On Fragility of Bubbles in Equilibrium Asset Pricing Models of Lucas-Type

On Fragility of Bubbles in Equilibrium Asset Pricing Models of Lucas-Type*

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Abstract

In this paper we study the existence of bubbles for pricing equilibria in a pure exchange economy à la Lucas, with infinitely lived homogeneous agents. The model is analyzed under fairly general assumptions: no restrictions either on the stochastic process governing dividends' distribution or on the utilities (possibly unbounded) are required. We prove that the pricing equilibrium is unique as long as the agents exhibit uniformly bounded relative risk aversion. A generic uniqueness result is also given regardless of agent's preferences. A few "pathological" examples of economies exhibiting pricing equilibria with bubble components are constructed. Finally, a possible relationship between our approach and the theory developed by Santos and Woodford on ambiguous bubbles is investigated. The whole discussion sheds more insight on the common belief that bubbles are a marginal phenomenon in such models.

JEL Classification Numbers: C61, C62, D51, G12.

1 Introduction

The main objective of this paper is to test how reasonable is the conjecture that multiple equilibria, or bubbles[1], are a negligible phenomenon in sequential equilibrium models of Lucas-type [15], with infinitely lived homogeneous agents. While we have not been lucky in proving that optimizing behavior of a representative agent with smooth preferences is enough to rule out bubbles, regardless of the random behavior of the economy, in the present work we definitely provide robust arguments that confine their appearance to a very restrictive class. This is achieved for a more general model than the original Lucas' one, in a framework similar to that developed in [11]. Two important aspects are implemented. First, no restriction on the probabilistic law of dividends is postulated (a similar setting can be found in [13] as well, but for different purposes). Second, nearly no boundedness assumption on both dividends and trading prices is assumed, as well as on utilities. The only hypotheses maintained are the differentiability of preferences and the zero short-sales constraint.

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[1]We are aware that the terminology here adopted may give rise to misinterpretations. Our model is a peculiar case in which the assets fundamental value is unambiguously defined, and thus bubble existence turns out to be equivalent to price indeterminacy. By slightly relaxing some assumptions (*e.g.* separability or differentiability of preferences), or by considering heterogeneity of individuals, indeterminacy and bubbles immediately get unrelated. For example, in [23] an economy is constructed such that the unique equilibrium is supported by positively priced fiat-money. We shall turn on the distinction between multiple equilibria and bubbles at the end.

Surprisingly, until recently the issue of bubbles in such a model has not attracted much attention. On the other hand, rather few examples of bubbles can be found in the literature. At least two reasons may explain this "lack of interest". First, since all bubble-producing factors are absent in Lucas-style models, it has been taken for granted that they should emerge only in rather special circumstances. This view-point is easily captured by consulting the by now wide literature on intertemporal asset pricing models (see [19], [4], [22], [6], [9], [14], [8], [16], [18], [10]). Perhaps, this intuition suggested that economies such that the equilibrium allocation is given by the initial resources deserved no further investigation, thus addressing more attention toward analyzing equilibrium models with heterogeneous agents as well as with various debt constraints.

A second reason for the scarce interest in studying price bubbles in Lucas-type models is perhaps due to the well known analytical difficulties in formulating some necessary condition of transversality at infinity (see Ekeland and Scheinkman [7] and Kamihigashi [12]). Results available in the literature show that this is a hard task in the stochastic setting, unless severe restrictions are imposed (see [24] and [21]).

Two recent papers stimulated the present research. Santos and Woodford [18] established general results on rational bubbles within a quite broad scenario of sequential equilibrium economies where traders have rational expectations. They proved that perpetual assets in non-zero net supply cannot give rise to unambiguous price bubbles and, in addition, to any sort of bubbles whenever preferences satisfy a certain property of discounting. However, it is important to stress that Santos and Woodford's analysis rests on the simplified assumption that the underlying stochastic environment has a tree structure with finitely many information sets at each instant of time. This allows them to provide an elegant theory of asset pricing which extends to an infinite-horizon dynamic context Kreps' arbitrage approach. Besides this, it is very important to remark that their analysis is somewhat different than ours: they are concerned with the issue of whether a given equilibrium involves a bubble component. Indeed, a natural consequence of dynamical incompleteness of markets is that the present value of the streams of future dividends is not uniquely determined, causing several complications and additional types of bubbles (the so-called ambiguous bubbles). We discuss some of these aspects that are in common with our results in Section 5.

The second paper, due to Kamihigashi [11], resembles closely our approach. To further strengthen the broadly accepted idea of marginality of bubbles, he provides a condition that assures the uniqueness of equilibrium not properly related to discounting properties of agent's preferences. To construct an example of multiple equilibria in a two-period economy where there are countably many states of the world, he needs to use a consumer's utility function that is unbounded. Moreover, he makes an important remark by observing that the presence of positive bubbles in his example is related to the violation of the Euler equation.

Owing to the generality of our setting, we must first study carefully the consistency of the model, in order to formulate suitable necessary and sufficient conditions for price equilibria to exist such that they include the fundamental values of assets. This is pursued in Section 2 where we show that the standard Euler equation is not necessary to construct the classical theory of assets equilibrium valuation. In place of the stochastic Euler equation, we shall utilize an Euler inequality as an optimality necessary condition. It is soon realized that the imposition of the Euler equations is not fully justified and may preclude potential bubbles.

Section 3, where the main results of this paper are given, is devoted to make precise the notion of "fragility" for potential multiple equilibria. We establish a result (Theorem 3) that characterizes potential price indeterminacy precisely as a borderline phenomenon: a slight modification of the amounts of assets, or that of dividends, has the effect that multiple equilibria disappear. To strengthen our argument, we then show that all preferences exhibiting uniformly bounded relative risk aversion fall outside the class of models having bubbles (Theorem 4).

It is natural, after having outlined multiplicity of equilibria as a possible outcome only in a non-generic set of economies, to devote our attention to the study of this "tiny" category. We are able to provide a rough classification of bubbles into two categories. All this is argued in Section 4 where we

construct two polar examples of bubbles. It turns out that agent's unbounded relative risk-aversion is the key ingredient in their construction.

Finally, Section 5 is dedicated to the already mentioned issue of ambiguous bubbles introduced by Santos and Woodford [18]. Two assumptions ruling out ambiguous bubbles and their connection with the theory developed in the previous sections are investigated. The first one generalizes an assumption on agent's impatience already studied in [18]. The second one is a transversality condition at infinity that turns out to be sufficient for non-existence of ambiguous bubbles. Their relation with Kamihigashi [11] uniqueness condition is discussed.

Most of the proofs are gathered in the final Appendix.

2 The Set-up

Let us formalize the model that closely follows [15]. There are k productive assets, each in fixed supply, that produce random quantities of a single perishable consumption good in all time periods. Consumers are identical in terms of preferences and endowments. At each trading time there are spot markets both for the consumption good and for shares in the assets. The uncertainty is modelled by a probability space $(\Omega, \mathbf{F}, \mu)$ where $\mathbf{F} = \{\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}\}$ is a filtration of σ-algebras describing the revelation of information. The asset dividends $\mathbf{d} = \left\{d_t(\omega) \in \mathbf{R}^k_+, \quad t = 0, 1, 2, ...\right\}$ form an $\mathbf{F}$-adapted process which represents the amount of the consumption good yielded by one unit of each single asset. The process $\mathbf{w} = \{w_t(\omega)\}$ is the non-negative $\mathbf{F}$-adapted process of exogenous endowments of the consumption good. We shall denote by $\mathbf{E}_t(\cdot)$ the conditional expectation[2] $\mathbf{E}(\cdot \mid \mathcal{F}_t)$.

Households' preferences are given by the separable life-time utility

$$\mathbf{E}_0 \sum_{t=0}^{\infty} u_t\left[c_t(\omega), \omega\right]$$

defined over the consumption processes $\mathbf{c} = \{c_t(\omega)\}$. Each instantaneous utility u_t is not necessarily uniform across states and the above series needs not be convergent. A process of assets holding strategy is denoted by $\mathbf{y} = \{y_t(\omega)\}$. The initial endowment of each asset is normalized to one, i.e., $y_0 = \mathrm{e} = (1, 1, ..., 1) \in \mathbf{R}^k$.

Here are the main assumptions to be effective throughout this paper. Even where it is not explicitly specified, properties pertaining all the functions involved must hold almost surely with respect to the measure μ. For vectors notation, a superscript will denote its component. For instance, $d^i_t(\omega)$ is the dividend payed by asset i, at epoch t when the state of the world is ω.

A. 1 $0 < d_t(\omega) \cdot \mathrm{e} + w_t(\omega) = \sum_{i=1}^k d^i_t(\omega) + w_t(\omega) < +\infty$ *a.s. for all* t.

A. 2 *For each* t*, utilities* $u_t(\cdot, \cdot)$ *are* $\mathcal{B}^1 \otimes \mathcal{F}_t$*-measurable, where* $\mathcal{B}^1$ *is the Borel* σ*-algebra in* $\mathbf{R}_+$*, and,* $u_t(\cdot, \omega)$ *are concave, strictly increasing and differentiable over* $\mathbf{R}_{++}$*, for each fixed* ω.

A.1 could be relaxed by admitting the total good supply $d_t \cdot \mathrm{e} + w_t$ to vanish with positive probability. However this requires the marginal utility to be finite at zero, thus generating some further formal complications. Needless to say, Assumption A.2 encompasses standard unbounded utilities, like logarithm, having $u_t(0, \omega) = -\infty$ with positive probability, as well as functions having infinite derivative at zero.

A contingent plan $(\mathbf{c}, \mathbf{y}) = \{c_t(\omega), y_t(\omega)\}$, $t \geq 0$, is said to be feasible if:

i) $c_t(\omega) \geq 0$ are $\mathcal{F}_t$-measurable variables for all $t \geq 0$;

ii) $y_t(\omega) \geq 0$ are $\mathcal{F}_{t-1}$-measurable for $t \geq 1$ and $y_0 = \mathrm{e} = (1, 1, ..., 1)$;

[2] In general, the initial σ-algebra $\mathcal{F}_0$ may not be the trivial one and thus the operator $\mathbf{E}_0$ does not agree necessarily with the expectation $\mathbf{E}$. This is not a merely empty generalization. It enables us to treat time t homogeneously. Every result obtained for $t = 0$ is immediately translated to any epoch t.

iii) $c_t(\omega) + p_t(\omega) \cdot [y_{t+1}(\omega) - y_t(\omega)] \leq d_t(\omega) \cdot y_t(\omega) + w_t(\omega)$ *a.s.* for $t \geq 0$.

