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No ideas

but in things

– William Carlos Williams

Mathematics is often considered to be a subject far removed from the mundane; Gregory Bateson called it a "rigorous fantasy…forever isolated by its axioms and definitions from the possibility of making an indicative statement about the real world" (Bateson, 2000, p. 428). The purpose of this chapter is to consider the nature of mathematics from an embodied perspective, in which mathematical ideas are assumed to be like other human conceptions, in that they emerge from the interaction between an individual and the world and among individuals through time. An additional aim is to examine how mathematical ideas are constructed and expressed, again working from an embodied perspective. Our thesis is that, far from being a lifeless abstraction, mathematics is a human cultural creation grounded in physical experience and expressed through multiple semiotic and bodily-based modalities. In this chapter, we examine the terms embodiment and multimodality as applied to mathematical thinking, and we analyze the affordances and constraints offered by different modalities in doing mathematics.

Attention to the role of the body in mathematics is consistent with the expanding scope of inquiry within the field of mathematics education, which initially investigated mathematics learning more or less at a distance, via surveys and written examinations, but later gathered data directly from individual learners (Kilpatrick, 1992). In recent decades, mathematics education experienced a "turn to the social" (Lerman, 2000), in which socio-cultural factors, including interaction and discourse, were acknowledged as essential. This was followed by a nascent "turn to the body" (Lakoff & Núñez, 2000), in which mathematical ideas, like other forms of cognition, are viewed as the product of embodied human existence (Johnson, 2007; Lakoff & Johnson, 1999).

From the perspective of embodiment, although mathematics may be socially constructed, this construction is not arbitrary or unconstrained, but rather is rooted in and shaped by the body (Núñez, Edwards, & Matos, 1999). The doing and communicating of mathematics is never a purely intellectual activity; it involves a wide range of bodily actions, from committing inscriptions to paper or typing equations into a computer, to speaking, listening, gesturing and gazing. Each of these different modalities offers a different set of potentialities to the person who is doing mathematics. Thus, the focus of this chapter is on the nature of embodiment and multimodality in mathematics, and how embodied resources can contribute to mathematical practice.

Embodied Mathematics

The idea that mathematical knowledge arises from experience in the physical world is not new. More than three decades ago, the mathematician and physicist Dirk Jan Struik (1986) distinguished between the symbolic forms of mathematics and its origins in "the world of experience:"

> Its abstract symbolism can blind us to the relationship it carries to the world of experience. Mathematics, born to this world, practised by members of this world

> with minds reflecting this world, must capture certain aspects of it—e.g., a "number," expressing correspondences between sets of different objects; or a "line," as the abstract of a rope, a particular type of edge, lane or way. The theorem you discover has not been hauled out of a chimerical world of ideas, but is a refined expression of a physical, biological, or societal property. (p. 286)

Yet, to many immersed in the layers of abstractions of mathematics, it may seem to be the least likely domain to claim a connection to the physical world, much less to the human body. After all, David Hilbert famously pointed out the abstract nature of mathematics when he said, "One must be able to say at all times – instead of points, straight lines, and planes – tables, chairs, and beer mugs" (Hilbert, cited in Reid, 1996, p. 57), implying that the "objects" of mathematics are but placeholders within an essential structure of logical relationships. In asking how an abstract mathematics is connected to the body, however, we may be posing the question the wrong way around. If we begin our analysis with the body instead of with the domain of mathematics, the question then becomes: What kind of mathematics can human beings create, with the kind of bodies, minds and brains that we possess?[1] In a sense, we already have the answer: It is the mathematics that we, as a species, have created (Lakoff & Núñez, 2000). But rather than starting with the finished (cultural) product and seeking bodily roots, it may be more helpful to begin with the body and trace a possible path whereby abstractions like mathematics can emerge.

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[1] This, of course, echoes Warren McCullough's question in his article "What is a number, that a man may know it, and a man, that he may know a number" (*General Semantics Bulletin* Nos. 26 and 27 (1961), 7-18), without, in our case, objectifying number.

