Single- and Cross-Generation Natural Hedging of Longevity and Financial Risk

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Abstract

This paper provides natural hedging strategies for life insurance and annuity businesses written on a single generation or on different generations in the presence of both longevity and interest rate risks. We obtain closed-form solutions for delta and gamma hedges against cohort-based longevity risk. We exploit the correlation between the mortality intensities of different generations and hedge the longevity risk of one cohort with products on other cohorts. An application with UK data on survivorship and bond dynamics shows that hedging is effective, even when rebalancing is infrequent.

Keywords: longevity risk, interest rate risk, delta-gamma hedging, natural hedging, cross-generation hedging.

1 Introduction

Longevity risk, i.e., the risk of unexpected changes in survivorship, is now perceived as an important threat to the safety of insurance companies and pension funds. Most actors in the financial market are long longevity risk. This has stimulated the transformation of contracts subject to longevity risk into an asset class, as originally suggested by Blake and Burrows (2001). The creation of q-forwards, s-forwards, longevity bonds and swaps represents a step in this direction, but this asset class is still in its infancy. In the meanwhile, insurance companies can benefit from natural hedging, i.e., from natural offsetting between the longevity risk exposures of death benefits and life contracts, such as annuities. The importance of exploiting this natural offsetting extends beyond theory. Cox and Lin (2007) find empirical evidence that insurers whose liability portfolios benefit from natural hedging have a competitive advantage and charge lower premiums. Despite being safe, sound and comparatively cheap, natural hedging is not trivial in the presence of longevity risk because the latter is difficult to capture per se in a parsimonious and manageable way and even more difficult to couple with a satisfactory model of financial risk, such as interest rate risk. However, the interactions between longevity and financial risk cannot be avoided from the perspective of immunization in the form of liability management, as the value of the reserves is subject to interest rate risk, and a fortiori, from the perspective of asset and liability management (ALM).

Natural hedging of longevity risk without financial risk has been recently addressed by Cox and Lin (2007), Wang et al. (2010), Gatzert and Wesker (2012) and Gatzert and Wesker (2013). Cox and Lin (2007), motivated by the empirical evidence mentioned above, propose the use of mortality swaps between annuity providers and life insurance writers. Wang et al. (2010) propose an immunization strategy that matches the duration and convexity of life insurance and annuity benefits. They

demonstrate that this strategy is effective in reducing longevity risk by calibrating it to US mortality data. However, they consider only liabilities, while we consider both assets and liabilities as well as financial risk. Gatzert and Wesker (2012) use simulations to select portfolios of policies that immunize the insurer's solvency against changes in mortality. Gatzert and Wesker (2013) consider the interactions among systematic, unsystematic, basis risk and adverse selection in determining the effectiveness of natural hedging.

Natural hedging with financial risk has been studied by Stevens et al. (2011). They show that financial risk has a clear impact on the overall initial riskiness of the annuity–life insurance mix. The effect of natural hedging may be overestimated when financial risk is ignored, affecting hedging possibilities . In their case, financial risk occurs only from potential losses from assets, while in our case, it affects both the assets and the fair value of liabilities.

We extend the existing literature in four directions. First, we model longevity and financial risk at the same time, and we assess their impact on the fair value of the insurer's net liabilities or reserves. We aim at hedging changes in reserves, at the first and second order approximations (delta and gamma hedging, respectively). The risk factors to hedge against are the differences between the mortality and interest rate intensities forecasted today and their actual realizations in the future. Second, to hedge liabilities, we let the insurer use new sales of insurance contracts, reinsurance and bonds, so we extend previous research by using both assets and liabilities for immunization. Third, we exploit hedging within a single generation and across generations (or across genders) to capture the fact that some products may be not marketed. For instance, death benefits for older generations may not be marketed. Thus, we develop a cohort-based mortality model, and we split the longevity risk factor of each generation into common and idiosyncratic parts. Fourth, we provide

all delta-gamma hedges in closed form. This enhances the computation and comprehension of the hedge drivers. Moreover, optimal hedges solve linear systems. Consequently, assessing whether reserves can be perfectly hedged (up to any chosen level of accuracy) and whether the mix of assets and/or liabilities that achieves the hedge is unique is a trivial matter. The framework can accommodate some important practical aspects, such as self-financing constraints, sales constraints and limited availability or absent products.

The UK-calibrated application that concludes this paper computes the hedging portfolios of an annuity. First, this example permits the comparison of the magnitudes of first and second order effects, i.e., the deltas and gammas, within a single type of risk, across risks and across generations. Second, the application illustrates the straightforward computation of sensitivities and hedges, given closed-form solutions and linear systems. Third, it demonstrates the effectiveness of delta and gamma hedging, even when the hedge is adjusted discretely in time, as occurs in practice, rather than renewed continuously. The application is extended to two generations to demonstrate the effectiveness of delta-gamma intra- and cross-generation hedging strategies, even in the presence of constraints on the products that can be sold or reinsured.

The paper is structured as follows: Section 2 reviews the mortality and interest rate model and examines the pricing and hedging of annuities and life insurance policies. Section 3 focuses on single-generation natural hedging. Section 4 presents hedging on a multiple-generation portfolio. Section 5 presents a hedge example calibrated to UK data. Assessment of hedge effectiveness is also provided. Section 6 concludes.

2 Longevity and interest rate risks

We place ourselves in a continuous-time framework, as suggested by Cairns et al. (2006a) and Cairns et al. (2008). In this framework, known as a "stochastic mortality" approach, we use a parsimonious, continuous-time model for cohort-specific mortality intensity that extends the classic Gompertz law and a benchmark model for interest rate risk, the Hull and White model. This section introduces the models for longevity and interest rate risk and delta-gamma hedging for a pure endowment, as obtained in Luciano et al. (2012).

2.1 Model for longevity risk

We consider the time of death of an individual the first jump time of a Poisson process with stochastic intensity, i.e., a Cox process. Let us introduce a probability space $(\Omega, \mathbf{F}, \mathbb{P})$ equipped with a filtration $\{\mathcal{F}_t : 0 \leq t \leq T\}$ that satisfies the usual properties of right-continuity and completeness.[1] Our approach is generation based. We use x to indicate both the group (generation, cohort or gender) and its age at time 0. The spot mortality intensity at calendar time t of an age belonging to generation x is denoted by $\lambda_x(t)$. We assume that under the historical $\mathbb{P}$ measure, $\lambda_x(t)$ follows an Ornstein-Uhlenbeck process without mean reversion (OU):

$$d\lambda_x(t) = a_x \lambda_x(t) dt + \sigma_x dW_x(t), \tag{1}$$

where $a_x > 0$, $\sigma_x \geq 0$, and W_x is a standard one-dimensional Brownian motion. Our choice of the OU process is motivated by its parsimony – very few parameters must be calibrated – and its appropriateness to fit cohort life tables because of

[1] This filtration reflects both mortality and financial information. For a discussion of its relationship to the natural filtration of the mortality intensity and interest rate processes as well as for the relevant change of measure, see Luciano et al. (2012).

its lack of mean reversion. It is an affine process for which we can find closed-form expressions for the survival probability. Moreover, it is a natural, stochastic extension of the Gompertz model for the force of mortality and is easy to interpret in light of traditional actuarial practice. Its major drawback is the positive probability that λ_x will become negative. However, in each application of our model, we verify that this probability is negligible and that the survival probability is decreasing over the duration of human life.[2] Together with the spot intensity, we consider the forward instantaneous intensity, denoted $f_x(t,T)$. This is the best forecast at time t of the spot intensity at T because it converges to it when the horizon of the forecast goes to zero, or $T \to t$: $f_x(t,t) = \lambda_x(t)$. Standard properties of affine processes allow us to represent the survival probability from time t to T as:

$$S_x(t,T) = \mathbb{E}\left[\exp\left(-\int_t^T \lambda_x(s)ds\right) \mid \mathcal{F}_t\right] = e^{\alpha_x(T-t)+\beta_x(T-t)\lambda_x(t)}, \tag{2}$$

$$\alpha_x(t) = \frac{\sigma_x^2}{2a_x^2}t - \frac{\sigma_x^2}{a_x^3}e^{a_x t} + \frac{\sigma_x^2}{4a_x^3}e^{2a_x t} + \frac{3\sigma_x^2}{4a_x^3}, \tag{3}$$

$$\beta_x(t) = \frac{1}{a_x}(1 - e^{a_x t}). \tag{4}$$

As due in a stochastic mortality environment, survival probabilities in the future $S_x(t,\cdot), t > 0$ are random variables at time 0. Following a hint in Jarrow and

[2]See Luciano and Vigna (2008). In that paper, the authors argue that the OU model, along with other non-mean reverting affine processes, meets all the criteria of a good mortality model developed by Cairns et al. (2006a), with the exception of strictly positive intensity. Indeed, it fits historical data well; its long-term future dynamics are biologically reasonable; it is convenient for pricing, valuation and hedging; and its long-term mortality improvements are not mean reverting. Most importantly for the case at hand, mortality-linked products can be priced using analytical methods.