Below we give the definition of Arrow-Radner sequential equilibrium, where Brock's [3] concept of weak maximality is adopted. To ease notation, from now on we will drop the argument ω of all the random functions under study. By abusing a bit notation, we shall also write $u_t(c_t)$ instead of $u_t(c_t(\omega), \omega)$ and the derivative $D_1 u_t(c,\omega)$ will be denoted by $u_t'(c)$. Symbols X^- and X^+ will denote the negative and the positive part of a random variable X, respectively. We also recall that, for non-negative random vectors $Y(\omega) \in \mathbf{R}^k$, the notation $\mathbf{E}_t(Y) < +\infty$ means $\mathbf{E}_t(|Y|) < +\infty$ or, equivalently, $\mathbf{E}_t(Y \cdot \mathrm{e}) < +\infty$. For a measurable set $A \in \mathcal{F}$, the indicator function of A will be denoted by $\mathbf{1}_A$.

Definition 1 *An equilibrium is an* $\mathbf{F}$*-adapted price process* $\mathbf{p}$ *such that:*

i) $0 \leq p_t < +\infty$ *a.s. for all* t*;*

and the plan $\mathbf{c}^* = \{c_t^*\} = \{d_t \cdot \mathrm{e} + w_t\}$, $\mathbf{y}^* = \{\mathrm{e}\}$ *satisfies the two conditions:*

ii) $\mathbf{E}_0 [u_t(c_t^*) - u_t(c_t)]^- < +\infty$ *a.s. for all* t*,*

iii) $\limsup_{N \to +\infty} \mathbf{E}_0 \sum_{t=0}^{N-1} [u_t(c_t^*) - u_t(c_t)] \geq 0$ *a.s.*

for any feasible plan $(\mathbf{c}, \mathbf{y})$ *where the* y_t*'s are essentially bounded, for each* $t \geq 1$*.*

Here we are given the standard notion of no-trade equilibrium in which agents hold their assets forever, and consume all their available income $d_t \cdot \mathrm{e} + w_t$ at each trading date. The framework adopted here is similar to that in [11], where a discounted, single-asset model has been analyzed. We want to remark that the restriction concerning boundedness of the y_t's is needed for technical reasons. It will only be effective whenever sufficient conditions of optimality are used (see proof of Theorem 1). It will also be seen that, at least for the fundamental prices, the plan $\mathbf{c}^*$ satisfies the stronger property of optimality $\liminf_{N \to +\infty} \mathbf{E}_0 \sum_{t=0}^{N-1} [u_t(c_t^*) - u_t(c_t)] \geq 0$. Henceforth, we shall always write $\{c_t^*\}$ to denote the equilibrium consumption allocation $\mathbf{c}^* = \{d_t \cdot \mathrm{e} + w_t\}$.

In the remaining part of this section, we build up the equilibrium analysis for our general framework. Though the whole discussion on the determination of pricing equilibrium as sum of its fundamental value and the speculative bubble is familiar in macroeconomics and finance (see Blanchard and Fischer [2]), we believe it is worth being reported here owing to the generality of our setting and the stress we shall put on a supermartingale property that turns out to characterize the bubble component. It is important remarking that we do not restrict prices to belong to some pre-chosen space as well as we maintain the weak notion of optimality. Both restrictions on prices or the use of stronger concepts of optimality might rule out possible pricing bubbles.

The starting point is formula (1) below, which turns out to be a short-run first-order condition, that takes the form of an Euler inequality rather than equality.

Proposition 1 *Under A.1-2, if* $\mathbf{p}$ *is an equilibrium then*

$$u_{t-1}'(c_{t-1}^*) p_{t-1} \geq \mathbf{E}_{t-1} \left[u_t'(c_t^*)(p_t + d_t) \right] \tag{1}$$

for $t \geq 1$*.*

One could ask in what cases equality in (1) is necessarily true. This requires to perform the left-hand derivative in the proof of this proposition. That is problematic insofar, once again, *a priori* restrictions on prices p_t are needed. It is not difficult to show that (1) holds with equality if the following two qualifications are satisfied:

$$p_t \cdot \mathrm{e} \leq M_t c_t^* \tag{2}$$

for some scalar M_t, and

$$\mathbf{E}_{t-1}\left[u_t\left(c_t^*\right)-u_t\left(\zeta d_t\cdot \mathrm{e}+w_t\right)\right]<+\infty \tag{3}$$

for some $\zeta<1$. For instance, either conditions are true when the states of the world are finite at trading date t. We shall also see later that a sufficient requirement for (3) is that u_t exhibits bounded relative risk-aversion.

Inequality (1) will be enough to build up pricing analysis within our general setting. As usual, by iterating (1) starting from t we get

$$u_t^{'}\left(c_t^*\right)p_t\geq \mathbf{E}_t\sum_{s=1}^{N}u_{t+s}^{'}\left(c_{t+s}^*\right)d_{t+s}+\mathbf{E}_t\left[u_{t+N}^{'}\left(c_{t+N}^*\right)p_{t+N}\right]$$

which, by taking the limsup over N, yields

$$u_t^{'}\left(c_t^*\right)p_t\geq \mathbf{E}_t\sum_{s=1}^{\infty}u_{t+s}^{'}\left(c_{t+s}^*\right)d_{t+s}+\limsup_{N\to+\infty}\mathbf{E}_t\left[u_{t+N}^{'}\left(c_{t+N}^*\right)p_{t+N}\right]. \tag{4}$$

Since $u_t^{'}\left(c_t^*\right)p_t<+\infty$, by (i) of Definition 1, and the last term is non-negative, we infer both conditions (5) and (6) displayed in the following proposition.

Proposition 2 *Under A.1-2, a necessary condition for equilibria to exist is*

$$\mathbf{E}_0\sum_{t=1}^{\infty}u_t^{'}\left(c_t^*\right)d_t<+\infty. \tag{5}$$

In this case

$$u_t^{'}\left(c_t^*\right)p_t\geq \mathbf{E}_t\sum_{s=1}^{\infty}u_{t+s}^{'}\left(c_{t+s}^*\right)d_{t+s} \tag{6}$$

for all $t\geq 0$.

In view of (6), let us define

$$u_t^{'}\left(c_t^*\right)p_t=\mathbf{E}_t\sum_{s=1}^{\infty}u_{t+s}^{'}\left(c_{t+s}^*\right)d_{t+s}+u_t^{'}\left(c_t^*\right)b_t \tag{7}$$

where the "bubble" component $b_t(\omega)\in\mathbf{R}_+^k$ is $\mathcal{F}_t$-measurable. Accordingly, we define the market fundamental (adapted) process $\mathbf{f}=\{f_t\}$ as

$$f_t=\frac{1}{u_t^{'}\left(c_t^*\right)}\mathbf{E}_t\sum_{s=1}^{\infty}u_{t+s}^{'}\left(c_{t+s}^*\right)d_{t+s}. \tag{8}$$

Thanks to (7), if $\mathbf{p}$ is an equilibrium, then $\mathbf{p}=\mathbf{f}+\mathbf{b}$, with $\mathbf{b}=\{b_t\}$. We have here the traditional definition of speculative bubble as the difference between the price of the asset and its fundamental value. Clearly, the fundamental price process $\mathbf{f}$ satisfies Euler inequality (1) with equality, i.e.:

$$u_{t-1}^{'}\left(c_{t-1}^*\right)f_{t-1}=\mathbf{E}_{t-1}\left[u_t^{'}\left(c_t^*\right)\left(f_t+d_t\right)\right]$$

while the non-negative price bubble $\mathbf{b}$ is a supermartingale.

$$u_{t-1}^{'}\left(c_{t-1}^*\right)b_{t-1}\geq \mathbf{E}_{t-1}\left[u_t^{'}\left(c_t^*\right)b_t\right]. \tag{9}$$

We obtain, in our general setting, the property that a bubble "never starts" in the rational expectations equilibrium (see [18]) but, as will be seen in Section 4, the possibility for a bubble component to exist and burst as time goes on, cannot be excluded.

It is interesting to sketch prices evolution according to the standard Euler equation, *i.e.,* as long as (1) holds with equality. In this case (4) turns into

$$u_t^{'}\left(c_t^*\right)p_t = \mathbf{E}_t\sum_{s=1}^{\infty}u_{t+s}^{'}\left(c_{t+s}^*\right)d_{t+s} \; + \lim_{N\to+\infty}\mathbf{E}_t\left[u_{t+N}^{'}\left(c_{t+N}^*\right)p_{t+N}\right]$$

and

$$u_t^{'}\left(c_t^*\right)b_t = \lim_{N\to+\infty}\mathbf{E}_t\left[u_{t+N}^{'}\left(c_{t+N}^*\right)p_{t+N}\right] \tag{10}$$

while the bubble process obeys the martingale difference equation

$$u_{t-1}^{'}\left(c_{t-1}^*\right)b_{t-1} = \mathbf{E}_{t-1}\left[u_t^{'}\left(c_t^*\right)b_t\right]$$

Clearly, in such a case the bubble component, if it exists, can never burst.

Next statement establishes fundamental values $\mathbf{f}$ to be an equilibrium, thus ensuring the sufficiency of (5) as well. This kind of results are usually proven by means of familiar sufficient conditions of transversality. Nonetheless, owing to the special nature of constraints, we prefer resorting to a more direct method. Details are reported in the Appendix. It should be noted that we do not assume the present value of future wealths $\mathbf{E}_0\sum_{t=1}^{\infty}u_t'\left(c_t^*\right)w_t$ to be finite.

