Cohomological Properties of Algebras and Coalgebras in Monoidal Categories with Applications

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Cohomological Properties of Algebras and Coalgebras in Monoidal Categories with Applications.

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Introduction

A monoidal (or tensor) category consists of a category $\mathcal{M}$ which is endowed with a distinguished object $\mathbf{1}$ (called *unit*) and with a functor $\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ (called *tensor product*) obeying some axioms that guarantee its associativity (usually up to an isomorphism) and the "compatibility" with $\mathbf{1}$ (see Definition 1.2.1). In this thesis we are concerned with two main types of monoidal categories: abelian and coabelian monoidal categories. An abelian (resp. coabelian) monoidal category consists of a monoidal category $(\mathcal{M}, \otimes, \mathbf{1})$ where the underline category $\mathcal{M}$ is abelian with right (resp. left) exact and additive tensor functors. The first obvious example of abelian monoidal category (this will be called the *classical case*) is the category of vector spaces over a field K, where K plays the role of the unit and the tensor product is the usual tensor product over K. There are many other examples of (co)abelian monoidal categories, as the categories of left, right and two-sided (co)modules over a Hopf algebra H, or the category of Yetter-Drinfeld modules over a Hopf algebra H with bijective antipode (see also Section 5.5 and Section 6.2). We are also interested in studying bialgebras in monoidal categories. In the classical case, when defining the notion of bialgebra, the canonical flip of tensor factors $V \otimes_K W \cong W \otimes_K V$ is used in the compatibility condition between multiplication and comultiplication. Such a morphism does not exist in an arbitrary monoidal category. A braided monoidal category, is a monoidal category such that, for every $X, Y \in \mathcal{M}$, there is a natural isomorphism $X \otimes Y \cong Y \otimes X$, called braiding, satisfying suitable conditions and which formalizes the flip of tensor factors. The category of Yetter-Drinfeld modules over a Hopf algebra H with bijective antipode is an example of a braided monoidal category. As we will explain in more details later, bialgebras in the braided monoidal category of Yetter-Drinfeld modules play a fundamental role in the classification of finite dimensional Hopf algebras. Monoidal Categories were introduced in 1963 by Bénabou [Be] (see also [McL2]). Braided monoidal categories were introduced by Joyal and Street in [JS], motivated by the theory of braids and links in topology.

The aims. The purpose of this thesis is to present in an unifying manner some recent results concerning the cohomological properties of algebras and coalgebras inside the framework of abelian monoidal categories and to exhibit some applications related to the classification of finite dimensional Hopf algebras problem. More precisely, we show how to introduce the Hochschild cohomology in an abelian monoidal category and classify (co)algebras of Hochschild dimension less or equal to 1. As an application we show how the notion of formally smooth (co)algebra in monoidal categories is useful to prove that certain Hopf algebras can be described by means of a bosonization type procedure. The quoted cohomological results can be also applied to prove that, as in the classical case, the tensor algebra $T_A(M)$, where A is a formally smooth algebra and M is a projective A-bimodule in a monoidal category $\mathcal{M}$, is itself formally smooth as an algebra in $\mathcal{M}$. Furthermore $T_H(M)$ can be endowed with a braided bialgebra structure whenever H is a braided bialgebra in a braided monoidal category $\mathcal{M}$ satisfying suitable assumptions and M is an object in ${}^H_H\mathcal{M}^H_H$. Similar results are obtained for the cotensor coalgebra $T^c_C(M)$, where C is a coalgebra in $\mathcal{M}$ and M is a bicomodule over C. The introduction of the cotensor coalgebra and the proof of its universal property in a monoidal category $\mathcal{M}$ is not immediate because of the lack of the notion of coradical for coalgebras in $\mathcal{M}$. Therefore new ideas are often required. The cotensor coalgebra $T^c_H(M)$ becomes a braided bialgebra when H is a braided bialgebra in a braided monoidal category $\mathcal{M}$ satisfying suitable assumptions and M is an object in ${}^H_H\mathcal{M}^H_H$. We provide a universal property both for $T_H(M)$ and $T^c_H(M)$ and we use it to prove that there exists a bialgebra homomorphism $F : T_H(M) \to T^c_H(M)$. In this way we can define a new bialgebra, that is $\mathrm{Im}(F)$, which is the braided analogue of the so called "bialgebra of type one" introduced by Nichols in [Ni] in the classical case. The H-coinvariant part of this bialgebra is called "Nichols algebra". "Nichols algebras", first appeared in [Ni], are an example of braided bialgebras of type one constructed in the braided monoidal category of Yetter-Drinfeld modules.

In this thesis, we also present a proof of the Heyneman-Radford Theorem for Monoidal Categories. The original Heyneman-Radford Theorem (see [HR, Proposition 2.4.2] or [Mo, Theorem 5.3.1, page 65]) is a very useful tool in classical Hopf algebra theory. We also point out that our proof is pretty different from the classical one and hence might be of some interest even in the classical case. An expected future application of this result is the following characterization for a "braided graded bialgebra" B in a monoidal category $\mathcal{M}$: B is a bialgebra of type one if and only if the natural bialgebra homomorphism $T_{B(0)}(B(1)) \to B$ is surjective, i.e. B is generated as an algebra by its components of degree 0 and 1.

Historical references. Let K be a field. Hochschild cohomology $\mathrm{H}^*(A, M)$ of a K-algebra A with coefficients in a A-bimodule M was introduced in [Ho] in order to classify, up to equivalence, all extensions of A with kernel M. Many other applications of this cohomology have been discovered since then. Let us mention here a few of them.
An algebra A is called separable if A is projective as an A-bimodule. Separable

algebras are characterized as those algebras A having Hochschild dimension zero, that is $\mathrm{H}^1(A, M) = 0$, for every A-bimodule M (se e.g. [We] and [CQ] for other properties of separable algebras).

The set of equivalence classes of Hochschild extensions of A with kernel M is in one-to-one correspondence with $\mathrm{H}^2(A, M)$. In particular, an algebra A has only trivial extensions exactly when $\mathrm{H}^2(A, M) = 0$, for any bimodule M, i.e. its Hochschild dimension is less than or equal to 1. These algebras, called formally smooth algebras, were introduced by J. Cuntz and D. Quillen in [CQ], where they are called quasi-free algebras and play the role of "functions algebras" of a "noncommutative smooth affine variety". Dually, in [JLMS] the notion of formally smooth coalgebra was introduced and characterized by means of a suitable coextension property.

In [AMS3], the authors introduced the Hochschild cohomology in the frame of monoidal categories, they investigated the properties of Hochschild cohomology of (co)algebras in an abelian monoidal category, and they proved that all properties of separable and formally smooth algebras and coalgebras, that we mentioned above, hold true in this wider context. The main applications of this approach are included in [AMS1] where, using the "categorical" version of Wedderburn-Malcev Theorem, besides other results, bialgebras with (dual) Chevalley property are characterized (see Theorem 6.8.6 and Theorem 6.8.7). In [Ar1] further results in terms of formally smooth (co)algebras instead of (co)separable (co)algebras are found (see Theorem 6.8.1 and Theorem 6.8.4). This results are used to prove that every connected Hopf algebra E over a field K with $\mathrm{char}\,(K) = 0$ has a weak projection $\pi : E \to K\,[x]$, for every non zero primitive element x of E.

For a classical proof of formal smoothness of the tensor algebra $T_A(M)$, where A is a formally smooth algebra and M is a projective bimodule over A, see [CQ, Proposition 5.3(3)]. In [JLMS] a similar result is provided for the cotensor coalgebra introduced by Nichols in [Ni]. It is then natural to wonder whether these two facts still hold for monoidal categories. This led in [AMS3] to the study of the cohomological properties of the tensor algebra. Moreover, in [AMS2] the notion of cotensor coalgebra was introduced for a given bicomodule over a coalgebra in an abelian monoidal category $\mathcal{M}$. More precisely, if $\mathcal{M}$ is also cocomplete, complete and AB5, such a cotensor coalgebra exists and satisfies a meaningful universal property which resembles the classical one (where the notion of coradical take a fundamental rule). Here the lack of the coradical filtration is filled by considering a direct limit of a filtration consisting of wedge products. In [AMS2], it is also proved that this coalgebra is formally smooth whenever the comodule is relative injective and the coalgebra itself is formally smooth.

Let H be a Hopf algebra over a field K and let M be a Hopf bimodule. Then the subalgebra $H[M]$ of the cotensor coalgebra $T^c_H(M)$ generated by H and M was firstly studied by Nichols and called "bialgebra of type one". The canonical inclusion $\sigma : H \hookrightarrow H[M]$ has a retraction $\pi : H[M] \to H$ which is a bialgebra homomorphism. Via π and σ it is possible to define an isomorphism of vector spaces (see 6.8.3) $H[M] \cong R \otimes_K H$ where $R = H[M]^{co(H)} = \{x \in H[M] | \sum x_{(1)} \otimes_K \pi(x_{(2)}) = x \otimes_K 1\}$ and $\sum x_{(1)} \otimes_K x_{(2)}$ is the Sweedler's sigma notation for the comultiplication of the bialgebra $H[M]$. R comes out to be a braided bialgebra in the monoidal category of Yetter-Drinfeld modules and is usually called a "Nichols algebra". Through the quoted isomorphism, $R \otimes_K H$ inherits a bialgebra structure depending only on π and σ. This is an example of the so called bosonization and is denoted by $R\#H$. Now, given a Hopf algebra E whose coradical H is a Hopf subalgebra (i.e. E has the dual Chevalley property), the associated graded coalgebra $gr(E)$ is a Hopf algebra whose coradical is still H. If $gr(E)$ is generated as an algebra by the components of degree 0 and 1, then it is a bialgebra of type one. This is the main point in the celebrated "Lifting method" by Andruskiewitsch and Schneider: the general principle they propose is first to analyze $R = gr(E)^{co(H)}$, then to transfer the information to $gr(E)$ by bosonization, and finally to lift it from $gr(E)$ to E via the filtration (see, e.g., [AG] and [AS]). This approach turned out to be very fruitful in the classification of finite dimensional pointed (i.e. all simple subcoalgebras are one-dimensional) Hopf algebras process.

The structure. In chapter 1 we recall the notion of monoidal category and the main tools we will use in the sequel. In particular we show how the notions of algebra, module over an algebra, coalgebra and comodule over a coalgebra, relative tensor and cotensor products, ideal and wedge product can be introduced in the general setting of monoidal categories. We also prove the Heyneman-Radford Theorem for Monoidal Categories.

In chapter 2 we deal with some results concerning the theory of relative left derived functors that will be necessary in defining and classifying the Hochschild cohomology in the frame of monoidal categories. We also recall and study the notion of relative projectivity and injectivity with a particular interest for those projective classes that are defined by means of suitable adjunctions related to the tensor functors.

In chapter 3, following [AMS3], we introduce and investigate the properties of Hochschild cohomology of algebras in an abelian monoidal category, and we show that many properties of separable and formally smooth algebras in the classical sense still hold true in this wider context.

In chapter 4 we introduce and study the properties of the tensor algebra inside the framework of monoidal categories. In particular, in section 4.3, the tensor algebra is endowed with a braided bialgebra structure that will be involved in the definition of a braided version of the notion of Bialgebra of type one.

In chapter 5, the concept of cotensor coalgebra for a given bicomodule M over a coalgebra C in an abelian monoidal category $\mathcal{M}$ is introduced and a universal property is given. We prove that this coalgebra is formally smooth whenever M

is relative injective and C is formally smooth. If $C = H$ is a braided bialgebra bialgebra inside a braided monoidal category $\mathcal{M}$ and M is an object in ${}^H_H\mathcal{M}^H_H$, the cotensor coalgebra is endowed with a braided bialgebra structure which is used to extend the notion of bialgebra of type one to the wider context of a braided monoidal category (see Definition 5.6.10). A universal property for the cotensor bialgebra is also given (see Theorem 5.6.8)

In chapter 6, following [Ar1], we provide a functorial characterization of *ad*-(co)invariant integrals and we show how the notion of formally smooth (co)algebra in monoidal categories is useful to prove that certain Hopf algebras can be described by means of a bosonizations type procedure. More precisely, we prove that given a bialgebra surjection $\pi : E \to H$ with nilpotent kernel such that H is a Hopf algebra which is formally smooth as a K-algebra, then π has a section which is a right H-colinear algebra homomorphism (Theorem 6.8.1). Moreover, if H is also endowed with an *ad*-invariant integral, then the section can be chosen to be H-bicolinear (Theorem 6.6.17). Dually, we prove that, if H is a Hopf subalgebra of a bialgebra E which is formally smooth as a K-coalgebra and such that $Corad(E) \subseteq H$, then E has a weak right projection onto H (Theorem 6.8.4). Furthermore, if H is also endowed with an *ad*-coinvariant integral, then the retraction can be chosen to be H-bilinear (Theorem 6.7.19).

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Chapter 1

Monoidal Categories

In this chapter we recall the notion of monoidal category and the main tools we will use in the sequel. In fact, the notion of monoidal category codifies in categorical terms the properties that allow to have an associative tensor product which is compatible with K in the category of vector spaces over a field K (this will be called the classical case). When working in a monoidal category, new techniques are required as objects need not to be sets. In this way the proofs are pretty different from the classical ones and hence might be of some interest even in the classical case. The number of different examples of monoidal categories is another reason for working in this wider context. In this way one can recover, in an unifying manner, many well known results. The notions of algebra, module over an algebra, coalgebra and comodule over a coalgebra, relative tensor and cotensor products, ideal and wedge product are introduced in the general setting of monoidal categories.

1.1 Preliminaries and notations.

A category will be denoted by $\mathcal{M}, \mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ while $\mathfrak{M} = \mathfrak{M}_K$ will denote the category of vector spaces over a field K. A will denote an algebra, R, S rings and H a Hopf algebra.
In a category $\mathcal{M}$ the set of morphisms from X to Y will be denoted by $\mathcal{M}(X,Y)$. If X is an object in $\mathcal{M}$, then the functor $\mathcal{M}(X,-)$ from $\mathcal{M}$ to $\mathfrak{Sets}$ associates to any morphism $u : U \to V$ in $\mathcal{M}$ the map that will be denoted by $\mathcal{M}(X,u)$. We say that a morphism $f : X \to Y$ in $\mathcal{M}$ *splits* (respectively *cosplits*) or has a section (resp. retraction) in $\mathcal{M}$ whenever there is a morphism $g : Y \to X$ such that $f \circ g = \mathrm{Id}_Y$ (resp. $g \circ f = \mathrm{Id}_X$). In this case we also say that f is a splitting (resp. cosplitting) morphism .
Throughout, K is a field and we write $\otimes$ for tensor product over K. We use Sweedler's notation for comultiplications $\Delta(c) = c_{(1)} \otimes c_{(2)} = c_1 \otimes c_2$, and the versions ${}^C\rho(x) = x_{<-1>} \otimes x_{<0>} = x_{-1} \otimes x_0$ and $\rho^C(x) = x_{<0>} \otimes x_{<1>} = x_0 \otimes x_1$ for

left and right comodules respectively (we omit the summation symbol for the sake of brevity).
Let X, Y be objects and $f : X \to Y$ be a morphism in a monoidal category $(\mathcal{M}, \otimes, \mathbf{1})$. Set

$$\begin{array}{cccc} X^{\otimes 0} = \mathbf{1}, & X^{\otimes 1} = X & \text{and} & X^{\otimes n} = X^{\otimes (n-1)} \otimes X, \text{ for every } n > 1 \\ f^{\otimes 0} = \mathrm{Id}_{\mathbf{1}}, & f^{\otimes 1} = f & \text{and} & f^{\otimes n} = f^{\otimes (n-1)} \otimes f, \text{ for every } n > 1. \end{array}$$

Let $[(X, i_X)]$ be a subobject of an object E in an abelian category $\mathcal{M}$, where $i_X = i_X^E : X \hookrightarrow E$ is a monomorphism and $[(X, i_X)]$ is the associated equivalence class. By abuse of language, we will say that (X, i_X) is a subobject of E and we will write $(X, i_X) = (Y, i_Y)$ to mean that $(Y, i_Y) \in [(X, i_X)]$. The same convention applies to cokernels. If (X, i_X) is a subobject of E then we will write $(E/X, p_X) = \mathrm{Coker}(i_X)$, where $p_X = p_X^E : E \to E/X$.
Let $(X_1, i_{X_1}^{Y_1})$ be a subobject of Y_1 and let $(X_2, i_{X_2}^{Y_2})$ be a subobject of Y_2. Let $x : X_1 \to X_2$ and $y : Y_1 \to Y_2$ be morphisms such that $y \circ i_{X_1}^{Y_1} = i_{X_2}^{Y_2} \circ x$. Then there exists a unique morphism, which we denote by $y/x = \frac{y}{x} : Y_1/X_1 \to Y_2/X_2$, such that $\frac{y}{x} \circ p_{X_1}^{Y_1} = p_{X_2}^{Y_2} \circ y$:

$$\begin{tikzcd} X_1 \arrow[r, hook, "i_{X_1}^{Y_1}"] \arrow[d, "x"'] & Y_1 \arrow[r, "p_{X_1}^{Y_1}"] \arrow[d, "y"'] & \frac{Y_1}{X_1} \arrow[d, "\frac{y}{x}"] \\ X_2 \arrow[r, hook, "i_{X_2}^{Y_2}"] & Y_2 \arrow[r, "p_{X_2}^{Y_2}"] & \frac{Y_2}{X_2} \end{tikzcd}$$

1.2 Monoidal Categories

***Definition* 1.2.1.** Recall that (see [Ka, Capitolo XI]) a *monoidal category* is a category $\mathcal{M}$ endowed with an object $\mathbf{1} \in \mathcal{M}$ (called *unit*), a functor $\otimes := \otimes_{\mathbf{1}} : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ (called *tensor product*), and functorial isomorphisms $a_{X,Y,Z} := {}^{\mathbf{1}}a^{\mathbf{1}}_{X,Y,Z} : (X \otimes_{\mathbf{1}} Y) \otimes_{\mathbf{1}} Z \to X \otimes_{\mathbf{1}} (Y \otimes_{\mathbf{1}} Z)$, $l_X := l_X^{\mathbf{1}} : \mathbf{1} \otimes_{\mathbf{1}} X \to X$, $r_X := r_X^{\mathbf{1}} : X \otimes_{\mathbf{1}} \mathbf{1} \to X$. The functorial isomorphism a is called *associativity constraint* and it satisfies the *Pentagon Axiom*, that is the following diagram is commutative, for every U, V, W, X

in $\mathcal{M}$:

$$\begin{tikzcd}[column sep=small]
 & ((U\otimes V)\otimes W)\otimes X \arrow[r, "\alpha_{U,V,W}\otimes X"] \arrow[dl, "\alpha_{U\otimes V,W,X}"'] & (U\otimes(V\otimes W))\otimes X \arrow[dr, "\alpha_{U,V\otimes W,X}"] & \\
(U\otimes V)\otimes(W\otimes X) \arrow[drr, "\alpha_{U,V,W\otimes X}"'] & & & U\otimes((V\otimes W)\otimes X) \arrow[dl, "U\otimes\alpha_{V,W,X}"] \\
 & & U\otimes(V\otimes(W\otimes X)) &
\end{tikzcd}$$

The functorial isomorphisms l and r are called respectively left and right *unit constraint* and they obey the *Triangle Axiom*, that is the following diagram is commutative, for every U, W in $\mathcal{M}$:

$$\begin{tikzcd}
(V\otimes \mathbf{1})\otimes W \arrow[rr, "a_{V,\mathbf{1},W}"] \arrow[dr, "r_V\otimes W"'] & & V\otimes(\mathbf{1}\otimes W) \arrow[dl, "V\otimes l_W"] \\
 & V\otimes W &
\end{tikzcd}$$

A monoidal category is called *strict* whenever the associativity constraint and the unit constraint are the respective identities.
It is well known that the Pentagon Axiom completely solves the consistency problem arising out of the possibility of going from $((U\otimes V)\otimes W)\otimes X$ to $U\otimes(V\otimes(W\otimes X))$ in two different ways (see [Mj1, page 420]). This allows the notation $X_1\otimes\cdots\otimes X_n$ forgetting the brackets for any object obtained from $X_1,\cdots X_n$ using $\otimes$. Also, as a consequence of the coherence theorem, the constraints take care of themselves and can then be omitted in any computation involving morphisms in $\mathcal{M}$.
Some examples of monoidal categories are included in Section 5.5 and in Section 6.2.

1.2.2. A *monoidal functor* $(F,\phi_0,\phi_2) : (\mathcal{M},\otimes,\mathbf{1},a,l,r)\to(\mathcal{M}',\otimes,\mathbf{1},a,l,r)$ between two monoidal categories consists of a functor $F:\mathcal{M}\to\mathcal{M}'$, an isomorphism $\phi_2(U,V):F(U\otimes V)\to F(U)\otimes F(V)$, natural in $U,V\in\mathcal{M}$, and an isomorphism $\phi_0:\mathbf{1}\to\mathbf{F}(\mathbf{1})$ such that the diagram

$$\begin{tikzcd}
(F(U)\otimes F(V))\otimes F(W) \arrow[r, "\phi_2(U,V)\otimes F(W)"] \arrow[d, "a_{F(U),F(V),F(W)}"] & F(U\otimes V)\otimes F(W) \arrow[r, "\phi_2(U\otimes V,W)"] & F((U\otimes V)\otimes W) \arrow[d, "F(a_{U,V,W})"] \\
F(U)\otimes(F(V)\otimes F(W)) \arrow[r, "F(U)\otimes\phi_2(V,W)"] & F(U)\otimes F(V\otimes W) \arrow[r, "\phi_2(U,V\otimes W)"] & F(U\otimes(V\otimes W))
\end{tikzcd}$$

is commutative, and the following conditions are satisfied:

$$F(l_U) \circ \phi_2(\mathbf{1}, U) \circ (\phi_0 \otimes F(U)) = l_{F(U)}, \quad F(r_U) \circ \phi_2(U, \mathbf{1}) \circ (F(U) \otimes \phi_0) = r_{F(U)}.$$

1.2.3. A *braided monoidal category* $(\mathcal{M}, c)$ is a monoidal category $(\mathcal{M}, \otimes, \mathbf{1})$ equipped with a *braiding* c, that is a natural isomorphism

$$c_{X,Y} : X \otimes Y \longrightarrow Y \otimes X$$

satisfying

$$c_{X \otimes Y, Z} = (c_{X,Z} \otimes Y)(X \otimes c_{Y,Z}) \qquad \text{and} \qquad c_{X, Y \otimes Z} = (Y \otimes c_{X,Z})(c_{X,Y} \otimes Z).$$

For further details on these topics, we refer to [Ka, Chapter XIII].

1.2.4. Algebras and Coalgebras. [Mj1, Definition 9.2.11] An associative **1**-algebra or simply an *algebra* A in a monoidal category $(\mathcal{M}, \otimes, \mathbf{1}, a, l, r)$ is a tern (A, m, u) where A is an object in $\mathcal{M}$ endowed with morphisms $m : A \otimes A \to A$ (multiplication) and a $u : \mathbf{1} \to A$ (unit) in $\mathcal{M}$ such that the diagrams

$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "A \otimes m"] \arrow[d, "m \otimes A"'] & A \otimes A \arrow[d, "m"] \\ A \otimes A \arrow[r, "m"] & m \end{tikzcd} \qquad \begin{tikzcd} \mathbf{1} \otimes A \arrow[r, "l_A"] \arrow[dr, "u \otimes A"'] & A & A \otimes \mathbf{1} \arrow[l, "r_A"'] \arrow[dl, "A \otimes u"] \\ & A \otimes A \arrow[u, "m"] & \end{tikzcd}$$

are commutative.
Let (A, m, u) and (A', m', u') be algebras in $\mathcal{M}$. A morphism $f : A \to A'$ is called an *algebra homomorphism* if $m' \circ (f \otimes f) = f \circ m$, and $f \circ u = u'$.
A *left A-module* is an object $M \in \mathcal{M}$ together with a morphism ${}^A\mu_M := \mu^l_M$: $A \otimes M \to M$ such that: ${}^A\mu_M \circ (A \otimes {}^A\mu_M) \circ a_{A,A,M} = {}^A\mu_M \circ (m \otimes M)$ and ${}^A\mu_M \circ (u \otimes M) = l_M$.
A morphism $f : M \to N$ between two left modules is called a *homomorphism of left A-modules* if ${}^A\mu_N \circ (A \otimes f) = f \circ {}^A\mu_M$. The category of left A-modules will be denoted by ${}_A\mathcal{M}$. The category $\mathcal{M}_A$ of right A-modules is introduced in a similar way.
An *A-bimodule* is a left and right A-module $(M, {}^A\mu_M, \mu^A_M)$ in $\mathcal{M}$ satisfying the following compatibility condition: ${}^A\mu_M \circ (A \otimes \mu^A_M) \circ a_{A,M,A} = \mu^A_M \circ ({}^A\mu_M \otimes A)$. ${}_A\mathcal{M}_A$ will denote the category of A-bimodules.
Recall that, given $V \in \mathcal{M}$ and $(M, {}^A\mu_M)$ a left A-module, the object $M \otimes V$ can be regarded as a left A-module via ${}^A\mu_{M \otimes V} := ({}^A\mu_M \otimes V) \circ a^{-1}_{A,M,V}$. Any algebra (A, m, u) can be considered as an A-bimodule by setting ${}^A\mu_A := \mu^A_M := m$.

An coassociative **1**-coalgebra or simply a *coalgebra* C in a monoidal category $(\mathcal{M}, \otimes, \mathbf{1}, a, l, r)$ is a tern (C, Δ, ε) where C is an object in $\mathcal{M}$ endowed with a comultiplication $\Delta : C \otimes C \to C$ and a counit $\varepsilon : C \to \mathbf{1}$ in $\mathcal{M}$ such that (C, Δ, ε) is an algebra in the dual monoidal category $\mathcal{M}^o$ of $\mathcal{M}$. Recall that $\mathcal{M}^o$ and $\mathcal{M}$ have the same objects but $\mathcal{M}^o(X, Y) = \mathcal{M}(Y, X)$ for any X, Y in $\mathcal{M}$. Given a coalgebra C in $\mathcal{M}$ one can define the categories of C-comodules ${}^C\mathcal{M}, \mathcal{M}^C, {}^C\mathcal{M}^C$ respectively as the categories of C-modules ${}_C(\mathcal{M}^o), (\mathcal{M}^o)_C, {}_C(\mathcal{M}^o)_C$.

1.2.5. Braided bialgebras. A *braided bialgebra* in a braided monoidal category $(\mathcal{M}, c)$, is a sextuple $(H, m, u, \Delta, \varepsilon)$ such that (H, m, u) is an algebra in $\mathcal{M}$, (H, Δ, ε) is a coalgebra in $\mathcal{M}$ and this two structure are compatible in the sense that the following diagrams

$$\begin{tikzcd}
 & H \otimes H \otimes H \otimes H \arrow[rr, "H\otimes c\otimes H"] & & H \otimes H \otimes H \otimes H \arrow[dr, "m\otimes m"] & \\
H \otimes H \arrow[ur, "\Delta\otimes\Delta"] \arrow[drr, "m"'] & & & & H \otimes H \\
 & & H \arrow[urr, "\Delta"'] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
B \otimes B \arrow[r, "\varepsilon\otimes\varepsilon"] \arrow[d, "m"'] & \mathbf{1} \otimes \mathbf{1} \arrow[d, "m_{\mathbf{1}}"] \\
B \arrow[r, "\varepsilon"] & \mathbf{1}
\end{tikzcd}$$

are commutative.

***Definition* 1.2.6.** Let H be a braided bialgebra in a braided monoidal category $(\mathcal{M}, c)$. An object in ${}^H_H\mathcal{M}^H_H$ is a 5-tuple $(M, \mu^r_M, \mu^l_M, \rho^r_M, \rho^l_M)$ such that

- $\left(M, \mu^r_M, \mu^l_M\right)$ is an H-bimodule;
- $\left(M, \rho^r_M, \rho^l_M\right)$ is an H-bicomodule;
- the following compatibility conditions are fulfilled:

$$\begin{aligned}
&(1.1) \qquad \rho^l_M \mu^l_M = \left(m_H \otimes \mu^l_M\right)(H \otimes c_{H,H} \otimes M)\left(\Delta_H \otimes \rho^l_M\right),\\
&(1.2) \qquad \rho^l_M \mu^r_M = \left(m_H \otimes \mu^r_M\right)(H \otimes c_{M,H} \otimes H)\left(\rho^l_M \otimes \Delta_H\right),\\
&(1.3) \qquad \rho^r_M \mu^l_M = \left(\mu^l_M \otimes m_H\right)(H \otimes c_{H,M} \otimes H)\left(\Delta_H \otimes \rho^r_M\right),\\
&(1.4) \qquad \rho^r_M \mu^r_M = \left(\mu^r_M \otimes m_H\right)(M \otimes c_{H,H} \otimes H)\left(\rho^r_M \otimes \Delta_H\right).
\end{aligned}$$

1.3 The relative tensor and cotensor functors

Let $(\mathcal{M}, \otimes, \mathbf{1}, a, l, r)$ be a monoidal category. For sake of simplicity, from now on, we will omit the associativity constraints.

Assume that $\mathcal{M}$ is abelian and let A be an algebra in $\mathcal{M}$. It can be proved (see [Ar2]) that ${}_A\mathcal{M}$ is an abelian category, whenever the functor $A \otimes (-) : \mathcal{M} \to \mathcal{M}$

is additive and right exact. In the case when both the functor $A \otimes (-) : \mathcal{M} \to \mathcal{M}$ and the functor $(-) \otimes A : \mathcal{M} \to \mathcal{M}$ are additive and right exact, then the category ${}_A\mathcal{M}_A$ is abelian too.

Since, sometimes, we have to work with more than one algebra in $\mathcal{M}$, and their bimodules, it is convenient to assume that $X \otimes (-) : \mathcal{M} \to \mathcal{M}$ and $(-) \otimes X : \mathcal{M} \to \mathcal{M}$ are additive and right exact, for any $X \in \mathcal{M}$. Hence we are led to the following definition.

***Definition* 1.3.1.** Let $\mathcal{M}$ be a monoidal category.
We say that $\mathcal{M}$ is an **abelian monoidal category** if $\mathcal{M}$ is abelian and $X \otimes (-) : \mathcal{M} \to \mathcal{M}$ and $(-) \otimes X : \mathcal{M} \to \mathcal{M}$ are additive and right exact, for any $X \in \mathcal{M}$.
We say that $\mathcal{M}$ is an **coabelian monoidal category** if $\mathcal{M}^o$ is an abelian monoidal category.

1.3.2. Let A be an algebra in an abelian monoidal category $\mathcal{M}$. The tensor product over A in $\mathcal{M}$ of a right A-module V and a left A-module W is defined to be the coequalizer:

$$(V \otimes A) \otimes W \rightrightarrows V \otimes W \xrightarrow{{}_A\chi_{V,W} = \chi_A(V,W)} V \otimes_A W \longrightarrow 0$$

Note that, since $\otimes$ preserves coequalizers, then $V \otimes_A W$ is also an A-bimodule, whenever V and W are A-bimodules. In fact there exists a functor

$$\otimes_A : {}_A\mathcal{M}_A \times {}_A\mathcal{M}_A \to {}_A\mathcal{M}_A$$

and morphisms ${}^A a^A, l^A, r^A$ that make the category $({}_A\mathcal{M}_A, \otimes_A, A, {}^A a^A, l^A, r^A)$ an abelian monoidal category (an algebra in this category will be called an A-*algebra*): see [AMS3, Theorem 1.12].

Dually, let C be a coalgebra in a coabelian monoidal category $\mathcal{M}$. The cotensor product over C in $\mathcal{M}$ of a right C-bicomodule V and a left C-comodule W is defined to be the equalizer:

$$0 \longrightarrow V \square_C W \xrightarrow{{}^C\varsigma_{V,W} = \varsigma_C(V,W)} V \otimes W \rightrightarrows V \otimes (C \otimes W)$$

Note that, since $\otimes$ preserves equalizers, then $V \square_C W$ is also a C-bicomodule, whenever V and W are C-bicomodules. In fact there exists of a functor

$$\square_C : {}^C\mathcal{M}^C \times {}^C\mathcal{M}^C \to {}^C\mathcal{M}^C$$

and morphisms ${}^C a^C, l^C, r^C$ that make the category $({}^C\mathcal{M}^C, \square_C, C, {}^C a^C, l^C, r^C)$ a coabelian monoidal category (a coalgebra in this category will be called a C-*coalgebra*).

What follows is a list of the most important monoidal categories meeting ours requirements.

1.4 Ideals

***Definition* 1.4.1.** An *ideal* of an algebra (A, m, u) in a monoidal category $(\mathcal{M}, \otimes, \mathbf{1})$ is a pair

$$(I, i_I)$$

where I is an A-bimodule and

$$i_I : I \to A$$

is a morphism of A-bimodules which is a monomorphism in $\mathcal{M}$.
A morphism $f : I \to J$ in ${}_A\mathcal{M}_A$, where I,J are two ideals, is called a *morphism of ideals* whenever

$$\begin{tikzcd} I \arrow[rr, "f"] \arrow[dr, "i_I"'] & & J \arrow[dl, "i_J"] \\ & A & \end{tikzcd}$$

f is a monomorphism in $\mathcal{M}$ as i_I is. Moreover f is unique, as i_J is a monomorphism. In this way we get the category of ideals of A which is denoted by $\mathcal{I}(A)$ and is a subcategory of ${}_A\mathcal{M}_A$.

***Remark* 1.4.2.** If $\mathcal{M}$ is an abelian monoidal category, then, for every f in ${}_A\mathcal{M}_A$, we have that f is a monomorphism in ${}_A\mathcal{M}_A$ iff it is a monomorphism in $\mathcal{M}$ so that, in this case, the ideals of A are exactly the subobjects of A in ${}_A\mathcal{M}_A$.

1.4.3. Next aim is to exhibit some examples of ideals and to prove that there exists a functor, called "product of two ideals" (see Proposition 1.4.8):

$$\begin{aligned} " \cdot " : \mathcal{I}(A) \times \mathcal{I}(A) &\to \mathcal{I}(A) \\ ((I, i_I), (J, i_J)) &\mapsto (IJ, i_{IJ}) \\ (f, g) &\mapsto f \cdot g, \end{aligned}$$

such that

$$(\mathcal{I}(A), \cdot, A)$$

is a monoidal category.

***Example* 1.4.4** (Kernel of an algebra homomorphism)**.** Let $f : A \to B$ be an algebra homomorphism in an abelian monoidal category $\mathcal{M}$. Since $\mathcal{M}$ is abelian, then $(K, i_K) = \mathrm{Ker}\,(f)$ exists in $\mathcal{M}$. The object B is a left A-bimodule via

$$\begin{aligned} \mu_B^l &: = m_B \circ (f \otimes B) : A \otimes B \to B \\ \mu_B^r &: = m_B \circ (B \otimes f) : B \otimes A \to B \end{aligned}$$

and f becomes a morphism in ${}_A\mathcal{M}_A$.
Moreover K can be endowed with a unique left A-bimodule structure such that the canonical injection $i_K : K \to A$ is a morphism in ${}_A\mathcal{M}_A$. Therefore (K, i_K) is an ideal of A.

***Proposition* 1.4.5.** *Let $\mathcal{M}$ be an abelian monoidal category. Let f be a morphism of A-bimodules. Let ${}_A\mathbb{H}_A : {}_A\mathcal{M}_A \to \mathcal{M}$ be the forgetful functor. Then*

1) $Ker({}_A\mathbb{H}_A(f))$ carries a natural A-bimodule structure (compatible with the definition map) that makes it the kernel of f in ${}_A\mathcal{M}_A$.

2) $Coker({}_A\mathbb{H}_A(f))$ carries a natural A-bimodule structure (compatible with the definition map) that makes it the cokernel of f in ${}_A\mathcal{M}_A$.

Proof. follows by [Ar2, proposition 3.3]. Note that here "abelian monoidal category" has a different meaning. □

***Proposition* 1.4.6.** *Let $(\mathcal{M}, \otimes, \mathbf{1})$ be an abelian monoidal category. Let (A, m, u) be an algebra in $\mathcal{M}$ and let $M \in {}_A\mathcal{M}_A$. Let $f : M \to A$ be a morphism in ${}_A\mathcal{M}_A$, where A is regarded as a bimodule via m. Then*

$$(Q, \pi) := Coker(f)$$

carries a unique algebra structure such that $\pi : A \to Q$ is an algebra homomorphism.

Proof. In view of Proposition 1.4.5, Q carries an A-bimodule structure

$$\left(Q, \mu_Q^l, \mu_Q^r\right)$$

such that π is a morphism of A-bimodules (see [Ar2, Proposition 4.3]).
By right exactness of the tensor functors, we have that

$$(Q \otimes Q, Q \otimes \pi) = \mathrm{Coker}(Q \otimes f).$$

Consider the following diagram:

$$\begin{tikzcd} Q \otimes M \arrow[r, "Q\otimes f"] & Q \otimes A \arrow[r, "Q\otimes\pi"] \arrow[d, "\mu_Q^r"'] & Q \otimes Q \arrow[r] \arrow[dl, dotted, "m_Q"] & 0 \\ & Q & & \end{tikzcd}$$

Since π is a morphism in $\mathcal{M}_A$ and f in ${}_A\mathcal{M}$, we have:

$$\mu_Q^r (Q \otimes f) (\pi \otimes M) = \mu_Q^r (\pi \otimes A) (A \otimes f) = \pi m (A \otimes f) = \pi f \mu_M^l = 0.$$

Now $\pi \otimes M$ is an epimorphism, as π is an epimorphism, so that

$$\mu_Q^r (Q \otimes f) = 0$$

By the universal property of the cokernel, there exists a unique morphism

$$m_Q : Q \otimes Q \to Q$$

in $\mathcal{M}$ such that $m_Q(Q \otimes \pi) = \mu^r_Q$. Set

$$u_Q := \pi u_A : \mathbf{1} \to Q.$$

We have

$$m_Q\left(\pi \otimes \pi\right) = m_Q\left(Q \otimes \pi\right)\left(\pi \otimes A\right) = \mu^r_Q\left(\pi \otimes A\right) = \pi \mu^r_A = \pi m_A.$$

Once proved that (Q, m_Q, u_Q) is an algebra in $\mathcal{M}$, the displayed relation implies that π is an algebra homomorphism. We have

$$\begin{aligned} m_Q\left(m_Q \otimes Q\right)\left(\pi \otimes \pi \otimes \pi\right) &= \pi m_A\left(m_A \otimes A\right) \\ &= \pi m_A\left(A \otimes m_A\right) = m_Q\left(Q \otimes m_Q\right)\left(\pi \otimes \pi \otimes \pi\right). \end{aligned}$$

By right exactness of the tensor functors, $(\pi \otimes \pi \otimes \pi)$ is an epimorphism and hence

$$m_Q\left(m_Q \otimes Q\right) = m_Q\left(Q \otimes m_Q\right).$$

Moreover

$$\begin{aligned} m_Q\left(Q \otimes u_Q\right)\left(\pi \otimes \mathbf{1}\right) &= m_Q\left(\pi \otimes \pi\right)\left(A \otimes u_A\right) \\ &= \pi m_A\left(A \otimes u_A\right) \\ &= \pi r_A = r_Q\left(\pi \otimes \mathbf{1}\right). \end{aligned}$$

Since $\pi \otimes \mathbf{1}$ is an epimorphism, we deduce that $m_Q\left(Q \otimes u_Q\right) = r_Q$. Similarly one proves that $m_Q\left(u_Q \otimes Q\right) = l_Q$ so that (Q, m_Q, u_Q) is an algebra in $\mathcal{M}$.

□

***Example* 1.4.7** (The product of two ideals)**.** Let $\mathcal{M}$ be an abelian monoidal category.
Let (I, i_I) and (J, i_J) be two ideals in (A, m, u). Set

$$\begin{gathered} m_{I,J} := m(i_I \otimes i_J) : I \otimes J \to A \\ \left(Q_{I,J}, \pi_{I,J}\right) = \operatorname{Coker}\left(m_{I,J}\right), \qquad \pi_{I,J} : A \to Q_{I,J} \\ \left(IJ, i_{IJ}\right) = \operatorname{Ker}\left(\pi_{I,J}\right) = \operatorname{Im}\left(m_{I,J}\right), \qquad i_{IJ} : IJ \to A \end{gathered}$$

Since $m_{I,J} \in {}_A\mathcal{M}_A$, by Proposition 1.4.6, $Q_{I,J}$ is an algebra and $\pi_{I,J}$ and algebra homomorphism.
By the previous example, we have that

$$\left(IJ, i_{IJ}\right)$$

is an ideal of A which is called *the product of I and J*.
Moreover, we have the following exact sequence:

$$0 \longrightarrow IJ \xrightarrow{i_{IJ}} A \xrightarrow{\pi_{I,J}} Q_{I,J} \longrightarrow 0. \tag{1.5}$$

Since $(IJ, i_{IJ}) = \mathrm{Ker}\,(\pi_{I,J})$ and $\pi_{I,J} m_{I,J} = 0$, by the universal property of the kernel, there is a unique morphism $\overline{m}_{I,J} : I \otimes J \to IJ$ such that the following diagram

$$\begin{tikzcd}
0 \arrow[r] & IJ \arrow[rr, "i_{IJ}"] & & A \arrow[r, "\pi_{I,J}"] & Q_{I,J} \arrow[r] & 0 \\
& & I \otimes J \arrow[ul, dotted, "\overline{m}_{I,J}"] \arrow[ur, "m_{I,J}"'] & & &
\end{tikzcd}$$

is commutative. Since $IJ = \mathrm{Im}(m_{I,J})$, it comes out that $\overline{m}_{I,J}$ is an epimorphism. Consider the case $I = A$.
Since i_J is a morphism in ${}_A\mathcal{M}$, we have

$$m_{A,J} = m \circ (\mathrm{Id}_A \otimes i_J) = i_J \circ \mu^l_J.$$

Since i_J is a monomorphism and μ^l_J an epimorphism, we deduce that

$$(AJ, i_{AJ}) = \mathrm{Im}\,(m_{A,J}) = (J, i_J)\,.$$

Analogously, in the case $J = A$, one has

$$(IA, i_{IA}) = (I, i_I)\,.$$

We need the following result.

***Proposition* 1.4.8.** *Let A be an algebra in an abelian monoidal category $\mathcal{M}$ and let $\mathcal{I}(A) \subseteq {}_A\mathcal{M}_A$ be the category of ideals of A. Then there exists a functor:*

$$\begin{gathered}
"\cdot" : \mathcal{I}(A) \times \mathcal{I}(A) \to \mathcal{I}(A) \\
((I, i_I), (J, i_J)) \mapsto (IJ, i_{IJ}) \\
(f, g) \mapsto f \cdot g
\end{gathered}$$

that will be called the product of two ideals functor. *Furthermore*

$$(\mathcal{I}(A), \cdot, A)$$

is a monoidal category.

Proof. Let

$$f : (I_1, i_{I_1}) \to (I_2, i_{I_2}) \qquad \text{and} \qquad g : (J_1, i_{J_1}) \to (J_2, i_{J_2})$$

be morphism of ideals of A. We have to define a morphism of ideals of A :

$$f \cdot g : (I_1 J_1, i_{I_1 J_1}) \to (I_2 J_2, i_{I_2 J_2})\,.$$

We have

$$m_{I_2,J_2} \circ (f \otimes g) = m_A \circ (i_{I_2} \otimes i_{J_2}) \circ (f \otimes g) = m_A \circ (i_{I_1} \otimes i_{J_1}) = m_{I_1,J_1} = \mathrm{Id}_A \circ m_{I_1,J_1}$$

so that, in the following diagram

$$\begin{tikzcd} I_1 \otimes J_1 \arrow[r, "m_{I_1,J_1}"] \arrow[d, "f\otimes g"'] & A \arrow[r, "\pi_{I_1,J_1}"] \arrow[d, equal] & Q_{I_1,J_1} \arrow[r] \arrow[d, dotted, "\pi"'] & 0 \\ I_2 \otimes J_2 \arrow[r, "m_{I_2,J_2}"] & A \arrow[r, "\pi_{I_2,J_2}"] & Q_{I_2,J_2} \arrow[r] & 0 \end{tikzcd}$$

the left square commutes. Since $(Q_{I_1,J_1}, \pi_{I_1,J_1}) = \mathrm{Coker}\,(m_{I_1,J_1})$ and

$$\pi_{I_2,J_2} \circ \mathrm{Id}_A \circ m_{I_1,J_1} = \pi_{I_2,J_2} \circ m_{I_2,J_2} \circ (f \otimes g) = 0,$$

by the universal property of the cokernel, there is a unique morphism

$$\pi : Q_{I_1,J_1} \to Q_{I_2,J_2}$$

such that the right square is also commutative. Furthermore, using the fact that π_{I_1,J_1} is an epimorphism, it is easy to check that π is an algebra homomorphism. Since $(I_2J_2, i_{I_2,J_2}) = \mathrm{Ker}\,(\pi_{I_2,J_2})$ and

$$\pi_{I_2,J_2} \circ \mathrm{Id}_A \circ i_{I_1J_1} = \pi \circ \pi_{I_1,J_1} \circ i_{I_1J_1} = 0$$

by the universal property of the kernel, there is a unique morphism

$$f \cdot g : I_1J_1 \to I_2J_2$$

such that

$$\begin{tikzcd} 0 \arrow[r] & I_1J_1 \arrow[r, "i_{I_1J_1}"] \arrow[d, "f\cdot g"'] & A \arrow[r, "\pi_{I_1,J_1}"] \arrow[d, equal] & Q_{I_1,J_1} \arrow[r] \arrow[d, dotted, "\pi"'] & 0 \\ 0 \arrow[r] & I_2J_2 \arrow[r, "i_{I_2J_2}"] & A \arrow[r, "\pi_{I_2,J_2}"] & Q_{I_2,J_2} \arrow[r] & 0 \end{tikzcd}$$

One can check that that $f \cdot g$ is a morphism of ideals $\mathcal{M}$. It remains to prove that $(\mathcal{I}(A), \cdot, A)$ is a monoidal category. The unit constraints are, by definition, the canonical morphisms

$$l_I : A \cdot I \longrightarrow I \qquad \text{and} \qquad r_I : I \cdot A \longrightarrow I,$$

for every ideal (I, i_I) of A. Furthermore, for every ideals I, J, K of A we have

$$\begin{aligned} \mathrm{Coker}\,[m \circ (i_{I,J} \otimes i_K)] &= \mathrm{Coker}\,[m \circ (i_{I,J} \otimes i_K) \circ (\overline{m}_{I,J} \otimes K)] \\ &= \mathrm{Coker}\,[m \circ (m \otimes R) \circ (i_I \otimes i_J \otimes i_K)] \\ &= \mathrm{Coker}\,[m \circ (R \otimes m) \circ (i_I \otimes i_J \otimes i_K)] \\ &= \mathrm{Coker}\,[m \circ (i_I \otimes i_{J,K}) \circ (I \otimes \overline{m}_{J,K})] \\ &= \mathrm{Coker}\,[m \circ (i_I \otimes i_{J,K})]\,, \end{aligned}$$

where $\overline{m}_{I,J} : I \otimes J \to IJ$ denotes the unique epimorphism defined by $i_{I,J} \circ \overline{m}_{I,J} = m(i_I \otimes i_J)$ (see Example 1.4.7), so that

$$\left((IJ)K, i_{(IJ)K}\right) = \mathrm{Im}\left[m \circ (i_{I,J} \otimes i_K)\right] = \mathrm{Im}\left[m \circ (i_I \otimes i_{J,K})\right] = \left(I(JK), i_{I(JK)}\right).$$

Therefore there is a (unique) homomorphism of ideals

$$\left((IJ)K, i_{(IJ)K}\right) \overset{a_{I,j,K}}{\longrightarrow} \left(I(JK), i_{I(JK)}\right). \tag{1.6}$$

Since a morphism of ideals is uniquely defined by its domain and codomain, it is clear that the pentagon and triangle axioms are fulfilled so that $(\mathcal{I}(A), \cdot, A)$ is a monoidal category. □

***Example* 1.4.9** (n-th power of ideals)**.** Let $\mathcal{M}$ be an abelian monoidal category and let (I, i_I) be an ideal of an algebra A in $\mathcal{M}$. Following the notations of Section 1.1 for "$\mathcal{M}$" $= ({}_A\mathcal{M}_A, \otimes_A, A)$, for every $n \in \mathbb{N}$, we can consider in view of Proposition 1.4.8

$$(I^n, i_{I^n}), \qquad \text{(the } n\text{-th power of } I)$$

where

$$I^0 := A \qquad \text{and} \qquad I^{n+1} = I^n I, \text{ for every } n \geq 0$$

The ideal I will be called *nilpotent* whenever

$$I^n = 0 \text{ for some } n \geq 1.$$

1.4.10. If (I, i_I) is an ideal then, by Proposition 1.4.6, there is a unique algebra structure on

$$\left(\frac{A}{I}, p_I\right) = \mathrm{Coker}(i_I),$$

such that the canonical projection $p_I : A \to \frac{A}{I}$ is an algebra map.
Let $i_I^J : (I, i_I) \to (J, i_J)$ is a morphism of ideals. We know there is a unique morphism $\frac{A}{i_I^J} : \frac{A}{I} \to \frac{A}{J}$ such that

$$\begin{tikzcd}
0 \arrow[r] & I \arrow[r, "i_I"] \arrow[d, "i_I^J"'] & A \arrow[r, "p_I"] \arrow[d, equal] & \frac{A}{I} \arrow[r] \arrow[d, dotted, "\frac{A}{i_I^J}"'] & 0 \\
0 \arrow[r] & J \arrow[r, "i_J"] & A \arrow[r, "p_J"] & \frac{A}{J} \arrow[r] & 0
\end{tikzcd}$$

It comes out that $\frac{A}{f}$ is an algebra homomorphism which is an epimorphism in $\mathcal{M}$. By the Snake Lemma, there exists a morphism $\omega : \mathrm{Ker}(\frac{A}{f}) \longrightarrow \frac{J}{I}$ such that the zig-zag sequence in the back is commutative:

$$\begin{tikzcd}
 & & 0 \arrow[rr] \arrow[dl] & & 0 \arrow[rr] \arrow[dl] & & \mathrm{Ker}(\frac{A}{i_I^J}) \arrow[dl, "u"'] \arrow[ddllll, dotted, "\omega"'] & \\
0 \arrow[r] & I \arrow[rr, "i_I"] \arrow[dd, "i_I^J"'] & & A \arrow[rr, "p_I"] \arrow[dd, "\mathrm{Id}_A"'] & & \frac{A}{I} \arrow[rr] \arrow[dd, "\frac{A}{i_I^J}"] & & 0 \\
 & & \frac{J}{I} \arrow[rr] \arrow[urrr, "\frac{i_J}{I}"] & & 0 \arrow[rr] & & 0 & \\
0 \arrow[r] & J \arrow[rr, "i_J"] \arrow[ur, "p_I^J"] & & A \arrow[rr, "p_J"] \arrow[ur] \arrow[uurr, bend left, "p_I"] & & \frac{A}{J} \arrow[rr] \arrow[ur] & & 0
\end{tikzcd}$$

Thus ω is an isomorphism. Furthermore ω makes the oblique triangle commutative, namely we have:

$$u \circ \omega = i_J/I,$$

where $u : \ker(\frac{A}{i_I^J}) \longrightarrow \frac{A}{I}$ denotes the canonical morphism defining $\ker(\frac{A}{i_I^J})$. In particular, since ω is an isomorphism, we can identify $\left(\ker(\frac{A}{i_I^J}), u\right)$ with $\left(\frac{J}{I}, \frac{i_J}{I}\right)$ so that we get the exact sequence

$$0 \to \frac{J}{I} \xrightarrow{\frac{i_J}{I}} \frac{A}{I} \xrightarrow{\frac{A}{i_I^J}} \frac{A}{J} \to 0.$$

1.4.11. Let (I, i_I) and (J, i_J) be ideals of an algebra A.
Then, by Proposition 1.4.8, we have two morphisms of ideals

$$i_{IJ}^I : IJ \to I, \qquad \text{and} \qquad i_{IJ}^J : IJ \to J$$

defined respectively by

$$i_{IJ}^I := \mathrm{Id}_I \cdot i_J, \qquad \text{and} \qquad i_{IJ}^I = i_I \cdot \mathrm{Id}_J.$$

This notation does not cause confusion in the case $I = J$ as, by uniqueness of morphisms of ideals, we have

$$\mathrm{Id}_I \cdot i_I = i_I \cdot \mathrm{Id}_I.$$

We have also

$$i_{IJ}^I \circ i_{(IJ)K}^{IJ} = i_{(IJ)K}^I$$

and the other analogue relations.

1.4.12. Let (I, i_I) be an ideal of an algebra A.
Let us define a morphism of ideals

$$i_I^n : I^{n+1} \to I^n,$$

for every $n \in \mathbb{N}$, by setting

$$i_I^n := i_{I^n I}^{I^n} = \mathrm{Id}_{I^n} \cdot i_I.$$

Note that, by uniqueness of morphisms of ideals, we have

$$i_I^0 := i_I \qquad \text{and} \qquad i_I^{n+1} = i_I^n \cdot \mathrm{Id}_I, \text{ for every } n \geq 0.$$

Moreover, as observed in 1.4.10, there is a unique algebra homomorphism

$$p_I^n = \frac{A}{i_I^n} : \frac{A}{I^{n+1}} \to \frac{A}{I^n}$$

which is an epimorphism in $\mathcal{M}$ and we have an exact sequence

$$0 \to \frac{I^n}{I^{n+1}} \xrightarrow{j_I^n} \frac{A}{I^{n+1}} \xrightarrow{p_I^n} \frac{A}{I^n} \to 0,$$

where $j_I^n = \frac{i_{I^n}}{I^{n+1}}$.

***Lemma* 1.4.13.** *[AMS3, the proof of Lemma 3.4] Let $\mathcal{M}$ be an abelian monoidal category.*
Let (A, m, u) be an algebra in $\mathcal{M}$, let (I, i_I) be an ideal in A and let $n \in \mathbb{N}^*$. Then

$$\left(\frac{I^n}{I^{n+1}}\right)^2 = 0.$$

Proof. By construction we have the following diagram with exact lines and commutative squares:

$$\begin{tikzcd}
0 \arrow[r] & I^n \arrow[r, "i_{I^n}"] \arrow[d, "q_{I^n}"'] & A \arrow[r, "p_{I^n}"] \arrow[d, "p_{I^{n+1}}"] & \frac{A}{I^n} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \frac{I^n}{I^{n+1}} \arrow[r, "j_I^n"] & \frac{A}{I^{n+1}} \arrow[r, "p_I^n"] & \frac{A}{I^n} \arrow[r] & 0
\end{tikzcd}$$

where q_{I^n} is the canonical projection.
We have to prove that $(I^n/I^{n+1})^2 = 0$ or, equivalently, that

$$m_{\frac{I^n}{I^{n+1}}, \frac{I^n}{I^{n+1}}} = m_{n+1} \circ (j_I^n \otimes j_I^n) = 0,$$

where m_i denotes the multiplication of A/I^i, for any natural number i.
Since q_{I^n} is an epimorphism in $\mathcal{M}$ and $(-) \otimes (-)$ is right exact in both variables,

then $q_{I^n} \otimes q_{I^n}$ is an epimorphism too.
Thus the required relation is equivalent to

$$m_{n+1} \circ (j_I^n q_{I^n} \otimes j_I^n q_{I^n}) = 0$$

Now, by uniqueness of morphisms of ideals, we get

$$i_{I^n} = i_I \circ i^I_{I^{n-1}I}.$$

Therfore, since $j_I^n q_{I^n} = p_{I^{n+1}} i_{I^n}$, we have

$$\begin{array}{rcl} m_{n+1} \circ (j_I^n q_{I^n} \otimes j_I^n q_{I^n}) & = & m_{n+1} \circ (p_{I^{n+1}} i_{I^n} \otimes p_{I^{n+1}} i_{I^n}) \\ & = & m_{n+1} \circ (p_{I^{n+1}} \otimes p_{I^{n+1}}) \circ (i_{I^n} \otimes i_{I^n}) \\ & = & p_{I^{n+1}} \circ m \circ (i_{I^n} \otimes i_{I^n}) \\ & = & p_{I^{n+1}} \circ m \circ (i_{I^n} \otimes i_I) \circ (I^n \otimes i^I_{I^{n-1}I}) \\ & = & p_{I^{n+1}} \circ m_{I^n,I} \circ (I^n \otimes i^I_{I^{n-1}I}) \\ (*) & = & p_{I^{n+1}} \circ i_{I^{n+1}} \circ \overline{m_{I^n,I}} \circ (I^n \otimes i^I_{I^{n-1}I}) = 0, \end{array}$$

where $(*)$ follows by

$$m_{I^n,I} = i_{I^{n+1}} \circ \overline{m_{I^n,I}}$$

which appeared in Example 1.4.7. □

1.5 Wedge product

In the classical case, the notion of wedge product (see [Mo, page 60]) plays a fundamental role in the study of coalgebras. In fact the coradical C_0 of a coalgebra C gives rise to the so called coradical filtration:

$$C_0 \subseteq C_0 \wedge_C C_0 \subseteq C_0 \wedge_C C_0 \wedge_C C_0 \subseteq \cdots \subseteq C,$$

which is exhaustive in the sense that its direct limit is C itself. The basic point when dealing with coalgebras in monoidal categories is that there is no notion of coradical. The idea then is to take a subcoalgebra D of a coalgebra C and to consider the coalgebra $\widetilde{D}$ which is the direct limit of the iterated wedge powers of D in C. Then the coalgebra D acts, in a certain sense, as the coradical of $\widetilde{D}$.
For this section we refer to [AMS2].

1.5.1. Let E be a coalgebra in a coabelian monoidal category $\mathcal{M}$. As in the case of vector spaces, we can introduce the wedge product of two subobjects X, Y of E in $\mathcal{M}$:

$$(X \wedge_E Y, i^E_{X \wedge Y}) := Ker[(p_X \otimes p_Y) \circ \triangle_E],$$

where $p_X : E \to E/X$ and $p_Y : E \to E/Y$ are the canonical quotient maps. In particular we have the following exact sequence:

$$0 \longrightarrow X \wedge_E Y \xrightarrow{i^E_{X \wedge Y}} E \xrightarrow{(p_X \otimes p_Y)\circ \triangle_E} E/X \otimes E/Y.$$

Consider the following commutative diagrams in $\mathcal{M}$

$$\begin{tikzcd} X_1 \arrow[r, hook, "i^{E_1}_{X_1}"] \arrow[d, "x"'] & E_1 \arrow[d, "e"] & Y_1 \arrow[r, hook, "i^{E_1}_{Y_1}"] \arrow[d, "y"'] & E_1 \arrow[d, "e"] \\ X_2 \arrow[r, hook, "i^{E_2}_{X_2}"'] & E_2 & Y_2 \arrow[r, hook, "i^{E_2}_{Y_2}"'] & E_2 \end{tikzcd}$$

where e is a coalgebra homomorphism. Then there is a unique morphism $x \wedge_e y : X_1 \wedge_{E_1} Y_1 \to X_2 \wedge_{E_2} Y_2$ such that the following diagram

$$\begin{tikzcd} X_1 \wedge_{E_1} Y_1 \arrow[r, "i^{E_1}_{X_1 \wedge_{E_1} Y_1}"] \arrow[d, dotted, "x \wedge_e y"'] & E_1 \arrow[d, "e"] \\ X_2 \wedge_{E_2} Y_2 \arrow[r, "i^{E_2}_{X_2 \wedge_{E_2} Y_2}"'] & E_2 \end{tikzcd}$$

commutes. In fact we have

$$\begin{aligned} & (p^{E_2}_{X_2} \otimes p^{E_2}_{Y_2}) \circ \Delta_{E_2} \circ e \circ i^{E_1}_{X_1 \wedge_{E_1} Y_1} \\ = \ & (p^{E_2}_{X_2} \otimes p^{E_2}_{Y_2}) \circ (e \otimes e) \circ \Delta_{E_1} \circ i^{E_1}_{X_1 \wedge_{E_1} Y_1} \\ = \ & (\frac{e}{x} \otimes \frac{e}{y}) \circ (p^{E_1}_{X_1} \otimes p^{E_1}_{Y_1}) \circ \Delta_{E_1} \circ i^{E_1}_{X_1 \wedge_{E_1} Y_1} = 0 \end{aligned}$$

so that, since $(X_2 \wedge_{E_2} Y_2, i^{E_2}_{X_2 \wedge_{E_2} Y_2})$ is the kernel of $(p^{E_2}_{X_2} \otimes p^{E_2}_{Y_2}) \circ \Delta_{E_2}$, we conclude.

Lemma 1.5.2. *Consider the following commutative diagrams in* $\mathcal{M}$

$$\begin{tikzcd} X_1 \arrow[r, hook, "i^{E_1}_{X_1}"] \arrow[d, "x"'] & E_1 \arrow[d, "e"] & Y_1 \arrow[r, hook, "i^{E_1}_{Y_1}"] \arrow[d, "y"'] & E_1 \arrow[d, "e"] \\ X_2 \arrow[r, hook, "i^{E_2}_{X_2}"'] \arrow[d, "x'"'] & E_2 \arrow[d, "e'"'] & Y_2 \arrow[r, hook, "i^{E_2}_{Y_2}"'] \arrow[d, "y'"'] & E_2 \arrow[d, "e'"'] \\ X_3 \arrow[r, hook, "i^{E_3}_{X_3}"'] & E_3 & Y_3 \arrow[r, hook, "i^{E_3}_{Y_3}"'] & E_3 \end{tikzcd}$$

where e and e' are coalgebra homomorphisms. Then we have

$$(x' \wedge_{e'} y') \circ (x \wedge_e y) = (x'x \wedge_{e'e} y'y) \tag{1.7}$$

Proof. : straightforward. $\square$

Lemma 1.5.3. *[AMS2, Lemma 2.16] Let E be a coalgebra in a coabelian monoidal category $\mathcal{M}$. Let $f : E \to L$ and $g : E \to M$ be morphism in $\mathcal{M}$. Then*

$$\mathrm{Ker}\,(f) \wedge_E \mathrm{Ker}\,(g) = Ker[(f \otimes g) \circ \triangle_E].$$

Proof. Let $(X, i_X) = \mathrm{Ker}\,(f)$ and let $(Y, i_Y) = \mathrm{Ker}\,(g)$. Let $p_X : E \to E/X$ and $p_Y : E \to E/Y$ be the canonical quotient maps. Since $f i_X = 0$, by the universal property of the cokernel, there exists a unique morphism

$$\gamma_X : E/X \to L$$

such that $\gamma_X p_X = f$:

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "i_X"] & E \arrow[r, "p_X"] \arrow[d, "f"'] & \frac{E}{X} \arrow[r] \arrow[dl, dotted, "\gamma_X"'] & 0 \\
& & L & &
\end{tikzcd}$$

Moreover, we have $(E/X, p_X) = \mathrm{coker}(i_X) = \mathrm{coker}(\mathrm{Ker}\,(f)) = \mathrm{coim}(f)$. As $\mathcal{M}$ is an abelian category, we have that $(E/X, \gamma_X) = \mathrm{Im}(f)$. In particular γ_X is a monomorphism. Analogously one gets a monomorphism $\gamma_Y : E/Y \to M$ such that $\gamma_Y p_Y = g$. Since $\mathcal{M}$ has left exact tensor functors, then $\gamma_X \otimes \gamma_Y$ is a monomorphism, so that, by definition, we get:

$$X \wedge_E Y := Ker[(p_X \otimes p_Y)\triangle_E] = Ker[(\gamma_X \otimes \gamma_Y)(p_X \otimes p_Y)\triangle_E] = Ker[(f \otimes g)\triangle_E].$$

$\square$

1.5.4. Let E be a coalgebra in a coabelian monoidal category $\mathcal{M}$. Recall that a subcoalgebra of E is a subobject (C, i_C^E) of E in $\mathcal{M}$ such that:

- C is a coalgebra in $\mathcal{M}$.
- $i_C^E : C \longrightarrow E$ is a coalgebra homomorphism in $\mathcal{M}$.

Given two subcoalgebras (C, i_C^E) and (D, i_D^E) of E, a morphism of subcoalgebras $f : (C, i_C^E) \longrightarrow (D, i_D^E)$ is a coalgebra homomorphism $f : C \longrightarrow D$ in $\mathcal{M}$ such that the following diagram

$$\begin{tikzcd}
C \arrow[rr, "f"] \arrow[dr, "i_C^E"'] & & D \arrow[dl, "i_D^E"] \\
& E &
\end{tikzcd}$$

commutes. Note that, since i_D^E is a monomorphism, f is uniquely defined by its domain and codomain.
Denote by $\mathcal{C}(E)$ the subcategory of $\mathcal{M}$ consisting in subcoalgebras of E and morphisms of subcoalgebras.