Turnbull (1994), we can write the survival probability as:

$$\begin{aligned} S_x(t,T) &= \frac{S_x(0,T)}{S_x(0,t)} \exp\left[-X_x(t,T)I(t) - Y_x(t,T)\right], \\ X_x(t,T) &= \frac{\exp(a_x(T-t)) - 1}{a_x}, \\ Y_x(t,T) &= -\sigma_x^2[1 - e^{2a_x t}]X_x(t,T)^2/(4a_x), \\ I(t) &= \lambda_x(t) - f_x(0,t). \end{aligned}$$

The term $I(t)$, the difference between the instantaneous mortality intensity at t and its forecast at time 0, is the longevity risk factor, i.e., the error in the forecast that exposes insurance companies and pension funds to longevity risk. At time 0, this is the only random quantity in future survival probabilities $S_x(t,T)$. Because $X_x(t,T) > 0$ and $I(t) = 0$ provides the "base" survival probability curve $S_x(t,T) = \frac{S_x(0,T)}{S_x(0,t)} \exp\left[-Y_x(t,T)\right]$, a positive $I(t)$ implies a survival curve lower than the base curve and vice versa. Indeed, $I(t)$ is greater than 0 if the realized mortality intensity is higher than its forecast, and consequently, the survival probability for every duration $T > t$ is lower than its forecast at time 0. Our hedging technique exploits the crucial feature that $I(\cdot)$ depends only on t. The same factor affects all survivals at each horizon T in the future.

2.2 Model for interest rate risk

While in the longevity domain we model spot intensities, in the financial domain we adopt the standard Heath, Jarrow and Morton framework (Heath et al. (1992)) and directly model the instantaneous forward rate $F(t,T)$, which is the rate that applies at instant T, as agreed upon at $t < T$.

Additionally, we assume that no arbitrages exist, and we start modeling directly

under the risk neutral measure equivalent to $\mathbb{P}$, which we call $\mathbb{Q}$.[3] We assume that the process for the forward interest rate $F(t,T)$, defined on the probability space $(\Omega, \mathbf{F}, \mathbb{Q})$, is the well-known Hull and White (1990) model with constant parameters:

$$dF(t,T) = -gF(t,T)dt + \Sigma e^{-g(T-t)}dW_F(t), \tag{5}$$

where $g > 0, \Sigma > 0$ and W_F is a univariate Brownian motion. The limit of the forward rate when $T \to t$ is the short rate that applies instantaneously at t, $r(t)$: $F(t,t) = r(t)$. The price of a zero-coupon bond issued at time t expiring at time T is:

$$B(t,T) = \mathbb{E}\left[\exp\left(-\int_t^T r(s)ds\right) \mid \mathcal{F}_t\right].$$

As with survival probabilities, bond prices at time t are random variables at time 0. They may be written as follows (see Jarrow and Turnbull (1994)):

$$\begin{aligned}
B(t,T) &= \frac{B(0,T)}{B(0,t)} \exp\left[-\bar{X}(t,T)K(t) - \bar{Y}(t,T)\right], \\
\bar{X}(t,T) &= \frac{1-\exp(-g(T-t))}{g}, \\
\bar{Y}(t,T) &= \frac{\Sigma^2}{4g}\left[1-\exp(-2gt)\right]\bar{X}^2(t,T), \\
K(t) &= r(t) - F(0,t).
\end{aligned}$$

As in the longevity case, the financial risk factor $K(t)$, which is the difference between the time-t spot and forward rates, is the only source of randomness that affects bonds. It is the same across all bond maturities T.

[3] That measure is unique, with one Wiener and one risky bond with non-degenerate volatility.

2.3 ALM and Greeks

In the presence of both longevity and interest rate risks, the fairly priced future reserves of every insurance product become stochastic because survival probabilities and bond prices are stochastic. This generates the need for liability hedging and opens the way to ALM. If we compute the fair value of the reserves and assume that insurance companies hedge this value, we implement liability hedging. If we assume that insurance companies set up a hedged portfolio under a self-financing constraint (which means that the premiums received for death or life benefits are used to build the hedge) and may include bonds in the menu of available assets, we perform asset and liability management.

To compute the fair value of an insurance liability, a change of probability measure is still needed. We assume that there exists a measure $\mathbb{Q}$ that allows the mortality intensity to remain affine.[4] This quite standard choice is equivalent to fixing a risk premium[5] $\theta_x(t) = \frac{q\lambda_x(t)}{\sigma_x}$, $q \in \mathbb{R}, q > -a_x$. We assume independence between longevity and financial risks after the change of measure. Thereby, we can provide expressions for the fair value of insurance liabilities as products of survival or death probabilities and discount factors.

Consider first a pure endowment contract starting at time 0 and paying one unit of account if the individual aged x is alive at time T. The fair value of such an insurance policy at time $t \geq 0$ is $Z_{Ex}(t,T)$. Assuming a single premium paid at the policy issue, Z_{Ex} is also the time-t reserve for the policy and the value that must

[4]With a slight abuse of notation, we also use $\mathbb{Q}$ to denote this measure. A more detailed discussion of the change of measure is provided in Luciano et al.(2012).

[5]Notice that, given the absence of a rich market for longevity bonds, there are no standard choices to apply in the choice of $\theta_x(t)$. See, for instance, the extensive discussion in Cairns et al. (2006b).

be hedged by the life office. We have:

$$\begin{aligned} Z_{Ex}(t,T) &= S_x(t,T)B(t,T) = \\ &= E_{\mathbb{Q}}\left[\exp\left(-\int_t^T \lambda_x(s)ds\right)|\mathcal{F}_t\right] E_{\mathbb{Q}}\left[\exp\left(-\int_t^T r(u)du\right)|\mathcal{F}_t\right] = \\ &= \frac{S_x(0,T)}{S_x(0,t)}\exp\left[-X_x(t,T)I(t) - Y_x(t,T)\right]\frac{B(0,T)}{B(0,t)}\exp\left[-\bar{X}(t,T)K(t) - \bar{Y}(t,T)\right], \end{aligned} \tag{6}$$

where the parameter a_x in X_x, Y_x has become $a'_x = a_x + q > 0$ to account for the measure change. For the sake of simplicity, we suppress the subscript x until Section 4, which introduces multiple cohorts. Using Ito's lemma, for a given t, we obtain the dynamics of the reserve Z_E as a function of the changes in the risk factors:

$$dZ_E = B\left(\Delta^M \Delta I + \frac{1}{2}\Gamma^M \Delta I^2\right) + S\left(\Delta^F \Delta K + \frac{1}{2}\Gamma^F \Delta K^2\right),$$

where

$$\begin{aligned} \Delta^M(t,T) &= \frac{\partial S}{\partial I} = -S(t,T)X(t,T) < 0, \\ \Gamma^M(t,T) &= \frac{\partial^2 S}{\partial I^2} = S(t,T)X^2(t,T) > 0, \\ \Delta^F(t,T) &= \frac{\partial B}{\partial K} = -B(t,T)\bar{X}(t,T) < 0, \\ \Gamma^F(t,T) &= \frac{\partial^2 B}{\partial K^2} = B(t,T)\bar{X}^2(t,T) > 0. \end{aligned}$$

To simplify the notation, we define the Greeks:

$$\Delta_E^M(t,T) = B(t,T)\Delta^M(t,T) < 0, \tag{7}$$

$$\Gamma_E^M(t,T) = B(t,T)\Gamma^M(t,T) > 0, \tag{8}$$

$$\begin{aligned} \Delta_E^F(t,T) &= S(t,T)\Delta^F(t,T) < 0, \\ \Gamma_E^F(t,T) &= S(t,T)\Gamma^F(t,T) > 0. \end{aligned}$$

