Total boundedness in metrizable spaces

TOTAL BOUNDEDNESS IN METRIZABLE SPACES

GERALD BEER, CAMILLO COSTANTINI, AND SANDRO LEVI

Abstract. We show that a metric space $\langle X, d\rangle$ is separable if and only if the bornology of its d-bounded subsets agrees with the bornology of ρ-totally bounded subsets with respect to some equivalent remetrization ρ. We also show that the bornology of d-totally bounded subsets agrees with the bornology of ρ-bounded subsets with respect to some equivalent remetrization if and only if the former bornology has a countable cofinal subfamily. Finally, we characterize those bornologies on a metrizable space that are bornologies of totally bounded sets as determined by some metric compatible with the topology.

1. Introduction

Let $\langle X, d\rangle$ be a metric space and for a nonempty subset A of X, let A^ε denote its ε-*enlargement* with respect to d, that is, $A^\varepsilon := \{x \in X : d(x, A) < \varepsilon\}$. A subset A of X is called *bounded* if for some $x_0 \in X$ and $r > 0$, we have $A \subseteq \{x_0\}^r$, whereas A is called *totally bounded* if $\forall \varepsilon > 0$, $\exists$ a finite subset F of X with $A \subseteq F^\varepsilon$. We denote the nonempty d-bounded subsets of X by $\mathbf{B}_d(X)$ and the nonempty d-totally bounded subsets by $\mathbf{TB}_d(X)$. Sometimes the families coincide, e.g., in finite dimensional Euclidean space.

In this paper, we address the following two questions:

(1) When does there exist an equivalent metric ρ such that $\mathbf{B}_d(X) = \mathbf{TB}_\rho(X)$?

(2) When does there exist an equivalent metric ρ such that $\mathbf{TB}_d(X) = \mathbf{B}_\rho(X)$?

Both questions have remarkably simple answers, with the class of metrics d satisfying (1) properly containing those that satisfy (2). Question (1) is less transparent than (2) and we display two separate paths to its resolution, one of which employs a natural embedding theorem for separable metric spaces into the sequence space $\mathbb{R}^{\mathbb{N}}$ equipped with product topology. Question (2) was actually settled by two of the authors in [6] as a byproduct of an investigation with very different goals, and here we supply a more traditional proof as well as some additional conditions. Finally, we characterize in two different ways those bornologies on a metrizable space that are bornologies of totally bounded sets as determined by some compatible metric. The first entails the existence of a certain kind of embedding, while the second is stated in terms of star-developments.

Date: April 27, 2010.

1991 *Mathematics Subject Classification.* Primary 54E35; Secondary 54E45.

Key words and phrases. bounded set, totally bounded set, bornology, boundedly compact metric space, completion, star-development, compatible normal sequence.

2. Preliminaries

If $\langle X, \mathcal{T}\rangle$ is a Hausdorff space and $A \subseteq X$, we write $\mathrm{cl}(A)$ and $\mathrm{int}(A)$ for the closure and interior of A respectively. We write $\mathcal{P}_0(X)$ for the nonempty subsets of X, and we denote the nonempty compact subsets of X by $\mathcal{K}_0(X)$. If $\mathcal{U}$ is a cover of X and $A \subseteq X$, we put $\mathrm{St}(A, \mathcal{U}) := \cup\{U \in \mathcal{U} : U \cap A \neq \emptyset\}$, of course writing $\mathrm{St}(x, \mathcal{U})$ for $\mathrm{St}(\{x\}, \mathcal{U})$. A cover $\mathcal{V}$ is said to *star-refine* $\mathcal{U}$ provided the associated cover $\{\mathrm{St}(V, \mathcal{V}) : V \in \mathcal{V}\}$ refines $\mathcal{U}$. As shown by A. Stone [17], the existence of an open star-refinement for each open cover of X is equivalent to paracompactness of $\langle X, \mathcal{T}\rangle$ [10].

By a *bornology* $\mathcal{B}$ on X [3, 5, 12], we mean a family of nonempty subsets of X that forms a cover of X and that is stable under finite unions and under taking nonempty subsets. The nonempty relatively compact subsets $\mathcal{RK}_0(X)$, i.e., the subsets with compact closure, form a typical bornology; obviously, $\mathcal{P}_0(X)$ is the largest. By a *base* for a bornology, we mean a subfamily $\mathcal{B}_0$ of $\mathcal{B}$ that is cofinal in $\mathcal{B}$ with respect to inclusion. For example, $\mathcal{K}_0(X)$ is a closed base for $\mathcal{RK}_0(X)$, and in the case of the bornology of bounded sets determined by a metric d, the set of all open (or closed) d-balls with fixed center and integral radius forms a countable base for the bornology. We denote the open ball with center x and radius $\alpha > 0$ in a metric space $\langle X, d\rangle$ by $S_d(x, \alpha)$ and the corresponding closed ball by $\overline{S}_d(x, \alpha)$.

Given a bornology $\mathcal{B}$ with a closed base on a Hausdorff space $\langle X, \mathcal{T}\rangle$, we can form the one-point extension of X associated with $\mathcal{B}$ by adjoining an ideal point $\{p\}$ to X and taking as open sets in the extension

$$\mathcal{T} \cup \{X \cup \{p\} \backslash B : B \in \mathcal{B} \text{ and } \mathrm{cl}(B) = B\}.$$

Obviously, if $\mathcal{B}_0$ is a closed base for the bornology, $\{X \cup \{p\} \backslash B : B \in \mathcal{B}_0\}$ forms a neighborhood base at the ideal point p. As expected, conditions for various separation axioms to hold, as well as metrizability and complete metrizability, have been identified (see, e.g., [4, 8, 9, 19]). As a particular case, when $\mathcal{B}$ is the bornology of sets with compact closure, we get a compact extension, which is Hausdorff provided X is locally compact [10]. More generally, starting with a bornology with closed base, the one-point extension is Hausdorff if and only if each point of X has a neighborhood belonging to the bornology (see, e.g., [19, pg. 821]).

We now focus on metrizable spaces. Let $\langle X, d\rangle$ be a metric space. We write $\langle \widetilde{X}, \widetilde{d}\rangle$ for its *completion*, which for definiteness may be viewed as the closure of $\langle X, d\rangle$ in the Banach space of bounded continuous real functions on X equipped with the usual supremum norm under the identification $x \leftrightarrow f_x$ where fixing $x_0 \in X$, $f_x := d(x, \cdot) - d(x_0, \cdot)$ [10, pg. 338]. The space is called *boundedly compact* if and only if each closed and bounded subset of X is compact. By a theorem of Vaughan [18], a metrizable space has a compatible boundedly compact metric if and only if the space is separable and locally compact.