***Proposition* 1.5.5.** *Let E be a coalgebra in an coabelian monoidal category $\mathcal{M}$ and let $\mathcal{C}(E) \subseteq \mathrm{Coalg}(\mathcal{M})$ be the category of subcoalgebras of E. Then there exists a functor:*

$$\begin{gathered} " \wedge_E " : \mathcal{C}(E) \times \mathcal{C}(E) \to \mathcal{C}(E) \\ ((C, i_C^E), (D, i_D^E)) \mapsto (C \wedge_E D, i_{C \wedge_E E}^E) \\ (f, g) \mapsto f \wedge_E g \end{gathered}$$

that will be called the wedge product of two subcoalgebras functor*. Furthermore*

$$(\mathcal{C}(E), \wedge_E, 0)$$

is a monoidal category.

Proof. Let (C, i_C^E) and (D, i_D^E) be subcoalgebras of E. By the foregoing, there exists a subobject $(C \wedge_E D, i_{C \wedge_E E}^E)$ of E. This is $Ker[(p_C \otimes p_D) \circ \triangle_E]$ by definition. Now, since $(E/C, p_C) = coker(i_C^E)$ and i_C^E is a morphism of coalgebras, by the dual of Example 1.4.4, we get that E/C is has a natural E-bicomodule structure such that p_C is a morphism of E-bicomodules. The same argument applies to p_D, so that $(p_C \otimes p_D) \circ \triangle_E$ comes out to be a morphism of E-bicomodules as a composition of morphisms of E-bicomodules. By the dual of Proposition 1.4.6

$$(C \wedge_E D, i_{C \wedge_E D}^E) := Ker[(p_C \otimes p_D) \circ \triangle_E]$$

carries a unique coalgebra structure such that $i_{C \wedge_E D}^E : C \wedge_E D \to E$ is a coalgebra homomorphism. Furthermore, given f and g morphisms in $\mathcal{C}(E)$, it is straightforward to check that the $f \wedge_E g \in \mathcal{C}(E)$. It remains to prove that $(\mathcal{C}(E), \wedge_E, 0)$ is a monoidal category. We have

$$\begin{aligned} (C \wedge_E 0, i_{C \wedge_E 0}^E) &= Ker[(p_C \otimes p_0) \circ \triangle_E] \\ &= Ker[(p_C \otimes E) \circ \triangle_E] \\ &= Ker[\rho_{E/C} \circ p_C] \\ &= Ker\,(p_C) = (C, i_C^E) \end{aligned}$$

so that there exists a unique subcoalgebra homomorphism

$$r_C : C \wedge_E 0 \to C.$$

Analogously one constructs $l_C : 0 \wedge_E C \to C$. Let (F, i_F^E) be a subcoalgebra of E.

By Lemma 1.5.3, we have:

$$\begin{aligned}
\left((C\wedge_E D)\wedge_E F, i^E_{(C\wedge_E D)\wedge_E F}\right) &= \mathrm{Ker}[(p_C\otimes p_D)\circ\triangle_E]\wedge_E \mathrm{Ker}\,(p_F)\\
&= \mathrm{Ker}\ \{[[(p_C\otimes p_D)\triangle_E]\otimes p_F]\,\Delta_E\}\\
&= \mathrm{Ker}\,[(p_C\otimes p_D\otimes p_F)\,(\triangle_E\otimes E)\,\Delta_E]\\
&= \mathrm{Ker}\,[(p_C\otimes p_D\otimes p_F)\,(E\otimes\triangle_E)\,\Delta_E]\\
&= \mathrm{Ker}\ \{[p_C\otimes[(p_D\otimes p_F)\triangle_E]]\,\Delta_E\}\\
&= \mathrm{Ker}\,(p_C)\wedge_E \mathrm{Ker}[(p_D\otimes p_F)\circ\triangle_E]\\
&= \left(C\wedge_E(D\wedge_E F)\,, i^E_{C\wedge_E(D\wedge_E F)}\right)
\end{aligned}$$

so that there exists a unique subcoalgebra homomorphism

$$a_{C,D,F}:(C\wedge_E D)\wedge_E F\to C\wedge_E(D\wedge_E F)\,.$$

Since a homomorphism of subcoalgebras is uniquely defined by its domain and codomain, it is clear that the pentagon and triangle axioms are fulfilled so that $(\mathcal{C}(E),\wedge_E,0)$ is a monoidal category. □

1.5.6. In view of Proposition 1.5.5, $(\mathcal{C}(E),\wedge_E,0)$ is a monoidal category so that we are led to use the notation of Section 1.1 defining the n-th wedge power

$$(D^{\wedge^n_C},\delta_n),$$

where $\delta_n := i^E_{D^{\wedge^n_C}}$, of a subcoalgebra (D,δ) of a coalgebra E in a coabelian monoidal category $\mathcal{M}$. Note that, by definition, we have $(D^{\wedge^0_C},\delta_0)=0$.

1.5.7. Let $(E,\Delta_E,\varepsilon_E)$ be a coalgebra in $\mathcal{M}$ and for every $n\in\mathbb{N}$, define the n-th iterated comultiplication of E,

$$\Delta^n_E : E\to E^{\otimes n+1},$$

by

$$\Delta^0_E=\mathrm{Id}_E,\quad \Delta^1_E=\Delta_E\quad\text{and}\quad \Delta^n_E=\left(\Delta^{\otimes n-1}_E\otimes E\right)\Delta_E,\ \text{for every } n>1.$$

***Proposition* 1.5.8.** *[AMS2, Proposition 2.17] Let $\delta : D\to E$ be a monomorphism which is a coalgebra homomorphism in a coabelian monoidal category $\mathcal{M}$. Set $(L,p)=\mathrm{coker}(\sigma)$. Then, for every $m,n\geq 1$, we have:*

$$(D^{\wedge^n_E},\delta_n) := \mathrm{Ker}\,(p^{\otimes n}\Delta^{n-1}_E) \tag{1.8}$$

$$D^{\wedge^m_E}\wedge_E D^{\wedge^n_E}=D^{\wedge^{m+n}_E}. \tag{1.9}$$

Proof. We prove (1.8) by induction on $n \geq 1$.
For $n = 1$ there is nothing to prove.
Let $n \geq 2$ and assume that $(D^{\wedge_E^{n-1}}, \delta_{n-1}) := \mathrm{Ker}\,(p^{\otimes n-1}\Delta_E^{n-2})$. By Lemma 1.5.3, we have:

$$\begin{aligned} D^{\wedge_E^n} &= D^{\wedge_E^{n-1}} \wedge_E D \\ &= \mathrm{Ker}\,(p^{\otimes n-1}\Delta_E^{n-2}) \wedge_E \mathrm{Ker}\,(p) \\ &= \mathrm{Ker}\,\left[(p^{\otimes n-1}\Delta_E^{n-2} \otimes p)\Delta_E\right] = \mathrm{Ker}\,(p^{\otimes n}\Delta_E^{n-1}). \end{aligned}$$

Finally, the two sides of (1.9) can be identified as $(\mathcal{C}(E), \wedge_E, 0)$ is a monoidal category (which was proved in Proposition 1.5.5). $\square$

Lemma 1.5.9. *[AMS2, Lemma 2.12] Let (C, Δ, ε) be a coalgebra in a cocomplete coabelian monoidal category $\mathcal{M}$ and let (M, ρ_M^r, ρ_M^l) be a C-bicomodule. Let $\delta : D \to E$ be a monomorphism which is a morphism of coalgebras such that the canonical morphism $\widetilde{\delta} : \widetilde{D} \to E$ of Notation 1.6.4 is a monomorphism. Then we have*

$$D^{\wedge_{\widetilde{D}}^n} = (D^{\wedge_E^n}, \xi_n).$$

Proof. Since, by definition, $(D^{\wedge_E^n}, \delta_n = \widetilde{\delta}\xi_n) := \mathrm{Ker}(p^{\otimes n}\Delta_E^{n-1})$ where $\widetilde{\delta}$ is a monomorphism, the following relation holds true

$$(D^{\wedge_E^n}, \xi_n) = \mathrm{Ker}(p^{\otimes n}\Delta_E^{n-1}\widetilde{\delta})$$

so that it remain so prove that $D^{\wedge_{\widetilde{D}}^n} = \mathrm{Ker}(p^{\otimes n}\Delta_E^{n-1}\widetilde{\delta})$. Recall that there exists a unique morphism $\frac{\widetilde{\delta}}{D} : \frac{\widetilde{D}}{D} \to \frac{E}{D}$ such that

$$\frac{\widetilde{\delta}}{D} \circ p_D^{\widetilde{D}} = p_D^E \circ \widetilde{\delta}.$$

Since $\widetilde{\delta}$ is a monomorphism, so is $\widetilde{\delta}/D$. Therefore, we have

$$\begin{aligned} D^{\wedge_{\widetilde{D}}^n} \quad &: = \mathrm{Ker}\left[\left(p_D^{\widetilde{D}}\right)^{\otimes n} \Delta_{\widetilde{D}}^{n-1}\right] \\ &= \mathrm{Ker}\left[\left(\frac{\widetilde{\delta}}{D}\right)^{\otimes n} \left(p_D^{\widetilde{D}}\right)^{\otimes n} \Delta_{\widetilde{D}}^{n-1}\right] \\ &= \mathrm{Ker}\left[\left(p_D^E\right)^{\otimes n} \widetilde{\delta}^{\otimes n}\Delta_{\widetilde{D}}^{n-1}\right] = \mathrm{Ker}(p^{\otimes n}\Delta_E^{n-1}\widetilde{\delta}). \end{aligned}$$

$\square$

1.6 Direct limits

In this section we deal with some properties of direct limits that will be useful in the study of coalgebras in a monoidal category.

Proposition 1.6.1. *Let $(\mathcal{M}, \otimes, \mathbf{1})$ be a coabelian monoidal category. Let (C, Δ, ε) be a coalgebra in $\mathcal{M}$ and let L be a C-bicomodule. Let $f : C \to L$ be a morphism in ${}^C\mathcal{M}^C$, where C is regarded as a bicomodule via Δ. Then*

$$(D, \delta) := Ker(f)$$

carries a natural coalgebra structure such that δ is a morphism of coalgebras.

Proof. is dual to Proposition 1.4.6. □

Proposition 1.6.2. *Let $\mathcal{M}$ be a monoidal category with direct limits.*
Let $((X_i)_{i\in\mathbb{N}}, (\xi_i^j)_{i,j\in\mathbb{N}})$ be a direct system in $\mathcal{M}$, where, for $i \leq j$, $\xi_i^j : X_i \to X_j$. Assume that X_i is a coalgebra and that ξ_i^j is a homomorphism of coalgebras for any $i, j \in \mathbb{N}$. Then $\varinjlim X_i$ carries a natural coalgebra structure that makes it the direct limit of $((X_i)_{i\in\mathbb{N}}, (\xi_i^j)_{i,j\in\mathbb{N}})$ as a direct system of coalgebras.

Proof. Let $(X_i, \Delta_{X_i}, \varepsilon_{X_i})$ be a coalgebra in $(\mathcal{M}, \otimes, \mathbf{1})$ for any $i \in \mathbb{N}$. Set $X := \varinjlim X_i$. Let $(\xi_i : X_i \to X)_{i\in\mathbb{N}}$ be the structural morphism of the direct limit, so that $\xi_j \xi_i^j = \xi_i$ for any $i \leq j$. We put

$$\Delta_i = (\xi_i \otimes \xi_i)\Delta_{X_i} : X_i \to X \otimes X, \text{ for any } i \in \mathbb{N}.$$

Since ξ_i^j is a homomorphism of coalgebras, one can prove that $\Delta_j \xi_i^j = \Delta_i$ so that there exists a unique morphism $\Delta : X \to X \otimes X$ such that

$$\Delta\xi_i = \Delta_i = (\xi_i \otimes \xi_i)\Delta_{X_i} \text{ for any } i \in \mathbb{N}. \tag{1.10}$$

It is easy to check that $(X \otimes \Delta)\Delta\xi_i = (\Delta \otimes X)\Delta\xi_i$ for every $i \in \mathbb{N}$ and hence, by the universal property of the direct limit, we get $(X \otimes \Delta)\Delta = (\Delta \otimes X)\Delta$. Now, as ξ_i^j is a homomorphism of coalgebras, $\varepsilon_{X_j}\xi_i^j = \varepsilon_{X_i}$. Hence, there exists a unique morphism $\varepsilon : X \to \mathbf{1}$ such that

$$\varepsilon\xi_i = \varepsilon_{X_i} \text{ for any } i \in \mathbb{N}. \tag{1.11}$$

Then we have $(X \otimes \varepsilon)\Delta\xi_i = r_X^{-1}\xi_i$, for any $i \in \mathbb{N}$ and hence, by the universal property of direct limits we deduce that $(X \otimes \varepsilon)\Delta = r_X^{-1}$. Analogously one gets $(\varepsilon \otimes X)\Delta = l_X^{-1}$. Thus (X, Δ, ε) is a coalgebra in $\mathcal{M}$. Note that relations (1.10) and (1.11) mean that $\xi_i : X_i \to X$ is a homomorphism of coalgebras.
Let now $(C, \Delta_C, \varepsilon_C)$ be a coalgebra in $\mathcal{M}$ and let $(f_i : X_i \to C)_{i\in\mathbb{N}}$ be a compatible

family of morphisms of coalgebras in $\mathcal{M}$. Since $(f_i : X_i \to C)_{i\in\mathbb{N}}$ is a compatible family of morphisms in $\mathcal{M}$, there exists a unique morphism $f : X \to C$ such that $f\xi_i = f_i$ for any $i \in \mathbb{N}$. We prove that f is a homomorphism of coalgebras. We have $(f \otimes f)\Delta\xi_i = \Delta_C f\xi_i$ and $\varepsilon_C f\xi_i = \varepsilon\xi_i$, for any $i \in \mathbb{N}$, and hence, by the universal property of the direct limit, we deduce that $(f \otimes f)\Delta = \Delta_C f$ and $\varepsilon_C f = \varepsilon$. □

***Proposition* 1.6.3.** *Let $\delta : D \to C$ be a monomorphism which is a homomorphism of coalgebras in an coabelian monoidal category $\mathcal{M}$. Then for any $i \leq j$ in $\mathbb{N}$ there is a (unique) morphism $\xi_i^j : D^{\wedge^i_C} \to D^{\wedge^j_C}$ such that*

$$\delta_j\xi_i^j = \delta_i. \tag{1.12}$$

Moreover ξ_i^j is a coalgebra homomorphism and $((D^{\wedge^i_C})_{i\in\mathbb{N}}, (\xi_i^j)_{i,j\in\mathbb{N}})$ is a direct system in $\mathcal{M}$ whose direct limit, if it exists, carries a natural coalgebra structure that makes it the direct limit of $((D^{\wedge^i_C})_{i\in\mathbb{N}}, (\xi_i^j)_{i,j\in\mathbb{N}})$ as a direct system of coalgebras.

Proof. Set $D^i := D^{\wedge^i_C}$ for any $i \in \mathbb{N}$. Consider the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & D^{i+1} \arrow[r, "\delta^{i+1}"] & C \arrow[r, "p^{\otimes i+1}\Delta_C^i"] & L^{\otimes i+1} \\
 & & D^i \arrow[u, "\delta_i"'] \arrow[ul, dotted, "\xi_i^{i+1}"] &
\end{tikzcd}$$

Let $i > 0$. Since $\delta_i = \mathrm{Ker}(p^{\otimes i}\Delta_C^{i-1})$ is a coalgebra homomorphism, we have:

$$\begin{aligned}
p^{\otimes i+1}\Delta_C^i\delta_i &= p^{\otimes i+1}(C \otimes \Delta_C^{i-1})\Delta_C\delta_i \\
&= p^{\otimes i+1}(C \otimes \Delta_C^{i-1})(\delta_i \otimes \delta_i)\Delta_{D^i} = (p\delta_i \otimes p^{\otimes i}\Delta_C^{i-1}\delta_i)\Delta_{D^i} = 0.
\end{aligned}$$

Then, for any $i \geq 1$, by the universal property of the kernel, there exists a unique morphism $\xi_i^{i+1} : D^i \to D^{i+1}$ such that $\delta_{i+1}\xi_i^{i+1} = \delta_i$. Set $\xi_0^1 = 0$ and for any $j > i$, define:

$$\xi_i^j = \xi_{j-1}^j\xi_{j-2}^{j-1}\cdots\xi_{i+1}^{i+2}\xi_i^{i+1} : D^i \to D^j.$$

In such a way we obviously obtain a direct system in $\mathcal{M}$. Let us prove that ξ_i^j is a homomorphism of coalgebras for any $j > i$. It is clearly sufficient to verify this for $j = i + 1$.
As δ_{i+1} and δ_i are coalgebra homomorphisms, we have

$$(\delta_{i+1}\otimes\delta_{i+1})\Delta_{D^{i+1}}\xi_i^{i+1} = \Delta_D\delta_{i+1}\xi_i^{i+1} = \Delta_D\delta_i = (\delta_i\otimes\delta_i)\Delta_{D^i} = (\delta_{i+1}\otimes\delta_{i+1})(\xi_i^{i+1}\otimes\xi_i^{i+1})\Delta_{D^i}.$$

Since the tensor functors are left exact, $\delta_{i+1} \otimes \delta_{i+1}$ is a monomorphism so that we get $\Delta_{D^{i+1}}\xi_i^{i+1} = (\xi_i^{i+1} \otimes \xi_i^{i+1})\Delta_{D^i}$. Moreover we have

$$\varepsilon_{D^{i+1}}\xi_i^{i+1} = \varepsilon_D\delta_{i+1}\xi_i^{i+1} = \varepsilon_D\delta_i = \varepsilon_{D^i}.$$

The last assertion follows by Proposition 1.6.2. □

***Notation* 1.6.4.** Let $\delta : D \to C$ be a homomorphism of coalgebras in a cocomplete coabelian monoidal category $\mathcal{M}$. By Proposition 1.6.3 $((D^{\wedge^i_C})_{i\in\mathbb{N}}, (\xi_i^j)_{i,j\in\mathbb{N}})$ is a direct system in $\mathcal{M}$ whose direct limit carries a natural coalgebra structure that makes it the direct limit of $((D^{\wedge^i_C})_{i\in\mathbb{N}}, (\xi_i^j)_{i,j\in\mathbb{N}})$ as a direct system of coalgebras.
From now on we set: $(\widetilde{D}_C, (\xi_i)_{i\in\mathbb{N}}) = \varinjlim (D^{\wedge^i_C})_{i\in\mathbb{N}}$, where $\xi_i : D^{\wedge^i_C} \to \widetilde{D}_C$ denotes the structural morphism of the direct limit. We simply write $\widetilde{D}$ if there is no danger of confusion. We note that, since $\widetilde{D}$ is a direct limit of coalgebras, the canonical (coalgebra) homomorphisms $(\delta_i : D^{\wedge^i_C} \to C)_{i\in\mathbb{N}}$, which are compatible by (1.12), factorize to a unique coalgebra homomorphism $\widetilde{\delta} : \widetilde{D} \to C$ such that $\widetilde{\delta}\xi_i = \delta_i$ for any $i \in \mathbb{N}$.

***Proposition* 1.6.5.** *Let $\mathcal{M}$ be a cocomplete abelian category. Let $(V_i)_{i\in\mathbb{N}}$ be a family of objects in $\mathcal{M}$ and let $(V, v_i) = \oplus_{i\in\mathbb{N}} V_i$ be the direct sum of the family $(V_i)_{i\in\mathbb{N}}$. Then*

$$(V, \nabla\left[(v_i)_{i=0}^n\right]) = \varinjlim (\oplus_{i=0}^n V_i),$$

where $\nabla\left[(v_i)_{i=0}^n\right] : \oplus_{i=0}^n V_i \to V$ denotes the codiagonal morphism associated to the family $(v_i)_{i=0}^n$.

Proof. Set $V^n := \oplus_{i=0}^n V_i$, for any $n \in \mathbb{N}$, and let $w_m^n : V^m \to V^n$ be the canonical inclusion for $m \leq n$. Let $(f_n : V^n \to X)_n$ be a compatible family of morphisms in $\mathcal{M}$, i.e. $f_n w_m^n = f_m$ for any $m \leq n$. Let $v_m^n : V_m \to V^n$ be the canonical inclusion for every $m \leq n$ and let $v_m^n = 0$ otherwise. Note that the morphism $\nabla\left[(v_i)_{i=0}^n\right] : V^n \to V$ is uniquely defined by the following relation:

$$\nabla\left[(v_i)_{i=0}^n\right] v_m^n = v_m, \text{ for every } m \leq n.$$

Observe that, for every $m \leq n \leq t$, we have

$$f_t v_m^t = f_t w_n^t v_m^n = f_n v_m^n$$

so that, by the universal property of the direct sum, there exists a unique morphism $f : V \to X$ such that

$$f v_m = f_n v_m^n, \tag{1.13}$$

for any $m \in \mathbb{N}$, where $n \in \mathbb{N}$ and $m \leq n$. Thus

$$f_n v_m^n = f v_m = f\nabla\left[(v_i)_{i=0}^n\right] v_m^n \text{ for every } m \leq n.$$

By the universal property of $V^n := \oplus_{i=0}^n V_i$, f_n is the unique morphism that composed with v_m^n gives $f_n v_m^n$ for any $m \leq n$. We get that

$$f_n = f\nabla\left[(v_i)_{i=0}^n\right], \text{ for every } n \in \mathbb{N}.$$

In order to conclude that $V = \varinjlim V^i$, it remains to prove that $f : V \to X$ is the unique morphism with this property. Let $g : V \to X$ be a morphism such that $f_n = g\nabla\left[(v_i)_{i=0}^n\right]$ for every $n \in \mathbb{N}$. Then

$$fv_m = f_n v_m^n = g\nabla\left[(v_i)_{i=0}^n\right] v_m^n = gv_m \text{ for every } m, n \in \mathbb{N}, m \leq n.$$

By uniqueness of f with respect to (1.13), we get $g = f$. $\square$

1.7 Cotensor product

1.7.1. Let $e : E_1 \to E_2$ be a coalgebra homomorphism in a coabelian monoidal category $\mathcal{M}$. Let $(V_1, \rho_{V_1}^{E_1})$ be a right E_1-comodule, let $(W_1, {}^{E_1}\rho_{W_1})$ be a left E_1-comodule, let $(V_2, \rho_{V_2}^{E_2})$ be a right E_2-comodule and let $(W_2, {}^{E_2}\rho_{W_2})$ be a left E_2-comodule. Let $v : V_1 \to V_2$ and $w : W_1 \to W_2$ be E_2-comodule homomorphisms (where V_1 and W_1 are regarded as E_2-comodules via e). Then there is a unique morphism $v\square_e w : V_1\square_{E_1} W_1 \to V_2\square_{E_2} W_2$ such that the following diagram

$$\begin{tikzcd}
V_1\square_{E_1} W_1 \arrow[r, "\varsigma_{E_1}(V_1,W_1)"] \arrow[d, dashed, "v\square_e w"'] & V_1 \otimes W_1 \arrow[d, "v\otimes w"] \\
V_2\square_{E_2} W_2 \arrow[r, "\varsigma_{E_2}(V_2,W_2)"'] & V_2 \otimes W_2
\end{tikzcd}$$

commutes. In fact we have

$$\begin{aligned}
& (\rho_{V_2}^{E_2} \otimes W_2) \circ (v \otimes w) \circ \varsigma_{E_1}(V_1, W_1) \\
= \; & (v \otimes E_2 \otimes w) \circ (\rho_{V_1}^{E_2} \otimes W_1) \circ \varsigma_{E_1}(V_1, W_1) \\
= \; & (v \otimes E_2 \otimes w) \circ [(V_1 \otimes e)\rho_{V_1}^{E_1} \otimes W_1] \circ \varsigma_{E_1}(V_1, W_1) \\
= \; & (v \otimes e \otimes w) \circ (\rho_{V_1}^{E_1} \otimes W_1) \circ \varsigma_{E_1}(V_1, W_1) \\
= \; & (v \otimes e \otimes w) \circ [V_1 \otimes {}^{E_1}\rho_{W_1}] \circ \varsigma_{E_1}(V_1, W_1) \\
= \; & (v \otimes E_2 \otimes w) \circ [V_1 \otimes (e \otimes W_1) \circ {}^{E_1}\rho_{W_1}] \circ \varsigma_{E_1}(V_1, W_1) \\
= \; & (v \otimes E_2 \otimes w) \circ (V_1 \otimes {}^{E_2}\rho_{W_1}) \circ \varsigma_{E_1}(V_1, W_1) \\
= \; & (V_2 \otimes {}^{E_2}\rho_{W_2}) \circ (v \otimes w) \circ \varsigma_{E_1}(V_1, W_1)
\end{aligned}$$

so that, since $(V_2\square_{E_2} W_2, \varsigma_{E_2}(V_2, W_2))$ is the equalizer of $\rho_{V_2}^{E_2} \otimes W_2$ and $V_2 \otimes {}^{E_2}\rho_{W_2}$, we conclude.
Note that if $E_1 = E_2 = E$ and $e = \mathrm{Id}_E$, one has

$$v\square_e w = v\square_E w.$$

***Lemma* 1.7.2.** *Let $e : E_1 \to E_2$ and $e' : E_2 \to E_3$ be coalgebra homomorphisms in $\mathcal{M}$. Let*

$$\begin{gathered}(V_1, \rho_{V_1}^{E_1}) \in \mathcal{M}^{E_1}, \quad (V_2, \rho_{V_2}^{E_2}) \in \mathcal{M}^{E_2}, \quad (V_3, \rho_{V_3}^{E_3}) \in \mathcal{M}^{E_3},\\ (W_1, {}^{E_1}\rho_{W_1}) \in {}^{E_1}\mathcal{M}, \quad (W_2, {}^{E_2}\rho_{W_2}) \in {}^{E_2}\mathcal{M}, \quad (W_3, {}^{E_3}\rho_{W_3}) \in {}^{E_3}\mathcal{M}.\end{gathered}$$

Let $v : V_1 \to V_2$ and $w : W_1 \to W_2$ be E_2-comodule homomorphisms (where V_1 and W_1 are regarded as E_2-comodules via e) and let $v' : V_2 \to V_3$ and $w' : W_2 \to W_3$ be E_3-comodule homomorphisms (where V_2 and W_2 are regarded as E_3-comodules via e'). Then

$$(v'\square_{e'}w') \circ (v\square_e w) = (v'v\square_{e'e}w'w). \tag{1.14}$$

Proof. : straightforward. $\square$

1.8 The Heyneman-Radford theorem for monoidal categories

This section is devoted to the proof of the Heyneman-Radford Theorem for Monoidal Categories. The original Heyneman-Radford's Theorem (see [HR, Proposition 2.4.2] or [Mo, Theorem 5.3.1, page 65]) is a very useful tool in classical Hopf algebra theory. We also point out that our proof is pretty different from the classical one and hence might be of some interest even in the classical case. We refer to [Ar3].

***Definition* 1.8.1.** Let E be a coalgebra and let $\delta : X \to E$ be a monomorphism in a coabelian monoidal category $\mathcal{M}$. Define the morphism

$$\alpha_X^E : E \to \frac{E}{X} \otimes \frac{E}{X}$$

by setting

$$\alpha_X^E = \left(p_X^E \otimes p_X^E\right) \circ \Delta_E.$$

Observe that $(X \wedge_E X, i_{X\wedge_E X}^E) = \mathrm{Ker}(\alpha_X^E)$.

***Lemma* 1.8.2.** *Let $\delta : D \to E$ and let $f : E \to C$ be coalgebra homomorphisms in a coabelian monoidal category $\mathcal{M}$. Assume that both δ and $f \circ \delta$ are monomorphism. Then the following diagram*

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d, "\alpha_D^E"'] & C \arrow[d, "\alpha_D^C"] \\ \frac{E}{D} \otimes \frac{E}{D} \arrow[r, "\frac{f}{D}\otimes\frac{f}{D}"] & \frac{C}{D} \otimes \frac{C}{D} \end{tikzcd}$$

is commutative.

Proof. Note that the notations E/D and C/D make sense as both δ and $f \circ \delta$ are monomorphisms. We have

$$\begin{aligned}\left(\frac{f}{D}\otimes\frac{f}{D}\right)\circ\alpha_D^E &= \left(\frac{f}{D}\otimes\frac{f}{D}\right)\circ\left(p_D^E\otimes p_D^E\right)\circ\Delta_E\\ &= \left(p_D^C\otimes p_D^C\right)\circ(f\otimes f)\circ\Delta_E\\ &= \left(p_D^C\otimes p_D^C\right)\circ\Delta_C\circ f=\alpha_D^C\circ f.\end{aligned}$$

□

Lemma 1.8.3. *[Ar3, Lemma 2.3] Let D and E be coalgebras in a coabelian monoidal category $\mathcal{M}$. Let $\delta : D \to E$ be a monomorphism which is a morphism of coalgebras in $\mathcal{M}$. Then, for every $n \in \mathbb{N}$, there exists a unique morphism $\tau_n : D^{n+1} \to D^n/D \otimes D^n/D$ such that the following diagram*

$$\begin{tikzcd} & D^{n+1} \arrow[dl, dotted, "\tau_n"'] \arrow[d, "\alpha_D^{D^{n+1}}"] \\ \frac{D^n}{D}\otimes\frac{D^n}{D} \arrow[r, "\frac{\xi_n^{n+1}}{D}\otimes\frac{\xi_n^{n+1}}{D}"'] & \frac{D^{n+1}}{D}\otimes\frac{D^{n+1}}{D}\end{tikzcd}$$

is commutative.

Proof. Consider the following exact sequence

$$0 \longrightarrow \frac{D^n}{D} \xrightarrow{\frac{\xi_n^{n+1}}{D}} \frac{D^{n+1}}{D} \xrightarrow{\frac{D^{n+1}}{\xi_1^n}} \frac{D^{n+1}}{D^n} \longrightarrow 0 \tag{1.15}$$

By applying the functor $D^{n+1}/D \otimes (-)$ we get

$$0 \longrightarrow \frac{D^{n+1}}{D}\otimes\frac{D^n}{D} \xrightarrow{\frac{D^{n+1}}{D}\otimes\frac{\xi_n^{n+1}}{D}} \frac{D^{n+1}}{D}\otimes\frac{D^{n+1}}{D} \xrightarrow{\frac{D^{n+1}}{D}\otimes\frac{D^{n+1}}{\xi_1^n}} \frac{D^{n+1}}{D}\otimes\frac{D^{n+1}}{D^n} \longrightarrow 0$$

We have

$$\begin{aligned}&\left(\frac{\delta_{n+1}}{D}\otimes\frac{\delta_{n+1}}{D^n}\right)\circ\left(\frac{D^{n+1}}{D}\otimes\frac{D^{n+1}}{\xi_1^n}\right)\circ\alpha_D^{D^{n+1}}\\ =\;&\left(\frac{\delta_{n+1}}{D}\otimes\frac{\delta_{n+1}}{D^n}\right)\circ\left(\frac{D^{n+1}}{D}\otimes\frac{D^{n+1}}{\xi_1^n}\right)\circ\left(p_D^{D^{n+1}}\otimes p_D^{D^{n+1}}\right)\circ\Delta_{D^{n+1}}\\ =\;&\left(\frac{\delta_{n+1}}{D}\otimes\frac{\delta_{n+1}}{D^n}\right)\circ\left(p_D^{D^{n+1}}\otimes p_{D^n}^{D^{n+1}}\right)\circ\Delta_{D^{n+1}}\\ =\;&\left(p_D^E\otimes p_{D^n}^E\right)\circ(\delta_{n+1}\otimes\delta_{n+1})\circ\Delta_{D^{n+1}}\\ =\;&\left(p_D^E\otimes p_{D^n}^E\right)\circ\Delta_E\circ\delta_{n+1}=0.\end{aligned}$$

In fact, by Proposition 1.5.8, $D^{n+1} = D \wedge_E D^n$. Since $\frac{\delta_{n+1}}{D} \otimes \frac{\delta_{n+1}}{D^n}$ is a monomorphism, we obtain

$$\left(\frac{D^{n+1}}{D} \otimes \frac{D^{n+1}}{\xi_1^n}\right) \circ \alpha_D^{D^{n+1}} = 0 \tag{1.16}$$

so that, as the above sequence is exact, by the universal property of kernels, there exists a unique morphism

$$\beta_n : D^{n+1} \to \frac{D^{n+1}}{D} \otimes \frac{D^n}{D}$$

such that

$$\left(\frac{D^{n+1}}{D} \otimes \frac{\xi_n^{n+1}}{D}\right) \circ \beta_n = \alpha_D^{D^{n+1}}. \tag{1.17}$$

By applying the functor $(-) \otimes D^n/D$ to (1.15), we get

$$0 \longrightarrow \frac{D^n}{D} \otimes \frac{D^n}{D} \xrightarrow{\frac{\xi_n^{n+1}}{D} \otimes \frac{D^n}{D}} \frac{D^{n+1}}{D} \otimes \frac{D^n}{D} \xrightarrow{\frac{D^{n+1}}{\xi_1^n} \otimes \frac{D^n}{D}} \frac{D^{n+1}}{D^n} \otimes \frac{D^n}{D} \longrightarrow 0.$$

We have

$$\begin{aligned}
& \left(\frac{D^{n+1}}{D^n} \otimes \frac{\xi_n^{n+1}}{D}\right) \circ \left(\frac{D^{n+1}}{\xi_1^n} \otimes \frac{D^n}{D}\right) \circ \beta_n \\
= \quad & \left(\frac{D^{n+1}}{\xi_1^n} \otimes \frac{D^{n+1}}{D}\right) \circ \left(\frac{D^{n+1}}{D} \otimes \frac{\xi_n^{n+1}}{D}\right) \circ \beta_n \\
\overset{(1.17)}{=} \quad & \left(\frac{D^{n+1}}{\xi_1^n} \otimes \frac{D^{n+1}}{D}\right) \circ \alpha_D^{D^{n+1}} = 0
\end{aligned}$$

where the last equality can be proved similarly to (1.16). Since $\frac{D^{n+1}}{D^n} \otimes \frac{\xi_n^{n+1}}{D}$ is a monomorphism we get

$$\left(\frac{D^{n+1}}{\xi_1^n} \otimes \frac{D^n}{D}\right) \circ \beta_n = 0$$

so that, as the previous sequence is exact, by the universal property of kernels there exists a unique morphism

$$\tau_n : D^{n+1} \to \frac{D^n}{D} \otimes \frac{D^n}{D}$$

such that

$$\left(\frac{\xi_n^{n+1}}{D} \otimes \frac{D^n}{D}\right) \circ \tau_n = \beta_n.$$

Finally we have

$$\begin{aligned}\left(\frac{\xi_n^{n+1}}{D}\otimes\frac{\xi_n^{n+1}}{D}\right)\circ\tau_n &= \left(\frac{D^{n+1}}{D}\otimes\frac{\xi_n^{n+1}}{D}\right)\circ\left(\frac{\xi_n^{n+1}}{D}\otimes\frac{D^n}{D}\right)\circ\tau_n\\ &= \left(\frac{D^{n+1}}{D}\otimes\frac{\xi_n^{n+1}}{D}\right)\circ\beta_n=\alpha_D^{D^{n+1}}.\end{aligned}$$

□

***Theorem* 1.8.4.** *[Ar3, Theorem 2.4] Let D and E be coalgebras in a cocomplete coabelian monoidal category $\mathcal{M}$ satisfying AB5. Let $\delta : D \to E$ be a monomorphism which is a morphism of coalgebras in $\mathcal{M}$ and keep the notations introduced in Notation 1.6.4.*
Let $f : E \to C$ be a coalgebra homomorphism and assume that

$$f\circ\delta_2 : D\wedge_E D\to C$$

is a monomorphism. Then the coalgebra homomorphism

$$f\circ\widetilde{\delta}:\widetilde{D}_E\to C$$

is a monomorphism.

Proof. Since $\mathcal{M}$ satisfies AB5, it is enough to prove that $f\circ\widetilde{\delta}\circ\xi_n = f\circ\delta_n$ is a monomorphism for every $n\in\mathbb{N}$.

For $n=0$, we have $f\circ\delta_0 = f\circ 0 = 0$ which is a monomorphism as $D^0 = 0$.

For $n=1$, we have $f\circ\delta_1 = f\circ\delta_2\circ\xi_1^2$ which is a monomorphism.

Let $n\geq 2$ and let us assume that $f\circ\delta_n$ is a monomorphism. Let us prove that $f\circ\delta_{n+1}$ is a monomorphism. Let $\lambda : X\to D^{n+1}$ be a morphism such that

$$f\circ\delta_{n+1}\circ\lambda = 0$$

and consider the following diagram

$$\begin{tikzcd}
 & D\wedge_{D^{n+1}}D \arrow[r, "D\wedge_{\delta_{n+1}}D"] \arrow[d, "i^{D^{n+1}}_{D\wedge_{D^{n+1}}D}"] & D\wedge_E D \arrow[d, "\delta_2"] & \\
X \arrow[ur, dotted, "\overline{\lambda}"] \arrow[r, "\lambda"] & D^{n+1} \arrow[r, "\delta_{n+1}"] \arrow[d, "\alpha_D^{D^{n+1}}"] \arrow[dl, "\tau_n"'] & E \arrow[r, "f"] \arrow[d, "\alpha_D^E"] & C \arrow[d, "\alpha_D^C"] \\
\frac{D^n}{D}\otimes\frac{D^n}{D} \arrow[r, "\frac{\xi_n^{n+1}}{D}\otimes\frac{\xi_n^{n+1}}{D}"'] & \frac{D^{n+1}}{D}\otimes\frac{D^{n+1}}{D} \arrow[r, "\frac{\delta_{n+1}}{D}\otimes\frac{\delta_{n+1}}{D}"'] & \frac{E}{D}\otimes\frac{E}{D} \arrow[r, "\frac{f}{D}\otimes\frac{f}{D}"'] & \frac{C}{D}\otimes\frac{C}{D}
\end{tikzcd}$$

where all the squares are commutative in view of Lemma 1.8.2 and the bottom triangle commutes in view of Lemma 1.8.3. We have

$$\begin{aligned}
&\left(\frac{f\delta_n}{D}\otimes\frac{f\delta_n}{D}\right)\circ\tau_n\circ\lambda\\
=\;&\left(\frac{f\delta_{n+1}\xi_n^{n+1}}{D}\otimes\frac{f\delta_{n+1}\xi_n^{n+1}}{D}\right)\circ\tau_n\circ\lambda\\
=\;&\left(\frac{f}{D}\otimes\frac{f}{D}\right)\circ\left(\frac{\delta_{n+1}}{D}\otimes\frac{\delta_{n+1}}{D}\right)\circ\left(\frac{\xi_n^{n+1}}{D}\otimes\frac{\xi_n^{n+1}}{D}\right)\circ\tau_n\circ\lambda\\
=\;&\alpha_D^C\circ f\circ\delta_{n+1}\circ\lambda=0.
\end{aligned}$$

Since $f\circ\delta_n$ is a monomorphism, we get that also $f\delta_n/D\otimes f\delta_n/D$ is a monomorphism so that we obtain

$$\tau_n\circ\lambda=0$$

and we have

$$\alpha_D^{D^{n+1}}\circ\lambda=\left(\frac{\xi_n^{n+1}}{D}\otimes\frac{\xi_n^{n+1}}{D}\right)\circ\tau_n\circ\lambda=0.$$

Thus, since $\left(D\wedge_{D^{n+1}}D, i^{D^{n+1}}_{D\wedge_{D^{n+1}}D}\right)=\mathrm{Ker}\left(\alpha_D^{D^{n+1}}\right)$, by the universal property of the kernel, there exists a unique morphis, $\overline{\lambda}:X\to D\wedge_{D^{n+1}}D$ such that

$$\lambda=i^{D^{n+1}}_{D\wedge_{D^{n+1}}D}\circ\overline{\lambda}.$$

Now we have

$$f\circ\delta_2\circ\left(D\wedge_{\delta_{n+1}}D\right)\circ\overline{\lambda}=f\circ\delta_{n+1}\circ\lambda=0.$$

Since $f\circ\delta_2$ and $D\wedge_{\delta_{n+1}}D$ are monomorphisms, we get that $\overline{\lambda}=0$ and hence $\lambda=0$. $\square$

Corollary 1.8.5. *(Heyneman-Radford) ([HR, Proposition 2.4.2] or [Mo, Theorem 5.3.1, page 65]) Let K be a field. Let E and C be K-coalgebras and let $f:E\to C$ be a coalgebra homomorphism such that $f_{|D\wedge_E D}$ is injective, where D is the coradical of E. Then f is injective.*

Proof. Since D is the coradical of E is well known that $(E,\mathrm{Id}_E)=(\widetilde{D}_E,\widetilde{\delta})$ (see e.g. [Sw, Corollary 9.0.4, page 185]). The conclusion follows by Theorem 1.8.4 applied in the case when $\mathcal{M}$ is the category of vector spaces over K. Observe that in this case "monomorphism" is equivalent to "injective". $\square$

Chapter 2

Relative projectivity and injectivity

In this chapter we deal with some results concerning the theory of relative left derived functors that will be used to define and classify the Hochschild cohomology in the frame of monoidal categories. We also recall and study the notion of relative projectivity and injectivity with a particular interest to those projective classes that are defined by means of suitable adjunctions related to the tensor functors.

2.1 Relative projectivity and injectivity

A main tool for introducing the Hochschild cohomology in the frame of monoidal categories is that of relative left derived functors. Most of the material introduced below can be found in [HS] and [We, Cap.8, page 279-281].

***Definitions* 2.1.1.** ([HS, Cap. IX, page 307-312]) Let $\mathfrak{C}$ be a category and let $\mathcal{H}$ be a class of morphisms in $\mathfrak{C}$.
An object $P \in \mathfrak{C}$ is called *f-projective*, where $f : C_1 \to C_2$ is a morphism in $\mathfrak{C}$, if

$$\mathrm{Hom}_{\mathfrak{C}}(P, f) : \mathrm{Hom}_{\mathfrak{C}}(P, C_1) \to \mathrm{Hom}_{\mathfrak{C}}(P, C_2) : g \mapsto f \circ g$$

is surjective.
P is *$\mathcal{H}$-projective* if it is f-projective for every $f \in \mathcal{H}$.
Define the *closure* of $\mathcal{H}$ by

$$\overline{\mathcal{H}} := \{f \in \mathfrak{C} \mid P \text{ is } \mathcal{H}\text{-projective } \Rightarrow P \text{ is } f\text{-projective, for every } P \in \mathfrak{C}\} \supseteq \mathcal{H}.$$

$\mathcal{H}$ is called *closed* if $\overline{\mathcal{H}} = \mathcal{H}$.
A closed class $\mathcal{H}$ is said to be *projective* if, for each object $C \in \mathfrak{C}$, there is a morphism $f : P \to C$ in $\mathcal{H}$ where P is $\mathcal{H}$-projective.
$\mathfrak{C}$ is called $\mathcal{H}$-semisimple, whenever every object in $\mathfrak{C}$ is $\mathcal{H}$-projective.

***Definitions* 2.1.2.** Assume now that $\mathfrak{C}$ is an abelian category.
Let $\mathcal{H}$ be a *closed* class of morphisms in $\mathfrak{C}$.
A morphism $f \in \mathfrak{C}$ is called *$\mathcal{H}$-admissible* if, in the canonical factorization

$$f = \mu \circ \xi,$$

where μ is a monomorphism and ξ is an epimorphism, we have $\xi \in \mathcal{H}$.
An exact sequence in $\mathfrak{C}$ is called *$\mathcal{H}$-exact* if all its morphisms are $\mathcal{H}$-admissible.
Finally, an *$\mathcal{H}$-projective resolution* of an object $C \in \mathfrak{C}$ is an $\mathcal{H}$-exact sequence

$$\cdots \longrightarrow P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} C \longrightarrow 0$$

such that P_n is $\mathcal{H}$-projective, for every $n \in \mathbb{N}$.

The whole theory of relative injectivity and its applications can be obtained by duality, i.e. by working in the opposite category of $\mathfrak{C}$ (note that the abelianity of a category is selfdual). Since this process is completely formal and does not require new ideas, when working with relative injectivity, we will just state the main results.

***Theorem* 2.1.3.** *Let $\mathfrak{C}$ be an abelian category and let $\mathcal{H}$ be a projective class of epimorphism in $\mathfrak{C}$. Then every object in $\mathfrak{C}$ admits an $\mathcal{H}$-projective resolution.*

Proof. The proof is similar to the classical one. Namely, let C be an object in $\mathfrak{C}$. Since $\mathcal{H}$ is a projective class of epimorphisms, there is an epimorphism $f_0 : P_0 \to C$, where P_0 is $\mathcal{H}$-projective. Set $d_0 = f_0$. Let $(K_1, i_1) = \mathrm{Ker}(d_0)$. Then there is an epimorphism $f_1 : P_1 \to K_1$ in $\mathcal{H}$, where P_1 is $\mathcal{H}$-projective. Set $d_1 = i_1 \circ f_1$. Proceeding in this way one gets a sequence

$$\begin{tikzcd}
\cdots \arrow[r, "d_{n+1}"] & P_n \arrow[rr, "d_n"] \arrow[dr, "f_n"] & & P_{n-1} \arrow[r, "d_{n-1}"] & \cdots \arrow[r, "d_2"] & P_1 \arrow[rr, "d_1"] \arrow[dr, "f_1"] & & P_0 \arrow[r, "d_0"] & C \arrow[r] & 0. \\
& & K_n \arrow[ur, hook, "i_n"'] & & & & K_1 \arrow[ur, hook, "i_1"'] & & &
\end{tikzcd}$$

Clearly d_n is $\mathcal{H}$-admissible for every $n \geqslant 0$ and $\mathrm{Im}(d_n) \simeq K_n = \mathrm{Ker}(d_{n-1})$, for every $n \geqslant 1$. Therefore the above is an $\mathcal{H}$-projective resolution of C. □

2.1.4. The theory of derived functors can be adapted to the relative context without difficulties. For details the reader is referred to [HS, page 308-309]. Let $\mathfrak{B}$, $\mathfrak{C}$ be abelian categories and let $\mathcal{H}$ be a projective class of epimorphism in $\mathfrak{B}$. By Theorem 2.1.3, every object in $\mathfrak{B}$ admits an $\mathcal{H}$-projective resolution.
Given a contravariant additive functor $\mathbf{T} : \mathfrak{B} \to \mathfrak{C}$ and given an $\mathcal{H}$-projective resolution

$$\mathbf{P}_\bullet \longrightarrow B \longrightarrow 0$$

of B, the object $\mathbf{H}^n(\mathbf{TP}_\bullet)$ depends only on B and yields an additive functor

$$\mathrm{R}^n_{\mathcal{H}}\mathbf{T} : \mathfrak{B} \to \mathfrak{C}, \qquad \mathrm{R}^n_{\mathcal{H}}\mathbf{T}\,(B) := \mathbf{H}^n(\mathbf{TP}_\bullet).$$

The functor $\mathrm{R}^n_{\mathcal{H}}\mathbf{T}$ is called the n-th *right $\mathcal{H}$-derived functor* of $\mathbf{T}$.

***Theorem* 2.1.5.** *[HS, see Theorem 2.1, page 309] Let $\mathfrak{B}, \mathfrak{C}$ be abelian categories, let $\mathcal{H}$ be a projective class of epimorphisms in $\mathfrak{B}$ and let*

$$0 \to B_1 \to B_2 \to B_3 \to 0$$

be a short $\mathcal{H}$-exact sequence in $\mathfrak{B}$.
Let $\mathbf{T} : \mathfrak{B} \to \mathfrak{C}$ be a contravariant additive functor.
Then for every $n \geq 0$ there exists a ***connecting homomorphism***

$$\omega_n : \mathrm{R}^n_{\mathcal{H}}\mathbf{T}\,(B_1) \to \mathrm{R}^{n+1}_{\mathcal{H}}\mathbf{T}\,(B_3)$$

such that the sequence

$$\begin{aligned} 0 \to \mathrm{R}^0_{\mathcal{H}}\mathbf{T}\,(B_3) \to \mathrm{R}^0_{\mathcal{H}}\mathbf{T}\,(B_2) \to \mathrm{R}^0_{\mathcal{H}}\mathbf{T}\,(B_1) \overset{\omega_0}{\to} \mathrm{R}^1_{\mathcal{H}}\mathbf{T}\,(B_3) \to \cdots \\ \cdots \overset{\omega_{n-1}}{\to} \mathrm{R}^n_{\mathcal{H}}\mathbf{T}\,(B_3) \to \mathrm{R}^n_{\mathcal{H}}\mathbf{T}\,(B_2) \to \mathrm{R}^n_{\mathcal{H}}\mathbf{T}\,(B_1) \overset{\omega_n}{\to} \mathrm{R}^{n+1}_{\mathcal{H}}\mathbf{T}\,(B_3) \to \cdots . \end{aligned}$$

is exact.

***Definition* 2.1.6.** Let $\mathfrak{B}$, $\mathfrak{C}$ be abelian categories and let $\mathcal{H}$ be a projective class of epimorphisms in $\mathfrak{B}$. Recall that a contravariant functor $\mathbf{T} : \mathfrak{B} \to \mathfrak{C}$ is called *left $\mathcal{H}$-exact* if, for every $\mathcal{H}$-exact sequence

$$B_1 \to B_2 \to B_3 \to 0,$$

the sequence $0 \to \mathbf{T}\,(B_3) \to \mathbf{T}\,(B_2) \to \mathbf{T}\,(B_1)$ is exact.

***Proposition* 2.1.7.** *([HS, pag. 311-312]) Let $\mathfrak{B}$, $\mathfrak{C}$ be abelian categories and let $\mathcal{H}$ be a projective class of epimorphisms in $\mathfrak{B}$.*
Let $\mathbf{T} : \mathfrak{B} \to \mathfrak{C}$ be a contravariant left $\mathcal{H}$-exact functor. Then:
1) $\mathbf{T}$ is additive.
2) There is a functorial isomorphism $\tau : \mathbf{T} \to \mathrm{R}^0_{\mathcal{H}}\mathbf{T}$.
3) $\mathrm{R}^n_{\mathcal{H}}\mathbf{T}\,(P) = 0$, for every $n > 0$ and for every $\mathcal{H}$-projective object P.

2.2 The Case of an Arbitrary Adjunction

***Theorem* 2.2.1.** *(see [Ar1, Theorem 2.2]) Let $\mathbb{H} : \mathfrak{B} \to \mathfrak{A}$ be a covariant functor and consider:*

$$\mathcal{P}_{\mathbb{H}} := \{f \in \mathfrak{B} \mid \mathbb{H}(f) \text{ splits in } \mathfrak{A}\}. \tag{2.1}$$

Let $\mathbb{T} : \mathfrak{A} \to \mathfrak{B}$ be a left adjoint of $\mathbb{H}$ and let $\varepsilon : \mathbb{T}\mathbb{H} \to \mathrm{Id}_{\mathfrak{B}}$ be the counit of the adjunction.
Then, for any object $P \in \mathfrak{B}$, the following assertions are equivalent:
 (a) *P is $\mathcal{P}_{\mathbb{H}}$-projective.*
 (b) *Every morphism $f : B \to P$ in $\mathcal{P}_{\mathbb{H}}$ has a section.*
 (c) *$\varepsilon_P : \mathbb{T}\mathbb{H}P \to P$ has a section $\beta : P \to \mathbb{T}\mathbb{H}P$, i.e. $\varepsilon_P \circ \beta = \mathrm{Id}_P$.*
 (d) *There is a split epimorphism $\pi : \mathbb{T}X \to P$ for a suitable object $X \in \mathfrak{A}$.*
In particular all objects of the form $\mathbb{T}X$, $X \in \mathfrak{A}$, are $\mathcal{P}_{\mathbb{H}}$-projective.
Moreover $\mathcal{P}_{\mathbb{H}}$ is a closed projective class.

Proof. Let $\eta : \mathrm{Id}_{\mathfrak{A}} \to \mathbb{H}\mathbb{T}$ be the unit of the adjunction.
$(a) \Rightarrow (b)$. Assume that $P \in \mathfrak{B}$ is $\mathcal{P}_{\mathbb{H}}$-projective i.e. that for every $f : B \to B'$ in $\mathcal{P}_{\mathbb{H}}$ and for every morphism $\gamma : P \to B'$, there exists a morphism $\beta : P \to B$ such that $\gamma = f \circ \beta$. In particular, for $B' := P$ and $\gamma := \mathrm{Id}_P$, there exists a morphism $\beta : P \to B$ such that $\mathrm{Id}_P = f \circ \beta$.
$(b) \Rightarrow (c)$. Since $\mathbb{H}(\varepsilon_B) \circ \eta_{\mathbb{H}B} = \mathrm{Id}_{\mathbb{H}B}$, we infer that $\mathbb{H}(\varepsilon_B)$ splits and hence the counit $\varepsilon_B : \mathbb{T}\mathbb{H}B \to B$ belongs to $\mathcal{P}_{\mathbb{H}}$ for any $B \in \mathfrak{B}$.
$(c) \Rightarrow (d)$. Obvious.
$(d) \Rightarrow (a)$. Let $f : B_1 \to B_2$ be in $\mathcal{P}_{\mathbb{H}}$ and denote by $g : \mathbb{H}B_2 \to \mathbb{H}B_1$ the section of $\mathbb{H}(f)$. Let $\gamma : P \to B_2$. Assume that $\pi : \mathbb{T}X \to P$ is a split morphism for a suitable object $X \in \mathfrak{A}$. Let $\sigma : P \to \mathbb{T}X$ be a section of π and $\tau : P \to B_1$ be defined by

$$P \xrightarrow{\sigma} \mathbb{T}X \xrightarrow{\mathbb{T}(\eta_X)} \mathbb{T}\mathbb{H}\mathbb{T}X \xrightarrow{\mathbb{T}\mathbb{H}(\pi)} \mathbb{T}\mathbb{H}P \xrightarrow{\mathbb{T}\mathbb{H}(\gamma)} \mathbb{T}\mathbb{H}B_2 \xrightarrow{\mathbb{T}(g)} \mathbb{T}\mathbb{H}B_1 \xrightarrow{\varepsilon_{B_1}} B_1.$$

We have

$$\begin{aligned} f\tau &= f\varepsilon_{B_1}\mathbb{T}(g)\,\mathbb{T}\mathbb{H}(\gamma)\,\mathbb{T}\mathbb{H}(\pi)\,\mathbb{T}(\eta_X)\,\sigma \\ &= \varepsilon_{B_2}\mathbb{T}\mathbb{H}(f)\,\mathbb{T}(g)\,\mathbb{T}\mathbb{H}(\gamma\pi)\,\mathbb{T}(\eta_X)\,\sigma \\ &= \varepsilon_{B_2}\mathbb{T}\left[\mathbb{H}(f)\,(g)\right]\mathbb{T}\mathbb{H}(\gamma\pi)\,\mathbb{T}(\eta_X)\,\sigma \\ &= \varepsilon_{B_2}\mathbb{T}\mathbb{H}(\gamma\pi)\,\mathbb{T}(\eta_X)\,\sigma \\ &= \gamma\pi\varepsilon_{\mathbb{T}X}\mathbb{T}(\eta_X)\,\sigma \\ &= \gamma\pi\sigma = \gamma \end{aligned}$$

and hence P is $\mathcal{P}_{\mathbb{H}}$-projective.

Since $\varepsilon_{\mathbb{T}X} \circ \mathbb{T}(\eta_X) = \mathrm{Id}_{\mathbb{T}X}$, by $(c) \Rightarrow (d) \Rightarrow (a)$, we have that $\mathbb{T}X$ is $\mathcal{P}_{\mathbb{H}}$-projective.

Let us prove that $\mathcal{P}_{\mathbb{H}}$ is closed. Let $f \in \overline{\mathcal{P}_{\mathbb{H}}}$, $f : B_1 \to B_2$. Since $\mathbb{T}\mathbb{H}B_2$ is $\mathcal{P}_{\mathbb{H}}$-projective, it is also f-projective i.e. $\mathfrak{B}(\mathbb{T}\mathbb{H}B_2, f)$ is surjective. In particular there exists a morphism $\nu : \mathbb{T}\mathbb{H}B_2 \to B_1$ such that $f \circ \nu = \varepsilon_{B_2}$, so that

$$\mathbb{H}(f) \circ \mathbb{H}(\nu) \circ \eta_{\mathbb{H}B_2} = \mathbb{H}(f \circ \nu) \circ \eta_{\mathbb{H}B_2} = \mathbb{H}(\varepsilon_{B_2}) \circ \eta_{\mathbb{H}B_2} = \mathrm{Id}_{\mathbb{H}B_2}$$

i.e. $\mathbb{H}(f)$ splits and hence $f \in \mathcal{P}_{\mathbb{H}}$.
The class $\mathcal{P}_{\mathbb{H}}$ is projective as, for every B in $\mathfrak{B}$, the morphism $\varepsilon_B : \mathbb{T}\mathbb{H}B \to B$ is in $\mathcal{P}_{\mathbb{H}}$ and $\mathbb{T}\mathbb{H}P$ is $\mathcal{P}_{\mathbb{H}}$-projective. $\square$

***Remark* 2.2.2.** We point out that the the class $\mathcal{P}_{\mathbb{H}}$, introduced in Theorem 2.2.1, need not to be a class of epimorphisms in general. In fact this is true if we also assume that the functor $\mathbb{H}$ is faithful (see 6.1.3). This will be the case in most of the examples we will consider. Thus our definition of projective class will agree with [HS, Cap. IX, page 307] where the class is always assumed to be a class of epimorphisms.

For completeness we include the dual statement of Theorem 2.2.1.

Theorem 2.2.3. *(see [Ar1, Theorem 2.3]) Let $\mathbb{T} : \mathfrak{A} \to \mathfrak{B}$ be a covariant functor and consider:*

$$\mathcal{I}_{\mathbb{T}} := \{g \in \mathfrak{A} \mid \mathbb{T}(g) \text{ cosplits in } \mathfrak{B}\} \tag{2.2}$$

Let $\mathbb{H} : \mathfrak{B} \to \mathfrak{A}$, be a right adjoint of $\mathbb{T}$ and let $\eta : \mathrm{Id}_{\mathfrak{A}} \to \mathbb{HT}$ be the unit of the adjunction.
Then, for any object $I \in \mathfrak{A}$, the following assertions are equivalent :
(*a*) *I is $\mathcal{I}_{\mathbb{T}}$-injective.*
(*b*) *every morphism $f : I \to A$ in $\mathcal{I}_{\mathbb{T}}$ has a retraction.*
(*c*) *$\eta_I : I \to \mathbb{HT}I$ has a retraction $\alpha : \mathbb{HT}I \to I$, i.e. $\alpha \circ \eta_I = \mathrm{Id}_I$.*
(*d*) *There is a cosplit morphism $i : I \to \mathbb{H}Y$ for a suitable object $Y \in \mathfrak{B}$.*
In particular all objects of the form $\mathbb{H}Y$, $Y \in \mathfrak{B}$, are $\mathcal{I}_{\mathbb{T}}$-injective.
Moreover $\mathcal{I}_{\mathbb{T}}$ is a closed injective class.

Theorem 2.2.4. *Let $\mathfrak{A}, \mathfrak{B}$ be a abelian categories. Let $\mathbb{H} : \mathfrak{B} \to \mathfrak{A}$ be a covariant functor. Assume that $\mathbb{H}$ is faithful. Let $\mathbb{T} : \mathfrak{A} \to \mathfrak{B}$ be a left adjoint of $\mathbb{H}$.*
Let $P_\bullet \xrightarrow{d_\bullet} B$ be a complex in $\mathfrak{B}$, where $P_{-1} = B$. Assume that, for every $n \in \mathbb{N}$, there is a morphism

$$s_n : \mathbb{H}(P_n) \to \mathbb{H}(P_{n+1})$$

such that

$$\mathbb{H}(d_0) \circ s_{-1} = \mathrm{Id}_{\mathbb{H}B} \quad \text{and} \quad \mathbb{H}(d_{n+1}) \circ s_n + s_{n-1} \circ \mathbb{H}(d_n) = \mathrm{Id}_{\mathbb{H}(P_n)}$$

i.e. an homotopy between the identity morphism of the complex
$\mathbb{H}(P_\bullet) \xrightarrow{\mathbb{H}(d_\bullet)} \mathbb{H}(B)$ and the zero morphism. Then $P_\bullet \xrightarrow{d_\bullet} B$ is a $\mathcal{P}_{\mathbb{H}}$-exact sequence.

Proof. Let

$$(K_n, i_n) = \mathrm{Ker}(d_n).$$

Since $d_n \circ d_{n+1} = 0$, for every $n \in \mathbb{N}$, by the universal property of kernels, there exists a unique morphism $p_n : P_{n+1} \to K_n$ such that

$$i_n \circ p_n = d_{n+1}.$$

We have

$$\begin{aligned}\mathbb{H}(i_n) &= \mathrm{Id}_{\mathbb{H}(P_n)} \circ \mathbb{H}(i_n) \\ &= [\mathbb{H}(d_{n+1}) \circ s_n + s_{n-1} \circ \mathbb{H}(d_n)] \circ \mathbb{H}(i_n) \\ &= \mathbb{H}(d_{n+1}) \circ s_n \circ \mathbb{H}(i_n) = \mathbb{H}(i_n) \circ \mathbb{H}(p_n) \circ s_n \circ \mathbb{H}(i_n)\end{aligned}$$

and hence

$$\mathbb{H}(i_n) = \mathbb{H}(d_{n+1}) \circ s_n \circ \mathbb{H}(i_n) = \mathbb{H}(i_n) \circ \mathbb{H}(p_n) \circ s_n \circ \mathbb{H}(i_n). \tag{2.3}$$

In view of [St, Proposition 9.4], $\mathbb{H}$ preserves limits and in particular kernels. Since i_n is a monomorphism, and hence a kernel in $\mathfrak{B}$, then $\mathbb{H}(i_n)$ is a kernel and hence a monomorphism in $\mathfrak{A}$. From the relations above, we get that

$$\mathbb{H}(p_n) \circ s_n \circ \mathbb{H}(i_n) = \mathrm{Id}_{\mathbb{H}(K_n)}.$$

i.e. $p_n \in \mathcal{P}_{\mathbb{H}}$ and hence d_{n+1} is admissible for every $n \in \mathbb{N}$. Note also that

$$\mathbb{H}(d_0) \circ s_{-1} = \mathrm{Id}_{\mathbb{H}B}$$

means that $d_0 \in \mathcal{P}_{\mathbb{H}}$ and hence d_0 is admissible as it is an epimorphism.
It remains to prove that the complex $P_\bullet \xrightarrow{d_\bullet} B$ is an exact sequence in $\mathfrak{B}$.
Let

$$(C_n, \pi_n) = \mathrm{Coker}(d_{n+1}).$$

We have to verify that

$$(K_n, i_n) = \mathrm{Im}(d_{n+1}) = \mathrm{Ker}(\pi_n).$$

By (2.3), we have

$$\mathbb{H}(\pi_n \circ i_n) = \mathbb{H}(\pi_n) \circ \mathbb{H}(i_n) = \mathbb{H}(\pi_n) \circ \mathbb{H}(d_{n+1}) \circ s_n \circ \mathbb{H}(i_n) = 0.$$

Since $\mathbb{H}$ is faithful, we get that $\pi_n \circ i_n = 0$.