Using those definitions, the following is the change in the pure endowment:

$$dZ_E = \Delta_E^M \Delta I + \frac{1}{2}\Gamma_E^M \Delta I^2 + \Delta_E^F \Delta K + \frac{1}{2}\Gamma_E^F \Delta K^2.$$

Let us consider an annuity – with annual installments R – issued at time 0 to an individual belonging to a certain generation. Assuming the payment of a single premium at policy inception, we obtain the prospective reserve Z_A from $t > 0$ to T:[6]

$$Z_A(t,T) = R\sum_{u=1}^{T-t} B_{t,u}S_{t,u},$$

where we use the short notation $B_{t,u}$ for $B(t,t+u)$. We use the same shortcut for S and the Greeks below. The change in the reserve for given t is straightforward to compute:

$$dZ_A = R\left[\Delta_A^M \Delta I + \frac{1}{2}\Gamma_A^M \Delta I^2 + \Delta_A^F \Delta K + \frac{1}{2}\Gamma_A^F \Delta K^2\right],$$

where

$$\begin{aligned}
\Delta_A^M(t,T) &= -\sum_{u=1}^{T-t} B_{t,u}S_{t,u}X_{t,u} = \sum_{u=1}^{T-t} \Delta_E^M(t,t+u) < 0,\\
\Gamma_A^M(t,T) &= \sum_{u=1}^{T-t} B_{t,u}S_{t,u}[X_{t,u}]^2 = \sum_{u=1}^{T-t} \Gamma_E^M(t,t+u) > 0,\\
\Delta_A^F(t,T) &= -\sum_{u=1}^{T-t} B_{t,u}S_{t,u}\bar{X}_{t,u} = \sum_{u=1}^{T-t} \Delta_E^F(t,t+u) < 0,\\
\Gamma_A^F(t,T) &= \sum_{u=1}^{T-t} B_{t,u}S_{t,u}[\bar{X}_{t,u}]^2 = \sum_{u=1}^{T-t} \Gamma_E^F(t,t+u) > 0.
\end{aligned}$$

As expected, deltas are negative and gammas are positive for both risks.

[6]The horizon T depends on the type of annuity issued, i.e., it is T for an annuity payable for T years and $\omega - x$ for a whole life annuity.

Let us consider now a life insurance issued at time 0. It has maturity T and sum assured C, paid at the end of the year of death. If a single premium is paid at policy inception, the following is the prospective reserve Z_D from $t > 0$ to horizon T:

$$Z_D(t,T) = C \sum_{u=1}^{T-t} B_{t,u}(S_{t,u-1} - S_{t,u}).$$

The change in the reserve Z_D at time t is:

$$dZ_D = C \left[\Delta_D^M \Delta I + \frac{1}{2}\Gamma_D^M \Delta I^2 + \Delta_D^F \Delta K + \frac{1}{2}\Gamma_D^F \Delta K^2 \right],$$

where

$$\Delta_D^M(t,T) = \sum_{u=1}^{T-t} B_{t,u}(\Delta_{t,u-1}^M - \Delta_{t,u}^M) > 0, \tag{9}$$

$$\Gamma_D^M(t,T) = \sum_{u=1}^{T-t} B_{t,u}(\Gamma_{t,u-1}^M - \Gamma_{t,u}^M) < 0, \tag{10}$$

$$\Delta_D^F(t,T) = \sum_{u=1}^{T-t} (S_{t,u-1} - S_{t,u})\, \Delta_{t,u}^F < 0,$$

$$\Gamma_D^F(t,T) = \sum_{u=1}^{T-t} (S_{t,u-1} - S_{t,u})\, \Gamma_{t,u}^F > 0.$$

As intuition would suggest, the Greeks for the longevity risks of life insurance and annuities have the opposite signs. If the actual mortality intensity is higher than forecasted ($\Delta I > 0$), life insurance contracts increase in value while endowments decrease, up to the first order. Second order derivatives also have opposite values because they mitigate the approximation given by the first derivative. The Greeks of life insurance and annuities with respect to financial risk have the same sign because they are all present values.

The Greeks with respect to the risk factors provide the exposure to mortality and

interest rate forecast errors in closed form. However, they do more: they create the possibility of hedging. If the sole aim is mortality risk hedging, one can create a hedged portfolio taking only short positions (i.e., issued policies). This is the case analyzed by Wang et al. (2010). If instead the aim is mortality and financial risk hedging using solely insurance products, a mix of short and long positions (i.e., issued policies and reinsurance purchased) is needed to neutralize the risk exposure. If the insurer cannot or does not want to buy reinsurance, natural hedging for mortality risk can be used; after creating a portfolio of issued policies, long and short positions can be taken in bonds to hedge financial risk. In this case, one should first hedge the mortality risk and then the interest rate risk because the insurance products used to hedge mortality risk produce additional interest rate risk.

3 Natural hedging within a single generation

To implement natural hedging, it is possible to compute the number of offsetting contracts in closed form. Consider an insurer who has issued m annuities with rate R and maturity T_1 as well as n life insurance policies with face value C and maturity T_2 on the same generation.[7] The change in the value of its liabilities $Z_H(t)$ for $t < \min\{T_1, T_2\}$, can be written as:

$$\begin{aligned} dZ_H(t) &= mdZ_A(t,T_1) + ndZ_D(t,T_2) = \\ &= (Rm\Delta_A^M + Cn\Delta_D^M)\Delta I(t) + \frac{1}{2}(Rm\Gamma_A^M + Cn\Gamma_D^M)\Delta I^2(t) + \\ &+ (Rm\Delta_A^F + Cn\Delta_D^F)\Delta K(t) + \frac{1}{2}(Rm\Gamma_A^F + Cn\Gamma_D^F)\Delta K^2(t), \end{aligned} \tag{11}$$

[7] Our results also apply when more life insurance contracts with different maturities T_i are considered, as the calibrated example in Section 5 shows.

where the coefficients of the changes in the risk factors are simply the weighted sums of the sensitivities of each liability. This follows from the fact that the risk factors depend only on t. What differs by product is the first and second order exposure – i.e., the Greeks.

Natural hedging can be achieved because it is possible to select m and n such that the coefficients of ΔI and ΔI^2 are both equal to 0:

$$\begin{cases} Rm\Delta_A^M + Cn\Delta_D^M = 0, \\ Rm\Gamma_A^M + Cn\Gamma_D^M = 0. \end{cases}$$

This is a linear system of two equations with two unknowns m and n. Negative solutions for m or n indicate that the insurer must take a short position or sell the contract. Positive solutions indicate long positions, i.e., reinsurance needs. Thus, longevity risk is hedged according to standard risk management techniques, up to first and second order approximations. As usual, because the risk factors are modeled in continuous time, a perfect hedge would require continuous adjustments of the positions. We show in Section 6 that delta-gamma strategies are quite robust to discrete implementation under realistic calibrations. The hedge described thus far entails only liabilities. We could extend it to assets by including bonds, which we do so in the application. We also consider the case of a self-financing hedge.

4 Natural hedging across generations

Suppose that we must hedge the liability of a pure endowment written on generation x with that written on another generation y. All generations are subject to the same financial risk factor because the term structure of interest rate is the same for all generations. We assume that each generation has its own intensity and,

therefore, its own longevity risk factor, but the intensities of two generations are instantaneously correlated. This captures the idea that the drivers of longevity risk, while not exactly the same across generations, are all affected by progress in medicine and improved welfare on the one hand and by possible pandemics on the other. Using the notation from Section 2, we have:

$$d\lambda_x = a'_x \lambda_x dt + \sigma_x dW_x(t), \tag{12}$$

$$d\lambda_y = a'_y \lambda_y dt + \sigma_y dW_y(t), \tag{13}$$

where the two Brownians have correlation coefficient ρ. Appendix A shows that, by re-parametrizing the two sources of risk through independent Wieners, one can isolate the mortality risk that affects generations x and y (the common risk) from the risk that affects only y (the idiosyncratic risk). We identify the common risk factor as $I(t)$ and denote the idiosyncratic risk factor describing the y-specific variations as $I'(t)$. Obviously, $I'(t)$ is instantaneously uncorrelated with $I(t)$, and the survival probability of generation y is affected by both I and I'. With this parametrization of the two factors, the Greeks for pure endowments, annuities and life insurance policies written on generation x, which we derived in the previous sections, still apply. We now provide the analogous Greeks for generation y. Appendix A shows that:

$$\begin{aligned} dS_y(t,T) &= \frac{\partial S_y(t,T)}{\partial t} dt + \Delta_y^{M,x}(t,T) dI + \Delta_y^{M,y}(t,T) dI' + \\ &+ \frac{1}{2}\Gamma_y^{M,x}(t,T) dI^2 + \frac{1}{2}\Gamma_y^{M,y}(t,T) dI'^2, \end{aligned} \tag{14}$$

where

$$\begin{aligned}
\Delta_y^{M,x}(t,T) &= \frac{\partial S_y(t,T)}{\partial \lambda_y}\rho\frac{\sigma_y}{\sigma_x} = -\rho\frac{\sigma_y}{\sigma_x}X_y(t,T)S_y(t,T) = \rho\frac{\sigma_y}{\sigma_x}\Delta_y^M(t,T), && (15)\\
\Delta_y^{M,y}(t,T) &= \frac{\partial S_y(t,T)}{\partial \lambda_y} = -X_y(t,T)S_y(t,T) = \Delta_y^M(t,T), && (16)\\
\Gamma_y^{M,x}(t,T) &= \left(\rho\frac{\sigma_y}{\sigma_x}\right)^2\frac{\partial^2 S_y(t,T)}{\partial^2 \lambda_y} = \left(\rho\frac{\sigma_y}{\sigma_x}\right)^2 X_y^2(t,T)S_y(t,T) = \left(\rho\frac{\sigma_y}{\sigma_x}\right)^2\Gamma_y^M(t,T), && (17)\\
\Gamma_y^{M,y}(t,T) &= \frac{\partial^2 S_y(t,T)}{\partial^2 \lambda_y} = X_y^2(t,T)S_y(t,T) = \Gamma_y^M(t,T). && (18)
\end{aligned}$$

Obviously, the Greeks of generation y with respect to the common risk factor x, $\Delta_y^{M,x}$ and $\Gamma_y^{M,x}$, depend on ρ. The sign of the first derivative with respect to the common risk $\Delta_y^{M,x}$ is opposite as the sign of the correlation coefficient ρ, while the derivative with respect to idiosyncratic risk $\Delta_y^{M,y}$ is negative, as expected. Both gamma coefficients are non-negative, as usual. The gamma with respect to the common risk $\Gamma_y^{M,x}$ is positive (and equals zero whenever ever ρ equals zero), while the gamma with respect to idiosyncratic risk $\Gamma_y^{M,y}$ is strictly positive.
Using notation similar to (7) and (8), the change in the reserve for a pure endowment on y is:

$$dZ_{Ey} = \Delta_{Ey}^{M,x}\Delta I + \frac{1}{2}\Gamma_{Ey}^{M,x}\Delta I^2 + \Delta_{Ey}^{M,y}\Delta I' + \frac{1}{2}\Gamma_{Ey}^{M,y}\Delta I'^2 + \Delta_{Ey}^{F}\Delta K + \frac{1}{2}\Gamma_{Ey}^{F}\Delta K^2.$$

The change in the reserve of an annuity on y is:

$$\begin{aligned}
dZ_{Ay} &= R\left[\Delta_{Ay}^{M,x}\Delta I + \frac{1}{2}\Gamma_{Ay}^{M,x}\Delta I^2 + \Delta_{Ay}^{M,y}\Delta I' + \frac{1}{2}\Gamma_{Ay}^{M,y}\Delta I'^2 + \Delta_{Ay}^{F}\Delta K + \frac{1}{2}\Gamma_{Ay}^{F}\Delta K^2\right],\\
\Delta_{Ay}^{M,j}(t,T) &= \sum_{u=1}^{T-t}\Delta_{Ey}^{M,j}(t,t+u),\\
\Gamma_{Ay}^{M,j}(t,T) &= \sum_{u=1}^{T-t}\Gamma_{Ey}^{M,j}(t,t+u) \geq 0,
\end{aligned}$$

with $j = x, y$. While the sign of $\Delta_{Ay}^{M,y}$ is negative as usual, the sign of $\Delta_{Ay}^{M,x}$ is opposite the sign of ρ. The gammas are strictly positive, with the exception of $\Gamma_{Ay}^{M,x}(t,T)$, which is null when $\rho = 0$.

The change in the reserve for life insurance is:

$$\begin{aligned} dZ_{Dy} &= C\left[\Delta_{Dy}^{M,x}\Delta I + \frac{1}{2}\Gamma_{Dy}^{M,x}\Delta I^2 + \Delta_{Dy}^{M,y}\Delta I' + \frac{1}{2}\Gamma_{Dy}^{M,y}\Delta I'^2 + \Delta_{Dy}^{F}\Delta K + \frac{1}{2}\Gamma_{Dy}^{F}\Delta K^2\right], \\ \Delta_{Dy}^{M,j}(t,T) &= \sum_{u=1}^{T-t} B_{t,u}(\Delta_{y}^{M,j}(t,t+u-1) - \Delta_{y}^{M,j}(t,t+u)), \\ \Gamma_{Dy}^{M,j}(t,T) &= \sum_{u=1}^{T-t} B_{t,u}(\Gamma_{y}^{M,j}(t,t+u-1) - \Gamma_{y}^{M,j}(t,t+u)), \end{aligned}$$

with $j = x, y$. For positive ρ, the comments on the sign of $\Delta_{Dy}^{M,x}(t,T)$ are the same as in (9). The opposite comments apply for negative correlations. Provided that $\rho \neq 0$, the comments on the sign of $\Gamma_{Dy}^{M,x}(t,T)$ are the same as in (10). The same comments as in (9) and (10) hold for the delta and gamma of generation y with respect to its factors $\Delta_{Dy}^{M,y}(t,T)$ and $\Gamma_{Dy}^{M,y}(t,T)$.

4.1 Hedging

If the insurer wants to hedge mortality risk on generation x but does not have enough insurance products on that generation, products on generation y can be used to offset exposure to the common risk factor. However, coverage of the common risk factor I implies exposing the portfolio to the idiosyncratic risk I' of generation y unless generations are perfectly correlated. This risk can be evaluated and can be either traded away by the insurer in a market or reinsured. Imagine an insurer who has issued n_H products on generation x with fair value Z_H. He can delta-gamma hedge his liability by assuming positions in n_i units of some other N instruments with fair value Z_i, $i = 1, ..., N$. The products available for hedging the longevity risk factor of each cohort can be written either on the same cohort as Z_H or on different ones. To simplify the notation, the index i of each product denotes both the type of product (i.e., E, A, D) and its maturity. As in the single-generation case, we interpret negative positions $n_i < 0$ as short positions on the corresponding

product and positive solutions $n_i > 0$ as reinsurance purchases.[8] Hedging portfolios are obtained by equating to zero $\Delta_{\Pi}^{M,j}$, Δ_{Π}^{F}, $\Gamma_{\Pi}^{M,j}$, and Γ_{Π}^{F}, where the subscript Π refers to the portfolio itself, the superscript M, j refers to the j-th longevity risk factor, $j = 1, ..., J$ is the cohort on which the product is written,[9] and F refers to the financial risk factor, which is unique across cohorts.

The quantities $n_i, i = 1, ..., N$ therefore solve the following system (for a given n_H):

$$\begin{cases} n_H \Delta_H^{M,j}(t, T_H) + \sum_{i=1}^{N} n_i \Delta_i^{M,j}(t, T_i) = 0, \ j = 1, ..., J. & (19) \\ n_H \Delta_H^{F}(t, T_H) + \sum_{i=1}^{N} n_i \Delta_i^{F}(t, T_i) = 0, & (20) \\ n_H \Gamma_H^{M,j}(t, T_H) + \sum_{i=1}^{N} n_i \Gamma_i^{M,j}(t, T_i) = 0, \ j = 1, ..., J. & (21) \\ n_H \Gamma_H^{F}(t, T_H) + \sum_{i=1}^{N} n_i \Gamma_i^{F}(t, T_i) = 0. & (22) \end{cases}$$

The expressions for the delta and gamma coefficients take the forms we derived in the previous sections, depending on the i-th product type (pure endowment, annuity, life insurance). When we simultaneously solve:

- the J equations (19), we delta hedge longevity risk;
- the $J + 1$ equations (19) and (20), we delta hedge both risks;
- the $2J$ equations (19) and (21), we delta-gamma hedge longevity risk;
- all $2+2J$ equations (19),(20),(21) and (22), we delta-gamma hedge both risks.