If $\langle X, d\rangle$ is a metric space, then its bornology of d-bounded subsets $\mathbf{B}_d(X)$ has a countable base, and $\forall B \in \mathbf{B}_d(X), \exists B_1 \in \mathbf{B}_d(X)$ with $\mathrm{cl}(B) \subseteq \mathrm{int}(B_1)$. As shown by Hu sixty years ago [14], these two properties of the family of metrically bounded sets are characteristic of bornologies on a metrizable space that are the bounded subsets with respect to some compatible metric. That is, if $\langle X, \mathcal{T}\rangle$ is a metrizable space, then a bornology $\mathcal{B}$ is the bornology of d-bounded subsets with respect to some compatible metric if and only if (1) $\mathcal{B}$ has a countable base, and (2) $\forall B \in \mathcal{B}, \exists B_1 \in \mathcal{B}$ with $\mathrm{cl}(B) \subseteq \mathrm{int}(B_1)$. That this result is not standard in topology texts save Hu's [15] is a little surprising. It turns out that if the underlying metrizable space is noncompact, there are in fact uncountably many distinct *metric bornologies* [3], that is, bornologies that arise as $\mathbf{B}_d(X)$ for some compatible metric d. In the final section of this article, we will present two separate sets of characteristic properties for bornologies of totally bounded sets with respect to some compatible metric.

For completeness, we list standards facts about the totally bounded subsets of a metric space $\langle X, d\rangle$, most of which should be well-known to the reader.

- $\mathbf{TB}_d(X)$ is a bornology with closed base;
- A is compact if and only if A is both complete and totally bounded;
- A is totally bounded if and only if each sequence in A has a Cauchy subsequence;
- A is totally bounded if and only if $\forall \varepsilon > 0$, there exists a finite subset F of A with $A \subseteq F^{\varepsilon}$;
- Each totally bounded subset A is separable;
- $\mathbf{TB}_d(X) \subseteq \mathbf{B}_d(X)$; equality holds if and only if $\langle \widetilde{X}, \widetilde{d}\rangle$ is boundedly compact;
- $\mathbf{TB}_d(X)$ is preserved under each uniformly equivalent remetrization.

3. Interchanging Bounded Sets With Totally Bounded Sets

We immediately settle our first question.

Theorem 3.1. *Let $\langle X, d\rangle$ be a metric space and let $x_0 \in X$. The following conditions are equivalent:*

(1) There exists an equivalent metric ρ such that $\mathbf{B}_d(X) = \mathbf{TB}_\rho(X)$;

(2) $\langle X, d\rangle$ is separable;

(3) There is a topological embedding φ of X into some metrizable space Y such that the family $\{cl_Y(\varphi(\overline{S}_d(x_0, n)) : n \in \mathbb{N}\}$ is cofinal in $\mathcal{K}_0(Y)$;

(4) There exists an equivalent metric ρ with $\mathbf{B}_d(X) = \mathbf{TB}_\rho(X) = \mathbf{B}_\rho(X)$.

Proof. (1) $\Rightarrow$ (2). From (1), X is a countable union of totally bounded sets and thus has a separable topology.

(2) $\Rightarrow$ (3). In the case that the metric d is bounded, by the Urysohn Embedding Theorem [21, pg. 166], there exists an embedding φ of X into $[0,1]^{\mathbb{N}}$. Let Y be the closure of $\varphi(X)$ in the product; choosing n with $X = \overline{S}_d(x_0; n)$, we have $Y = \mathrm{cl}_Y(\varphi(\overline{S}_d(x_0, n))$, and so this set alone is cofinal in $\mathcal{K}_0(Y)$.

If d is unbounded, let $\{x_i : i \in \mathbb{N}\}$ be dense in X. For each positive integer i, let $f_i : X \to \mathbb{R}$ be defined by $f_i(x) = d(x, x_i)$. Suppose $C \neq \emptyset$ is a closed subset of X and $x \notin C$; choosing x_i with $d(x, x_i) < d(x_i, C)$, we have $f_i(x) \notin \mathrm{cl}(f_i(C))$. As $\{f_i : i \in \mathbb{N}\}$ separates points from closed sets

$$x \mapsto \langle f_i(x)\rangle_{i=1}^{\infty}$$

is an embedding φ of X into $\mathbb{R}^{\mathbb{N}}$ equipped with the product topology [15, 21]. We claim that $Y := \mathrm{cl}(\varphi(X))$ equipped with the relative topology satisfies the required conditions.

Let d^* be a metric compatible with the product topology on $\mathbb{R}^{\mathbb{N}}$ (see, e.g., [21, pg. 162]). Let $n \in \mathbb{N}$ be arbitrary; $\forall i \in \mathbb{N},\ f_i(\overline{S}_d(x_0, n))$ is bounded, so by the Tychonoff Theorem, $\mathrm{cl}_Y(\varphi(\overline{S}_d(x_0, n))$ is compact as it is contained in a product of closed intervals. It remains to show that $\{\mathrm{cl}_Y(\varphi(\overline{S}_d(x_0, n)) : n \in \mathbb{N}\}$ is cofinal in $\mathcal{K}_0(Y)$. If not, suppose $K \in \mathcal{K}_0(Y)$ is contained in none of these. For each $n \in \mathbb{N}$, take $y_n \in K \backslash \mathrm{cl}_Y(\varphi(\overline{S}_d(x_0, n))$ and then $x_n \in X$ with $d(x_n, x_0) > n$ and $d^*(y_n, \varphi(x_n)) < \frac{1}{n}$. By the compactness of K and the metrizability of Y, some subsequence $\langle y_{n_k} \rangle_{k=1}^{\infty}$ of $\langle y_n \rangle_{n=1}^{\infty}$ must d^*-converge to some point of K. Denoting the limit by y_0, we also have $\langle \varphi(x_{n_k}) \rangle_{k=1}^{\infty}$ convergent to y_0. But this is impossible, as for each fixed $i \in \mathbb{N}$, we have

$$\lim_{k \to \infty} f_i(x_{n_k}) = \lim_{k \to \infty} d(x_{n_k}, x_i) = \infty.$$

(3) $\Rightarrow$ (4). We now identify X with $\varphi(X)$. Note that X must be dense in Y, else for some $y_0 \in Y,\ y_0 \notin \mathrm{cl}_Y(X)$. Thus, the compact set $\{y_0\}$ fails to be contained in any set of the form $\mathrm{cl}_Y(\overline{S}_d(x_0, n))$, and this violates condition (3). Since $\mathcal{K}_0(Y)$ has a countable cofinal subfamily and Y is first countable, by a theorem of Arens [2], Y is locally compact, and since Y is locally compact, separable and metrizable, it has a compatible boundedly compact metric $\widetilde{\rho}$ [18]. Let us denote by ρ the trace of $\widetilde{\rho}$ on $X \times X$ (note the appropriateness of the notation). By the sixth bulleted property of total boundedness listed in the previous section, we have $\mathbf{TB}_\rho(X) = \mathbf{B}_\rho(X)$. It remains to show that $\mathbf{TB}_\rho(X) = \mathbf{B}_d(X)$.