Theorem 1 *Under A.1-2 and the additional condition* $\mathbf{E}_0\left[u_t'\left(c_t^*\right)w_t\right] < \infty$ *for all* $t \geq 0$*, a necessary and sufficient condition for an equilibrium to exist is that*

$$\mathbf{E}_0\sum_{t=1}^{\infty}u_t^{'}\left(c_t^*\right)d_t < +\infty. \tag{11}$$

An equilibrium is given by the market fundamental values:

$$f_t = \frac{1}{u_t^{'}\left(c_t^*\right)}\mathbf{E}_t\sum_{s=1}^{\infty}u_{t+s}^{'}\left(c_{t+s}^*\right)d_{t+s}.$$

3 Main Results

In this section we present results ruling out the emergence of multiple equilibria. The first sufficient criterion has been established by Kamihigashi [11]. Its proof reflects the intuition that, if a bubble occurred, an infinitely lived consumer could gain by permanently reducing his holding of the asset. To be more specific, it allows for a uniform downward perturbation within the feasible set without facing an infinite loss.

Theorem 2 *A sufficient condition for the fundamental price* $\mathbf{f}$ *given in (8) to be the unique equilibrium is that for some scalar* $0 < \zeta < 1$

$$\mathbf{E}_0\sum_{t=1}^{\infty}\left[u_t\left(d_t\cdot\mathrm{e}+w_t\right) - u_t\left(\zeta d_t\cdot\mathrm{e}+w_t\right)\right] < +\infty \tag{12}$$

To illustrate the strength of (12), consider the standard case in which the utilities are discounted, *i.e.*, $u_t\left(c\right) = \beta^t u\left(c\right)$. If the preference function u is bounded, then (12) is trivially true. Therefore, possible violation to (12) requires u to be unbounded. We refer to [11] (Theorem 5.1) for several assumptions on $u\left(c\right)$ guaranteeing qualification (12).

The following criterion will play a central role in proving Theorems 3 and 4.

Corollary 1 *A sufficient condition for (12) is*

$$\mathbf{E}_0 \sum_{t=1}^{\infty} u_t^{'} \left(\zeta d_t \cdot \mathrm{e} + w_t\right) d_t < +\infty \tag{13}$$

for some scalar $0 < \zeta < 1$.

This corollary is an immediate consequence of concavity of u_t's that entails

$$u_t \left(d_t \cdot \mathrm{e} + w_t\right) - u_t \left(\zeta d_t \cdot \mathrm{e} + w_t\right) \leq (1-\zeta) u_t^{'} \left(\zeta d_t \cdot \mathrm{e} + w_t\right) d_t \cdot \mathrm{e}$$

and thus (13) implies (12).

It is worth also noticing that, again in force of concavity,

$$u_t \left(d_t \cdot \mathrm{e} + w_t\right) - u_t \left(\zeta d_t \cdot \mathrm{e} + w_t\right) \geq (1-\zeta) \, u_t^{'} \left(c_t^*\right) d_t \cdot \mathrm{e}$$

which reveals (12) to be sufficient for (11). Therefore, Kamihigashi's criterion implies existence and uniqueness simultaneously.

A slight modification of the proof of Theorem 2 establishes a further specification of (12) focussing on a single asset i and upon the occurrence of some event $A \in \mathcal{F}_s$.

Proposition 3 *If for an event* $A \in \mathcal{F}_s$, *one has*

$$\mathbf{E}_s \sum_{t=s+1}^{\infty} \mathbf{1}_A \left[u_t \left(c_t^*\right) - u_t \left(c_t^* - (1-\zeta) \, d_t^i\right)\right] < +\infty \tag{14}$$

for some scalar $0 < \zeta < 1$, *then the bubble component, for the* i^{th} *component* p_t^i, *vanishes after epoch* s, *as long as* A *occurs. That is,* $b_t^i(\omega) = 0$, *for* $t \geq s$ *and almost all* $\omega \in A$.

A remarkable consequence of Proposition 3 is the absence of a positive bubble component for *fiat money assets*, *i.e.*, assets for which $d_t^i(\omega) \equiv 0$ for all t. In fact, in such a case, (14) is trivially true, irrespectively of agent's preferences. This extends Corollary 3.2 in [18] to our setting.

The intuition behind next fragility result rests behind the evident similarity between necessary condition for existence of at least one equilibrium (11) and sufficient condition for uniqueness (13). The idea of fragility can be easily grasped through two parallel arguments: consider the economy parametrized either on the initial assets endowment $y_0 = \mathrm{e}$ or on the dividend stream $\mathbf{d} = \{d_t\}$, then any slight perturbation of the parameter forces a possible bubble to disappear.

Theorem 3 *If a price bubble occurs for an initial endowment* $y_0 = \mathrm{v} \in \mathbf{R}_{++}^k$, *then there is only one equilibrium for each initial endowment* $\overline{\mathrm{v}} \gg \mathrm{v}$, *while there are no equilibria at all for each endowment* $\underline{\mathrm{v}} \ll \mathrm{v}$. *Likewise, if for some dividend sequence* $\mathbf{d} = \{d_t\}$ *a bubble arises, then there is only one equilibrium for dividends* $\{\zeta d_t\}$, *with* $\zeta > 1$, *and there are no equilibria if* $\zeta < 1$.

This theorem amounts to saying that uniqueness and non-existence of equilibrium are the sole robust configurations, while bubbles are a borderline phenomenon that may only arise when the set of equilibria is about to become empty. To further illustrate this point, let us focus on economies parametrized with respect to dividends $\{\zeta d_t\}$, where $\zeta \in (0, \infty)$. For sake of simplicity, assume $k = 1$ and $\mathcal{F}_0$ to be the trivial algebra. Consider the function

$$J(\zeta) = \mathbb{E}_0 \sum_{t=1}^{\infty} u_t' \left(\zeta d_t + w_t\right) d_t$$

which turns out to be non-increasing and right-continuous. If $J(\zeta) < +\infty$ for all ζ, there is always uniqueness. Likewise, $J(\zeta) = +\infty$ for all ζ, implies no equilibria. Therefore, the only interesting case

happens when there is a jump from $+\infty$ to a finite value at some (unique) critical level ζ_b, which is the unique parameter value such that bubbles may arise. Note also that, as long as $J(\zeta_b) = +\infty$, a sudden change through the two stable configurations is witnessed and no bubble can occur. However, the existence of a threshold ζ_b, such that $J(\zeta_b) < +\infty$ (*i.e.* such that (13) fails), is not enough to generate a bubble: (12) must be violated as well.

We have not been able to single out classes of models assuring the existence of bubbles along this argument. For instance, the two-period economies studied in Example 1 of the next section enjoy this property, but other examples analyzed in [17] lead to different outcomes. On the other hand, an important class where condition (12) is not necessary at all has to be mentioned: in the deterministic model bubbles can never arise, regardless of (12). This case will be briefly recovered in Section 5.

Let us end this section by presenting another strong argument in favor of bubble fragility. Unlike Theorem 3, formulated without resorting to any specification of agent's preferences, next statement is related to agent's risk aversion. We give a sufficient condition for (13) in the spirit of assumptions (L2) and (U2) in [11] that implies to be $J(\zeta_b) = +\infty$ for the critical value (if any).

Theorem 4 *Assume preferences u_t exhibit uniformly bounded relative risk aversion, i.e.,*

$$-\frac{u_t''(c)\,c}{u_t'(c)} \leq R \tag{15}$$

for all $c \geq 0$, $t \geq 0$ and for some scalar R. Then pricing equilibrium is uniquely determined, whenever it exists[3]*.*

Note that this result encompasses almost all agent preferences in conceivable economic models.

4 Bubbles Examples

All the criteria formulated in the previous section are only sufficient conditions, hence, it is not clear at this stage whether or not examples of economies in which equilibrium valuations contain positive bubbles actually exist. Theorems 3 and 4 make clear that their construction is not a simple matter.

In view of Theorem 2, the first step in trying to construct bubbles is the following rough classification distinguishing two polar cases in which sufficient criterion (12) is violated.

1. For all $\zeta < 1$,
$$\mathbf{E}_0 \sum_{t=1}^{\infty} [u_t(c_t^*) - u_t(\zeta d_t \cdot \mathrm{e} + w_t)] = +\infty$$
with positive probability, but there is a time $N > 1$ and some constant $\zeta < 1$ such that
$$\mathbf{E}_{N-1} \sum_{t=N}^{\infty} [u_t(c_t^*) - u_t(\zeta d_t \cdot \mathrm{e} + w_t)] < +\infty.$$

2. For all time $N \geq 1$ and all $\zeta < 1$,
$$\mathbf{E}_{N-1} \sum_{t=N}^{\infty} [u_t(c_t^*) - u_t(\zeta d_t \cdot \mathrm{e} + w_t)] = +\infty$$
with positive probability.

[3] It must be emphasized that twice differentiability hypothesis on preferences is not necessary at all. It is sufficient that some R exists such that $u_t'(c,\omega)c^R$ are nondecreasing for all t and for *a.e.* ω.

By virtue of Proposition 3, economies falling into the first category, exhibit prices having $b_t = 0$ for $t \geq N-1$, and therefore bubbles, if any, must eventually burst after some time. Not surprisingly, it turns out that their occurrence is related to the violation of the Euler equation. Models with bubble component belonging to the second class, seem less dependent on violation of Euler equality, as it will emerge clearly in Example 2 below.

Despite of a somewhat common view-point, we show that bubbles do occur, that is, the borderline set described in Theorem 3 may be non-empty. An example of bubbles of the first type has already been given by [11]. Here, we propose a general construction.

Example 1 Let $(\Omega, \mathcal{F}, \mu)$ be a probability space modelling the world states. The uncertainty is completely revealed at time $t = 1$. Therefore, $\mathcal{F}_0 = \{\emptyset, \Omega\}$, and $\mathcal{F}_t = \mathcal{F}$ for $t \geq 1$. The dividends of a single asset are $d_0 > 0$, $d_1(\omega) > 0$ and $d_t(\omega) = 0$ for $t \geq 2$. Endowments are $w_0 = w_1 = 0$ and $w_t = \overline{w} > 0$ for $t \geq 2$. Agent's preferences are given by $u_t = \beta^t v(c)$. Regarding to the utility v, it is assumed to satisfy the two requirements

$$\begin{aligned} \mathbf{E}\left[v'(d_1)d_1\right] &< +\infty \\ \mathbf{E}\left[v(d_1) - v(\zeta d_1)\right] &= +\infty \end{aligned} \tag{16}$$

for all $\zeta < 1$. The fundamental values turn out to be

$$f_0 = \frac{\beta}{v'(d_0)}\mathbf{E}\left[v'(d_1)d_1\right] \qquad \text{and} \qquad f_t(\omega) = 0 \text{for } t > 0.$$

Clearly, a consequence of Proposition 3 is that no bubbles can arise for $t \geq 1$.