[8] Alternatively, we can interpret positive positions as need for mortality-linked contracts, such as survivor bonds and other derivatives.

[9] Notice that the number of longevity risk factors to hedge against is the same as the number of generations in the portfolio.

In all cases we perform liability hedging. ALM occurs when we either require the portfolio to be self-financing or add a bond. In the former case, a further equation must be added to the system. Because we have assumed that a single fair premium is paid at policy issuance, self-financing strategies are characterized by the self-financing constraint:

$$n_H Z_H + \sum_{i=1}^{N} n_i Z_i = 0. \tag{23}$$

The self-financing constraint equates the inflows from sales and the outflows for asset purchases; it means that the premiums are used to buy hedging instruments. To include a zero-coupon bond,[10] it is sufficient to expand our notation to include a contract Z_{N+1} whose value is the discount factor (or to imagine a fake generation that has a constant survival probability equal to one). Because bonds are unaffected by mortality risk, their Greeks with respect to mortality are null, and bonds can be used as hedging instruments for financial risk.

We obtain a unique solution to the system of equations that solves the delta or delta-gamma hedging problem if the matrix of the n_i coefficients is full rank and the number of hedging instruments equals this rank. This imposes a restriction on how many life insurance liabilities (and bonds, when they are admitted) are used for coverage:

- $N = J$ instruments for delta hedging longevity risk;
- $N = J + 1$ instruments for delta hedging both risks;
- $N = 2J$ instruments for delta-gamma hedging longevity risk;
- $N = 2 + 2J$ instruments for delta-gamma hedging both risks.

[10]Because any bond can be stripped into zero-coupon bonds, with some additional notation, we can also include coupon bonds.

If one accepts multiple solutions, this restriction can be relaxed. A further instrument is required if the portfolio must be self-financing.

As anticipated in Section 2, one can also implement a two-step procedure that requires solving two systems of equations sequentially. First, neutralize the Greeks with respect to mortality using insurance products. Then, given the insurance portfolio, use bonds to cover the resulting financial risk. In the special case in which the term structure of interest rates is flat and interest rate risk is absent, one can obviously omit the second step.

4.2 Practical issues

The delta-gamma hedging procedure proposed in the previous sections appears relatively simple, but its implementation must cope with some important practical issues. It might be difficult for a life office to sell the exact number of policies required for hedging purposes and/or to buy the exact amount of reinsurance. If the desired number of policies does not match the availability of potential customers in the market, the system of equations (19)–(23) should be accompanied by some additional constraints. The additional restrictions may imply that the system has no longer a solution. In this case, it is necessary to include additional hedging instruments to obtain at least one solution to the system.

If reinsurance is not available, the delta-gamma hedging procedure should be performed in two steps, as described in Section 2. The two-step procedure provides more freedom to choose insurance products and may avoid reinsurance. The following simple example illustrates this.

Suppose that a life office sells one annuity written on generation x ($n_H = -1$) and wants to hedge it with the appropriate number $(n_1, n_2)$of life insurance policies with different maturities issued on the same generation. The numbers that achieve

delta-gamma hedging solve the following system:

$$\begin{cases} -\Delta_H + n_1\Delta_1 + n_2\Delta_2 = 0, \\ -\Gamma_H + n_1\Gamma_1 + n_2\Gamma_2 = 0. \end{cases} \tag{24}$$

Suppose that we want to avoid reinsurance, i.e., we want both n_1 and n_2 to be negative. Assuming that $\frac{\Delta_1}{\Gamma_1} \neq \frac{\Delta_2}{\Gamma_2}$, it can be proved that the goal is achieved if and only if the following inequality holds:

$$\min\left\{\frac{\Delta_1}{\Gamma_1}, \frac{\Delta_2}{\Gamma_2}\right\} < \frac{\Delta_H}{\Gamma_H} < \max\left\{\frac{\Delta_1}{\Gamma_1}, \frac{\Delta_2}{\Gamma_2}\right\}. \tag{25}$$

This means that the ratio between delta and gamma of the annuity lies between the corresponding ratios of the life insurance policies. In Section 5.3, we provide an example in which condition (25) is satisfied, and the quantities n_1 and n_2 are negative.

Finally, reinsurance is costly. We should separate two cases: non-self-financing and self-financing strategies. In the first case, reinsurance premiums do not enter the system; hence, they do not affect the solution of the system directly. The hedging strategy does not change, even though its total cost is higher. In the second case, reinsurance premiums can be introduced into the model by adding the corresponding loadings to the fair value Z_i of the hedging instruments in equation (23). In this case, the solution to the system is affected, but the method remains valid.

5 UK-calibrated application

In Sections 5.1 and 5.2, we calibrate the model to UK data and present the corresponding hedge ratios. In Section 5.3, we compute the portfolio mix of annuities

and life insurance contracts within each generation that immunizes the portfolio up to the first and second order. We also assess the effectiveness of the hedging strategies using Monte Carlo simulation. In Section 5.4, we consider cross-generation immunization.

5.1 Calibration

In calibrating the model to UK data, we assume that the risk premium on longevity risk is null: $q = 0$, so that $a' = a$.[11] We calibrate the parameters of the mortality intensity processes using UK cohort tables taken from the Human Mortality Database. We consider contracts written on the lives of male individuals who were 35 (generation 1973, which we call x), 65 (generation 1943, which we call y) and 75 (generation 1933, which we call z) years old on 31/12/2008. We calibrate the OU model to the data using a standard least squares method. We denote the set of n observed survival probabilities for each generation $j = x, y, z$ and different horizons $\tau = 1, 2, ..., n$ as ${}_\tau\tilde{p}_j$. We jointly estimate the parameters and instantaneous correlations of the three generations. The calibrated values of a_j, σ_j and vector $\rho = [\rho_{xy}, \rho_{xz}]$ minimize the error:

$$\sum_{j=x,y,z} \sum_{\tau=1}^{n} ({}_\tau\tilde{p}_j - S_j(\tau; a_j, \sigma_j, \rho))^2,$$

where $S_j(\tau; a_j, \sigma_j, \rho)$ are the theoretical survival probabilities for τ years. We derive them following the framework described in Duffie et al. (2000), as explained in Appendix B. The value of $\lambda_j(0)$ is equal to $-\ln p_j$. We select twenty years of observations ($n = 20$), using 31/12/1988 as the observation starting point. It follows

[11] This assumption will be easily removed by calibrating the model parameters to actual mortality derivative prices as soon as a liquid market for them exists. Alternatively, we could parametrize the results to a hypothetical, positive risk premium.

Table 1: Calibrated parameters

Mortality intensity				
	a_{GEN}	σ_{GEN}	$\lambda_{GEN}(0)$	$\rho_{x,GEN}$
$GEN:x$	0.0809	0.0000325	0.000396	1
$GEN:y$	0.0801	0.0001987	0.002919	0.9919
$GEN:z$	0.0750	0.0005970	0.0087	0.9971
Spot interest rate				
g	Σ	θ	$r(0)$	Calibration error
0.0244	0.0217	0.2432	0.0153	0.000346101

that the initial ages for generations x, y, and z are 15, 45, and 55, respectively. Table 1 reports the result of our calibration. The overall calibration error is $7.72 * 10^{-5}$.

The calibrated parameters satisfy the sufficient condition for biological reasonableness for the OU model, as in Luciano and Vigna (2008), and the shape of the predicted survival curve is reasonable (decreasing up to the terminal age $\omega = 120$). However, because the model is used to predict survival probabilities that are in the distant future, survivorship at older ages is forecasted with some degree of uncertainty.[12] Although the OU model does not guarantee positive mortality intensity, we find that the probabilities of negative λ_x, λ_y, λ_z are negligible given the parameters in Table 1. At the one-year horizon considered in our simulations, the probability of negative intensity is 10^{-37} for generation x, 10^{-53} for generation y and 10^{-52} for generation z.

In the application that follows, we consider longevity risk alone as well as longevity and interest rate risks together. To prepare to introduce interest rate risk, we calibrate the constant parameter Hull-White model to UK government bonds on

[12]It is well known that no parsimonious mortality model describes mortality over the whole life span. Hence, survival probabilities at older ages may not be forecasted properly. The use of a two-factor model is likely to improve on this point. Alternatively, the calibration of the parameters a_x and σ_x could be performed using projected mortality tables rather than observed figures, as in Luciano and Vigna (2008). This would imply calibrating the parameters of the OU mortality model using data forecasted by other mortality models (those underlying the projected tables).