Suppose first $B \subseteq X$ is ρ-totally bounded. As $\langle Y, \widetilde{\rho} \rangle$ is complete, the set $\mathrm{cl}_Y(B)$ is compact. By cofinality, choose $n \in \mathbb{N}$ with $\mathrm{cl}_Y(B) \subseteq \mathrm{cl}_Y(\overline{S}_d(x_0, n))$. This of course implies that

$$B \subseteq \mathrm{cl}_X(B) \subseteq \mathrm{cl}_X(\overline{S}_d(x_0, n)) = \overline{S}_d(x_0, n),$$

and it follows that $\mathbf{TB}_\rho(X) \subseteq \mathbf{B}_d(X)$. For the reverse inclusion, if $B \in \mathbf{B}_d(X)$, we choose $n \in \mathbb{N}$ with $B \subseteq \overline{S}_d(x_0, n)$; as $B \subseteq \mathrm{cl}_Y(\overline{S}_d(x_0, n))$ we see that B is $\widetilde{\rho}$-totally bounded. As approximations by finite sets can be done from the inside to determine total boundedness, ρ-total boundedness of B follows.

(4) $\Rightarrow$ (1). This is trivial □

Corollary 3.2. *Let $\langle X, \mathcal{T} \rangle$ be a metrizable space. The following conditions are equivalent:*

(1) $\langle X, \mathcal{T} \rangle$ is separable and noncompact;

(2) There exists a compatible metric ρ for $\langle X, \mathcal{T} \rangle$ whose completion is unbounded and boundedly compact.

Proof. As is well-known [10, pg. 347], $\langle X, \mathcal{T} \rangle$ is noncompact if and only if there exists an unbounded compatible metric (this follows immediately from the failure of the space to be pseudo-compact). Since (2) implies X is separable, (2) $\Rightarrow$ (1) holds. For (1) $\Rightarrow$ (2), starting with an unbounded metric d for $\langle X, \mathcal{T} \rangle$, by the sixth bulleted property for total boundedness, the metric ρ described in condition (4) of Theorem 3.1 does the job. $\square$

Theorem 3.1 warrants commentary. First, identifying X with $\varphi(X)$, an embedding of the form described in condition (3) of Theorem 3.1 has these noteworthy sequential properties:

(i) Each bounded sequence in X has a cluster point in Y;

(ii) Each sequence in X convergent to a point of Y must be bounded in $\langle X, d\rangle$

Condition (ii) means in particular that a sequence in X that converges to infinity in d-distance cannot cluster in Y. It is left as a routine exercise to the reader to show that conditions (i) + (ii) + density of X in Y are together equivalent to condition (3) of Theorem 3.1.

Second, we note that while we used a boundedly compact compatible metric for Y in the proof of (3) $\Rightarrow$ (4), the proof shows that any complete compatible metric will work to get $\mathbf{B}_d(X) = \mathbf{TB}_\rho(X)$, and it is possible that the bounded sets for a different complete metric can properly contain its totally bounded sets. That is, it is not necessary that $\mathbf{B}_\rho(X) = \mathbf{TB}_\rho(X)$ holds in order that $\mathbf{B}_d(X) = \mathbf{TB}_\rho(X)$ hold. For example, in the plane with Euclidean metric d, the metric

$$\rho((x_1, y_1), (x_2, y_2)) := \min\{1, d((x_1, y_1), (x_2, y_2))\} + |y_2 - y_1|,$$

being uniformly equivalent to d, is complete, and so by the seventh bulleted property for total boundedness has the same totally bounded sets as d. However, $\{(x, 0) : x \in \mathbb{R}\} \in \mathbf{B}_\rho(\mathbb{R}^2)$, as it has ρ-diameter one. As we shall presently see, there are uncountably many compatible metrics on $\mathbb{R}^2$ determining distinct metric bornologies each of whose induced totally bounded sets coincide with $\mathbf{B}_d(\mathbb{R}^2)$.

Finally, the assumption of metrizability of Y in condition (3) cannot be removed. This is because the assumption that $\mathcal{K}_0(Y)$ has a countable cofinal subfamily does not guarantee metrizability of Y, even if each compact subspace with the relative topology is metrizable and the space is assumed to be Tychonoff. To see this, consider the Hilbert space ℓ_2 equipped with the weak topology. The weakly compact sets here are the weakly closed norm bounded subsets, each of which is metrizable by reflexivity and separability [13, pg. 72]. While the closed balls with fixed center and integral radius are cofinal in the weakly compact subsets, the space itself is not metrizable.

Theorem 3.1 involves the existence of a certain embedding, a theme which will be taken up again in this paper. We note that there a direct proof of (2) $\Rightarrow$ (4) which is more in the spirit of the standard proof of Hu's characterization of metric bornologies [3, 14, 15]: add the continuous pseudometric $|d(x, x_0) - d(w, x_0)|$ to a compatible totally bounded metric for X to produce an equivalent metric ρ satisfying (4). The details are left to the interested reader.

We now turn to our second question. We first notice that separability of $\langle X, d\rangle$ is not sufficient. For example, consider the completely metrizable space of irrationals $\mathbb{R}\backslash\mathbb{Q}$ with the topology it inherits from $\mathbb{R}$, equipped with a complete metric d. If a remetrization ρ produced a metric with $\mathbf{B}_\rho(\mathbb{R}\backslash\mathbb{Q}) = \mathbf{TB}_d(\mathbb{R}\backslash\mathbb{Q})$, then $\mathbf{TB}_d(\mathbb{R}\backslash\mathbb{Q})$ would have a countable closed base. But each closed totally bounded subset in this setting must be compact and thus has empty interior. By Baire's theorem, $\mathbb{R}\backslash\mathbb{Q}$

cannot be a countable union of such sets, and we have a contradiction. A similar analysis applies to the space ℓ_2 with the metric induced by $||\cdot||_2$.