We show the existence of a positive bubble component at $t = 0$. Since $p_t = f_t = 0$ for $t \geq 1$, if we set $y_1 = 1 + \delta$, consumptions are $c_0 = d_0 - p_0\delta$, $c_1 = (1+\delta)d_1$, $c_t = \overline{w}$, for $t \geq 2$. By evaluating the objective function over this consumption plan, it is immediate to see that p_0 will be an equilibrium if the convex function

$$\varphi(\delta) = v(d_0) - v(d_0 - p_0\delta) + \beta\mathbf{E}\left[v(d_1) - v((1+\delta)d_1)\right]$$

defined over the interval $-1 \leq \delta < d_0/p_0$, achieves its minimum at $\delta = 0$. In view of (16), $\varphi(\delta) = +\infty$ if $\delta < 0$ and $\varphi(0) = 0$. Thanks to convexity, the optimum lies at zero whenever $\varphi_+^{'}(0) \geq 0$. Simple calculations lead to $D_+\varphi(0) = v^{'}(d_0)p_0 - \beta E\left[v^{'}(d_1)d_1\right] \geq 0$, which amounts to $p_0 \geq f_0$. Consequently, any price $p_0 \geq f_0$ is an equilibrium and the Euler equation is violated when $p_0 > f_0$. Observe that here the violation of the Euler equation is due the failure of (3), while (2) remains true.

To complete this example, we need to specify functions v satisfying both conditions in (16). Set $\Omega = \{1, 2, ...\}$ and $\mathcal{F} = 2^{\Omega}$. In view of Theorem 4, a good candidate turns out to be $v(c) = -e^{\frac{1}{c}}$, exhibiting unbounded relative risk aversion close to the origin. Let $d_1(n) = n^{-1}$ be the dividend payed at epoch 1 by the asset and μ_n be any probability defined over states satisfying

$$\mu_n \sim \frac{e^{-n}}{n^{2+\alpha}}$$

as $n \to \infty$ and with $\alpha > 0$. The first hypothesis of (16) becomes

$$\mathbf{E}\left[v'(d_1)d_1\right] = \mathbf{E}\left(e^{\frac{1}{d_1}}d_1^{-1}\right) = \sum_{n=1}^{\infty} e^n n\mu_n < +\infty$$

where the series converges since $e^n n\mu_n$ is asymptotically equivalent to $n^{-(1+\alpha)}$. Regarding to the second one of (16), observe that

$$\mathbf{E}\left[v(d_1) - v(\zeta d_1)\right] = \mathbf{E}\left(e^{\frac{1}{\zeta d_1}} - e^{\frac{1}{d_1}}\right) = \sum_{n=1}^{\infty}\left(e^{\frac{n}{\zeta}} - e^n\right)\mu_n$$

where the terms of this series are asymptotically equivalent to

$$n^{-(2+\alpha)}\left[e^{n\left(\zeta^{-1}-1\right)}-1\right]$$

which go to infinity as $n \to \infty$, and so $E\left[v(d_1)-v(\zeta d_1)\right]=+\infty$.

Obviously, this bubble is not robust at all. To test its fragility, set the dividend to be $d_1(n)=\zeta n^{-1}$. Then the fundamental values are the unique equilibrium for $\zeta>1$, while, there are no equilibria if $\zeta<1$, since the first of (16) fails.

It should be noted that the assumption of having infinitely many observable states at time $t=1$ becomes crucial in order to violate the Euler equation. Since Example 1 represents essentially a two-period economy, here the "infinity" feature of the economy is spread along states over a single period. On the contrary, next example features the existence of second-kind bubbles by means of a truly infinite-horizon economy with finitely many information nodes at every trading date. As a consequence, Euler equation can never be violated. In this circumstances, multiple equilibria may arise by modeling agents having increasing relative risk-aversion over time.

Example 2 *(Bubbles and Petersburg assets)* Let $\Omega=\{1,2,...\}$ and the σ-algebra $\mathcal{F}_t$ be generated by the finite partition $\{1\}$, $\{2\}$, ..., $\{t\}$, $\{t+1,\, t+2,\, \ldots\}$. If we set the dividends of a single perpetual asset to be $d_t(\omega)>0$ for $\omega=t$ and $d_t(\omega)=0$ otherwise, it is easy to realize that for any equilibrium one has $p_t(\omega)=0$ for all $\omega \leq t$, while $p_t(\omega)>0$ for $\omega \geq t+1$, no matter whatever preferences and exogenous resources are given.

Given that the informational structure is finite at any trading date, it is simpler to describe it by means of an event-date tree. Among the $t+1$ information sets of $\mathcal{F}_t$, we label $s^t=\{t+1,t+2,....\}$ and $m^t=\{t\}$. The remaining nodes will be little relevant and thus we do not assign them any particular symbol. With this notation at hand, all nodes s^t have two immediate successors s^{t+1} and m^{t+1}, while all others nodes have only one immediate successor. According to this notation, we have $d\left(m^t\right)>0$ and $d(\cdot)=0$ elsewhere, while $p\left(s^t\right)>0$ and $p(\cdot)=0$ elsewhere. We now specialize the elements of this tree. The probability measure will be defined through the uniform transition probabilities

$$\pi\left(m^{t+1} \mid s^t\right)=\pi\left(s^{t+1} \mid s^t\right)=1/2.$$

The agents' preferences are

$$u\left(m^t, c\right) \equiv v_t(c)=-2^t t^{-2-\alpha} c^{-t}$$

with $\alpha>0$, and linear elsewhere. More specifically, $u\left(s^t, c\right)=\beta^t c$ with $0<\beta<1$. Clearly, preferences display unbounded relative risk aversion along states m^t since v_t's relative risk-aversion index equals $t+1$. The dividends payed by the asset are $d\left(m^t\right)=1$, for all $t \geq 1$, and 0 elsewhere. At each date t, the endowments are $w\left(m^t\right)=0$ and $w(\cdot)=\overline{w}>0$ at each other node.

Condition (12) fails, given that for $\zeta<1$

$$\mathbf{E}_0 \sum_{t=1}^{\infty}\left[v_t(1)-v_t(\zeta)\right]=\sum_{t=1}^{\infty} t^{-2-\alpha}\left(\zeta^{-t}-1\right)=\infty \tag{17}$$

We can easily calculate the Euler equation along states s^t. By using the shorthand $p_t \equiv p\left(s^t\right)$

$$p_t=(1/2)\left[\beta p_{t+1}+\beta^{-t} v_{t+1}'(1)\right]. \tag{18}$$

By iterating (18), we get

$$p_t=\beta^{-t} \sum_{s=1}^{\infty} 2^{-s} v_{t+s}'(1)+\lim_{n \to \infty} 2^{-n} \beta^n p_{t+n}$$

where the first addendum is its fundamental value and the second is the bubble component. Note that this series converges since

$$\sum_{s=1}^{\infty} 2^{-s} v_{t+s}'(1)=2^t \sum_{s=1}^{\infty}(t+s)^{-1-\alpha}<\infty$$

and the bubble component obeys the martingale law $b_{t+1} = 2\beta^{-1} b_t$.

Next proposition states formally that such trading prices with positive bubble component are consistent with equilibrium requirements.

Proposition 4 *All prices $p_t = f_t + b_t$ with bubble component growing along states s^t according to the difference equation $b_{t+1} = 2\beta^{-1} b_t$, with $b_0 \geq 0$, are equilibria.*

The structure of this event-date tree resembles the Petersburg game and it could slightly be modified to get several more elaborated examples. Note, for instance, that here the bubble component will burst with probability 1 (although it grows exponentially). However, it is not difficult to modify the probability law over Ω so that the bubble does not burst with positive probability. In the original working paper version of this article [17], we derive another example of Petersburg asset, having countably many nodes at every date, that exhibits a bubble due to the continuous violation of the Euler equation at each trading date. This confirms our intuition on pursuing an Euler inequality rather than equality to develop the equilibrium theory of Section 2.

5 Ambiguous Bubbles

It is well known from the finite-horizon theory that state-prices can be determined by non-existence of opportunities for pure intertemporal arbitrage profits. In our general probabilistic structure, this can be taken into account by conveniently adopting the following terminology. Given an equilibrium price process $\mathbf{p}$, an adapted sequence $a_t(\omega)$ of *strictly* positive functions will be termed a *(pseudo) state-prices* consistent with $\mathbf{p}$, if

$$a_t p_t = \mathbf{E}_t \left[a_{t+1} \left(p_{t+1} + d_{t+1} \right) \right] \tag{19}$$

holds for all $t \geq 0$. Strictly speaking, the a_t's are not the traditional state-prices of Finance, because they are distorted by the probability law. However, there is a one-to-one correspondence with state prices, as long as the stochastic process is given through finite information nodes. In fact, in this case (19) amounts to

$$a\left(s^t\right) p\left(s^t\right) = \sum_{s^{t+1}|s^t} \pi\left(s^{t+1} \mid s^t\right) a\left(s^{t+1}\right) \left[p\left(s^{t+1}\right) + d\left(s^{t+1}\right) \right]$$

where $\pi\left(s^{t+1} \mid s^t\right)$ is the transition probability and s^t, s^{t+1} are adjacent nodes (we are using here the notation in [18]). After multiplying by $\mu\left(s^t\right)$, we get

$$\overline{a}\left(s^t\right) p\left(s^t\right) = \sum_{s^{t+1}|s^t} \overline{a}\left(s^{t+1}\right) \left[p\left(s^{t+1}\right) + d\left(s^{t+1}\right) \right],$$

which is the traditional intertemporal no-arbitrage equation and $\overline{a}\left(s^t\right) = a\left(s^t\right) \mu\left(s^t\right)$ are the familiar state-prices. Clearly, formulation (19) suits better when the states are not necessarily finite.

