentions declared, there are frequent cases of 'missing'- that is tacitly used - propositions in the works of the Peano School. Besides, the contrast between tacit and explicit regards the very adoption of the symbols, because some of Peano's collaborators confine the ideography to the private sphere of their research, masking its use in publications. As regards the historical aspects as well, that which is explicit in the Peanian written production is comprised of only a set of notes which includes bibliographical references and transcriptions of extracts from sources. What is implicit concerns all the rest, that

is 1) the majority of primary and secondary literature used and 2) if there exists a non-naive historiographical conception underlying the compilation of these notes. Further, as regards this kind of extra-mathematical elements, it is necessary to bear in mind that a stereotyping of roles within the Peano School becomes evident fairly early on. According to it, G. Vailati was 'the philosopher', G. Vacca 'the historian', M. Pieri 'the geometer', and so on. This makes it even more difficult to distinguish the tacit from the explicit dimension, because some of Peano's collaborators were prone to remain silent about certain components of their own research, leaving to their colleagues, who were considered the 'specialists' in one area or another, the task of explaining them.

In light of this overall picture, first of all we determine how, in the particular case-study of the Peanian logical-foundational studies, the terms of the tension tacit *vs* explicit are to be specified, taking into consideration the fact that, with this expression, we allude to a very composite body of knowledge, constructed by a community of scholars.

As far as the side of explicit is concerned, the problem consists in choosing a printed work that is, or can be considered as, representative of the activities carried out by the School of Peano. Among the possible alternatives [1] we have chosen the three last editions of the *Formulaire de Mathématiques* (1901, 1902-03, 1908) to represent the explicit side of the dichotomy because

- this is the only work that, at least along a general line, can be considered representative of the School of Peano *as a whole*; [2]

- this is the work for which we possess the majority of the unpublished sources, which makes it possible to identify some of the tacit aspects of the mathematics produced in the Peano School;

- both inside and outside the School, this treatise was indicated as the *summa* of the research of Peano and his collaborators.

Instead, we have chosen to entrust the opposite side of the tension, that is the tacit aspect, to the oral dimension that surrounds the editing of the *Formulaire*. In fact, the close relationships that mature between the members of a working group are based upon everyday contact which is essentially oral in nature and lead

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[1]They include, for example, the journals *Rivista di Matematica* (1891-1906), *Schola et Vita* (1925-1939) and some works by individual members of the Peano's School, such as the inaugural lecture given by Pieri in Catania, *Uno sguardo al nuovo indirizzo logico-matematico delle scienze deduttive* (Annuario R. Univ. Catania, 1906-07, p. 21-82), or the article by A. Padoa, *Logica matematica e matematica elementare* (Atti II Congresso Associazione Mathesis, Livorno, 1902, p. 186-200), called 'the manifesto of the Italian logicians'.

[2]In effect, according to U. Cassina, the Peano School comprised some forty researchers, but only a part of those took part in editing the *Formulaire*. These are the ones that we indicate as the 'first generation' of Peano's students, including G. Vailati, G. Vacca, C. Burali-Forti, A. Padoa, M. Pieri, the second generation being constituted, with few exceptions, by the Peano's interlinguistic collaborators and by the group of his female students who worked with him during the last years of his life. Further, some students of this first generation provided just very marginal additions to the *Formulaire*; others stopped the collaboration after the first edition, other ones after the third or the fourth.

inevitably to the creation and the socialization of a massive amount of tacit mathematical knowledge. Further, these oral testimonies, and in particular the conversations which many of Peano students remember vividly, can be reconstructed today thanks to the correspondences among Peano, Vacca, Vailati, Pieri, Padoa and Burali-Forti, the manuscripts conserved in the *Peano-Vacca Archive* in Turin and the volumes of the *Formulaire* with Peano and Vacca's autograph notes.

Once our dichotomy is fixed in terms of orality *vs* publication in the last three editions of the *Formulaire*, the mathematics produced in the Peano School becomes a collection of tacit elements. Thus we discover that remaining altogether implicit are the proofs of the Cantor-Bernstein theorem devised by Vacca with the help of Vailati and Burali-Forti, and the systems of postulates of arithmetic proposed by Vacca and Pieri. Examples of this same tenor could be listed by the dozen.