Of course Hu's two conditions provide necessary and sufficient conditions for $\mathbf{TB}_d(X)$ to be a metric bornology. But in the case of bornologies of totally bounded sets, the second condition is redundant, as we intend to show. First, we give a lemma that we will use in the proof.

Lemma 3.3. *Let $\langle X, d\rangle$ be a metric space, let B be a totally bounded subset of X, and let E be a dense subset of X. Then there exists a totally bounded subset E_0 of E with $B \subseteq cl(E_0)$.*

Proof. Let C be the closure of B in the completion of X. Since E is also dense in $\langle \widetilde{X}, \widetilde{d}\rangle$, we can choose for each $n \in \mathbb{N}$ an irreducible finite subset E_n of E with $C \subseteq E_n^{\frac{1}{n}}$ (the enlargement is taken in the completion). This means that reciprocally, $E_n \subseteq C^{\frac{1}{n}}$. We intend to show that $C \cup \bigcup_{n=1}^{\infty} E_n$ is totally bounded in $\langle \widetilde{X}, \widetilde{d}\rangle$.

To see this, let $\varepsilon > 0$ be arbitrary and choose $k \in \mathbb{N}$ with $\frac{1}{k} < \varepsilon$. Since C is compact, $E_k^{\frac{1}{k}}$ contains some enlargement C^{δ} where $\delta \in (0,1)$. By irreducibility, whenever $n > \delta^{-1}$, we have $E_n \subseteq C^{\delta}$, and as a result, choosing $n_0 > \delta^{-1}$, we have

$$C \cup \bigcup_{n=1}^{\infty} E_n \subseteq E_k^{\varepsilon} \cup (\bigcup_{n=1}^{n_0} E_n)^{\varepsilon}.$$

This proves $C \cup \bigcup_{n=1}^{\infty} E_n$ is $\widetilde{d}$-totally bounded, and so its subset $\cup_{n=1}^{\infty} E_n$ is $\widetilde{d}$-totally bounded and thus is d-totally bounded. Our construction gives $B \subseteq C \subseteq \mathrm{cl}_{\widetilde{X}}(\cup_{n=1}^{\infty} E_n)$, and so $B \subseteq \mathrm{cl}_X(\cup_{n=1}^{\infty} E_n)$. $\square$

We will employ our lemma only in the case that B is compact.

Theorem 3.4 (cf. [6, Theorem 4.12])**.** *Let $\langle X, d\rangle$ be a metric space. The following conditions are equivalent:*

(1) $\mathbf{TB}_d(X)$ has a countable base;

(2) There exists a metric ρ equivalent to d for which $\mathbf{TB}_d(X) = \mathbf{B}_\rho(X)$;

(3) The one-point extension of X associated with $\mathbf{TB}_d(X)$ is metrizable;

(4) The one-point extension of X associated with $\mathbf{TB}_d(X)$ has a countable neighborhood base at the ideal point.

Proof. The implications (3) $\Rightarrow$ (4) and (4) $\Rightarrow$ (1) are obvious, and as the one-point extension corresponding to a metric bornology is always metrizable [4], we get (2) $\Rightarrow$ (3). Thus we must only establish (1) $\Rightarrow$ (2).

Let $\langle \widetilde{X}, \widetilde{d}\rangle$ be the completion of $\langle X, d\rangle$ and let $\{B_n : n \in \mathbb{N}\}$ be cofinal in $\mathbf{TB}_d(X)$ with respect to inclusion. We claim that the family of compact sets $\{\mathrm{cl}_{\widetilde{X}}(B_n) : n \in \mathbb{N}\}$ is cofinal in $\mathcal{K}_0(\widetilde{X})$. To see this, let K be an arbitrary compact subset of $\langle \widetilde{X}, \widetilde{d}\rangle$. By Lemma 3.3 and the density of X in $\widetilde{X}$, there exists $E_0 \subseteq X$ with $E_0 \in \mathbf{TB}_{\widetilde{d}}(\widetilde{X})$ and $K \subseteq \mathrm{cl}_{\widetilde{X}}(E_0)$. Clearly $E_0 \in \mathbf{TB}_d(X)$, so for some n, $E_0 \subseteq B_n$, and the claim is established.

Arguing now just as in the proof of Theorem 3.1, there is an equivalent boundedly compact metric $\widetilde{\rho}$ for $\widetilde{X}$, and if ρ is its trace on $X \times X$, we get $\mathbf{TB}_d(X) = \mathbf{TB}_\rho(X)$ so that $\mathbf{TB}_d(X) = \mathbf{B}_\rho(X)$. $\square$

If $\mathbf{TB}_d(X)$ has a countable base, by Hu's Theorem, it must have a (countable) open base. As noticed in [6] using unrelated methods, actually more can be shown: for each $B \in \mathbf{TB}_d(X)$ there exists $\varepsilon > 0$ with $B^\varepsilon \in \mathbf{TB}_d(X)$, that is, the bornology of totally bounded sets is *stable under small enlargements.*

We further note the interesting lack of symmetry in Theorems 3.1 and 3.4: in the first, the conditions given are purely topological, whereas in the second, the conditions given are bornological.

Earlier in this section, we saw that two equivalent unbounded metrics with different metric bornologies could determine the same bornology of nonempty totally bounded sets. We aim to show just how robust this phenomenon is. This falls out of the following result of independent interest.

Theorem 3.5. *Let $\langle X, d\rangle$ be an unbounded metric space. Then there exists a family of metrics $\{\rho_r : r \in \mathbb{R}\}$ on X each uniformly equivalent to d such that whenever $r_1 \in \mathbb{R}, r_2 \in \mathbb{R}$, and $r_1 \neq r_2$, then $\mathbf{B}_{\rho_{r_1}}(X) \neq \mathbf{B}_{\rho_{r_2}}(X)$.*

Proof. The proof of Hu's Theorem [14] shows that whenever $\{C_n : n \in \mathbb{N}\}$ is a family of nonempty closed sets such that (i) $\forall n,\ X\backslash C_n \neq \emptyset$, (ii) $\cup_{n=1}^\infty C_n = X$, and (iii) $\forall n \in \mathbb{N},\ C_n \subseteq \text{int}\ (C_{n+1})$, then $\{C_n : n \in \mathbb{N}\}$ is a base for the bornology of bounded sets as determined by some unbounded metric ρ equivalent to d. Further, ρ can be chosen to be uniformly equivalent to d if and only if $\exists\delta > 0,\ \forall n \in \mathbb{N},\ \exists k \in \mathbb{N}$ with $C_n^\delta \subseteq C_k$, where the enlargement is taken with respect to d [3, Theorem 4.2].