The theory developed Section 2 can be easily embedded into this approach that uses (19). For instance, we would have

$$a_0 p_0 = \mathbf{E}_0 \sum_{t=1}^{\infty} a_t d_t + \lim_{N \to \infty} \mathbf{E}_0 \left[a_N p_N \right] \tag{20}$$

which might generate a different splitting between the fundamental solution and the bubble component, with respect to the classical decomposition discussed in Section 2.

According to [18], a pricing equilibrium $\mathbf{p}$ (possibly the unique one) is said ambiguously to involve a speculative bubble if one has $\lim_{t\to\infty} \mathbf{E}_0 \left[a_t p_t \right] = 0$ for some state-price process a_t, while $\lim_{t\to\infty} \mathbf{E}_0 \left[a'_t p_t \right] > 0$ for some other process a'_t. On the contrary, an equilibrium $\mathbf{p}$ involves unambiguously no bubble, provided that $\lim_{t\to\infty} \mathbf{E}_0 \left[a_t p_t \right] = 0$ holds regardless of state-price processes a_t which are chosen.

As a matter of fact, the only example in [18] of bubbles for Lucas' models (Example 4.5) is an economy exhibiting an ambiguous bubble, as the bubble component depends crucially on different state-prices adopted. To see why this sort of bubble must be considered outside the theory discussed in the previous sections, it suffices observing that the equilibrium so constructed is unique and (12) is fulfilled. Example 3 presented later will display similar features (another example generalizing the binomial tree of [18] is reported in [17]).

In the following we investigate the relationship between Kamihigashi's condition (12) and other two criteria that exclude the occurrence of ambiguous bubbles. We treat separately each single asset and, since its price may vanish with positive probability, we make use of the following notation. For a given equilibrium and for a fixed asset i, let $P_t^i = \{p_t^i = 0\}$ be the zero-price event belonging to $\mathcal{F}_t$. Clearly, from (19), it turns out that $P_t^i \subset P_{t+1}^i$.

The first assumption is related to agent's preferences.

A. 3 *There is a non-negative scalar sequence $\{\sigma_t\}$, having the following properties:*

i)

$$\sum_{t=1}^{\infty} \sigma_t = +\infty$$

ii) for every integer s and $A \in \mathcal{F}_s$ with $\mu(A) > 0$ and $A \subset \left(P_s^i\right)^c$, there exists a scalar $\zeta = \zeta(s, A)$, with $0 < \zeta < \sigma_s^{-1}$, depending on s and A and such that the consumption stream $\widetilde{\mathbf{c}} = \{\widetilde{c}_t\}$, defined as

$$\widetilde{c}_t = \begin{cases} c_t^*, \text{ for } 0 \le t \le s-1, \\ c_t^* + \zeta d_t^i, \text{ for } t = s, \\ c_t^* - \zeta \sigma_s d_t^i, \text{ for } t \ge s+1 \end{cases}$$

overtakes $\mathbf{c}^*$ over A. That is:

$$\begin{array}{ll} \liminf_{N \to +\infty} & \mathbf{E}_0 \sum_{t=0}^{N-1} \mathbf{1}_A \left[u_t(\widetilde{c}_t) - u_t(c_t^*)\right] = \\ & \mathbf{E}_0 \sum_{t=0}^{\infty} \mathbf{1}_A \left[u_t(\widetilde{c}_t) - u_t(c_t^*)\right] > 0. \end{array}$$

The second assumption does not rely on preferences, but it is directly constructed along a given equilibrium.

A. 4 *For each fixed asset i*

$$\lim_{T \to \infty} \sum_{t=0}^{T} \frac{d_t^i}{p_t^i} = +\infty \tag{21}$$

holds uniformly (adopting the convention $\sum_{t=0}^{T} d_t^i / p_t^i = +\infty$ over P_t^i).

A few comments are in order. A.3 is closely related to the assumption A.2 on agent's impatience postulated by Santos and Woodford [18] as well as the uniform lower bound on impatience assumption in Magill and Quinzii [16]. Indeed Theorem 3.3 in [18] on non-occurrence of bubbles, regardless of the state prices chosen, basically rests on their assumption on impatience. Note that our A.3 is considerably weaker that theirs. A.4 is a transversality condition at infinity related to the exclusion of rolling-over debts in Ponzi strategies. It is also linked to Cass' efficient condition (see [5]): in the deterministic setting (21) amounts to saying that the equilibrium allocation is efficient. However, this is no longer true in the stochastic framework[4].

Let us establish at once the relationship between these two different assumptions and (12).

Proposition 5 *A.3 $\Longrightarrow$ (12). A.3 $\Longrightarrow$ A.4.*

[4] We are indebted to an anonymous referee to have drawn our attention on this interpretation. See also [17] for further qualifications.

Caution is needed to interpret these implications. A.3 is a property on the preferences, while A.4 is a transversality hypothesis on one single equilibrium. Implication A.3 $\Longrightarrow$ A.4 means that all the price equilibria will satisfy that transversality condition, provided A.3 is true. The proof of this part is given in the Appendix. The first implication deserves explanation as well. From (i) of A.3, some time N will exist so that $\sigma_N > 0$. If we set $A = \Omega$ in (ii), we can infer

$$\mathbf{E}_0 \sum_{t \geq N+1} \left[u_t\left(c_t^*\right) - u_t\left(c_t^* - \zeta \sigma_N d_t^i\right)\right] < \mathbf{E}_0 \left[u_N\left(c_N^* + \zeta d_N^i\right) - u_N\left(c_N^*\right)\right] < \infty$$

where the last inequality is true in force of (ii) of Definition 1. Therefore, (14), which is a specification of (12), is valid at least for $t \geq N+1$.

Next statement is our main result on non-existence of ambiguous bubbles.

Theorem 5 *If a pricing equilibrium* $\mathbf{p}$ *satisfies A.4, then* p_t^i *unambiguously involve no bubble.*

It is worth observing that A.3, while more restrictive than (12), guarantees our desired property of non-existence of ambiguous bubbles. On the contrary, next example shows that (12) does not guarantee this property. Furthermore, it also shows that, despite of having (a bit improperly) labelled A.3 as a condition on impatience, as a matter of fact it involves a more complicated interplay between the discounting and the nature of dividend process.

Example 3 *(Petersburg asset and ambiguous bubbles)* The asset structure is similar to that of Example 2. $\Omega = \{1, 2, \ldots\}$ and $\mathcal{F}_t$ are generated by the finite set partition $\{1\}, \{2\}, \ldots, \{t\}, \{t+1, t+2, \ldots\}$. Let $\mu(n) > 0$, for all n, be an assigned probability over Ω. Once again, the dividends of a single asset be $d_t(\omega) > 0$ for $\omega = t$ and $d_t(\omega) = 0$. We know that for any pricing equilibrium one has $p_t(\omega) = 0$ for all $\omega \leq t$, while $p_t(\omega) > 0$ for $\omega \geq t+1$, irrespectively of preferences and exogenous resources.

The series (21) goes to infinity trivially, but the limit is never uniform across states since

$$\mu\left(\sum_{t=1}^{T} \frac{d_t}{p_t} \geq N\right) = \sum_{n=1}^{T} \mu(n)$$

for all $N > 0$, and consequently A.4 fails. Therefore, in view of Proposition 5, A.3 fails as well. Note that this price process could well be the unique equilibrium determined by one agent having preferences satisfying (12). Now we show that there is always a valuation bubble in this class of economies.

To see this is convenient to adopt the tree notation of Example 2. With the notation $s^t = \{t+1, t+2, \ldots.\}$ and $m^t = \{t\}$, we must specify the state-price sequence $a\left(s^t\right)$ along states s^t. Elsewhere, the state prices can be any. Let us fix any number $0 \leq \Delta < p\left(s^0\right)$ and any sequence $a\left(m^t\right)$ such that $\sum_{t=1}^{\infty} a\left(m^t\right) d\left(m^t\right) < \infty$. If we define

$$a\left(s^t\right) = p\left(s^t\right)^{-1} \left[\Delta\left[p\left(s^0\right) - \Delta\right]^{-1} \sum_{s=1}^{\infty} a\left(m^s\right) d\left(m^s\right) + \sum_{s=t+1}^{\infty} a\left(m^s\right) d\left(m^s\right)\right]$$

it is readily seen that such state prices are consistent with the equilibrium since

$$a\left(s^t\right) p\left(s^t\right) = a\left(s^{t+1}\right) p\left(s^{t+1}\right) + a\left(m^{t+1}\right) d\left(m^{t+1}\right)$$

Moreover,

$$\lim_{t \to \infty} \frac{a\left(s^t\right) p\left(s^t\right)}{a\left(s^0\right)} = \Delta,$$

which proves the existence of a positive bubble along the states s^t. Clearly this bubble is ambiguous since it disappears by setting $\Delta = 0$. In this circumstance, the prices equal the fundamental values according to a certain state-price process (of course, this last assertion is also a consequence of Theorem 3.1 in [18]).

A few words on the context where the various assumptions discussed in this section can be used are needed. Actually, they play significantly different roles. The state-price approach addresses more general intertemporal pricing models than the one examined in the previous sections. Preferences must not necessarily be time-separable and differentiable. Moreover, the method (Theorem 5 in particular) applies also to heterogeneous agents models. To stay inside our original focus, it must be assumed that the set of state-prices contains the process $a_t = u'_t(c^*_t)$ (namely, the Euler equation must hold). Under this hypothesis, any price equilibrium $\mathbf{p}$ with respect to some state-price that satisfies A.4 is the market fundamental value which unambiguously involves no bubble. On the other hand, whenever A.3 holds there is a unique price equilibrium which does not contain a bubble component regardless of the chosen state-price process.

Let us conclude with a simple but interesting application of the discussion above: the assets pricing in deterministic sequential markets. As already mentioned, assumption A.4 boils down to the condition of efficiency. Hence, A.4 must hold true along any price equilibrium. Consequently, Theorem 5 implies that no ambiguous bubble may arise with no uncertainty. We give a formal statement of this since it is a straightforward and meaningful consequence of the approach based on A.4. Moreover, our statement is slightly more general than other results available in literature. It could be derived from Theorem 3.1 in [18] and from Theorem 6.1 in [10] as well, though both require the present value of the aggregate endowment to be finite, an assumption not required in our statement.