Let now $\beta : P_n \to X$ be a morphism in $\mathfrak{B}$ such that $\beta \circ i_n = 0$. Then

$$\beta \circ d_{n+1} = \beta \circ i_n \circ p_n = 0.$$

Since $(C_n, \pi_n) =$Coker(d_{n+1}), there is a unique morphism $\gamma : C_n \to X$ such that

$$\gamma \circ \pi_n = \beta.$$

In this way we have proved that

$$(C_n, \pi_n) = \mathrm{Coker}(i_n).$$

Since $\mathfrak{B}$ is an abelian category and i_n is a monomorphism, this is equivalent to $(K_n, i_n) = \mathrm{Ker}(\pi_n)$. □

2.3 Some adjunctions associated to the tensor functor

2.3.1. Let (A, m, u) be an algebra in $(\mathcal{M}, \otimes, \mathbf{1}, a, l, r)$. We have the functors

$$
\begin{aligned}
{}_A\mathbb{T} : \mathcal{M} \to {}_A\mathcal{M} \text{ where } {}_A\mathbb{T}(X) := A \otimes X \text{ and } {}_A\mathbb{T}(f) := A \otimes f,\\
\mathbb{T}_A : \mathcal{M} \to \mathcal{M}_A \text{ where } \mathbb{T}_A(X) := X \otimes A \text{ and } \mathbb{T}_A(f) := f \otimes A,\\
{}_A\mathbb{T}_A : \mathcal{M} \to {}_A\mathcal{M}_A \text{ where } {}_A\mathbb{T}_A(X) := A \otimes (X \otimes A) \text{ and } {}_A\mathbb{T}_A(f) := A \otimes (f \otimes A),
\end{aligned}
$$

with their right adjoint (see [AMS3, Proposition 1.6]) ${}_A\mathbb{H}, \mathbb{H}_A, {}_A\mathbb{H}_A$, respectively, that forget the module structures. Then the adjunctions $(\mathbb{T}_A, \mathbb{H}_A)$, $({}_A\mathbb{T}, {}_A\mathbb{H})$ and $({}_A\mathbb{T}_A, {}_A\mathbb{H}_A)$, give rise to the following classes of **epimorphisms**:

$$
\begin{aligned}
\mathcal{P}_A := \mathcal{P}_{\mathbb{H}_A} = \{g \in \mathcal{M}_A \mid g \text{ splits in } \mathcal{M}\},\\
{}_A\mathcal{P} := \mathcal{P}_{{}_A\mathbb{H}} = \{g \in {}_A\mathcal{M} \mid g \text{ splits in } \mathcal{M}\},\\
\mathcal{P} := \mathcal{P}_{{}_A\mathbb{H}_A} = \{g \in {}_A\mathcal{M}_A \mid g \text{ splits in } \mathcal{M}\}.
\end{aligned}
$$

Proposition 2.3.2. *[AMS3, Proposition 1.6]*
a) ${}_AT$ is a left adjoint of ${}_AU : {}_A\mathcal{M} \to \mathcal{M}$, the functor that "forgets" the module structure.

b) T_A is a left adjoint of $U_A : \mathcal{M}_A \to \mathcal{M}$, the functor that "forgets" the module structure.

c) ${}_AT_A$ is a left adjoint of ${}_AU_A : {}_A\mathcal{M}_A \to \mathcal{M}$, the functor that "forgets" the bimodule structure.

Proof. a) To prove that ${}_AT$ is a left adjoint of ${}_AU : {}_A\mathcal{M} \to \mathcal{M}$ we need morphisms:

$$
\begin{tikzcd}
{}_A\mathcal{M}(A \otimes X, M) \arrow[r, shift left, "\phi_l(X,M)"] & \mathcal{M}(X, M) \arrow[l, shift left, "\psi_l(X,M)"]
\end{tikzcd}
$$

which are mutual inverses that are natural in X and M. We define $\phi_l(X, M)(f) := f(u \otimes X)l_X^{-1}$ and $\psi_l(X, M)(g) := \mu(A \otimes g)$, where μ is the module structure of M. It is easy to prove that $\psi_l(X, M)(g)$ is a morphism of left modules, and that $\psi_l(X, M)$ is the inverse of $\phi_l(X, M)$.

b) The isomorphisms

$$
\begin{tikzcd}
\mathcal{M}_A(X \otimes A, M) \arrow[r, shift left, "\phi_r(X,M)"] & \mathcal{M}(X, M) \arrow[l, shift left, "\psi_r(X,M)"]
\end{tikzcd}
$$

are now given by $\phi_r(X, M)(f) := f(X \otimes u)r_X^{-1}$ and $\psi_r(X, M)(g) := \mu(g \otimes A)$, where μ is the module structure of M.

c) The isomorphisms

$$
{}_A\mathcal{M}_A\left((A \otimes X) \otimes A, M\right) \underset{\psi(X,M)}{\overset{\phi(X,M)}{\rightleftarrows}} \mathcal{M}(X, M)
$$

are obtained by combining the isomorphisms constructed above:

$$
\phi(X, M) = \phi_l(X, M)\phi_r(A \otimes X, M),
$$

and similarly for $\psi(X, M)$. For future references, we explicitly write them down:

$$
\begin{aligned}
&(2.4) \qquad & \phi(X, M)(f) &= f(A \otimes X \otimes u)r_{A\otimes X}^{-1}(u \otimes X)l_X^{-1}, \\
&(2.5) \qquad & \psi(X, M)(g) &= \mu_M^r(\mu_M^l \otimes A)(A \otimes g \otimes A),
\end{aligned}
$$

where μ_r and μ_l give respectively the right and left A-module structures of M. □

***Corollary* 2.3.3.** *Let $(\mathcal{M}, \otimes, \mathbf{1})$ be an abelian monoidal category. The functors ${}_AT$, T_A and ${}_AT_A$ are additive and preserve colimits. In particular they are right exact.*

Proof. In view of the fact that the tensor product is an additive functor in both variables, all the functors that appear in Proposition 2.3.2 are additive and the adjunctions themselves are additive too. □

By applying Theorem 2.2.1 in the case of the adjunction $({}_A\mathbb{T}_A, {}_A\mathbb{H}_A)$, we deduce the following result.

***Theorem* 2.3.4.** *Let P be an object in ${}_A\mathcal{M}_A$, the following assertions are equivalent:*

(a) P is $\mathcal{P}$-projective.

(b) Every morphism $f : M \to P$ in $\mathcal{P}$ has a section.

(c) $\varepsilon_P = \mu_P^r(\mu_P^l \otimes A) : A \otimes P \otimes A \to P$ has a section $\beta : P \to A \otimes P \otimes A$ in ${}_A\mathcal{M}_A$, i.e. $\varepsilon_P\beta = \mathrm{Id}_P$.

(d) There is a split epimorphism $\pi : A \otimes X \otimes A \to P$ in ${}_A\mathcal{M}_A$ for a suitable object $X \in \mathcal{M}$.

In particular all objects of the form $A \otimes X \otimes A$, $X \in \mathcal{M}$, are $\mathcal{P}$-projective.

Moreover $\mathcal{P}$ is a closed projective class of epimorphisms.

Chapter 3

Hochschild cohomology

In this chapter, we introduce and investigate the properties of Hochschild cohomology of algebras in an *abelian monoidal category* (see Definition 1.3.1), and we will show that several properties of separable and formally smooth algebras in the classical sense still hold true in this wider context. The multitude of interesting examples is one of the explanations for our interest in defining Hochschild cohomology of algebras in abelian monoidal categories. In this way we will recover, in an unifying manner, many well known results regarding apparently different variants of Hochschild cohomology. The main applications of our work on Hochschild cohomology are included in [AMS1]. In that paper, using the "categorical" version of Wedderburn-Malcev Theorem, besides other results, we characterize bialgebras with (dual) Chevalley property (see Theorem 6.8.6 and Theorem 6.8.7).

3.1 Hochschild cohomology

3.1.1. Let A be an algebra in an abelian monoidal category $(\mathcal{M}, \otimes, \mathbf{1})$, and let $\mathbb{T}_A$ and $\mathbb{H}_A$ be the functors defined in 2.3.1. For every $(M, \mu_M^r) \in \mathcal{M}_A$ let us consider the complex $(\beta_*(A, M), d_*)$, where we set

$$\beta_n(A, M) = \begin{cases} 0, & \text{for } n < -1; \\ M, & \text{for } n = -1; \\ M \otimes A^{\otimes n+1} & \text{for } n > -1; \end{cases}$$

and $d_n : \beta_n(A, M) \to \beta_{n-1}(A, M)$ is defined by

$$d_n = \sum_{i=0}^{n} (-1)^i \partial_i^n,$$

where

$$\partial_i^n = \begin{cases} M \otimes A^{\otimes n-i-1} \otimes m \otimes A^{\otimes i}, & \text{for } 0 \leq i < n; \\ \mu_M^r \otimes A^{\otimes n}, & \text{for } i = n. \end{cases} \tag{3.1}$$

Observe that, for $n \geq 0$ we have

$$\begin{aligned} \beta_n(A,M) = \beta_{n-1}(A,M) \otimes A = \mathbb{T}_A\mathbb{H}_A(\beta_{n-1}(A,M)), \\ \partial_i^n = \mu^r_{\beta_{n-i-1}(A,M)} \otimes A^{\otimes i} = \beta_{i-1}(A, \mu^r_{\beta_{n-i-1}(A,M)}). \end{aligned}$$

Moroever, for $n \geq 0$ and $i \geq 1$, we have

$$\partial_i^n = \partial_{i-1}^{n-1} \otimes A.$$

***Theorem* 3.1.2.** *Let M be an object in $\mathcal{M}$. We have that:*
i) If $M \in \mathcal{M}_A$ then $(\beta_*(A,M), d_*)$ is a $\mathcal{P}_A$-projective resolution of M.
ii) If $M \in {}_A\mathcal{M}_A$ then $(\beta_*(A,M), d_*)$ is a $\mathcal{P}$-projective resolution of M.

Proof. Let us check that $(\beta_*(A,M), d_*)$ is a complex, i.e. $d_n \circ d_{n+1} = 0$, for every $n \in \mathbb{N}$.
If $n = 0$, we have

$$\begin{aligned} d_n \circ d_{n+1} &= d_0 \circ d_1 \\ &= \partial_0^0 \circ \left(\partial_0^1 - \partial_1^1\right) \\ &= \mu^r_M \circ (M \otimes m) - \mu^r_M \circ (\mu^r_M \otimes A) = 0 \end{aligned}$$

Let $n > 0$. Assume $d_t \circ d_{t+1} = 0$ for every $0 \leq t \leq n-1$ and let us prove that $d_n \circ d_{n+1} = 0$. First of all, we have

$$\begin{aligned} d_n &= \sum_{i=0}^{n} (-1)^i \partial_i^n \\ &= \partial_0^n + \sum_{i=1}^{n} (-1)^i \partial_i^n \\ &= \partial_0^n + \sum_{i=1}^{n} (-1)^i \left(\partial_{i-1}^{n-1} \otimes A\right) \\ &= \partial_0^n + \left[\left(\sum_{i=1}^{n} (-1)^i \partial_{i-1}^{n-1}\right) \otimes A\right] \\ &= \partial_0^n + \left[\left(\sum_{u=0}^{n-1} (-1)^{u+1} \partial_u^{n-1}\right) \otimes A\right] \\ &= \partial_0^n + \left[-\left(\sum_{u=0}^{n-1} (-1)^u \partial_u^{n-1}\right) \otimes A\right] \\ &= \partial_0^n - \left(d_{n-1} \otimes A\right). \end{aligned}$$

so that

$$\begin{aligned}
d_n \circ d_{n+1} &= [\partial_0^n - (d_{n-1} \otimes A)] \circ \left[\partial_0^{n+1} - (d_n \otimes A)\right] \\
&= \partial_0^n \circ \partial_0^{n+1} - \partial_0^n \circ (d_n \otimes A) - (d_{n-1} \otimes A) \circ \partial_0^{n+1} + (d_{n-1} \otimes A) \circ (d_n \otimes A) \\
&= \partial_0^n \circ \partial_0^{n+1} - \partial_0^n \circ (d_n \otimes A) - (d_{n-1} \otimes A) \circ \partial_0^{n+1} \\
&= \partial_0^n \circ \partial_0^{n+1} - \partial_0^n \circ [(\partial_0^n - (d_{n-1} \otimes A)) \otimes A] - (d_{n-1} \otimes A) \circ \partial_0^{n+1} \\
&= \partial_0^n \circ \partial_0^{n+1} - \partial_0^n \circ (\partial_0^n \otimes A) + \partial_0^n \circ (d_{n-1} \otimes A \otimes A) - (d_{n-1} \otimes A) \circ \partial_0^{n+1} \\
&= \left(M \otimes A^{\otimes n-1} \otimes m\right) \circ \left(M \otimes A^{\otimes n} \otimes m\right) + \\
&\quad - \left(M \otimes A^{\otimes n-1} \otimes m\right) \circ \left(M \otimes A^{\otimes n-1} \otimes m \otimes A\right) + \\
&\quad + \left(M \otimes A^{\otimes n-1} \otimes m\right) \circ (d_{n-1} \otimes A \otimes A) - (d_{n-1} \otimes A) \circ \left(M \otimes A^{\otimes n} \otimes m\right) \\
&= 0
\end{aligned}$$

Thus $(\beta_*(A, M), d_*)$ is a complex.

Since $\beta_n(A, M) = \mathbb{T}_A \mathbb{H}_A(\beta_{n-1}(A, M))$, for every $n \in \mathbb{N}$, by Theorem 2.3.4, one has that $\beta_n(A, M)$ is $\mathcal{P}_A$-projective.

For every $n \geq -1$, let $s_n : \beta_n(A, M) \to \beta_{n+1}(A, M)$ be the morphism in $\mathcal{M}$ defined by:

$$s_n = (\beta_n(A, M) \otimes u) \circ r^{-1}_{\beta_n(A,M)},$$

where $u : \mathbf{1} \to A$ is the unit of A.

We have

$$\begin{aligned}
d_0 \circ s_{-1} &= d_0 \circ (\beta_{-1}(A, M) \otimes u) \circ r^{-1}_{\beta_{-1}(A,M)} \\
&= \mu^r_M \circ (M \otimes u) \circ r^{-1}_M = \mathrm{Id}_M = \mathrm{Id}_{\beta_{-1}(A,M)}
\end{aligned}$$

Moreover, for any $n \geq 0$, we have

$$\begin{aligned}
& d_{n+1} \circ s_n \\
= \; & \left[\partial_0^{n+1} - (d_n \otimes A)\right] \circ (\beta_n(A, M) \otimes u) \circ r^{-1}_{\beta_n(A,M)} \\
= \; & \partial_0^{n+1} \circ (\beta_n(A, M) \otimes u) \circ r^{-1}_{\beta_n(A,M)} - (d_n \otimes A) \circ (\beta_n(A, M) \otimes u) \circ r^{-1}_{\beta_n(A,M)} \\
= \; & \left(M \otimes A^{\otimes n} \otimes m\right) \circ (\beta_n(A, M) \otimes u) \circ r^{-1}_{\beta_n(A,M)} - (\beta_{n-1}(A, M) \otimes u) \circ r^{-1}_{\beta_{n-1}(A,M)} \circ d_n \\
= \; & ((\beta_{n-1}(A, M) \otimes m) \circ (\beta_{n-1}(A, M) \otimes A \otimes u) \circ r^{-1}_{\beta_n(A,M)} - s_{n-1} \circ d_n \\
= \; & \left[\beta_{n-1}(A, M) \otimes \left(m \circ (A \otimes u) \circ r_A^{-1}\right)\right] - s_{n-1} \circ d_n = \mathrm{Id}_{\beta_n(A,M)} - s_{n-1} \circ d_n.
\end{aligned}$$

Then, we showed that:

$$d_0 \circ s_{-1} = \mathrm{Id}_A \quad \text{and} \quad d_{n+1} \circ s_n + s_{n-1} \circ d_n = \mathrm{Id}_{\beta_n(A,M)}.$$

Observe now that $(\beta_*(A, M), d_*)$ is a complex in $\mathcal{M}_A$ (resp. ${}_A\mathcal{M}_A$) whenever M is an object in $\mathcal{M}_A$ (resp. ${}_A\mathcal{M}_A$). Since $\mathbb{H}_A : \mathcal{M}_A \to \mathcal{M}$ (resp. ${}_A\mathbb{H}_A : {}_A\mathcal{M}_A \to \mathcal{M}$) is faithful, we can apply Theorem 2.2.4 to conclude that $(\beta_*(A, M), d_*)$ is a $\mathcal{P}_A$-projective (resp. $\mathcal{P}$-projective) resolution of M. □

***Definition* 3.1.3.** As in the classical case, the exact complex $(\beta_*(A, M), d_*)$ will be called the *bar resolution* of M.

3.1.4. By Theorem 2.3.4, $\mathcal{P}$ is a projective class of epimorphisms so that, in view of Theorem 2.1.3, any object in ${}_A\mathcal{M}_A$ admits a $\mathcal{P}$-projective resolution. Furthermore, as in the non-relative case, one can prove that such a resolution is unique up to a homotopy.

Following 2.1.4, we can now consider, for every $M \in {}_A\mathcal{M}_A$, the right $\mathcal{P}$-derived functors $\mathrm{R}^*_{\mathcal{P}} F_M$ of $F_M := {}_A\mathcal{M}_A(-, M)$.

***Definition* 3.1.5.** For every $M, N \in {}_A\mathcal{M}_A$, we set:

$$\mathrm{Ext}^*_{\mathcal{P}}(N, M) = \mathrm{R}^*_{\mathcal{P}} F_M(N). \tag{3.2}$$

The following well known result can be proved as in the non-relative case.

***Proposition* 3.1.6.** *Let (A, m, u) be an algebra in an abelian monoidal category $(\mathcal{M}, \otimes, \mathbf{1})$ and let $N \in_A \mathcal{M}_A$. The following assertions are equivalent:*
(a) N is $\mathcal{P}$-projective.
(b) $\mathrm{Ext}^1_{\mathcal{P}}(N, M) = 0$, *for all $M \in {}_A\mathcal{M}_A$.*
(c) $\mathrm{Ext}^n_{\mathcal{P}}(N, M) = 0$, *for all $M \in {}_A\mathcal{M}_A$, and $n > 0$.*

***Definition* 3.1.7.** Let (A, m, u) be an algebra in an abelian monoidal category $(\mathcal{M}, \otimes, \mathbf{1})$, and let M be an A-bimodule.
The *Hochschild cohomology* of A with coefficients in M is:

$$\mathrm{H}^*(A, M) = \mathrm{Ext}^*_{\mathcal{P}}(A, M).$$

The *Hochschild dimension* of A is

$$\mathrm{Hdim}(A) = \min \left\{ n \in \mathbb{N} \mid \mathrm{H}^{n+1}(A, M) = 0, \forall M \in {}_A\mathcal{M}_A \right\}$$

if it exists. If such an n does not exist, we will say that the Hochschild dimension of A is infinite.

3.1.8. In order to compute $\mathrm{H}^*(A, M)$ we shall apply the functor ${}_A\mathcal{M}_A(-, M)$ to the bar resolution $\beta_*(A, A)$ which is, by Theorem 3.1.2, a $\mathcal{P}$-projective resolution of A:

$$\cdots \longrightarrow A^{\otimes n+2} \xrightarrow{d_n} A^{\otimes n+1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} A \otimes A \otimes A \xrightarrow{d_1} A \otimes A \xrightarrow{d_0} A \longrightarrow 0$$

For any morphism f in $\mathcal{M}$ let us denote ${}_A\mathcal{M}_A(f, M)$ by $\widehat{f}$. Thus we have the following complex:

$$0 \longrightarrow {}_A\mathcal{M}_A(A, M) \xrightarrow{\widehat{d_0}} {}_A\mathcal{M}_A(A \otimes A, M) \xrightarrow{\widehat{d_1}} {}_A\mathcal{M}_A(A \otimes A \otimes A, M) \xrightarrow{\widehat{d_2}} \cdots .$$

Take the notations of Proposition 2.3.2. For every $n > 1$, we denote by b^{n-1} the unique map that makes the following diagram commutative:

$$\begin{tikzcd} {}_A\mathcal{M}_A(A \otimes A^{\otimes n} \otimes A, M) \arrow[r, "\phi(A^{\otimes n},M)"] & \mathcal{M}(A^{\otimes n}, M) \\ {}_A\mathcal{M}_A(A \otimes A^{\otimes n-1} \otimes A, M) \arrow[u, "\widehat{d}_n"] & \mathcal{M}(A^{\otimes n-1}, M) \arrow[l, "\psi(A^{\otimes n-1},M)"] \arrow[u, "b^{n-1}"'] \end{tikzcd}$$

and we set

$$b^0 := \phi(A, M) \circ \widehat{d_1} \circ \widehat{\gamma} \circ \psi(\mathbf{1}, M),$$

where $\gamma : A \otimes A :\longrightarrow A \otimes \mathbf{1} \otimes A$ is the canonical isomorphism.
In this way, we obtain the so called *standard complex*:

$$0 \longrightarrow \mathcal{M}(\mathbf{1}, M) \xrightarrow{b^0} \mathcal{M}(A, M) \xrightarrow{b^1} \mathcal{M}(A \otimes A, M) \xrightarrow{b^2} \mathcal{M}(A \otimes A \otimes A, M) \xrightarrow{b^3} \cdots .$$

By definition

$$\mathrm{H}^n(A, M) = \frac{\mathrm{Ker}\,(b^n)}{\mathrm{Im}\,(b^{n-1})}$$

For every $f \in \mathcal{M}(\mathbf{1}, M)$, set

$$b^0_0\,(f) := \mu^r_M(f \otimes A)l^{-1}_A \qquad \text{and} \qquad b^0_1\,(f) := \mu^l_M(A \otimes f)r^{-1}_A$$

while, for every $n > 0$ and $f \in \mathcal{M}(A^{\otimes n}, M)$, set:

$$b^n_i(f) = \begin{cases} \mu^r_M \circ (f \otimes A), & i = 0; \\ f \circ (A^{\otimes n-i} \otimes m \otimes A^{\otimes i-1}), & i = 1, \ldots, n; \\ \mu^l_M \circ (A \otimes f)\,, & i = n+1. \end{cases}$$

It can be easily proved that

$$b^n_i = \phi(A^{\otimes n+1}, M) \circ \widehat{\partial^{n+1}_i} \circ \psi(A^{\otimes n}, M)$$

and

$$b^n(f) = \sum\nolimits_{i=0}^{n+1} (-1)^i b^n_i(f), \qquad \text{for every } n \geq 0.$$

In particular, for $n \in \{0, 1, 2\}$ the differentials b^n are given by:

$$\begin{aligned} b^0(f) &= \mu^r_M(f \otimes A)l^{-1}_A - \mu^l_M(A \otimes f)r^{-1}_A; \\ b^1(f) &= \mu^r_M(f \otimes A) - fm + \mu^l_M(A \otimes f); \\ b^2(f) &= \mu^r_M(f \otimes A) - f(A \otimes m) + f(m \otimes A) - \mu^l_M(A \otimes f). \end{aligned}$$

***Definition* 3.1.9.** The abelian group $\mathrm{Ker}\,(b^n)$ is also denoted by $\mathrm{Z}^n(A, M)$ and its elements are called *n-cocycles*. The abelian group $\mathrm{Im}\,(b^{n-1})$ is also denoted by $\mathrm{Z}^n(A, M)$ and its elements are called of *n-coboundaries*.
A 1-cocycle is also called a *derivation* of A with values in the A-bimodule M.

3.2 Separable algebras

***Remark* 3.2.1.** Let (A, m, u) be an algebra in a monoidal category $\mathcal{M}$. The multiplication m always has a section in ${}_A\mathcal{M}$ and in $\mathcal{M}_A$, namely $A \otimes u$ and respectively $u \otimes A$. In general, m has no section in ${}_A\mathcal{M}_A$. We are going to characterize those algebras whose multiplication has an A-bilinear section.

***Lemma* 3.2.2.** *Let (A, m, u) be a separable algebra in a monoidal category $(\mathcal{M}, \otimes, \mathbf{1})$. If $(M, \mu^l_M, \mu^r_M) \in {}_A\mathcal{M}_A$, then μ^l_M and μ^r_M split in ${}_A\mathcal{M}_A$.*

Proof. We will only prove that μ^r_M splits in ${}_A\mathcal{M}_A$, for μ^l_M we can proceed analogously.
Let $\sigma : A \to A \otimes A$ be a section of m in ${}_A\mathcal{M}_A$.

Let $\gamma_r : M \to M \otimes A$ be defined by

$$\gamma_r = (\mu^r_M \otimes A)(M \otimes \sigma)(M \otimes u) r^{-1}_M.$$

Using the fact that M is an A-bimodule and the naturality of the right unit constraint, it is easy to check that γ_r is a section of μ^r_M in $\mathcal{M}_A$. Let us prove that γ_r is also left A-linear. Since σ is right A-linear, we have $(A \otimes m)(\sigma \otimes A) = \sigma m$ so that

$$\begin{aligned}
(M \otimes m)(\gamma_r \otimes A) &= (M \otimes m)(\mu^r_M \otimes A \otimes A)(M \otimes \sigma \otimes A)(M \otimes u \otimes A)\left(r^{-1}_M \otimes A\right) \\
&= (\mu^r_M \otimes A)\left[M \otimes (A \otimes m)(\sigma \otimes A)(u \otimes A)\right]\left(M \otimes l^{-1}_A\right) \\
&= (\mu^r_M \otimes A)\left[M \otimes \sigma m (u \otimes A) l^{-1}_A\right] \\
&= (\mu^r_M \otimes A)(M \otimes \sigma).
\end{aligned}$$

On the other hand, since σ is left A-linear, we have $(m \otimes A)(A \otimes \sigma) = \sigma m$ so that

$$\begin{aligned}
\gamma_r \mu^r_M &= (\mu^r_M \otimes A)(M \otimes \sigma)(M \otimes u) r^{-1}_M \mu^r_M \\
&= (\mu^r_M \otimes A)(M \otimes \sigma)(M \otimes u)(\mu^r_M \otimes \mathbf{1}) r^{-1}_{M \otimes A} \\
&= \left[\mu^r_M (\mu^r_M \otimes A) \otimes A\right](M \otimes A \otimes \sigma)(M \otimes A \otimes u)\left(M \otimes r^{-1}_A\right) \\
&= \left[\mu^r_M (M \otimes m) \otimes A\right]\left[M \otimes (A \otimes \sigma)(A \otimes u) r^{-1}_A\right] \\
&= (\mu^r_M \otimes A)\left[M \otimes (m \otimes A)(A \otimes \sigma)(A \otimes u) r^{-1}_A\right] \\
&= (\mu^r_M \otimes A)\left[M \otimes \sigma m (A \otimes u) r^{-1}_A\right] \\
&= (\mu^r_M \otimes A)(M \otimes \sigma) = (M \otimes m)(\gamma_r \otimes A).
\end{aligned}$$

□

***Theorem* 3.2.3.** *[AMS3, Theorem 1.30] Let (A, m, u) be an algebra in an abelian monoidal category $(\mathcal{M}, \otimes, \mathbf{1})$. The following assertions are equivalent:*

(a) m splits in ${}_A\mathcal{M}_A$.

(b) A is $\mathcal{P}$-projective.
(c) $\mathrm{H}^1(A, M) = 0$, *for all* $M \in {}_A\mathcal{M}_A$.
(d) $\mathrm{H}^n(A, M) = 0$, *for all* $n > 0$ *and for all* $M \in_A\mathcal{M}_A$.
(e) Any morphism in ${}_A\mathcal{M}_A$ *splits in* ${}_A\mathcal{M}_A$ *whenever it splits in* $\mathcal{M}$.
(f) The category ${}_A\mathcal{M}_A$ *is* $\mathcal{P}$*-semisimple.*

Proof. (a) $\Rightarrow$ (f) Let $(M, \mu_M^l, \mu_M^r) \in {}_A\mathcal{M}_A$. By the previous lemma there are $s_l : M \to A \otimes M$ and $s_r : M \to M \otimes A$ sections in ${}_A\mathcal{M}_A$ of μ_M^l and μ_M^r, respectively. Then $(s_l \otimes A)s_r$ is a section of $\mu_M^r(\mu_M^l \otimes A)$ in ${}_A\mathcal{M}_A$. It follows that M is a direct summand of $(A \otimes M) \otimes A$, which proves that M is $\mathcal{P}$-projective.

The other implications follow as in the classical case. □

***Definition* 3.2.4.** Any algebra (A, m, u) in an abelian monoidal category $(\mathcal{M}, \otimes, \mathbf{1})$, satisfying one of the conditions of Theorem 3.2.3, is called *separable.*

***Corollary* 3.2.5.** *An algebra* (A, m, u) *in an abelian monoidal category* $(\mathcal{M}, \otimes, \mathbf{1})$ *is separable iff* $\mathrm{Hdim}(A) = 0$.

3.3 Hochschild extensions of algebras in a monoidal category

Our goal in this section is to classify Hochschild extensions of an algebra A (defined in an appropriate way) by using the second Hochschild cohomology group $\mathrm{H}^2(A, -)$. This classification will be used in the next section to investigate algebras of Hochschild dimension 1.

First some definitions and preliminary results.

***Definition* 3.3.1.** Let A and B be two algebras in a monoidal category $(\mathcal{M}, \otimes, \mathbf{1})$. A morphism $\sigma : B \to A$ in $\mathcal{M}$ is called *unital* if $\sigma u_B = u_A$, where u_A and u_B are the units of A and B, respectively. Moreover, if $f : A \to B$ is a morphism of algebras in $\mathcal{M}$ we shall say that σ is an *unital section* of f if $f\sigma = \mathrm{Id}_B$ and σ is an unital morphism.

Let (E, m_E, u_E) be an algebra in $\mathcal{M}$. If $i : X \longrightarrow E$ is a monomorphism in $\mathcal{M}$ then we will write $X^2 = 0$ in the case when $m_E(i \otimes i) = 0$.

***Lemma* 3.3.2.** *[AMS3, Lemma 2.3] Let* (A, m, u) *and* (E, m_E, u_E) *be algebras. Let* $\pi : E \to A$ *be a morphism of algebras in* $(\mathcal{M}, \otimes, \mathbf{1})$ *that has a section* $\sigma : A \to E$ *in* $\mathcal{M}$. *Let* $K = Ker(\pi)$ *and assume that* $K^2 = 0$.

a) We have:

$$m_E\,(\sigma u \otimes \sigma u)\, l_{\mathbf{1}}^{-1} = 2\sigma u - u_E. \tag{3.3}$$

b) The morphism $\sigma' := 2\sigma - m_E\,(\sigma\otimes\sigma)\,(A\otimes u)\,r_A^{-1}$ *is a unital section of* π.

c) Let $\mu_K^l : A\otimes K \to K$ *and* $\mu_K^r : K\otimes A \to K$ *be the maps uniquely defined by:*

$$i\mu_K^l = m_E(\sigma\otimes i), \tag{3.4}$$

$$i\mu_K^r = m_E(i\otimes\sigma), \tag{3.5}$$

where $i : K \to E$ *is the canonical inclusion. Then* $\left(K, \mu_K^l, \mu_K^r\right)$ *is an* A*-bimodule and* μ_K^l *and* μ_K^r *do not depend on the choice of the section* σ.

Proof. a) The relation $\pi\,(\sigma u - u_E) = 0$ tells us that there exists a unique morphism $\lambda : \mathbf{1} \to K$ so that

$$\sigma u - u_E = i\lambda. \tag{3.6}$$

On the other hand, $K^2 = 0$ so that $m_E(i\otimes i) = 0$. We get $m_E[(\sigma u - u_E)\otimes(\sigma u - u_E)]l_{\mathbf{1}}^{-1} = 0$. Hence:

$$m_E(\sigma u\otimes\sigma u)l_{\mathbf{1}}^{-1} - m_E(u_E\otimes\sigma u)l_{\mathbf{1}}^{-1} - m_E(\sigma u\otimes u_E)l_{\mathbf{1}}^{-1} + m_E(u_E\otimes u_E)l_{\mathbf{1}}^{-1} = 0.$$

For any morphism $f : \mathbf{1} \to E$, we have $m_E(f\otimes u_E)l_{\mathbf{1}}^{-1} = f$ and $m_E(u_E\otimes f)l_{\mathbf{1}}^{-1} = f$ (note that $r_{\mathbf{1}} = l_{\mathbf{1}}$). It results $m_E\,(\sigma u\otimes\sigma u)\,l_{\mathbf{1}}^{-1} - \sigma u - \sigma u + u_E = 0$, so relation (3.3) is proved.

b) Straightforward computation. We have $\pi\sigma' = \mathrm{Id}_A$ as $m_A\,(A\otimes u)\,r_A^{-1} = \mathrm{Id}_A$ and σ is a section of π. One can prove easily that σ' is unital by using the definition of σ', the fact that the right unit constraint is functorial, the equality $r_{\mathbf{1}} = l_{\mathbf{1}}$ and relation (3.3).

c) The relation $\pi[m_E\,(\sigma\otimes\sigma) - \sigma m] = m\,(\pi\sigma\otimes\pi\sigma) - \pi\sigma m = 0$ tells us that there exists a unique morphism $\omega : A\otimes A \to K$ such that

$$i\omega = m_E\,(\sigma\otimes\sigma) - \sigma m. \tag{3.7}$$

The relation $\pi m_E\,(\sigma\otimes E)\,(A\otimes i) = m_A\,(\pi\sigma\otimes\pi)\,(A\otimes i) = 0$ tells us that there exists a unique morphism $\mu_K^l : A\otimes K \to K$ such that $i\mu_K^l = m_E(\sigma\otimes i)$. Analogously one gets that there exists a morphism $\mu_K^r : K\otimes A \to K$, uniquely defined by (3.5). By definition of μ_K^l and using (3.7), we have

$$\begin{aligned} i\mu_K^l(A\otimes\mu_K^l) &= m_E[\sigma\otimes m_E(\sigma\otimes i)] \\ &= m_E[m_E(\sigma\otimes\sigma)\otimes i] = m_E((i\omega+\sigma m)\otimes i) = i\mu_K^l(m\otimes K). \end{aligned}$$

Moreover, by (3.6), we obtain

$$i\mu_K^l(u\otimes K) = m_E(\sigma u\otimes i) = m_E(i\lambda + u_E\otimes i) = m_E(u_E\otimes i)l_K^{-1}l_K = il_K$$

Analogously we get $i\mu_K^r(\mu_K^r \otimes A) = i\mu_K^r(K \otimes m)$ and $i\mu_K^r(K \otimes u) = ir_K$. Since i is a monomorphism we deduce that

$$\begin{array}{rl} \mu_K^l(u \otimes K) = l_K, & \qquad \mu_K^r(K \otimes u) = r_K, \\ \mu_K^l(A \otimes \mu_K^l) = \mu_K^l(m \otimes K), & \qquad \mu_K^r(\mu_K^r \otimes A) = \mu_K^r(K \otimes m). \end{array}$$

Furthermore,

$$\begin{array}{rcl} i\mu_K^l(A \otimes \mu_K^r) & = & m_E[\sigma \otimes m_E(i \otimes \sigma)] = m_E\,(E \otimes m_E)\,(\sigma \otimes i \otimes \sigma), \\ i\mu_K^r(\mu_K^l \otimes A) & = & m_E[m_E(\sigma \otimes i) \otimes \sigma] = m_E\,(m_E \otimes E)\,(\sigma \otimes i \otimes \sigma). \end{array}$$

so $i\mu_K^l(A \otimes \mu_K^r) = i\mu_K^r(\mu_K^l \otimes A)$. We conclude that (K, μ_K^l, μ_K) is an A-bimodule.

We now prove that μ_K^l does not depend on the choice of σ. Let $\tau : A \to E$ be another section of π in $\mathcal{M}$ and let $\gamma_l : A \otimes K \to K$ be the associated left module structure. As $\pi\,(\sigma - \tau) = 0$ there exists a unique morphism $\nu : A \to K$ such that $i\nu = \sigma - \tau$. Then

$$i(\mu_K^l - \gamma_l) = i\mu_K^l - i\gamma_l = m_E(\sigma \otimes i) - m_E(\tau \otimes i) = m_E[(\sigma - \tau) \otimes i] = m_E(i\nu \otimes i) = 0,$$

so $\mu_K^l = \gamma_l$, as i is a monomorphism. Analogously, μ_K^r does not depend on σ. $\square$

***Definitions* 3.3.3.** 1) Let (A, m, u) be an algebra in an abelian monoidal category $(\mathcal{M}, \otimes, \mathbf{1})$ and let (M, μ_M^l, μ_M^r) be an A-bimodule.
A *Hochschild extension* (E) of A with kernel M is an exact sequence in $\mathcal{M}$:

$$(E) \qquad\qquad 0 \longrightarrow M \xrightarrow{\ i\ } E \xrightarrow{\ \pi\ } A \longrightarrow 0$$

that satisfies the following conditions:

a) π is has a section σ in $\mathcal{M}$;
b) (E, m_E, u_E) is an algebra in $\mathcal{M}$ and π is an algebra homomorphism;
c) $M^2 = 0$, that is $m_E\,(i \otimes i) = 0$;
d) the morphisms μ_K^l and μ_M^r fulfill relations (3.4) and (3.5), i.e.

$$i\mu_M^l = m_E(\sigma \otimes i) \qquad \text{and} \qquad i\mu_M^r = m_E(i \otimes \sigma).$$

2) Two Hochschild extensions of A:

$$\begin{array}{c} 0 \longrightarrow M \xrightarrow{\ i\ } E \xrightarrow{\ \pi\ } A \longrightarrow 0 \\ 0 \longrightarrow M \xrightarrow{\ i'\ } E' \xrightarrow{\ \pi'\ } A \longrightarrow 0 \end{array}$$

with kernel M are *equivalent* if there is a morphism of algebras $f : E \to E'$ such that $\pi' f = \pi$ and $fi = i'$.

3) An extension $\pi : E \to A$ is a *trivial extension* whenever it admits a section that is an algebra homomorphism.

***Remarks* 3.3.4.** 1) Let (E) be a Hochschild extension of A with kernel M. Since $M^2 = 0$, by the previous lemma, one can define another bicomodule structure on M, by choosing an arbitrary section σ of π in $\mathcal{M}$. The third condition from the definition of Hochschild extensions means that this new structure and (M, μ_M^l, μ_M^r) coincide.

2) By the short 5-Lemma (see [McL1, Lemma 1, page 198]), f is always an isomorphism of algebras.

3) Let $\pi : E \to A$ be a morphism of algebras in $(\mathcal{M}, \otimes, \mathbf{1})$ that has a section $\sigma : A \to E$ in $\mathcal{M}$. Let $(\ker(\pi), i)$ be the kernel of π and assume that $(\ker(\pi))^2 = 0$. By Lemma 3.3.2,

$$0 \longrightarrow \ker(\pi) \xrightarrow{\ i\ } E \xrightarrow{\ \pi\ } A \longrightarrow 0$$

is a Hochschild extension.

***Lemma* 3.3.5.** *[AMS3, Lemma 2.6] Let (A, m, u) be an algebra in an abelian monoidal category $\mathcal{M}$ and let $(M, \mu_M^r, \mu_M^l) \in {}_A\mathcal{M}_A$. Suppose that $\omega : A \otimes A \to M$ is a morphism in $\mathcal{M}$. Define $m_\omega : (A \oplus M) \otimes (A \oplus M) \to A \oplus M$ and $u_\omega : \mathbf{1} \to A \oplus M$ by setting:*

$$\begin{aligned} m_\omega &= i_A m\,(p_A \otimes p_A) + i_M \left[\mu_M^r\,(p_M \otimes p_A) + \mu_M^l\,(p_A \otimes p_M) - \omega\,(p_A \otimes p_A)\right], \\ u_\omega &= i_A u + i_M \omega(u \otimes u) l_{\mathbf{1}}^{-1}, \end{aligned}$$

where i_A, i_M are the canonical injections in $A \oplus M$ and p_A, p_M are the canonical projections. Then m_ω is an associative multiplication if and only if ω is a Hochschild 2-cocycle. Moreover, in this case, $(A \oplus M, m_\omega, u_\omega)$ is an algebra and

$$0 \longrightarrow M \xrightarrow{\ i_M\ } A \oplus M \xrightarrow{\ p_A\ } A \longrightarrow 0$$

is a Hochschild extension of A with kernel (M, i_M). This extension will be denoted by (E_ω).

Proof. First we want to show that ω is a 2-cocycle if and only if m_ω is associative, i.e. we have

$$\mu_M^l\,(A \otimes \omega) - \omega(m \otimes A) + \omega\,(A \otimes m) - \mu_M^r(\omega \otimes A) = 0 \tag{3.8}$$

if and only if $m_\omega\,(E_\omega \otimes m_\omega) = m_\omega\,(m_\omega \otimes E_\omega)$. In fact the last relation holds true if and only if

$$p_A m_\omega\,(E_\omega \otimes m_\omega) = p_A m_\omega\,(m_\omega \otimes E_\omega) \quad \text{and} \quad p_M m_\omega\,(E_\omega \otimes m_\omega) = p_M m_\omega\,(m_\omega \otimes E_\omega)\,.$$

A straightforward, but tedious, computation shows us that:

$$p_A m_\omega \left(E_\omega \otimes m_\omega\right) = m(A \otimes m)\left(p_A \otimes p_A \otimes p_A\right),$$
$$p_A m_\omega \left(m_\omega \otimes E_\omega\right) = m(m \otimes A)\left(p_A \otimes p_A \otimes p_A\right),$$
$$p_M m_\omega \left(E_\omega \otimes m_\omega\right) = f - \mu_M^l(A \otimes \omega)\left(p_A \otimes p_A \otimes p_A\right) - \omega(A \otimes m)\left(p_A \otimes p_A \otimes p_A\right),$$
$$p_M m_\omega \left(m_\omega \otimes E_\omega\right) = g - \mu_M^r(\omega \otimes A)(p_A \otimes p_A \otimes p_A) - \omega(m \otimes A)(p_A \otimes p_A \otimes p_A),$$

where:

$$f = \left(\begin{array}{c} \mu_M^r(M \otimes m)\left(p_M \otimes p_A \otimes p_A\right) + \mu_M^l(A \otimes \mu_M^r)\left(p_A \otimes p_M \otimes p_A\right) + \\ +\mu_M^l(A \otimes \mu_M^l)\left(p_A \otimes p_A \otimes p_M\right) \end{array}\right),$$
$$g = \left(\begin{array}{c} \mu_M^r(\mu_M^r \otimes A)(p_M \otimes p_A \otimes p_A) + \mu_M^r(\mu_M^l \otimes A)\left(p_A \otimes p_M \otimes p_A\right) + \\ \mu_M^l(m \otimes M)\left(p_A \otimes p_A \otimes p_M\right) \end{array}\right).$$

As M is an A-bimodule we get $f = g$.
Therefore $p_M m_\omega \left(E_\omega \otimes m_\omega\right) = p_M m_\omega \left(m_\omega \otimes E_\omega\right)$ if and only if

$$(3.9)\quad [\mu_M^l(A\otimes\omega)+\omega(m\otimes A)]\left(p_A \otimes p_A \otimes p_A\right) = [\mu_M^r(\omega\otimes A)+\omega(m\otimes A)](p_A\otimes p_A\otimes p_A).$$

Furthermore, this relation holds if and only if ω is a 2-cocycle (the direct implication follows by composing (3.9) with $i_A\otimes i_A\otimes i_A$ to the right, and the converse is obvious). In conclusion, the multiplication on E_ω is associative if and only if ω is a 2-cocycle.

For proving that u_ω is the unit of E_ω we proceed similarly. We need the following equalities:

$$(3.10)\qquad p_A m_\omega(E_\omega \otimes u_\omega) = p_A r_{E_\omega} \quad \text{and} \quad p_M m_\omega\left(E_\omega \otimes u_\omega\right) = p_M r_{E_\omega},$$
$$(3.11)\qquad p_A m_\omega\left(u_\omega \otimes E_\omega\right) = p_A l_{E_\omega} \quad \text{and} \quad p_M m_\omega\left(u_\omega \otimes E_\omega\right) = p_M l_{E_\omega}.$$

We will prove only (3.10), the proof of (3.11) being left to the reader. First we notice that we have $p_A r_{E_\omega} = r_A(p_A \otimes \mathbf{1})$ and $p_M r_{E_\omega} = r_M(p_M \otimes \mathbf{1})$, as the unit constraint r is a natural morphism. Furthermore, by the definition of m_ω and u_ω, we get:

$$p_A m_\omega(E_\omega \otimes u_\omega) = m(p_A \otimes p_A u_\omega) = m(p_A \otimes u) = m(A \otimes u)(p_A \otimes \mathbf{1}) = r_A(p_A \otimes \mathbf{1}),$$

so we have the first equality of (3.10). We still have to prove the second relation of (3.10). We have:

$$\begin{array}{rcl} p_M m_\omega(E_\omega \otimes u_\omega) & = & \mu_M^r(p_M \otimes p_A u_\omega) + \mu_M^l(p_A \otimes p_M u_\omega) - \omega(p_A \otimes p_A u_\omega) \\ & = & \mu_M^r(p_M \otimes u) + \mu_M^l[p_A \otimes \omega(u \otimes u) l_{\mathbf{1}}^{-1}] - \omega(p_A \otimes u). \end{array}$$

Obviously, $\mu_M^r(p_M \otimes u) = r_M(p_M \otimes \mathbf{1})$.
Thus, to conclude it is enough to show that $\mu_M^l[p_A \otimes \omega(u \otimes u) l_{\mathbf{1}}^{-1}] = \omega(p_A \otimes u)$.

Indeed,

$$\begin{aligned}\omega(m \otimes A)\,(A \otimes u \otimes u) &= \omega(m\,(A \otimes u) \otimes u) = \omega(r_A \otimes u),\\ \omega\,(A \otimes m)\,(A \otimes u \otimes u) &= \omega\,(A \otimes m\,(u \otimes u))\\ &= \omega[A \otimes m\,(u \otimes A)\,(\mathbf{1} \otimes u)]\\ &= \omega(A \otimes l_A)(A \otimes \mathbf{1} \otimes u).\end{aligned}$$

On the other hand, by the triangle axiom we have $A \otimes l_A = r_A \otimes A$, so that:

$$\omega\,(A \otimes m)\,(A \otimes u \otimes u) = \omega(r_A \otimes A)\,(A \otimes \mathbf{1} \otimes u) = \omega(r_A \otimes u).$$

We deduce

$$\omega(m \otimes A)\,(A \otimes u \otimes u) = \omega\,(A \otimes m)\,(A \otimes u \otimes u)\,.$$

Therefore, if we compose (3.8) with $A \otimes u \otimes u$ to the right, we obtain:

$$\mu^l_M[A \otimes \omega\,(u \otimes u)] = \mu^r_M[\omega\,(A \otimes u) \otimes u] = \omega\,(A \otimes u)\,r_{A\otimes\mathbf{1}}.$$

Hence:

$$\mu^l_M[p_A \otimes \omega\,(u \otimes u)\,l_{\mathbf{1}}^{-1}] = \mu^l_M[A \otimes \omega\,(u \otimes u)](p_A \otimes l_{\mathbf{1}}^{-1}) = \omega\,(A \otimes u)\,r_{A\otimes\mathbf{1}}(p_A \otimes l_{\mathbf{1}}^{-1}).$$

Finally,

$$\omega\,(A \otimes u)\,r_{A\otimes\mathbf{1}}(p_A \otimes l_{\mathbf{1}}^{-1}) = \omega\,(A \otimes u)\,r_{A\otimes\mathbf{1}}(A \otimes l_{\mathbf{1}}^{-1})(p_A \otimes \mathbf{1}) = \omega\,(p_A \otimes u)$$

as $r_{A\otimes\mathbf{1}} = A \otimes l_{\mathbf{1}}$, by the triangle axiom. □

***Definitions* 3.3.6.** a) The Hochschild extension $p_A : E_\omega \to A$, introduced in the lemma above, is called the *Hochschild extension associated to* ω.

b) If (A, m_A, u_A) and (E, m_E, u_E) are algebras and $\sigma : A \to E$ is a morphism in $\mathcal{M}$, we define the *curvature* of σ to be the morphism:

$$\theta_\sigma : A \otimes A \to E, \quad \theta_\sigma := \sigma m_A - m_E(\sigma \otimes \sigma) \tag{3.12}$$

***Proposition* 3.3.7.** *[AMS3, Proposition 2.8] Let $\pi : E \to A$ be a Hochschild extension of A with kernel (M, i), let $\sigma : A \to E$ be a section of π and let θ_σ be the curvature of σ. Then there is a unique morphism $\omega : A \otimes A \to M$, such that $i\omega = \theta_\sigma$. Moreover, ω is a 2-cocycle whose class $[\omega] \in \mathrm{H}^2(A, M)$ does not depend on the choice of σ. If $p_A : E_\omega \to A$ is the Hochschild extension associated to ω, the morphism*

$$f_\omega := \sigma p_A + ip_M : E_\omega \to E$$

defines an equivalence of Hochschild extensions.

Proof. The morphism π is an algebra homomorphism, and hence $\pi\theta_\sigma = 0$. Thus there exists a unique morphism $\omega : A \otimes A \to M$ such that $i\omega = \theta_\sigma$. Let μ_M^l and μ_M^r be the morphisms that define the module structure of M and let m_A and m_E be the multiplications of A and E respectively. By formulas (3.4), (3.5) and the construction of ω we have:

$$ib^2(\omega) = m_E(\theta_\sigma \otimes \sigma) - \theta_\sigma(A \otimes m_A) + \theta_\sigma(m_A \otimes A) - m_E(\sigma \otimes \theta_\sigma).$$

Thus, by the definition of curvature θ_σ we get $ib^2(\omega) = 0$. Since i is a monomorphism, we obtain $b^2(\omega) = 0$, that is ω is a cocycle.

Let $\sigma' : A \to E$ be another section of π. Since $\pi(\sigma - \sigma') = 0$, there exists a unique morphism $\tau : A \to M$ such that $i\tau = \sigma - \sigma'$. Let ω' be the 2-cocycle associated to σ'. Since μ_M^l and μ_M^r are independent of the choice of the section, the relation (3.4) holds true if we replace σ by σ'. Hence by definition of b^1, equation $i\omega' = \theta_{\sigma'}$ and construction of τ we get:

$$i\left(\omega' - b^1(\tau)\right) = \left\{ \begin{array}{c} \sigma' m_A - m_E(\sigma' \otimes \sigma') - m_E[\sigma' \otimes (\sigma - \sigma')] + \\ +\sigma m_A - \sigma' m_A - m_E[(\sigma - \sigma') \otimes \sigma] \end{array} \right\}.$$

We deduce $i(\omega' - b^1(\tau)) = \sigma m_A - m_E(\sigma \otimes \sigma) = \theta_\sigma = i\omega$, so that $\omega' = b^1(\tau) + \omega$. Thus $[\omega] = [\omega']$.

It remains to show that f_ω is an equivalence of extensions. First, f_ω is a morphism of algebras. Indeed, we have:

$$\begin{aligned} f_\omega m_\omega &= [\sigma p_A + ip_M]\, m_\omega \\ &= \sigma m_A(p_A \otimes p_A) + i[\mu_M^r(p_M \otimes p_A) + \mu_M^l(p_A \otimes p_M) - \omega(p_A \otimes p_A)] \\ &= m_E \left[\begin{array}{c} (i \otimes \sigma)(p_M \otimes p_A) + (\sigma \otimes i)(p_A \otimes p_M) + \\ + (\sigma \otimes \sigma)(p_A \otimes p_A) + (i \otimes i)(p_M \otimes p_M) \end{array} \right] \\ &= m_E(f_\omega \otimes f_\omega), \end{aligned}$$

as $m_E(i \otimes i) = 0$. Similarly, we have:

$$\begin{aligned} f_\omega u_\omega &= [\sigma p_A + ip_M]\, u_\omega \\ &= \sigma u + i\omega(u \otimes u)\, l_{\mathbf{1}}^{-1} = \sigma u + \sigma m_A(u \otimes u)\, l_{\mathbf{1}}^{-1} - m_E(\sigma u \otimes \sigma u)\, l_{\mathbf{1}}^{-1} \\ &= 2\sigma u - m_E(\sigma u \otimes \sigma u)\, l_{\mathbf{1}}^{-1} = u_E, \end{aligned}$$

where for the last equality we used (3.3). Finally, one can check easily that $\pi f_\omega = p_A$ and $f_\omega i_M = i$, so f_ω is an equivalence of Hochschild extensions. $\square$

***Definitions* 3.3.8.** With the notations of the previous Proposition, ω is called *the* 2-*cocycle associated to* σ, while the class $[\omega]$ is called the *cohomology class associated to the Hochschild extension* $\pi : E \to A$.

***Lemma* 3.3.9.** *Let $\omega : A \otimes A \to M$ be a 2-cocycle and let $p_A : E_\omega \to A$ be the Hochschild extension associated to ω. Then the cohomology class associated to the Hochschild extension $p_A : E_\omega \to A$ is exactly $[\omega]$.*

Proof. Since $i_A : A \to E_\omega$ is a section of p_A, we have:

$$\theta_{i_A} = i_A m_A - m_\omega (i_A \otimes i_A) = i_A m_A - i_A m_A + i_M \omega = i_M \omega.$$

Thus, in view of Proposition 3.3.7, the cohomology class associated to this extension is $[\omega]$. □

3.3.10. Let A be an algebra and let M be an A-bimodule. If $\pi : E \to A$ is a Hochschild extension, we will denote by $[E]$ the class of all Hochschild extensions equivalent to it. We define:

$$\operatorname{Ext}(A, M) := \{[E] \mid E \to A \text{ is a Hochschild extension of } A \text{ with kernel } M\}.$$

***Proposition* 3.3.11.** *[AMS3, Proposition 2.12] Let A be an algebra and let M be an A-bimodule. If $\omega, \omega' : A \otimes A \to M$ are 2-cocycles, then:*

$$[\omega] = [\omega'] \Longleftrightarrow [E_\omega] = [E_{\omega'}].$$

Moreover, if $[\omega] = 0$, i.e. there exists a morphism $\tau : A \to M$ such that $\omega = b^1(\tau)$, then the morphism $\sigma := i_A + i_M \tau : A \to E_\omega$ is an algebra homomorphism which is a section of $p_A : E_\omega \to A$.

Proof. Suppose that $[E_\omega] = [E_{\omega'}]$. Therefore, there exists an algebra homomorphism $g : E_\omega \to E_{\omega'}$ which is an equivalence of Hochschild extensions, that is $p_A g = p_A$ and $g i_M = i_M$.
As $g i_A$ is a section of $p_A : E_{\omega'} \to A$, we have:

$$\theta'_{g i_A} = g i_A m_A - m_{\omega'} (g i_A \otimes g i_A) = g \left[i_A m_A - m_\omega (i_A \otimes i_A) \right] = g i_M \omega = i_M \omega,$$

so that, by definition, the cohomology class associated to $(E_{\omega'})$ is $[\omega]$. On the other hand, by Lemma 3.3.9, the cohomology class of $(E_{\omega'})$ is $[\omega']$. Thus $[\omega] = [\omega']$.

If $[\omega] = [\omega']$, there exists a morphism $\tau : A \to M$ such that $\omega = \omega' + b^1(\tau)$. The morphism $\sigma := i_A + i_M \tau : A \to E_\omega$ is a section of $p_A : E_\omega \to A$. Let μ_M^l and μ_M^r be the morphisms that define the module structure of M. Thus:

$$\begin{aligned}\theta_\sigma &= \sigma m_A - m_\omega (\sigma \otimes \sigma) = i_A m_A + i_M \tau m_A - i_A m_A - i_M \left[\mu_M^r (\tau \otimes A) + \mu_M^l (A \otimes \tau) - \omega \right] \\ &= i_M \left[\tau m_A - \mu_M^r (\tau \otimes A) - \mu_M^l (A \otimes \tau) + \omega \right] = i_M(-b^1(\tau) + \omega) = i_M \omega'.\end{aligned}$$

Applying Proposition 3.3.7 to the Hochschild extension $p_A : E_\omega \to A$, we get that there is an equivalence between $(E_{\omega'})$ and (E_ω), namely $f_{\omega'} = \sigma p_A + i_M p_M : E_{\omega'} \to E_\omega$. Therefore $[E_\omega]$ and $[E_{\omega'}]$ are equal.

If $\omega' = 0$, then $i_A : A \to E_{\omega'}$ is clearly an algebra homomorphism which is a section of the projection $p_A : E_{\omega'} \to A$. Now, $\sigma := i_A + i_M\tau$ is a section of $p_A : E_\omega \to A$ so that the morphism $f_{\omega'} = \sigma p_A + ip_M : E_{\omega'} \to E_\omega$ is an algebra homomorphism. Since $f_{\omega'} i_A = \sigma$, we conclude. □

***Theorem* 3.3.12.** *[AMS3, Theorem 2.13] Let A be an algebra and let M be an A-bimodule. The map:*

$$\Phi : \mathrm{H}^2(A, M) \to \mathrm{Ext}\,(A, M)\,,$$

where $\Phi\,([\omega]) := [E_\omega]\,,$ is well−defined and is a bijection.

Proof. Φ is well−defined and bijective by Proposition 3.3.11 and Proposition 3.3.7. □

***Lemma* 3.3.13.** *Let A and B be algebras in a monoidal category $(\mathcal{M}, \otimes, \mathbf{1})$ and let $f : A \to B$ be an algebra homomorphism. Let M be an $B-$bimodule. Let $\omega_B \in \mathcal{M}(B \otimes B, M)$ and $\omega_A := \omega_B(f \otimes f) \in \mathcal{M}(A \otimes A, M)$. Then, if we regard M as an A-bimodule via f, then $b^2(\omega_A) = b^2(\omega_B)(f \otimes f \otimes f)$. In particular ω_A is a 2-cocycle whenever ω_B is.*

Proof. Let μ_M^r: $M \otimes B \to M$ and μ_M^l: $B \otimes M \to M$ be the morphisms defining the module structure of M. Then the left A-module structure on M is given by $\widetilde{\mu_M^l} := \mu_M^l(f \otimes M)$. The map $\widetilde{\mu_M^r}$, giving the right A-module structure of M, is defined similarly. Hence the relation

$$b^2(\omega_B)(f \otimes f \otimes f) = b^2(\omega_A).$$

follows by the definitions of b^2, ω_A, $\widetilde{\mu_M^l}$, $\widetilde{\mu_M^r}$ and the fact that f is an algebra homomorphism. □

***Proposition* 3.3.14.** *[AMS3, Proposition 2.15] Let A and B be algebras in a monoidal category $(\mathcal{M}, \otimes, \mathbf{1})$ and let $f : A \to B$ be an algebra homomorphism. Let $\pi : E \to B$ be a Hochschild extension of B with kernel (M, i), let $\sigma : B \to E$ be a section of π and let $\omega_B : B \otimes B \to M$ the associated 2-cocycle. Let $\omega_A := \omega_B(f \otimes f)$. If $p_A : E_{\omega_A} \to A$ is the Hochschild extension associated to ω_A, then the morphism*

$$\pi_f := \sigma f p_A + ip_M : E_{\omega_A} \to E$$

defines an algebra homomorphism such that the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "i_M"] \arrow[d, equal] & E_{\omega_A} \arrow[r, "p_A"] \arrow[d, dotted, "\pi_f"'] & A \arrow[r] \arrow[d, "f"'] & 0 \\
0 \arrow[r] & M \arrow[r, "i"] & E \arrow[r, "\pi"] & B \arrow[r] & 0
\end{tikzcd}$$

Proof. Denote the canonical injections (respectively projections) in E_{ω_A} by $i_A,\ i_M$ (respectively $p_A,\ p_M$). Let j_B, j_M (respectively q_B, q_M) be the canonical injections (respectively projections) in E_{ω_B}. By Proposition 3.3.7, the morphism

$$f_{\omega_B} := \sigma q_B + i q_M : E_{\omega_B} \to E$$

defines an equivalence of Hochschild extensions. It is clear that the following diagram

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "i_M"] \arrow[d, equal] & E_{\omega_A} \arrow[r, "p_A"] \arrow[d, dotted, "\xi"'] & A \arrow[r] \arrow[d, "f"'] & 0 \\ 0 \arrow[r] & M \arrow[r, "j_M"] & E_{\omega_B} \arrow[r, "q_B"] & B \arrow[r] & 0 \end{tikzcd}$$

commutes, where $\xi := j_B f p_A + j_M p_M : E_{\omega_A} \to E_{\omega_B}$. If ξ is an algebra homomorphism, then $f_{\omega_B}\xi = (\sigma q_B + i q_M)\xi = \sigma f p_A + i p_M$ is an algebra homomorphism, and hence π_f satisfies the required properties. Thus, let us check that ξ is an algebra homomorphism. Let $\mu_M^r : M \otimes B \to M$ and $\mu_M^l : B \otimes M \to M$ be the morphisms defining the B-module structures of M. If $\zeta = m_{\omega_B}(\xi \otimes \xi)$ we have:

$$\begin{aligned} \zeta &= m_{\omega_B}[(j_B f p_A \otimes j_B f p_A) + (j_B f p_A \otimes j_M p_M) + (j_M p_M \otimes j_B f p_A) + (j_M p_M \otimes j_M p_M)] \\ &= j_B m_B (f p_A \otimes f p_A) - j_M \omega_B (f p_A \otimes f p_A) + j_M \mu_M^l (f p_A \otimes p_M) + j_M \mu_M^r (p_M \otimes f p_A) \\ &= j_B f m_A (p_A \otimes p_A) - j_M \omega_A (p_A \otimes p_A) + j_M \widetilde{\mu_M^l} (p_A \otimes p_M) + j_M \widetilde{\mu_M^r} (p_M \otimes p_A) = \xi m_{\omega_A}. \end{aligned}$$

Moreover, $\xi u_{\omega_A} = j_B f u_A + j_M \omega_A (u_A \otimes u_A)\, l_{\mathbf{1}}^{-1} = j_B u_B + j_M \omega_B (f u_A \otimes f u_A)\, l_{\mathbf{1}}^{-1}$ $= u_{\omega_B}$. □

***Corollary* 3.3.15.** *Let $\pi : E \to B$ be a Hochschild extension of B with kernel (M, i), let $\sigma : B \to E$ be a section of π. Let $\omega_B : B \otimes B \to M$ be the 2-cocycle associated to σ, let $f : A \to B$ be an algebra homomorphism and let $\omega_A = \omega_B(f \otimes f)$. If there exists a morphism $\tau : A \to M$ such that $\omega_A = b^1(\tau)$, i.e. $[\omega_A] = 0$, then the morphism*

$$\overline{f} = \sigma f + i\tau : A \to E$$

defines a morphism of algebras such that $\pi \overline{f} = f$.

Proof. Since $[\omega_A] = 0$, by Proposition 3.3.11, the morphism $\sigma' := i_A + i_M \tau : A \to E_{\omega_A}$ is an algebra homomorphism which is a section of $p_A : E_{\omega_A} \to A$. By Proposition 3.3.14, then the morphism

$$\pi_f := \sigma f p_A + i p_M : E_{\omega_A} \to E$$

defines an algebra homomorphism such that $\pi\pi_f = f p_A$. Then the morphism $\overline{f} := \pi_f \sigma'$ is an algebra map such that $\pi \overline{f} = \pi \pi_f \sigma' = f p_A \sigma' = f$. □

3.4 Formally smooth algebras

The starting point is the basic observation included in the following Lemma.

***Lemma* 3.4.1.** *Let $\mathcal{M}$ be an abelian monoidal category. Let (A, m, u) be an algebra in $\mathcal{M}$, let (I, i_I) be an ideal in A and let $n \in \mathbb{N}^*$. With the notations of section 1.4 and in particular of 1.4.12, if the canonical morphism $p_I^n : A/I^{n+1} \to A/I^n$ splits in $\mathcal{M}$ then the sequence*

$$0 \to \frac{I^n}{I^{n+1}} \xrightarrow{j_I^n} \frac{A}{I^{n+1}} \xrightarrow{p_I^n} \frac{A}{I^n} \to 0,$$

defines a Hochschild extension of $\frac{A}{I^n}$ with kernel $\frac{I^n}{I^{n+1}}$.

Proof. follows in view of 1.4.12, Lemma 1.4.13 and 3) of Remarks 3.3.4. □

***Definition* 3.4.2.** Let $\mathcal{M}$ be an abelian monoidal category.
Let $(E_n, p_n)_{n\in\mathbb{N}^*}$ be a sequence of morphisms in $\mathcal{M}$

$$\cdots \xrightarrow{p_{n+1}} E_{n+1} \xrightarrow{p_n} E_n \xrightarrow{p_{n-1}} \cdots \xrightarrow{p_2} E_2 \xrightarrow{p_1} E_1. \tag{3.13}$$

We say that $(E_n, p_n)_{n\in\mathbb{N}^*}$ is an *inverse system of extensions* if

- p_n is an algebra homomorphism,
- $(\mathrm{Ker}(p_n))^2 = 0$, for any $n \geq 1$.

We say that $(E_n, p_n)_{n\in\mathbb{N}^*}$ is an *inverse system of Hochschild extensions* if

- $(E_n, p_n)_{n\in\mathbb{N}^*}$ is an inverse system of extensions,
- p_n has a section in $\mathcal{M}$, for any $n \geq 1$.

We say that an inverse system of Hochschild extensions $(E_n, p_n)_{n\in\mathbb{N}^*}$ *has an inverse limit* if $\varprojlim E_n$ exists in the category $\mathfrak{Alg}(\mathcal{M})$ of algebras in $\mathcal{M}$.