Since X is not d-bounded, there exists a sequence $\langle x_j\rangle$ in X with distinct terms such that $\forall j \in \mathbb{N},\ \overline{S}_d(x_j, j) \cap \{x_k : k \neq j\} = \emptyset$. It is well-known that there exists a family of infinite subsets $\{\mathbb{M}_r : r \in \mathbb{R}\}$ of $\mathbb{N}$ each two members of which have finite intersection (see, e.g., [11, 16]). In particular, this almost disjoint family of subsets has cardinality of the continuum $\mathfrak{c}$. For each $r \in \mathbb{R}$ and $n \in \mathbb{N}$ put

$$C_{r,n} := \bigcup_{j \in \mathbb{M}_r} \overline{S}_d(x_j, n).$$

Notice that for each index r, each infinite subset of $\{x_n : n \in \mathbb{M}_r\}$ has infinite diameter. It follows that for each $n \in \mathbb{N}$, the family $\{\overline{S}_d(x_j, n) : j \in \mathbb{M}_r\}$ is locally finite, whence each set $C_{r,n}$ is closed. Further, as $C_{r,n}$ fails to contain x_j whenever $j > n$ and $j \notin \mathbb{M}_r$, we see that $X\backslash C_{n,r}$ is nonempty. Finally, whenever $\delta < 1$ we have $C_{r,n}^\delta \subseteq C_{r,n+1}$. By [3, Theorem 4.2], for each $r \in \mathbb{R}$ there exists a metric ρ_r uniformly equivalent to d such that $\{C_{r,n} : n \in \mathbb{N}\}$ is base for $\mathbf{B}_{\rho_r}(X)$. However, if $r_1 \in \mathbb{R}$ and $r_2 \in \mathbb{R}$ are distinct, by construction $\{x_j : j \in \mathbb{M}_{r_1}\}\backslash\{x_j : j \in \mathbb{M}_{r_2}\}$ is infinite. It is clear that $\{x_j : j \in \mathbb{M}_{r_1}\}$ is not contained in $C_{r_2,n}$ for any $n \in \mathbb{N}$. This means that $\mathbf{B}_{\rho_{r_1}}(X) \nsubseteq \mathbf{B}_{\rho_{r_2}}(X)$. □

Remark 3.6. If we do not assume that the initial metric d on X is unbounded, then there may be no unbounded metric uniformly equivalent to d, e.g., consider any compact metric space, or for our metric space the unit ball in ℓ_2.

Corollary 3.7. *Let $\langle X, d\rangle$ be an unbounded metric space. Then there exists a family of compatible metrics $\{\rho_r : r \in \mathbb{R}\}$ on X such that $\forall r \in \mathbb{R},\ \mathbf{TB}_{\rho_r}(X) = \mathbf{TB}_d(X)$ while whenever $r_1 \in \mathbb{R}, r_2 \in \mathbb{R}$, and $r_1 \neq r_2$, then $\mathbf{B}_{\rho_{r_1}}(X) \neq \mathbf{B}_{\rho_{r_2}}(X)$.*

Proof. Uniformly equivalent metrics determine the same totally bounded sets. □

Corollary 3.7 leads one to ask: under what circumstances does a particular metric bornology correspond to a family of bornologies of totally bounded subsets having cardinality of the continuum? Our final result of this section completely resolves this question.

Theorem 3.8. *Let $\langle X, d\rangle$ be a metric space. The following conditions are equivalent:*

(1) $\langle X, d\rangle$ is not boundedly compact;

(2) There exists a metric ρ on X equivalent to d such that $\mathbf{B}_\rho(X) = \mathbf{B}_d(X)$, yet $\mathbf{TB}_\rho(X) \neq \mathbf{TB}_d(X)$;

(3) There is a family of compatible metrics $\{\rho_r : r \in \mathbb{R}\}$ on X such that $\forall r \in \mathbb{R},\ \mathbf{B}_{\rho_r}(X) = \mathbf{B}_d(X)$ while whenever $r_1 \in \mathbb{R}, r_2 \in \mathbb{R}$, and $r_1 \neq r_2$, then $\mathbf{TB}_{\rho_{r_1}}(X) \neq \mathbf{TB}_{\rho_{r_2}}(X)$.

Proof. (1) ⇒ (3). Let $B \in \mathbf{B}_d(X)$ be closed but noncompact, and let $\langle x_n\rangle$ be a sequence in B with distinct terms without a cluster point. Let $\{\mathbb{M}_r : r \in \mathbb{R}\}$ be the almost disjoint family prescribed in the proof of Theorem 3.5. For each $r \in \mathbb{R}$ define a metric δ_r on the closed set $\{x_n : n \in \mathbb{N}\}$ as follows:

$$\delta_r(x_j, x_n) := \begin{cases} 0 & \text{if } j = n \\ \max\{\frac{1}{j+1}, \frac{1}{n+1}\} & \text{if } j \neq n \text{ and } \{j, n\} \subseteq \mathbb{M}_r \\ 1 & \text{otherwise} \end{cases}.$$

The discreteness of $\{x_n : n \in \mathbb{N}\}$ implies that the metric is compatible with the relative topology. Since $\{x_n : n \in \mathbb{N}\}$ is a closed subset of X, by the Hausdorff extension theorem [10, pg. 369], there exists a compatible metric δ_r^* on X whose restriction to $\{x_n : n \in \mathbb{N}\}$ coincides with δ_r.

Now if $f : X \to \mathbb{R}$ is arbitrary and $A \in \mathcal{P}_0(X)$, then $f(A)$ is bounded if and only if $\sup\{|f(a_1) - f(a_2)| : a_1 \in A, a_2 \in A\} < \infty$. As the distance functional $d(\cdot, B)$ restricted to A is bounded if and only if $A \in \mathbf{B}_d(X)$, for each $r \in \mathbb{R}$ the metric $\rho_r : X \times X \to [0, \infty)$ defined by

$$\rho_r(x, w) := \min\{1, \delta_r^*(x, w)\} + |d(x, B) - d(w, B)|$$

satisfies $\mathbf{B}_{\rho_r}(X) = \mathbf{B}_d(X)$. By the continuity of $d(\cdot, B)$, the metric ρ_r is equivalent to δ_r^* and thus is a compatible metric.