Proposition 6 *Any price sequential equilibrium in markets without uncertainty satisfies A.4 and therefore Theorem 5 applies.*

The last result clearly implies uniqueness of price equilibrium in the differentiable case. As long as non-differentiability is assumed, like in Gilles-LeRoy example (see [9], [11] and [17]), equilibrium indeterminacy is possible, with no bubbles involved. As said at the beginning, this is one case where the two concepts, bubbles and multiple equilibria, get totally unrelated.

6 Concluding Remarks

Is the issue of price bubbles in intertemporal capital asset pricing models with one representative agent basically closed? The answer is yes, as long as one follows the traditional valuation by means of fundamental value. We have indeed added further strong arguments in favor of fragility of occurrence of bubbles. Of course, some theoretical issues remain still open and interesting enough to deserve further investigation. One of them is the identification of classes of models in which sufficient criterion (12) turns out to be necessary as well. In view of Theorem 3, this would allow a characterization for models with a unique equilibrium for $\zeta > 1$, no equilibria for $\zeta < 1$ and an equilibrium involving some bubble component for $\zeta = 1$. Actually, some examples studied in [17] tell us that formal elaborations in this direction are a difficult task.

As long as one tries to encompass the theory developed by Santos and Woodford [18] into this point of view, several technical difficulties arise. Their Theorem 3.1 on non-existence of unambiguous bubbles must definitely be regarded as one of the main contributions on bubbles fragility, and we do not know to what extent it could be modified to fit our setting. The roots of our treatment rest upon the asymptotic behavior of series (21) which is closely related to the exclusion of Ponzi schemes. Up to our knowledge, such an approach seems novel and this series happens to exhibit a strong relationship with the exclusion of valuation bubbles, as established in Theorem 5.

Let us add some more comments on the construction of bubbles. As it has been argued in Section 4, the occurrence of bubbles seems to be another form of paradox related to the economics of infinity (some kind of paradoxes has well been described in Shell [20]). Roughly speaking, we have met two paradoxes of infinity.

The first one, the milder one, arises when an infinite number of states of the world is observable, at least at some trading date. The traditional first order condition (Euler equation), valid for a uniformly interior equilibrium, is no longer necessarily true. This has led to the construction of the so-called

bursting bubbles that violate the Euler equation (Example 1). It must be underlined that this kind of bubbles is not related to the infinite-horizon setting. They do survive in finite-horizon economies (Kamihigashi's example is just performed for a two-period economy). It should also be noted that the violation of the Euler equation here has nothing to do with the violation of the Euler equations in heterogeneous agents models with debts constraints. We have in mind Bewley's [1] consumption smoothing example with positively valued fiat money (see [14] and [18]). There, it is not possible to uniformly perturb downward the equilibrium trading plan, because the borrowing constraint is binding and, consequently, the failure of the Euler equation is the rule.

The second type of bubbles requires a truly infinite-horizon economy (Example 2). In this case no violation of the Euler equation is required. The paradox of infinity here is originated by violating the already cited principle asserting that whenever a bubble occurred, an infinitely lived agent might gain by permanently reducing the asset holding. For instance, if the agent has increasing relative risk-aversion through time[5], this rule may be no longer true.

7 Appendix

The short-run optimality conditions stated in Proposition 1 require a preliminary lemma.

Lemma 1 *Under A.1-2, if* $\mathbf{p}$ *is a pricing equilibrium process, then*

$$\left\{u_{t-1}^{'}\left(c_{t-1}^{*}\right)p_{t-1}-\mathbf{E}_{t-1}\left[u_{t}^{'}\left(c_{t}^{*}\right)\left(d_{t}+p_{t}\right)\right]\right\}\cdot y\geq 0 \tag{22}$$

for all $t\geq 1$ *and for all random vectors* $y(\omega)\geq 0$, $\mathcal{F}_{t-1}$*-measurable, essentially bounded and such that* $p_{t-1}\cdot y\leq\gamma c_{t-1}^{*}$, *for some number* $\gamma>0$, *depending on* y.

Proof. Fix $y(\omega)$ and consider the function

$$J(\varepsilon)=u_{t-1}\left(c_{t-1}^{*}\right)-u_{t-1}\left(c_{t-1}\right)+\mathbf{E}_{t-1}\left[u_{t}\left(c_{t}^{*}\right)-u_{t}\left(c_{t}\right)\right]$$

for $0\leq\varepsilon\leq\varepsilon_{0,}$ with ε_0 small enough and where

$$\begin{aligned}y_{t}&=\mathrm{e}+\varepsilon y\\ c_{t-1}&=c_{t-1}^{*}-\varepsilon p_{t-1}\cdot y\\ c_{t}&=c_{t}^{*}+\varepsilon\left(p_{t}+d_{t}\right)\cdot y\end{aligned}$$

By construction, $y_t>0$, $c_t>0$ and in force of (ii), $c_{t-1}>0$ as long as $\varepsilon<\gamma^{-1}$. Clearly $J(0)=0$. It is readily seen that such perturbation must leave the agent worse-off. Therefore, it must be $J(\varepsilon)\geq 0$. Moreover, $J(\varepsilon)$ is well defined because $J(\varepsilon)<+\infty$; this is true as $c_t\geq c_t^*$ and thus $u_t(c_t^*)-u_t(c_t)\leq 0$. Take a decreasing sequence $\varepsilon_n\to 0$ and consider the sequence $\varepsilon_n^{-1}\left[J(\varepsilon_n)-J(0)\right]\geq 0$. The limit $J_{+}^{'}(0)$ must be non-negative, provided it does exist. It is immediately seen that

$$J_{+}^{'}(0)=u_{t-1}^{'}\left(c_{t-1}^{*}\right)p_{t-1}\cdot y-\mathbf{E}_{t-1}\left[u_{t}^{'}\left(c_{t}^{*}\right)\left(p_{t}+d_{t}\right)\cdot y\right]$$

where the second addendum holds by the monotone convergence theorem, since the functions $\varepsilon_n^{-1}\left[u_t(c_t^*)-u_t(c_t)\right]\leq 0$ converge to $u_t^{'}(c_t^*)(p_t+d_t)\cdot y$ decreasingly in force of concavity of u_t. From $J_{+}^{'}(0)\geq 0$, (22) follows. ■

Proof of Proposition 1. Fix an integer n and define the event $A_n=\{\omega:|p_{t-1}(\omega)|\leq n$ and $c_{t-1}^{*}(\omega)\geq n^{-1}\}\in\mathcal{F}_{t-1}$. As $n\to\infty$, $A_n\uparrow\Omega\backslash N$, where $\mu(N)=0$. Consider the function $y(\omega)=\mathbf{1}_{A_n}v$,

[5]Note also that the focus on risk-aversion in our model has a respectable antecedent in Lucas' paper, where the relation between asset prices elasticity and relative risk aversion is pointed out.

where $v \in \mathbf{R}_{+}^{k}$ is any fixed vector. This $\mathcal{F}_{t-1}$-measurable function meets assumptions (i) and (ii) of Lemma 1 and thus we can write:

$$\mathbf{1}_{A_n}\left\{u_{t-1}^{'}\left(c_{t-1}^{*}\right)p_{t-1}-\mathbf{E}_{t-1}\left[u_{t}^{'}\left(c_{t}^{*}\right)\left(p_{t}+d_{t}\right)\right]\right\}\cdot v\geq 0.$$

As vector $v \geq 0$ is arbitrary, it follows

$$u_{t-1}^{'}\left(c_{t-1}^{*}\right)p_{t-1}\geq \mathbf{E}_{t-1}\left[u_{t}^{'}\left(c_{t}^{*}\right)\left(p_{t}+d_{t}\right)\right]$$

for all $\omega \in A_n$. As n goes to infinity, it is true almost surely and this completes the proof. ■

Proof of Theorem 1. Observe that $\mathbf{E}_0\left[u^{'}\left(c_t^*\right)p_t\right] < +\infty$ for any price sequence satisfying (1), provided that $p_0 < +\infty$, as it has been assumed. This has two implications. First, $\mathbf{E}_0\left[u^{'}\left(c_t^*\right)p_t\cdot y\right] < +\infty$, for all essentially bounded functions $y\left(\omega\right)$. Second, $\mathbf{E}_s\left[u^{'}\left(c_t^*\right)p_t\cdot y\right] < +\infty$ for $s \geq 0$, as well. Let (c_t, y_t) be any feasible consumption-portfolio plan for a price process satisfying (1). Multiplying the budget constraint by $u_t^{'} \equiv u_t^{'}\left(c_t^*\right)$, we obtain

$$u_t^{'}c_t \leq u_t^{'}p_t\cdot\left(y_t-y_{t+1}\right)+u_t^{'}d_t\cdot y_t+u_t^{'}w_t$$

Taking the expected value and exploiting (1)

$$\begin{aligned}\mathbf{E}_{t-1}\left[u_t^{'}c_t\right] &\leq \mathbf{E}_{t-1}\left[\left(p_t+d_t\right)u_t^{'}\right]\cdot y_t-\mathbf{E}_{t-1}\left[u_t^{'}p_t\cdot y_{t+1}\right]+\mathbf{E}_{t-1}\left[u_t^{'}w_t\right]\\ &\leq u_{t-1}^{'}p_{t-1}\cdot y_t-\mathbf{E}_{t-1}\left[u_t^{'}p_t\cdot y_{t+1}\right]+\mathbf{E}_{t-1}\left[u_t^{'}w_t\right].\end{aligned}$$

Note that index t is taken greater than 0 and $\mathbf{E}_{t-1}\left[u_t^{'}p_t\cdot y_{t+1}\right]$ is finite, as y_{t+1} is assumed to be essentially bounded. Taking now the expected value $\mathbf{E}_0$ and summing up from $t = 1$ to $t = N$

$$\begin{aligned}\mathbf{E}_0\textstyle\sum_{t=1}^{N}u_t^{'}c_t \leq\ & u_0^{'}p_0\cdot y_1-\mathbf{E}_0\left[u_N^{'}p_N\cdot y_{N+1}\right]\\ &+\mathbf{E}_0\textstyle\sum_{t=1}^{N}u_t^{'}w_t\leq u_0^{'}p_0\cdot y_1+\mathbf{E}_0\sum_{t=1}^{N}u_t^{'}w_t.\end{aligned}$$