***Remark* 3.4.3.** 1) Let $(E_n, p_n)_{n\in\mathbb{N}^*}$ be an inverse system of Hochschild extensions. In view of Remarks 3.3.4 each exact sequence

$$0 \longrightarrow \mathrm{Ker}(p_n) \xrightarrow{i} E_{n+1} \xrightarrow{p_n} E_n \longrightarrow 0$$

defines a Hochschild extension of E_n with kernel $\mathrm{Ker}(p_n)$. This justifies the above terminology.
2) We point out that, if $\mathcal{M}$ is an abelian monoidal category and the inverse limit $\varprojlim E_n$ exists in $\mathcal{M}$, then it can be endowed with a natural algebra structure in such a way that it is the inverse limit in the category $\mathfrak{Alg}(\mathcal{M})$ of algebras in $\mathcal{M}$. Therefore, in this case, $(E_n, p_n)_{n\in\mathbb{N}^*}$ has an inverse limit.

***Example* 3.4.4.** Let I be an ideal of an algebra E in an abelian monoidal category and let
$$p_I^n : \frac{E}{I^{n+1}} \to \frac{E}{I^n}$$
the morphism defined in 1.4.13. In view of Lemma 3.4.1, we have that
$$(\frac{E}{I^n}, p_I^n)_{n\in\mathbb{N}^*},$$
defines an inverse system of extensions which is called the *$I-$adic inverse system.* This is not an inverse system of Hochschild extensions unless each p_I^n has a section in $\mathcal{M}$.

***Definition* 3.4.5.** Let A, E, B be algebras and let $\pi : E \to B$ be an algebra homomorphism in a monoidal category $\mathcal{M}$. We say that A *has the lifting property with respect to* π whenever the canonical map
$$\mathrm{Hom}_{\mathrm{alg}}(A,\pi) : \mathrm{Hom}_{\mathrm{alg}}(A,E) \to \mathrm{Hom}_{\mathrm{alg}}(A,B) : f \mapsto \pi \circ f$$
is surjective. This means that every algebra homomorphism $g : A \to B$ can be lifted to an algebra homomorphism $f : A \to E$ that makes the following diagram
$$\begin{tikzcd} & A \arrow[dl, dotted, "f"'] \arrow[d, "g"] \\ E \arrow[r, "\pi"'] & B \end{tikzcd}$$
commutative.

***Theorem* 3.4.6.** *[AMS3, Theorem 3.8] Let $(\mathcal{M}, \otimes, \mathbf{1})$ be an abelian monoidal category. Let (A, m, u) be an algebra in $\mathcal{M}$. Then the following conditions are equivalent:*

(a) A has the lifting property with repect to every algebra homomorphism $\pi : E \to B$ that splits in $\mathcal{M}$ and such that $(Ker(\pi))^2 = 0$.

(b) A has the lifting property with repect to the canonical morphism
$$\varprojlim E_n \to E_1$$
for every inverse system of Hochschild extensions $(E_n, p_n)_{n\in\mathbb{N}^*}$ which has inverse limit $\varprojlim E_n$.

(c) A has the lifting property with repect to the canonical morphism
$$\varprojlim \frac{E}{I^n} \to \frac{E}{I}$$
for any algebra E in $\mathcal{M}$ and any ideal I of E such that $(E/I^n, p_I^n)_{n\in\mathbb{N}^*}$ is an inverse system of Hochschild extensions which has inverse limit $\varprojlim E/I^n$.

(d) Any Hochschild extension of A is trivial.

(e) $\mathrm{H}^2(A, M) = 0$, for every $M \in {}_A\mathcal{M}_A$.

Proof. (a) $\Rightarrow$ (b) Let $(E_n, p_n)_{n\in\mathbb{N}^*}$ be an inverse system of Hochschild extensions that has an inverse limit $\varprojlim E_n$. Let $f : A \to E_1$ be an algebra homomorphism. Since $p_n : E_{n+1} \to E_n$ is by hypothesis a Hochschild extension, for every $n \geq 1$, we can construct inductively a morphism of algebras $f_n : A \to E_n$ such that $f := f_1$ and $f_n = p_n f_{n+1}$. We deduce that there is an algebra homomorphism $g : A \to \varprojlim E_n$ such that $q_1 g = f$, where $q_n : \varprojlim E_n \to E_n$ are the canonical morphisms coming from the definition of the inverse limit in a category.

(b) $\Rightarrow$ (c) Obvious.

(c) $\Rightarrow$ (d) Let $M \in {}_A\mathcal{M}_A$ and let

$$0 \longrightarrow M \xrightarrow{i} E \xrightarrow{\pi} A \longrightarrow 0$$

be a Hochschild extension of A with kernel M. Let $p_M : E \to E/M$ be the canonical projection and let $f : E/M \to A$ be the canonical isomorphism such that $fp_M = \pi$. Now $p_M : E \to E/M$ is a Hochschild extension of E/M with kernel M. Since $M^2 = 0$, it is clear that $(E, p_M) = \varprojlim E/M^n$ so that there exists $g \in \mathrm{Hom}_{\mathrm{alg}}(A, E)$ such that $p_M g = f^{-1}$. Thus

$$\pi g = f p_M g = f f^{-1} = \mathrm{Id}_A.$$

(d) $\Rightarrow$ (e) Let $M \in {}_A\mathcal{M}_A$ and let

$$0 \longrightarrow M \xrightarrow{i} E \xrightarrow{\pi} A \longrightarrow 0$$

be a Hochschild extension of A with kernel M. By the definition of trivial extension, there exists a section $\sigma : A \to E$ of π that is an algebra homomorphism. If ω is the cocycle associated to σ then

$$i\omega = \theta_\sigma = \sigma m_A - m_E\,(\sigma \otimes \sigma) = 0.$$

Thus $\omega = 0$ so, by Proposition 3.3.7, we have $[E] = [E_0]$. Therefore, $\mathrm{Ext}\,(A, M) = \{[E_0]\}$ and hence, by Theorem 3.3.12, we get $\mathrm{H}^2\,(A, M) = 0, \forall M \in {}_A\mathcal{M}_A$.

(e) $\Rightarrow$ (a) Since $(\mathrm{Ker}(\pi))^2 = 0$ it results that $\pi : E \to B$ is a Hochschild extension of B with kernel $\mathrm{Ker}(\pi)$. The conclusion follows by Corollary 3.3.15. $\square$

***Definition* 3.4.7.** Any algebra (A, m, u) in an abelian monoidal category $(\mathcal{M}, \otimes, \mathbf{1})$, satisfying one of the conditions of Theorem 3.4.6, is called *formally smooth.*

***Corollary* 3.4.8.** *Any separable algebra in $\mathcal{M}$ is formally smooth.*

***Corollary* 3.4.9.** *Let (A, m, u) be an algebra in an abelian monoidal category $(\mathcal{M}, \otimes, \mathbf{1})$. Then the following assertions are equivalent:*

(a) A is formally smooth.

(b) Ker(m) is $\mathcal{P}$-projective.

Proof. Let $(L, j) := \mathrm{Ker}(m)$ and let us consider the exact sequence:

$$0 \longrightarrow L \xrightarrow{j} A \otimes A \xrightarrow{m} A \longrightarrow 0.$$

We know that m has a section in $\mathcal{M}$ so that $m \in \mathcal{P}$. Given any $M \in {}_A\mathcal{M}_A$, we apply the functor $F := {}_A\mathcal{M}_A(-, M)$ to the sequence above and find:

$$\mathrm{Ext}^1_{\mathcal{P}}(A \otimes A, M) \longrightarrow \mathrm{Ext}^1_{\mathcal{P}}(L, M) \longrightarrow \mathrm{Ext}^2_{\mathcal{P}}(A, M) \longrightarrow \mathrm{Ext}^2_{\mathcal{P}}(A \otimes A, M).$$

Since $A \otimes A$ is $\mathcal{P}$-projective, we get that $\mathrm{Ext}^1_{\mathcal{P}}(L, M) \simeq \mathrm{Ext}^2_{\mathcal{P}}(A, M) = \mathrm{H}^2(A, M)$. We conclude by applying Proposition 3.1.6 and Theorem 3.4.6. □

***Theorem* 3.4.10.** *[AMS3, Theorem 3.13] Let (A, m, u) be an algebra in an abelian monoidal category $(\mathcal{M}, \otimes, \mathbf{1})$. Then the following conditions are equivalent:*

(a) A is formally smooth.

(b) The canonical morphism $\varprojlim E/I^n \to A$ has an algebra homomorphism section, where I is an ideal in an algebra E such that $E/I \simeq A$ (as algebras) and $(E/I^n, p^n_I)_{n\in\mathbb{N}^*}$ is an inverse system of Hochschild extensions which has inverse limit $\varprojlim E/I^n$.

(c) Let $\pi : E \to A$ be an epimorphism in $\mathcal{M}$. If π is an algebra homomorphism, the kernel I of π is nilpotent and $(E/I^n, p^n_I)_{n\in\mathbb{N}^*}$ is an inverse system of Hochschild extensions, then π has an algebra homomorphism section.

Proof. (a) $\Rightarrow$ (b) Let $f : E/I \to A$ be an isomorphism of algebras. Let $q_n : \varprojlim E/I^n \to E/I^n$ be the canonical map, coming from the definition of inverse limit. Then the morphism $\varprojlim E/I^n \to A$ is the composition of f and q_1. Since A is formally smooth, condition (c) of Theorem 3.4.6 holds true. Hence, there is an algebra homomorphism $g : A \to \varprojlim E/I^n$ such that $q_1 g = f^{-1}$. Thus g is a section of $\varprojlim E/I^n \to A$.

(b) $\Rightarrow$ (c) Let $i_I : I \to E$ be the inclusion, let $p_I : E \to E/I$ be the projection and let $N \geq 2$ be a natural number such that $I^N = 0$. Then $p^n_I : E/I^{n+1} \to E/I^n$ is the identity morphism of E, for $n \geq N$. On the other hand, since (A, π) is the cokernel of i_I, there is an isomorphism of algebras $f : E/I \to A$ such that $\pi = f p_I$. As $(E/I^n, p^n_I)_{n\in\mathbb{N}^*}$ is an inverse inverse system of Hochschild extensions then, by assumption, $\varprojlim E/I^n \to A$ has an algebra homomorphism section. Obviously E, together the canonical morphisms $p_{I^n} : E \to E/I^n$, is the inverse limit of the $I-$adic inverse system. Thus, in this case, $q_1 = p_I$ so the canonical map $\varprojlim E/I^n \to A$ is $f p_I = \pi$. Thus π splits.

(c) $\Rightarrow$ (a) Let $\pi : E \to A$ be a Hochschild extension. Since $(\mathrm{Ker}(\pi))^2 = 0$ then $I = \mathrm{Ker}(\pi)$ is nilpotent and $p^n_I : E/I^{n+1} \to E/I^n$ is the identity morphism of E, for $n \geq 2$. In particular, $p^1_I = p_I$. Let $f : E/I \to A$ be the algebra isomorphism such that $f p_I = \pi$. We deduce that p^1_I splits as, by definition π, does. Obviously, for any $n \geq 2$, we have $p^n_I = \mathrm{Id}_{E/I^n}$, so $(E/I^n, p^n_I)_{n\in\mathbb{N}^*}$ is an inverse system of Hochschild extensions. Thus π has an algebra homomorphism section. □

3.5 Coseparable and formally smooth coalgebras

The whole theory of Hochschild cohomology for coalgebras and its application to coseparability and formal smoothness can be obtained from our general framework by duality, i.e. by working in the dual category of $(\mathcal{M}, \otimes, \mathbf{1}, a, l, r)$. Since this process is completely formal and does not require new ideas we will just state the main results.

3.5.1. Let (C, Δ, ε) be a coalgebra in $(\mathcal{M}, \otimes, \mathbf{1}, a, l, r)$. Like in the dual case, we have the functors

$$^C\mathbb{H} : \mathcal{M} \to {}^C\mathcal{M} \text{ where } {}^C\mathbb{H}(X) := C \otimes X \text{ and } {}^C\mathbb{H}(f) := C \otimes f,$$
$$\mathbb{H}^C : \mathcal{M} \to \mathcal{M}^C \text{ where } \mathbb{H}^C(X) := X \otimes C \text{ and } \mathbb{H}^C(f) := f \otimes C,$$
$$^C\mathbb{H}^C : \mathcal{M} \to {}^C\mathcal{M}^C \text{ where } {}^C\mathbb{H}^C(X) := C \otimes (X \otimes C) \text{ and } {}^C\mathbb{H}^C(f) := C \otimes (f \otimes C),$$

with their left adjoint ${}^C\mathbb{T}, \mathbb{T}^C, {}^C\mathbb{T}^C$, respectively, that forget the comodule structures. Then the adjunctions $({}^C\mathbb{T}, {}^C\mathbb{H})$, $(\mathbb{T}^C, \mathbb{H}^C)$ and $({}^C\mathbb{T}^C, {}^C\mathbb{H}^C)$ gives rise to the following classes of **monomorphisms**:

$$^C\mathcal{I} := \mathcal{I}_{^C\mathbb{T}} = \{g \in {}^C\mathcal{M} \mid g \text{ cosplits in } \mathcal{M}\},$$
$$\mathcal{I}^C := \mathcal{I}_{\mathbb{T}^C} = \{g \in \mathcal{M}^C \mid g \text{ cosplits in } \mathcal{M}\},$$
$$\mathcal{I} := \mathcal{I}_{^C\mathbb{T}^C} = \{g \in {}^C\mathcal{M}^C \mid g \text{ cosplits in } \mathcal{M}\}.$$

3.5.2. Now, for any C-bicomodule $M \in {}^C\mathcal{M}^C$, we define the Hochschild cohomology of C with coefficients in M by:

$$\mathrm{H}^*(M, C) = \mathrm{Ext}^*_{\mathcal{I}}(M, C),$$

where $\mathrm{Ext}^*_{\mathcal{I}}(M, -)$ are the relative left derived functors of ${}^C\mathcal{M}^C(M, -)$. Note that $\mathrm{H}^*(M, C)$ is the Hochschild cohomology of the algebra C with the coefficients in M (regarded as objects in $\mathcal{M}^\circ$).

***Theorem* 3.5.3** (dual to Theorem 3.2.3)**.** *Let (C, Δ, ε) be a coalgebra in a coabelian monoidal category $\mathcal{M}$. The following assertions are equivalent:*
(a) Δ cosplits in ${}^C\mathcal{M}^C$.
(b) C is $\mathcal{I}-$injective.
(c) $\mathrm{H}^1(M, C) = 0$, for all $M \in {}^C\mathcal{M}^C$.
(d) $\mathrm{H}^n(M, C) = 0$, for all $M \in {}^C\mathcal{M}^C$ and $n > 0$.
(e) Any morphism in ${}^C\mathcal{M}^C$ cosplits in ${}^C\mathcal{M}^C$ whenever it cosplits in $\mathcal{M}$.
(f) The category ${}^C\mathcal{M}^C$ is $\mathcal{I}$-cosemisimple (i.e. every object in ${}^C\mathcal{M}^C$ is $\mathcal{I}-$injective).

***Definition* 3.5.4.** Any coalgebra (C, Δ, ε) in a coabelian monoidal category $(\mathcal{M}, \otimes, \mathbf{1})$, satisfying one of the conditions of Theorem 3.5.3, is called *coseparable.*

3.5.5. Let (C, Δ, ε) be an coalgebra in $(\mathcal{M}, \otimes, \mathbf{1})$ and let $\left(L, \rho_L^l, \rho_L^r\right)$ be a C-bicomodule. By the dual of Definition 3.1.9, we have that a morphism

$$\zeta : L \to C \otimes C$$

is a Hochschild 2-cocyle whenever

$$b^2(\zeta) = (\zeta \otimes C) \circ \rho_L^r - (C \otimes \Delta) \circ \zeta + (\Delta \otimes C) \circ \zeta - (C \otimes \zeta) \circ \rho_L^l$$

is zero.

***Definition* 3.5.6** (dual to Definition 3.3.3)**.** Let $(C, \Delta_C, \varepsilon_C)$ be a coalgebra in a coabelian monoidal category $(\mathcal{M}, \otimes, \mathbf{1})$ and let (L, ρ_L^l, ρ_L^r) be a C-bicomodule. A *Hochschild coextension* (E) of C with cokernel L, is an exact sequence in $\mathcal{M}$:

$$(E) \qquad 0 \longrightarrow C \xrightarrow{\sigma} E \xrightarrow{p} L \longrightarrow 0$$

that satisfies the following conditions:

a) σ has a retraction π in $\mathcal{M}$;
b) $(E, \Delta_E, \varepsilon_E)$ is a coalgebra in $\mathcal{M}$ and σ is a coalgebra homomorphism;
c) $C \wedge_E C = E$, that is $(p \otimes p)\Delta = 0$;
d) the morphisms ρ_L^l and ρ_L^r fulfill the following relations

$$\rho_L^l p = (\pi \otimes p)\Delta_E \qquad \text{and} \qquad \rho_L^r p = (p \otimes \pi)\Delta_E.$$

The following result will lead to the definition of a coalgebra structure for the cotensor coalgebra.

***Lemma* 3.5.7.** *Let (C, Δ, ε) be a coalgebra in a coabelian monoidal category $\mathcal{M}$ and let $(L, \rho_L^r, \rho_L^l) \in {}^C\mathcal{M}^C$. Suppose that $\zeta : L \to C \otimes C$ is a morphism in $\mathcal{M}$. Define $\Delta_\zeta : C \oplus L \to (C \oplus L) \otimes (C \oplus L)$ and $\varepsilon_\zeta : C \oplus L \to \mathbf{1}$ by setting:*

$$\Delta_\zeta = (i_C \otimes i_C)\, \Delta p_C + \left[(i_L \otimes i_C)\, \rho_L^r + (i_C \otimes i_L)\, \rho_L^l - (i_C \otimes i_C)\, \zeta\right] p_L, \tag{3.14}$$

$$\varepsilon_\zeta = \varepsilon p_C + l_{\mathbf{1}}\, (\varepsilon \otimes \varepsilon)\, \zeta p_L, \tag{3.15}$$

where i_C, i_L are the canonical injections in $C \oplus L$ and p_C, p_L are the canonical projections. Then Δ_ζ is a coassociative comultiplication if and only if ζ is a Hochschild 2-cocycle. Moreover, in this case, $(C \oplus L, \Delta_\zeta, \varepsilon_\zeta)$ is a coalgebra and

$$(E) \qquad 0 \longrightarrow C \xrightarrow{i_C} C \oplus L \xrightarrow{p_L} L \longrightarrow 0$$

is a Hochschild coextension of C with cokernel (L, p_L). This coextension will be denoted by E_ζ.

Proof. The dual of an abelian monoidal category is a coabelian monoidal category. The conclusion follows by applying Lemma 3.3.5. □

***Definition* 3.5.8.** Let $\mathcal{M}$ be a coabelian monoidal category.
Let $\left((E_i)_{i\in\mathbb{N}}, \left(\eta_i^j\right)_{i,j\in\mathbb{N}}\right)$ be a sequence of morphisms in $\mathcal{M}$

$$E_1 \xrightarrow{\eta_1^2} E_2 \xrightarrow{\eta_2^3} \cdots \xrightarrow{\eta_{n-1}^n} E_n \xrightarrow{\eta_n^{n+1}} E_{n+1} \xrightarrow{\eta_{n+1}^{n+2}} \cdots . \tag{3.16}$$

We say that $\left((E_i)_{i\in\mathbb{N}}, \left(\eta_i^j\right)_{i,j\in\mathbb{N}}\right)$ is a *direct system of coextensions* if

- i_n is a coalgebra homomorphism,
- $E_n \wedge_{E_{n+1}} E_n = E_n$, for any $n \geq 1$.

We say that $\left((E_i)_{i\in\mathbb{N}}, \left(\eta_i^j\right)_{i,j\in\mathbb{N}}\right)$ is a *direct system of Hochschild coextensions* if

- $\left((E_i)_{i\in\mathbb{N}}, \left(\eta_i^j\right)_{i,j\in\mathbb{N}}\right)$ is a direct system of extensions,
- η_i^{i+1} has a retraction in $\mathcal{M}$, for any $n \geq 1$.

We say that a direct system of Hochschild coextensions $\left((E_i)_{i\in\mathbb{N}}, \left(\eta_i^j\right)_{i,j\in\mathbb{N}}\right)$ *has a direct limit* if $\varinjlim E_n$ exists in the category $\mathfrak{Coalg}(\mathcal{M})$ of coalgebras in $\mathcal{M}$.

***Definition* 3.5.9.** Lat C, E, D be coalgebras and let $\delta : D \to E$ be a coalgebra homomorphism in a monoidal category $\mathcal{M}$. We say that C *has the extension property with respect to* δ whenever the canonical map

$$\mathrm{Hom}_{\mathrm{coalg}}(\delta, C) : \mathrm{Hom}_{\mathrm{coalg}}(E, C) \to \mathrm{Hom}_{\mathrm{coalg}}(D, C) : f \mapsto f \circ \delta$$

is surjective. This means that every coalgebra homomorphism $g : D \to C$ can be extended to a coalgebra homomorphism $f : E \to C$ that makes the following diagram

$$\begin{tikzcd} D \arrow[r, "\delta"] \arrow[d, "g"'] & E \arrow[dl, dotted, "f"'] \\ C & \end{tikzcd}$$

commutative.

***Theorem* 3.5.10** (dual to Theorem 3.4.6)**.** *[AMS3, Theorem 4.16] Let* $(\mathcal{M}, \otimes, \mathbf{1})$ *be a coabelian monoidal category. Let* (C, Δ, ε) *be a coalgebra in* $\mathcal{M}$*. Then the following conditions are equivalent:*

(a) C *has has the extension property with respect to every coalgebra homomorphism* $\delta : D \to E$ *that cosplits in* $\mathcal{M}$ *and such that* $D \wedge_E D = E$*.*

(b) C *has has the extension property with respect to the canonical morphism*

$$E_1 \to \varinjlim E_n$$

for every direct system of Hochschild coextensions $\left((E_i)_{i\in\mathbb{N}}, (\eta_i^j)_{i,j\in\mathbb{N}}\right)$ *which has direct limit* $\varinjlim E_n$*.*

(c) C *has has the extension property with respect to the canonical morphism*

$$D \to \varinjlim D^{\wedge_E^n}$$

for any coalgebra E *in* $\mathcal{M}$ *and any subcoalgebra* $(D, \delta : D \to E)$ *of* E *such that* $\left(\left(D^{\wedge_E^i}\right)_{i\in\mathbb{N}}, (\xi_i^j)_{i,j\in\mathbb{N}}\right)$ *is a direct system of Hochschild coextensions which has direct limit* $\varinjlim D^{\wedge_E^n}$*.*

(d) Any Hochschild coextension of C *is trivial.*

(e) $\mathrm{H}^2(M, C) = 0$*, for any* $M \in {}^C\mathcal{M}^C$*.*

***Definition* 3.5.11.** Any coalgebra (C, Δ, ε) in a coabelian monoidal category $(\mathcal{M}, \otimes, \mathbf{1})$, satisfying one of the conditions of Theorem 3.5.10, is called *formally smooth.*

***Corollary* 3.5.12.** *Any coseparable coalgebra in* $\mathcal{M}$ *is formally smooth.*

***Corollary* 3.5.13.** *Let* (C, Δ, ε) *be a coalgebra in a coabelian monoidal category* $(\mathcal{M}, \otimes, \mathbf{1})$*. Then the following assertions are equivalent:*

(a) C *is formally smooth.*

(b) $\operatorname{Coker}\Delta$ *is* $\mathcal{I}$*-injective.*

***Definition* 3.5.14.** Let E be a coalgebra in $\mathcal{M}$ and let (C, δ), where $\delta : C \to E$, be a subcoalgebra of E. We will say that C is *conilpotent* in E if there is n such that $\delta_n : C^{\wedge_E^n} \to E$ is an isomorphism.

***Theorem* 3.5.15.** *Let* (C, Δ, ε) *be a coalgebra in a coabelian monoidal category* $(\mathcal{M}, \otimes, \mathbf{1})$*. Then the following conditions are equivalent:*

(a) C *is formally smooth.*

(b) The canonical morphism $C \to \varinjlim C^{\wedge_E^n}$ *has a coalgebra homomorphism retraction, where* E *is a coalgebra endowed with a coalgebra homomorphism* $\delta : C \to E$ *which is a monomorphism in* $\mathcal{M}$ *and such that* $\left(\left(C^{\wedge_E^i}\right)_{i\in\mathbb{N}}, (\xi_i^j)_{i,j\in\mathbb{N}}\right)$ *is a direct system of Hochschild coextensions which has direct limit* $\varinjlim C^{\wedge_E^n}$*.*

(c) Let $\delta : C \to E$ be a monomorphism in $\mathcal{M}$. If δ is a coalgebra homomorphism, C is conilpotent and $\left(\left(C^{\wedge^i_E}\right)_{i\in\mathbb{N}}, \left(\xi_i^j\right)_{i,j\in\mathbb{N}}\right)$ is a direct system of Hochschild coextensions, then δ has a coalgebra homomorphism retraction.

Chapter 4

The tensor algebra

In this chapter we introduce the tensor algebra inside the framework of monoidal categories. As in the classical case, we prove that the tensor algebra $T_A(M)$, where A is a formally smooth algebra and M is a projective A-bimodule in a monoidal category $\mathcal{M}$, is itself formally smooth as an algebra in $\mathcal{M}$. Furthermore $T_H(M)$ can be endowed with a braided bialgebra structure whenever H is a braided bialgebra in a braided monoidal category $\mathcal{M}$ satisfying suitable assumptions and M is an object in ${}^H_H\mathcal{M}^H_H$. This structure will be used in chapter 5 in the definition of a braided version of the notion of Bialgebra of type one (see Definition 5.6.10).

4.1 The algebra structure and the universal property

***Definition* 4.1.1.** Let $(\mathcal{M}, \otimes, \mathbf{1})$ be a *cocomplete* (i.e. $\mathcal{M}$ has arbitrary coproducts) abelian monoidal category.
We say that *the tensor product commutes with direct sums* whenever

$$Y \otimes (\oplus_{i\in I} X_i) = \oplus_{i\in I}(Y \otimes X_i) \qquad \text{and} \qquad (\oplus_{i\in I} X_i) \otimes Y = \oplus_{i\in I}(X_i \otimes Y),$$

for any $Y \in \mathcal{M}$ and for any family $(X_i)_{i\in I}$ in $\mathcal{M}$.

4.1.2. Let $(\mathcal{M}, \otimes, \mathbf{1})$ be a cocomplete abelian monoidal category such that the tensor product commutes with direct sums. For any object $X \in \mathcal{M}$, we can define

$$T = T(X) \qquad \text{(the tensor algebra of } X\text{)}.$$

Let

$$m_{p,q} : X^{\otimes p} \otimes X^{\otimes q} \to X^{\otimes p+q}, \text{ for every } p, q \in \mathbb{N}$$

be the canonical isomorphism, which is unique by Coherence Theorem.
Still by Coherence Theorem one has

$$m_{p+q,r}[m_{p,q} \otimes X^{\otimes r}] = m_{p,q+r}[X^{\otimes p} \otimes m_{q,r}]. \tag{4.1}$$

We now define

$$T(X) := \oplus_{p \in \mathbb{N}} X^{\otimes p}$$

and

$$m_T := \oplus_{p \in \mathbb{N}} \left(\nabla[(m_{i,j})_{i+j=p}] \right) : T(X) \otimes T(X) \to T(X),$$

where $\nabla[(m_{i,j})_{i+j=p}] : \oplus_{i+j=p} (X^{\otimes i} \otimes X^{\otimes j}) \longrightarrow X^{\otimes p}$ denotes the codiagonal morphism associated to the family $(m_{i,j})_{i+j=p}$. Note that this makes sense. In fact, since the tensor product commutes with direct sums, we have

$$T(X) \otimes T(X) = \oplus_{p \in \mathbb{N}} \left[\oplus_{i+j=p} \left(X^{\otimes i} \otimes X^{\otimes j} \right) \right].$$

If

$$i_p : X^{\otimes p} \to T(X)$$

is the canonical injection, then m_T is uniquely defined by

$$m_T \circ (i_p \otimes i_q) = i_{p+q} \circ m_{p,q}, \text{ for every } p, q \in \mathbb{N}. \tag{4.2}$$

By (4.1) it results immediately that $m_{T(X)}$ is associative. Moreover,

$$(T(X), m_T, i_0)$$

is an algebra in $\mathcal{M}$.

In analogy to the case of vector spaces, the tensor algebra has the following universal property.

***Theorem* 4.1.3** (Universal property of the tensor algebra)**.** *Let* $(\mathcal{M}, \otimes, \mathbf{1})$ *be a cocomplete abelian monoidal category such that the tensor product commutes with direct sums.*
Let $X \in \mathcal{M}$*, let* A *be an algebra in* $\mathcal{M}$ *and let* $f : X \to A$ *be a morphism in* $\mathcal{M}$*. Then there is a unique algebra homomorphism* $\overline{f} : T(X) \to A$ *such that*

$$\begin{tikzcd} X \arrow[rr, "i_1"] \arrow[dr, "f"'] & & T(X) \arrow[dl, dotted, "\overline{f}"] \\ & A & \end{tikzcd}$$

4.1.4. Let (A, m_A, u_A) be an algebra in a cocomplete abelian monoidal category $(\mathcal{M}, \otimes, \mathbf{1})$ such that the tensor product commutes with direct sums.
As explained in 1.3.2, we have that $({}_A\mathcal{M}_A, \otimes_A, A)$ is an abelian monoidal category. Furthermore one can see that ${}_A\mathcal{M}_A$ has also arbitrary direct sums which commute with $\otimes_A$.
Therefore we can consider, in the monoidal category $({}_A\mathcal{M}_A, \otimes_A, A)$, the tensor algebra of an arbitrary A-bimodule M. We will denote it by

$$(T = T_A(M), \overline{m_T}, \overline{u_T}).$$

Note that

$$T = \oplus_{p\in\mathbb{N}} M^{\otimes_A p}$$

$$\overline{m_T} : T \otimes_A T \to T \qquad \text{and} \qquad \overline{u_T} : A \to T.$$

Set

$$\begin{aligned} m_T &= \overline{m_T} \circ {}_A\chi_{T,T} : T \otimes T \to T \qquad \text{and} \\ u_T &= \overline{u_T} \circ u_A : \mathbf{1} \to T, \end{aligned}$$

where ${}_A\chi_{T,T} : T \otimes T \longrightarrow T \otimes_A T$ is the canonical morphism introduced in 1.3.2. Then (T, m_T, u_T) is an algebra in $(\mathcal{M}, \otimes, \mathbf{1})$.

We are now able to state the Universal property of the relative tensor algebra.

***Theorem* 4.1.5** (Universal property of the relative tensor algebra)**.** *Let* $(\mathcal{M}, \otimes, \mathbf{1})$ *be a* cocomplete *abelian monoidal category such that the tensor product commutes with direct sums.*
Let A, B *be algebras in* $\mathcal{M}$ *and let* $f_A : A \to B$ *be an algebra homomorphism.*
Let $M \in {}_A\mathcal{M}_A$*, and let* $f_M : M \to B$ *be a morphism in* ${}_A\mathcal{M}_A$*, where* $B \in {}_A\mathcal{M}_A$ *via* f_0.
Then there is a unique algebra homomorphism $f : T_A(M) \to B$ *such that*

$$\begin{tikzcd} M \arrow[r, "i_1"] \arrow[d, "f_M"'] & T_A(M) \arrow[dl, dotted, "f"] \\ B & A \arrow[l, "f_A"] \arrow[u, "i_0"'] \end{tikzcd}$$

where $i_0 : A \to T_A(M)$ *and* $i_1 : M \to T_A(M)$ *are the canonical injections.*

Proof. Using the fact that f_A is an algebra homomorphism, one can prove that the multiplication of B factors to a unique morphism $\overline{m_B} : B \otimes_A B \longrightarrow B$ such that $\overline{m_B} \circ {}_A\chi_{B,B} = m_B$. Furthermore $(B, \overline{m_B}, f_A)$ comes out to be an algebra in the monoidal category $({}_A\mathcal{M}_A, \otimes_A, A)$. Since $(T_A(M), \overline{m_T}, \overline{u_T})$ is the tensor algebra in this category, by its universal property, there is a unique algebra homomorphism $f : (T_A(M), \overline{m}_T, \overline{u_T}) \longrightarrow (B, \overline{m}_B, f_A)$ in ${}_A\mathcal{M}_A$ such that

$$f \circ i_1 = f_M.$$

Moreover, if $T = T_A(M)$, we have $f \circ i_0 = f \circ \overline{u_T} = f_A$. Finally

$$\begin{aligned} f \circ m_T &= f \circ \overline{m}_T \circ {}_A\chi_{T,T} \\ &= \overline{m}_B \circ (f \otimes_A f) \circ {}_A\chi_{T,T} \\ &= \overline{m}_B \circ {}_A\chi_{B,B} \circ (f \otimes_A f) = m_B \circ (f \otimes_A f), \\ f \circ u_T &= f \circ \overline{u_T} \circ u_A = f_A \circ u_A = u_B, \end{aligned}$$

so that $f : (T_A(M), m_T, u_T) \longrightarrow (B, m_B, u_B)$ is an algebra homomorphism in $\mathcal{M}$. □

4.2 Formal smoothness

In the classical case it is well known (see e.g. [CQ, Proposition 5.3]) that the tensor algebra $T_A(M)$, where A is a formally smooth (quasi-free in terminology of [CQ]) algebra and M is a projective A-bimodule, is itself formally smooth. The following theorem states that this result still holds true in monoidal categories.

***Theorem* 4.2.1.** *[AMS3, Proposition 3.29] Let A be an algebra in a cocomplete abelian monoidal category $\mathcal{M}$ such that the tensor product commutes with direct sums. If*
- *A is a formally smooth algebra in $\mathcal{M}$,*
- *M is a $\mathcal{P}$-projective A-bimodule in the sense of section 2.3,*

then the tensor algebra $T_A(M)$ is also formally smooth as an algebra in $\mathcal{M}$.

Proof. Let $\pi : E \to T_A(M)$ be a Hochschild extension of $T_A(M)$ in $\mathcal{M}$. Since A is formally smooth, by the first condition from Theorem 3.4.6, there exists an algebra homomorphism $g_0 : A \to E$ such that $\pi g_0 = i_0$, where $i_0 : A \to T_A(M)$ is the canonical inclusion. The objects E and $T_A(M)$ have a natural A-bimodule structure induced by g_0 and i_0, respectively. Thus π becomes a morphisms of A-bimodules so that $\pi \in \mathcal{P}$. Let $i_1 : M \to T_A(M)$ be the canonical inclusion. Since M is $\mathcal{P}$-projective and $\pi \in \mathcal{P}$, there exists a morphism of A-bimodules $g_1 : M \to E$ such that $\pi g_1 = i_1$.

$$\begin{tikzcd}
 & A \arrow[dl, dotted, "g_0"'] \arrow[d, "i_0"] & \\
E \arrow[r, "\pi"] & T_A(M) \arrow[r] \arrow[l, dotted, bend left, "g"] & 0 \\
 & M \arrow[u, "i_1"'] \arrow[ul, dotted, "g_1"] &
\end{tikzcd}$$

By the universal property of $T_A(M)$, there exists a unique algebra homomorphism $g : T_A(M) \to E$ such that $gi_0 = g_0$ and $gi_1 = g_1$. Then $\pi g i_0 = \pi g_0 = i_0$ and $\pi g i_1 = \pi g_1 = i_1$, so $\pi g = \mathrm{Id}_{T_A(M)}$. This means that π is a trivial Hochschild extension. $\square$

***Corollary* 4.2.2.** *If (A, m, u) is a formally smooth algebra, the tensor algebra $T_A(Ker(m))$ is also formally smooth. If A is separable, the tensor algebra $T_A(M)$ is formally smooth, for any $M \in {}_A\mathcal{M}_A$.*

Proof. Apply Corollary 3.4.9 and Theorem 3.2.3. $\square$

***Remark* 4.2.3.** Let A be an algebra in a cocomplete abelian monoidal category $\mathcal{M}$ such that the tensor product commutes with direct sums. By the universal property of the tensor algebra it results that $T(X)$ is formally smooth, for any object X in $\mathcal{M}$. Thus $T_A(M)$ is always formally smooth as an algebra in $({}_A\mathcal{M}_A, \otimes_A, A)$ for *any* A-bimodule M, and even if A *is not* formally smooth in $\mathcal{M}$.

4.3 Braided bialgebra Structure

Next aim is to provide a braided bialgebra structure (see 1.2.5) for the tensor algebra inside a braided monoidal category.

***Theorem* 4.3.1.** *Let H be a braided bialgebra in a cocomplete abelian braided monoidal category $(\mathcal{M}, c)$. Assume that the tensor product commutes with direct sums.*
Let $(M, \mu_M^r, \mu_M^l, \rho_M^r, \rho_M^l)$ be in ${}_H^H\mathcal{M}_H^H$. Let $T = T_H(M)$ be the tensor algebra. Then there are unique coalgebra homomorphisms

$$\Delta_T : T \to T \otimes T \qquad \textit{and} \qquad \varepsilon_T : T \to \mathbf{1}$$

such that the diagrams

(4.3)
$$\begin{tikzcd} M \arrow[r, "i_1"] \arrow[d, "(i_0\otimes i_1)\rho^l_M+(i_1\otimes i_0)\rho^r_M"'] & T \arrow[dl, dotted, "\Delta_T"'] \\ T\otimes T & H \arrow[l, "(i_0\otimes i_0)\Delta_H"] \arrow[u, "i_0"'] \end{tikzcd} \qquad \begin{tikzcd} M \arrow[r, "i_1"] \arrow[d, "0"'] & T \arrow[dl, dotted, "\varepsilon_T"'] \\ \mathbf{1} & H \arrow[l, "\varepsilon_H"] \arrow[u, "i_0"'] \end{tikzcd}$$

are commutative, where $i_n : M^{\otimes_H n} \to T$ *denotes the canonical injection. Moreover* $(T, m_T, u_T, \Delta_T, \varepsilon_T)$ *is a braided bialgebra in* $\mathcal{M}$*.*

Proof. First of all recall that $(T \otimes T, m_{T\otimes T}, u_{T\otimes T})$ is an algebra where

$$\begin{aligned} m_{T\otimes T} &: T\otimes T\otimes T\otimes T \overset{T\otimes c_{T,T}\otimes T}{\longrightarrow} T\otimes T\otimes T\otimes T \overset{m_T\otimes m_T}{\longrightarrow} T\otimes T, \\ u_{T\otimes T} &: \mathbf{1} \overset{(m_{\mathbf{1}})^{-1}}{\longrightarrow} \mathbf{1}\otimes\mathbf{1} \overset{u_T\otimes u_T}{\longrightarrow} T\otimes T. \end{aligned}$$

Set

$$\begin{aligned} f_H &: = (i_0\otimes i_0)\,\Delta_H : H \to T\otimes TE, \\ f_M &: = (i_0\otimes i_1)\,\rho^l_M + (i_1\otimes i_0)\,\rho^r_M : M \to T\otimes T. \end{aligned}$$

Then f_H is a algebra homomorphism:

$$\begin{aligned} f_H m_H &= (i_0\otimes i_0)\,\Delta_H m_H \\ &= (i_0\otimes i_0)\,(m_H\otimes m_H)\,(H\otimes c_{H,H}\otimes H)\,(\Delta_H\otimes\Delta_H) \\ &\overset{(4.2)}{=} (m_T\otimes m_T)\,(i_0\otimes i_0\otimes i_0\otimes i_0)\,(H\otimes c_{H,H}\otimes H)\,(\Delta_H\otimes\Delta_H) \\ &= (m_T\otimes m_T)\,(T\otimes c_{T,T}\otimes T)\,(i_0\otimes i_0\otimes i_0\otimes i_0)\,(\Delta_H\otimes\Delta_H) \\ &= m_{T\otimes T}\,(f_H\otimes f_H) \end{aligned}$$

Moreover f_M is a morphism of left H-modules

$$\begin{aligned} f_M\mu^l_M &= (i_0\otimes i_1)\,\rho^l_M\mu^l_M + (i_1\otimes i_0)\,\rho^r_M\mu^l_M \\ &\overset{(1.1),(1.3)}{=} (i_0\otimes i_1)\left(m_H\otimes\mu^l_M\right)(H\otimes c_{H,H}\otimes M)\left(\Delta_H\otimes\rho^l_M\right) + \\ &\quad + (i_1\otimes i_0)\left(\mu^l_M\otimes m_H\right)(H\otimes c_{H,M}\otimes H)\,(\Delta_H\otimes\rho^r_M) \\ &= (m_T\otimes m_T)\,(i_0\otimes i_0\otimes i_0\otimes i_1)\,(H\otimes c_{H,H}\otimes M)\left(\Delta_H\otimes\rho^l_M\right) + \\ &\quad + (m_T\otimes m_H)\,(i_0\otimes i_1\otimes i_0\otimes i_0)\,(H\otimes c_{H,M}\otimes H)\,(\Delta_H\otimes\rho^r_M) \\ &= (m_T\otimes m_T)\,(T\otimes c_{T,T}\otimes T)\,(i_0\otimes i_0\otimes i_0\otimes i_1)\left(\Delta_H\otimes\rho^l_M\right) + \\ &\quad + (m_T\otimes m_H)\,(T\otimes c_{T,T}\otimes T)\,(i_0\otimes i_0\otimes i_1\otimes i_0)\,(\Delta_H\otimes\rho^r_M) \\ &= m_{T\otimes T}\left[f_H\otimes(i_0\otimes i_1)\,\rho^l_M\right] + m_{T\otimes T}\left[f_H\otimes(i_1\otimes i_0)\,\rho^r_M\right] \\ &= m_{T\otimes T}\left[f_H\otimes f_M\right]. \end{aligned}$$

Analogously $f_M\mu_M^r = m_{T\otimes T}(f_M \otimes f_H)$, i.e. f_M is a morphism of right H-modules and hence a morphism of H-bicomodules. By applying Theorem 4.1.5 to the case $B = T \otimes T$ we get an algebra homomorphism

$$\Delta_T = f : T \to T \otimes T$$

such that the left side of 4.3 is commutative. We have

$$\begin{aligned}
(\Delta_T \otimes T)\,\Delta_T i_0 &= (\Delta_T \otimes T)\, f_H \\
&= (\Delta_T \otimes T)\,(i_0 \otimes i_0)\,\Delta_H \\
&= (f_H \otimes i_0)\,\Delta_H \\
&= (i_0 \otimes i_0 \otimes i_0)\,(\Delta_H \otimes H)\,\Delta_H
\end{aligned}$$

Analogously $(T \otimes \Delta_T)\,\Delta_T i_0 = (i_0 \otimes i_0 \otimes i_0)\,(H \otimes \Delta_H)\,\Delta_H$ and hence

$$(\Delta_T \otimes T)\,\Delta_T i_0 = (T \otimes \Delta_T)\,\Delta_T i_0.$$

Moreover

$$\begin{aligned}
&(\Delta_T \otimes T)\,\Delta_T i_1 \\
=\ &(\Delta_T \otimes T)\, f_M \\
=\ &(\Delta_T \otimes T)\,(i_0 \otimes i_1)\,\rho_M^l + (\Delta_T \otimes T)\,(i_1 \otimes i_0)\,\rho_M^r \\
=\ &(f_H \otimes i_1)\,\rho_M^l + (f_M \otimes i_0)\,\rho_M^r \\
=\ &(i_0 \otimes i_0 \otimes i_1)\,(\Delta_H \otimes M)\,\rho_M^l + (i_0 \otimes i_1 \otimes i_0)\left(\rho_M^l \otimes H\right)\rho_M^r + \\
&+ (i_1 \otimes i_0 \otimes i_0)\,(\rho_M^r \otimes H)\,\rho_M^r
\end{aligned}$$

Analogously

$$\begin{aligned}
&(T \otimes \Delta_T)\,\Delta_T i_1 \\
=\ &(i_0 \otimes i_0 \otimes i_1)\left(H \otimes \rho_M^l\right)\rho_M^l + (i_0 \otimes i_1 \otimes i_0)\,(H \otimes \rho_M^r)\,\rho_M^l \\
&+ (i_1 \otimes i_0 \otimes i_0)\,(M \otimes \Delta_H)\,\rho_M^r
\end{aligned}$$

so that

$$(\Delta_T \otimes T)\,\Delta_T i_1 = (T \otimes \Delta_T)\,\Delta_T i_1.$$

Since $T \otimes T \otimes T$ is an algebra and $(T \otimes \Delta_T)\,\Delta_T$ is an algebra homomorphism, then by uniqueness in the universal property of tensor algebra (Theorem 4.1.5), we have

$$(\Delta_T \otimes T)\,\Delta_T = (T \otimes \Delta_T)\,\Delta_T.$$

Set

$$\begin{aligned}
f'_H &: = \varepsilon_H : H \to \mathbf{1}, \\
f'_M &: = 0 : M \to \mathbf{1}.
\end{aligned}$$

Then f'_H is an algebra homomorphism and f'_M is a morphism of H-bimodules. By applying Theorem 4.1.5 to the case $B = \mathbf{1}$ we get an algebra homomorphism

$$\varepsilon_T = f' : T \to \mathbf{1}$$

such that the right side of 4.3 is commutative.We have

$$\begin{aligned}
(\varepsilon_T \otimes T)\,\Delta_T i_0 &= (\varepsilon_T \otimes T)\, f_H \\
&= (\varepsilon_T \otimes T)\,(i_0 \otimes i_0)\,\Delta_H \\
&= (f'_H \otimes i_0)\,\Delta_H \\
&= (\varepsilon_H \otimes i_0)\,\Delta_H \\
&= (\mathbf{1} \otimes i_0)\, l_H^{-1} \\
&= l_T^{-1} i_0
\end{aligned}$$

and

$$\begin{aligned}
(\varepsilon_T \otimes T)\,\Delta_T i_1 &= (\varepsilon_T \otimes T)\, f_M \\
&= (\varepsilon_T \otimes T)\,(i_0 \otimes i_1)\,\rho_M^l + (\varepsilon_T \otimes T)\,(i_1 \otimes i_0)\,\rho_M^r \\
&= (f'_H \otimes i_1)\,\rho_M^l + (f'_M \otimes i_0)\,\rho_M^r \\
&= (\varepsilon_H \otimes i_1)\,\rho_M^l \\
&= (\mathbf{1} \otimes i_1)\, l_M^{-1} = l_T^{-1} i_1
\end{aligned}$$

Since $\mathbf{1} \otimes T$ is an algebra and $(\varepsilon_T \otimes T)\,\Delta_T$ is an algebra homomorphism, then by uniqueness in the universal property of tensor algebra (Theorem 4.1.5), we have

$$(\varepsilon_T \otimes T)\,\Delta_T = l_T^{-1}.$$

Analogously $(T \otimes \varepsilon_T)\,\Delta_T = r_T^{-1}$. Thus $(T, m_T, u_T, \Delta_T, \varepsilon_T)$ is a braided bialgebra in $\mathcal{M}$. □

We are now able to state the universal property of the tensor bialgebra.

Theorem 4.3.2. *Let H be a braided bialgebra in a cocomplete abelian braided monoidal category $(\mathcal{M}, c)$. Assume that the tensor product commutes with direct sums.*
Let $(M, \mu_M^r, \mu_M^l, \rho_M^r, \rho_M^l)$ be in ${}_H^H\mathcal{M}_H^H$. Let $T = T_H(M)$ be the tensor algebra.
Let E be a braided bialgebra in $\mathcal{M}$.
Let $f_H : H \to E$ be a bialgebra homomorphism and let $f_M : M \to E$ be a morphism of H-bimodules, where E is a bimodule via f_H. Assume that

$$\Delta_E f_M = (f_H \otimes f_M)\,\rho_M^l + (f_M \otimes f_H)\,\rho_M^r$$

(i.e. f_M is a coderivation of E with domain the E-bicomodule M, where M is regarded as a bicomodule via f_H). Then there is a unique algebra homomorphism $f : T_H(M) \to E$ such that $fi_0 = f_H$ and $fi_1 = f_M$, where $i_n : M^{\otimes_H n} \to T_H(M)$ denotes the canonical injection.

$$\begin{tikzcd} M \arrow[r, "i_1"] \arrow[d, "f_M"'] & T_H(M) \arrow[dl, dotted, "f"] \\ E & H \arrow[l, "f_H"] \arrow[u, "i_0"'] \end{tikzcd}$$

Moreover f is a bialgebra homomorphism.

Proof. By Theorem 4.1.5, there is a unique algebra homomorphism $f : T \to E$ such that $fi_0 = f_H$ and $fi_1 = f_M$. By Theorem 4.3.1, we have

$$(f \otimes f)\,\Delta_T i_0 = (f \otimes f)\,(i_0 \otimes i_0)\,\Delta_H = (f_H \otimes f_H)\,\Delta_H = \Delta_E f_H = \Delta_E f i_0$$

and

$$\begin{aligned} (f \otimes f)\,\Delta_T i_1 &= (f \otimes f)\left[(i_0 \otimes i_1)\,\rho_M^l + (i_1 \otimes i_0)\,\rho_M^r\right] \\ &= (fi_0 \otimes fi_1)\,\rho_M^l + (fi_1 \otimes fi_0)\,\rho_M^r \\ &= (f_H \otimes f_M)\,\rho_M^l + (f_M \otimes f_H)\,\rho_M^r \\ &= \Delta_E f_M = \Delta_E f i_1 \end{aligned}$$

From $\Delta_E f_M = (f_H \otimes f_M)\,\rho_M^l + (f_M \otimes f_H)\,\rho_M^r$, we get

$$\begin{aligned} \varepsilon_E f_M &= m_{\mathbf{1}}\,(\varepsilon_E \otimes \varepsilon_E)\,\Delta_E f_M \\ &= m_{\mathbf{1}}\,(\varepsilon_E \otimes \varepsilon_E)\,(f_H \otimes f_M)\,\rho_M^l + m_{\mathbf{1}}\,(\varepsilon_E \otimes \varepsilon_E)\,(f_M \otimes f_H)\,\rho_M^r \\ &= m_{\mathbf{1}}\,(\varepsilon_H \otimes \varepsilon_E f_M)\,\rho_M^l + m_{\mathbf{1}}\,(\varepsilon_E f_M \otimes \varepsilon_H)\,\rho_M^r \\ &= \varepsilon_E f_M + \varepsilon_E f_M \end{aligned}$$

so that

$$\varepsilon_E f_M = 0.$$

Hence, by Theorem 4.1.5, we have

$$\begin{aligned} \varepsilon_E f i_0 &= \varepsilon_E f_H = \varepsilon_H = \varepsilon_T i_0, \\ \varepsilon_E f i_1 &= \varepsilon_E f_M = 0 = \varepsilon_T i_1. \end{aligned}$$

Since $(f \otimes f)\,\Delta_T, \Delta_E f : T \to E \otimes E$ and $\varepsilon_E f, \varepsilon_T : T \to \mathbf{1}$ are algebra homomorphisms, as a composition of algebra homomorphisms, and since

$$(f \otimes f)\,\Delta_T i_n = \Delta_E f i_n \qquad \text{and} \qquad \varepsilon_E f i_n = \varepsilon_T i_n$$

for $n = 0, 1$, by uniqueness in Theorem 4.1.5, we get that $(f \otimes f)\,\Delta_T = \Delta_E f$ and $\varepsilon_E f = \varepsilon_T$ i.e. that f is a coalgebra homomorphism. $\square$

Chapter 5

Cotensor coalgebras

5.1 Preliminaries and notations

Let C be a coalgebra over a field K and let M be a C-bicomodule. The cotensor coalgebra $T^c_C(M)$ was introduced by Nichols in [Ni] as a main tool to construct some new Hopf algebras that he called "bialgebras of type one". These bialgebras can be reconstructed, via a bosonization procedure, from the so called Nichols algebras, which are essentially the H-coinvariant parts of the bialgebras of type one, in the case when $C = H$ is a Hopf algebra and M is a Hopf bimodule. Nichols algebras, also named quantum symmetric algebras in [Ro], have been deeply investigated and appear as a main step in the classification of finite dimensional Hopf algebras problem (see, e.g., [AG] and [AS]). In fact, in the case that $C = H$ is a Hopf algebra and M is a Hopf H-bimodule, the cotensor coalgebra $T^c_C(M)$ is a bialgebra that is called "quantum shuffle Hopf algebra" by Rosso in [Ro] where some fundamental properties of this bialgebra and of its coinvariant Hopf algebra are investigated. The coalgebra of paths of a quiver Q is an instance of a cotensor coalgebra. Namely let Q_0 be the set of vertices and let Q_1 be the set of arrows of Q. Then $M = KQ_1$ is a C-bicomodule where $C = KQ_0$ is equipped with its natural coalgebra structure. The cotensor coalgebra $T^c_C(M)$ is the path coalgebra of the quiver Q. In [CR], Cibils and Rosso provide the classification of path coalgebras which admit a graded Hopf algebra structure, allowing the quiver to be infinite. On the other hand, in [JLMS], hereditary coalgebras with coseparable coradical are characterized by means of a suitable cotensor coalgebra. Moreover it is proved that if C is a formally smooth coalgebra and M is $\mathcal{I}$-injective then $T^c_C(M)$ is formally smooth.
In this chapter the notion of cotensor coalgebra in an abelian monoidal category is introduced. We would like to outline that this fact is not immediate. In fact the notion of coradical plays a fundamental role in the usual definition for coalgebras over a field (see [Ni]) while we have no coradical substitution here. Also, having developed in [AMS3] the notion of formally smooth coalgebras for abelian monoidal

categories, we wanted to obtain the second quoted result of [JLMS] in this more general setting, namely we prove Theorem 5.4.8 which states that, in a cocomplete and complete abelian monoidal category $\mathcal{M}$ satisfying $AB5$, with left and right exact tensor functors and such that denumerable coproducts commute with $\otimes$, the cotensor coalgebra $T_C^c(M)$ is formally smooth whenever C is a formally smooth coalgebra in $\mathcal{M}$ and M is an $\mathcal{I}$-injective C-bicomodule in $\mathcal{M}$. We point out that in [Ar4] the other quoted result of [JLMS] was investigated in the frame of coabelian monoidal categories. In fact hereditary coalgebras that are the direct limit $\widetilde{D}$ of a filtration consisting of wedge products of a subcoalgebra D, where D is a coseparable coalgebra in $\mathcal{M}$, are characterized by means of a cotensor coalgebra: more precisely, under suitable assumptions, $\widetilde{D}$ is hereditary if and only if it is formally smooth if and only if it is the cotensor coalgebra $T_D^c(D \wedge D/D)$ if and only if it is a cotensor coalgebra $T_D^c(N)$, where N is a certain D-bicomodule in $\mathcal{M}$.
In this chapter, we also provide a braided bialgebra structure for the cotensor coalgebra inside a braided monoidal category. This structure is used to extend the notion of bialgebra of type one, introduced in the classical case by Nichols in [Ni], to the wider context of a braided monoidal category (see Definition 5.6.10). A universal property for the cotensor bialgebra is also proven (see Theorem 5.6.8)

We will write $\square$ instead of $\square_C$, whenever there is no danger of misunderstanding.

***Notations* 5.1.1.** Let (C, Δ, ε) be a coalgebra in a coabelian monoidal category $\mathcal{M}$ and let (M, ρ_M^r, ρ_M^l) be a $C-$bicomodule. As observed in 1.3.2, $({}^C\mathcal{M}^C, \square, C)$ defines a monoidal category. In view of Notation 1.6.4, we will write

$$M^{\square 0} = C, M^{\square 1} = M \quad \text{and} \quad M^{\square n} = M^{\square n-1} \square M \text{ for any } n > 1.$$

Define $(C^n(M))_{n\in\mathbb{N}}$ by

$$C^0(M) = 0, C^1(M) = C \quad \text{and} \quad C^n(M) = C^{n-1}(M) \oplus M^{\square n-1} \text{ for any } n > 1.$$

Let $\sigma_i^{i+1} : C^i(M) \to C^{i+1}(M)$ be the canonical inclusion and for any $j > i$, define:

$$\sigma_i^j = \sigma_{j-1}^j \sigma_{j-2}^{j-1} \cdots \sigma_{i+1}^{i+2} \sigma_i^{i+1} : C^i(M) \to C^j(M).$$

Then $((C^i(M))_{i\in\mathbb{N}}, (\sigma_i^j)_{i,j\in\mathbb{N}})$ is a direct system in $\mathcal{M}$.
When the category $\mathcal{M}$ is also cocomplete, we define

$$T_C^c(M) = \bigoplus_{n\in\mathbb{N}} M^{\square n} = C \oplus M \oplus M^{\square 2} \oplus M^{\square 3} \oplus \cdots .$$

Throughout let

$$\begin{aligned} \pi_n^m &: C^n(M) \twoheadrightarrow C^m(M) \ (m \le n), \qquad \pi_n : T_C^c(M) \twoheadrightarrow C^n(M), \\ p_n^m &: C^n(M) \twoheadrightarrow M^{\square m} \ (m < n), \qquad p_n : T_C^c(M) \twoheadrightarrow M^{\square n}, \end{aligned}$$

be the canonical projections and let

$$\begin{array}{rcl} \sigma_m^n & : & C^m(M) \hookrightarrow C^n(M)\ (m \leq n), \qquad \sigma_n : C^n(M) \hookrightarrow T_C^c(M), \\ i_m^n & : & M^{\square m} \hookrightarrow C^n(M)\ (m < n), \qquad i_m : M^{\square m} \hookrightarrow T_C^c(M), \end{array}$$

be the canonical injection for any $m, n \in \mathbb{N}$. For technical reasons we set $\pi_n^m = 0$, $\sigma_m^n = 0$ for any $n < m$ and $p_n^m = 0$, $i_m^n = 0$ for any $n \leq m$. Then, we have the following relations:

$$p_n \sigma_k = p_k^n, \qquad p_n i_k = \delta_{n,k} \mathrm{Id}_{M^{\square k}}, \qquad \pi_n i_k = i_k^n.$$

Moreover, we have:

$$\begin{array}{lll} \pi_n^m \sigma_k^n = \sigma_k^m, \text{ if } k \leq m \leq n, & \text{and} & \pi_n^m \sigma_k^n = \pi_k^m, \text{ if } m \leq k \leq n, \\ p_n^m \pi_k^n = p_k^m, \text{ if } m < n \leq k, & \text{and} & \sigma_n^m i_k^n = i_k^m, \text{ if } k < n \leq m, \\ p_n^m \sigma_k^n = p_k^m, \text{ if } m < k \leq n, & \text{and} & \pi_n^m i_k^n = i_k^m, \text{ if } k < m \leq n, \\ p_n^m \pi_n = p_m, \text{ if } m < n, & \text{and} & \sigma_n i_m^n = i_m, \text{ if } m < n, \\ \pi_n \sigma_k = \sigma_k^n, \text{ if } k \leq n, & \text{and} & \pi_n \sigma_k = \pi_k^n, \text{ if } n \leq k, \\ p_n^m i_m^n = \mathrm{Id}_{M^{\square m}}, \text{ if } m < n. & & \end{array}$$

In the other cases, these compositions are zero.

***Corollary* 5.1.2.** *Let (C, Δ, ε) be a coalgebra in a cocomplete abelian monoidal category $\mathcal{M}$ and let (M, ρ_M^r, ρ_M^l) be a C-bicomodule. Then*

$$(T_C^c(M), (\sigma_n)_{n \in \mathbb{N}}) = \varinjlim C^n(M).$$

Proof. It follows by Proposition 1.6.5, once observed that $C^n(M) = \oplus_{m=0}^{n-1} M^{\square m}$ and $\sigma_n = \nabla \left[(i_m)_{m=0}^{n-1} \right]$. □

5.2 The coalgebra structure

5.2.1. Note that $M^{\square n}$ is a C-bicomodule via

$$\rho_n^l := \rho_M^l \square M^{\square n-1} \qquad \text{and} \qquad \rho_n^r := M^{\square n-1} \square \rho_M^r.$$

Our next aim is to define, for any $n \in \mathbb{N} \setminus \{0\}$, a Hochschild 2-cocycle

$$\zeta^n : M^{\square n} \to C^n(M) \otimes C^n(M).$$

Then we will apply Lemma 3.5.7 to obtain that, for any $n > 0$, $C^{n+1}(M) = C^n(M) \oplus M^{\square n}$ can be endowed with a coalgebra structure $(C^{n+1}(M), \Delta_{\zeta^n}, \varepsilon_{\zeta^n})$ in $\mathcal{M}$ such that the canonical inclusion $\sigma_n^{n+1} : C^n(M) \to C^{n+1}(M)$ is a Hochschild coextension of $C^n(M)$ with cokernel $M^{\square n}$. Then, by Proposition 1.6.2, $T_C^c(M)$ will carry a natural coalgebra structure that makes it the direct limit of $((C^i(M))_{i\in\mathbb{N}},(\sigma_i^j)_{i,j\in\mathbb{N}})$ as a direct system of coalgebras.

Let $\varsigma_M = \varsigma_C(M,M) : M\Box M \to M\otimes M$ be the canonical inclusion and define

$$\zeta^1 = 0 \quad \text{and} \quad \zeta^n = -\sum_{t=1}^{n-1}(i_t^n \otimes i_{n-t}^n)(M^{\Box t-1}\Box\varsigma_M\Box M^{\Box n-1-t}), \forall n>1. \tag{5.1}$$

where we identify $C\Box X$ and $X\Box C$ with X, for any C-bicomodule X.

***Proposition* 5.2.2.** *[AMS2, Proposition 2.8] Let (C,Δ,ε) be a coalgebra in a cocomplete coabelian monoidal category $\mathcal{M}$ and let (M,ρ_M^r,ρ_M^l) be a C-bicomodule. Let*

$$\Delta(1)=\Delta, \qquad \text{and} \qquad \varepsilon(1)=\varepsilon,$$
$$\overline{\rho}_1^l=\rho_M^l, \qquad \text{and} \qquad \overline{\rho}_1^r=\rho_M^r,$$

and for every $n\geq 2$ set

$$\Delta(n)=\Delta_{\zeta^{n-1}}, \qquad \text{and} \qquad \varepsilon(n)=\varepsilon_{\zeta^{n-1}},$$

as defined in (3.14), (3.15), and let

$$\overline{\rho}_n^l=(\sigma_1^n\otimes M^{\Box n})(\rho_M^l\Box M^{\Box n-1}), \qquad \text{and} \qquad \overline{\rho}_n^r=(M^{\Box n}\otimes\sigma_1^n)(M^{\Box n-1}\Box\rho_M^r).$$

Then, for any $n\geq 1$, we have

a) $(C^n(M),\Delta(n),\varepsilon(n))$ is a coalgebra.

b) $(M^{\Box n},\overline{\rho}_n^l,\overline{\rho}_n^r)$ is a bicomodule over the coalgebra $C^n(M)$ such that the morphism $\zeta^n : M^{\Box n}\to C^n(M)\otimes C^n(M)$, given by (5.1), defines a Hochschild 2-cocycle.

c) $\varepsilon(n)=\varepsilon_C\pi_n^1$.

d) For every $1\leq t\leq n-1$ we have

$$\Delta(n)i_t^n=(i_t^n\otimes\sigma_t^n)\,\overline{\rho}_t^r+(\sigma_t^n\otimes i_t^n)\,\overline{\rho}_t^l-(\sigma_t^n\otimes\sigma_t^n)\,\zeta^t. \tag{5.2}$$

e) $\Delta(n)$ fulfils the following relations:

$$\begin{aligned}
\Delta(n)i_0^n &= (i_0^n\otimes i_0^n)\,\Delta\\
\Delta(n)i_1^n &= (i_1^n\otimes i_0^n)\,\rho_M^r+(i_0^n\otimes i_1^n)\,\rho_M^l\\
\textit{and for } 2 &\leq t\leq n-1\\
\Delta(n)i_t^n &= (i_t^n\otimes i_0^n)\,(M^{\Box t-1}\Box\rho_M^r)+(i_0^n\otimes i_t^n)\,(\rho_M^l\Box M^{\Box t-1})+\\
&\quad+\sum_{r=1}^{t-1}(i_r^n\otimes i_{t-r}^n)(M^{\Box r-1}\Box\varsigma_M\Box M^{\Box t-1-r}).
\end{aligned}$$

Proof. Set $C^n = C^n(M)$ for any $n \geq 1$. Recall that $M^{\square n}$ is a C-bicomodule via

$$\rho_n^l := \rho_M^l \square M^{\square n-1}, \qquad \text{and} \qquad \rho_n^r := M^{\square n-1} \square \rho_M^r.$$

Let us prove all the statements of the theorem by induction on $n \geq 1$.