Now if we can prove that whenever r_1 and r_2 are distinct real numbers, we have $\mathbf{TB}_{\rho_{r_1}}(X) \neq \mathbf{TB}_{\rho_{r_2}}(X)$, then we will be done. By construction, $\{x_n : n \in \mathbb{M}_{r_1}\}$ contains an infinite subset $E(r_1, r_2)$ disjoint from $\{x_n : n \in \mathbb{M}_{r_2}\}$. Observe that $\{x_n : n \in \mathbb{M}_{r_1}\}$ is the range of a ρ_{r_1}-Cauchy sequence because ρ_{r_1} restricted to B agrees with $\min\{\delta_{r_1}^*, 1\}$. Thus $\{x_n : n \in \mathbb{M}_{r_1}\}$ and therefore its subset $E(r_1, r_2)$ are ρ_{r_1}-totally bounded. On the other hand, the set $E(r_1, r_2)$ is ρ_{r_2}-uniformly discrete, for whenever $\{x, w\} \subseteq E(r_1, r_2)$, we compute

$$\rho_{r_2}(x, w) = \min\{\delta_{r_2}^*(x, w), 1\} = \min\{\delta_{r_2}(x, w), 1\} = 1.$$

Thus, $E(r_1, r_2)$ as a subset of $\{x_n : n \in \mathbb{M}_{r_1}\}$ belongs to $\mathbf{TB}_{\rho_{r_1}}(X) \backslash \mathbf{TB}_{\rho_{r_2}}(X)$ as required.

(3) ⇒ (2). This is obvious.

(2) $\Rightarrow$ (1). Suppose condition (1) fails, i.e, d is boundedly compact. If ρ is equivalent to d and $\mathbf{B}_\rho(X) = \mathbf{B}_d(X)$, then the ρ-bounded sets are just the relatively compact subsets of X. Since always $\mathcal{RK}_0(X) \subseteq \mathbf{TB}_\rho(X)$, we get

$$\mathbf{TB}_\rho(X) = \mathcal{RK}_0(X) = \mathbf{TB}_d(X).$$

Thus condition (2) fails. □

4. Which Bornologies are Bornologies of Totally Bounded Sets?

Here we produce two separate sets of necessary and sufficient conditions for a bornology on a metrizable space to be a bornology of totally bounded sets with respect to some compatible metric. The first set of conditions we give are not internal conditions as are those given in Hu's characterization of metric bornologies; rather, they are of the flavor of condition (3) of Theorem 3.1.

Theorem 4.1. *Let $\mathcal{B}$ be a family of nonempty subsets of a metrizable space $\langle X, \mathcal{T}\rangle$. Then $\mathcal{B} = \mathbf{TB}_d(X)$ for some compatible metric d if and only if there exists an embedding ψ of X into a completely metrizable space Y with the following property:*

$$(\sharp) \qquad \mathcal{B} = \{E \in \mathcal{P}_0(X) : \psi(E) \text{ is relatively compact in } Y\}.$$

Proof. For necessity, assume for some compatible d that $\mathcal{B} = \mathbf{TB}_d(X)$. Let ψ be the inclusion of $\langle X, d\rangle$ into $\langle \widetilde{X}, \widetilde{d}\rangle$. It is obvious that ($\sharp$) holds for this embedding. For sufficiency, assume that such an embedding exists and put $W := \text{cl}_Y(\psi(X))$. As a closed subspace of Y, W is completely metrizable itself. Let $\widetilde{d}$ be a compatible complete metric for W and let d be its trace on $\psi(X) \times \psi(X)$. If $E \in \mathcal{B}$, then $\text{cl}_Y(\psi(E)) = \text{cl}_W(\psi(E))$ is compact, so $\psi(E)$ is $\widetilde{d}$-totally bounded, and since $\psi(E) \subseteq \psi(X)$, we get $\psi(E) \in \mathbf{TB}_d(\psi(X))$. On the other hand, if $\psi(E) \in \mathbf{TB}_d(\psi(X))$, then by completeness $\text{cl}_W(\psi(E)) = \text{cl}_Y(\psi(E))$ is compact, and so E belongs to $\mathcal{B}$. □

Notice in the proof of Theorem 4.1 that it is not necessary to assume in advance that $\mathcal{B}$ is a bornology; the reader is invited to derive this directly from condition ($\sharp$). Our first example shows the necessity of assuming that Y is completely metrizable.

Example 4.2. Consider $\mathbb{Q}$ as a subspace of $\mathbb{R}$; as Baire's theorem fails in $\mathbb{Q}$, the space is not completely metrizable. Put $\mathcal{B} := \{E \in \mathcal{P}_0(\mathbb{Q}) : \text{cl}_{\mathbb{Q}}(E) \text{ is compact}\}$. Evidently, the identity map on $\mathbb{Q}$ satisfies ($\sharp$). But the family of nonempty relatively compact subsets a metric space forms a bornology of totally bounded sets if and only if the space is completely metrizable, as argued in Corollary 4.6 *infra*.

In our next example we show that ($\sharp$) cannot be replaced by this weaker statement: ($\triangleright$) $\forall E \in \mathcal{B},\ \text{cl}_Y(\psi(E))$ is compact. This is true even if we assume $\mathcal{B}$ is a bornology.

Example 4.3. Consider [0,1] equipped with the usual topology; by its compactness, $\mathbf{TB}_d([0,1]) = \mathcal{P}_0([0,1])$ whenever d is a compatible metric. But any bornology on $[0,1]$ having as a member a dense subset of $[0,1]$ satisfies condition ($\triangleright$) with respect to the identity map, e.g., the bornology consisting of all sets of the form $F \cup Q$ where F is a nonempty finite subset of $[0,1]$ and Q is a possibly empty subset of $\mathbb{Q}$.

It is clear that the argument used in establishing (2) $\Rightarrow$ (3) in Theorem 3.1 can be easily adjusted to show that ($\sharp$) holds for the specific embedding φ described therein, yielding a direct proof of (2) $\Rightarrow$ (1). We next present a different application of Theorem 4.1.