By adding the first term $u_0^{'}c_0 \leq u_0^{'}p_0\cdot(\mathrm{e}-y_1)+u_0^{'}d_0\cdot\mathrm{e}+u_0^{'}w_0$, and by using (7), we get

$$\mathbf{E}_0\sum_{t=0}^{N}u_t^{'}c_t\leq u_0^{'}b_0\cdot\mathrm{e}+\mathbf{E}_0\sum_{t=0}^{N}u_t^{'}c_t^{*}+\mathbf{E}_0\sum_{t=N+1}^{\infty}u_t^{'}d_t\cdot\mathrm{e}$$

that is true for all N, for any feasible consumption sequence and where b_0 is the price bubble at epoch 0. To conclude, from the concavity property $u_t\left(c_t^*\right)-u_t\left(c_t\right)\geq u_t^{'}\left(c_t^*\right)\left(c_t^*-c_t\right)$, it follows that

$$\mathbf{E}_0\sum_{t=0}^{N}\left[u_t\left(c_t^*\right)-u_t\left(c_t\right)\right]\geq\mathbf{E}_0\sum_{t=0}^{N}u_t^{'}\left(c_t^*\right)\left(c_t^*-c_t\right)\geq -u_0^{'}\left(b_0\cdot\mathrm{e}\right)-\mathbf{E}_0\sum_{t=N+1}^{\infty}u_t^{'}d_t\cdot\mathrm{e}$$

and, in force of (11),

$$\liminf_{N\to\infty}\mathbf{E}_0\sum_{t=0}^{N}\left[u_t\left(c_t^*\right)-u_t\left(c_t\right)\right]\geq -\left(b_0\cdot\mathrm{e}\right)u_0^{'}$$

Therefore, if the market prices agree with the fundamental values f_t, that is $b_0 = 0$, we obtain the desired property of optimality. ■

Proof of Theorem 2. It follows the same line of the proof of Lemma 4.1 in [11] and therefore we shall only sketch it. Consider the asset holding strategy $\mathbf{y}^1$ defined by $y_0^1 = \mathrm{e}$ and $y_t^1 = \mathrm{e} - \varepsilon v$ for $t \geq 1$, where v is a fixed vector in $\mathbf{R}_{++}^{k}$ and ε satisfies the condition $0 < \varepsilon v \leq (1-\zeta)\mathrm{e}$, being ζ

defined in (12). Let $\mathbf{c}^1$ be the corresponding consumption stream. Now, for $\alpha \in (0,1)$, define the plan $\mathbf{y}^\alpha = (1-\alpha)\,\mathrm{e} + \alpha \mathbf{y}^1$ with the relative consumptions $\mathbf{c}^\alpha = (1-\alpha)\mathbf{c}^* + \alpha \mathbf{c}$. By concavity,

$$\alpha^{-1}\left[u_t\left(c_t^*\right) - u_t\left(c_t^\alpha\right)\right] \leq u_t\left(c_t^*\right) - u_t\left(c_t^1\right).$$

Since $c_t^\alpha \leq c_t^*$, from $t \geq 1$ on, sums are increasing and, by taking the limit as $N \to \infty$ and then expectation, the following inequalities are true:

$$0 \leq \mathbf{E}_0 \sum_{t=0}^{\infty} \alpha^{-1}\left[u_t\left(c_t^*\right) - u_t\left(c_t^\alpha\right)\right] \leq \mathbf{E}_0 \sum_{t=0}^{\infty} \left[u_t\left(c_t^*\right) - u_t\left(c_t^1\right)\right] < +\infty$$

where the first is due to optimality of plan $\mathbf{c}^*$ (see (*iii*) of Definition 1), whilst the second is valid by (12). As $\alpha \downarrow 0$, the functions above increase, therefore, through repeated applications of the monotone convergence theorem, we get

$$u_0^{'}\left(c_0^*\right) p_0 \leq \mathbf{E}_0 \sum_{t=1}^{\infty} u_t^{'}\left(c_t^*\right) d_t. \tag{23}$$

In view of (6), (23) yields $p_0 = f_0$, and the proof is complete because, from (9), $b_0 = 0$ implies $b_t = 0$ for all t. ■

Proof of Proposition 3. It is a straightforward variant of Theorem 2. It will suffice to consider the assets holding strategy $\mathbf{y}^i$ defined by $y_0^i = \mathrm{e}$ and $y_t^i = \mathrm{e} - \varepsilon \mathrm{e}_i$ for $t \geq 1$, where $\varepsilon > 0$ is sufficiently small and e_i is the vector having zero components but the ith, which equals one. The corresponding consumption stream $\mathbf{c}^i$ is given by $c_0^i = c_0^* + \varepsilon p_0^i$ and $c_t^i = c_t^* - \varepsilon d_t^i$ for $t \geq 1$. Remaining steps closely follows those of the preceding proof. ■

Proof of Theorem 3. Suppose there is some bubble when the initial assets supply is $y_0 = \mathrm{v} \in \mathbf{R}_{++}^k$. Since (11) must be fulfilled, one has

$$\mathbf{E}_0 \sum_{t=1}^{\infty} u_t^{'}\left(d_t \cdot \mathrm{v} + w_t\right) d_t < +\infty$$

Take any initial vector $\overline{\mathrm{v}}$ such that $\overline{\mathrm{v}} \gg \mathrm{v}$. Then, there is some $\zeta < 1$ for which $\zeta\overline{\mathrm{v}} \gg \mathrm{v}$. Monotonicity of u_t' implies

$$\mathbf{E}_0 \sum_{t=1}^{\infty} u_t^{'}(\zeta d_t \cdot \overline{\mathrm{v}} + w_t) d_t < +\infty$$

which is the sufficient condition (12) for the equilibrium with initial asset holding $\overline{\mathrm{v}}$ to be unique. Likewise, assume that, for $y_0 = \underline{\mathrm{v}} \ll \mathrm{v}$, equilibria do exist, hence

$$\mathbf{E}_0 \sum_{t=1}^{\infty} u_t^{'}(d_t \cdot \underline{\mathrm{v}} + w_t) d_t < +\infty$$

which in turn entails

$$\mathbf{E}_0 \sum_{t=1}^{\infty} u_t^{'}(\xi d_t \cdot \underline{\mathrm{v}} + w_t) d_t < +\infty$$

for all $\xi > 1$. By picking $\xi > 1$ and $\zeta < 1$ so that $\underline{\mathrm{v}} \ll \xi\underline{\mathrm{v}} \ll \zeta\mathrm{v} \ll \mathrm{v}$,

$$\mathbf{E}_0 \sum_{t=0}^{\infty} u_t^{'}\left(\zeta d_t \cdot \mathrm{v} + w_t\right) d_t < +\infty$$

must hold. But this contradicts the assumption that some bubble occur for $y_0 = \mathrm{v}$. Clearly, a similar line of reasoning applies for a perturbation of the dividends d_t. ■

Proof of Theorem 4. Let us utilize sufficient condition (13) formulated in Corollary 1. We first observe that if there is a constant $M(\zeta)$, independent of t, such that

$$u_t^{'}(\zeta c+h) \leq M(\zeta) u_t^{'}(c+h) \tag{24}$$

for some $\zeta < 1$ and for all $c \geq 0$, $h \geq 0$, $t \geq 0$, then (11) implies (13). In force of (15), the function $u_t^{'}(c+h)\, c^R$ is non-decreasing, as can be checked by calculating its derivative. Thus $u_t^{'}(\zeta c+h)\, \zeta^R \leq u_t^{'}(c+h)$ for $\zeta \leq 1$, and (24) is valid by setting $M(\zeta) = \zeta^{-R}$. ■

Proof of Proposition 4. Denote by $y_{t+1} \equiv y\left(s^t\right)$ a feasible trading plan. It finances consumptions

$$c\left(s^t\right) = p_t\left(y_t - y_{t+1}\right) + \overline{w}, \qquad c\left(m^t\right) = y_t$$

and $c(\cdot) = \overline{w}$ elsewhere. Furthermore

$$\begin{array}{ll} \mathbf{E}_0 \sum_{t=0}^{N}\left[u_t\left(c_t^*\right) - u_t\left(c_t\right)\right] = & \sum_{t=1}^{N} 2^{-t}\left[v_t(1) - v_t\left(y_t\right)\right] + \\ & \sum_{t=0}^{N} 2^{-t}\beta^t p_t\left(y_{t+1} - y_t\right) \end{array} \tag{25}$$

The proof will be accomplished by considering strategies y_t separately in the two following exhaustive classes: a) $\limsup_{t\to\infty} y_t \geq 1$; b) $\limsup_{t\to\infty} y_t < 1$.