If $n = 1$, then $C^1 = C$ is a coalgebra and $M^{\square 1} = M$ is a C^1-bicomodule by hypothesis. Obviously $\zeta^1 = 0$ fulfills (5.5). We have $\varepsilon(1) = \varepsilon = \varepsilon_C \pi_1^1$ and, since $i_0^1 = \mathrm{Id}_C$ and $i_1^1 = 0$, we get

$$\Delta(1) i_0^1 = \Delta = \left(i_0^1 \otimes i_0^1\right) \Delta, \qquad \Delta(1) i_1^1 = 0 = \left(i_1^1 \otimes i_0^1\right) \rho_M^r + \left(i_0^1 \otimes i_1^1\right) \rho_M^l.$$

Let $n \geq 2$. Assume all the assertions hold true for any $1 \leq t < n$.
Thus $(C^{n-1}, \Delta(n-1), \varepsilon(n-1))$ is a coalgebra in $\mathcal{M}$, $(M^{\square n-1}, \overline{\rho}_{n-1}^l, \overline{\rho}_{n-1}^r)$ is a C^{n-1}-bicomodule and ζ^{n-1} is a 2-cocycle. By Lemma 3.5.7 applied to "C" $= C^{n-1}$ and "M" $= M^{\square n-1}$, then $(C^n, \Delta(n), \varepsilon(n))$ is a coalgebra. Moreover $\sigma_{t-1}^t : C^{t-1} \to C^t$ is a Hochschild coextension of C^{t-1} with cokernel $M^{\square t-1}$ for any $1 \leq t < n$.
Since $(M^{\square n}, \rho_n^l, \rho_n^r)$ is a C-bicomodule and $\sigma_1^n : C \to C^n$ is a coalgebra homomorphism (as a composition of coalgebra homomorphism), then $(M^{\square n}, \overline{\rho}_n^l, \overline{\rho}_n^r)$ is a C^n-bicomodule, where

$$\overline{\rho}_n^l = (\sigma_1^n \otimes M^{\square n})(\rho_M^l \square M^{\square n-1}), \qquad \text{and} \qquad \overline{\rho}_n^r = (M^{\square n} \otimes \sigma_1^n)(M^{\square n-1} \square \rho_M^r).$$

Recall that, by definition, we have:

$$\begin{aligned}\Delta(n) &= \Delta_{\zeta^{n-1}} \\ &= \left(\sigma_{n-1}^n \otimes \sigma_{n-1}^n\right) \Delta(n-1) \pi_n^{n-1} + \begin{bmatrix} \left(i_{n-1}^n \otimes \sigma_{n-1}^n\right) \overline{\rho}_{n-1}^r + \\ + \left(\sigma_{n-1}^n \otimes i_{n-1}^n\right) \overline{\rho}_{n-1}^l + \\ - \left(\sigma_{n-1}^n \otimes \sigma_{n-1}^n\right) \zeta^{n-1} \end{bmatrix} p_n^{n-1},\end{aligned}$$

For any $0 \leq t \leq n-1$, we have that

$$\begin{aligned} & \left[\left(i_{n-1}^n \otimes \sigma_{n-1}^n\right) \overline{\rho}_{n-1}^r + \left(\sigma_{n-1}^n \otimes i_{n-1}^n\right) \overline{\rho}_{n-1}^l - \left(\sigma_{n-1}^n \otimes \sigma_{n-1}^n\right) \zeta^{n-1}\right] p_n^{n-1} i_t^n \\ = & \left[\left(i_{n-1}^n \otimes \sigma_{n-1}^n\right) \overline{\rho}_{n-1}^r + \left(\sigma_{n-1}^n \otimes i_{n-1}^n\right) \overline{\rho}_{n-1}^l - \left(\sigma_{n-1}^n \otimes \sigma_{n-1}^n\right) \zeta^{n-1}\right] \delta_{t,n-1} \end{aligned}$$

so that we obtain

$$(5.3) \qquad \begin{aligned} \Delta(n) i_t^n = & \left(\sigma_{n-1}^n \otimes \sigma_{n-1}^n\right) \Delta(n-1) \pi_n^{n-1} i_t^n + \\ & + \begin{bmatrix} \left(i_{n-1}^n \otimes \sigma_{n-1}^n\right) \overline{\rho}_{n-1}^r + \left(\sigma_{n-1}^n \otimes i_{n-1}^n\right) \overline{\rho}_{n-1}^l \\ - \left(\sigma_{n-1}^n \otimes \sigma_{n-1}^n\right) \zeta^{n-1} \end{bmatrix} \delta_{t,n-1}. \end{aligned}$$

For $t = 0 < n-1$, we have:

$$\Delta(n) i_0^n = \left(\sigma_{n-1}^n \otimes \sigma_{n-1}^n\right) \Delta(n-1) i_0^{n-1},$$

so that, inductively we get:

$$\Delta(n)i_0^n = \left(\sigma_{n-1}^n \otimes \sigma_{n-1}^n\right)\left(i_0^{n-1} \otimes i_0^{n-1}\right)\Delta_C = \left(i_0^n \otimes i_0^n\right)\Delta_C.$$

Let us prove that for every $t \in \mathbb{N}$, such that $0 < t \leq n-1$ we have (5.2).
Now, we apply (5.3). If $t = n-1$, we get

$$\Delta(n)i_t^n = \left(i_{n-1}^n \otimes \sigma_{n-1}^n\right)\overline{\rho}_{n-1}^r + \left(\sigma_{n-1}^n \otimes i_{n-1}^n\right)\overline{\rho}_{n-1}^l - \left(\sigma_{n-1}^n \otimes \sigma_{n-1}^n\right)\zeta^{n-1}.$$

If $0 < t < n-1$, we get

$$\begin{aligned}
\Delta(n)i_t^n &= \left(\sigma_{n-1}^n \otimes \sigma_{n-1}^n\right)\Delta(n-1)i_t^{n-1} \\
&= \left(\sigma_{n-1}^n \otimes \sigma_{n-1}^n\right)\left[\begin{array}{c} \left(i_t^{n-1} \otimes \sigma_t^{n-1}\right)\overline{\rho}_t^r + \left(\sigma_t^{n-1} \otimes i_t^{n-1}\right)\overline{\rho}_t^l + \\ -\left(\sigma_t^{n-1} \otimes \sigma_t^{n-1}\right)\zeta^t \end{array}\right] \\
&= \left(i_t^n \otimes \sigma_t^n\right)\overline{\rho}_t^r + \left(\sigma_t^n \otimes i_t^n\right)\overline{\rho}_t^l - \left(\sigma_t^n \otimes \sigma_t^n\right)\zeta^t.
\end{aligned}$$

Thus we have obtained (5.2). Note that, for $t = 1$, by definition of $\overline{\rho}_1^l, \overline{\rho}_1^r$ and since $\zeta^1 = 0$, one gets

$$\Delta(n)i_1^n = \left(i_1^n \otimes \sigma_1^n\right)\overline{\rho}_1^r + \left(\sigma_1^n \otimes i_1^n\right)\overline{\rho}_1^l = \left(i_1^n \otimes i_0^n\right)\rho_M^r + \left(i_0^n \otimes i_1^n\right)\rho_M^l.$$

For $2 \leq t \leq n-1$, by definition of $\overline{\rho}_n^l, \overline{\rho}_n^r$ and of ζ^t, from (5.2), we get:

$$\begin{aligned}
\Delta(n)i_t^n &= \left(i_t^n \otimes \sigma_t^n\right)(M^{\square t} \otimes \sigma_1^t)(M^{\square t-1}\square\rho_M^r) + \\
&\quad + \left(\sigma_t^n \otimes i_t^n\right)(\sigma_1^t \otimes M^{\square t})(\rho_M^l \square M^{\square t-1}) + \\
&\quad + \sum_{r=1}^{t-1}\left(\sigma_t^n \otimes \sigma_t^n\right)\left(i_r^t \otimes i_{t-r}^t\right)(M^{\square r-1}\square\varsigma_M\square M^{\square t-1-r}),
\end{aligned}$$

so that we obtain:

$$\begin{aligned}
\Delta(n)i_t^n &= \left(i_t^n \otimes i_0^n\right)(M^{\square t-1}\square\rho_M^r) + \left(i_0^n \otimes i_t^n\right)(\rho_M^l\square M^{\square t-1}) + \\
&\quad + \sum_{r=1}^{t-1}(i_r^n \otimes i_{t-r}^n)(M^{\square r-1}\square\varsigma_M\square M^{\square t-1-r}).
\end{aligned}$$

Moreover, by definition, we have:

$$\varepsilon(n) = \varepsilon_{\zeta^{n-1}} = \varepsilon(n-1)\pi_n^{n-1} + l_{\mathbf{1}}[\varepsilon(n-1) \otimes \varepsilon(n-1)]\zeta^{n-1}p_n^{n-1}. \tag{5.4}$$

Since $n \geq 2$ and $\varepsilon(n-1) = \varepsilon_C\pi_{n-1}^1$, by (5.4), we have

$$\begin{aligned}
\varepsilon(n) &= \varepsilon_C\pi_{n-1}^1\pi_n^{n-1} + l_{\mathbf{1}}(\varepsilon_C\pi_{n-1}^1 \otimes \varepsilon_C\pi_{n-1}^1)\zeta^{n-1}p_n^{n-1} \\
&= \varepsilon_C\pi_n^1 + l_{\mathbf{1}}(\varepsilon_C \otimes \varepsilon_C)(\pi_{n-1}^1 \otimes \pi_{n-1}^1)\zeta^{n-1}p_n^{n-1}.
\end{aligned}$$

By definition of ζ^{n-1}, if $n=2$ we have $\zeta^1=0$ so that $(\pi^1_{n-1}\otimes\pi^1_{n-1})\zeta^{n-1}=0$. If $n\geq 3$, we have

$$(\pi^1_{n-1}\otimes\pi^1_{n-1})\zeta^{n-1}=\sum_{t=1}^{n-2}(\pi^1_{n-1}\otimes\pi^1_{n-1})(i^{n-1}_t\otimes i^{n-1}_{n-1-t})(M^{\square t-1}\square\varsigma_M\square M^{\square n-2-t})=0.$$

We conclude that $(\pi^1_n\otimes\pi^1_n)\zeta^{n-1}=0$ for any $n\geq 2$ and hence $\varepsilon(n)=\varepsilon_C\pi^1_n$.

Recall that ζ^n is a 2-cocycle means that it verifies the following relation:

$$0=b^2(\zeta^n)=(\zeta^n\otimes C^n)\overline{\rho}^r_n-[C^n\otimes\Delta(n)]\zeta^n+[\Delta(n)\otimes C^n]\zeta^n-(C^n\otimes\zeta^n)\overline{\rho}^l_n. \tag{5.5}$$

Now, since $\sigma^n_1=i^n_0$ and $n\geq 2$, we have

$$\begin{aligned}
&-(\zeta^n\otimes C^n)\overline{\rho}^r_n\\
=\;&\sum_{t=1}^{n-1}[(i^n_t\otimes i^n_{n-t})(M^{\square t-1}\square\varsigma_M\square M^{\square n-1-t})\otimes C^n](M^{\square n}\otimes\sigma^n_1)(M^{\square n-1}\square\rho^r_M)\\
=\;&\sum_{t=1}^{n-1}(i^n_t\otimes i^n_{n-t}\otimes\sigma^n_1)(M^{\square t-1}\square\varsigma_M\square M^{\square n-1-t}\otimes C)(M^{\square n-1}\square\rho^r_M)\\
=\;&\sum_{t=1}^{n-1}\left(i^n_t\otimes i^n_{n-t}\otimes i^n_0\right)(M^{\square t}\otimes M^{\square n-1-t}\square\rho^r_M)(M^{\square t-1}\square\varsigma_M\square M^{\square n-1-t})
\end{aligned}$$

and

$$
\begin{aligned}
&-[C^n\otimes\Delta(n)]\zeta^n\\
&=[C^n\otimes\Delta(n)]\sum_{t=1}^{n-1}(i_t^n\otimes i_{n-t}^n)(M^{\square t-1}\square\varsigma_M\square M^{\square n-1-t})\\
&=\sum_{t=1}^{n-1}(i_t^n\otimes\Delta(n)i_{n-t}^n)(M^{\square t-1}\square\varsigma_M\square M^{\square n-1-t})\\
&=\sum_{t=1}^{n-1}\left\{i_t^n\otimes\left[\begin{array}{c}\left(i_{n-t}^n\otimes i_0^n\right)(M^{\square n-t-1}\square\rho_M^r)+\\ +\left(i_0^n\otimes i_{n-t}^n\right)(\rho_M^l\square M^{\square n-t-1})+\\ +\sum_{r=1}^{n-t-1}(i_r^n\otimes i_{n-t-r}^n)(M^{\square r-1}\square\varsigma_M\square M^{\square n-t-1-r})\end{array}\right]\right\}(M^{\square t-1}\square\varsigma_M\square M^{\square n-1-t})\\
&=\sum_{t=1}^{n-1}\left(i_t^n\otimes i_{n-t}^n\otimes i_0^n\right)(M^{\square t}\otimes M^{\square n-t-1}\square\rho_M^r)(M^{\square t-1}\square\varsigma_M\square M^{\square n-1-t})+\\
&+\sum_{t=1}^{n-1}\left(i_t^n\otimes i_0^n\otimes i_{n-t}^n\right)(M^{\square t}\otimes\rho_M^l\square M^{\square n-t-1})(M^{\square t-1}\square\varsigma_M\square M^{\square n-1-t})+\\
&+\sum_{t=1}^{n-1}\sum_{r=1}^{n-t-1}(i_t^n\otimes i_r^n\otimes i_{n-t-r}^n)(M^{\square t}\otimes M^{\square r-1}\square\varsigma_M\square M^{\square n-t-1-r})(M^{\square t-1}\square\varsigma_M\square M^{\square n-1-t})\\
&=-(\zeta^n\otimes C^n)\overline{\rho}_n^r+\\
&+\sum_{t=1}^{n-1}\left(i_t^n\otimes i_0^n\otimes i_{n-t}^n\right)(M^{\square t}\otimes\rho_M^l\square M^{\square n-t-1})(M^{\square t-1}\square\varsigma_M\square M^{\square n-1-t})+\\
&+\sum_{t=1}^{n-1}\sum_{r=1}^{n-t-1}(i_t^n\otimes i_r^n\otimes i_{n-t-r}^n)(M^{\square t}\otimes M^{\square r-1}\square\varsigma_M\square M^{\square n-t-1-r})(M^{\square t-1}\square\varsigma_M\square M^{\square n-1-t})
\end{aligned}
$$

Analogously one has

$$-(C^n\otimes\zeta^n)\overline{\rho}_n^l=\sum_{t=1}^{n-1}(i_0^n\otimes i_t^n\otimes i_{n-t}^n)(\rho_M^l\square M^{\square t-1}M^{\square n-t})(M^{\square t-1}\square\varsigma_M\square M^{\square n-1-t})$$

and

$$
\begin{aligned}
&[\Delta(n)\otimes C^n]\zeta^n\\
=\;&-(C^n\otimes\zeta^n)\overline{\rho}_n^l+\\
&+\sum_{t=1}^{n-1}\left(i_t^n\otimes i_0^n\otimes i_{n-t}^n\right)(M^{\square t-1}\square\rho_M^r\otimes M^{\square n-t})(M^{\square t-1}\square\varsigma_M\square M^{\square n-1-t})+\\
&+\sum_{t=1}^{n-1}\sum_{r=1}^{t-1}(i_r^n\otimes i_{t-r}^n\otimes i_{n-t}^n)(M^{\square r-1}\square\varsigma_M\square M^{\square t-1-r}\otimes M^{\square n-t})(M^{\square t-1}\square\varsigma_M\square M^{\square n-1-t}).
\end{aligned}
$$

Now, for any $1 \leq t \leq n-1$, by definition of ς_M we have

$$\begin{aligned}
& \left(i_t^n \otimes i_0^n \otimes i_{n-t}^n\right)(M^{\square t-1}\square\rho_M^r \otimes M^{\square n-t})(M^{\square t-1}\square\varsigma_M\square M^{\square n-1-t}) \\
= & \left(i_t^n \otimes i_0^n \otimes i_{n-t}^n\right)(\left[M^{\square t-1}\square(\rho_M^r \otimes M)\varsigma_M\square M^{\square n-1-t}\right] \\
= & \left(i_t^n \otimes i_0^n \otimes i_{n-t}^n\right)(\left[M^{\square t-1}\square(M \otimes \rho_M^l)\varsigma_M\square M^{\square n-1-t}\right] \\
= & \left(i_t^n \otimes i_0^n \otimes i_{n-t}^n\right)(M^{\square t} \otimes \rho_M^l\square M^{\square n-1-t})(M^{\square t-1}\square\varsigma_M\square M^{\square n-1-t})
\end{aligned}$$

an also

$$\begin{aligned}
& \sum_{t=1}^{n-1}\sum_{r=1}^{t-1}(i_r^n \otimes i_{t-r}^n \otimes i_{n-t}^n)(M^{\square r-1}\square\varsigma_M\square M^{\square t-1-r} \otimes M^{\square n-t})(M^{\square t-1}\square\varsigma_M\square M^{\square n-1-t}) \\
= & \sum_{t=1}^{n-1}\sum_{r=1}^{t-1}(i_r^n \otimes i_{t-r}^n \otimes i_{n-t}^n)(M^{\square r} \otimes M^{\square t-1-r}\square\varsigma_M\square M^{\square n-1-t})(M^{\square r-1}\square\varsigma_M\square M^{\square n-1-r}) \\
= & \sum_{t=1}^{n-1}\sum_{\substack{r+j=t \\ r,j>0}}(i_r^n \otimes i_j^n \otimes i_{n-t}^n)(M^{\square r} \otimes M^{\square j-1}\square\varsigma_M\square M^{\square n-1-t})(M^{\square r-1}\square\varsigma_M\square M^{\square n-1-r}) \\
= & \sum_{\substack{r+j+k=n \\ k,r,j>0}}(i_r^n \otimes i_j^n \otimes i_k^n)(M^{\square r} \otimes M^{\square j-1}\square\varsigma_M\square M^{\square k-1})(M^{\square r-1}\square\varsigma_M\square M^{\square n-1-r}) \\
= & \sum_{t=1}^{n-1}\sum_{\substack{r+k=t \\ r,k>0}}(i_t^n \otimes i_r^n \otimes i_k^n)(M^{\square t} \otimes M^{\square r-1}\square\varsigma_M\square M^{\square k-1})(M^{\square t-1}\square\varsigma_M\square M^{\square n-1-t}) \\
= & \sum_{t=1}^{n-1}\sum_{r=1}^{n-t-1}(i_t^n \otimes i_r^n \otimes i_{n-t-r}^n)(M^{\square t} \otimes M^{\square r-1}\square\varsigma_M\square M^{\square n-t-1-r})(M^{\square t-1}\square\varsigma_M\square M^{\square n-1-t})
\end{aligned}$$

Then ζ^n satisfies (5.5). $\square$

Theorem 5.2.3. *[AMS2, Theorem 2.9] Let (C, Δ, ε) be a coalgebra in a cocomplete coabelian monoidal category $\mathcal{M}$ and let (M, ρ_M^r, ρ_M^l) be a C-bicomodule. $(T_C^c(M), (\sigma_i)_{i\in\mathbb{N}})$ carries a natural coalgebra structure that makes it the direct limit of $((C^i(M))_{i\in\mathbb{N}}, (\sigma_i^j)_{i,j\in\mathbb{N}})$ as a direct system of coalgebras.*

Proof. By Proposition 5.2.2, for any $n \in \mathbb{N} \setminus \{0\}$, the canonical inclusion $\sigma_{n-1}^n : C^{n-1}(M) \to C^n(M)$ is a Hochschild coextension of $C^{n-1}(M)$ with cokernel $M^{\square n-1}$. In particular σ_{n-1}^n is a coalgebra homomorphism for any $n \in \mathbb{N}$. Then, by 5.1.1, σ_m^n is a coalgebra homomorphism for any $m, n \in \mathbb{N}$. Now, in view of Corollary 5.1.2 and Proposition 1.6.2, $T_C^c(M)$ carries a natural coalgebra structure that makes it the direct limit of $((C^i(M))_{i\in\mathbb{N}}, (\sigma_i^j)_{i,j\in\mathbb{N}})$ as a direct system of coalgebras. $\square$

Lemma 5.2.4. *Let (C, Δ, ε) be a coalgebra in a cocomplete coabelian monoidal category $\mathcal{M}$ and let (M, ρ_M^r, ρ_M^l) be a C-bicomodule. Let $T^c := T_C^c(M)$. Then,*

$$\varepsilon_{T^c} i_t = \delta_{t,0} \varepsilon_C,$$

for every $t \in \mathbb{N}$, and Δ_{T^c} fulfils the following relations:

$$\begin{array}{rcl} \Delta_{T^c} i_0 & = & (i_0 \otimes i_0)\,\Delta \\ \Delta_{T^c} i_1 & = & (i_1 \otimes i_0)\,\rho_M^r + (i_0 \otimes i_1)\,\rho_M^l \\ \textit{and for } 2 & \leq & t \\ \Delta_{T^c} i_t & = & (i_t \otimes i_0)\,(M^{\square t-1} \square \rho_M^r) + (i_0 \otimes i_t)\,(\rho_M^l \square M^{\square t-1}) + \\ & & + \displaystyle\sum_{r=1}^{t-1} (i_r \otimes i_{t-r})(M^{\square r-1} \square \varsigma_M \square M^{\square t-1-r}). \end{array}$$

Proof. By construction, the counit Δ_{T^c} of T^c is uniquely defined by the following relation

$$\varepsilon_{T^c} \sigma_t = \varepsilon\,(t)\,, \text{ for every } t \in \mathbb{N}.$$

By Proposition 5.2.2, for every $t \in \mathbb{N}$, we have

$$\varepsilon_{T^c} i_t = \Delta_{T^c} \sigma_{t+1} i_t^{t+1} = \varepsilon\,(t+1)\, i_t^{t+1} = \varepsilon_C \pi_{t+1}^1 i_t^{t+1} = \delta_{t,0} \varepsilon_C i_t^1 = \delta_{t,0} \varepsilon_C.$$

By construction, the comultiplication Δ_{T^c} of T^c is uniquely defined by the following relation

$$\Delta_{T^c} \sigma_t = (\sigma_t \otimes \sigma_t)\,\Delta\,(t)\,, \text{ for every } t \in \mathbb{N}.$$

For every $t \in \mathbb{N}$, we have

$$\Delta_{T^c} i_t = \Delta_{T^c} \sigma_{t+1} i_t^{t+1} = (\sigma_{t+1} \otimes \sigma_{t+1})\,\Delta\,(t+1)\, i_t^{t+1}.$$

From this equality and by Proposition 5.2.2, we get

$$\Delta_{T^c} i_0 = (\sigma_1 \otimes \sigma_1)\,\Delta\,(1)\, i_0^1 = (\sigma_1 \otimes \sigma_1)\left(i_0^1 \otimes i_0^1\right)\Delta = (i_0 \otimes i_0)\,\Delta,$$

and also

$$\begin{array}{rcl} \Delta_{T^c} i_1 & = & (\sigma_2 \otimes \sigma_2)\,\Delta\,(2)\, i_1^2 \\ & = & (\sigma_2 \otimes \sigma_2)\left(i_1^2 \otimes i_0^2\right)\rho_M^r + \left(i_0^2 \otimes i_1^2\right)\rho_M^l \\ & = & (i_1 \otimes i_0)\,\rho_M^r + (i_0 \otimes i_1)\,\rho_M^l \end{array}$$

and, for every $2 \leq t$, we get

$$\begin{aligned}
\Delta_{T^c} i_t &= (\sigma_{t+1} \otimes \sigma_{t+1}) \, \Delta \, (t+1) \, i_t^{t+1} \\
&= (\sigma_{t+1} \otimes \sigma_{t+1}) \begin{bmatrix} \left(i_t^{t+1} \otimes i_0^{t+1}\right) (M^{\Box t-1} \Box \rho_M^r) + \\ + \left(i_0^{t+1} \otimes i_t^{t+1}\right) (\rho_M^l \Box M^{\Box t-1}) + \\ + \sum_{r=1}^{t-1} (i_r^{t+1} \otimes i_{t-r}^{t+1})(M^{\Box r-1} \Box \varsigma_M \Box M^{\Box t-1-r}) \end{bmatrix} \\
&= (i_t \otimes i_0) \, (M^{\Box t-1} \Box \rho_M^r) + (i_0 \otimes i_t) \, (\rho_M^l \Box M^{\Box t-1}) + \\
&\quad + \sum_{r=1}^{t-1} (i_r \otimes i_{t-r})(M^{\Box r-1} \Box \varsigma_M \Box M^{\Box t-1-r}).
\end{aligned}$$

$\Box$

5.3 The universal property

Next aim is to prove the universal property of the cotensor coalgebra.

Lemma 5.3.1. *[AMS2, Lemma 2.10] Let (C, Δ, ε) be a coalgebra in a cocomplete coabelian monoidal category $\mathcal{M}$ and let (M, ρ_M^r, ρ_M^l) be a C-bicomodule. Let $T^c := T_C^c(M)$. Then, for any $m, n \geq 1$ the following relations hold true:*

$$(p_m \otimes p_n)\Delta_{T^c} = (M^{\Box m-1} \Box \varsigma_M \Box M^{\Box n-1}) p_{m+n}, \text{ for any } m, n \geq 1; \tag{5.6}$$

$$(p_m \otimes p_0)\Delta_{T^c} = (M^{\Box m-1} \Box \rho_M^r) p_m, \text{ for any } m \geq 1; \tag{5.7}$$

$$(p_0 \otimes p_n)\Delta_{T^c} = (\rho_M^l \Box M^{\Box n-1}) p_n, \text{ for any } n \geq 1; \tag{5.8}$$

$$(p_0 \otimes p_0)\Delta_{T^c} = \Delta_C p_0. \tag{5.9}$$

Proof. By Lemma 5.2.4Then, for $k = 0, 1$ we have respectively

$$\begin{aligned}
(p_m \otimes p_n)\Delta_{T^c} i_0 &= (p_m \otimes p_n) \, (i_0 \otimes i_0) \, \Delta \\
&= \delta_{m,0}\delta_{n,0}\Delta = \begin{cases} 0, \text{ for } m \geq 1 \text{ and } n \geq 1; \\ \delta_{m,0}\Delta, \text{ for } n = 0; \\ \delta_{n,0}\Delta, \text{ for } m = 0; \end{cases} \\
(p_m \otimes p_n)\Delta_{T^c} i_1 &= (p_m \otimes p_n) \left[(i_1 \otimes i_0) \, \rho_M^r + (i_0 \otimes i_1) \, \rho_M^l \right] \\
&= \delta_{m,1}\delta_{n,0}\rho_M^r + \delta_{m,0}\delta_{n,1}\rho_M^l = \begin{cases} 0, \text{ for } m \geq 1 \text{ and } n \geq 1; \\ \delta_{m,1}\rho_M^r, \text{ for } n = 0; \\ \delta_{n,1}\rho_M^l, \text{ for } m = 0; \end{cases}
\end{aligned}$$

while, for $t \geq 2$,we have:

$$\begin{aligned}
\Delta_{T^c} i_t &= (i_t \otimes i_0) \, (M^{\Box t-1} \Box \rho_M^r) + (i_0 \otimes i_t) \, (\rho_M^l \Box M^{\Box t-1}) + \\
&\quad + \sum_{r=1}^{t-1} (i_r \otimes i_{t-r})(M^{\Box r-1} \Box \varsigma_M \Box M^{\Box t-1-r}).
\end{aligned}$$

$$\begin{aligned}
(p_m \otimes p_n)\Delta_{T^c} i_t &= (p_m \otimes p_n)\left[\begin{array}{c} (i_t \otimes i_0)\,(M^{\square t-1}\square \rho_M^r)+ \\ +\,(i_0 \otimes i_t)\,(\rho_M^l \square M^{\square t-1})+ \\ +\sum_{r=1}^{t-1}(i_r \otimes i_{t-r})(M^{\square r-1}\square \varsigma_M \square M^{\square t-1-r}) \end{array}\right] \\
&= \delta_{m,t}\delta_{n,0}(M^{\square t-1}\square \rho_M^r) + \delta_{m,o}\delta_{n,t}(\rho_M^l \square M^{\square t-1}) + \\
&\quad + \sum_{r=1}^{t-1} \delta_{m,r}\delta_{n,t-r}(M^{\square r-1}\square \varsigma_M \square M^{\square t-1-r}) \\
&= \begin{cases} \delta_{n,t-m}(M^{\square m-1}\square \varsigma_M \square M^{\square n-1}), \text{ for } m \geq 1 \text{ and } n \geq 1; \\ \delta_{0,t-m}(M^{\square t-1}\square \rho_M^r), \text{ for } n = 0; \\ \delta_{n,t}(\rho_M^l \square M^{\square t-1}), \text{ for } m = 0; \end{cases}
\end{aligned}$$

so that, for any $t \geq 0$, we get:

$$\begin{aligned}
(p_m \otimes p_n)\Delta_{T^c} i_t &= \delta_{n+m,t}(M^{\square m-1}\square \varsigma_M \square M^{\square n-1}) \\
&= (M^{\square m-1}\square \varsigma_M \square M^{\square n-1})p_{m+n} i_t, \text{ for any } m, n \geq 1; \\
(p_m \otimes p_0)\Delta_{T^c} i_t &= \delta_{m,t}(M^{\square t-1}\square \rho_M^r) = (M^{\square m-1}\square \rho_M^r)p_m i_t, \text{ for any } m \geq 1; \\
(p_0 \otimes p_n)\Delta_{T^c} i_t &= \delta_{n,t}(\rho_M^l \square M^{\square t-1}) = (\rho_M^l \square M^{\square n-1})p_n i_t, \text{ for any } n \geq 1; \\
(p_0 \otimes p_0)\Delta_{T^c} i_t &= \delta_{0,t}\Delta(1) = \Delta_C p_0 i_t.
\end{aligned}$$

Therefore, we conclude. □

***Proposition* 5.3.2.** *[AMS2, Proposition 2.11] Let (C, Δ, ε) be a coalgebra in a cocomplete coabelian monoidal category $\mathcal{M}$ and let (M, ρ_M^r, ρ_M^l) be a C-bicomodule. Let $T^c := T_C^c(M)$. Let E be a coalgebra and let $\alpha : E \to T^c$ and $\beta : E \to T^c$ be coalgebra homomorphisms. If $p_1\alpha = p_1\beta$, then $p_n\alpha = p_n\beta$ for any $n \geq 1$.*

Proof. Let us prove it by induction on $n \geq 1$, the case $n = 1$ being true by assumption. Thus let $n \geq 2$ be such that $p_n\alpha = p_n\beta$. Then

$$\begin{aligned}
(\varsigma_M \square M^{\square n-1})p_{n+1}\alpha &\overset{(5.6)}{=} (p_1 \otimes p_n)\Delta_{T^c}\alpha \\
&= (p_1\alpha \otimes p_n\alpha)\Delta_{T^c} \\
&= (p_1\beta \otimes p_n\beta)\Delta_{T^c} \\
&= (p_1 \otimes p_n)\Delta_{T^c}\beta \overset{(5.6)}{=} (\varsigma_M \square M^{\square n-1})p_{n+1}\beta.
\end{aligned}$$

Since ς_M is a monomorphism, then, by left exactness of the tensor functors, $\varsigma_M \square M^{\square n-1}$ is a monomorphism too, so that $p_{n+1}\alpha = p_{n+1}\beta$. □

Our next aim is to prove the universal property of $T_C^c(M)$.

***Theorem* 5.3.3.** *[AMS2, Theorem 2.13] Let (C, Δ, ε) be a coalgebra in a cocomplete coabelian monoidal category $\mathcal{M}$ and let (M, ρ_M^r, ρ_M^l) be a C-bicomodule. Let δ :*

$D \to E$ be a monomorphism which is a homomorphism of coalgebras such that the canonical morphism $\widetilde{\delta} : \widetilde{D} \to E$ of Notation 1.6.4 is a monomorphism. Let $f_C : \widetilde{D} \to C$ be a coalgebra homomorphism and let $f_M : \widetilde{D} \to M$ be a morphism of C-bicomodules such that $f_M \xi_1 = 0$, where $\widetilde{D}$ is a C-bicomodule via f_C. Then there is a unique morphism $f : \widetilde{D} \to T^c_C(M)$ such that

$$\begin{tikzcd} D^{\wedge^n_E} \arrow[r, "\xi_n"] \arrow[d, "f_n"'] & \widetilde{D} \arrow[d, dotted, "f"] \\ C^n(M) \arrow[r, "\sigma_n"] & T^c_C(M) \end{tikzcd}$$

is commutative for any $n \in \mathbb{N}$, where

$$f_n = \sum_{t=0}^{n} i^n_t f_M^{\square t} \overline{\Delta}_{\widetilde{D}}^{t-1} \xi_n \tag{5.10}$$

and $\overline{\Delta}^n_{\widetilde{D}} : \widetilde{D} \to \widetilde{D}^{\square n+1}$ is the n^{th}-iteration of $\overline{\Delta}_{\widetilde{D}}$ ($\overline{\Delta}^{-1}_{\widetilde{D}} = f_C, \overline{\Delta}^0_{\widetilde{D}} = Id_{\widetilde{D}}, \overline{\Delta}^1_{\widetilde{D}} = \overline{\Delta}_{\widetilde{D}} : \widetilde{D} \to \widetilde{D} \square \widetilde{D}$).
Moreover:

1) *f is a coalgebra homomorphism;*
2) *the following diagram is commutative*

$$\begin{tikzcd} T^c_C(M) \arrow[r, "p_1"] \arrow[d, "p_0"'] & M \\ C & \widetilde{D} \arrow[l, "f_C"] \arrow[u, "f_M"'] \arrow[ul, "f"] \end{tikzcd}$$

where $p_n : T^c_C(M) \to M^{\square n}$ denotes the canonical projection.
Furthermore, any coalgebra homomorphism $f : \widetilde{D} \to T^c_C(M)$ that fulfils 2) satisfies the following relation:

$$p_k f = f_M^{\square k} \overline{\Delta}_{\widetilde{D}}^{k-1} \text{ for any } k \in \mathbb{N}. \tag{5.11}$$

Proof. Set $T^c := T^c_C(M)$. By Proposition 1.5.8), if we denote by (L, p) the cokernel of $\delta : D \to E$ in $\mathcal{M}$, we have

$$(D^{\wedge^n_E}, \delta_n) := \mathrm{Ker}\,(p^{\otimes n} \Delta_E^{n-1})$$

for any $n \in \mathbb{N} \setminus \{0\}$, where $\Delta^n : E \to E^{\otimes n+1}$ is the n^{th} iterated comultiplication of E ($\Delta^0 = \mathrm{Id}_E, \Delta^1 := \Delta_E$). Since $f_C \xi_1 : D \to C$ is a coalgebra homomorphism, then D becomes a C-bicomodule and ξ_1 a morphism of C-bicomodules. Set $C^n = C^n(M)$ and $D^n = D^{\wedge^n_E}$ for any $n \in \mathbb{N} \setminus \{0\}$. Define $f_n : D^n \to C^n$, for every $n \in \mathbb{N}$, as in

(5.10).
Note that, for every $n \geq 1$, $i_n^n = 0$ so that we have

$$f_n = \sum_{t=0}^{n-1} i_t^n f_M^{\square t} \overline{\Delta}_{\widetilde{D}}^{t-1} \xi_n.$$

Let us prove that $f_{n+1}\xi_n^{n+1} = \sigma_n^{n+1} f_n$ for any $n \in \mathbb{N}$. We have that

$$\begin{aligned} f_{n+1}\xi_n^{n+1} &= \sum_{t=0}^{n} i_t^{n+1} f_M^{\square t} \overline{\Delta}_{\widetilde{D}}^{t-1} \xi_{n+1}\xi_n^{n+1} \\ &= \sum_{t=0}^{n} i_t^{n+1} f_M^{\square t} \overline{\Delta}_{\widetilde{D}}^{t-1} \xi_n \\ &= \sum_{t=0}^{n-1} \sigma_n^{n+1} i_t^n f_M^{\square t} \overline{\Delta}_{\widetilde{D}}^{t-1} \xi_n + i_n^{n+1} f_M^{\square n} \overline{\Delta}_{\widetilde{D}}^{n-1} \xi_n \\ &= \sigma_n^{n+1} f_n + i_n^{n+1} f_M^{\square n} \overline{\Delta}_{\widetilde{D}}^{n-1} \xi_n. \end{aligned}$$

Let $(\widetilde{L}, \widetilde{p})$ be the cokernel of $\xi_1 : D \to \widetilde{D}$ in $\mathcal{M}$. Let $\varsigma_{\widetilde{L}} : \widetilde{L} \square \widetilde{L} \to \widetilde{L} \otimes \widetilde{L}$ be the canonical injection. Define $\varsigma_{\widetilde{L}}^n : \widetilde{L}^{\square n} \to \widetilde{L}^{\otimes n}$, for every $n \in \mathbb{N}$, by setting $\varsigma_{\widetilde{L}}^0 = \mathrm{Id}_C$, $\varsigma_{\widetilde{L}}^1 = \mathrm{Id}_{\widetilde{L}}$, $\varsigma_{\widetilde{L}}^2 = \varsigma_{\widetilde{L}}$ and $\varsigma_{\widetilde{L}}^n = (\widetilde{L}^{\otimes n-2} \otimes \varsigma_{\widetilde{L}})(\varsigma_{\widetilde{L}}^{n-1} \square \widetilde{L})$ for any $n > 2$. Since the tensor functors are left exact, $\varsigma_{\widetilde{L}}^n$ is a monomorphism. By Lemma 1.5.9, we have

$$(D^{\wedge_E^n}, \xi_n) = D^{\wedge_{\widetilde{D}}^n} = \mathrm{Ker}\,(\widetilde{p}^{\otimes n} \Delta_{\widetilde{D}}^{n-1})$$

Thus, for any $n \geq 1$, we have

$$\mathrm{Ker}\,(\widetilde{p}^{\otimes n} \Delta_{\widetilde{D}}^{n-1}) = \mathrm{Ker}\,(\varsigma_{\widetilde{L}}^n \circ \widetilde{p}^{\square n} \overline{\Delta}_{\widetilde{D}}^{n-1}) = \mathrm{Ker}\,(\widetilde{p}^{\square n} \overline{\Delta}_{\widetilde{D}}^{n-1}).$$

so that we get

$$\widetilde{p}^{\square n} \overline{\Delta}_{\widetilde{D}}^{n-1} \xi_n = 0. \tag{5.12}$$

Now, since $f_M \xi_1 = 0$, there is a unique morphism of C-bicomodules $\lambda : \widetilde{L} \to M$ such that $\lambda \widetilde{p} = f_M$. Thus:

$$f_M^{\square n} \overline{\Delta}_{\widetilde{D}}^{n-1} \xi_k = \lambda^{\square n} (\widetilde{p}^{\square n} \overline{\Delta}_{\widetilde{D}}^{n-1} \xi_n) \xi_k^n \overset{(5.12)}{=} 0, \text{ for any } k \leq n. \tag{5.13}$$

We conclude that $f_{n+1}\xi_n^{n+1} = \sigma_n^{n+1} f_n$ for any $n \in \mathbb{N}$ so that $(\sigma_n f_n : D^n \to T^c)_n$ is a compatible family of morphisms in $\mathcal{M}$. Thus there is a unique morphism $f : \widetilde{D} \to T^c$ such that

$$f \xi_k = \sigma_k f_k, \text{ for any } k \in \mathbb{N}.$$

We have that

$$p_k^n \sum_{t=0}^{k} i_t^k f_M^{\square t} \overline{\Delta}_{\widetilde{D}}^{t-1} \xi_k = f_M^{\square n} \overline{\Delta}_{\widetilde{D}}^{n-1} \xi_k \tag{5.14}$$

for any $0 \leq n < k$. Note that, for $k \leq n$, $p_k^n = 0$ and the right side of (5.14) is zero by (5.13). Thus, the relation above holds true for any $k, n \in \mathbb{N}$. Then we get:

$$p_n f \xi_k = p_n \sigma_k f_k = p_k^n f_k = p_k^n \sum_{t=0}^{k} i_t^k f_M^{\square t} \overline{\Delta}_{\widetilde{D}}^{t-1} \xi_k = f_M^{\square n} \overline{\Delta}_{\widetilde{D}}^{n-1} \xi_k, \text{ for any } k, n \in \mathbb{N}.$$

We conclude that $p_n f = f_M^{\square n} \overline{\Delta}_E^{n-1}$, for any $n \in \mathbb{N}$. In particular, for $n = 0, 1$, we get $p_0 f = \overline{\Delta}_{\widetilde{D}}^{-1} = f_C$ and $p_1 f = f_M \overline{\Delta}_{\widetilde{D}}^{0} = f_M$.
We have now to prove that f is a coalgebra homomorphism. Let us check that $f_n : D^n \to C^n$ is a coalgebra homomorphism for every $n \in \mathbb{N}$.
For $n = 0$, $f_n = 0$ and there is nothing to prove.
Assume $n \geq 1$. By Proposition 5.2.2, we get:

$$\begin{aligned}
& \Delta(n) f_n \\
= \;& \sum_{t=0}^{n} \Delta(n) i_t^n f_M^{\square t} \overline{\Delta}_{\widetilde{D}}^{t-1} \xi_n \\
= \;& \Delta(n) i_0^n f_M^{\square 0} \overline{\Delta}_{\widetilde{D}}^{-1} \xi_n + \Delta(n) i_1^n f_M^{\square 1} \overline{\Delta}_{\widetilde{D}}^{0} \xi_n + \sum_{t=2}^{n} \Delta(n) i_t^n f_M^{\square t} \overline{\Delta}_{\widetilde{D}}^{t-1} \xi_n \\
= \;& \Delta(n) i_0^n f_C \xi_n + \Delta(n) i_1^n f_M \xi_n + \sum_{t=2}^{n} \Delta(n) i_t^n f_M^{\square t} \overline{\Delta}_{\widetilde{D}}^{t-1} \xi_n \\
= \;& (i_0^n \otimes i_0^n) \Delta_C f_C \xi_n + \left[(i_1^n \otimes i_0^n) \rho_M^r + (i_0^n \otimes i_1^n) \rho_M^l \right] f_M \xi_n + \\
& + \sum_{t=2}^{n} \left[\begin{array}{c} (i_t^n \otimes i_0^n) \left(M^{\square t-1} \square \rho_M^r \right) + \\ + (i_0^n \otimes i_t^n) \left(\rho_M^l \square M^{\square t-1} \right) + \\ \sum_{r=1}^{t-1} (i_r^n \otimes i_{t-r}^n) (M^{\square r-1} \square \varsigma_M \square M^{\square t-1-r}) \end{array} \right] f_M^{\square t} \overline{\Delta}_{\widetilde{D}}^{t-1} \xi_n \\
= \;& (i_0^n f_C \otimes i_0^n f_C) \Delta_{\widetilde{D}} \xi_n + \left[(i_1^n \otimes i_0^n) \rho_M^r + (i_0^n \otimes i_1^n) \rho_M^l \right] f_M \xi_n + \\
& + \sum_{t=2}^{n} \left[(i_t^n \otimes i_0^n) \left(M^{\square t-1} \square \rho_M^r \right) + (i_0^n \otimes i_t^n) \left(\rho_M^l \square M^{\square t-1} \right) \right] f_M^{\square t} \overline{\Delta}_{\widetilde{D}}^{t-1} \xi_n + \\
& + \sum_{t=2}^{n} \sum_{r=1}^{t-1} (i_r^n \otimes i_{t-r}^n) (M^{\square r-1} \square \varsigma_M \square M^{\square t-1-r}) f_M^{\square t} \overline{\Delta}_{\widetilde{D}}^{t-1} \xi_n
\end{aligned}$$

$$
\begin{aligned}
&= (i_0^n f_C \otimes i_0^n f_C)\Delta_{\widetilde{D}}\xi_n + \\
&+ \sum_{t=1}^{n} \left[(i_t^n \otimes i_0^n)\,(M^{\Box t-1}\Box \rho_M^r) + (i_0^n \otimes i_t^n)\,(\rho_M^l \Box M^{\Box t-1}) \right] f_M^{\Box t}\overline{\Delta}_{\widetilde{D}}^{t-1}\xi_n + \\
&+ \sum_{t=2}^{n}\sum_{r=1}^{t-1} (i_r^n \otimes i_{t-r}^n)(M^{\Box r-1}\Box \varsigma_M \Box M^{\Box t-1-r}) f_M^{\Box t}\overline{\Delta}_{\widetilde{D}}^{t-1}\xi_n.
\end{aligned}
$$

Now:

$$
\begin{aligned}
& \sum_{t=1}^{n} (i_t^n \otimes i_0^n)\,(M^{\Box t-1}\Box \rho_M^r) f_M^{\Box t}\overline{\Delta}_{\widetilde{D}}^{t-1}\xi_n \\
= & \sum_{t=1}^{n} (i_t^n \otimes i_0^n)\,(f_M^{\Box t-1}\Box \rho_M^r f_M)\overline{\Delta}_{\widetilde{D}}^{t-1}\xi_n \\
= & \sum_{t=1}^{n} (i_t^n \otimes i_0^n)\,[f_M^{\Box t-1}\Box (f_M \otimes C)\rho_{\widetilde{D}}^r]\overline{\Delta}_{\widetilde{D}}^{t-1}\xi_n \\
= & \sum_{t=1}^{n} (i_t^n \otimes i_0^n)\,(f_M^{\Box t-1}\Box f_M \otimes C)(\widetilde{D}^{\Box t-1}\Box \rho_{\widetilde{D}}^r)\overline{\Delta}_{\widetilde{D}}^{t-1}\xi_n \\
= & \sum_{t=1}^{n} (i_t^n \otimes i_0^n)\,(f_M^{\Box t} \otimes C)[\widetilde{D}^{\Box t-1}\Box (\widetilde{D} \otimes f_C)\Delta_{\widetilde{D}}]\overline{\Delta}_{\widetilde{D}}^{t-1}\xi_n \\
= & \sum_{t=1}^{n} (i_t^n \otimes i_0^n)\,(f_M^{\Box t} \otimes f_C)[\widetilde{D}^{\Box t-1}\Box \Delta_{\widetilde{D}}]\overline{\Delta}_{\widetilde{D}}^{t-1}\xi_n \\
= & \sum_{t=1}^{n} (i_t^n \otimes i_0^n)\,(f_M^{\Box t} \otimes f_C)[\overline{\Delta}_{\widetilde{D}}^{t-1} \otimes \widetilde{D}]\Delta_{\widetilde{D}}\xi_n \\
= & \sum_{t=1}^{n} \left(i_t^n f_M^{\Box t}\overline{\Delta}_{\widetilde{D}}^{t-1} \otimes i_0^n f_C \right) \Delta_{\widetilde{D}}\xi_n
\end{aligned}
$$

Analogously one gets

$$
\sum_{t=1}^{n} (i_0^n \otimes i_t^n)\,(\rho_M^l \Box M^{\Box t-1}) f_M^{\Box t}\overline{\Delta}_{\widetilde{D}}^{t-1}\xi_n = \sum_{t=1}^{n} \left(i_0^n f_C \otimes i_t^n f_M^{\Box t}\overline{\Delta}_{\widetilde{D}}^{t-1} \right) \Delta_{\widetilde{D}}\xi_n
$$

Moreover we have:

$$
\begin{aligned}
& \sum_{t=2}^{n}\sum_{r=1}^{t-1}(i_r^n \otimes i_{t-r}^n)(M^{\square r-1}\square\varsigma_M\square M^{\square t-1-r})f_M^{\square t}\overline{\Delta}_{\widetilde{D}}^{t-1}\xi_n \\
= & \sum_{t=2}^{n}\sum_{r=1}^{t-1}(i_r^n \otimes i_{t-r}^n)(f_M^{\square r} \otimes f_M^{\square t-r})(\widetilde{D}^{\square r-1}\square\varsigma_{\widetilde{D}}\square\widetilde{D}^{\square t-1-r})\overline{\Delta}_{\widetilde{D}}^{t-1}\xi_n \\
= & \sum_{t=2}^{n}\sum_{r=1}^{t-1}(i_r^n f_M^{\square r} \otimes i_{t-r}^n f_M^{\square t-r})(\overline{\Delta}_{\widetilde{D}}^{r-1} \otimes \overline{\Delta}_{\widetilde{D}}^{t-1-r})\Delta_{\widetilde{D}}\xi_n \\
= & \sum_{t=2}^{n}\sum_{r=1}^{t-1}(i_r^n f_M^{\square r}\overline{\Delta}_{\widetilde{D}}^{r-1} \otimes i_{t-r}^n f_M^{\square t-r}\overline{\Delta}_{\widetilde{D}}^{t-1-r})\Delta_{\widetilde{D}}\xi_n
\end{aligned}
$$

So we get

$$
\begin{aligned}
\Delta(n)f_n = \; & (i_0^n f_C \otimes i_0^n f_C)\Delta_{\widetilde{D}}\xi_n + \\
& + \sum_{t=1}^{n}\left(i_t^n f_M^{\square t}\overline{\Delta}_{\widetilde{D}}^{t-1} \otimes i_0^n f_C\right)\Delta_{\widetilde{D}}\xi_n + \\
& + \sum_{t=1}^{n}\left(i_0^n f_C \otimes i_t^n f_M^{\square t}\overline{\Delta}_{\widetilde{D}}^{t-1}\right)\Delta_{\widetilde{D}}\xi_n + \\
& + \sum_{t=2}^{n}\sum_{r=1}^{t-1}(i_r^n f_M^{\square r}\overline{\Delta}_{\widetilde{D}}^{r-1} \otimes i_{t-r}^n f_M^{\square t-r}\overline{\Delta}_{\widetilde{D}}^{t-1-r})\Delta_{\widetilde{D}}\xi_n
\end{aligned}
$$

On the other hand we have

$$
\begin{aligned}
& (f_n \otimes f_n)\Delta_{D^n} \\
= & (\sum_{t=0}^{n} i_t^n f_M^{\square t}\overline{\Delta}_{\widetilde{D}}^{t-1}\xi_n \otimes \sum_{k=0}^{n} i_k^n f_M^{\square k}\overline{\Delta}_{\widetilde{D}}^{k-1}\xi_n)\Delta_{D^n} \\
= & (\sum_{t=0}^{n} i_t^n f_M^{\square t}\overline{\Delta}_{\widetilde{D}}^{t-1} \otimes \sum_{k=0}^{n} i_k^n f_M^{\square k}\overline{\Delta}_{\widetilde{D}}^{k-1})\Delta_{\widetilde{D}}\xi_n \\
= & (i_0^n f_M^{\square 0}\overline{\Delta}_{\widetilde{D}}^{-1} \otimes i_0^n f_M^{\square 0}\overline{\Delta}_{\widetilde{D}}^{-1})\Delta_{\widetilde{D}}\xi_n + \\
& + \sum_{t=1}^{n}(i_t^n f_M^{\square t}\overline{\Delta}_{\widetilde{D}}^{t-1} \otimes i_0^n f_M^{\square 0}\overline{\Delta}_{\widetilde{D}}^{-1})\Delta_{\widetilde{D}}\xi_n + \\
& + \sum_{k=1}^{n}(i_0^n f_M^{\square 0}\overline{\Delta}_{\widetilde{D}}^{0-1} \otimes i_k^n f_M^{\square k}\overline{\Delta}_{\widetilde{D}}^{k-1})\Delta_{\widetilde{D}}\xi_n \\
& +(\sum_{t=1}^{n} i_t^n f_M^{\square t}\overline{\Delta}_{\widetilde{D}}^{t-1} \otimes \sum_{k=1}^{n} i_k^n f_M^{\square k}\overline{\Delta}_{\widetilde{D}}^{k-1})\Delta_{\widetilde{D}}\xi_n
\end{aligned}
$$

$$\begin{aligned}
&= (i_0^n f_C \otimes i_0^n f_C)\Delta_{\widetilde{D}}\xi_n + \sum_{t=1}^{n}(i_t^n f_M^{\Box t}\overline{\Delta}_{\widetilde{D}}^{t-1} \otimes i_0^n f_C)\Delta_{\widetilde{D}}\xi_n + \\
&\quad + \sum_{k=1}^{n}(i_0^n f_C \otimes i_k^n f_M^{\Box k}\overline{\Delta}_{\widetilde{D}}^{k-1})\Delta_{\widetilde{D}}\xi_n + \\
&\quad + \sum_{t=1}^{n}\sum_{k=1}^{n}(i_t^n f_M^{\Box t}\overline{\Delta}_{\widetilde{D}}^{t-1} \otimes i_k^n f_M^{\Box k}\overline{\Delta}_{\widetilde{D}}^{k-1})\Delta_{\widetilde{D}}\xi_n \\
&= \Delta(n)f_n - \sum_{t=2}^{n}\sum_{r=1}^{t-1}(i_r^n f_M^{\Box r}\overline{\Delta}_{\widetilde{D}}^{r-1} \otimes i_{t-r}^n f_M^{\Box t-r}\overline{\Delta}_{\widetilde{D}}^{t-1-r})\Delta_{\widetilde{D}}\xi_n + \\
&\quad + \sum_{t=1}^{n}\sum_{k=1}^{n}(i_t^n f_M^{\Box t}\overline{\Delta}_{\widetilde{D}}^{t-1} \otimes i_k^n f_M^{\Box k}\overline{\Delta}_{\widetilde{D}}^{k-1})\Delta_{\widetilde{D}}\xi_n.
\end{aligned}$$

Now

$$\begin{aligned}
&(i_t^n f_M^{\Box t}\overline{\Delta}_{\widetilde{D}}^{t-1} \otimes i_k^n f_M^{\Box k}\overline{\Delta}_{\widetilde{D}}^{k-1})\Delta_{\widetilde{D}}\xi_n \\
&= (i_t^n f_M^{\Box t} \otimes i_k^n f_M^{\Box k})(\overline{\Delta}_{\widetilde{D}}^{t-1} \otimes \overline{\Delta}_{\widetilde{D}}^{k-1})\Delta_{\widetilde{D}}\xi_n \\
&= (i_t^n f_M^{\Box t} \otimes i_k^n f_M^{\Box k})(\widetilde{D}^{\Box t-1}\Box\varsigma_{\widetilde{D}}\Box\widetilde{D}^{\Box k-1})\overline{\Delta}_{\widetilde{D}}^{t+k-1}\xi_n \\
&= (i_t^n \otimes i_k^n)(f_M^{\Box t} \otimes f_M^{\Box k})(\widetilde{D}^{\Box t-1}\Box\varsigma_{\widetilde{D}}\Box\widetilde{D}^{\Box k-1})\overline{\Delta}_{\widetilde{D}}^{t+k-1}\xi_n \\
&= (i_t^n \otimes i_k^n)(M^{\Box t-1}\Box\varsigma_M\Box M^{\Box k-1})f_M^{\Box t+k}\overline{\Delta}_{\widetilde{D}}^{t+k-1}\xi_n.
\end{aligned}$$

By (5.13), the last term is zero whenever $t+k>n$, so that:

$$\begin{aligned}
&\sum_{t=1}^{n}\sum_{k=1}^{n}(i_t^n f_M^{\Box t}\overline{\Delta}_{\widetilde{D}}^{t-1} \otimes i_k^n f_M^{\Box k}\overline{\Delta}_{\widetilde{D}}^{k-1})\Delta_{\widetilde{D}}\xi_n \\
&= \sum_{\substack{1\le t,k\le n \\ t+k\le n}}(i_t^n f_M^{\Box t}\overline{\Delta}_{\widetilde{D}}^{t-1} \otimes i_k^n f_M^{\Box k}\overline{\Delta}_{\widetilde{D}}^{k-1})\Delta_{\widetilde{D}}\xi_n \\
&= \sum_{t=2}^{n}\sum_{r=1}^{t-1}(i_r^n f_M^{\Box r}\overline{\Delta}_{\widetilde{D}}^{r-1} \otimes i_{t-r}^n f_M^{\Box t-r}\overline{\Delta}_{\widetilde{D}}^{t-1-r})\Delta_{\widetilde{D}}\xi_n,
\end{aligned}$$

and hence

$$(f_n \otimes f_n)\Delta_{D^n} = \Delta(n)f_n.$$

Furthermore $\varepsilon(0)f_0 = 0 = \varepsilon_{D^0}$, while, for every $n \geq 1$, we have

$$\begin{aligned}
\varepsilon(n)f_n &= \sum_{t=0}^{n} \varepsilon(n) i_t^n f_M^{\square t} \overline{\Delta}_{\widetilde{D}}^{t-1} \xi_n \\
&= \sum_{t=0}^{n} \varepsilon_C \pi_n^1 i_t^n f_M^{\square t} \overline{\Delta}_{\widetilde{D}}^{t-1} \xi_n \\
&= \varepsilon_C \pi_n^1 i_0^n f_M^{\square 0} \overline{\Delta}_{\widetilde{D}}^{0-1} \xi_n \\
&= \varepsilon_C i_0^1 f_C \xi_n = \varepsilon_C f_C \xi_n = \varepsilon_{D^n}.
\end{aligned}$$

We conclude that f_n is a coalgebra homomorphism. Now, by construction, f is the unique morphism such that $f\xi_k = \sigma_k f_k$, for any $k \in \mathbb{N}$. By Proposition 1.6.3, $((D^{\wedge^i_C})_{i\in\mathbb{N}}, (\xi_i^j)_{i,j\in\mathbb{N}})$ is a direct system in $\mathcal{M}$ whose direct limit $\widetilde{D}$ carries a natural coalgebra structure that makes it the direct limit of $((D^{\wedge^i_C})_{i\in\mathbb{N}}, (\xi_i^j)_{i,j\in\mathbb{N}})$ as a direct system of coalgebras. Since σ_k is a coalgebra homomorphism so is $\sigma_k f_k$ and hence f is a coalgebra homomorphism.
Assume now that $g : E \to T^c$ is another coalgebra homomorphism such that $p_0 g = f_C$ and $p_1 g = f_M$. Then, by Proposition 5.3.2, we have $p_n g = p_n f$ for any $n \in \mathbb{N}$. □

***Lemma* 5.3.4.** *Let $(X_i)_{i\in\mathbb{N}}$ be a family of objects in a cocomplete and complete abelian category $\mathcal{M}$ satisfying AB5. Let Y be an object in $\mathcal{M}$ and $f : Y \to \oplus_{i\in\mathbb{N}} X_i$ be a morphism such that*

$$p_k f = 0 \text{ for any } k \in \mathbb{N},$$

where $p_k : \oplus X_i \to X_k$ denotes the canonical projection. Then $f = 0$.

Proof. By [Po, Corollary 8.10, page 61], $\mathcal{M}$ is a C_2-category so that the conclusion follows by [Po, Proposition, page 54]. □

***Corollary* 5.3.5.** *Let (C, Δ, ε) be a coalgebra in a cocomplete complete coabelian monoidal category $\mathcal{M}$ satisfying AB5 and let (M, ρ_M^r, ρ_M^l) be a C-bicomodule. Let $T^c := T_C^c(M)$. Let E be a coalgebra and let $\alpha : E \to T^c$ and $\beta : E \to T^c$ be coalgebra homomorphisms.*
Then $\alpha = \beta$ whenever $p_n\alpha = p_n\beta$, for $n = 0, 1$.

Proof. follows by Theorem 5.3.2 and Lemma 5.3.4. □

***Theorem* 5.3.6** (The universal property of cotensor coalgebra)**.** *[AMS2, Theorem 2.15] Let (C, Δ, ε) be a coalgebra in a cocomplete and complete abelian monoidal category $\mathcal{M}$ satisfying AB5. Let (M, ρ_M^r, ρ_M^l) be a C-bicomodule. Let $\delta : D \to E$ be a monomorphism which is a homomorphism of coalgebras. Let $f_C : \widetilde{D} \to C$ be a coalgebra homomorphism and let $f_M : \widetilde{D} \to M$ be a morphism of C-bicomodules such that $f_M\xi_1 = 0$, where $\widetilde{D}$ is a bicomodule via f_C. Then there is a unique coalgebra* *homomorphism* $f : \widetilde{D} \to T_C^c(M)$ *such that* $p_0 f = f_C$ *and* $p_1 f = f_M$*, where* $p_n : T_C^c(M) \to M^{\Box n}$ *denotes the canonical projection.*

$$\begin{tikzcd}
T_C^c(M) \arrow[r, "p_1"] \arrow[d, "p_0"'] & M & \\
C & \widetilde{D} \arrow[l, "f_C"] \arrow[u, "f_M"'] \arrow[ul, dotted, "f"] & D \arrow[l, "\xi_1"] \arrow[ul, "0"']
\end{tikzcd}$$

Proof. Since $\mathcal{M}$ satisfies AB5, the morphism $\widetilde{\delta} : \widetilde{D} \to E$ of Notation 1.6.4 is a monomorphism, so that, by applying Theorem 5.3.3, there is a coalgebra homomorphism $f : \widetilde{D} \to T_C^c(M)$ such that $p_0 f = f_C$ and $p_1 f = f_M$. The uniqueness follows by Corollary 5.3.5. $\square$

The following result describes completely the wedge powers of C as a subcoalgebra of the cotensor coalgebra $T_C^c(M)$, where M is a C-bicomodule.

***Theorem* 5.3.7.** *[AMS2, Theorem 2.18] Let* (C, Δ, ε) *be a coalgebra in a cocomplete and complete coabelian monoidal category* $\mathcal{M}$ *satisfying AB*5*. Let* (M, ρ_M^r, ρ_M^l) *be a* C*-bicomodule. Let* $T^c := T_C^c(M)$*. Then*

$$(C^n(M), \sigma_n) = C^{\wedge_{T^c}^n},$$

for every $n \in \mathbb{N}$.

Proof. Let $n \in \mathbb{N}$. Let

$$\lambda_a^n : M^{\Box a} \to \oplus_{b \geq n} M^{\Box b}$$

be defined by

$$\lambda_a^n = \begin{cases} \text{the canonical injection} & \text{if } a \geq n, \\ 0 & \text{otherwise.} \end{cases}$$

Define

$$\nu_n : \oplus_{a \geq n} M^{\Box a} \to T^c$$

as the codiagonal map of the family $(i_a)_{a \geq n}$ so that we have

$$\nu_n \circ \lambda_a^n = \begin{cases} i_a & \text{for every } a \geq n \\ 0 & \text{otherwise.} \end{cases}$$

Define

$$\tau_n : T^c \to \oplus_{a \geq n} M^{\Box a}$$

as the codiagonal map of the family $(\lambda_a^n)_{a \in \mathbb{N}}$, that is

$$\tau_n i_a = \lambda_a^n, \text{ for every } a \in \mathbb{N}.$$

Thus, we have
$$\tau_n\nu_n\lambda_a^n = \tau_n i_a = \lambda_a^n \text{ for } a \geq n,$$
so that
$$\tau_n\nu_n = \mathrm{Id}_{\oplus_{a\geq n}M^{\square a}} \text{ for every } n \in \mathbb{N}.$$
Let $C^n = C^n(M)$ for every $n \in \mathbb{N}$. Let us prove the following sequence
$$0 \to C^n \xrightarrow{\sigma_n} T^c \xrightarrow{\tau_n} \oplus_{a\geq n}M^{\square a} \to 0$$
is exact. We check that $\left(\oplus_{a\geq n}M^{\square a}, \tau_n\right) = \mathrm{Coker}\,(\sigma_n)$.
Since $\tau_n\nu_n = \mathrm{Id}$, it is clear that τ_n is an epimorphism and that ν_n is a monomorphism. From
$$\tau_n\sigma_n i_a^n = \tau_n i_a = \lambda_a^n = 0, \text{ for every } 0 \leq a \leq n-1,$$
we deduce that
$$\tau_n\sigma_n = 0 \text{ for every } n \in \mathbb{N}.$$
Let $f : T^c \to X$ be a morphism such that $f\sigma_n = 0$ for every $n \in \mathbb{N}$. Thus, for every $0 \leq a \leq n-1$, we have
$$f i_a = f\sigma_n i_a^n = 0$$
Set
$$\overline{f} = f\nu_n$$
and let us prove that $f = \overline{f}\tau_n$. From
$$\overline{f}\tau_n i_a = f\nu_n\lambda_a^n = \begin{cases} f i_a & \text{for every } a \geq n \\ 0 & \text{otherwise.} \end{cases}$$
we deduce that $\overline{f}\tau_n i_a = f i_a$, for every $a \in \mathbb{N}$, and hence $\overline{f}\tau_n = f$.