Proposition 4.4. *Let $\mathcal{S}(X)$ be the bornology of nonempty separable subsets of a metrizable space $\langle X, \mathcal{T}\rangle$. Then $\mathcal{S}(X)$ is the bornology of totally bounded subsets with respect to some compatible metric if and only if X itself is separable.*

Proof. If X is separable, then $\mathcal{S}(X) = \mathcal{P}_0(X)$, so that if d is a compatible totally bounded metric, we have $\mathbf{TB}_d(X) = \mathcal{P}_0(X)$. Conversely, suppose now that $\mathcal{S}(X)$ is the bornology of totally bounded subsets with respect to some compatible metric, yet X is not separable. Let Y and ψ be as guaranteed by Theorem 4.1 with respect to the family $\mathcal{S}(X)$. Of course, $\mathrm{cl}_Y(\psi(X))$ is not compact; thus, there exists a sequence $\langle y_n\rangle$ in $\mathrm{cl}_Y(\psi(X))$ without a cluster point. Without loss of generality we can assume that each y_n actually belongs to $\psi(X)$. Choosing for each $n \in X$, $x_n \in X$ with $\psi(x_n) = y_n$, we obtain a separable subset $\{x_n : n \in \mathbb{N}\}$ mapped by ψ to a set failing to be relatively compact, which is a contradiction. $\square$

We now turn to an internal characterization of bornologies that are bornologies of totally bounded sets. One version of the Alexandroff-Urysohn metrization theorem - the first general metrization theorem [1] - is this: a Hausdorff (or even T_0) space $\langle X, \mathcal{T}\rangle$ is metrizable if and only if there exists a sequence of open covers $\langle \mathcal{U}_n\rangle$ of X with the following two properties [21, pg. 167]:

(1) $\forall n \in \mathbb{N},\ \mathcal{U}_{n+1}$ star-refines $\mathcal{U}_n$;

(2) $\forall x \in X, \{\mathrm{St}(x, \mathcal{U}_n) : n \in \mathbb{N}\}$ is a neighborhood base for $\mathcal{T}$ at x.

A sequence of open covers satisfying both (1) and (2) is called a *star-development* or a *compatible normal sequence* for X. An analysis of a standard proof of the metrization theorem leads easily to our internal characterization.

Theorem 4.5. *Let $\langle X, \mathcal{T}\rangle$ be a metrizable space and let $\mathcal{B}$ be a family of nonempty subsets of X. Then $\mathcal{B} = \mathbf{TB}_d(X)$ for some compatible metric d if and only if there is a star-development $\langle \mathcal{U}_n\rangle$ for X such that*

$$(\vdash) \qquad \mathcal{B} = \{E \in \mathcal{P}_0(X) : \forall n \in \mathbb{N},\ \mathcal{U}_n \textit{ admits a finite subcover of } E\}.$$

Proof. If $\mathcal{B} = \mathbf{TB}_d(X)$ for some compatible metric d, then let $\mathcal{U}_n = \{S_d(x, \frac{1}{3^n}) : x \in X\}$ for every $n \in \mathbb{N}$. It is easily shown that $\langle \mathcal{U}_n\rangle$ is a star-development of X. If $E \in \mathcal{B}$, then for each n, E can be covered by finitely many d-balls of radius of radius $\frac{1}{3^n}$, i.e., by finitely many elements of $\mathcal{U}_n$, and if for each n, E can be covered by finitely many members of $\mathcal{U}_n$, then almost by definition, E is d-totally bounded.

For sufficiency, first note that if $\mathcal{B}$ satisfies ($\vdash$), then $\mathcal{B}$ is a bornology. The existence of a star-development $\langle \mathcal{U}_n\rangle$ allows for the construction of a compatible metric d such that $\forall n \geq 2$, both of the following conditions hold [21, pg. 167]:

(i) $\mathcal{U}_n$ refines $\{S_d(x, \frac{1}{2^{n-1}}) : x \in X\}$;

(ii) $\{S_d(x, \frac{1}{2^n}) : x \in X\}$ refines $\mathcal{U}_{n-1}$.

Suppose $\mathcal{B}$ satisfies ($\vdash$). Fix $E \in \mathcal{B}$, let $\varepsilon > 0$ be arbitrary and choose n with $2^{1-n} < \varepsilon$. By ($\vdash$) choose $\{U_1, U_2, \ldots, U_{k_n}\} \subseteq \mathcal{U}_n$ with $E \subseteq \cup_{j=1}^{k_n} U_j$ and then by

(i) choose $\forall j \leq k_n,\ x_j$ with $U_j \subseteq S_d(x_j, \frac{1}{2^{n-1}})$. This yields $E \subseteq \{x_1, x_2, \ldots x_{k_n}\}^\varepsilon$ and so $E \in \mathbf{TB}_d(X)$. On the other hand, if E is d-totally bounded and $n \in \mathbb{N}$ is arbitrary, we can choose a finite subset F of X with $E \subseteq F^{2^{-n-1}}$. Using (ii), this immediately yields $E \in \mathcal{B}$. □

As an immediate corollary we obtain a curious characterization of those metrizable spaces that are completely metrizable.

Corollary 4.6. *Let $\langle X, \mathcal{T}\rangle$ be a metrizable space. The following conditions are equivalent:*

(1) $\langle X, \mathcal{T}\rangle$ is completely metrizable;

(2) The bornology of nonempty relatively compact subsets is a bornology of totally bounded subsets with respect to some compatible metric;

(3) There exists a star-development $\langle \mathcal{U}_n\rangle$ for X such that a subset A of X is relatively compact if and only if $\forall n \in \mathbb{N}$, $\mathcal{U}_n$ admits a finite subcover of A.

Proof. The implication (1) ⇒ (2) is obvious while (2) ⇔ (3) follows from Theorem 4.5. Finally, for (2) ⇒ (1), if $\langle X, \mathcal{T}\rangle$ fails to be completely metrizable, then for each compatible metric d, there is a d-Cauchy sequence that fails to converge, and so its set of terms while d-totally bounded fails to be relatively compact. □

With respect to the bornology of separable subsets that we dealt with in Proposition 4.4, even more can be said than what Theorem 4.5 literally yields.

We acknowledge the possibility that the equivalence of conditions (2) and (3) in Proposition 4.7 below may be known, but we have not been able to find a reference in the literature.

Proposition 4.7. *Let $\mathcal{S}(X)$ be the family of nonempty separable subsets of a metrizable space $\langle X, \mathcal{T}\rangle$. The following conditions are equivalent:*

(1) $\mathcal{S}(X)$ is the bornology of totally bounded subsets with respect to some compatible metric;

(2) $\langle X, \mathcal{T}\rangle$ is separable;

(3) There exists a star-development $\langle \mathcal{U}_n\rangle$ for X such that for each n, $\mathcal{U}_n$ is a finite family of sets.