By means of the inequality

$$v_t(1) - v_t\left(y_t\right) \geq v_t^{\prime}(1)\left(1 - y_t\right)$$

the right-hand side of (25) is greater than

$$\begin{array}{l} \sum_{t=1}^{N} 2^{-t+1}\beta^{t-1}\left[p_{t-1} - 2^{-1}\beta^{1-t}v_t^{\prime}(1) - 2^{-1}\beta p_t\right]\left(y_t - 1\right) \\ +2^{-N}\beta^N p_N\left(y_{N+1} - 1\right) = 2^{-N}\beta^N p_N\left(y_{N+1} - 1\right) \end{array}$$

where equality holds thanks to (18). Since $2^{-N}\beta^N p_N \to b_0 \geq 0$, in the case (a)

$$\limsup_{N\to\infty} 2^{-N}\beta^N p_N\left(y_{N+1} - 1\right) \geq 0$$

and our claim is proven. Consider now case (b). Taking limits in (25), we get

$$\sum_{t=1}^{\infty} 2^{-t}\left[v_t(1) - v_t\left(y_t\right)\right] - \sum_{t=0}^{\infty} 2^{-t}\beta^t p_t\left(y_t - y_{t+1}\right)$$

provided that the two series make sense. The first series diverges. In fact, it turns out to be definitively $y_t \leq \zeta$ for some $\zeta < 1$ and the series diverges by virtue of (17). Thus, our claim will be true provided that the second series does not diverge. On the other hand,

$$\begin{array}{ll} \sum_{t=0}^{\infty} 2^{-t}\beta^t p_t\left(y_t - y_{t+1}\right) & = \sum_{t=0}^{\infty} 2^{-t}\beta^t p_t\left(1 - y_{t+1}\right) - \sum_{t=1}^{\infty} 2^{-t}\beta^t p_t\left(1 - y_t\right) \\ & = \sum_{t=1}^{\infty} 2^{-t+1}\beta^{t-1}\left(p_{t-1} - 2^{-1}\beta p_t\right)\left(1 - y_t\right) \\ & = \sum_{t=1}^{\infty} 2^{-t} v_t^{\prime}(1)\left(1 - y_t\right) \\ & \leq \sum_{t=1}^{\infty} 2^{-t} v_t^{\prime}(1) < \infty \end{array}$$

where the third equality uses (18), and the desired result is proven. ■

Proof of Proposition 5. To show the implication A.3 $\Longrightarrow$ A.4, we claim the events

$$A_t = \left\{d_t^i / p_t^i < \sigma_t\right\} \subset \left(P_t^i\right)^c$$

to be μ-negligible for all t. Arguing by contradiction, suppose that $\mu\left(A_s\right) > 0$ for some s. Then, picking ζ, relatively to A_s as in A.3, one can rewrite this event as $A_s = \left\{\zeta d_s^i - \zeta\sigma_s p_s^i < 0\right\}$ or, equivalently, in vector notation

$$A_s = \left\{\zeta d_s^i + p_s \cdot \left[\left(\mathrm{e} - \zeta\sigma_s \mathrm{e}_i\right) - \mathrm{e}\right] < 0\right\} \tag{26}$$

where e_i denote the $\mathbf{R}^k$ vector with all null entries but the ith equals 1.

We now construct a plan $\{\widetilde{c}_t, \widetilde{y}_t\}$ as follows: $\{\widetilde{c}_t, \widetilde{y}_t\} = \{c_t^*, \mathrm{e}\}$ for all $\omega \in \Omega$, if $t < s$ and for $\omega \notin A_s$ if $t \geq s$. If $\omega \in A_s$, then $\{\widetilde{c}_s, \widetilde{y}_s\} = \left\{c_s^* + \zeta d_s^i, \mathrm{e}\right\}$ and $\{\widetilde{c}_t, \widetilde{y}_t\} = \left\{c_t^* - \zeta\sigma_s d_t^i, \mathrm{e} - \zeta\sigma_s \mathrm{e}_i\right\}$ for $t \geq s+1$. By using (26),

$$\begin{array}{ll} \left(c_s^* + \zeta d_s^i\right) + p_s \cdot [(\mathrm{e} - \zeta\sigma_s \mathrm{e}_i) - \mathrm{e}] \leq c_s^* = d_s \cdot \mathrm{e} + w_s & \text{and} \\ c_t^* - \zeta\sigma_s d_t^i + p_t \cdot 0 = d_t \cdot (\mathrm{e} - \zeta\sigma_s \mathrm{e}_i) + w_t & \text{for } t \geq s+1. \end{array}$$

and thus $\{\widetilde{c}_t, \widetilde{y}_t\}$ is feasible. By construction we have

$$\mathbf{E}_0 \sum_{t=0}^{N-1} \left[u_t\left(\widetilde{c}_t\right) - u_t\left(c_t^*\right)\right] = \mathbf{E}_0 \sum_{t=0}^{N-1} \mathbf{1}_{A_s} \left[u_t\left(\widetilde{c}_t\right) - u_t\left(c_t^*\right)\right]$$

for all $N \geq 1$. Taking the liminf, A.3 entails

$$\liminf_{N \to +\infty} \sum_{t=0}^{N-1} \mathbf{E}_0 \left[u_t\left(\widetilde{c}_t\right) - u_t\left(c_t^*\right)\right] > 0$$

which contradicts weak optimality of plan $\{c_t^*, \mathrm{e}\}$. Concluding, $\mu\left(A_t\right) = 0$, for all t and, consequently, $d_t^i / p_t^i \geq \sigma_t$ for almost all $\omega \in \Omega$. This implies our assert. ■

To prove Theorem 5, some more notation and one preliminary lemma are required. Let us introduce the scalar sequence $\widehat{y}_t^i$ defined recursively as:

$$\widehat{y}_{t+1}^i = \left(1 + \frac{d_t^i}{p_t^i}\right) \widehat{y}_t^i. \tag{27}$$

and where the initial condition $\widehat{y}_0^i$ is assumed to be a strictly positive scalar. Clearly, $\widehat{y}_{t+1}^i$ is well defined over $\left(P_t^i\right)^c$.

Lemma 2 *Let $\{p_t, a_t\}$ be two $\mathbf{F}$-adapted processes satisfying (19). Given the sequence $\widehat{y}_t^i$ defined above, if we agree upon setting $p_t^i \widehat{y}_{t+1}^i = 0$ over P_t^i, then the process $a_t p_t^i \widehat{y}_{t+1}^i$ is a supermartingale.*

Proof. Set $C_t^i = \left(P_t^i\right)^c$. By definition,

$$\mathbf{E}_{t-1}\left[a_t p_t^i \widehat{y}_{t+1}^i\right] = \mathbf{E}_{t-1}\left[\mathbf{1}_{C_t^i} a_t p_t^i \widehat{y}_{t+1}^i\right]$$

According to (27)

$$\mathbf{1}_{C_t^i}(p_t^i \widehat{y}_{t+1}^i) = \mathbf{1}_{C_t^i}(p_t^i + d_t^i)\widehat{y}_t^i$$

that, by means of (19), leads to

$$\begin{array}{rcl} \mathbf{E}_{t-1}\left[a_t p_t^i \widehat{y}_{t+1}^i\right] & = & \mathbf{E}_{t-1}\left[a_t \mathbf{1}_{C_t^i}(p_t^i + d_t^i)\widehat{y}_t^i\right] \leq \\ \mathbf{E}_{t-1}\left[a_t(p_t^i + d_t^i)\right] \widehat{y}_t^i & = & a_{t-1} p_{t-1}^i \widehat{y}_t^i \end{array}$$

as was to be shown. ■

Proof of Theorem 5. We shall extend the sequence $\widehat{y}_{t+1}^i$, defined in (27), by setting $\widehat{y}_{t+1}^i = \infty$ over P_t^i. Clearly, $\widehat{y}_{t+1}^i$ is an extended-value and increasing sequence. According to (27),

$$\widehat{y}_{t+1}^i = \prod_{k=0}^{t} \left(1 + \frac{d_k^i}{p_k^i}\right) \widehat{y}_0^i.$$

By means of the first of the inequalities

$$1+\sum_{t=0}^{N}\alpha_t \leq \prod_{t=0}^{N}(1+\alpha_t) \leq \exp\left(\sum_{t=0}^{N}\alpha_t\right) \tag{28}$$

that holds for all sequences of scalars $\alpha_t \geq 0$ and all $N \geq 1$, from (21) it follows that the sequence $\widehat{y}_t^i \to \infty$ uniformly.

Fix any pseudo state-prices a_t consistent with $\mathbf{p}$. We know from Lemma 2 that $a_t p_t^i \widehat{y}_{t+1}^i$ is a supermartingale. Hence, $\mathbf{E}_0\left[a_t p_t^i \widehat{y}_{t+1}^i\right] \leq a_0 p_0^i \widehat{y}_1^i$. This means $\mathbf{E}_0\left[\mathbf{1}_{C_t^i} a_t p_t^i \widehat{y}_{t+1}^i\right] \leq a_0 p_0^i \widehat{y}_1^i$, where $C_t^i = \left(P_t^i\right)^c$. Since the sequence $\widehat{y}_t^i$ diverges uniformly, for any N we can find a time T so that $\widehat{y}_{t+1}^i \geq N$ for all $t \geq T$. Hence, $\mathbf{E}_0\left[\mathbf{1}_{C_t^i} a_t p_t^i\right] \leq N^{-1} a_0 p_0^i \widehat{y}_1^i$. On the other hand, $\mathbf{E}_0\left[\mathbf{1}_{P_t^i} a_t p_t^i\right] = 0$, which gives $\mathbf{E}_0\left[a_t p_t^i\right] \leq N^{-1} a_0 p_0^i \widehat{y}_1^i$ and, in turn, $\mathbf{E}_0\left[a_t p_t^i\right] \to 0$ as $t \to \infty$. Now, in view of (20), we can infer that the bubble component relative to the selected state-prices vanishes. ■

Proof of Proposition 6. Let us focus on a single asset. If prices p_t^i eventually vanish, nothing is to be proved. Hence set $p_t^i > 0$ for all t. Arguing by contradiction, suppose A.4 fails, namely

$$\sum_{t=0}^{\infty}\frac{d_t^i}{p_t^i} = M < +\infty.$$

From the second inequality in (28), it follows

$$\prod_{t=0}^{\infty}\left(1+\frac{d_t^i}{p_t^i}\right) \leq \exp\left(\sum_{t=0}^{\infty}\frac{d_t^i}{p_t^i}\right) = e^M$$

Therefore, the deterministic increasing sequence $\widehat{y}_t^i$, defined in (27) is bounded by $e^M \widehat{y}_0^i$. So, if we take $\widehat{y}_0^i < e^{-M}$, we shall have $\widehat{y}_t^i < 1$ for all t. We claim that this leads to a contradiction. It suffices constructing the following plan. If e_i denotes the vector having zero components but the ith equal to one, then let

$$\widetilde{y}_t = \mathrm{e} - g_t \widehat{y}_t^i \mathrm{e}_i$$

where $g_0 = 0$ and $g_t = 1$ for all $t \geq 1$. It is feasible because $\widetilde{y}_t > 0$ and finances consumptions

$$\widetilde{c}_t = c_t^* + (g_{t+1} - g_t)\, p_t^i \widehat{y}_{t+1}^i$$

Clearly $\widetilde{c}_0 = c_0^* + p_0^i \widehat{y}_1^i$, while $\widetilde{c}_t = c_t^*$ for all $t \geq 1$. This contradicts the optimality of c_t^*. ■

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