Let us prove that $C^n = C^{\wedge^n_{T^c}}$, for every $n \in \mathbb{N}$.
The case $n = 0$ is trivial. Let us prove the equality above for every $n \geq 1$ by induction on n.
If $n = 1$, by definition, we have $C^1 = C = C^{\wedge^1_{T^c}}$.
Let $n \geq 2$ and assume that $C^{\wedge^{n-1}_{T^c}} = C^{n-1}$. By Proposition 1.5.8 and Lemma 1.5.3, we have
$$C^{\wedge^n_{T^c}} = C^{\wedge^{n-1}_{T^c}} \wedge_{T^c} C^{\wedge^1_{T^c}} = C^{n-1}\wedge_{T^c}C^1 = \mathrm{Ker}\,(\tau_{n-1})\wedge_{T^c}\mathrm{Ker}\,(\tau_1) = \mathrm{Ker}\,[(\tau_{n-1}\otimes\tau_1)\,\Delta_{T^c}]$$
so that
$$0 \to C^{\wedge^n_{T^c}} \hookrightarrow T^c \xrightarrow{(\tau_{n-1}\otimes\tau_1)\Delta_{T^c}} \left(\oplus_{a\geq n-1}M^{\square a}\right) \otimes \left(\oplus_{a\geq 1}M^{\square a}\right)$$
is an exact sequence. In order to conclude, it is enough to check that the following sequence
$$0 \to C^n \xrightarrow{\sigma_n} T^c \xrightarrow{(\tau_{n-1}\otimes\tau_1)\Delta_{T^c}} \left(\oplus_{a\geq n-1}M^{\square a}\right) \otimes \left(\oplus_{a\geq 1}M^{\square a}\right)$$

is exact. For every $0 \leq a \leq n-1$, we have

$$(\tau_{n-1} \otimes \tau_1)\, \Delta_{T^c} \sigma_n i_a^n = (\tau_{n-1} \otimes \tau_1)\, (\sigma_n \otimes \sigma_n)\, \Delta\,(n)\, i_a^n$$

By Proposition 5.2.2 we can write

$$\Delta\,(n)\, i_a^n = \sum_{r=0}^{a} (i_r^n \otimes i_{a-r}^n) f_{r,a-r}$$

where $f_{i,j} : M^{\square a} \to M^{\square i} \otimes M^{\square j}$ are suitable morphisms. Thus we get

$$\begin{aligned}
(\tau_{n-1} \otimes \tau_1)\, \Delta_{T^c} \sigma_n i_a^n &= \sum_{r=0}^{a} (\tau_{n-1} \otimes \tau_1)\, (\sigma_n \otimes \sigma_n)\, (i_r^n \otimes i_{a-r}^n) f_{r,a-r} \\
&= \sum_{r=0}^{a} (\tau_{n-1} i_r \otimes \tau_1 i_{a-r})\, f_{r,a-r} = \sum_{r=0}^{a} \left(\lambda_r^{n-1} \otimes \lambda_{a-r}^1\right) f_{r,a-r} = 0
\end{aligned}$$

Therefore $(\tau_{n-1} \otimes \tau_1)\, \Delta_{T^c} \sigma_n = 0$, for every $n \in \mathbb{N}$.
Let $g : Y \to T^c$ be a morphism such that

$$(\tau_{n-1} \otimes \tau_1)\, \Delta_{T^c} g = 0.$$

Now, for every $c \in \mathbb{N}$ and for every $a \geq b$, we have

$$p_a \nu_b \tau_b i_c = p_a \nu_b \lambda_c^b = \begin{cases} p_a i_c & \text{for every } c \geq b \\ 0 = p_a i_c & \text{otherwise.} \end{cases}$$

so that

$$p_a \nu_b \tau_b = p_a, \ \text{for every } a \geq b.$$

Thus, for every $a \geq n-1$ and $b \geq 1$, by Lemma 5.3.1, we have

$$0 = (p_a \nu_{n-1} \tau_{n-1} \otimes p_b \nu_1 \tau_1)\, \Delta_{T^c} g = (p_a \otimes p_b)\, \Delta_{T^c} g = \left(M^{\square a-1} \square \varsigma_M \square M^{\square b}\right) p_{a+b} g.$$

By left exactness of the tensor functors, $M^{\square a-1} \square \varsigma_M \square M^{\square b}$ is a monomorphism so that

$$p_{a+b} g = 0.$$

We conclude that

$$p_c g = 0, \ \text{for every } c \geq n.$$

Set

$$\overline{g} = \pi_n g.$$

and let us prove that $g = \sigma_n \overline{g}$. By Lemma 5.3.4 this is the case if and only if

$$p_a g = p_a \sigma_n \overline{g}, \ \text{for every } a \in \mathbb{N}.$$

We have

$$p_a \sigma_n \overline{g} = p_n^a \pi_n g = \begin{cases} p_a g & \text{for every } a < n \\ 0 = p_a g & \text{otherwise.} \end{cases}$$

$\square$

5.4 Formal smoothness

The main aim of this section is to prove Theorem 5.4.8 which is the "cotensor" analogue of Theorem 4.2.1.

***Proposition* 5.4.1.** *[AMS2, Proposition 3.4] Let $i \in \{1,2\}$. Let $f_i : X_i \to Y_i$ be morphisms in an abelian monoidal category $\mathcal{M}$. Let $\sigma_i : Y_i \to X_i$ such that $f_i\sigma_i = \mathrm{Id}_{Y_i}$. Then*

$$Ker(f_1 \otimes f_2) = [\mathrm{Ker}\,(f_1) \otimes X_2] + [X_1 \otimes \mathrm{Ker}\,(f_2)].$$

Proof. Let $(K_i, k_i) = \mathrm{Ker}\,(f_i)$ for $i = 1,2$. Let $\nu_1 : K_1 \otimes X_2 \to (K_1 \otimes X_2) \oplus (X_1 \otimes K_2)$ and $\nu_2 : X_1 \otimes K_2 \to (K_1 \otimes X_2) \oplus (X_1 \otimes K_2)$ be the canonical inclusions. Then, by the universal property of coproducts, there is a unique morphism $\tau : (K_1 \otimes X_2) \oplus (X_1 \otimes K_2) \to X_1 \otimes X_2$ such that

$$\tau\nu_1 = k_1 \otimes X_2 \qquad \text{and} \qquad \tau\nu_2 = X_1 \otimes k_2. \tag{5.15}$$

By definition, one has $(K_1 \otimes X_2) + (X_1 \otimes K_2) = \mathrm{Im}(\tau) = \mathrm{Ker}\,(\pi)$, where $(C, \pi) = \mathrm{coker}(\tau)$.
Thus, in order to prove our statement, we will show that $(C, \pi) = (Y_1 \otimes Y_2, f_1 \otimes f_2)$.
By (5.15), we have

$$\begin{aligned} (f_1 \otimes f_2)\tau\nu_1 &= (f_1 \otimes f_2)(k_1 \otimes X_2) = 0, \\ (f_1 \otimes f_2)\tau\nu_2 &= (f_1 \otimes f_2)(X_1 \otimes k_2) = 0, \end{aligned}$$

so that $(f_1 \otimes f_2)\tau = 0$. By the universal property of cokernels, we obtain a unique morphism $\alpha : C \to Y_1 \otimes Y_2$ such that $\alpha\pi = f_1 \otimes f_2$.
Define $\beta : Y_1 \otimes Y_2 \to C$ by $\beta := \pi(\sigma_1 \otimes \sigma_2)$. Let us prove that β is a two-sided inverse of α. Clearly one has

$$\alpha\beta = \alpha\pi(\sigma_1 \otimes \sigma_2) = (f_1 \otimes f_2)(\sigma_1 \otimes \sigma_2) = Id_{X_1 \otimes X_2}.$$

Now, since $f_i\sigma_i = \mathrm{Id}_{Y_i}$, there is a unique morphism $\rho_i : X_i \to K_i$ such that $\rho_i k_i = \mathrm{Id}_{K_i}$ and

$$k_i\rho_i + \sigma_i f_i = \mathrm{Id}_{X_i}, \text{ for any } i \in \{1,2\}. \tag{5.16}$$

Then we have:

$$\begin{aligned}
\beta\alpha\pi &= \beta(f_1 \otimes f_2)\\
&= \pi(\sigma_1 f_1 \otimes \sigma_2 f_2)\\
&\overset{(5.16)}{=} \pi[\sigma_1 f_1 \otimes (\mathrm{Id}_{X_2} - k_2\rho_2)]\\
&= \pi(\sigma_1 f_1 \otimes \mathrm{Id}_{X_2}) - \pi(\sigma_1 f_1 \otimes k_2\rho_2)\\
&= \pi[(\mathrm{Id}_{X_1} - k_1\rho_1) \otimes \mathrm{Id}_{X_2}] - \pi(X_1 \otimes k_2)(\sigma_1 f_1 \otimes \rho_2)\\
&\overset{(5.15)}{=} \pi - \pi(k_1\rho_1 \otimes \mathrm{Id}_{X_2}) - \pi\tau\nu_2(\sigma_1 f_1 \otimes \rho_2)\\
&= \pi - \pi(k_1 \otimes \mathrm{Id}_{X_2})(\rho_1 \otimes \mathrm{Id}_{X_2})\\
&\overset{(5.15)}{=} \pi - \pi\tau\nu_1(\rho_1 \otimes \mathrm{Id}_{X_2})\\
&= \pi
\end{aligned}$$

Since π is an epimorphism we conclude that $\beta\alpha = \mathrm{Id}_C$ and hence that α is an isomorphism. Thus $(C,\pi) = (Y_1 \otimes Y_2, f_1 \otimes f_2)$. $\square$

***Proposition* 5.4.2.** *Let $\delta : D \to C$ be a morphism that cosplits in $\mathcal{M}$. If δ is a coalgebra homomorphism, then we have*

$$D^{\wedge_C^2} = D \wedge_C D = \Delta_C^{-1}(D \otimes C + C \otimes D). \tag{5.17}$$

Proof. Set $(L,p) = \mathrm{coker}(\sigma)$. Let

$$(D^{\wedge_C^n}, \delta_n) := \mathrm{Ker}\,(p^{\otimes n}\Delta_C^{n-1}).$$

We have

$$\begin{aligned}
D^{\wedge_C^2} &= \mathrm{Ker}\,[(p \otimes p)\Delta_C]\\
&\overset{(*)}{=} \Delta_C^{-1}[\mathrm{Ker}\,(p \otimes p)]\\
&\overset{(**)}{=} \Delta_C^{-1}\{[\mathrm{Ker}\,(p) \otimes C] + [C \otimes \mathrm{Ker}\,(p)]\} = \Delta_C^{-1}[(D \otimes C) + (C \otimes D)].
\end{aligned}$$

where in (*) we have applied ([St, Proposition 5.1, page 90]) and in (**) Proposition 5.4.1. $\square$

Proof. Since

$$p_i'\sigma\alpha_i = p_i'\alpha_i'\sigma_i = 0,$$

by the universal property of cokernels, there is a unique morphism $\tau_i : L_i \to L_i'$ such that $\tau_i p_i = p_i'\sigma$. Then we have:

$$(p_1' \otimes p_2')\Delta_{A'}\sigma\lambda_{1,2} = (p_1' \otimes p_2')(\sigma \otimes \sigma)\Delta_A\lambda_{1,2} = (\tau_1 \otimes \tau_2)(p_1 \otimes p_2)\Delta_A\lambda_{1,2} = 0.$$

By the universal property of kernels, there is a unique morphism $\lambda : X_1 \wedge_A X_2 \to X_1' \wedge_{A'} X_2'$ such that $\lambda_{1,2}'\lambda = \sigma\lambda_{1,2}$. Clearly, as σ and $\lambda_{1,2}$ are monomorphisms, λ is a monomorphism too. $\square$

Lemma 5.4.3. *Let $((X_i)_{i\in\mathbb{N}},(\xi_i^j)_{i,j\in\mathbb{N}})$ and let $((Y_i)_{i\in\mathbb{N}},(\zeta_i^j)_{i,j\in\mathbb{N}})$ be direct systems in a monoidal category $\mathcal{M}$, where, for $i\leq j$, $\xi_i^j : X_i \to X_j$ and $\zeta_i^j : Y_i \to Y_j$. Let $\sigma : A \to B$ be a coalgebra homomorphism and let $(\alpha_i : X_i \to A)_{i\in\mathbb{N}}$ and $(\beta_i : Y_i \to B)_{i\in\mathbb{N}}$ be compatible families of morphisms in $\mathcal{M}$. Let $\lambda_i : X_i \to Y_i$ be a morphism such that $\beta_i\lambda_i = \sigma\alpha_i$, for any $i\in\mathbb{N}$. If β_i is a monomorphism, for any $i\in\mathbb{N}$, we have that $(\lambda_i : X_i \to Y_i)_{i\in\mathbb{N}}$ is a direct system of morphisms in $\mathcal{M}$.*

Proof. For any $i\leq j$, we have that:

$$\beta_j\lambda_j\xi_i^j = \sigma\alpha_j\xi_i^j = \sigma\alpha_i = \beta_i\lambda_i = \beta_j\zeta_i^j\lambda_i.$$

Since β_j is a monomorphism for any $j\in\mathbb{N}$, we conclude that $\lambda_j\xi_i^j = \zeta_i^j\lambda_i$ i.e. that $(\lambda_i : X_i \to Y_i)_{i\in\mathbb{N}}$ is a direct system of morphisms in $\mathcal{M}$. □

Lemma 5.4.4. *Let $\mathcal{M}$ be a cocomplete monoidal category with left exact direct limits. Let $((X_i)_{i\in\mathbb{N}},(\xi_i^j)_{i,j\in\mathbb{N}})$ be a direct system in $\mathcal{M}$, where, for $i\leq j$, $\xi_i^j : X_i \to X_j$. Let $\gamma : \mathbb{N}\to\mathbb{N}$ be an injection. Then $((X_{\gamma(i)})_{i\in\mathbb{N}},(\xi_{\gamma(i)}^{\gamma(j)})_{i,j\in\mathbb{N}})$ is a direct system in $\mathcal{M}$. Let $(X,(\lambda_i)_{i\in\mathbb{N}}) = \varinjlim X_{\gamma(i)}$, where $\lambda_i : X_{\gamma(i)} \to X$ for any $i\in\mathbb{N}$. Then $(X,(\xi_i)_{i\in\mathbb{N}}) = \varinjlim X_i$, where $\xi_i : X_i \to X$ is defined by $\xi_i := \lambda_j\xi_i^{\gamma(j)} : X_i \to X$, where $j\in\mathbb{N}$ is such that $\gamma(j)\geq i$.*

Proof. Clearly $((X_{\gamma(i)})_{i\in\mathbb{N}},(\xi_{\gamma(i)}^{\gamma(j)})_{i,j\in\mathbb{N}})$ is a direct system. Let us prove the last assertion. Let $j,j'\in\mathbb{N}$ such that $\gamma(j')\geq\gamma(j)\geq i$. Then

$$\lambda_{j'}\xi_i^{\gamma(j')} = \lambda_{j'}\xi_{\gamma(j)}^{\gamma(j')}\xi_i^{\gamma(j)} = \lambda_j\xi_i^{\gamma(j)},$$

so that ξ_i is well defined. Note that

$$\xi_{\gamma(j)} = \lambda_j\xi_{\gamma(j)}^{\gamma(j)} = \lambda_j. \tag{5.18}$$

Moreover, for any $i\leq j$, and for any $t\in\mathbb{N}$ such that $\gamma(t)\geq j$ we have:

$$\xi_j\xi_i^j = \lambda_t\xi_j^{\gamma(t)}\xi_i^j = \lambda_t\xi_i^{\gamma(t)} = \xi_i,$$

so that $(\xi_i : X_i \to X)_{i\in\mathbb{N}}$ is a direct system of morphisms. Let now $(f_i : X_i \to Y)_{i\in\mathbb{N}}$ be a compatible family of morphisms in $\mathcal{M}$. Then $(f_{\gamma(i)} : X_{\gamma(i)} \to Y)_{i\in\mathbb{N}}$ is a compatible family of morphisms in $\mathcal{M}$ so that there exists a unique morphism $f : X \to Y$ such that $f\lambda_i = f_{\gamma(i)}$ for any $i\in\mathbb{N}$. For any $i\in\mathbb{N}$ and for any $j\in\mathbb{N}$ is such that $\gamma(j)\geq i$,we obtain

$$f\xi_i = f\lambda_j\xi_i^{\gamma(j)} = f_{\gamma(i)}\xi_i^{\gamma(j)} = f_i.$$

Let $g : X \to Y$ be another morphism such that $g\xi_i = f_i$. Then we have

$$g\lambda_i \overset{(5.18)}{=} g\xi_{\gamma(i)} = f_{\gamma(i)}.$$

By uniqueness of f we get $g = f$, so that $(X,(\xi_i)_{i\in\mathbb{N}}) = \varinjlim X_i$. □

Lemma 5.4.5. *Let $(\mathcal{M}, \otimes, \mathbf{1})$ be a cocomplete abelian monoidal category satisfying $AB5$, with left exact direct limits and left and right exact tensor functors. Let $(\mathbb{I}, \leq)$ be a directed partially ordered set. Let $((X_i)_{i\in\mathbb{I}}, (\xi_i^j)_{i,j\in\mathbb{I}})$ be a direct system in $\mathcal{M}$, where, for $i \leq j$, $\xi_i^j : X_i \to X_j$. Let $(w_i : X_i \to W)_{i\in\mathbb{I}}$ be a compatible family of monomorphisms in $\mathcal{M}$. Let $(X, (\xi_i)_{i\in\mathbb{I}}) = \varinjlim X_i$. Let $w : X \to W$ be the unique morphism such that $w\xi_i = w_i$ for every i. Let $\xi : \oplus X_i \to X$ be the unique morphism such that $\xi\varepsilon_i = \xi_i$ for any $i \in \mathbb{I}$ and let $\omega : \oplus X_i \to W$ be the unique morphism such that $\omega\varepsilon_i = w_i$ for any $i \in \mathbb{I}$, where $\varepsilon_i : X_i \to \oplus X_i$ is the canonical inclusion. Then:*

$$w\xi = \omega.$$

Moreover w is a monomorphism and ξ is an epimorphism.

Proof. Since $w\xi_i = w_i$, the ξ_i's are monomorphisms. Clearly we have

$$w\xi\varepsilon_i = w\xi_i = w_i = \omega\varepsilon_i \text{ for any } i \in \mathbb{I}$$

and hence

$$w\xi = \omega$$

Moreover, regarding $(w_i : X_i \to W)_{i\in\mathbb{I}}$ as a direct system of monomorphism, in view of $AB5$, we have that w is a monomorphism and ξ is an epimorphism. □

Lemma 5.4.6. *Let $(\mathcal{M}, \otimes, \mathbf{1})$ be a cocomplete abelian monoidal category satisfying $AB5$ and with left and right exact tensor functors. Let $(\mathbb{I}, \leq)$ be a directed partially ordered set. Let $((X_i)_{i\in\mathbb{I}}, (\xi_i^j)_{i,j\in\mathbb{I}})$ be a direct system in $\mathcal{M}$, where, for $i \leq j$, $\xi_i^j :$ $X_i \to X_j$. If $\oplus X_i$ commutes with $\otimes$, then $\varinjlim X_i$ does.*

Proof. Let $((X_i)_{i\in\mathbb{I}}, (\xi_i^j)_{i,j\in\mathbb{I}})$ be a direct system in $\mathcal{M}$, where, for $i \leq j$, $\xi_i^j : X_i \to X_j$. Let $(X, (\xi_i)_{i\in\mathbb{I}}) = \varinjlim X_i$ and let Y be an object in $\mathcal{M}$. By [St, Lemma 1.2, page 115], the ξ_i's are monomorphisms. Also, by the universal property of the coproduct, there is a unique morphism $\xi : \oplus X_i \to X$ such that

$$\xi\varepsilon_i = \xi_i \text{ for any } i \in \mathbb{I} \tag{5.19}$$

where $\varepsilon_i : X_i \to \oplus X_i$ is the canonical inclusion. Moreover, ξ is an epimorphism. Assume that

$$((\oplus X_i) \otimes Y, \varepsilon_i \otimes Y) = \oplus(X_i \otimes Y), \tag{5.20}$$

Let $\gamma_i : X_i \otimes Y \to \varinjlim(X_i \otimes Y)$ be the canonical morphism. By the universal property of coproduct and by (5.20), there is a unique morphism $\gamma : (\oplus X_i) \otimes Y \to \varinjlim(X_i \otimes Y)$ such that

$$\gamma(\varepsilon_i \otimes Y) = \gamma_i \text{ for any } i \in \mathbb{I}. \tag{5.21}$$

In an analogous way, by the universal property of direct limits, there is a unique morphism $\Lambda : \varinjlim(X_i \otimes Y) \to X \otimes Y$ such that

$$\Lambda\gamma_i = \xi_i \otimes Y \text{ for any } i \in \mathbb{I}. \tag{5.22}$$

It is easy to see that we can apply Lemma 5.4.5 to the present situation and get:

$$\Lambda\gamma = \xi \otimes Y.$$

where Λ is a monomorphism and γ is an epimorphism. Moreover, since the tensor functor is left exact and ξ is an epimorphism, we get that $\xi \otimes Y$ is an epimorphism. Hence Λ is an epimorphism too. $\square$

***Theorem* 5.4.7.** *[AMS2, Theorem 3.11] Let (C, Δ, ε) be a coalgebra in a cocomplete abelian monoidal category $\mathcal{M}$ satisfying $AB5$, with left and right exact tensor functors. Assume that denumerable coproducts commute with $\otimes$. Let $\alpha : C \to A$ and $\sigma : A \to B$ be monomorphisms which are coalgebra homomorphisms and let $\beta = \sigma\alpha$. Assume that σ cosplits in $\mathcal{M}$. Let $p_\alpha = \mathrm{coker}(\alpha)$ in $\mathcal{M}$, let $(C^{\wedge^n_A}, \alpha_n) := \mathrm{Ker}\,(p_\alpha^{\otimes n}\Delta_A^{n-1})$ and assume that α_n cosplits in $\mathcal{M}$, for every $n \in \mathbb{N}$. If $\widetilde{C}_A = A$ and $B = A^{\wedge_B 2}$, then $\widetilde{C}_B = B$.*

Proof. For any morphism η we set $(L_\eta, p_\eta) = \mathrm{coker}(\eta)$ in $\mathcal{M}$. By Proposition 5.4.2, we get

$$B = A^{\wedge_B 2} \overset{(5.17)}{=} \Delta_B^{-1}(A \otimes B + B \otimes A).$$

Let

$$(C^{\wedge^n_A}, \alpha_n) := \mathrm{Ker}\,(p_\alpha^{\otimes n}\Delta_A^{n-1}) \qquad \text{and} \qquad (C^{\wedge^n_B}, \beta_n) := \mathrm{Ker}\,(p_\beta^{\otimes n}\Delta_B^{n-1}).$$

By assumption $(A, (\alpha_n)_{n\in\mathbb{N}}) = \widetilde{C}_A = \varinjlim C^{\wedge^n_A}$. Then, by Lemma 5.4.6, we obtain:

$$A \otimes B = (\varinjlim C^{\wedge^n_A}) \otimes B = \varinjlim(C^{\wedge^n_A} \otimes B).$$

We have:

$$\begin{aligned}
B &= \Delta_B^{-1}[\varinjlim(C^{\wedge^m_A} \otimes B) + \varinjlim(B \otimes C^{\wedge^n_A})] \\
&= \Delta_B^{-1}\varinjlim[(C^{\wedge^n_A} \otimes B) + (B \otimes C^{\wedge^n_A})] \\
&= \varinjlim\Delta_B^{-1}[(C^{\wedge^n_A} \otimes B) + (B \otimes C^{\wedge^n_A})] \\
&= \varinjlim(C^{\wedge^n_A} \wedge_B C^{\wedge^n_A})
\end{aligned}$$

where in the second equality we have used that in an $AB5$-category direct limits of direct systems of subobjects are just sums of their respective families; in the third we have used a well known property of $AB5$-categories (see [St, Proposition

1.1, page 114]); in the last equality we have used Proposition 5.4.2 in the case $\delta = \sigma\alpha_n : C^{\wedge^n_A} \to B$. Following 1.5.1, one defines inductivelly the morphism

$$C^{\wedge^n_\sigma} : C^{\wedge^n_A} \to C^{\wedge^n_B}$$

which is uniquely defined by

$$\beta_n C^{\wedge^n_\sigma} = \sigma\alpha_n. \tag{5.23}$$

By Lemma 5.4.3, $(C^{\wedge^n_\sigma} : C^{\wedge^n_A} \to C^{\wedge^n_B})_{i\in\mathbb{N}}$ is a direct system of monomorphisms in $\mathcal{M}$. Let $m \leq n$. Note that, if we denote by $\xi^n_{A,m} : C^{\wedge^m_A} \to C^{\wedge^n_A}$ and by $\xi^n_{B,m} : C^{\wedge^m_B} \to C^{\wedge^n_B}$ the canonical morphisms, this means that $C^{\wedge^n_\sigma}\xi^n_{A,m} = \xi^n_{B,m}C^{\wedge^m_\sigma}$.
Let $(L_n, p_n) = \mathrm{coker}(\sigma_n\alpha_n)$ and $(L'_n, p'_n) = \mathrm{coker}(\beta_n)$. Let also

$$\begin{aligned} (C^{\wedge^n_A} \wedge_B C^{\wedge^n_A}, \lambda_n) &= \mathrm{Ker}\,[(p_n \otimes p_n)\Delta_B] \qquad \text{and} \\ (C^{\wedge^n_B} \wedge_B C^{\wedge^n_B}, \lambda'_n) &= \mathrm{Ker}\,[(p'_n \otimes p'_n)\Delta_B]. \end{aligned}$$

Then

$$\lambda'_n \circ \left(C^{\wedge^n_\sigma} \wedge_B C^{\wedge^n_\sigma}\right) = \lambda_n.$$

By Lemma 5.4.3, $(C^{\wedge^n_\sigma} \wedge_B C^{\wedge^n_\sigma} : C^{\wedge^n_A} \wedge_B C^{\wedge^n_A} \to C^{\wedge^n_B} \wedge_B C^{\wedge^n_B})_{n\in\mathbb{N}}$ is a direct system of monomorphisms in $\mathcal{M}$. Since, by the foregoing, $B = \varinjlim(C^{\wedge^n_A} \wedge_B C^{\wedge^n_A})$, by Lemma 5.6.1, applied in the case $\gamma_i = C^{\wedge^i_\sigma} \wedge_B C^{\wedge^i_\sigma}$ for any $i \in \mathbb{N}$, we obtain that

$$(B, (\lambda'_n)_{n\in\mathbb{N}}) = \varinjlim(C^{\wedge^n_B} \wedge_B C^{\wedge^n_B}).$$

As

$$(C^{\wedge^n_B} \wedge_B C^{\wedge^n_B}, \lambda'_n) \overset{(1.9)}{=} (C^{\wedge^{2n}_B}, \beta_{2n})$$

we obtain

$$(B, (\beta_{2n})_{n\in\mathbb{N}}) = \varinjlim C^{\wedge^{2n}_B}.$$

Now, apply Lemma 5.4.4 in the case when $\gamma : \mathbb{N} \to \mathbb{N}$ is defined by setting $\gamma(n) = 2n$ for every $n \in \mathbb{N}$. Then we get

$$(B, (\beta_n)_{n\in\mathbb{N}}) = \varinjlim C^{\wedge_B n} = \widetilde{C}_B.$$

□

***Theorem* 5.4.8.** *[AMS2, Theorem 4.15] Let (C, Δ, ε) be a formally smooth coalgebra in a cocomplete and complete coabelian monoidal category $\mathcal{M}$ satisfying $AB5$, with left and right exact tensor functors. Assume that denumerable coproducts commute with $\otimes$. Let (M, ρ^r_M, ρ^l_M) be an $\mathcal{I}$-injective C-bicomodule. Then the cotensor coalgebra $T^c_C(M)$ is formally smooth.*

Proof. We will prove that any Hochschild coextension of $T^c := T^c_C(M)$ is trivial. Let $\sigma : T^c \to B$ be a Hochschild coextension of T^c. Since the canonical projection $p_0 : T^c \to C$ is a coalgebra homomorphism and C is formally smooth, by $a)$ of Theorem 3.5.10, there exists a coalgebra homomorphism $g_0 : B \to C$ such that $g_0\sigma = p_0$. Then B is a C-bicomodule via g_0. Moreover σ becomes a morphism of C-bicomodules. Since M is $\mathcal{I}$-injective and the canonical projection $p_1 : T^c \to M$ is a morphism of C-bicomodules, then there is a morphism of C-bicomodules $g_1 : B \to M$ such that $g_1\sigma = p_1$.

$$\begin{tikzcd}
 & M & \\
0 \arrow[r] & T^c_C(M) \arrow[u, "p_1"] \arrow[d, "p_0"'] \arrow[r, "\sigma"'] & B \arrow[ul, dotted, "g_1"'] \arrow[dl, dotted, "g_0"] \arrow[l, dotted, bend right, "\pi"'] \\
 & C &
\end{tikzcd}$$

Since, by Proposition 5.3.7, we have

$$\widetilde{C}_{T^c} = \varinjlim C^{\wedge^n_{T^c}} = \varinjlim C^n(M) = T^c$$

We are going to apply the universal property of cotensor coalgebra. In order to do it we have to check that the "coradical condition" is fulfilled. Since $\sigma_n : C^n \to T^c$ cosplits and since, by definition of Hochschild coextension, $B = (T^c)^{\wedge^2_B}$ and σ cosplits, then by Theorem 5.4.7 applied to the case "α" $= i_0 : C \to T^c$ the canonical inclusion and "σ" $= \sigma$, we have $\widetilde{C}_B = B$. Now we have

$$g_1\sigma i_0 = p_1 i_0 = 0.$$

Therefore we can apply Theorem 5.3.6 in the case when "C" $=$ "D" $= C$, "M" $= M$, "E" $= B$ and "δ" $= \sigma i_0$ in order to obtain a unique coalgebra homomorphism $f : B \to T^c$ such that $p_0 f = g_0$ and $p_1 f = g_1$. Then we have

$$p_0 f\sigma = g_0\sigma = p_0, \qquad \text{and} \qquad p_1 f\sigma = g_1\sigma = p_1.$$

By Corollary 5.3.5, we conclude that $f\sigma = \mathrm{Id}_{T^c}$. □

5.5 Examples

We now provide a number of examples of abelian monoidal categories for which our results apply. These categories are all Grothendieck categories and hence cocomplete and complete abelian categories satisfying AB5.

Let B be a bialgebra over a field K.

• *The category* ${}_B\mathfrak{M} = ({}_B\mathfrak{M}, \otimes_K, K)$, *of all left modules over* B. The tensor $V \otimes W$ of two left B-modules is an object in ${}_B\mathfrak{M}$ via the diagonal action; the unit is K regarded as a left B-module via ε_B.

• *The category* ${}_B\mathfrak{M}_B = ({}_B\mathfrak{M}_B, \otimes_K, K)$, *of all two-sided modules over* B. The tensor $V \otimes W$ of two B-bimodules carries, on both sides, the diagonal action; the unit is K regarded as a B-bimodule via ε_B.

• *The category* ${}^B\mathfrak{M} = ({}^B\mathfrak{M}, \otimes_K, K)$, *of all left comodules over* B. The tensor product $V \otimes W$ of two left B-comodules is an object in ${}^B\mathfrak{M}$ via the diagonal coaction; the unit is K regarded as a left B-comodule via the map $k \mapsto 1_B \otimes k$.

• *The category* ${}^B\mathfrak{M}^B = ({}^B\mathfrak{M}^B, \otimes_K, K)$ *of all two-sided comodules over* B. The tensor $V \otimes W$ of two B-bicomodules carries, on both sides, the diagonal coaction; the unit is K regarded as a B-bicomodule via the maps $k \mapsto 1_B \otimes k$ and $k \mapsto k \otimes 1_B$.

• Let H be a Hopf algebra over a field K with bijective antipode.
The category ${}_H^H\mathcal{YD} = ({}_H^H\mathcal{YD}, \otimes_K, K)$ *of left Yetter-Drinfeld modules over* H. Recall that an object V in ${}_H^H\mathcal{YD}$ is a left H-module and a left H-comodule satisfying, for any $h \in H, v \in V$, the compatibility condition:

$$\sum (h_{(1)}v)_{<-1>}h_{(2)} \otimes (h_{(1)}v)_{<0>} = \sum h_{(1)}v_{<-1>} \otimes h_{(2)}v_{<0>}$$

where $\Delta_H(h) = \sum h_{(1)} \otimes h_{(2)}$ and $\rho(v) = \sum v_{<-1>} \otimes v_{<0>}$ denote the comultiplication of H and the left H-comodule structure of V respectively (we used Sweedler notation).

The tensor product $V \otimes W$ of two Yetter-Drinfeld modules is an object in ${}_H^H\mathcal{YD}$ via the diagonal action and the codiagonal coaction; the unit in ${}_H^H\mathcal{YD}$ is K regarded as a left H-comodule via the map $x \mapsto 1_H \otimes x$ and as a left H-module via the counit ε_H.
The category ${}_H^H\mathcal{YD}$ is a braided monoidal category where, for every $V, W \in {}_H^H\mathcal{YD}$, the braiding $c_{V,W} : V \otimes W \to W \otimes V$ is defined by setting:

$$c_{V,W}(v \otimes w) = \sum v_{<-1>}w \otimes v_{<0>},$$

for every $v \in V$ and $w \in W$.

• *The category* ${}_Q\mathfrak{M} = ({}_Q\mathfrak{M}, \otimes_K, K)$, *of all left modules over a quasi-bialgebra* Q *over a field* K (see [Ka, Definition XV.1.1, page 368]).

• *The category* ${}_Q^Q\mathcal{YD}$ *of Yetter-Drinfeld modules over a finite dimensional quasi-Hopf algebra* Q (see [Mj2]). In fact, since H is finite dimensional, this category is isomorphic to the category of left modules over the quantum double $D(H)$.

5.6 Braided bialgebra structure

The main aim of this section is to provide a braided bialgebra structure for the cotensor coalgebra inside a braided monoidal category. This structure is used to extend the notion of bialgebra of type one, introduced in the classical case by Nichols in [Ni], to the wider context of a braided monoidal category (see Definition 5.6.10). A universal property for the cotensor bialgebra is also proven (see Theorem 5.6.8).

***Lemma* 5.6.1.** *Let $\mathcal{M}$ be a monoidal category with left exact direct limits. Let $((A_i)_{i\in\mathbb{N}}, (\alpha_i^j)_{i,j\in\mathbb{N}})$ and $((B_i)_{i\in\mathbb{N}}, (\beta_i^j)_{i,j\in\mathbb{N}})$ be direct systems in $\mathcal{M}$, where, for $i \leq j$, $\alpha_i^j : A_i \to A_j$ and $\beta_i^j : B_i \to B_j$. Let $(\gamma_i : A_i \to B_i)_{i\in\mathbb{N}}$ be a direct system of monomorphisms. Let $(A, (\alpha_i)_{i\in\mathbb{N}}) = \varinjlim A_i$ and let $(\beta_i : B_i \to A)_{i\in\mathbb{N}}$ be a compatible family of monomorphisms such that $\beta_i\gamma_i = \alpha_i$ for any $i \in \mathbb{N}$. Then $(A, (\beta_i)_{i\in\mathbb{N}}) = \varinjlim B_i$.*

Proof. Since direct limits are left exact in $\mathcal{M}$, the canonical morphism $\varinjlim \beta_i : \varinjlim B_i \to A$ is a monomorphism. Moreover, since $\varinjlim \beta_i \circ \varinjlim \gamma_i = \varinjlim \alpha_i = \mathrm{Id}_A$, we have that $\varinjlim \beta_i$ is also an epimorphism and hence an isomorphism. $\square$

5.6.2. Let $(\mathcal{M}, \otimes, \mathbf{1})$ be a cocomplete coabelian monoidal category. Recall that a *graded* coalgebra in $\mathcal{M}$ is a coalgebra (B, Δ, ε) endowed with a family (B_i, β_i) of subobjects of B, such that

$$B = \oplus_{i\in\mathbb{N}} B_i$$

and there exists a family $(\Delta_i)_{i\in\mathbb{N}}$ of morphisms

$$\Delta_i : B_i \to (B \otimes B)_i = \oplus_{a+b=i} (B_a \otimes B_b),$$

such that

$$\Delta\beta_i = \nabla\left[(\beta_a \otimes \beta_b)_{a+b=i}\right]\Delta_i$$

and

$$\varepsilon\beta_i = 0, \text{ for every } i \geq 1.$$

Here $\nabla\left[(\beta_a \otimes \beta_b)_{a+b=i}\right]$ denotes the codiagonal morphism associated to the family $(\beta_a \otimes \beta_b)_{a+b=i}$.
It follows that $(B_0, \Delta_0, \varepsilon_0)$ is a coalgebra in $\mathcal{M}$, where $\varepsilon_0 : B_0 \to \mathbf{1}$ is defined by setting

$$\varepsilon_0 = \varepsilon\beta_0.$$

Moreover β_0 is a coalgebra homomorphism.

***Proposition* 5.6.3.** *[AMS2, Proposition 3.3] Let $(\mathcal{M}, \otimes, \mathbf{1})$ be a cocomplete coabelian monoidal category. Let $B = \oplus_{i\in\mathbb{N}} B_i$ be a graded coalgebra. Denote by (L, p) the cokernel of β_0 in $\mathcal{M}$. Then*

$$p^{\otimes n+1}\Delta_B^n\beta_b = 0, \text{ for every } 0 \leq b \leq n. \tag{5.24}$$

Moreover

$$B = \varinjlim (B_0^{\wedge_B^i})_{i\in\mathbb{N}}.$$

Proof. Denote by $\beta_i : B_i \to B$ the canonical inclusion and denote by $\tau_i : B \to B_i$ the canonical projection, for every $i \in \mathbb{N}$. Since β_0 is a coalgebra homomorphism and β_0 is a monomorphism, we can consider

$$(B_0^{\wedge_B^n}, \delta_n) := \mathrm{Ker}(p^{\otimes n}\Delta_B^{n-1}).$$

Denote by $\xi_i^j : B_0^{\wedge_B^i} \to B_0^{\wedge_B^j}$ the canonical inclusion, for every $j \geq i$. Hence we have

$$\delta_j \xi_i^j = \delta_i.$$

In order to prove (5.24), we proceed by induction on $n \geq 0$. For $n = 0$, then $b = 0$ and we have

$$p^{\otimes n+1}\Delta_B^n \beta_b = p\Delta_B^0 \beta_0 = p\beta_0 = 0.$$

Let $n \geq 1$ and assume $p^{\otimes i+1}\Delta_B^i \beta_j = 0$, for every $0 \leq j \leq i \leq n-1$. For every $0 \leq c \leq n$, we have

$$\begin{aligned} p^{\otimes n+1}\Delta_B^n \beta_c &= \left(p^{\otimes n-1} \otimes p^{\otimes 2}\right)\left(\Delta_B^{n-2} \otimes \Delta_B\right)\Delta_B \beta_c \\ &= \left(p^{\otimes n-1}\Delta_B^{n-2} \otimes p^{\otimes 2}\Delta_B\right)\nabla\left[(\beta_a \otimes \beta_b)_{a+b=c}\right]\Delta_c \\ &= \nabla\left[\left(p^{\otimes n-1}\Delta_B^{n-2}\beta_a \otimes p^{\otimes 2}\Delta_B \beta_b\right)_{a+b=c}\right]\Delta_c = 0. \end{aligned}$$

By definition of $(B_0^{\wedge_B^n}, \delta_n)$, there exists a unique morphism

$$\gamma_n : \oplus_{i=0}^n B_i \to B_0^{\wedge_B^{n+1}}$$

such that

$$\delta_{n+1}\gamma_n = \nabla\left[(\beta_i)_{i=0}^n\right].$$

Since each β_i cosplits, then $\nabla\left[(\beta_i)_{i=0}^n\right]$ is a monomorphism. Thus also γ_n is a monomorphism. Denote by $\beta_a^b : \oplus_{i=0}^a B_i \to \oplus_{i=0}^b B_i$ the canonical injection when $a \leq b$. Then we have

$$\delta_{n+2}\gamma_{n+1}\beta_n^{n+1} = \nabla\left[(\beta_i)_{i=0}^{n+1}\right]\beta_n^{n+1} = \nabla\left[(\beta_i)_{i=0}^n\right] = \delta_{n+1}\gamma_n = \delta_{n+2}\xi_{n+1}^{n+2}\gamma_n$$

Since δ_{n+2} is a monomorphism, we get that

$$\gamma_{n+1}\beta_n^{n+1} = \xi_{n+1}^{n+2}\gamma_n$$

for every $n \in \mathbb{N}$. Thus $(\gamma_n : \oplus_{i=0}^n B_i \to B_0^{\wedge_B^{n+1}})_{n\in\mathbb{N}}$ defines a direct system of monomorphisms in $\mathcal{M}$. Now, as, by Proposition 1.6.5, $(B, (\nabla\left[(\beta_i)_{i=0}^n\right])_{n\in\mathbb{N}}) = \varinjlim (\oplus_{i=0}^n B_i)$, by Lemma 5.6.1 we have that $(B, (\delta_n)_{n\in\mathbb{N}}) = \varinjlim (B_0^{\wedge_B^i})_{i\in\mathbb{N}}$. $\square$

***Proposition* 5.6.4.** *Let $(\mathcal{M}, c)$ be a cocomplete coabelian braided monoidal category and let $B = \oplus_{n\in\mathbb{N}} B_n$ be a graded coalgebra in $\mathcal{M}$. Assume that the tensor product commutes with direct sums. Then $(B \otimes B, \Delta_{B\otimes B}, \varepsilon_{B\otimes B})$ is a graded coalgebra where*

$$\begin{aligned}\Delta_{B\otimes B} &: B \otimes B \xrightarrow{\Delta_B\otimes\Delta_B} B \otimes B \otimes B \otimes B \xrightarrow{B\otimes c_{B,B}\otimes B} B \otimes B \otimes B \otimes B,\\ \varepsilon_{B\otimes B} &: B \otimes B \xrightarrow{\varepsilon_B\otimes\varepsilon_B} \mathbf{1} \otimes \mathbf{1} \xrightarrow{m_\mathbf{1}} \mathbf{1}.\end{aligned}$$

and with graduation given by $(B \otimes B)_n = \bigoplus_{a+b=n} B_a \otimes B_b$.

Proof. It is well known that $(B \otimes B, \Delta_{B\otimes B}, \varepsilon_{B\otimes B})$ is a coalgebra (it is dual to [Mj1, Lemma 9.2.12, page 438]). Let us check the part of the statement concerning the graduation. Since B is a graded coalgebra, there exists a family $(\Delta_n)_{n\in\mathbb{N}}$ of morphism, $\Delta_n : B_n \to (B \otimes B)_n$ such that

$$\Delta_B \beta_n = \nabla \left[(\beta_a \otimes \beta_b)_{a+b=n}\right] \Delta_n,$$

where $\beta_n : B_n \to B$ and $\nabla \left[(\beta_a \otimes \beta_b)_{a+b=n}\right] : (B \otimes B)_n \to B\otimes B$ denote the canonical injection. Let

$$\Delta_{i,j} := (p_i \otimes p_j)\, \Delta_{i+j} : B_{i+j} \to B_i \otimes B_j,$$

for every $i, j \in \mathbb{N}$. We have

$$\Delta_B \beta_n = \nabla \left[(\beta_i \otimes \beta_j)_{i+j=a}\right] \Delta_n = \sum_{i+j=n} (\beta_i \otimes \beta_j)\, \Delta_{i,j}.$$

Since the tensor product commutes with direct sums, we can write freely

$$\begin{aligned}\left[(B \otimes B) \otimes (B \otimes B)\right]_n &= \oplus_{a+b=n} \left[(B \otimes B)_a \otimes (B \otimes B)_b\right]\\ &= \oplus_{i+j+u+v=n} (B_i \otimes B_j \otimes B_u \otimes B_v)\end{aligned}$$

Via this identification we have

$$\nabla \left\{\nabla \left[(\beta_i \otimes \beta_j)_{i+j=a}\right] \otimes \nabla \left[(\beta_u \otimes \beta_v)_{u+v=b}\right]\right\} = \nabla \left[(\beta_i \otimes \beta_j \otimes \beta_u \otimes \beta_v)_{i+j+u+v=n}\right].$$

Denote by

$$\beta_{i,j,u,v} : B_i \otimes B_j \otimes B_u \otimes B_v \to \oplus_{i'+j'+u'+v'=n} (B_{i'} \otimes B_{j'} \otimes B_{u'} \otimes B_{v'})$$

the canonical injection and define $\Delta_n^{B\otimes B} : (B \otimes B)_n \to \left[(B \otimes B) \otimes (B \otimes B)\right]_n$ by

$$\Delta_n^{B\otimes B} := \nabla \left[\sum_{\substack{i+j=a\\ u+v=b}} \beta_{i,j,u,v} \left(B_i \otimes c_{B_j,B_u} \otimes B_v\right) (\Delta_{i,j} \otimes \Delta_{u,v})\right]_{a+b=n}.$$

We have

$$
\begin{aligned}
& \nabla\left\{\nabla\left[(\beta_i\otimes\beta_j)_{i+j=a}\right]\otimes\nabla\left[(\beta_u\otimes\beta_v)_{u+v=b}\right]\right\}\circ\Delta_n^{B\otimes B} \\
= \; & \nabla\left[(\beta_i\otimes\beta_j\otimes\beta_u\otimes\beta_v)_{i+j+u+v=n}\right]\circ \\
& \circ\nabla\left[\sum_{\substack{i+j=a\\u+v=b}}\beta_{i,j,u,v}\left(B_i\otimes c_{B_j,B_u}\otimes B_v\right)\left(\Delta_{i,j}\otimes\Delta_{u,v}\right)\right]_{a+b=n} \\
= \; & \nabla\left[\begin{array}{c}\sum_{\substack{i+j=a\\u+v=b}}\nabla\left[(\beta_i\otimes\beta_j\otimes\beta_u\otimes\beta_v)_{i+j+u+v=n}\right]\circ\\ \circ\beta_{i,j,u,v}\left(B_i\otimes c_{B_j,B_u}\otimes B_v\right)\left(\Delta_{i,j}\otimes\Delta_{u,v}\right)\end{array}\right]_{a+b=n} \\
= \; & \nabla\left[\sum_{\substack{i+j=a\\u+v=b}}(\beta_i\otimes\beta_u\otimes\beta_j\otimes\beta_v)\left(B_i\otimes c_{B_j,B_u}\otimes B_v\right)\left(\Delta_{i,j}\otimes\Delta_{u,v}\right)\right]_{a+b=n} \\
= \; & \nabla\left[\sum_{\substack{i+j=a\\u+v=b}}(B\otimes c_{B,B}\otimes B)(\beta_i\otimes\beta_j\otimes\beta_u\otimes\beta_v)\left(\Delta_{i,j}\otimes\Delta_{u,v}\right)\right]_{a+b=n} \\
= \; & (B\otimes c_{B,B}\otimes B)\,\nabla\left[\sum_{\substack{i+j=a\\u+v=b}}(\beta_i\otimes\beta_j)\,\Delta_{i,j}\otimes(\beta_u\otimes\beta_v)\,\Delta_{u,v}\right]_{a+b=n} \\
= \; & (B\otimes c_{B,B}\otimes B)\,\nabla\left[\Delta_B\beta_a\otimes\Delta_B\beta_b\right]_{a+b=n} \\
= \; & (B\otimes c_{B,B}\otimes B)\,(\Delta_B\otimes\Delta_B)\,\nabla\left[(\beta_a\otimes\beta_b)_{a+b=n}\right] \\
= \; & \Delta_{B\otimes B}\nabla\left[(\beta_a\otimes\beta_b)_{a+b=n}\right].
\end{aligned}
$$

Moreover we have

$$
\begin{aligned}
\varepsilon_{B\otimes B}\circ\nabla\left[(\beta_a\otimes\beta_b)_{a+b=n}\right] &= m_{\mathbf{1}}\left(\varepsilon_B\otimes\varepsilon_B\right)\nabla\left[(\beta_a\otimes\beta_b)_{a+b=n}\right] \\
&= m_{\mathbf{1}}\nabla\left[(\varepsilon_B\beta_a\otimes\varepsilon_B\beta_b)_{a+b=n}\right]=0,
\end{aligned}
$$

for every $n\geq 1$. Since the tensor product commutes with direct sums, we can write

$$B\otimes B=(\oplus_{a\in\mathbb{N}}B_a)\otimes(\oplus_{b\in\mathbb{N}}B_b)=\oplus_{n\in\mathbb{N}}\left(B\otimes B\right)_n$$

so that $B\otimes B$ is a graded coalgebra. $\square$

***Proposition* 5.6.5.** *Let $(C,\Delta_C,\varepsilon_C)$ be a coalgebra in a cocomplete coabelian monoidal category $\mathcal{M}$ and let (M,ρ_M^r,ρ_M^l) be a C-bicomodule. Let $T^c=T_H^c(M)$ be the cotensor coalgebra. Then $T^c=\oplus_{n\in\mathbb{N}}T_n^c$ is a graded coalgebra where $T_n^c=M^{\square_H n}$.*

Proof. By Proposition 5.2.2 we can write

$$\Delta(n)\, i_t^n = \sum_{i+j=t} (i_i^n \otimes i_j^n) f_{i,j}, \text{ for every } n \geq 1 \text{ and } 0 \leq t \leq n,$$

where $f_{i,j} : M^{\square_H i+j} \to M^{\square_H i} \otimes M^{\square_H j}$ are suitable morphisms. Denote by

$$\beta_{i,j} : T_i^c \otimes T_j^c \to (T^c \otimes T^c)_{i+j} = \bigoplus_{a+b=i+j} T_a^c \otimes T_b^c$$

the canonical injection. Define

$$\Delta_n := \sum_{i+j=n} \beta_{i,j} f_{i,j} : T_n^c \to (T^c \otimes T^c)_n$$

Then

$$\begin{aligned} \nabla\left[(i_a^n \otimes i_b^n)_{a+b=n}\right] \Delta_n &= \nabla\left[(i_a^n \otimes i_b^n)_{a+b=n}\right] \sum_{i+j=n} \beta_{i,j} f_{i,j} \\ &= \sum_{i+j=n} \nabla\left[(i_a^n \otimes i_b^n)_{a+b=n}\right] \beta_{i,j} f_{i,j} = \sum_{i+j=n} (i_i \otimes i_j)\, f_{i,j}. \end{aligned}$$

By construction, Δ_{T^c} is uniquely defined by $\Delta_{T^c}\sigma_i = (\sigma_i \otimes \sigma_i)\, \Delta(i)$, for every $i \in \mathbb{N}$, so that

$$\begin{aligned} \Delta_{T^c} i_n &= \Delta_{T^c} \sigma_{n+1} i_n^{n+1} \\ &= (\sigma_{n+1} \otimes \sigma_{n+1})\, \Delta(n+1)\, i_n^{n+1} \\ &= (\sigma_{n+1} \otimes \sigma_{n+1}) \sum_{i+j=n} (i_i^{n+1} \otimes i_j^{n+1}) f_{i,j} \\ &= \sum_{i+j=n} (\sigma_{n+1} i_i^{n+1} \otimes \sigma_{n+1} i_j^{n+1}) f_{i,j} \\ &= \sum_{i+j=n} (i_i \otimes i_j) f_{i,j} = \nabla\left[(i_a^n \otimes i_b^n)_{a+b=n}\right] \Delta_n. \end{aligned}$$

By construction ε_{T^c} is uniquely defined by $\varepsilon_{T^c}\sigma_i = (\sigma_i \otimes \sigma_i)\, \varepsilon(i)$, for every $i \in \mathbb{N}$, so that

$$\varepsilon_{T^c} i_n = \varepsilon_{T^c} \sigma_{n+1} i_n^{n+1} = (\sigma_{n+1} \otimes \sigma_{n+1})\, \varepsilon(n+1)\, i_n^{n+1} = (\sigma_{n+1} \otimes \sigma_{n+1})\, \varepsilon_C \pi_{n+1}^1 i_n^{n+1} = 0$$

for every $n \geq 1$. □

Theorem 5.6.6. *Let H be a braided bialgebra in a cocomplete and complete coabelian braided monoidal category $(\mathcal{M}, c)$ satisfying $AB5$. Assume that the tensor*

product commutes with direct sums.
Let $(M, \mu_M^r, \mu_M^l, \rho_M^r, \rho_M^l)$ be in ${}_H^H\mathcal{M}_H^H$. Let $T^c = T_H^c(M)$ be the cotensor coalgebra. Then there are unique coalgebra homomorphisms

$$m_{T^c} : T^c \otimes T^c \to T^c \qquad \text{and} \qquad u_{T^c} : \mathbf{1} \to T^c$$

such that the diagrams

(5.25)
$$\begin{tikzcd} T^c \arrow[r, "p_1"] \arrow[d, "p_0"'] & M \\ H & T^c \otimes T^c \arrow[l, "m_H(p_0\otimes p_0)"] \arrow[ul, dotted, "m_{T^c}"'] \arrow[u, "\mu_M^l(p_0\otimes p_1)+\mu_M^r(p_1\otimes p_0)"'] \end{tikzcd} \qquad \begin{tikzcd} T^c \arrow[r, "p_1"] \arrow[d, "p_0"'] & M \\ H & \mathbf{1} \arrow[l, "u_H"] \arrow[ul, dotted, "u_{T^c}"'] \arrow[u, "0"'] \end{tikzcd}$$

are commutative, where $p_n : T^c \to M^{\Box_H n}$ denotes the canonical projection. Moreover $(T^c, m_{T^c}, u_{T^c}, \Delta_{T^c}, \varepsilon_{T^c})$ is a braided bialgebra in $\mathcal{M}$.

Proof. First of all recall that $(E = T^c \otimes T^c, \Delta_E, \varepsilon_E)$ is a coalgebra where

$$\begin{aligned} \Delta_E &: T^c \otimes T^c \xrightarrow{\Delta_{T^c}\otimes\Delta_{T^c}} T^c \otimes T^c \otimes T^c \otimes T^c \xrightarrow{T^c\otimes c_{T^c,T^c}\otimes T^c} T^c \otimes T^c \otimes T^c \otimes T^c, \\ \varepsilon_E &: T^c \otimes T^c \xrightarrow{\varepsilon_{T^c}\otimes\varepsilon_{T^c}} \mathbf{1} \otimes \mathbf{1} \xrightarrow{m_{\mathbf{1}}} \mathbf{1}. \end{aligned}$$

By Proposition 5.6.5, $T^c = \oplus_{n\in\mathbb{N}} T_n^c$ is a graded coalgebra where $T_n^c = M^{\Box_H n}$. Then, by Proposition 5.6.4, $E = \oplus_{n\in\mathbb{N}} E_n$ is a graded coalgebra where

$$E_n = \bigoplus_{i=0}^{n} T_n^c \otimes T_{n-i}^c$$

By Proposition 5.6.3, we have

(5.26)
$$E = \varinjlim (E_0^{\wedge_E^i})_{i\in\mathbb{N}} = \varinjlim ((T_0^c \otimes T_0^c)^{\wedge_E^i})_{i\in\mathbb{N}} = \varinjlim ((H \otimes H)^{\wedge_E^i})_{i\in\mathbb{N}}$$

Set

$$\begin{aligned} f_H &: = m_H (p_0 \otimes p_0) : E \to H, \\ f_M &: = \mu_M^l (p_0 \otimes p_1) + \mu_M^r (p_1 \otimes p_0) : E \to M. \end{aligned}$$

Then f_H is a coalgebra homomorphism:

$$\begin{aligned} \Delta_H f_H &= \Delta_H m_H (p_0 \otimes p_0) \\ &= (m_H \otimes m_H)(H \otimes c_{H,H} \otimes H)(\Delta_H \otimes \Delta_H)(p_0 \otimes p_0) \\ &\overset{(5.9)}{=} (m_H \otimes m_H)(H \otimes c_{H,H} \otimes H)(p_0 \otimes p_0 \otimes p_0 \otimes p_0)(\Delta_{T^c} \otimes \Delta_{T^c}) \\ &= (m_H \otimes m_H)(p_0 \otimes p_0 \otimes p_0 \otimes p_0)(T^c \otimes c_{T^c,T^c} \otimes T^c)(\Delta_{T^c} \otimes \Delta_{T^c}) \\ &= (f_H \otimes f_H)\Delta_E. \end{aligned}$$

Moreover f_M is a morphism of left H-comodules

$$\begin{aligned}
\rho_M^l f_M &= \rho_M^l \mu_M^l (p_0 \otimes p_1) + \rho_M^l \mu_M^r (p_1 \otimes p_0) \\
&\overset{(1.1),(1.2)}{=} \left(m_H \otimes \mu_M^l\right)(H \otimes c_{H,H} \otimes M)\left(\Delta_H \otimes \rho_M^l\right)(p_0 \otimes p_1) + \\
&\quad + (m_H \otimes \mu_M^r)(H \otimes c_{M,H} \otimes H)\left(\rho_M^l \otimes \Delta_H\right)(p_1 \otimes p_0) \\
&\overset{(5.9),(5.8)}{=} \left(m_H \otimes \mu_M^l\right)(H \otimes c_{H,H} \otimes M)(p_0 \otimes p_0 \otimes p_0 \otimes p_1)(\Delta_{T^c} \otimes \Delta_{T^c}) + \\
&\quad + (m_H \otimes \mu_M^r)(H \otimes c_{M,H} \otimes H)(p_0 \otimes p_1 \otimes p_0 \otimes p_0)(\Delta_{T^c} \otimes \Delta_{T^c}) \\
&= \left(m_H \otimes \mu_M^l\right)(p_0 \otimes p_0 \otimes p_0 \otimes p_1)(T^c \otimes c_{T^c,T^c} \otimes T^c)(\Delta_{T^c} \otimes \Delta_{T^c}) + \\
&\quad + (m_H \otimes \mu_M^r)(p_0 \otimes p_0 \otimes p_1 \otimes p_0)(T^c \otimes c_{T^c,T^c} \otimes T^c)(\Delta_{T^c} \otimes \Delta_{T^c}) \\
&= \left[f_H \otimes \mu_M^l (p_0 \otimes p_1)\right]\Delta_E + \left[f_H \otimes \mu_M^r (p_1 \otimes p_0)\right]\Delta_E \\
&= (f_H \otimes f_M)\,\Delta_E.
\end{aligned}$$

Analogously $\rho_M^r f_M = (f_M \otimes f_H)\,\Delta_E$, i.e. f_M is a morphism of right H-comodules and hence a morphism of H-bicomodules. Moreover $f_M (i_0 \otimes i_0) = 0$. By applying Theorem 5.3.3 to the case "C" $= H$, "D" $= H \otimes H$ and "δ" $= i_0 \otimes i_0 : H \otimes H \to E$, using the maps f_H and f_M above, since, by (5.26), we have

$$\left(\widetilde{D}, \widetilde{\delta}\right) = (E, \mathrm{Id}_E)\,,$$

we get a coalgebra homomorphism

$$m_{T^c} = f : E \to T^c$$

such that the left side of 5.25 is commutative. We have

$$\begin{aligned}
p_0 m_{T^c} (m_{T^c} \otimes T^c) &= f_H (m_{T^c} \otimes T^c) \\
&= m_H (p_0 \otimes p_0)(m_{T^c} \otimes T^c) \\
&= m_H (f_H \otimes p_0) \\
&= m_H (m_H \otimes H)(p_0 \otimes p_0 \otimes p_0)\,.
\end{aligned}$$

Analogously $p_0 m_{T^c} (T^c \otimes m_{T^c}) = m_H (H \otimes m_H)(p_0 \otimes p_0 \otimes p_0)$ and hence

$$p_0 m_{T^c} (m_{T^c} \otimes T^c) = p_0 m_{T^c} (T^c \otimes m_{T^c})\,.$$

Moreover

$$\begin{aligned}
&\ p_1 m_{T^c} (m_{T^c} \otimes T^c) \\
=&\ f_M (m_{T^c} \otimes T^c) \\
=&\ \mu_M^l (p_0 \otimes p_1)(m_{T^c} \otimes T^c) + \mu_M^r (p_1 \otimes p_0)(m_{T^c} \otimes T^c) \\
=&\ \mu_M^l (f_H \otimes p_1) + \mu_M^r (f_M \otimes p_0) \\
=&\ \mu_M^l (m_H \otimes M)(p_0 \otimes p_0 \otimes p_1) + \\
&+ \mu_M^r \left(\mu_M^l \otimes H\right)(p_0 \otimes p_1 \otimes p_0) + \mu_M^r (\mu_M^r \otimes H)(p_1 \otimes p_0 \otimes p_0)\,.
\end{aligned}$$

Analogously

$$\begin{aligned}
& p_1 m_{T^c}\left(T^c \otimes m_{T^c}\right) \\
= & \mu_M^l\left(H \otimes \mu_M^l\right)\left(p_0 \otimes p_0 \otimes p_1\right)+\mu_M^l\left(H \otimes \mu_M^r\right)\left(p_0 \otimes p_1 \otimes p_0\right) \\
& +\mu_M^r\left(M \otimes m_H\right)\left(p_1 \otimes p_0 \otimes p_0\right)
\end{aligned}$$

so that

$$p_1 m_{T^c}\left(m_{T^c} \otimes T^c\right)=p_1 m_{T^c}\left(T^c \otimes m_{T^c}\right).$$

Since $T^c \otimes T^c \otimes T^c$ is a coalgebra and $m_{T^c}\left(m_{T^c} \otimes T^c\right)$ and $m_{T^c}\left(T^c \otimes m_{T^c}\right)$ are coalgebra homomorphisms, then by Corollary 5.3.5, we have

$$m_{T^c}\left(m_{T^c} \otimes T^c\right)=m_{T^c}\left(T^c \otimes m_{T^c}\right).$$

Set

$$\begin{aligned}
f_H^{\prime} & := u_H: \mathbf{1} \rightarrow H, \\
f_M^{\prime} & := 0: \mathbf{1} \rightarrow M.
\end{aligned}$$

Then $f_H^{\prime}$ is a coalgebra homomorphism and $f_M^{\prime}$ is a morphism of H-bicomodules. By applying Theorem 5.3.3 to the case $D=\mathbf{1}$ and $\delta=\mathrm{Id}_{\mathbf{1}}: \mathbf{1} \rightarrow \mathbf{1}$, since

$$\left(\widetilde{D}, \widetilde{\delta}\right)=\left(\mathbf{1}, \mathrm{Id}_{\mathbf{1}}\right)$$

we get a coalgebra homomorphism

$$u_{T^c}=f^{\prime}: \mathbf{1} \rightarrow T^c$$

such that the right side of 5.25 is commutative.We have

$$\begin{aligned}
p_0 m_{T^c}\left(u_{T^c} \otimes T^c\right) & =f_H\left(u_{T^c} \otimes T^c\right) \\
& =m_H\left(p_0 \otimes p_0\right)\left(u_{T^c} \otimes T^c\right) \\
& =m_H\left(u_H \otimes H\right)\left(\mathbf{1} \otimes p_0\right) \\
& =l_H\left(\mathbf{1} \otimes p_0\right)=p_0 l_{T^c}
\end{aligned}$$

and

$$\begin{aligned}
p_1 m_{T^c}\left(u_{T^c} \otimes T^c\right) & =f_M\left(u_{T^c} \otimes T^c\right) \\
& =\mu_M^l\left(p_0 \otimes p_1\right)\left(u_{T^c} \otimes T^c\right)+\mu_M^r\left(p_1 \otimes p_0\right)\left(u_{T^c} \otimes T^c\right) \\
& =\mu_M^l\left(u_H \otimes M\right)\left(\mathbf{1} \otimes p_1\right) \\
& =l_M\left(\mathbf{1} \otimes p_1\right)=p_1 l_{T^c}.
\end{aligned}$$

Since $\mathbf{1} \otimes T^c$ is a coalgebra and $m_{T^c}(u_{T^c} \otimes T^c)$ and r_{T^c} are coalgebra homomorphisms, then by Corollary 5.3.5, we have

$$m_{T^c}(u_{T^c} \otimes T^c) = l_{T^c}.$$

Analogously $m_{T^c}(T^c \otimes u_{T^c}) = r_{T^c}$. Thus $(T^c, m_{T^c}, u_{T^c}, \Delta_{T^c}, \varepsilon_{T^c})$ is a braided bialgebra in $\mathcal{M}$. □

***Remark* 5.6.7.** Let M be an object in a cocomplete and complete coabelian braided monoidal category $(\mathcal{M}, c)$ satisfying $AB5$. Assume that the tensor product commutes with direct sums.
By applying Theorem 5.6.6 to the case $H = \mathbf{1}$ we endow the cotensor coalgebra $T^c = T^c_H(M)$ with an algebra structure such that T^c becomes a braided bialgebra. This algebra structure is the braided analogue of the so called "Shuffle Algebra" in the category of vector spaces.