Proof. We already know that conditions (1) and (2) are equivalent. For (3) ⇒ (2), it is immediate from (3) that

$$\{E \in \mathcal{P}_0(X) : \forall n \in \mathbb{N},\ \mathcal{U}_n \text{ admits a finite subcover of } E\}$$

coincides with $\mathcal{P}_0(X)$. Thus, by Theorem 4.5, X must be totally bounded with respect to some compatible metric and so X is separable.

For (2) ⇒ (3), topologically embed $\langle X, \mathcal{T}\rangle$ into the compact metrizable space $Y := [0,1]^{\mathbb{N}}$. Identify X with its image, and let ρ be metric compatible with the product topology for Y. Let $\varepsilon_1 > 0$ be arbitrary. By compactness take F_1 finite in the product such that $Y = \cup_{y \in F_1} S_\rho(y, \varepsilon_1)$. Again by compactness, let $\lambda_1 > 0$ be a Lebesgue number for the cover, and then put $\varepsilon_2 := \min\{\lambda_1, \frac{\varepsilon_1}{3}\}$. Let $F_2 \subseteq Y$ be finite with $Y = \cup_{y \in F_2} S_\rho(y, \varepsilon_2)$. Let $\lambda_2 > 0$ be a Lebesgue number for this second cover, and then put $\varepsilon_3 := \min\{\lambda_2, \frac{\varepsilon_2}{3}\}$. Continuing in this way, the sequence of finite open covers $\langle \mathcal{U}_n \rangle$ of X, where for each n,

$$\mathcal{U}_n := \{S_\rho(y, \varepsilon_n) \cap X : y \in F_n\}$$

is easily shown to be a star-development of X. □

It is noteworthy that in the proof of Theorem 4.5, we use star-developments to construct a totally bounded metric with certain properties, whereas in the proof of Proposition 4.7, we do just the opposite. We also mention that we have been able to derive the equivalence of conditions (1) and (2) in Theorem 3.1 from Theorem 4.5; however, we have not inserted our argument here, as it requires some nontrivial techniques to obtain, from a given cover, a star-refinement with suitable supplementary properties. So in total, we know of four distinct paths to a resolution of question (1) posed in the introduction.

Given a Tychonoff space $\langle X, \mathcal{T} \rangle$ and a compatible diagonal uniformity $\mathcal{U}$, one calls a subset B of X $\mathcal{U}$-*totally bounded* [10] if for each entourage U there exists a finite subset F of X with $B \subseteq U(F)$; this of course agrees with our definition in the case of metric uniformities. Motivated by applications to topological vector spaces, boundedness of B is usually defined as follows [7]: $B \subseteq X$ is $\mathcal{U} - bounded$ if $\forall U \in \mathcal{U}$, there exists a finite set F and $n \in \mathbb{N}$ with $B \subseteq U^n(F)$. This definition is problematic for us in that it does not subsume boundedness in metric spaces (consider an infinite set with the zero-one metric). Nevertheless, one may ask analagous questions in this framework. Vroegrijk [20] has recently shown that these two notions coincide under an appropriate re-uniformization of $\langle X, \mathcal{T} \rangle$, and has furthermore exhibited internal conditions on a bornology on a Tychonoff space that make it a bornology of bounded/totally bounded sets with respect to some compatible uniformity.

References

1. P. Alexandroff and P. Urysohn, *Une condition nécéssaire et suffisante pour qu'une classe (L) soit une classe (D)*, C. R. Acad. Sci. Paris **177** (1923), 1274-1277.
2. R. Arens, *A topology for spaces of transformations*, Annals of Math. **47** (1946), 480-495.
3. G. Beer, *On metric boundedness structures*, Set-Valued Anal. **7** (1999), 195-208.
4. G. Beer, *On convergence to infinity*, Monat. Math. **129** (2000), 267-280.
5. G. Beer and S. Levi, *Total boundedness and bornologies*, Top. Appl. **156** (2009), 1271-1288.
6. G. Beer and S. Levi, *Uniform continuity, uniform convergence, and shields*, to appear, Set-Valued and Variational Anal.
7. N. Bourbaki, *Topology générale*, Hermann, Paris, 1965.
8. A. Caterino and S. Guazzone, *Extensions of unbounded topological spaces*, Rend. Sem. Mat. Univ. Padova **100** (1998), 123-135.
9. A. Caterino, T. Panduri, and M. Vipera, *Boundedness, one-point extensions, and B-extensions*, Math. Slovaca. **58** (2008), 101-114.
10. R. Engelking, *General topology*, Polish Scientific Publsihers, Warsaw, 1977.
11. L. Gillman and M. Jerison, *Rings of continuous functions*, Van Nostrand, Princeton, N. J., 1960.
12. H. Hogbe-Nlend, *Bornologies and functional analysis*, North-Holland, Amsterdam, 1977.
13. R. Holmes, *Geometric functional analysis*, Springer-Verlag, New York, 1975.
14. S.-T. Hu, *Boundedness in a topological space*, J. Math Pures Appl. **228** (1949), 287-320.
15. S.-T. Hu, *Introduction to general topology*, Holden-Day, San Francisco, 1966.
16. K. Kunen, *Set theory. An introduction to independence proofs.* Studies in logic and the foundations of mathematics **102**, North-Holland, Amsterdam, 1980.
17. A. Stone, *Paracompactness and product spaces*, Bull. Amer. Math Soc. **54** (1948), 977-982.
18. H. Vaughan, *On locally compact metrizable spaces*, Bull. Amer. Math. Soc. **43** (1937), 532-535.

19. M. Vipera, *Some results on sequentially compact extensions*, Comment. Math. Univ. Carolinae **39** (1998), 819-831.
20. T. Vroegrijk, *Uniformizable and realcompact bornological universes*, preprint.
21. S. Willard, *General topology*, Addison-Wesley, Reading, MA, 1970.

Department of Mathematics, California State University Los Angeles, 5151 State University Drive, Los Angeles, California 90032, USA
E-mail address: gbeer@cslanet.calstatela.edu

Dipartimento di Matematica dell' Univerità di Torino, Via Carlo Alberto 10, 10123 Torino, Italy
E-mail address: camillo.costantini@unito.it

Dipartimento di Matematica e Applicazioni, Universita' di Milano-Bicocca, Via Cozzi 53, 20125 Milano, Italy
E-mail address: sandro.levi@unimib.it