Table 2: Greeks for annuities and life insurance policies

	Whole life ann.	10-yr LI	12-yr LI	15-yr LI	20-yr LI	25-yr LI
Gen. 1973 (x)						
Price		2.52	3.14	4.13	5.98	8.08
Δ_i^M		6282.31	7793.43	10188.18	14492.64	19154.94
Γ_i^M		-457732.60	-734018.14	-1344487.87	-3114396.87	-6385279.86
Δ_i^F		-13.49	-19.70	-31.50	-58.36	-95.47
Γ_i^F		87.79	150.17	290.55	683.80	1339.05
Gen. 1943 (y)						
Price	12.68	16.60	20.15	25.37	33.43	40.16
Δ_i^M	-1411.54	5153.97	6070.33	7250.65	8532.00	8799.38
Γ_i^M	479726.3	-365650.98	-553680.14	-915343.48	-1689235.25	- 2505336.59
Δ_i^F	-105.87	-86.49	-122.03	-183.87	-300.95	-418.91
Γ_i^F	1270.02	553.62	910.38	1644.84	3351.31	5423.88
Gen. 1933 (z)						
Price	8.87	36.88	42.81	50.27	58.77	63.07
Δ_i^M	-494.96	3276.24	3525.16	3637.64	3311.63	2742.82
Γ_i^M	65834.52	-201349.93	-274028.32	-377954.11	-486175.47	-473502.73
Δ_i^F	-57.37	-181.47	-240.85	-328.95	-451.38	-526.27
Γ_i^F	542.04	1121.42	1716.88	2760.36	4531.13	5837.12

2/1/2009. We use data on the nominal government spot yield curve provided by the Bank of England. We obtain the zero-coupon prices on that date and calibrate the parameters of the interest rate dynamics using the least squares method, i.e., minimizing the squared differences between observed and fitted bond prices. The results are presented in Table 1.

5.2 Greeks

We first analyze each generation separately. Consider whole life annuities with a unit benefit on generations y and z and life insurance policies with different maturities on all three generations. The life insurance contracts (LIs) have sum insured of $C = 100$ and maturities of 10, 12, 15, 20, and 25 years. Table 2 summarizes the reserves and Greeks of the policies.

The table shows that the prices of LIs written on the youngest generation x are approximately 5 and 10 times smaller than those on y and z, respectively, because

the death probabilities of generation x are much smaller than the others. The annuity price of generation z is lower than y because that generation has a shorter life expectancy at policy inception.

It is evident from the table that, within each generation, the deltas and gammas with respect to longevity are greater (in absolute value terms) than the Greeks for financial risk. This happens for every product and maturity and is consistent with the result in Luciano et al. (2012) for pure endowments. Despite the relative magnitude of the Greeks, a one standard deviation change in the financial risk factor has a greater impact than a comparable change in the longevity risk factor, especially over the short term because the standard deviation of the financial risk factor is higher than that of the mortality risk factor.[13]

If we consider a single insurance product and compare the Greeks across generations we notice the following:

– the sensitivity to mortality risk decreases in absolute value when moving from younger to older generations;

– the sensitivity to financial risk increases in absolute value when moving from younger to older generations for LIs and decreases for annuities.

The Greeks with respect to the longevity risks of annuities and LIs have opposite signs, while the sensitivities with respect to financial risk match in sign. Thus, there exists no portfolio mix of annuities and life insurance contracts without long positions that is able to neutralize the exposure against both longevity and interest

[13]For instance, consider a one-year change in the two risk factors equal to one standard deviation: $\Delta I = 0.0002$ and $\Delta K = 0.0217$. The effect on the value of an annuity written on generation y is

$$\Delta^M \Delta I + \frac{1}{2}\Delta I^2 + \Delta^F \Delta K + \frac{1}{2}\Delta K^2 = -2.57.$$

The first two terms, which represent the change in annuity value due to mortality risk, sum to -0.28, while the last two terms, which represent the value change due to interest rate risk, sum to -2.29. Overall, the effect of mortality risk, though non-negligible, is smaller than that due to interest rate risk despite the magnitude of the longevity Greeks.

rate risks –either reinsurance or bonds are needed.

5.3 Intra-generational hedging

Consider an insurer who has issued a whole life annuity with a unit benefit on an individual belonging to generation y, that is, $n_H = -1$. Using insurance contracts and bonds, he aims to achieve instantaneous neutrality to longevity and interest rate shocks. We compute the hedging coefficients, i.e., the positions the insurer must hold, when he uses LIs on the same cohort y as the annuitant and different hedging objectives (listed in the first column of Table 3). We assume the existence of enough instruments to provide a unique solution to the hedging system. The objectives are specified in the first column of Table 3 and include coverage of the first or second order of different risks, with or without self-financing. Empty cells refer to redundant hedging instruments, which are not used in the hedging strategy. The rest of Table 3 reports the quantities n_i needed for hedging. The last column contains the initial value of the hedged portfolio. This value is the sum of the premiums collected from the sale of policies net of reinsurance costs, where the value of self-financing strategies is zero.

Given the set of LIs with maturities ranging from 10 to 25 years, the first row of the table shows the only case in which a portfolio of policies issued by the insurer is naturally hedged (without resorting to reinsurance or bonds), which is delta-hedging of longevity risk only, as in Wang et al. (2010). Delta hedging of the mortality risk of the annuity is accomplished by issuing 0.27 10-yr LIs. When we create the other hedging strategies, i.e., we want to neutralize exposure to both sources of risk or establish self-financing strategies, we must combine short positions with at least one long position on a contract. This is needed to make the portfolio self-financing or

Table 3: Hedging strategies for an annuity on generation y

Strategy/Instrument	10-yr LI	12-yr LI	15-yr LI	20-yr LI	25-yr LI	Value
D M	-0.27					17.22
D MF	-7.84	6.43				13.42
DG M	3.36	-3.09				23.25
DG MF	-27.04		59.54	-52.09	17.12	5.00
D M SF	-34.49	29.05				0
D MF SF	26.25		-36.98	15.40		0
DG M SF	603.32	-807.15	246.70			0
DG MF SF	-197.89	322.69	-132.32	-12.02	13.82	0

In the table, D stands for delta hedging, DG for delta-gamma hedging, M for longevity only, MF for both longevity and financial risk neutralization, and SF for self-financing. Empty cells refer to instruments that are not used in the strategy; their holding is set to zero. The Value column refers to the fair value of the liability portfolio, which equals the premiums collected when setting up the position.

to neutralize exposure with respect to financial risk, which increases when adding the LI policies issued (together with the annuity) to the portfolio. For instance, line 2 of Table 3 reports the delta hedging portfolio for longevity and financial risks. Reinsurance on 6.43 12-yr LIs is needed to reach the objective. If we consider a strategy involving 10- and 30-yr LIs, it is possible to design a delta-gamma hedging strategy for longevity risk without resorting to reinsurance, as described in Section 4.2. Indeed, in this case, condition (25) is satisfied:

$$\frac{\Delta_1}{\Gamma_1} = -0.0141 < \frac{\Delta_H}{\Gamma_H} = -0.00294 < \frac{\Delta_2}{\Gamma_2} = -0.00268.$$

The delta-gamma non-self-financing hedging strategy (not reported in Table 3) consists in issuing 0.03 10-yr LIs and 0.15 30-yr LIs.

As explained in Sections 2.3 and 4.1, it is also possible to substitute some hedging instruments with bonds. The use of bonds in this strategy has the advantage of not requiring reinsurance on LIs. For instance, delta hedging of both longevity and financial risks can be achieved by using one LI and one bond from the interest rate market rather than two LIs. Such a hedging portfolio (not reported in Table 3)

is made by issuing 0.27 10-yr LIs (as in longevity only delta hedging) and buying 20.60 10-yr zero-coupon bonds. The value of this portfolio, 2.63, is smaller than that of natural hedging without bonds, 13.42 (line 2 of Table 3).

5.3.1 Hedge effectiveness

This section shows the effectiveness of some of the hedging portfolios reported in Table 3 in reducing longevity and financial risks. We compare non-hedged, delta hedged, and delta-gamma hedged portfolios. For illustrative purposes, we consider the following three portfolios:

1. Portfolio 1 (NH): Non-hedged, self-financing portfolio containing 1 ($m = -1$) issued annuity (not reported in Table 3).