***Theorem* 5.6.8.** *Let H be a braided bialgebra in a cocomplete and complete coabelian braided monoidal category $(\mathcal{M}, c)$ satisfying $AB5$. Assume that the tensor product commutes with direct sums.*
Let $(M, \mu^r_M, \mu^l_M, \rho^r_M, \rho^l_M)$ be in ${}^H_H\mathcal{M}^H_H$. Let $T^c = T^c_H(M)$ be the cotensor coalgebra.
Let $\delta : D \to E$ be a monomorphism which is a homomorphism of coalgebras. Assume that there exist morphisms

$$m_{\widetilde{D}} : \widetilde{D} \otimes \widetilde{D} \to \widetilde{D} \qquad \textit{and} \qquad u_{\widetilde{D}} : \mathbf{1} \to \widetilde{D}$$

such that $\left(\widetilde{D}, m_{\widetilde{D}}, u_{\widetilde{D}}, \Delta_{\widetilde{D}}, \varepsilon_{\widetilde{D}},\right)$ is a braided bialgebra in $\mathcal{M}$.
Let $f_H : \widetilde{D} \to H$ be a bialgebra homomorphism and let $f_M : \widetilde{D} \to M$ be a morphism of H-bicomodules such that $f_M \xi_1 = 0$, where $\widetilde{D}$ is a bicomodule via f_H. Assume that

$$f_M m_{\widetilde{D}} = \mu^l_M (f_H \otimes f_M) + \mu^r_M (f_M \otimes f_H)$$

(i.e. f_M is a derivation of $\widetilde{D}$ with values in the $\widetilde{D}$-bimodule M, where M is regarded as a bimodule via f_H). Then there is a unique coalgebra homomorphism $f : \widetilde{D} \to T^c_H(M)$ such that $p_0 f = f_H$ and $p_1 f = f_M$, where $p_n : T^c_C(M) \to M^{\square n}$ denotes the canonical projection.

$$\begin{tikzcd}
T^c_H(M) \arrow[r, "p_1"] \arrow[d, "p_0"'] & M & \\
H & \widetilde{D} \arrow[l, "f_H"] \arrow[u, "f_M"'] \arrow[ul, dotted, "f"] & D \arrow[l, "\xi_1"] \arrow[ul, "0"']
\end{tikzcd}$$

Moreover f is a bialgebra homomorphism.

Proof. By Theorem 5.3.6, there is a unique coalgebra homomorphism $f : \widetilde{D} \to T^c_H(M)$ such that $p_0 f = f_H$ and $p_1 f = f_M$. By Theorem 5.6.6, we have

$$p_0 m_{T^c}(f \otimes f) = m_H(p_0 \otimes p_0)(f \otimes f) = m_H(f_H \otimes f_H) = f_H m_{\widetilde{D}} = p_0 f m_{\widetilde{D}},$$

and

$$\begin{aligned} p_1 m_{T^c}(f \otimes f) &= \left[\mu^l_M(p_0 \otimes p_1) + \mu^r_M(p_1 \otimes p_0)\right](f \otimes f) \\ &= \mu^l_M(p_0 f \otimes p_1 f) + \mu^r_M(p_1 f \otimes p_0 f) \\ &= \mu^l_M(f_H \otimes f_M) + \mu^r_M(f_M \otimes f_H) \\ &= f_M m_{\widetilde{D}} = p_1 f m_{\widetilde{D}}. \end{aligned}$$

From $f_M m_{\widetilde{D}} = \mu^l_M(f_H \otimes f_M) + \mu^r_M(f_M \otimes f_H)$, we get

$$\begin{aligned} f_M u_{\widetilde{D}} &= f_M m_{\widetilde{D}}\left(u_{\widetilde{D}} \otimes u_{\widetilde{D}}\right) m_{\mathbf{1}}^{-1} \\ &= \mu^l_M\left(f_H u_{\widetilde{D}} \otimes f_M u_{\widetilde{D}}\right) m_{\mathbf{1}}^{-1} + \mu^r_M\left(f_M u_{\widetilde{D}} \otimes f_H u_{\widetilde{D}}\right) m_{\mathbf{1}}^{-1} \\ &= \mu^l_M\left(u_H \otimes f_M u_{\widetilde{D}}\right) m_{\mathbf{1}}^{-1} + \mu^r_M\left(f_M u_{\widetilde{D}} \otimes u_H\right) m_{\mathbf{1}}^{-1} \\ &= f_M u_{\widetilde{D}} + f_M u_{\widetilde{D}} \end{aligned}$$

so that

$$f_M u_{\widetilde{D}} = 0.$$

Hence, by Theorem 5.3.6, we have

$$\begin{aligned} p_0 f u_{\widetilde{D}} &= f_H u_{\widetilde{D}} = u_H = p_0 u_{T^c} \\ p_1 f u_{\widetilde{D}} &= f_M u_{\widetilde{D}} = 0 = p_1 u_{T^c} \end{aligned}$$

Since $m_{T^c}(f \otimes f), f m_{\widetilde{D}} : \widetilde{D} \otimes \widetilde{D} \to T^c$ and $f u_{\widetilde{D}}, u_{T^c} : \mathbf{1} \to T^c$ are coalgebra homomorphisms, as a composition of coalgebra homomorphisms, and since

$$p_i m_{T^c}(f \otimes f) = p_i f m_{\widetilde{D}} \qquad \text{and} \qquad p_i f u_{\widetilde{D}} = p_i u_{T^c}$$

for $i = 0, 1$, then, by Corollary 5.3.5, we get that $m_{T^c}(f \otimes f) = f m_{\widetilde{D}}$ and $f u_{\widetilde{D}} = u_{T^c}$ i.e. that f is an algebra homomorphism. □

Theorem 5.6.9. *Let H be a braided bialgebra in a cocomplete and complete abelian coabelian braided monoidal category $(\mathcal{M}, c)$ satisfying $AB5$. Assume that the tensor product commutes with direct sums.*
Let $(M, \mu^r_M, \mu^l_M, \rho^r_M, \rho^l_M)$ be in ${}^H_H\mathcal{M}^H_H$.
Then there is a unique algebra homomorphism $F : T_H(M) \to T^c_H(M)$ such that

$Fi_0 = i_0^c$ and $Fi_1 = i_1^c$, where $i_n : M^{\otimes_H n} \to T_H(M)$ and $i_n^c : M^{\square_H n} \to T_H^c(M)$ denote the canonical injections.

$$\begin{tikzcd} M \arrow[r, "i_1"] \arrow[d, "i_1^c"'] & T_H(M) \arrow[dl, dotted, "F"] \\ T_H^c(M) & H \arrow[l, "i_0^c"] \arrow[u, "i_0"'] \end{tikzcd}$$

Moreover F is a bialgebra homomorphism.

Proof. Let $T = T_H(M)$ and $T^c = T_H^c(M)$. In view of Lemma 5.2.4, $i_0^c : H \to T^c$ is a coalgebra homomorphism. Moreover we have

$$\begin{aligned} p_0 m_{T^c}(i_0^c \otimes i_0^c) &= m_H (p_0 \otimes p_0)(i_0^c \otimes i_0^c) = m_H = p_0 i_0^c m_H, \\ p_1 m_{T^c}(i_0^c \otimes i_0^c) &= \left[\mu_M^l (p_0 \otimes p_1) + \mu_M^r (p_1 \otimes p_0)\right](i_0^c \otimes i_0^c) = 0 = p_1 i_0^c m_H \end{aligned}$$

so that, by Corollary 5.3.5, we get $m_{T^c}(i_0^c \otimes i_0^c) = i_0^c m_H$. Analogously, from

$$\begin{aligned} p_0 u_T &= u_H = p_0 i_0^c u_H, \\ p_1 u_T &= 0 = p_1 i_0^c u_H, \end{aligned}$$

we get that $u_T = i_0^c u_H$. Then i_0^c is also an algebra homomorphism and hence a bialgebra homomorphism. We have

$$\begin{aligned} p_0 m_{T^c}(i_1^c \otimes i_0^c) &= m_H (p_0 \otimes p_0)(i_1^c \otimes i_0^c) = 0 = p_0 i_1^c \mu_M^r \\ p_1 m_{T^c}(i_1^c \otimes i_0^c) &= \left[\mu_M^l (p_0 \otimes p_1) + \mu_M^r (p_1 \otimes p_0)\right](i_1^c \otimes i_0^c) = \mu_M^r = p_1 i_1^c \mu_M^r \end{aligned}$$

and hence $m_{T^c}(i_1^c \otimes i_0^c) = i_1^c \mu_M^r$ which means that i_1^c is a morphism of right H-bimodules. Similarly one gets that i_1^c is a morphism of left H-bimodules. The conclusion follows by Theorem 4.3.2, once proved that i_1^c is a coderivation. But this holds true in view of Lemma 5.2.4. □

***Definition* 5.6.10.** Take the notations and assumptions of Theorem 5.6.9. Following [Ni, page 1533], let

$$(H[M], i_{H[M]}) = \mathrm{Im}(F),$$

where $i_{H[M]} : H[M] \to T_H^c(M)$. Since F is a morphism of braided bialgebras and $\mathcal{M}$ is an abelian category, then $H[M]$ can be endowed with unique braided bialgebra structure such that $i_{H[M]}$ is a bialgebra homomorphism. This will be called *the braided bialgebra of type one associated to H and M.*

Chapter 6

Applications to the theory of Hopf algebras

A bialgebra with a projection is a bialgebra E over a field K endowed with a Hopf algebras H and two bialgebra maps $\sigma : H \to E$ and $\pi : E \to H$ such that $\pi \circ \sigma = \mathrm{Id}_H$. In [Rad2], M. D. Radford describes the structure of bialgebras with a projection: E can be decomposed as the smash product of H by the (right) H-coinvariant part of E which comes out to be a braided bialgebra in the monoidal category ${}_H^H\mathcal{YD}$ of Yetter-Drinfeld modules over H. This construction appeared as an important tool in the classification of finite dimensional Hopf algebras. It is meaningful that, even relaxing some assumption on π (as was done by P. Schauenburg in [Sch1]) or on σ (see 6.8.3 which is from [AMS1]), it is possible to reconstruct E by means of a suitable bosonization type procedure. An occurrence of this situation appeared in [AMS1], where it is shown that if E is a bialgebra such that $H = E/J$ is a quotient Hopf algebra of E which is semisimple, J denoting the Jacobson radical of E, then the canonical Hopf projection $\pi : E \to H$ admits a left H-colinear algebra section $\sigma : H \to E$. Furthermore (see Theorem 6.8.6) this section can be chosen to be H-bicolinear, whenever H is also cosemisimple. In [AMS1] also the dual situation of a bialgebra E whose coradical, say H, is a Hopf subalgebra is described. In this case there is a retraction π of the canonical injection σ which is a left H-linear (bilinear if H is also semisimple as in Theorem 6.8.7) coalgebra map. These results are achieved by means of the characterization of (co)separable (co)algebras in the framework of monoidal categories that was developed in [AMS3] and is here included in Chapter 3.

In [SVO], D. Ştefan and F. Van Oystaeyen provided a generalization of Wedderburn-Malcev theorem for finite dimensional H-comodule algebras where H is endowed with an ad-invariant integral (see Definition 6.6.1).

In this chapter, following [Ar1], we provide a functorial characterization of ad-(co)invariant integrals and we show how the notion of formally smooth (co)algebra is useful to prove that certain Hopf algebras can be described by means of a bosonizations type procedure. More precisely, we prove that given a bialgebra surjection $\pi : E \to H$ with nilpotent kernel such that H is a Hopf algebra which is formally smooth as a K-algebra, then π has a section which is a right H-colinear algebra homomorphism (Theorem 6.8.1). Moreover, if H is also endowed with an ad-invariant integral, then the section can be chosen to be H-bicolinear (Theorem 6.6.17). Dually, we prove that, if H is a Hopf subalgebra of a bialgebra E which is formally smooth as a K-coalgebra and such that $Corad(E) \subseteq H$, then E has a weak right projection onto H (Theorem 6.8.4). Furthermore, if H is also endowed with an ad-coinvariant integral, then the retraction can be chosen to be H-bilinear (Theorem 6.7.19).

6.1 Separable functors and relative projectivity

Separable functors were introduced by C. Năstăsescu, M. Van den Bergh and F. Van Oystaeyen in [NVdB]. As we will see in Theorem 6.6.7 and Theorem 6.7.7, the existence of ad-(co)invariant integrals is characterized by means of the separability of suitable functors.

6.1.1. Let $\mathbb{U} : \mathfrak{B} \to \mathfrak{A}$ be a covariant functor. We have functors

$$Hom_{\mathfrak{B}}(\bullet, \bullet), Hom_{\mathfrak{A}}(\mathbb{U}(\bullet), \mathbb{U}(\bullet)) : \mathfrak{B}^{op} \times \mathfrak{B} \to \underline{\underline{\mathfrak{Sets}}}$$

and a natural transformation

$$\mathcal{U} : Hom_{\mathfrak{B}}(\bullet, \bullet) \to Hom_{\mathfrak{A}}(\mathbb{U}(\bullet), \mathbb{U}(\bullet)),$$
$$\mathcal{U}_{B_1,B_2}(f) := \mathbb{U}(f) \text{ for all objects } B_1, B_2 \in \mathfrak{B}.$$

We say that $\mathbb{U}$ is *faithful* (*full*) whenever the map $\mathcal{U}_{B_1,B_2}$ is injective (surjective) for all objects $B_1, B_2 \in \mathfrak{B}$. The functor $\mathbb{U}$ is called *separable* if $\mathcal{U}$ splits, that is there is a natural transformation $\mathcal{P} : Hom_{\mathfrak{A}}(\mathbb{U}(\bullet), \mathbb{U}(\bullet)) \to Hom_{\mathfrak{B}}(\bullet, \bullet)$ such that $\mathcal{P} \circ \mathcal{U} = \mathbf{1}_{Hom_{\mathfrak{B}}(\bullet,\bullet)}$, the identity natural transformation on $Hom_{\mathfrak{B}}(\bullet, \bullet)$.
It is proved in [Raf, page 1446] that this definition is consistent with the one given in [NVdB] in the following more explicit form.
For all objects $B_1, B_2 \in \mathfrak{B}$ there is a map $\mathcal{P}_{B_1,B_2} : Hom_{\mathfrak{A}}(\mathbb{U}B_1, \mathbb{U}B_2) \to Hom_{\mathfrak{B}}(B_1, B_2)$ such that:

$S1)$ $\mathcal{P}_{B_1,B_2}(\mathbb{U}(f)) = f$, for any $f \in Hom_{\mathfrak{B}}(B_1, B_2)$;

$S2)$ $\mathcal{P}_{B_1,B_2}(l) \circ f = g \circ \mathcal{P}_{B_1,B_2}(h)$ for every commutative diagram in $\mathfrak{A}$ of type:

$$\begin{tikzcd} \mathbb{U}(B_1) \arrow[r, "h"] \arrow[d, "\mathbb{U}(f)"'] & \mathbb{U}(B_2) \arrow[d, "\mathbb{U}(g)"] \\ \mathbb{U}(B_3) \arrow[r, "l"'] & \mathbb{U}(B_4) \end{tikzcd}$$

***Remark* 6.1.2.** Let $\alpha : X \to Y$ be a morphism in $\mathfrak{B}$. If $\mathbb{U}$ is a faithful functor, then, α is an epimorphism (resp. monomorphism) whenever $\mathbb{U}(\alpha)$ is.

Let us recall some well known property on separable functors.

Lemma 6.1.3. *[NVdB, Proposition 1.2] Let $\mathbb{U} : \mathfrak{B} \to \mathfrak{A}$ be a covariant separable functor and let $\alpha : X \to Y$ be a morphism in $\mathfrak{B}$. If $\mathbb{U}(\alpha)$ has a section h (resp. a retraction l) in $\mathfrak{A}$, then α has a section (retraction) in $\mathfrak{B}$.*

Lemma 6.1.4. *Let $F : \mathfrak{A} \to \mathfrak{B}$ and $G : \mathfrak{B} \to \mathfrak{C}$ be covariant functors. Then $\mathcal{P}_F \subseteq \mathcal{P}_{GF}$ and $\mathcal{I}_F \subseteq \mathcal{I}_{GF}$. Moreover the equalities hold whenever G is separable.*

Theorem 6.1.5. *Consider functors $\mathbb{T} : \mathfrak{A} \to \mathfrak{B}$ and $\mathbb{H} : \mathfrak{B} \to \mathfrak{C}$. Then, we have that:*
1) *If $\mathbb{T}$ and $\mathbb{H}$ are separable, then $\mathbb{H} \circ \mathbb{T}$ is also separable.*
2) *If $\mathbb{H} \circ \mathbb{T}$ is separable, then $\mathbb{T}$ is separable.*
3) *If $\mathfrak{C} = \mathfrak{A}$ and $(\mathbb{T}, \mathbb{H})$ is a category equivalence, then $\mathbb{T}$ and $\mathbb{H}$ are both separable.*

Proof. See [CMZ, Proposition 46 and Corollary 9]. □

We quote from [Raf] the so called Rafael Theorem:

Theorem 6.1.6. *(see [Raf, Theorem 1.2]) Let $(\mathbb{T}, \mathbb{H})$ be an adjunction, where $\mathbb{T} : \mathfrak{A} \to \mathfrak{B}$ and $\mathbb{H} : \mathfrak{B} \to \mathfrak{A}$. Then we have:*

1) $\mathbb{T}$ is separable iff the unit $\eta : \mathrm{Id}_{\mathfrak{A}} \to \mathbb{H}\mathbb{T}$ of the adjunction cosplits, i.e. there exists a natural transformation $\mu : \mathbb{H}\mathbb{T} \to \mathrm{Id}_{\mathfrak{A}}$ such that $\mu \circ \eta = \mathrm{Id}_{\mathrm{Id}_{\mathfrak{A}}}$, the identity natural transformation on $\mathrm{Id}_{\mathfrak{A}}$.

2) $\mathbb{H}$ is separable iff the counit $\varepsilon : \mathbb{T}\mathbb{H} \to \mathrm{Id}_{\mathfrak{B}}$ of the adjunction splits, i.e. there exists a natural transformation $\sigma : \mathrm{Id}_{\mathfrak{B}} \to \mathbb{T}\mathbb{H}$ such that $\varepsilon \circ \sigma = \mathrm{Id}_{\mathrm{Id}_{\mathfrak{B}}}$, the identity natural transformation on $\mathrm{Id}_{\mathfrak{B}}$.

Corollary 6.1.7. *Let $(\mathbb{T}, \mathbb{H})$ be an adjunction, where $\mathbb{T} : \mathfrak{A} \to \mathfrak{B}$ and $\mathbb{H} : \mathfrak{B} \to \mathfrak{A}$. Then we have:*

1) $\mathbb{H}$ separable $\Rightarrow$ any object in $\mathfrak{B}$ is $\mathcal{P}_{\mathbb{H}}$-projective.

2) $\mathbb{T}$ separable $\Rightarrow$ any object in $\mathfrak{A}$ is $\mathcal{I}_{\mathbb{T}}$-injective.

Proof. 1) Let B be an object in $\mathfrak{B}$. Since $\mathbb{H}(\varepsilon_B) \circ \eta_{\mathbb{H}B} = \mathrm{Id}_{\mathbb{H}B}$ and $\mathbb{H}$ is separable, by Lemma 6.1.3, ε_B has a section in $\mathfrak{B}$. By Theorem 2.2.1, B is $\mathcal{P}_{\mathbb{H}}$-projective.
2) follows analogously by Lemma 6.1.3 and Theorem 2.2.3 once we observe that $\varepsilon_{\mathbb{T}A} \circ \mathbb{T}(\eta_A) = \mathrm{Id}_{\mathbb{T}A}$ for any $A \in \mathfrak{A}$. □

We are now ready to prove the main theorem of this section, that investigates whether a functor F (resp. F') preserves and reflects relative projective (resp. injective) objects.

***Theorem* 6.1.8.** *[Ar1, Theorem 3.8] Let* $(\mathbb{T},\mathbb{H})$ *and* $(\mathbb{T}',\mathbb{H}')$ *be adjunctions and assume that, in the following diagrams,* $\mathbb{T}'\circ F'$ *and* $F\circ\mathbb{T}$ *(respectively* $F'\circ\mathbb{H}$ *and* $\mathbb{H}'\circ F$*) are naturally equivalent:*

$$\begin{tikzcd} \mathfrak{A} \arrow[r, "F'"] \arrow[d, "\mathbb{T}"'] & \mathfrak{A}' \arrow[d, "\mathbb{T}'"] \\ \mathfrak{B} \arrow[r, "F"'] & \mathfrak{B}' \end{tikzcd} \qquad \begin{tikzcd} \mathfrak{A} \arrow[r, "F'"] & \mathfrak{A}' \\ \mathfrak{B} \arrow[r, "F"'] \arrow[u, "\mathbb{H}"'] & \mathfrak{B}' \arrow[u, "\mathbb{H}'"] \end{tikzcd}$$

Let P *be an object in* $\mathfrak{B}$ *and let* I *be an object in* $\mathfrak{A}$*. We have:*

a) P *is* $\mathcal{P}_{\mathbb{H}}$*-projective* $\Longrightarrow$ $F(P)$ *is* $\mathcal{P}_{\mathbb{H}'}$*-projective; the converse is true whenever* F *is separable.*

a^{op}) I *is* $\mathcal{I}_{\mathbb{T}}$*-injective* $\Longrightarrow$ $F'(I)$ *is* $\mathcal{I}_{\mathbb{T}'}$*-injective; the converse is true whenever* F' *is separable.*

Proof. a) Let $\varepsilon:\mathbb{TH}\to\mathrm{Id}_{\mathfrak{B}}$ be the counit of the adjunction $(\mathbb{T},\mathbb{H})$.
Assume that P is $\mathcal{P}_{\mathbb{H}}$-projective. Then, by Theorem 2.2.1, $\varepsilon_P:\mathbb{TH}P\to P$ has a section $\beta:P\to\mathbb{TH}P$, i.e. $\varepsilon_P\circ\beta=\mathrm{Id}_P$. Since $F(\beta)$ is a section of $F(\varepsilon_P):\mathbb{T}'\mathbb{H}'FP\sim F\mathbb{TH}P\to FP$, by applying Theorem 2.2.1 to the adjunction $(\mathbb{T}',\mathbb{H}')$ in the case when $X=\mathbb{H}'FP$ and to the split morphism $F(\varepsilon_P)$, we conclude that FP is $\mathcal{P}_{\mathbb{H}'}$-projective.
Conversely, assume FP $\mathcal{P}_{\mathbb{H}'}$-projective and F separable. Let $\eta:\mathrm{Id}_{\mathfrak{B}}\to\mathbb{HT}$ be the unit of the adjunction $(\mathbb{T},\mathbb{H})$. Thus $\mathbb{H}(\varepsilon_P)\circ\eta_{\mathbb{H}P}=\mathrm{Id}_{\mathbb{H}P}$ and hence $F'(\eta_{\mathbb{H}P})$ is a section of $F'\mathbb{H}(\varepsilon_P)$. Then also $\mathbb{H}'F(\varepsilon_P)$ has a section, so that $F(\varepsilon_P):F\mathbb{TH}P\to FP$ belongs to $\mathcal{P}_{\mathbb{H}'}$. As FP is $\mathcal{P}_{\mathbb{H}'}$-projective, by Theorem 2.2.1, we get a section in $\mathfrak{B}'$ of $F(\varepsilon_P)$. Since F is separable, by Lemma 6.1.3, we conclude that ε_P splits in $\mathfrak{B}$: hence P is $\mathcal{P}_{\mathbb{H}}$-projective.
a^{op}) follows dually. $\square$

6.2 Examples of "good" monoidal categories

For the reader sake we recall the special features of the following examples where H is a Hopf algebra. Some of them are already included in Section 5.5 but in a more general form.

• The category $(\mathfrak{M}_K,\otimes_K,K)$ of all modules over a field K.

Let $(H,m_H,u_H,\Delta_H,\varepsilon_H,S)$ be a Hopf algebra over field K. Then we have the following categories (see [Sch2] for more details).
• The category ${}_H\mathfrak{M}=({}_H\mathfrak{M},\otimes_K,K)$, of all left modules over H: the unit K is a left H-module via ε_H and the tensor $V\otimes W$ of two left H-modules can be regarded as an object in ${}_H\mathfrak{M}$ via the diagonal action. Analogously the category $\mathfrak{M}_H$ can be introduced.

• The category ${}_H\mathfrak{M}_H = ({}_H\mathfrak{M}_H, \otimes_K, K)$, of all two-sided modules over H: the unit K is a H-bimodule via ε_H and the tensor $V \otimes W$ of two H-bimodules carries, on both sides, the diagonal action.

We can dualize all the structures given for modules in order to obtain categories of comodules.
• The category ${}^H\mathfrak{M} = ({}^H\mathfrak{M}, \otimes_K, K)$, of all left comodules over H: the unit K is a left H-comodule via the map $k \mapsto 1_H \otimes k$ and the tensor product $V \otimes W$ of two left H-comodules can be regarded as an object in ${}^H\mathfrak{M}$ via the codiagonal coaction. Analogously the category $\mathfrak{M}^H$ can be introduced.
• The category ${}^H\mathfrak{M}^H = ({}^H\mathfrak{M}^H, \otimes_K, K)$ of all two-sided comodules over H: the unit K is a H-bicomodule via the maps $k \mapsto 1_H \otimes k$ and $k \mapsto k \otimes 1_H$; the tensor $V \otimes W$ of two H-bicomodules carries, on both sides, the codiagonal coaction.

We provide a list of the monoidal categories we need in the sequel. They are "good" in the sense that they are abelian or coabelian monoidal categories.

As observed in 1.3.2, given an algebra A in an abelian monoidal category $(\mathcal{M}, \otimes, \mathbf{1})$, we can construct the monoidal category of A-bimodules $({}_A\mathcal{M}_A, \otimes_A, A)$.
Applying this (in particular for $A := H$) to the categories $(\mathfrak{M}_K, \otimes_K, K)$, $(\mathfrak{M}^H, \otimes_K, K)$, $({}^H\mathfrak{M}, \otimes_K, K)$ and $({}^H\mathfrak{M}^H, \otimes_K, K)$, we obtain respectively:
• ${}_A\mathfrak{M}_A = ({}_A\mathfrak{M}_A, \otimes_A, A)$, ${}_A\mathfrak{M}_A^H = ({}_A\mathfrak{M}_A^H, \otimes_A, A)$, ${}_A^H\mathfrak{M}_A = ({}_A^H\mathfrak{M}_A, \otimes_A, A)$, ${}_A^H\mathfrak{M}_A^H = ({}_A^H\mathfrak{M}_A^H, \otimes_A, A)$.

Given a coalgebra C in a coabelian monoidal category $(\mathcal{M}, \otimes, \mathbf{1})$, we can construct the monoidal category of C-bicomodules $({}^C\mathcal{M}^C, \square_C, C)$.
Applying this (in particular for $C := H$) to the categories $(\mathfrak{M}_K, \otimes_K, K)$, $(\mathfrak{M}_H, \otimes_K, K)$, $({}_H\mathfrak{M}, \otimes_K, K)$ and $({}_H\mathfrak{M}_H, \otimes_K, K)$, we obtain respectively:
• ${}^C\mathfrak{M}^C = ({}^C\mathfrak{M}^C, \square_C, C)$, ${}^C\mathfrak{M}_H^C = ({}^C\mathfrak{M}_H^C, \square_C, C)$, ${}_H^C\mathfrak{M}^C = ({}_H^C\mathfrak{M}^C, \square_C, C)$, ${}_H^C\mathfrak{M}_H^C = ({}_H^C\mathfrak{M}_H^C, \square_C, C)$.

It is well known that $({}_H^H\mathfrak{M}_H^H, \otimes_H, H)$ and $({}_H^H\mathfrak{M}_H^H, \square_H, H)$ are equivalent monoidal categories (see [Sch2, Theorem 5.7]).

We now consider the categories of Yetter-Drinfeld modules over H. Recall that a *twisted antipode* for H is an antipode $\overline{S}$ for H^{op} (and hence also for H^{cop}). One can check that S^{-1} is a twisted antipode whenever S is bijective. If H is commutative or cocommutative then $S^2 = S \circ S = \mathrm{Id}_H$ and consequently $\overline{S} = S$.
• The category ${}_H^H\mathcal{YD} = ({}_H^H\mathcal{YD}, \otimes_K, K)$, of all left-left Yetter-Drinfeld modules over H: the unit K is a left H-comodule via the map $k \mapsto 1_H \otimes k$ and a left H-module via ε_H; the tensor product $V \otimes W$ of two left-left Yetter-Drinfeld modules can be regarded as an object in ${}_H^H\mathcal{YD}$ via the diagonal action and the codiagonal coaction. Recall that an object V in ${}_H^H\mathcal{YD}$ is a left H-module and a left H-comodule satisfying,

for any $h \in H, v \in V$, the compatibility condition:

$$\begin{aligned} (h_1v)_{<-1>}h_2 \otimes (h_1v)_{<0>} &= h_1v_{<-1>} \otimes h_2v_{<0>} \qquad \text{or} \\ (hv)_{<-1>} \otimes (hv)_{<0>} &= h_1v_{<-1>}S(h_3) \otimes h_2v_{<0>}. \end{aligned}$$

Analogously the categories $\mathcal{YD}_H^H$, ${}_H\mathcal{YD}^H$ and ${}^H\mathcal{YD}_H$ can be defined. The compatibility conditions are respectively:

$$\begin{gathered} (vh_2)_{<0>} \otimes h_1(vh_2)_{<1>} = v_{<0>}h_1 \otimes v_{<1>}h_2, \\ (h_2v)_{<0>} \otimes (h_2v)_{<1>}h_1 = h_1v_{<0>} \otimes h_2v_{<1>}, \\ h_2(vh_1)_{<-1>} \otimes (vh_1)_{<0>} = v_{<-1>}h_1 \otimes v_{<0>}h_2, \end{gathered}$$

or equivalently

$$\begin{gathered} (vh)_{<0>} \otimes (vh)_{<1>} = v_{<0>}h_2 \otimes S(h_1)v_{<1>}h_3, \\ (hv)_{<0>} \otimes (hv)_{<1>} = h_2v_{<0>} \otimes h_3v_{<1>}\overline{S}(h_1), \\ (vh)_{<-1>} \otimes (vh)_{<0>} = \overline{S}(h_3)v_{<-1>}h_1 \otimes v_{<0>}h_2, \end{gathered}$$

for all $h \in H, v \in V$ and where in the last two cases the right conditions are available when H has a twisted antipode $\overline{S}$.
The categories of Yetter-Drinfeld modules over a Hopf algebra with bijective antipode are Grothendieck categories.

6.3 Further results on separable algebras

Let us recall the following result that holds true for unitary rings.

***Proposition* 6.3.1.** *[NVdB, Proposition 1.3] For any ring homomorphism $i : S \to R$, the following are equivalent:*
(1) *R is separable in $({}_S\mathfrak{M}_S, \otimes_S, S)$, i.e. R/S is separable.*
(2) *The restriction of scalars functor ${}_R\mathfrak{M} \to {}_S\mathfrak{M}$ is separable.*
(3) *The restriction of scalars functor $\mathfrak{M}_R \to \mathfrak{M}_S$ is separable.*

As we will explain in Remark 6.3.4, the previous result, in general, can not be extended to algebras in a monoidal category.

***Lemma* 6.3.2.** *Let A be a separable algebra in a monoidal category $\mathcal{M}$. The following assertions hold true:*

1) The forgetful functor ${}_A\mathbb{H} : {}_A\mathcal{M} \to \mathcal{M}$ is separable. In particular, any left A-module $(M, {}^A\mu_M)$ is ${}_A\mathcal{P}$-projective. Moreover if M is an A-bimodule, the multiplication ${}^A\mu_M : A \otimes M \to M$ has a section ${}^A\sigma_M$ which is A-bilinear and natural in M.

2) The forgetful functor $\mathbb{H}_A : \mathcal{M}_A \to \mathcal{M}$ is separable. In particular, any right A-module (M, μ_M^A) is $\mathcal{P}_A$-projective. Moreover if M is an A-bimodule, the multiplication $\mu_M^A : M \otimes A \to M$ has a section σ_M^A which is A-bilinear and natural in M.

Proof. 1) By assumption, the multiplication m of A admits a section $\nu : A \to A \otimes A$ in ${}_A\mathcal{M}_A$. Let $(M, {}^A\mu_M)$ be a left A-module and consider the morphism ${}^A\sigma_M : M \to A \otimes M$ defined by ${}^A\sigma_M := (A \otimes l_M^A) \circ (\nu \otimes_A M) \circ (l_M^A)^{-1}$, where $l_M^A : A \otimes_A M \to M$ is the quotient of ${}^A\mu_M$. Obviously ${}^A\sigma_M \in {}_A\mathcal{M}$ (note that ${}^A\sigma_M \in {}_A\mathcal{M}_A$ whenever $M \in {}_A\mathcal{M}_A$). Moreover, from ${}^A\mu_M \circ (A \otimes l_M^A) = l_M^A \circ (m \otimes_A M)$, we get: ${}^A\mu_M \circ {}^A\sigma_M = l_M^A \circ (m \otimes_A M) \circ (\nu \otimes_A M) \circ (l_M^A)^{-1} = \mathrm{Id}_M$.
Thus ${}^A\mu_M : A \otimes M \to M$ admits a section in ${}_A\mathcal{M}$. Since ${}^A\mu$ is the counit of the adjunction $({}_A\mathbb{T}, {}_A\mathbb{H})$, and ${}^A\sigma_M$ defines a natural transformation ${}^A\sigma : \mathrm{Id}_{{}_A\mathcal{M}} \to {}_A\mathbb{T}_A\mathbb{H}$, we get, by Theorem 6.1.6, that ${}_A\mathbb{H}$ is separable. Note that, by Corollary 6.1.7, if the forgetful functor ${}_A\mathbb{H} : {}_A\mathcal{M} \to \mathcal{M}$ is separable, then any left A-module is ${}_A\mathcal{P}$-projective.
2) follows analogously. □

***Proposition* 6.3.3.** *Let H be a Hopf algebra with antipode S over a field K. The forgetful functors $\mathfrak{M}_H^H \to \mathfrak{M}^H$ and ${}^H\mathfrak{M}_H^H \to {}^H\mathfrak{M}^H$ are separable.*

Proof. Composing the functor $(-)^{coH} : \mathfrak{M}^H \to \mathfrak{M}_K$ with the forgetful functor $\mathfrak{M}_H^H \to \mathfrak{M}^H$, one gets the Sweedler's equivalence of categories $(-)^{coH} : \mathfrak{M}_H^H \to \mathfrak{M}_K$. Since, by Theorem 6.1.6, this functor is separable, by Theorem 6.1.5, the forgetful functor $\mathfrak{M}_H^H \to \mathfrak{M}^H$ is separable too.
Composing the functor $(-)^{coH} : {}^H\mathfrak{M}^H \to {}^H\mathfrak{M}$ with the forgetful functor ${}^H\mathfrak{M}_H^H \to {}^H\mathfrak{M}^H$, one gets the Sweedler's equivalence of categories $(-)^{coH} : {}^H\mathfrak{M}_H^H \to {}^H\mathfrak{M}$. As in the first part, we conclude that the forgetful functor ${}^H\mathfrak{M}_H^H \to {}^H\mathfrak{M}^H$ is separable. □

***Remark* 6.3.4.** By Lemma 6.3.2, the forgetful functor $\mathbb{H}_A : \mathcal{M}_A \to \mathcal{M}$ is separable for any separable algebra A in a monoidal category $\mathcal{M}$. The converse does not hold true. In fact, when $\mathcal{M} = \mathfrak{M}^H$ and $A = H$, the functor $\mathbb{H}_A$ is always separable (Proposition 6.3.3), but A is separable in $\mathcal{M}$ if and only if H is a semisimple algebra ([AMS1, Proposition 2.11]).

***Proposition* 6.3.5.** *Let A be an algebra in a monoidal category $\mathcal{M}$. The following assertions are equivalent:*

(a) A is separable in $\mathcal{M}$.
(b) The forgetful functor ${}_A\mathbb{H}_A : {}_A\mathcal{M}_A \to \mathcal{M}$ is separable.
(c) Any A-bimodule is $\mathcal{P}$-projective.
(d) The A-bimodule A is $\mathcal{P}$-projective.

Proof. $(a) \Rightarrow (b)$ If $(M, {}^A\mu_M, \mu_M^A)$ is an A-bimodule, by Lemma 6.3.2, there are A-bilinear natural sections ${}^A\sigma_M$ and σ_M^A, respectively of ${}^A\mu_M$ and μ_M^A. The morphism $\sigma_M := ({}^A\sigma_M \otimes A) \circ \sigma_M^A : M \to A \otimes M \otimes A$ is a section in ${}_A\mathcal{M}_A$ of the counit $\varepsilon_M := \mu_M^A \circ ({}^A\mu_M \otimes A) : A \otimes M \otimes A \to M$ of the adjunction $({}_A\mathbb{T}_{A}, {}_A\mathbb{H}_A)$. Since σ_M is natural in M, we get a natural transformation $\sigma : \mathrm{Id}_{\mathcal{M}} \to {}_A\mathbb{T}_{AA}\mathbb{H}_A$ such that $\varepsilon \circ \sigma = \mathrm{Id}_{\mathrm{Id}_{\mathcal{M}}}$. We conclude by Theorem 6.1.6.
$(b) \Rightarrow (c)$ follows by Corollary 6.1.7.
$(c) \Rightarrow (d)$ Obvious.
$(d) \Rightarrow (a)$ Since A is $\mathcal{P}$-projective, the multiplication $m : A \otimes A \to A$, that is a morphism in $\mathcal{P}$, admits a section $\sigma : A \to A \otimes A$ in ${}_A\mathcal{M}_A$. □

***Corollary* 6.3.6.** *Let A be a separable algebra in $\mathfrak{M}_K$. Then any left A-module is projective in A-$\mathfrak{Mod}$. Hence any left A-module is also injective in A-$\mathfrak{Mod}$ and A is semisimple. Moreover any A-bimodule is projective in ${}_A\mathfrak{M}_A$ and hence any A-bimodule is injective in ${}_A\mathfrak{M}_A$.*

Proof. Since $\mathcal{M} = \mathfrak{M}_K$, any epimorphism in $\mathcal{M}$ splits. So a left A-module is ${}_A\mathcal{E}$-projective iff it is projective in A-$\mathfrak{Mod}$ in the usual sense. The right and two-sided cases follow analogously. □

6.3.7. Let $(F', \phi_0, \phi_2) : (\mathcal{M}, \otimes, \mathbf{1}, a, l, r) \to (\mathcal{M}', \otimes, \mathbf{1}, a, l, r)$ be a monoidal functor between two monoidal categories, where $\phi_2(U, V) : F'(U \otimes V) \to F'(U) \otimes F'(V)$, for any $U, V \in \mathcal{M}$ and $\phi_0 : \mathbf{1} \to F'(\mathbf{1})$. Let (A, m, u) be an algebra in $\mathcal{M}$. It is well known that $(A', m_{A'}, u_{A'}) := (F'(A), m_{F'(A)}, u_{F'(A)})$ is an algebra in $\mathcal{M}'$, where

$$\begin{aligned} m_{F'(A)} &:= F'(A) \otimes F'(A) \xrightarrow{\phi_2(A,A)} F'(A \otimes A) \xrightarrow{F'(m)} F'(A) \\ u_{F'(A)} &:= \mathbf{1}' \xrightarrow{\phi_0} F'(\mathbf{1}) \xrightarrow{F'(u)} F'(A). \end{aligned}$$

Consider the functor $F : {}_A\mathcal{M}_A \to {}_{A'}\mathcal{M}'_{A'}$ defined by

$$F((M, {}^A\mu_M, \mu_M^A)) = (F'(M), {}^{A'}\mu_{F'(M)}, \mu_{F'(M)}^{A'}),$$

where

$$\begin{aligned} {}^{A'}\mu_{F'(M)} &:= F'(A) \otimes F'(M) \xrightarrow{\phi_2(A,M)} F'(A \otimes M) \xrightarrow{F'({}^A\mu_M)} F'(M) \\ \mu_{F'(M)}^{A'} &:= F'(M) \otimes F'(A) \xrightarrow{\phi_2(M,A)} F'(M \otimes A) \xrightarrow{F'(\mu_M^A)} F'(M). \end{aligned}$$

Let us study a particular case of Theorem 6.1.8.

***Proposition* 6.3.8.** *Let $\mathcal{M}$ and $\mathcal{M}'$ be abelian monoidal categories. Let A, A', F' and F as in 6.3.7. Then, in the following diagrams, $\mathbb{T}' \circ F'$ and $F \circ \mathbb{T}$ are naturally* *equivalent and* $F' \circ \mathbb{H} = \mathbb{H}' \circ F$*:*

$$\begin{tikzcd} \mathcal{M} \arrow[r, "F'"] \arrow[d, "\mathbb{T}"'] & \mathcal{M}' \arrow[d, "\mathbb{T}'"] \\ {}_A\mathcal{M}_A \arrow[r, "F"'] & {}_{A'}\mathcal{M}'_{A'} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{M} \arrow[r, "F'"] & \mathcal{M}' \\ {}_A\mathcal{M}_A \arrow[r, "F"'] \arrow[u, "\mathbb{H}"] & {}_{A'}\mathcal{M}'_{A'} \arrow[u, "\mathbb{H}'"'] \end{tikzcd}$$

where $(\mathbb{T}, \mathbb{H})$ *is the adjunction* $({}_A\mathbb{T}_A, {}_A\mathbb{H}_A)$ *defined in 2.3.1, and* $(\mathbb{T}', \mathbb{H}')$ *is analogously defined.*
We have that:

$P \in {}_A\mathcal{M}_A$ *is* $\mathcal{P}_{\mathbb{H}}$*-projective* $\Longrightarrow F(P) \in {}_{A'}\mathcal{M}'_{A'}$ *is* $\mathcal{P}_{\mathbb{H}'}$*-projective; the converse is true whenever* F *is separable.*
In particular we obtain that:

i) A *is separable in* $\mathcal{M} \Longrightarrow A'$ *is separable in* $\mathcal{M}'$ *(i.e.* $\mathbb{H}$ *is separable* $\Longrightarrow \mathbb{H}'$ *is separable); the converse is true whenever* F *is separable.*

ii) If F' *preserves kernels, then:* A *is formally smooth in* $\mathcal{M} \Longrightarrow A'$ *is formally smooth in* $\mathcal{M}'$*; the converse is true whenever* F *is separable.*

Proof. Define $\alpha_M : F'(A) \otimes F'(M) \otimes F'(A) \to F'(A \otimes M \otimes A)$ by $\alpha_M = \phi_2(A \otimes M, A)[\phi_2(A, M) \otimes F'(A)]$, for any $M \in \mathcal{M}$. Then $(\alpha_M)_{M \in \mathcal{M}}$ defines a natural equivalence $\alpha : \mathbb{T}'\mathbb{F}' \to \mathbb{F}\mathbb{T}$.
The first assertion holds by Theorem 6.1.8.
ii) By Proposition 6.3.5, A is separable in $\mathcal{M}$ iff $A \in {}_A\mathcal{M}_A$ is $\mathcal{P}_{\mathbb{H}}$-projective iff the functor $\mathbb{H}$ is separable. Analogously A' is separable in $\mathcal{M}'$ iff $A' \in {}_{A'}\mathcal{M}'_{A'}$ is $\mathcal{P}_{\mathbb{H}'}$-projective iff the functor $\mathbb{H}'$ is separable. Since $A' = F(A)$, we conclude by the first part.
iii) Let $(\Omega^1(A), j) = \ker(m_A)$ in $\mathcal{M}$. Since F' preserves kernels, we get that

$$(\Omega^1(A'), j') := \mathrm{Ker}(m_{A'}) = (F'(\Omega^1(A), \phi_2(A, A)F'(j))$$

in $\mathcal{M}'$. Observe that, $\Omega^1(A') = \ker(m_{A'}) = \ker[F'(m)\phi_2(A, A)]$. Now, if we regard regard $\Omega^1(A)$ as an A-bimodule via the structures induced by m_A and $\Omega^1(A')$ as an A'-bimodule via the structures induced by $m_{A'}$, we obtain that $\Omega^1(A') = F(\Omega^1(A))$. By definition, A is formally smooth in $\mathcal{M}$ iff $\Omega^1 A \in {}_A\mathcal{M}_A$ is $\mathcal{P}_{\mathbb{H}}$-projective. Analogously A' is formally smooth in $\mathcal{M}'$ iff $\Omega^1(A') \in {}_{A'}\mathcal{M}'_{A'}$ is $\mathcal{P}_{\mathbb{H}'}$-projective. Since $\Omega^1(A') = F(\Omega^1(A))$, we conclude by the first part. □

***Examples* 6.3.9.** Let H be a Hopf algebra over a field K. With hypotheses and notations of Proposition 6.3.8, let $\mathcal{M}' := \mathfrak{M}_K$. We want to apply the previous result in the particular case when $\mathcal{M} := ({}^H\mathfrak{M}^H, \otimes, K), (\mathfrak{M}^H, \otimes, K)$ or $({}^H\mathfrak{M}, \otimes, K)$. Let A be an algebra in $\mathcal{M}$.
1) $\mathcal{M} := {}^H\mathfrak{M}^H$. The forgetful functor $F_1 : {}^H_A\mathfrak{M}^H_A \to {}_A\mathfrak{M}_A$ has a right adjoint $G_1 : {}_A\mathfrak{M}_A \to {}^H_A\mathfrak{M}^H_A$, $G_1(M) = H \otimes M \otimes H$, where $G_1(M)$ is a bicomodule via $\Delta_H \otimes M \otimes H$

and $H \otimes M \otimes \Delta_H$, and it is a bimodule with diagonal actions. For any $M \in {}^H_A\mathfrak{M}^H_A$ the unit of the adjunction is the map $\eta_M : M \to H \otimes M \otimes H, \eta_M = ({}^H\rho_M \otimes H) \circ \rho^H_M$.
2) $\mathcal{M} := \mathfrak{M}^H$. The forgetful functor $F_r : {}_A\mathfrak{M}^H_A \to {}_A\mathfrak{M}_A$ has a right adjoint $G_r : {}_A\mathfrak{M}_A \to {}_A\mathfrak{M}^H_A$, $G_r(M) = M \otimes H$, where $G_r(M)$ is a comodule via $M \otimes \Delta_H$, and it is a bimodule with diagonal actions. For any $M \in {}_A\mathfrak{M}^H_A$ the unit of the adjunction is the map $\eta_M : M \to M \otimes H, \eta_M = \rho^H_M$.
3) $\mathcal{M} := {}^H\mathfrak{M}$. As in example 2), one can introduce the forgetful functor $F_l : {}^H_A\mathfrak{M}_A \to {}_A\mathfrak{M}_A$ and its right adjoint G_l.
In the case $A = H$ we set $(F_2, G_2) := (F_1, G_1)$.
The forgetful functor $F_b : {}^H_H\mathfrak{M}^H_H \to {}_H\mathfrak{M}^H_H$ has a right adjoint $G_b : {}_H\mathfrak{M}^H_H \to {}^H_H\mathfrak{M}^H_H$, $G_b(M) = H \otimes M$, where $G_b(M)$ is a bicomodule via $\Delta_H \otimes M$ and $M \otimes \rho^H_M$, and it is a bimodule with diagonal action.
The forgetful functor $F_a : {}_H\mathfrak{M}^H_H \to {}_H\mathfrak{M}_H$, is nothing but F_r in the case $A = H$. Then it has a right adjoint $G_a : {}_H\mathfrak{M}_H \to {}_H\mathfrak{M}^H_H$, which is G_r for $A = H$.
Note that the forgetful functor $F_2 : {}^H_H\mathfrak{M}^H_H \to {}_H\mathfrak{M}_H$ can be decomposed as $F_2 = F_a \circ F_b$.

In view of Examples 6.3.9, we obtain the following crucial result:

***Theorem* 6.3.10.** *Let H be a Hopf algebra over a field K and let $\mathcal{M}$ denote one of the categories ${}^H\mathfrak{M}^H, \mathfrak{M}^H, {}^H\mathfrak{M}$. Let A be an algebra in $\mathcal{M}$ and consider the forgetful functors $\mathbb{H} : {}_A\mathcal{M}_A \to \mathcal{M}$, $\mathbb{H}' : {}_A\mathfrak{M}_A \to \mathfrak{M}_K$ and $F : {}_A\mathcal{M}_A \to {}_A\mathfrak{M}_A$.*
We have that:

$P \in {}_A\mathcal{M}_A$ is $\mathcal{P}_{\mathbb{H}}$-projective $\Longrightarrow$ P is $\mathcal{P}_{\mathbb{H}'}$-projective as an object in ${}_A\mathfrak{M}_A$; the converse is true whenever F is separable.
In particular we obtain that:

i) A is separable as an algebra in $\mathcal{M}$ $\Longrightarrow$ A is separable as an algebra in $\mathfrak{M}_K$; the converse is true whenever F is separable.

ii) A is formally smooth as an algebra in $\mathcal{M}$ $\Longrightarrow$ A is formally smooth as an algebra in $\mathfrak{M}_K$; the converse is true whenever F is separable.

Proof. Apply Proposition 6.3.8 in the case when $\mathcal{M}' = \mathfrak{M}_K$, and $F' : \mathcal{M} \to \mathfrak{M}_K$ is the forgetful functor. $\square$

***Remark* 6.3.11.** The separability of the functor F in Theorem 6.3.10 has a relevant interest. Conditions for this separability to hold can be found in Lemma 6.6.6 and Theorem 6.6.7.

6.4 Further results on coseparable coalgebras

The whole theory of Hochschild cohomology for coalgebras and its application to coseparability and formal smoothness can be obtained from our general framework by duality, i.e. by working in the dual category of $(\mathcal{M}, \otimes, \mathbf{1}, a, l, r)$. Since this process is completely formal and does not require new ideas we will just state the main results.

***Proposition* 6.4.1.** *Let H be a Hopf algebra with antipode S over a field K. The forgetful functors $\mathfrak{M}_H^H \to \mathfrak{M}_H$ and ${}_H\mathfrak{M}_H^H \to {}_H\mathfrak{M}_H$ are separable.*

Proof. is dual to Proposition 6.3.3. □

***Proposition* 6.4.2.** *Let C be a coalgebra in a monoidal category $\mathcal{M}$. The following assertions are equivalent:*

(*a*) *C is coseparable in $\mathcal{M}$.*
(*b*) *The forgetful functor ${}^C\mathbb{T}^C : {}^C\mathcal{M}^C \to \mathcal{M}$ is separable.*
(*c*) *Any C-bicomodule is ${}^C\mathcal{I}^C$-injective.*
(*d*) *The C-bicomodule C is ${}^C\mathcal{I}^C$-injective.*

***Corollary* 6.4.3.** *Any coseparable coalgebra in a monoidal category $\mathcal{M}$ is formally smooth.*

***Corollary* 6.4.4.** *Let C be a coseparable coalgebra in $\mathfrak{M}_K$. Then any left C-comodule is injective in C-$\mathfrak{Comod}$. Hence any left C-comodule is also projective in C-$\mathfrak{Comod}$ and C is cosemisimple. Moreover any C-bicomodule is injective in ${}^C\mathfrak{M}^C$ and hence any C-bicomodule is projective in ${}^C\mathfrak{M}^C$.*

6.4.5. Let $(F', \phi_0, \phi_2) : (\mathcal{M}, \otimes, \mathbf{1}, a, l, r) \to (\mathcal{M}', \otimes, \mathbf{1}, a, l, r)$ be a monoidal functor between two monoidal categories, where $\phi_2(U, V) : F'(U \otimes V) \to F'(U) \otimes F'(V)$, for any $U, V \in \mathcal{M}$ and $\phi_0 : \mathbf{1} \to \mathbf{F}'(\mathbf{1})$. Let (C, Δ, ε) is a coalgebra in $\mathcal{M}$. It is well known that $(F'(C), \Delta_{F'(C)}, \varepsilon_{F'(C)})$ is a coalgebra in $\mathcal{M}'$, where

$$\begin{aligned} \Delta_{F'(C)} &:= F'(C) \stackrel{F'(\Delta)}{\longrightarrow} F'(C \otimes C) \stackrel{\phi_2^{-1}(C,C)}{\longrightarrow} F'(C) \otimes F'(C) \\ \varepsilon_{F'(C)} &:= F'(C) \stackrel{F'(\varepsilon)}{\longrightarrow} F'(\mathbf{1}) \stackrel{\phi_0^{-1}}{\longrightarrow} \mathbf{1}'. \end{aligned}$$

Consider the functor $F : {}^C\mathcal{M}^C \to {}^{C'}\mathcal{M}'^{C'}$ defined by

$$F((M, {}^C\rho_M, \rho_M^C)) = (F'(M), {}^{C'}\rho_{F'(M)}, \rho_{F'(M)}^{C'}),$$

where

$$\begin{aligned} {}^{C'}\rho_{F'(M)} &:= F'(C) \stackrel{F'({}^C\rho_M)}{\longrightarrow} F'(C \otimes M) \stackrel{\phi_2^{-1}(C,M)}{\longrightarrow} F'(C) \otimes F'(M) \\ \rho_{F'(M)}^{C'} &:= F'(C) \stackrel{F'(\rho_M^C)}{\longrightarrow} F'(M \otimes C) \stackrel{\phi_2^{-1}(M,C)}{\longrightarrow} F'(M) \otimes F'(C). \end{aligned}$$

***Proposition* 6.4.6.** *Let $\mathcal{M}$ and $\mathcal{M}'$ be abelian monoidal categories. Let C, C', F' and F as in Example 6.4.5. Then, in the following diagrams, $\mathbb{H}' \circ G'$ and $G \circ \mathbb{H}$ are naturally equivalent and $G' \circ \mathbb{T} = \mathbb{T}' \circ G$:*

$$\begin{tikzcd} {}^{C}\mathcal{M}^{C} \arrow[r, "G"] \arrow[d, "\mathbb{T}"'] & {}^{C'}\mathcal{M}'^{C'} \arrow[d, "\mathbb{T}'"] \\ \mathcal{M} \arrow[r, "G'"'] & \mathcal{M}' \end{tikzcd} \qquad \begin{tikzcd} {}^{C}\mathcal{M}^{C} \arrow[r, "G"] & {}^{C'}\mathcal{M}'^{C'} \\ \mathcal{M} \arrow[r, "G'"'] \arrow[u, "\mathbb{H}"] & \mathcal{M}' \arrow[u, "\mathbb{H}'"'] \end{tikzcd}$$

where $(\mathbb{T}, \mathbb{H})$ is the adjunction $({}^{C}\mathbb{T}^{C}, {}^{C}\mathbb{H}^{C})$ defined in 3.5.1, and $(\mathbb{T}', \mathbb{H}')$ is analogously defined.
We have that:

$I \in {}^{C}\mathcal{M}^{C}$ is $\mathcal{I}_{\mathbb{T}}$-injective $\Longrightarrow G(I) \in {}^{C'}\mathcal{M}'^{C'}$ is $\mathcal{I}_{\mathbb{T}'}$-injective; the converse is true whenever G is separable.
In particular we obtain that:

i) C is coseparable in $\mathcal{M} \Longrightarrow C'$ is coseparable in $\mathcal{M}'$ (i.e. $\mathbb{T}$ is separable $\Longrightarrow$ $\mathbb{T}'$ is separable); the converse is true whenever G is separable.

ii) If G' preserves cokernels, then: C is formally smooth in $\mathcal{M} \Longrightarrow C'$ is formally smooth in $\mathcal{M}'$; the converse is true whenever G is separable.

Proof. Dual to Proposition 6.3.8. □

***Examples* 6.4.7.** Let H be a Hopf algebra over a field K. With hypotheses and notations of Proposition 6.4.6, let $\mathcal{M}' := \mathfrak{M}_K$. We want to apply the previous result in the particular case when $\mathcal{M} := ({}_H\mathfrak{M}_H, \otimes, K), (\mathfrak{M}_H, \otimes, K)$ or $({}_H\mathfrak{M}, \otimes, K)$. Let C be a coalgebra in $\mathcal{M}$.
1) $\mathcal{M} := {}_H\mathfrak{M}_H$. The forgetful functor $G^1 : {}^{C}_{H}\mathfrak{M}^{C}_{H} \to {}^{C}\mathfrak{M}^{C}$ has a left adjoint $F^1 : {}^{C}\mathfrak{M}^{C} \to {}^{C}_{H}\mathfrak{M}^{C}_{H}$, $F^1(M) = H \otimes M \otimes H$, where $F^1(M)$ is a bimodule via $m_H \otimes M \otimes H$ and $H \otimes M \otimes m_H$, and it is a bicomodule with codiagonal coactions. For any $M \in {}^{C}_{H}\mathfrak{M}^{C}_{H}$ the counit of the adjunction is the map $\varepsilon_M : H \otimes M \otimes H \to M, \varepsilon_M = \mu^H_M \circ ({}^H\mu_M \otimes H)$.
2) $\mathcal{M} := \mathfrak{M}_H$. The forgetful functor $G^r : {}^{C}\mathfrak{M}^{C}_{H} \to {}^{C}\mathfrak{M}^{C}$ has a left adjoint $F^r : {}^{C}\mathfrak{M}^{C} \to {}^{C}\mathfrak{M}^{C}_{H}$, $F^r(M) = M \otimes H$, where $F^r(M)$ is a module via $M \otimes m_H$, and it is a bicomodule with codiagonal coactions. For any $M \in {}^{C}\mathfrak{M}^{C}_{H}$ the counit of the adjunction is the map $\varepsilon_M : M \otimes H \to M, \varepsilon_M = \mu^H_M$.
3) $\mathcal{M} := {}_H\mathfrak{M}$. As in example 2), one can introduce the forgetful functor $G^l : {}^{C}_{H}\mathfrak{M}^{C} \to {}^{C}\mathfrak{M}^{C}$ and its left adjoint F^l.
In the case $C = H$ we set $(F^2, G^2) := (F^1, G^1)$.
The forgetful functor $G^a : {}^{H}\mathfrak{M}^{H}_{H} \to {}^{H}\mathfrak{M}^{H}$ is nothing but G^r in the case $C = H$. Then it has a left adjoint $F^a : {}^{H}\mathfrak{M}^{H} \to {}^{H}\mathfrak{M}^{H}_{H}$, which is F^r for $C = H$.
The forgetful functor $G^b : {}^{H}_{H}\mathfrak{M}^{H}_{H} \to {}^{H}\mathfrak{M}^{H}_{H}$ has a left adjoint $F^b : {}^{H}\mathfrak{M}^{H}_{H} \to {}^{H}_{H}\mathfrak{M}^{H}_{H}$, $F^b(M) = H \otimes M$, where $F^b(M)$ is a bimodule via $m_H \otimes M$ and $H \otimes \mu^H_M$, and it is a bicomodule with codiagonal coactions.

Note that the forgetful functor $G^2 : {}_H^H\mathfrak{M}_H^H \to {}^H\mathfrak{M}^H$ can be decomposed as $G^2 = G^a \circ G^b$.

In view of Examples 6.4.7, we obtain the following crucial result:

***Theorem* 6.4.8.** *Let H be a Hopf algebra over a field K and let $\mathcal{M}$ denote one of the categories ${}_H\mathfrak{M}_H, \mathfrak{M}_H, {}_H\mathfrak{M}$. Let C be a coalgebra in $\mathcal{M}$ and consider the forgetful functors $\mathbb{T} : {}^C\mathcal{M}^C \to \mathcal{M}$, $\mathbb{T}' : {}^C\mathfrak{M}^C \to \mathfrak{M}_K$ and $G : {}^C\mathcal{M}^C \to {}^C\mathfrak{M}^C$.*
We have that:

$I \in {}^C\mathcal{M}^C$ is $\mathcal{I}_{\mathbb{T}}$-injective $\Longrightarrow$ I is $\mathcal{I}_{\mathbb{T}'}$-injective as an object in ${}^C\mathfrak{M}^C$; the converse is true whenever G is separable.
In particular we obtain that:

i) C is coseparable as a coalgebra in $\mathcal{M}$ $\Longrightarrow$ C is coseparable as a coalgebra in $\mathfrak{M}_K$; the converse is true whenever G is separable.

ii) C is formally smooth as a coalgebra in $\mathcal{M}$ $\Longrightarrow$ C is formally smooth as a coalgebra in $\mathfrak{M}_K$; the converse is true whenever G is separable.

***Remark* 6.4.9.** The separability of the functor F in Theorem 6.4.8 has a relevant interest. Conditions for this separability to hold can be found in Lemma 6.7.6 and Theorem 6.7.7.

6.5 Some adjunctions and integrals

6.5.1. Let H be a Hopf algebra with antipode S over a field K and set:

$$h \triangleright x := h_1 x S(h_2) \qquad \text{and} \qquad x \triangleleft h := S(h_1) x h_2$$
$${}^H\varrho(h) := h_1 S(h_3) \otimes h_2 \qquad \text{and} \qquad \varrho^H(h) := h_2 \otimes S(h_1) h_3$$

for all $h, x \in H$. It is easy to check that $\triangleright$ defines a left module action of H on itself called *left adjoint action* and that ${}^H\varrho$ defines a left comodule coaction of H on itself called *left adjoint coaction*. Analogously $\triangleleft$ gives rise to the right adjoint action and ϱ^H to the right adjoint coaction.
If S is bijective, we can consider the following actions and coactions of H on itself:

$$h \blacktriangleright x := h_2 x S^{-1}(h_1) \qquad \text{and} \qquad x \blacktriangleleft h := S^{-1}(h_2) x h_1$$
$$\overline{\varrho}^H(h) = h_2 \otimes h_3 S^{-1}(h_1) \qquad \text{and} \qquad {}^H\overline{\varrho}(h) := S^{-1}(h_3) h_1 \otimes h_2.$$

The structures above provide two different ways of looking at H as an object in the categories of Yetter-Drinfeld modules. In fact, if Δ_H is the comultiplication and m_H is the multiplication of H, then H can be regarded as an object in ${}_H^H\mathcal{YD}$, $\mathcal{YD}_H^H$, ${}_H\mathcal{YD}^H$, ${}^H\mathcal{YD}_H$ respectively via:

$$(\triangleright, \Delta_H), (\triangleleft, \Delta_H), (\blacktriangleright, \Delta_H), (\blacktriangleleft, \Delta_H) \qquad \text{or}$$
$$(m_H, {}^H\varrho), (m_H, \varrho^H), (m_H, \overline{\varrho}^H), (m_H, {}^H\overline{\varrho}).$$

6.5.2. The adjunctions.
The actions recalled in 6.5.1 are closely linked to the categories of Yetter-Drinfeld modules. We now consider some adjunctions involving these modules that will be very useful in finding equivalent conditions to the existence of an *ad*-invariant integral.
1) The forgetful functor $F_3 : {}_H^H\mathcal{YD} \to {}_H\mathfrak{M}$ has a right adjoint $G_3 : {}_H\mathfrak{M} \to {}_H^H\mathcal{YD}, G(M) = H \otimes M$, where $G(M)$ is a comodule via $\Delta_H \otimes M$ and a module via the action: $h \cdot (l \otimes m) = h_1 l S(h_3) \otimes h_2 m$. For any $M \in {}_H^H\mathcal{YD}$ the unit of the adjunction is the map $\eta_M : M \to H \otimes M, \eta_M = {}^H\rho_M$.
2) The forgetful functor $F_4 : \mathcal{YD}_H^H \to \mathfrak{M}_H$ has a right adjoint $G_4 : \mathfrak{M}_H \to \mathcal{YD}_H^H, G_4(M) = M \otimes H$, where $G_4(M)$ is a comodule via $M \otimes \Delta_H$ and a module via the action: $(m \otimes l) \cdot h = mh_2 \otimes S(h_1) l h_3$. For any $M \in \mathcal{YD}_H^H$ the unit of the adjunction is the map $\eta_M : M \to M \otimes H, \eta_M = \rho_M^H$.
3) Assume H has bijective antipode. The forgetful functor $F_5 : {}_H\mathcal{YD}^H \to {}_H\mathfrak{M}$ has a right adjoint $G_5 : {}_H\mathfrak{M} \to {}_H\mathcal{YD}^H, G_5(M) = M \otimes H$, where $G_5(M)$ is a comodule via $M \otimes \Delta_H$ and a module via the action: $h \cdot (l \otimes m) = h_2 l \otimes h_3 m S^{-1}(h_1)$. For any $M \in {}_H\mathcal{YD}^H$ the unit of the adjunction is the map $\eta_M : M \to M \otimes H, \eta_M = \rho_M^H$.
4) Assume H has bijective antipode. The forgetful functor $F_6 : {}^H\mathcal{YD}_H \to \mathfrak{M}_H$ has a right adjoint $G_6 : \mathfrak{M}_H \to {}^H\mathcal{YD}_H, G_6(M) = H \otimes M$, where $G_6(M)$ is a comodule via $\Delta_H \otimes M$ and a module via the action: $(l \otimes m) \cdot h = S^{-1}(h_3) l h_1 \otimes m h_2$. For any $M \in {}^H\mathcal{YD}_H$ the unit of the adjunction is the map $\eta_M : M \to H \otimes M, \eta_M = {}^H\rho_M$.