It is also possible to discern a criterion, obviously tacit, used by Peano to determine, from among the proposals submitted to him, what to promote to the level of publication in the *Formulaire*. The tacit dimension is in fact more evident with regard to meta-theoretical problems such as the criteria for choosing the primitive concepts and propositions; the consistence and independence of the axioms; the ways of schematizing language between the opposing poles of natural language and ideographical symbolism; the sensitivity with regard to adherence to physical or psychological reality of mathematical concepts, as opposed to their abstract and formalist connotations, etc. The divergences regarding these questions - or at least the differences in opinion among Peano's collaborators - are remarkable. In the absence of an agreement, Peano seem usually choose to gloss over such reflections in the *Formulaire*, to the point of rendering the positions of individuals indistinguishable, or even leaving them entirely implicit, omitting reference to the literature that lay behind them, which also differed notably from student to student.[3]

In light of this context, we illustrate in the second part of the talk:

- an example of a result by Padoa (his method for proving the independence of the arithmetical postulates), which was promoted in its essence to the explicit degree, after having circulated for a very brief time in tacit form;

- an example of a critical comment, by Vailati, about the chapter *Logique* of the *Formulaire*, which remained tacit in its entirety forever, in spite of attempts to render it explicit in the two last editions of the treatise;

- an example of a contribution by Pieri and Padoa (an hypothetical-deductive system for the Euclidean geometry based on two primitive concepts), which remained tacit almost completely and almost forever, even though it had

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[3]There was a circulation of volumes and articles that were shared by Peano's entourage. Not by chance, the same publications by L. Couturat, E. Huntington, B. Russell, among the few to be cited in the *Formulaire*, are present in the personal libraries of Peano, Vacca and Vailati. Flanking this, however, was a set of readings (G. Frege, D. Hilbert, F. Brentano, A. Naville, ...) recommended by the individual members of the School, appreciated by *some* but not *all* of the colleagues. In a few cases this second type of references was noted by Peano in his *marginalia* and even more rarely it reached the level of explicit quotations in the *Formulaire*.

arrived 'a step away' from being made explicit in the fourth edition of the *Formulaire.*

Taking into account these examples, the question arises whether Burali-Forti, Padoa, Vacca and Vailati's allusions to Peano's 'bad tendencies' towards his collaborators are true or not. According to our analysis, it appears plausible to maintain that Peano was led to render and maintain explicit in the *Formulaire* all and only those results (his own as well as others) that entered into the spirit of the treatise as *he* conceived it and that *he* held to be best from the scientific or didactic point of view, inviting the individual contributors themselves to make explicit, through articles, the contributions that had not found a place in the treatise. Faced with proposals for substantial modifications, Peano could become quite cutting. The fact that the dissemination of the views with respect to meta-mathematical problems is prevalently entrusted to the tacit dimension is further ascribable to the cultural context, which made it politically opportune to emphasize some implications while downplaying others, and which drove the School of Peano to make explicit only what was widely shared, in the attempt to consolidate the image of a cohesive group of researchers.

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Tacit Knowledge in Mathematics: Definition, Types, Examples
Herbert Breger

Tacit knowledge in mathematics is defined to be knowledge that is essential for the understanding of a mathematical theory although it cannot be deduced from the axioms resp. the presuppositions which were generally accepted at that time[1][2][3]. Three arguments were given in order to show that there is tacit knowledge in this sense. Firstly, a computer to which the axioms of a particular theory are given cannot produce a textbook of that theory. Secondly, experts use undefined words like beautiful, natural, deep, profound, exotic, elegant in order to express statements about the importance of a notion or a proof or on the structure of a theory. This is knowledge on the meta level. Thirdly, it is well-known in mathematical education that understanding a proof line by line does not necessarily imply to understand the proof as a whole.

In the second part of the talk a number of types of situations in which tacit knowledge occurs were discussed. There is a tacit knowledge of axiomatisation (illustrated with the examples of Eilenberg and Steenrod's axiomatisation of algebraic topology and Klein and Lie's considerations of groups as well as Eilenberg and Mac Lane's introduction of natural transformations). Historians are well aware of the problems of hindsight which originate from seeing a theory in the past with