2. Portfolio 2 (DH): Self-financing longevity and financial delta-hedged portfolio containing the annuity and the LIs with the maturities and positions as in line 6 of Table 3.

3. Portfolio 3 (DGH): Self-financing longevity and financial delta-gamma-hedged portfolio containing the annuity and the LIs as in line 8 of Table 3.

After the hedge has been established, the approximate value of the change in the overall reserve (portfolio value) is given by:

$$\begin{aligned} dZ_H &= \left(-\Delta_A^M + \sum_{i=1}^{N} n_i \Delta_i^M(t,T_i)\right) \Delta I(t) + \frac{1}{2}\left(-\Gamma_A^M + \sum_{i=1}^{N} n_i \Gamma_i^M(t,T_i)\right) \Delta I(t)^2 + \\ &+ \left(-\Delta_A^F + \sum_{i=1}^{N} n_i \Delta_i^F(t,T_i)\right) \Delta K(t) + \frac{1}{2}\left(-\Gamma_A^F + \sum_{i=1}^{N} n_i \Gamma_i^F(t,T_i)\right) \Delta K(t)^2, \end{aligned}$$

where N is the number of hedging instruments involved in the strategy, which is 0 in the NH, 3 in the DH, 5 in the DGH portfolios.

In the DH portfolio, only the coefficients multiplying $\Delta I(t)$ and $\Delta K(t)$ are nullified, while in the DGH portfolio, the coefficients multiplying $\Delta I(t)^2$ and $\Delta K(t)^2$

Table 4: Mean and Standard deviation of the hedging error after 3 months and 1 year

	Mean (Std Deviation)	
Strategy	3-month	1-year
Portfolio 1 (NH)	0.90 (0.69)	1.88 (1.33)
Portfolio 2 (DH)	0.29 (0.32)	0.91 (1.28)
Portfolio 3 (DGH)	0.25 (0.20)	0.83 (0.64)

are set tozero as well. Thus, the approximate instantaneous change of the unhedged portfolio is:

$$dZ_H^{NH} = -\Delta_A^M \Delta I(t) - \frac{1}{2}\Gamma_A^M \Delta I(t)^2 - \Delta_A^F \Delta K(t) - \frac{1}{2}\Gamma_A^F \Delta K(t)^2,$$

and the change in the reserve of the DH portfolio is:

$$dZ_H^{DH} = \frac{1}{2}\left(-\Gamma_A^M + \sum_{i=1}^{3} n_i \Gamma_i^M(t,T_i)\right)\Delta I(t)^2 + \frac{1}{2}\left(-\Gamma_A^F + \sum_{i=1}^{N} n_i \Gamma_i^F(t,T_i)\right)\Delta K(t)^2.$$

The approximate change of the DGH portfolio is null: $dZ_H^{DGH} = 0$. Hence, we would expect the change in the reserve of a non-hedged portfolio to be larger than that of a delta-hedged portfolio, that we obviously expect to be larger than that of a delta-gamma hedged portfolio, the last being null. To check this, we have simulated 100000 realizations of the stochastic interest rate $r(t+dt)$ and the stochastic mortality intensity $\lambda(t+dt)$ using the calibrated parameters. In each of the 100000 scenarios, we have computed the change in value of the reserve for the three portfolios over the time horizons of 3 months and 1 year. In Table 4, we collect the means and standard deviations of the absolute change in their portfolio reserves (hedging error).

As expected, over both horizons, the means and standard deviations are lowest when the portfolio is delta-gamma hedged. The reduction in the variability of the

Table 5: Stress test on the means and standard deviations of hedging error after 1 year

Strategy	a		σ		$\lambda(0)$	
	+10%	-10%	+10%	-10%	+10%	-10%
	Mean (Std Deviation)					
Portfolio 1 (NH)	1.88 (1.33)	1.89 (1.33)	1.89 (1.33)	1.88 (1.33)	1.96 (1.30)	1.86 (1.41)
Portfolio 3 (DGH)	0.88 (0.67)	0.79 (0.62)	0.86 (0.69)	0.80 (0.60)	1.66 (0.96)	0.71 (0.72)

position after a 3-month period is substantial (0.69 in the NH portfolio, 0.20 in the DGH portfolio). The one-year standard deviation of the DGH strategy is 0.64 vs. 1.33 for the unhedged portfolio.

Because we use a continuous-time theoretical model, the results are likely to improve using a rebalancing interval shorter than 3 months. However, when rebalancing the strategy at intervals shorter than one year, seasonality effects, which are not captured by our mortality model, may intervene. The table shows that even if our strategy performs best when rebalanced at shorter time intervals, it still leads to relevant hedging gains when rebalanced at longer time spans.

The variability reduction obtained through the DGH portfolio, together with the lower average hedging error, is robust to calibration error in the mortality model. Table 5 shows the standard deviation of the strategy when the true value of each parameter is 10% above or below the value used in computing the strategy. The strategy is most sensitive to an error in the choice of $\lambda(0)$, when the true initial instantaneous mortality intensity is higher than assumed when the hedge is calculated. Still, even if the error is as high as 10%, the DGH strategy outperforms the unhedged portfolio.

Table 6: Hedging strategies for an annuity on generation y using products written on generation x

Strategy/Instrument	10-yr LI_x	12-yr LI_x	15-yr LI_x	20-yr LI_x	Value
D M				-0.45	15.30
D MF	-11.94			4.57	15.44
DG M	18.33			-8.55	17.54
DG MF	2202.73	-3874.76	2007.37	-282-95	17.80
D M SF	-233.75			100.73	0
D MF SF	1196.14		-1497.71	533.77	0

In the table, D stands for delta hedging, DG for delta-gamma hedging, M for longevity only, MF for both longevity and financial risk neutralization, and SF for self-financing. Empty cells refer to instruments that are not used in the strategy; their holding is set to zero. The Value column refers to the fair value of the liability portfolio, which equals the premiums collected when setting up the position.

5.4 Cross-generational natural hedging

We now consider the case in which the insurer has a portfolio including products issued on generations x and y and wants to hedge the longevity risk of the second generation using exclusively products on the first. In our example, the insurer has issued an annuity on the older cohort y ($n_H = -1$) and wants to use life insurance policies with different maturities written on the lives of the younger generation x as hedging instruments.

Using cross-generational hedging, we exploit the correlation between the dynamics of the intensities of these two generations to hedge against the common risk factor. Table 6 reports the optimal hedging strategies to achieve different goals. As in Section 5.3, we assume that there are enough products to guarantee the existence and uniqueness of the hedging strategy.

To compare different possibilities associated with the same hedging objective, we identify and compare three types of hedging portfolios. The common hedging objective is delta hedging of mortality and financial risk. The first portfolio is reported in line 2 of Table 3 and includes only products written on the lives of individuals belonging to generation y. The second portfolio is reported in line

2 of Table 6 in which the annuity on y is hedged by LIs on x only. The third portfolio includes products on y and bonds, as described at the end of Section 5.3. Idiosyncratic risk is involved, as the value of the second portfolio is affected by a change in the risk factor I'. The first and third portfolios are unaffected by idiosyncratic risk.

If enough products on generation y are available, namely, 2 LIs, there is no need to hedge cross-generationally. When no sale or reinsurance on LIs on generation y is feasible, the insurer can construct a delta hedged portfolio using LIs written on generation x. He should issue 11.94 10-yr LIs and buy reinsurance on 4.57 20-yr LIs. In this example, the practical application of natural cross-generation hedging yields a portfolio whose value of liabilities is slightly higher than that of the intra-generational portfolio (15.44 vs. 13.42). To compare the effectiveness of the hedges, we compute the value change of the three portfolios after a shock on the three risk factors as large as their instantaneous volatility ($\Delta I = -0.000198$, $\Delta I' = -0.000025$, $\Delta K = -0.0217$). Table 7 reports the effects of such changes on the values of the portfolios and isolates the impact of each risk factor realization. In particular, the column labeled "effect of ΔI" reports the quantity $\Delta_{\Pi}^{M,j}\Delta I + \frac{1}{2}\Gamma_{\Pi}^{M,j}\Delta I^2$, where $\Delta_{\Pi}^{M,j}$ and $\Gamma_{\Pi}^{M,j}$ are the delta and the gamma of the whole portfolio with respect to changes in I, and so on.