Consider now the dual version of this functors.

1^{op}) The forgetful functor $G^3 : {}_H^H\mathcal{YD} \to {}^H\mathfrak{M}$ has a left adjoint $F^3 : {}^H\mathfrak{M} \to {}_H^H\mathcal{YD}, F^3(M) = H \otimes M$, where $F^3(M)$ is a module via $m_H \otimes M$ and a comodule via the coaction: ${}^H\rho(h \otimes m) = h_1 m_{-1} S(h_3) \otimes h_2 \otimes m_0$. For any $M \in {}_H^H\mathcal{YD}$ the counit of the adjunction is the map $\varepsilon_M : H \otimes M \to M, \varepsilon_M = {}^H\mu_M$.
2^{op}) The forgetful functor $G^4 : \mathcal{YD}_H^H \to \mathfrak{M}^H$ has a left adjoint $F^4 : \mathfrak{M}^H \to \mathcal{YD}_H^H, F^4(M) = M \otimes H$, where $F^4(M)$ is a module via $M \otimes m_H$ and a comodule via the coaction: $\rho^H(m \otimes h) = m_0 \otimes h_2 \otimes S(h_1) m_1 h_3$. For any $M \in \mathcal{YD}_H^H$ the counit of the adjunction is the map $\varepsilon_M^4 : M \otimes H \to M, \varepsilon_M^4 = \mu_M^H$.
3^{op}) Assume H has bijective antipode. The forgetful functor $G^5 : {}_H\mathcal{YD}^H \to \mathfrak{M}^H$ has a left adjoint $F^5 : \mathfrak{M}^H \to {}_H\mathcal{YD}^H, F^5(M) = H \otimes M$, where $F^5(M)$ is a module via $m_H \otimes M$ and a comodule via the coaction: $\rho^H(h \otimes m) = h_2 \otimes m_0 \otimes h_3 m_1 S^{-1}(h_1)$. For any $M \in {}_H\mathcal{YD}^H$ the counit of the adjunction is the map $\varepsilon_M : H \otimes M \to M, \varepsilon_M = {}^H\mu_M$.
4^{op}) Assume H has bijective antipode. The forgetful functor $G^6 : {}^H\mathcal{YD}_H \to {}^H\mathfrak{M}$ has a left adjoint $F^6 : {}^H\mathfrak{M} \to {}^H\mathcal{YD}_H, F^6(M) = M \otimes H$, where $F^6(M)$ is a module via $M \otimes m_H$ and a comodule via the coaction: ${}^H\rho(m \otimes h) = S^{-1}(h_3) m_{-1} h_1 \otimes m_0 \otimes h_2$. For any $M \in {}^H\mathcal{YD}_H$ the counit of the adjunction is the map $\varepsilon_M : M \otimes H \to M, \varepsilon_M = \mu_M^H$.

6.5.3. Integrals. Let K be any field. An augmented K-algebra (A, m, u, p) is a K-algebra (A, m, u) endowed with an algebra homomorphism $p : A \to K$ called augmentation of A. An element $x \in A$ is a *left integral in* A, whenever $a \cdot_A x = p(a)\, x$, for every $a \in A$. The definition of a right integral in A is analogous. A is called *unimodular*, whenever the space of left and right integrals in A coincide. A left (resp. right) integral x in A is called a *left (resp. right) total integral in* A, whenever $p(x) = 1_K$.

Let $(H, m_H, u_H, \Delta_H, \varepsilon_H)$ be a bialgebra.

1) $(H, m_H, u_H, \varepsilon_H)$ is an augmented algebra. Then a left integral in H is an element $t \in H$ such that $h \cdot_H t = \varepsilon_H(h)\, t$, for every $h \in H$. Moreover t is total whenever $\varepsilon_H(t) = 1_K$. It can be proved that $S_H(t)$ is a right integral in H when t is a left integral in H and that $t = S_H(t)$ under the further hypothesis that t is total. In particular a left total integral is a right total integral. The converse also holds true so that one can speak of total integral in H without specifying "left" or "right".

2) $(H^*, m_{H^*}, u_{H^*}, \varepsilon_{H^*})$ is an augmented algebra. Then a left integral in H^* is an element $\lambda \in H^*$, that is a K-linear map $f\lambda = f(1_H)\, \lambda$, for every $f \in H^*$. Moreover λ is total, whenever $\lambda(1_H) = 1_K$. It is clear that $\lambda \in H^*$ is a left (resp. right) integral in H^* if and only if $h_1\lambda(h_2) = 1_H\lambda(h)$ (resp. $\lambda(h_1)h_2 = \lambda(h)1_H$) for every $h \in H$. By arguments similar to the ones used in 1), one can speak of total integral in H^* without specifying "left" or "right".

If H is finite dimensional, H^* becomes a Hopf algebra: in particular one can consider the notion of left integral in $(H^*)^*$ in the sense of 2). By means of the isomorphism

$$H \to H^{**} : h \longmapsto \left(\begin{array}{c} H^* \to K \\ f \longmapsto f(h) \end{array} \right),$$

one can check that a left integral in H^{**} is nothing but a left integral in H in the sense of 1): thus there is no danger of confusion.

For the reader's sake, we outline the following facts.

Theorem 6.5.4. *Let H be a Hopf algebra with antipode S over any field K. Then we have:*

1) There exists a total integral $t \in H$ (i.e. H is semisimple) if and only if H is separable.

2) There exists a total integral $\lambda \in H^*$ (i.e. H is cosemisimple) if and only if H is coseparable.

Proof. 1) " $\Leftarrow$ " Let $\sigma : H \to H \otimes H$ an H-bilinear section of the multiplication m and set $t_\sigma := (H \otimes \varepsilon_H)\sigma(1_H) \in H$. Then t_σ is a total integral.
" $\Rightarrow$ " Let $t \in H$ be a total integral. Since t is a left integral and Δ_H is an homomorphism of algebras, we have:

$$ht_1 \otimes S(t_2) = h_1t_1 \otimes S(h_2t_2)h_3 = \varepsilon_H(h_1)t_1 \otimes S(t_2)h_2 = t_1 \otimes S(t_2)h, \forall h \in H, \tag{6.1}$$

so that the map $\sigma_t : H \to H \otimes H : h \mapsto ht_1 \otimes S(t_2)$ is H-bilinear. Moreover $m_H\sigma_t(h) = ht_1S(t_2) = h\varepsilon_H(t) = h$, so that σ_t is an H-bilinear section of m_H and H is separable by definition.
2) " $\Leftarrow$ " Let $\theta : H \otimes H \to H$ an H-bicolinear retraction of the comultiplication Δ and set $\lambda_\theta := \varepsilon_H\theta(- \otimes 1_H) \in H^*$. Then λ_θ is a total integral.
" $\Rightarrow$ " Let integral $\lambda \in H^*$ be a left integral such that $\lambda(1_H) = 1$. Since λ is a left integral and m is an homomorphism of coalgebras, we have:
(6.2)

$$x_1\lambda(x_2S(y)) = x_1S(y_2)\lambda(x_2S(y_1))y_3 = (xS(y_1))_1\lambda((xS(y_1))_2)y_2 = \lambda(xS(y_1))y_2,$$

for every $x, y \in H$, so that the map $\theta_\lambda : H \otimes H \to H : x \otimes y \mapsto x_1\lambda(x_2S(y))$ is H-bicolinear. Moreover $\theta_\lambda\Delta(h) = h_1\lambda(h_2S(h_3)) = h\lambda(1_H) = h$, so that θ_λ is an H-bicolinear retraction of the comultiplication Δ and H is coseparable by definition.

□

6.6 Ad-invariant integrals

Next aim is to characterize the existence of a so called *ad*-invariant integral.
A remarkable fact is that any semisimple and cosemisimple Hopf algebra H over a field K admits such an integral (see Theorem 6.8.5).

***Definition* 6.6.1.** [SVO, Definition 1.11] Let H be a Hopf algebra with antipode S over any field K and let $\lambda \in H^*$.
λ will be called an *ad-invariant integral* whenever:
a) $h_1\lambda(h_2) = 1_H\lambda(h)$ for all $h \in H$ (i.e. λ is a left integral in H^*);
b) $\lambda(h_1xS(h_2)) = \varepsilon(h)\lambda(x)$, for all $h, x \in H$ (i.e. λ is left linear with respect to $\triangleright$);
c) $\lambda(1_H) = 1_K$.

***Lemma* 6.6.2.** *An element $\lambda \in H^*$ is an ad-invariant integral if and only if it is a retraction of the unit $u_H : K \to H$ of H in ${}^H_H\mathcal{YD}$, where H is regarded as an object in the category via the left adjoint action $\triangleright$ and the comultiplication Δ_H.*

***Examples* 6.6.3.** 1) Let G be an arbitrary group an let KG be the group algebra associated. Let $\lambda : KG \to K$ be defined by $\lambda(g) = \delta_{e,g}$ (the Kronecker symbol), where e denotes the neutral element of G. Then λ is an *ad*-invariant integral for KG (see [SVO, Corollary 2.8]).
2) Every commutative cosemisimple Hopf algebra has an *ad*-invariant integral.
3) As we will see in Lemma 6.8.5, any semisimple and cosemisimple Hopf algebra has an *ad*-invariant integral.

***Remark* 6.6.4.** It is known (see [DNR, Theorem 5.3.2 and Proposition 5.5.3]) that, for any Hopf algebra H with a total integral $\lambda \in H^*$, the K-linear space of left and right integrals in H^* are both one dimensional and hence both generated by λ. Hence there can be only one *ad*-invariant integral, namely the unique total integral.

The following lemma shows that in the definition of *ad*-invariant integral we can choose $\triangleleft, \blacktriangleright$ or $\blacktriangleleft$ instead of $\triangleright$. Since λ is in particular a total integral, it is both a left and a right integral. Thus it is the same to have a retraction of u_H in ${}^H_H\mathcal{YD}, \mathcal{YD}^H_H, {}_H\mathcal{YD}^H$ or ${}^H\mathcal{YD}_H$.

Lemma 6.6.5. *Let H be a Hopf algebra with antipode S over any field K and let $\lambda \in H^*$ be a total integral. Then the following are equivalent:*

(1) *λ is left linear with respect to $\triangleright$.*
(2) *λ is right linear with respect to $\triangleleft$.*
(3) *λ is left linear with respect to $\blacktriangleright$.*
(4) *λ is right linear with respect to $\blacktriangleleft$.*

Proof. We have that λ is both a left integral and a right integral for H^*.
Since λ is a total integral S is bijective (see [DNR, Corollary 5.4.6]) and hence it makes since to consider S^{-1}.
(1) $\Rightarrow$ (2) Observe that: $S(x \triangleleft h) = S(S(h_1)xh_2) = S(h)_1 S(x) S[S(h)_2] = S(h) \triangleright S(x)$.
Thus, since $\lambda = \lambda S$ and λ is left linear with respect to $\triangleright$, we get $\lambda(x \triangleleft h) = \lambda S(x \triangleleft h) = \lambda(S(h) \triangleright S(x)) = \varepsilon S(h)\lambda(S(x)) = \varepsilon(h)\lambda S(x) = \varepsilon(h)\lambda(x)$ that is λ is right linear with respect to $\triangleleft$.
(2) $\Rightarrow$ (1) follows analogously once proved the relation $S(h \triangleright x) = S(x) \triangleleft S(h)$.
(1) $\Rightarrow$ (3) We have: $S[h \blacktriangleright S^{-1}(x)] = S[h_2 S^{-1}(x) S^{-1}(h_1)] = h_1 x S(h_2) = h \triangleright x$.
Then, since $\lambda = \lambda S$ and λ is left linear with respect to $\triangleright$, we have $\lambda(h \blacktriangleright x) = \lambda S(h \blacktriangleright S^{-1}S(x)) = \lambda(h \triangleright S(x)) = \varepsilon(h)\lambda S(x) = \varepsilon(h)\lambda(x)$ i.e. λ is left linear with respect to $\blacktriangleright$.
(3) $\Rightarrow$ (1) Since λ is left linear with respect to $\blacktriangleright$ one has $\lambda(h \triangleright x) = \lambda S[h \blacktriangleright S^{-1}(x)] = \lambda[h \blacktriangleright S^{-1}(x)] = \varepsilon(h)\lambda S S^{-1}(x) = \varepsilon(h)\lambda(x)$ i.e. λ is left linear with respect to $\triangleright$.
(1) $\Leftrightarrow$ (4) Analogous to (1) $\Leftrightarrow$ (3) by means of $S^{-1}[S(x) \triangleleft h] = x \blacktriangleleft h$. $\square$

The following result improves [AMS1, Theorem 2.29].

Lemma 6.6.6. *Let H be a Hopf algebra with antipode S over a field K. Assume there exists an ad-invariant integral $\lambda \in H^*$. Then we have that:*

i) The forgetful functor ${}_A\mathfrak{M}^H_A \to {}_A\mathfrak{M}_A$ is separable for any algebra A in $\mathfrak{M}^H$.
ii) The forgetful functor ${}^H_A\mathfrak{M}_A \to {}_A\mathfrak{M}_A$ is separable for any algebra A in ${}^H\mathfrak{M}$.
iii) The forgetful functor ${}^H_A\mathfrak{M}^H_A \to {}_A\mathfrak{M}_A$ is separable for any algebra A in ${}^H\mathfrak{M}^H$.

Proof. *i)* By Examples 6.3.9, the forgetful functor $F_r : {}_A\mathfrak{M}^H_A \to {}_A\mathfrak{M}_A$ has a right adjoint $G_r : {}_A\mathfrak{M}_A \to {}_A\mathfrak{M}^H_A, G_8(M) = M \otimes H$. Thus by Theorem 6.1.6, F_r is separable if and only if the unit $\eta^H : \mathrm{Id}_{{}_A\mathfrak{M}^H_A} \to G_r F_r$ of the adjunction cosplits, i.e. there exists a natural transformation $\mu^H : G_rF_r \to \mathrm{Id}_{{}_A\mathfrak{M}_A^H}$ such that $\mu_M^H \circ \eta_M^H = \mathrm{Id}_M$ for any M in ${}_A\mathfrak{M}_A^H$. Let us define:

$$\mu_M^H : M \otimes H \to M, \quad \mu_M^H(m \otimes h) = m_0\lambda(m_1S(h)).$$

Obviously $(\mu_M^H)_{M \in {}_A\mathfrak{M}_A^H}$ is a functorial morphism.
Let us check that μ_M^H is a morphism in ${}_A\mathfrak{M}_A^H$, i.e. a morphism of A-bimodules and of H-bicomodules. Since $\mu_M^A \in \mathfrak{M}^H$, we have: $\mu_M^H((m\otimes h)a) = m_0a_0\lambda(m_1a_1S(a_2)S(h)) = \mu_M^H(m \otimes h)a$.
Since ${}^A\mu_M \in \mathfrak{M}^H$ and as λ satisfies relation b) of Definition 6.6.1, we get that μ_M^H is also left A-linear: $\mu_M^H(a(m \otimes h)) = a_0m_0\lambda(a_1 \rhd m_1S(h)) = a\mu_M^H(m \otimes h)$.
By (6.2), we have: $\lambda(xS(y_1))y_2 = x_1\lambda(x_2S(y)), \forall x, y \in H$. Thus we get also the right H-collinearity of μ_M^H: $(\mu_M^H \otimes H)\rho^H(m \otimes h) = m_0 \otimes \lambda(m_1S(h_1))h_2 = m_0 \otimes m_1\lambda(m_2S(h)) = \rho^H\mu_M^H(m \otimes h)$.
It remains to prove that μ_M^H is a retraction of η_M^H: $\mu_M^H\eta_M^H(m) = m_0\lambda(m_1S(m_2)) = m\lambda(1_H) = m$.
ii) Analogous to *i*) by setting ${}^H\mu_M(h \otimes m) = \lambda(hS(m_{-1}))m_0$.
iii) We have to construct a functorial retract of $(\eta_M)_{M \in {}_A^H\mathfrak{M}_A^H}$,where $\eta_M = ({}_M^H\eta \otimes H) \circ \eta_M^H$. By the previous part, there are a functorial retraction $(\mu_M^H)_{M \in {}_A\mathfrak{M}_A^H}$ of $(\sigma_M^H)_{M \in {}_A\mathfrak{M}_A^H}$ and a functorial retract $({}^H\mu_M)_{M \in {}_A^H\mathfrak{M}_A}$ of $({}^H\sigma_M)_{M \in {}_A^H\mathfrak{M}_A}$. Let us define the morphism $\mu_M : H \otimes M \otimes H \to M$ by $\mu_M = \mu_M^H \circ ({}^H\mu_M \otimes H)$. Obviously it is a retraction of σ_M in ${}_A\mathfrak{M}_A^H$. It is easy to prove that $\mu_M = {}^H\mu_M \circ (H \otimes \mu_M^H)$: hence one gets that μ_M is a morphism in ${}_A^H\mathfrak{M}_A^H$. □

Theorem 6.6.7. *[Ar1, Theorem 5.11] Let H be a Hopf algebra with antipode S over a field K. The following assertions are equivalent:*

(1) *There is an ad-invariant integral $\lambda \in H^*$.*
(2) *The forgetful functor ${}_A^H\mathfrak{M}_A^H \to {}_A\mathfrak{M}_A$ is separable for any algebra A in ${}^H\mathfrak{M}^H$.*
(3) *The forgetful functor ${}_H^H\mathfrak{M}_H^H \to {}_H\mathfrak{M}_H$ is separable.*
(3*b*) *H is coseparable in $({}_H\mathfrak{M}_H, \otimes, K)$.*
(4) *The forgetful functor ${}_H^H\mathcal{YD} \to {}_H\mathfrak{M}$ is separable.*
(4*b*) *K is $\mathcal{I}_F$-injective where F is the forgetful functor of* (4).

Proof. (1) ⇒ (2) follows by Lemma 6.6.6.
(2) ⇒ (3). Obvious.
(3) ⇔ (3*b*). It is just Proposition 6.4.2 applied to $\mathcal{M} = ({}_H\mathfrak{M}_H, \otimes, K)$.
(3) ⇒ (4). Take the notations of Examples 6.3.9 and 6.5.2. Since $F_2 : {}_H^H\mathfrak{M}_H^H \to {}_H\mathfrak{M}_H$ is separable and $F_2 = F_a \circ F_b$, where $F_b : {}_H^H\mathfrak{M}_H^H \to {}_H\mathfrak{M}_H^H$ and F_a: ${}_H\mathfrak{M}_H^H \to {}_H\mathfrak{M}_H$, then, by Theorem 6.1.5, F_b is separable. Consider the inverses $(F')^{-1}$ and F^{-1} respectively of the functors $F' = (-)^{coH} : {}_H^H\mathfrak{M}_H^H \to {}_H^H\mathcal{YD}$ and $F = (-)^{coH} : {}_H\mathfrak{M}_H^H \to {}_H\mathfrak{M}$ (these are category equivalences; see [Sch2, Theorem 5.7]). One can easily check that $F^{-1} \circ F_3 = F_b \circ (F')^{-1}$. By Theorem 6.1.5, $(F')^{-1}$ is separable so that $F^{-1} \circ F_3$,

and hence F_3, is a separable functor.
(4) $\Rightarrow$ (4b). By Corollary 6.1.7 the separability of $F_3 : {}_H^H\mathcal{YD} \to {}_H\mathfrak{M}$ (that has G_3 as aright adjoint) implies that any object in ${}_H^H\mathcal{YD}$, in particular K, is $\mathcal{I}_{F_3}$-injective.
(4b) $\Rightarrow$ (1). Observe that u_H can be regarded as a morphism in ${}_H^H\mathcal{YD}$, once H is regarded as an object in ${}_H^H\mathcal{YD}$ via the action $\triangleright$ (defined in 6.5.1) and the coaction given by the comultiplication Δ. In particular, u_H belongs to $\mathcal{I}_{F_3}$: in fact the counit ε_H of H is a left linear retraction of $F_3(u_H)$. Hence, since K is $\mathcal{I}_{F_3}$-injective, there is $\lambda : H \to K$ in ${}_H^H\mathcal{YD}$ such that $\lambda \circ u_H = \mathrm{Id}_K$, i.e., by Lemma 6.6.2, an ad-invariant integral. $\square$

***Remark* 6.6.8.** The following assertions are all equivalent to the existence of an ad-invariant integral $\lambda \in H^*$.

(5) The forgetful functor $\mathcal{YD}_H^H \to \mathfrak{M}_H$ is separable.
(6) The forgetful functor ${}_H\mathcal{YD}^H \to {}_H\mathfrak{M}$ is separable and S is bijective.
(7) The forgetful functor ${}^H\mathcal{YD}_H \to \mathfrak{M}_H$ is separable and S is bijective.
(8) K is $\mathcal{I}_F$-injective where F is the forgetful functor of (5),(6) or (7).

In fact, note that ${}_H^H\mathfrak{M}_H^H \simeq \mathcal{YD}_H^H$. Since λ is in particular a total integral, the antipode S is bijective and hence, by [Sch2, Corollary 6.4], we can also assume ${}^H\mathcal{YD}_H \simeq {}_H^H\mathfrak{M}_H^H \simeq {}_H\mathcal{YD}^H$. Now, by means of Lemma 6.6.5, one can proceed like in the proof of Theorem 6.6.7.

6.6.9. Let H be a Hopf algebra with bijective antipode (e.g. H f.d.). For every $h, h' \in H$ and $f, f' \in H^*$, define

$$(h \rightharpoonup f)(h') = f(h'h) \quad \text{and} \quad (f \leftharpoonup h)(h') = f(hh'),$$
$$h \rightharpoondown f = \sum h_{(1)} \rightharpoonup f \leftharpoonup S^{-1}\left(h_{(2)}\right) \quad \text{and} \quad f \leftharpoondown h = \sum S^{-1}\left(h_{(1)}\right) \rightharpoonup f \leftharpoonup h_{(2)}.$$

The Drinfeld Double $D(H) = H^{*cop} \bowtie H$ (see [Mo, Definition 10.3.5, page 188]) is a Hopf algebra that has $H^{*cop} \otimes H$ as its underlying vector space. The multiplication is given by

$$(f \bowtie h)(f' \bowtie h') = \sum f(h_{(1)} \rightharpoondown f'_{(2)}) \bowtie (h_{(2)} \leftharpoondown f'_{(1)})f$$

for all $h, h' \in H$ and $f, f' \in H^*$ with identity

$$1_{D(H)} = \varepsilon_H \bowtie 1_H.$$

The comultiplication is given by

$$\Delta_{D(H)}(f \bowtie h) = \sum \left(f_{(1)} \bowtie h_{(1)}\right) \otimes \left(f_{(2)} \bowtie h_{(2)}\right).$$

The counit is

$$\varepsilon_{D(H)}(f \bowtie h) = f(1_H)\,\varepsilon_H(h).$$

The antipode is defined by

$$\begin{aligned} S_{D(H)}(f \bowtie h) &= \sum \left[S_H\left(h_{(2)}\right) \rightharpoonup S_{H^*}\left(f_{(1)}\right) \right] \otimes \left[f_{(2)} \rightharpoonup S_H\left(h_{(1)}\right) \right] \\ &= \sum \left[S_{H^*}\left(f_{(2)}\right) \leftharpoonup h_{(1)} \right] \otimes \left[S_H\left(h_{(2)}\right) \leftharpoonup S_{H^*}\left(f_{(1)}\right) \right]. \end{aligned}$$

***Theorem* 6.6.10.** *[Ar1, Theorem 5.13] Let H be a finite dimensional Hopf algebra with antipode S over a field K and let $D(H)$ be the Drinfeld Double. The following assertions are equivalent:*

(i) There is an ad-invariant integral $\lambda \in H^*$.

(ii) The forgetful functor ${}_{D(H)}\mathfrak{M} \to {}_H\mathfrak{M}$ is separable.

(iii) $D(H)$ is separable in $({}_H\mathfrak{M}_H, \otimes_H, H)$, i.e. $D(H)/H$ is separable.

Proof. $(i) \Leftrightarrow (ii)$. Since H is finite dimensional, it has bijective antipode. Hence we have ${}_H^H\mathcal{YD} \simeq {}_H\mathcal{YD}^H \simeq {}_{D(H)}\mathfrak{M}$. By Theorem 6.6.7, (i) holds iff the forgetful functor ${}_H^H\mathcal{YD} \to {}_H\mathfrak{M}$ is separable iff ${}_{D(H)}\mathfrak{M} \to {}_H\mathfrak{M}$ is separable.
$(ii) \Leftrightarrow (iii)$. follows by Proposition 6.3.1 applied to the ring homomorphism $H \to D(H) = H^{*cop} \bowtie H : h \mapsto \varepsilon_H \bowtie h$. □

***Proposition* 6.6.11.** *Let H be a Hopf algebra with an ad-invariant integral $\lambda \in H^*$ and let $\mathcal{M} = \mathfrak{M}^H, {}^H\mathfrak{M}, {}^H\mathfrak{M}^H$. For any algebra A in $\mathcal{M}$, we have:*

i) A is separable as an algebra in $\mathcal{M}$ iff it is separable as an algebra in $\mathfrak{M}_K$.

ii) A is formally smooth as an algebra in $\mathcal{M}$ iff it is formally smooth as an algebra in $\mathfrak{M}_K$.

Proof. Since H has an *ad*-invariant integral λ, by Lemma 6.6.6, the forgetful functor $F : {}_A\mathcal{M}_A \to {}_A\mathfrak{M}_A$ is separable. By Theorem 6.3.10 we conclude. □

***Proposition* 6.6.12.** *Let H be a Hopf algebra and let A and E be algebras in $\mathcal{M} = \mathfrak{M}^H, {}^H\mathfrak{M}$. Let $\pi : E \to A$ be an algebra homomorphism in $\mathcal{M}$ which is surjective. Assume that A is formally smooth as an algebra in $\mathcal{M}$ and that the kernel of π is a nilpotent ideal. Given an algebra homomorphism $f : H \to A$ in $\mathcal{M}$, then π has a section which is an algebra homomorphism in $\mathcal{M}$.*

Proof. $\mathcal{M} = \mathfrak{M}^H$) Let I denote the kernel of π and assume there is an $n \in \mathbb{N}$ such that $I^n = 0$. First of all let us observe that, since π is a morphism in $\mathfrak{M}^H$, I is a subobject of E in $\mathfrak{M}^H$. Hence, for every $r > 0$, I^r is a subobject of E and the canonical maps $E/I^{r+1} \to E/I^r$ are morphisms in $\mathfrak{M}^H$.
Now, the object I^r/I^{r+1} has a natural module structure over $E/I \simeq A$, and hence, via f, a module structure over H. With respect to this structure I^r/I^{r+1} is an object in $\mathfrak{M}_H^H$. Via the category equivalences $\mathfrak{M}_H^H \simeq {}_K\mathfrak{M}$, we get that I^r/I^{r+1} is a cofree right comodule i.e. $I^r/I^{r+1} \simeq V \otimes H$ in $\mathfrak{M}_H^H$, for a suitable $V \in {}_K\mathfrak{M}$. In particular I^r/I^{r+1} is an injective comodule, so any canonical map $E/I^{r+1} \to E/I^r$

has a section in $\mathfrak{M}^H$.
By Theorem 3.4.10, we conclude.
$\mathcal{M} = {}^H\mathfrak{M}$) follows analogously. $\square$

***Example* 6.6.13.** Let H be a Hopf algebra and assume that H is formally smooth in $\mathfrak{M}^H$. Then, by Corollary 3.4.9 and Theorem 4.2.1, the tensor algebra $T := T_H(\mathrm{Ker}(m_H))$ is formally smooth as an algebra in the monoidal category $\mathfrak{M}^H$. Assume that $\pi : E \to T$ is an epimorphism that is also a morphism of algebras in $\mathfrak{M}^H$ such that $I := \mathrm{Ker}(\pi)$ is a nilpotent coideal. By Proposition 6.6.12, applied in the case when $f : H \to T$ is the canonical injection, π has a section which is an algebra homomorphism in $\mathfrak{M}^H$. (In particular also the projection $E \to T \to H$ has a section which is an algebra homomorphism in $\mathfrak{M}^H$). Observe that T is not semisimple in general because its dimension needs not to be finite.

***Theorem* 6.6.14.** *[Ar1, Theorem 5.18] Let H be a Hopf algebra and let E be an algebra in $\mathcal{M} = \mathfrak{M}^H, {}^H\mathfrak{M}, {}^H\mathfrak{M}^H$. Let $\pi : E \to H$ be an algebra homomorphism in $\mathcal{M}$ which is surjective. Assume that H is formally smooth as an algebra in $\mathcal{M}$ and that the kernel I of π is a nilpotent ideal. Then π has a section which is an algebra homomorphism in $\mathcal{M}$ for*
a) $\mathcal{M} = \mathfrak{M}^H$ or ${}^H\mathfrak{M}$.
b) $\mathcal{M} = {}^H\mathfrak{M}^H$ if any canonical map $E/I^{r+1} \to E/I^r$ splits in $\mathcal{M}$.

Proof. Since π is a morphism in $\mathcal{M}$, the kernel I of π is a subobject of E in $\mathcal{M}$. Hence, for every $r > 0$, I^r is a subobject of E and the canonical maps $E/I^{r+1} \to E/I^r$ are morphisms in $\mathcal{M}$.
a) Apply Proposition 6.6.12 in the case when $E := H$ and $f := \mathrm{Id}_H$.
b) follows easily by Theorem 3.4.6. $\square$

Proposition 6.6.12 studies the existence in $\mathcal{M} = \mathfrak{M}^H, {}^H\mathfrak{M}$ of algebra sections of morphisms of algebras $\pi : E \to A$ where A is a formally smooth algebra in $\mathcal{M}$ endowed with a morphism of algebras $f : H \to A$ in $\mathcal{M}$. The following results show that the existence of *ad*-invariant integrals provides such a section in $\mathcal{M} = \mathfrak{M}^H, {}^H\mathfrak{M}, {}^H\mathfrak{M}^H$ (without f).

***Lemma* 6.6.15.** *Let H be a Hopf algebra with a total integral $\lambda \in H^*$. Then any epimorphism in $\mathcal{M} = \mathfrak{M}^H, {}^H\mathfrak{M}, {}^H\mathfrak{M}^H$ has a section in $\mathcal{M}$.*

Proof. Since λ is a total integral in H^*, then, by Theorem 6.5.4, H is coseparable in $\mathfrak{M}_K$. Therefore any right (resp. left, two-sided) H-comodule is projective (see Corollary 6.4.4). In particular any epimorphism in $\mathcal{M}$ has a section in $\mathcal{M}$. $\square$

***Theorem* 6.6.16.** *[Ar1, Theorem 5.20] Let H be a Hopf algebra with an ad-invariant integral $\lambda \in H^*$. Let A and E be algebras in $\mathcal{M} = \mathfrak{M}^H\ {}^H\mathfrak{M}, {}^H\mathfrak{M}^H$. Let $\pi : E \to A$ be an algebra homomorphism in $\mathcal{M}$ which is surjective. Assume that*

A is formally smooth as an algebra in $\mathfrak{M}_K$ and that the kernel of π is a nilpotent ideal. Then π has a section which is an algebra homomorphism in $\mathcal{M}$.

Proof. By Proposition 6.6.11, A is formally smooth as an algebra in $\mathcal{M}$. Let $n \geq 1$ such that $I^n = 0$, where $I = \mathrm{Ker}(\pi)$. Since, in particular, λ is a total integral, by Lemma 6.6.15, any epimorphism in the category $\mathcal{M}$ splits in $\mathcal{M}$. Thus, for every $r = 1, \cdots, n-1$ the canonical morphism $\pi_r : E/I^r \to E/I^{r+1}$ has a section in the category $\mathcal{M}$. We can now conclude by applying Theorem 3.4.6 to the homomorphism of algebras $\pi : E \to A$. □

***Theorem* 6.6.17.** *[Ar1, Theorem 5.21] Let H be a Hopf algebra with an ad-invariant integral and such that H is formally smooth as an algebra in ${}_K\mathfrak{M}$. Let E be an algebra in $\mathcal{M} = \mathfrak{M}^H, {}^H\mathfrak{M}, {}^H\mathfrak{M}^H$. Let $\pi : E \to H$ be a algebra homomorphism in $\mathcal{M}$ which is surjective and with nilpotent kernel. Then π has a section which is an algebra homomorphism in $\mathcal{M}$.*

***Remark* 6.6.18.** By Proposition 6.6.11, if H is a Hopf algebra with an *ad*-invariant integral and H is formally smooth as an algebra in $({}_K\mathfrak{M}, \otimes, K)$, then it is formally smooth as an algebra in $(\mathfrak{M}^H, \otimes, K)$. Then the case $\mathcal{M} = \mathfrak{M}^H$ (analogously $\mathcal{M} = {}^H\mathfrak{M}$) of the above corollary can be also deduced by Theorem 6.6.14.

6.7 Ad-coinvariant integrals

We want now to treat the dual of all the results of the previous section. We just state the main results that can be proved analogously.

First of all we characterize the existence of a so called *ad*-coinvariant integral.
A remarkable fact is that any semisimple and cosemisimple Hopf algebra H over a field K admits such an integral (see Theorem 6.8.5).

***Definition* 6.7.1.** Let H be a Hopf algebra with antipode S over any field K and let $t \in H$.
t will be called an *ad-coinvariant integral* whenever:
a) $ht = \varepsilon_H(h)t$ for all $h \in H$ (i.e. t is a left integral in H);
b) $t_1 S(t_3) \otimes t_2 = 1_H \otimes t$, (i.e. t is left coinvariant with respect to ${}^H\varrho$);
c) $\varepsilon_H(t) = 1_K$.
Therefore we have:

***Lemma* 6.7.2.** *An element $t \in H$ is an ad-coinvariant integral if and only if the map $\tau : K \to H : k \mapsto kt$ is a section of the counit $\varepsilon_H : H \to K$ of H in ${}^H_H\mathcal{YD}$, where H is regarded as an object in the category via the left adjoint coaction ${}^H\varrho$ and the multiplication m_H.*

***Example* 6.7.3.** 1) Let G be a finite group an let K^G be the algebra of functions from G to K. Then K^G becomes a Hopf algebra which is dual to the group algebra KG. From Example 6.6.3, we infer that K^G has an *ad*-coivariant integral, namely the map $G \to K : g \mapsto \delta_{e,g}$ (the Kronecker symbol), where e denotes the neutral element of G.
2) Every cocommutative semisimple Hopf algebra has an *ad*-coinvariant integral.
3) As we will see in Lemma 6.8.5, any semisimple and cosemisimple Hopf algebra has an *ad*-coinvariant integral .

***Remark* 6.7.4.** It is known that, for any Hopf algebra H with a total integral $t \in H$, the K-linear spaces of left and right integrals in H are both one dimensional and so both generated by t. Hence there can be only one *ad*-coinvariant integral, namely the unique total integral.

The following lemma shows that in the definition of *ad*-coinvariant integral we can choose $\varrho^H, \overline{\varrho}^H$ or ${}^H\overline{\varrho}$ instead of ${}^H\varrho$. Since t is in particular a total integral, it is both a left integral and a right integral. Thus it is the same to have a retraction of ε_H in ${}_H^H\mathcal{YD}, \mathcal{YD}_H^H, {}_H\mathcal{YD}^H$ or ${}^H\mathcal{YD}_H$.

Lemma 6.7.5. *Let H be a Hopf algebra with antipode S over any field K and let $t \in H$ be a total integral. Then the following are equivalent:*
(1) *t is left coinvariant with respect to ${}^H\varrho$.*
(2) *t is right coinvariant with respect to ϱ^H.*
(3) *t is right coinvariant with respect to $\overline{\varrho}^H$.*
(4) *t is left coinvariant with respect to ${}^H\overline{\varrho}$.*

Proof. Analogous to 6.6.5. □

Lemma 6.7.6. *Let H be a Hopf algebra with antipode S over a field K. Assume there exists an ad-coinvariant integral $t \in H$. Then we have that:*
i) The forgetful functor ${}^C\mathfrak{M}_H^C \to {}^C\mathfrak{M}^C$ is separable for any coalgebra C in $\mathfrak{M}_H$.
ii) The forgetful functor ${}_H^C\mathfrak{M}^C \to {}^C\mathfrak{M}^C$ is separable for any coalgebra C in ${}_H\mathfrak{M}$.
iii) The forgetful functor ${}_H^C\mathfrak{M}_H^C \to {}^C\mathfrak{M}^C$ is separable for any coalgebra C in ${}_H\mathfrak{M}_H$.

Proof. We proceed as in the proof of Lemma 6.6.6.
i) By Examples 6.4.7, the forgetful functor $G^r : {}^C\mathfrak{M}_H^C \to {}^C\mathfrak{M}^C$ has a right adjoint $F^r : {}^C\mathfrak{M}^C \to {}^C\mathfrak{M}_H^C, F^r(M) = M \otimes H$. Thus by Theorem 6.1.6, G^r is separable if and only if the counit $\varepsilon^H : F^rG^r \to \mathrm{Id}_{{}^C\mathfrak{M}_H^C}$ of the adjunction splits, i.e. there exists a natural transformation $\sigma^H : \mathrm{Id}_{{}^C\mathfrak{M}_H^C} \to F^rG^r$ such that $\varepsilon_M^H \circ \sigma_M^H = \mathrm{Id}_M$ for any M in ${}^C\mathfrak{M}_H^C$. Using (6.1), one can easily check that the following map works: $\sigma_M^H : M \to M \otimes H$, $\sigma_M^H(m) = mt_1 \otimes S(t_2)$.
ii) Analogous to *i*) by setting ${}^H\sigma_M(m) = t_1 \otimes S(t_2)m$.
iii) Define $\sigma_M := ({}^H\sigma_M \otimes H) \circ \sigma_M^H : M \to H \otimes M \otimes H$. □

We can now consider the main result concerning *ad*-coinvariant integrals. The equivalence (1) $\Leftrightarrow$ (3b) was proved in a different way in [AMS1, Proposition 2.11].

***Theorem* 6.7.7.** *[Ar1, Theorem 7.7] Let H be a Hopf algebra with antipode S over a field K. The following assertions are equivalent:*

(1) *There is an ad-coinvariant integral $t \in H$.*

(2) *The forgetful functor ${}_H^C\mathfrak{M}_H^C \to {}^C\mathfrak{M}^C$ is separable for any coalgebra C in ${}_H\mathfrak{M}_H$.*

(3) *The forgetful functor ${}_H^H\mathfrak{M}_H^H \to {}^H\mathfrak{M}^H$ is separable.*

(3b) *H is separable in $({}^H\mathfrak{M}^H, \otimes, K)$.*

(4) *The forgetful functor ${}_H^H\mathcal{YD} \to {}^H\mathfrak{M}$ is separable.*

(4b) *K is $\mathcal{E}_G$-projective where G is the forgetful functor of* (4).

Proof. Analogous to that of Theorem 6.6.7. □

***Remark* 6.7.8.** The following assertions are all equivalent to the existence of an *ad*-coinvariant integral $t \in H$:

(5) The forgetful functor $\mathcal{YD}_H^H \to \mathfrak{M}^H$ is separable.

(6) The forgetful functor ${}_H\mathcal{YD}^H \to \mathfrak{M}^H$ is separable and S is bijective.

(7) The forgetful functor ${}^H\mathcal{YD}_H \to {}^H\mathfrak{M}$ is separable and S is bijective.

(8) K is $\mathcal{E}_G$-projective where G is the forgetful functor of (5),(6) or (7).

In fact, note that ${}_H^H\mathfrak{M}_H^H \simeq \mathcal{YD}_H^H$. Since t is in particular a total integral, the antipode S is bijective and hence, by [Sch2, Corollary 6.4], we can also assume ${}^H\mathcal{YD}_H \simeq {}_H^H\mathfrak{M}_H^H \simeq {}_H\mathcal{YD}^H$. Now, by means of Lemma 6.7.5. one prove the above equivalences.

***Theorem* 6.7.9.** *[Ar1, Theorem 7.9] Let H be a finite dimensional Hopf algebra with antipode S over a field K and let $D(H)$ be the Drinfeld Double. The following assertions are equivalent:*

(i) *There is an ad-coinvariant integral $t \in H$.*

(ii) *The forgetful functor $\mathfrak{M}^{D(H)^*} \to \mathfrak{M}^H$ (equiv. ${}_{D(H)}\mathfrak{M} \to {}_{H^*}\mathfrak{M}$) is separable.*

(iii) *$D(H)^*$ is coseparable in $({}^H\mathfrak{M}^H, \Box_H, H)$ (equiv. $D(H)/H^*$ is separable).*

Proof. dual to Theorem 6.6.10. □

Proposition 6.7.10. *Let H be a Hopf algebra with an ad-coinvariant integral t and let $\mathcal{M} = \mathfrak{M}_H, {}_H\mathfrak{M}, {}_H\mathfrak{M}_H$. For any coalgebra C in $\mathcal{M}$, we have:*

i) C is coseparable as a coalgebra in $\mathcal{M}$ iff it is coseparable as a coalgebra in $\mathfrak{M}_K$.

ii) C is formally smooth as a coalgebra in $\mathcal{M}$ iff it is formally smooth in $\mathfrak{M}_K$.

Proof. Since H has an *ad*-coinvariant integral t, by Lemma 6.7.6, the forgetful functor $G : {}^C\mathcal{M}^C \to {}^C\mathfrak{M}^C$ is separable. By Theorem 6.4.8 we conclude. □

***Remark* 6.7.11.** Let $\mathcal{M}$ be one of the monoidal categories $\mathfrak{M}_K, \mathfrak{M}_H, {}_H\mathfrak{M}$ or ${}_H\mathfrak{M}_H$. Let C be a subcoalgebra of a coalgebra E in $\mathcal{M}$. Then $C^{\wedge^1_E} \subseteq \cdots \subseteq C^{\wedge^n_E} \subseteq C^{\wedge^{n+1}_E} \subseteq \cdots \subseteq E$.
Moreover, by [Sw, Remark and Proposition, page 226], one has that $\cup_{n\in\mathbb{N}} C^{\wedge^n_E} = E$ if and only if $Corad(E) \subseteq C$. Note that $\cup_{n\in\mathbb{N}} C^{\wedge^n_E} = \varinjlim C^{\wedge^i_E}$.

***Theorem* 6.7.12.** *[Ar1, Theorem 7.15] Let H be a Hopf algebra. Let C be a subcoalgebra of a coalgebra E in $\mathcal{M} = \mathfrak{M}_K, \mathfrak{M}_H, {}_H\mathfrak{M}$ or ${}_H\mathfrak{M}_H$. Assume that C is formally smooth as a coalgebra in $\mathcal{M}$ and that $Corad(E) \subseteq C$. If any inclusion map $i_r : C^{\wedge^r_E} \to C^{\wedge^{r+1}_E}$ cosplits in $\mathcal{M}$, then there exists a coalgebra homomorphism $\pi : E \to C$ in $\mathcal{M}$ such that $\pi_{|C} = \mathrm{Id}_C$.*

Proof. As observed in Remark 6.7.11, we have $E = \cup_{n\in\mathbb{N}} C^{\wedge^n_E} = \varinjlim C^{\wedge^i_E}$. The conclusion follows by applying Theorem 3.5.15. $\square$

***Proposition* 6.7.13.** *Let H be a Hopf algebra. Let C be a subcoalgebra of a coalgebra E in $\mathcal{M} = \mathfrak{M}_H, {}_H\mathfrak{M}$. Assume that C is formally smooth as a coalgebra in $\mathcal{M}$ and that $Corad(E) \subseteq C$. Given a coalgebra homomorphism $g : C \to H$ in $\mathcal{M}$, then there exists a coalgebra homomorphism $\pi : E \to C$ in $\mathcal{M}$ such that $\pi_{|C} = \mathrm{Id}_C$.*

Proof. $\mathcal{M} = \mathfrak{M}_H$) In order to apply Theorem 6.7.12, we have only to prove that any inclusion map $C^{\wedge^n_E} \hookrightarrow C^{\wedge^{n+1}_E}$ cosplits in $\mathfrak{M}_H$. Since

$$C^{\wedge^{n+1}_E} = C^{\wedge^n_E} \wedge_E C = C \wedge_E C^{\wedge^n_E} = \Delta_E^{-1}(E \otimes C + C^{\wedge^n_E} \otimes E),$$

the quotient $C^{\wedge^{n+1}_E}/C^{\wedge^n_E}$ becomes a right C-comodule in $\mathfrak{M}_H$ via the map ρ_n^C, given by $x + C^{\wedge^n_E} \mapsto (x_1 + C^{\wedge^n_E}) \otimes x_2$. Since $g : C \to H$ is a morphism of coalgebras in $\mathfrak{M}_H$, then $(\mathrm{Id} \otimes g) \circ \rho_n^C$ is a right H-comodule structure map for $C^{\wedge^{n+1}_E}/C^{\wedge^n_E}$ that is right H-linear. Thus $C^{\wedge^{n+1}_E}/C^{\wedge^n_E}$ becomes an object in $\mathfrak{M}_H^H$: by the fundamental theorem for Hopf modules ($\mathfrak{M}_H^H \simeq {}_K\mathfrak{M}$), we get that $C^{\wedge^{n+1}_E}/C^{\wedge^n_E} \simeq V \otimes H$ in $\mathfrak{M}_H^H$, for a suitable $V \in {}_K\mathfrak{M}$, i.e. $C^{\wedge^{n+1}_E}/C^{\wedge^n_E}$ is a free right H-module. In particular $C^{\wedge^{n+1}_E}/C^{\wedge^n_E}$ is a projective right H-module, so that the inclusion map $i : C^{\wedge^n_E} \hookrightarrow C^{\wedge^{n+1}_E}$ has a retraction in $\mathfrak{M}_H$.
$\mathcal{M} = {}_H\mathfrak{M}$) follows analogously. $\square$

***Example* 6.7.14.** Let H be a Hopf algebra and assume that H is formally smooth as a coalgebra in $\mathfrak{M}_H$. Then, by Corollary 3.5.13 and Theorem 5.4.8, the cotensor coalgebra $T := T_H^c(\mathrm{Coker}(\Delta_H))$ is formally smooth as a coalgebra in the monoidal category $\mathfrak{M}_H$. Assume that $\sigma : T \to E$ is an monomorphism that is also a morphism of coalgebras in $\mathfrak{M}_H$. By Proposition 6.7.13, applied in the case when $g : T \to H$ is the canonical projection, σ has a retraction which is a coalgebra homomorphism in $\mathfrak{M}_H$. (In particular also the injection $H \to T \to E$ has a retraction which is a coalgebra homomorphism in $\mathfrak{M}_H$). Observe that T is not cosemisimple in general, because its coradical is included in H (see Theorem 5.3.7 and [Sw, Proposition 11.1.1, page 226]).

***Theorem* 6.7.15.** *[Ar1, Theorem 7.17] Let H be a Hopf algebra which is a subcoalgebra of a coalgebra E in $\mathcal{M} = \mathfrak{M}_H, {}_H\mathfrak{M}$ or ${}_H\mathfrak{M}_H$. Assume that H is formally smooth as a coalgebra in $\mathcal{M}$ and that $Corad(E) \subseteq H$. Then there exists a coalgebra homomorphism $\pi : E \to H$ in $\mathcal{M}$ such that $\pi_{|H} = \mathrm{Id}_H$ for*

a) $\mathcal{M} = \mathfrak{M}_H$ or ${}_H\mathfrak{M}$.

b) $\mathcal{M} = {}_H\mathfrak{M}_H$ if any inclusion map $H^{\wedge_E^n} \hookrightarrow H^{\wedge_E^{n+1}}$ cosplits in $\mathcal{M}$.

Proof. $H^{\wedge_E^n}$ is a subcoalgebra of E in $\mathcal{M}$ and the inclusion map $H^{\wedge_E^n} \hookrightarrow H^{\wedge_E^{n+1}}$ is obviously a morphism in $\mathcal{M}$.
a) Apply Proposition 6.7.13 in the case when $C := H$ and $g := \mathrm{Id}_H$.
b) Apply Theorem 6.7.12 in the case when $C = H$. □

***Examples* 6.7.16.** Let E be a coalgebra in the category of vector spaces. Let $C = Corad(E)$. In this case, the sequence $(C^{\wedge_E^n})_{n\in\mathbb{N}}$ is simply denoted by $(E_n)_{n\in\mathbb{N}}$ and it is the so called coradical filtration of E.
Let H be a Hopf algebra. Assume that E is a coalgebra in $\mathcal{M} = \mathfrak{M}_H, {}_H\mathfrak{M}_H$ and that $H = C = Corad(E)$. We have two cases.
$\mathcal{M} = {}_H\mathfrak{M}_H$) If any inclusion $E_n \hookrightarrow E_{n+1}$ cosplits in ${}_H\mathfrak{M}_H$ and H is formally smooth as a coalgebra in ${}_H\mathfrak{M}_H$, then, by Theorem 6.7.15, there is an homomorphisms of coalgebras $\pi : E \to H$ in ${}_H\mathfrak{M}_H$ such that $\pi_{|H} = \mathrm{Id}_H$.
$\mathcal{M} = \mathfrak{M}_H$) By [AMS1, Theorem 2.11], since H is cosemisimple in $\mathfrak{M}_K$, then H is coseparable in $\mathfrak{M}_H$. In particular H is formally smooth as a coalgebra in $\mathfrak{M}_H$. Again, by Theorem 6.7.15, there is an homomorphisms of coalgebras $\pi : E \to H$ in $\mathfrak{M}_H$ such that $\pi_{|H} = \mathrm{Id}_H$ (see also [AMS1, Theorem 2.17]).

Proposition 6.7.13 studies the existence in $\mathcal{M} = \mathfrak{M}_H, {}_H\mathfrak{M}$ of coalgebra retractions of coalgebras inclusion $C \hookrightarrow E$ where C is a formally smooth coalgebras in $\mathcal{M}$ endowed with a morphism of coalgebras $g : C \to H$ in $\mathcal{M}$. The following results show that the existence of ad-coinvariant integrals provide such a section in $\mathcal{M} = \mathfrak{M}_H, {}_H\mathfrak{M}, {}_H\mathfrak{M}_H$ (without g).

***Lemma* 6.7.17.** *Let H be a Hopf algebra with a total integral $t \in H$. Then any monomorphism in $\mathcal{M} = \mathfrak{M}_H, {}_H\mathfrak{M}, {}_H\mathfrak{M}_H$ has a retraction in $\mathcal{M}$.*

Proof. Since t is a total integral in H, then H is separable by Theorem 6.5.4-2). Therefore any right (resp. left, two-sided) H-module is injective (see Corollary 6.3.6). In particular any monomorphism in $\mathcal{M}$ has a retraction in $\mathcal{M}$. □

***Theorem* 6.7.18.** *[Ar1, Theorem 7.20] Let H be a Hopf algebra with an ad-coinvariant integral $t \in H$. Let C be a subcoalgebra of a coalgebra E in $\mathcal{M} =$*

$\mathfrak{M}_H, {}_H\mathfrak{M}, {}_H\mathfrak{M}_H$. Assume that C is formally smooth as a coalgebra in $\mathfrak{M}_K$ and that $Corad(E) \subseteq C$. Then there exists a coalgebra homomorphism $\pi : E \to C$ in $\mathcal{M}$ such that $\pi_{|C} = \mathrm{Id}_C$.

Proof. By Proposition 6.7.10, C is formally smooth as a coalgebra in $\mathcal{M}$. Since t is in particular a total integral in H, by Lemma 6.7.17, any monomorphism in $\mathcal{M}$, in particular the inclusion map $C^{\wedge^n_E} \hookrightarrow C^{\wedge^{n+1}_E}$ for any $n \in \mathbb{N}$, has a retraction in $\mathcal{M}$. Now apply Theorem 6.7.12. □

***Theorem* 6.7.19.** *[Ar1, Theorem 7.21] Let H be a Hopf algebra with an ad-coinvariant integral and such that H is formally smooth as a coalgebra in $\mathfrak{M}_K$. If H is a subcoalgebra of a coalgebra E in $\mathcal{M} = \mathfrak{M}_H, {}_H\mathfrak{M}, {}_H\mathfrak{M}_H$ and $Corad(E) \subseteq H$, then there exists a coalgebra homomorphism $\pi : E \to H$ in $\mathcal{M}$ such that $\pi_{|H} = \mathrm{Id}_H$.*

***Remark* 6.7.20.** By Proposition 6.7.10, if H is a Hopf algebra with an ad-coinvariant integral and H is formally smooth as a coalgebra in $(\mathfrak{M}_K, \otimes, K)$, then it is formally smooth as a coalgebra in $(\mathfrak{M}_H, \otimes, K)$. Then the case $\mathcal{M} = \mathfrak{M}_H$ (analogously $\mathcal{M} = {}_H\mathfrak{M}$) of the above corollary can be also deduced by Theorem 6.7.15.

The following result provides a significant example of Hopf algebra endowed with both an ad-invariant and ad-coinvariant integral.

6.8 Splitting morphism of bialgebras

We now give some application of the previous results.

***Theorem* 6.8.1.** *[Ar1, Theorem 5.32] Let H be a Hopf algebra and let E be a bialgebra. Let $\pi : E \to H$ be a bialgebra homomorphism which is surjective. Assume that H is formally smooth as an algebra in $\mathfrak{M}_K$ and that the kernel I of π is a nilpotent ideal. Then π has a section which is an algebra homomorphism in $\mathfrak{M}^H$ (resp. ${}^H\mathfrak{M}$).*

Proof. In view of [Ar1, Proposition 5.27], H is formally smooth as an algebra in $\mathfrak{M}_K$ if and only if it is formally smooth as an algebra in $\mathfrak{M}^H$ (resp. ${}^H\mathfrak{M}$). Since H as a coalgebra is a quotient of E, then E carries a unique H-bicomodule structure that makes of π a coalgebra homomorphism in ${}^H\mathfrak{M}^H$. By Theorem 6.6.14, we conclude. □

***Definition* 6.8.2.** [Sch1, Definition 5.1] Let E be a bialgebra and let H be a Hopf subalgebra of E. Recall that a *weak right (resp. left) projection* (onto H) is a retraction $\pi : E \to H$ for the inclusion map which is a right (resp. left) H-linear coalgebra map. We call π a *weak two-sided projection*, whenever π is also right H-linear.

6.8.3. A bialgebra with a projection is a bialgebra E over a field K endowed with a Hopf algebras H and two bialgebra maps $\sigma : H \to E$ and $\pi : E \to H$ such that $\pi \circ \sigma = \mathrm{Id}_H$. In [Rad2], M. D. Radford describes the structure of bialgebras with a projection: E can be decomposed as the smash product of H with the (right) H-coinvariant part of E which comes out to be a braided bialgebra in the monoidal category ${}_H^H\mathcal{YD}$ of Yetter-Drinfeld modules over H. This construction appeared as an important tool in the classification of finite dimensional Hopf algebras. It is meaningful that, even relaxing some assumption on π (as was done by P. Schauenburg in [Sch1]) or on σ (see [AMS1]), it is possible to reconstruct E by means of a suitable bosonization type procedure.

More precisely, let E be a bialgebra and let H be a Hopf subalgebra of E. Denote by $\sigma : H \to E$ the inclusion. Assume there exists a weak right projection $\pi : E \to H$ of E onto H (the dual situation can be treated analogously). Consider the space of right H-coinvariant elements of E:

$$R = E^{co(H)} = \left\{ e \in E \mid \sum e_{(1)} \otimes \pi\left(e_{(2)}\right) = e \otimes 1_H \right\}$$

Set

$$\tau : E \to R, \tau(e) = \sum e_{(1)} \sigma S \pi\left(e_{(2)}\right)$$

which is a well defined map as

$$\begin{aligned}
(E \otimes \pi)\Delta_E \tau(e) &= \sum {e_{(1)}}_{(1)} \sigma S \pi\left(e_{(2)}\right)_{(1)} \otimes \pi\left[{e_{(1)}}_{(2)} \sigma S \pi\left(e_{(2)}\right)_{(2)}\right] \\
&= \sum {e_{(1)}}_{(1)} \sigma S \pi\left({e_{(2)}}_{(2)}\right) \otimes \pi\left[{e_{(1)}}_{(2)} \sigma S \pi\left({e_{(2)}}_{(1)}\right)\right] \\
&= \sum e_{(1)} \sigma S \pi\left(e_{(4)}\right) \otimes \pi\left[e_{(2)} \sigma S \pi\left(e_{(3)}\right)\right] \\
&= \sum e_{(1)} \sigma S \pi\left(e_{(4)}\right) \otimes \pi\left(e_{(2)}\right) S \pi\left(e_{(3)}\right) \\
&= \sum e_{(1)} \sigma S \pi\left(e_{(3)}\right) \otimes \pi\left(e_{(2)}\right)_{(1)} S\left[\pi\left(e_{(2)}\right)_{(2)}\right] \\
&= \sum e_{(1)} \sigma S \pi\left(e_{(3)}\right) \otimes \varepsilon_H \pi\left(e_{(2)}\right) 1_H \\
&= \tau(e) \otimes 1_H
\end{aligned}$$

and hence $\tau(e) \in R$, for every $e \in E$. The map

$$\epsilon : R \otimes H \to E,\ \epsilon(r \otimes h) = r\sigma(h)$$

is an isomorphism of K-vector spaces, the inverse being defined by

$$\epsilon^{-1} : E \to R \otimes H,\ \epsilon^{-1}(e) = \sum \tau\left(e_{(1)}\right) \otimes \pi\left(e_{(2)}\right).$$

In fact we have

$$\begin{aligned}\tau\left[a\sigma\left(h\right)\right] &= \sum a_{(1)}\sigma\left(h_{(1)}\right)\sigma S_H\pi\left[a_{(2)}\sigma\left(h_{(2)}\right)\right]\\ &= \sum a_{(1)}\sigma\left(h_{(1)}\right)\sigma S_H\left[\pi\left(a_{(2)}\right)h_{(2)}\right]\\ &= \tau\left(a\right)\varepsilon_H\left(h\right)\end{aligned}$$

and hence

$$\begin{aligned}\epsilon^{-1}\epsilon(r\otimes h) &= \epsilon^{-1}\left[r\sigma(h)\right]\\ &= \sum \tau\left[r_{(1)}\sigma(h_{(1)})\right]\otimes\pi\left[r_{(2)}\sigma(h_{(2)})\right]\\ &= \sum \tau\left(r_{(1)}\right)\varepsilon_H(h_{(1)})\otimes\pi\left(r_{(2)}\right)h_{(2)}\\ &= \sum \tau\left(r_{(1)}\right)\otimes\pi\left(r_{(2)}\right)h\\ &= \tau\left(r\right)\otimes\pi\left(1_H\right)h\\ &= r\otimes h\end{aligned}$$

and

$$\begin{aligned}\epsilon\epsilon^{-1}(e) &= \sum \epsilon\left[\tau\left(e_{(1)}\right)\otimes\pi\left(e_{(2)}\right)\right]\\ &= \sum \tau\left(e_{(1)}\right)\sigma\pi\left(e_{(2)}\right)\\ &= \sum e_{(1)}\sigma S\pi\left(e_{(2)}\right)\sigma\pi\left(e_{(3)}\right)\\ &= \sum e_{(1)}\sigma\left\{S\left[\pi\left(e_{(2)}\right)_{(1)}\right]\pi\left(e_{(2)}\right)_{(2)}\right\}\\ &= \sum e_{(1)}\sigma\left\{\varepsilon_H\pi\left(e_{(2)}\right)1_H\right\}\\ &= e.\end{aligned}$$

Clearly E defines, via ϵ, a bialgebra structure on $R\otimes H$ that will depend on the chosen σ and π. This bialgebra structure has been described in [Sch1, Section 5] and in [Sch3, Section 5]. If π is also left H-linear (i.e. it is a week two-sided projection) then (see [AMS1, Theorem 3.64]) to such an (E,π,σ) one associates a quadruple (R,u,m,ξ) (called dual Yetter-Drinfeld quadruple), where R, as defined above, is a coalgebra in the monoidal category $({}^H_H\mathcal{YD},\otimes,K)$ and $u:K\to R$, $m:R\otimes R\to R$, and $\xi:R\otimes R\to H$ are K-linear maps satisfying ten equalities. Then E can be reconstructed by these data. In fact the bialgebra E is isomorphic to $R\#_\xi H$ which is $R\otimes H$ endowed with a suitable bialgebra structure that depends on the dual Yetter-Drinfeld quadruple: this structure on $R\otimes H$ can be somehow regarded as a deformation of the usual bosonization structure recalled above via ξ.

Theorem 6.8.4. *[Ar1, Theorem 7.36] Let H be a Hopf subalgebra of a bialgebra E. Assume that H is formally smooth as a coalgebra in $\mathfrak{M}_K$ and that $Corad(E)\subseteq H$. Then E has a weak right (resp. left) projection onto H.*

Proof. In view of [Ar1, Proposition 7.27], H is formally smooth as a coalgebra in $\mathfrak{M}_K$ if and only if it is formally smooth as a coalgebra in $\mathfrak{M}_H$ (resp. ${}_H\mathfrak{M}$). Since H is a subalgebra of E, then E carries a unique H-bimodule structure that makes of H a subcoalgebra of E in ${}_H\mathfrak{M}_H$. By Theorem 6.7.15, we conclude. □

***Theorem* 6.8.5.** *[AMS1, Theorem 2.27] Let H be a semisimple and cosemisimple Hopf algebra over a field K. Then there are:*
1) *an ad-invariant integral $\lambda \in H^*$;*
2) *an ad-coinvariant integral $t \in H$.*

Proof. First let us note that any semisimple Hopf algebra is finite dimensional (see [Mo]).
Since H is semisimple and cosemisimple, by [Rad1, Proposition 7], the Drinfeld double $D(H)$ is semisimple. By a result essentially due to Majid (see [Mo, Proposition 10.6.16]), and by [RT, Proposition 6], we get that the category ${}_H^H\mathcal{YD} \simeq {}_{D(H)}\mathfrak{M}$ is semisimple. Then the counit $\varepsilon : H \to K$ has a section in ${}_H^H\mathcal{YD}$ so that, by Lemma 6.7.2, there is an *ad*-coinvariant integral. Analogously the unit $u : K \to H$ has a retraction in ${}_H^H\mathcal{YD}$ so that, by Lemma 6.6.2, there is an *ad*-invariant integral. □

***Theorem* 6.8.6.** *[AMS1, Theorem 2.28] Let E be a Hopf algebra such that J, the Jacobson radical of E, is a nilpotent coideal in E. Assume that $H := E/J$ is both semisimple and cosemisimple (e.g. H is semisimple over a field of characteristic 0). Then there is an algebra homomorphism $\sigma : H \to E$ in ${}^H\mathfrak{M}^H$ such that $\pi\sigma = Id_H$, where $\pi : E \to H$ denotes the canonical projection.*

Proof. Since H is semisimple and cosemisimple, in view of Theorem 6.8.5, there is an *ad*-invariant integral. Since H as a coalgebra is a quotient of E, then E carries a unique H-bicomodule structure that makes of π a coalgebra homomorphism in ${}^H\mathfrak{M}^H$. By Theorem 6.5.4, H is separable as an algebra in ${}_K\mathfrak{M}$ so that, in view of Corollary 3.4.8, it is formally smooth as an algebra in ${}_K\mathfrak{M}$. By Theorem 6.6.17, we conclude. □

***Theorem* 6.8.7.** *[AMS1, Theorem 2.35] Let E be a Hopf algebra such that H, the coradical of E, is a Hopf subalgebra. Assume that H is semisimple as an algebra (e.g. H is f.d. over a field of characteristic 0). Then E has a weak two-sided projection onto H.*

Proof. Since H is semisimple and cosemisimple, in view of Theorem 6.8.5, there is an *ad*-coinvariant integral. Since H is a subalgebra of E, then E carries a unique H-bimodule structure that makes of H a subcoalgebra of E in ${}_H\mathfrak{M}_H$. By Theorem 6.5.4, H is coseparable as a coalgebra in ${}_K\mathfrak{M}$ so that, in view of Corollary 3.5.12, it is formally smooth as a coalgebra in ${}_K\mathfrak{M}$. By Theorem 6.7.19, we conclude. □

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