As in Table 4, the difference between unhedged and hedged portfolios is remarkable: the order of magnitude of the change in portfolio value of the unhedged portfolio is at least 10 times larger than in the hedged portfolios. In all portfolios, the changes are non-null because they are delta hedged, and second order terms affect the change of the reserve. The table highlights that, even if the forecast error on interest rate risk is the main source of hedging error for the unhedged portfolio, unexpected changes in mortality rates can contribute substantially to portfolio variability. Realized

Table 7: Effect of a realized change in the risk factors on 3 portfolios

Portfolio/Risk factor	Effect of ΔI	Effect of $\Delta I'$	Effect of ΔK	Total
Unhedged	-2.29%	0	-20.48%	-22.76%
Single-generation	0.30%	0	0.01%	0.31%
Cross-generation	1.80%	-0.46%	1.19%	2.54%
Bond	-0.43%	0	-2.11%	-2.55%

The table shows the percentage change in the unhedged portfolio and three hedging portfolios following realized changes in the risk factors equal to
$\Delta I = -0.000198, \Delta I' = -0.000025, and \Delta K = -0.0217.$

mortality intensity is lower than predicted and interest rates decrease, leading to a substantial and unexpected 22.76% increase in the fair value of the annuity, mainly (approximately 90%) due to interest rate forecast error. This is the change in the value of an unhedged position. In contrast, 96% of the variation of the single-generation portfolio is due to the change ΔI. In the cross-generation portfolio, we see the effect of idiosyncratic risk: even if correlation is very close to 1, the effect of the change in I' (-0.46%) is notable, being (in absolute terms) nearly one-third of the change due to I (1.80%). The within-generation strategy performs best, as the change in the portfolio value is 0.31%. Compared with the unhedged portfolio, the cross-generation hedging strategy greatly reduces the value change to 2.54% with respect to the unhedged portfolio. The hedging strategy including the bond leads to a slightly higher percentage portfolio value change of -2.55%. The increase in the value of the long position on the bond is indeed offset by a larger increase in the value of the insurance liabilities. Consequently, it performs worse than the portfolios that include insurance liabilities only.

6 Conclusions

In this paper, we have studied the natural hedging of financial and longevity risks using the delta-gamma technique. We have assumed a continuous-time, cohort-based model for longevity risk that generalizes the classic Gompertz law and a standard stochastic interest rate model (Hull-White).

We have extended the existing literature by analyzing financial and longevity risks at the same time by providing closed-form hedges and by considering intra-generation as well as cross-generation hedges. We have obtained portfolios that are immunized to longevity and financial risks up to the second order, and we have clarified the role of natural hedging between annuities and life insurance policies. Our numerical application to UK data achieves different goals. First, it permits comparison of financial and longevity sensitivities (the Greeks) within and across generations. Second, it demonstrates natural hedging up to the second order in closed form. Third, it allows discussion of intra- and cross-generational hedging. In particular, we demonstrate how to exploit the fact that survival probabilities of a cohort are correlated with the longevity risk factors of another cohort; when there are not enough products written on one generation, products written on other cohorts help complete the longevity market and make natural hedging feasible. This is particularly important when hedging annuities written on older people, such as pensioners. Last but not least, we assess the effectiveness of natural hedging strategies; Monte Carlo simulations show that the change in the reserve of a non-hedged portfolio is higher than that of a delta- or delta-gamma-hedged portfolio.

Appendix A

This Appendix obtains the risk factors against which to hedge in the presence of two correlated Brownian motions that affect the intensities of two generations/genders.

We can write the dynamics of the x and y generations' intensities in (12) and (13) in terms of two independent Brownian motions, $\tilde{W}_x$ and $\tilde{W}_y$:

$$\begin{aligned} d\lambda_x(t) &= a'_x\lambda_x(t)dt + \sigma_x d\tilde{W}_x(t), \\ d\lambda_y(t) &= a'_y\lambda_y(t)dt + \sigma_y(\rho d\tilde{W}_x(t) + \sqrt{1-\rho^2}d\tilde{W}_y(t)). \end{aligned}$$

Because from (26) $d\tilde{W}_x(t) = (d\lambda_x(t) - a'_x\lambda_x(t)dt)/\sigma_x$, we can rewrite $d\lambda_y$ as:

$$d\lambda_y(t) = \rho\frac{\sigma_y}{\sigma_x}d\lambda_x(t) + (a'_y\lambda_y(t) - \rho\frac{\sigma_y}{\sigma_x}a'_x\lambda_x(t))dt + \sqrt{1-\rho^2}\sigma_y d\tilde{W}_y(t).$$

The dynamics $d\lambda_y(t)$ depend on $d\lambda_x(t) = dI(t)$ and on $d\lambda'_y(t)$, defined as:

$$d\lambda'_y(t) = (a'_y\lambda_y(t) - \rho\frac{\sigma_y}{\sigma_x}a'_x\lambda_x(t))dt + \sqrt{1-\rho^2}\sigma_y d\tilde{W}_y(t).$$

Applying Ito's lemma and rearranging, we have the following expression for dS_y:

$$\begin{aligned} dS_y(t,T) &= \frac{\partial S_y(t,T)}{\partial t}dt + \frac{\partial S_y(t,T)}{\partial \lambda_y}\rho\frac{\sigma_y}{\sigma_x}d\lambda_x + \frac{\partial S_y(t,T)}{\partial \lambda_y}d\lambda'_y + \\ &+ \frac{1}{2}\left(\rho\frac{\sigma_y}{\sigma_x}\right)^2\frac{\partial^2 S_y(t,T)}{\partial \lambda_y^2}d\lambda_x d\lambda_x + \frac{1}{2}\frac{\partial^2 S_y(t,T)}{\partial \lambda_y^2}d\lambda'_y d\lambda'_y. \end{aligned}$$

S_y depends on the two risk factors identified above: change in the mortality intensity of cohort x, $d\lambda_x$, and change in the mortality intensity of λ_y, which is uncorrelated with the dynamics of the mortality of cohort x, $d\lambda'_y$. Notice also that $dI(t) = d\lambda_x(t)$. Defining as Greeks the coefficients of the first and second order changes in the risk factors – according to (15), (16), (17), (18) – and setting $dI' = d\lambda'_y$, we obtain

$dS_y(t,T)$ as in (14) in the text.

Appendix B

The processes $\lambda_x(t)$ and $\lambda_y(t)$ can be reparametrized as follows:

$$\begin{aligned} d\lambda_x(t) &= a_x\lambda_x(t)dt + \sigma_x dW_x(t), \\ d\lambda_y(t) &= a_y\lambda_y(t)dt + \sigma_{xy}dW_x(t) + \sigma_{yy}dW_y(t). \end{aligned} \tag{26}$$

Applying the framework of Duffie et al. (2000), it is possible to calculate the survival probabilities for generations x and y, yielding:

$$S_x(0,t) = \mathbb{E}\left[e^{-\int_0^t \lambda_x(s)ds}|\mathcal{F}_0\right] = e^{\alpha^x(t)+\beta_1^x(t)\lambda_x(0)+\beta_2^x(t)\lambda_y(0)},$$

with

$$\begin{cases} \alpha^x(t) = \frac{\sigma_x^2}{2a_x^2}\left(t - 2\frac{e^{a_x t}-1}{a_x} + \frac{e^{2a_x t}-1}{2a_x}\right), \\ \beta_1^x(t) = \frac{1-e^{a_x t}}{a_x}, \\ \beta_2^x(t) = 0, \end{cases}$$

and

$$S_y(0,t) = \mathbb{E}\left[e^{-\int_0^t \lambda_y(s)ds}|\mathcal{F}_0\right] = e^{\alpha^y(t)+\beta_1^y(t)\lambda_x(0)+\beta_2^y(t)\lambda_y(0)},$$

with

$$\begin{cases} \alpha^y(t) = \frac{\sigma_{xy}^2+\sigma_{yy}^2}{2a_y^2}\left(t - 2\frac{e^{a_y t}-1}{a_y} + \frac{e^{2a_y t}-1}{2a_y}\right), \\ \beta_1^y(t) = 0, \\ \beta_2^y(t) = \frac{1-e^{a_y t}}{a_y}. \end{cases}$$

Finally, according to (26), the instantaneous correlation between $d\lambda_x$ and $d\lambda_y$ is given by:

$$Corr\left(d\lambda_x(t), d\lambda_y(t)\right) = \rho_{xy} = \frac{\sigma_{xy}}{\sqrt{\sigma_{xy}^2 + \sigma_{yy}^2}}.$$

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