Nemirovsky (2003) states that, "mathematical abstractions grow to a large extent out of bodily activities having the potential to refer to things and events as well as to be self-referential" (p. 103). Thus, the essential starting place for understanding cognition is in its relationship to the body. An important point to remember is that both evolutionarily and developmentally, the mind has evolved to control the body, within specific and ever changing external circumstances. It did not evolve primarily in order to process symbols or engage in purely intellectual thought. Clark (2001) contrasts a view of the mind as a processor of symbols (common in "first generation" cognitive science, Gallese & Lakoff, 2005) with a contemporary embodied view:

> In place of the intellectual engine cogitating in a realm of detailed inner models, we confront the embodied, embedded agent acting as an equal partner in adaptive responses which draw on the resources of mind, body, and world…The idea here is that the brain should not be seen primarily as a locus of inner *descriptions* of external states of affairs; rather, it should be seen as a locus of inner *structures* that act as operators upon the world via their role in determining actions. (Clark, 2001, p. 47)

In other words, the mind is first a controller of bodily action. Whatever else it may be is built on that foundation. Since bodily action often involves the manipulation of objects in the world, including, importantly, the control and manipulation of one's own body, this experience constitutes a universal source domain for constructing many kinds of understandings, including mathematical ones. The significance of our experiences with objects as a source domain for mathematical understandings will be discussed next.

Physical Objects and Mathematical Objects

Our initial human experiences, both developmentally and evolutionarily, are with physical objects rather than symbols. That is, for an infant (or a human being before language was "invented"), interaction with the world does not take place through formal, conventional symbols, but through things. The emergence of language and symbols, in the individual and the species, occurs only after a sustained period of contact with objects via physical actions. Thus, the "primal" knowledge we gain when we learn to physically manipulate objects is available to all human beings (including practitioners of mathematics) as they engage with less concrete entities. Although experience with objects as a source domain in mathematics may be difficult to discern, precisely because such experience is so ubiquitous, an examination of mathematics discourse and texts reveals its influence. When a teacher tells a student to "carry the 1" or to "integrate the function," he or she is implicitly assuming that the student can think about a number or a function as a "thing" that can be acted on. Even as researchers, we tend to talk about mathematical "entities" or "objects," which, from the point of view of formal mathematics is inappropriate, since these "entities" are explicitly stated to be non-corporeal and their characteristics are based solely on logical properties and definitions rather than worldly attributes. Font and his colleagues discuss the differences between physical objects, which are ostensive (i.e., to which one can point, Quine, 1950), and mathematical objects, which are not (Font, Godino, Planas, & Acevedo, 2010):

> In mathematical discourse it is considered, whether explicitly or implicitly, that mathematical objects exist in a special way (non-ostensive, virtual, ideal, mental, abstract, general, etc., depending on the theoretical perspective) that is different from the way in which physical objects exist, and which particularly differs from

> the material symbols that represent them…speaking about the existence of mathematical objects, as objects that exist in a form that is different from that of their material symbols, is essentially a metaphorical question. (p. 15)

Font and his colleagues (Font et al, 2009, 2010) spell out the manner in which teachers, learners and mathematicians talk about and "manipulate" non-ostensive mathematical entities as if they were things. This occurs through a metaphor or conceptual mapping between the source domain of physical objects and the target domain of mathematical entities, a mapping that projects the inferential structure of the source domain onto the target domain (Lakoff, 1992). Thus, we are able to talk about an entire range of mathematical "entities" (e.g., sums, integrals, groups) as if they were objects with independent existence (rather than socially-shared labels for mentally-constructed patterns and structures).

Font and colleagues (Font et al., 2009) call this mapping the "Object Metaphor" or the "mathematical entities as physical objects" metaphor (p. 15). Note that this is an unconscious conceptual metaphor (or a single scope blend, see Fauconnier & Turner, 2002). Undertaking what they call "mathematical idea analysis," Lakoff and Núñez (2000) have identified a wide range of such metaphors within mathematical discourse. Not only does the object metaphor show itself in speech and written text, but is also found in gesture and imagery. When discussing various mathematical entities, we may point to locations in gesture space, or delineate regions on a piece of paper, actions which would not make sense unless we were thinking of these entities as ostensive objects. The object metaphor is a foundational metaphor in mathematical thought and activity, and its pervasiveness (yet "invisibility") provides an important example of the way that physical embodiment is deeply woven into mathematics.

Thus, from the perspective of embodiment, our interactions with the world involving physical objects provide the input space or source domain for the kind of non-physical entities we later "create" by proposing and accepting shared definitions in mathematics. Another expression of the object metaphor can be found in the often-studied cognitive transformation of mathematical actions or processes into so-called "objects" (e.g., Dubinsky & Harel, 1992; Sfard, 1994; Tall, Thomas, Davis, Gray, & Simpson, 2000). Just as in the physical world, we might engage in an action or process called "walking," and later talk about going for a "walk" (where "a walk" is now considered a "thing" rather than an action), in mathematics, many mathematical concepts are first experienced as processes, and later referenced as objects. For example, students learn first to solve a problem (a process), but later can step back to consider whether their solution (an entity) is optimal. Algebra students first engage in plotting and connecting points on a grid, and later think of the outcome of these processes as an object, a mathematical function. This is a very important conceptual capability when doing mathematics; without it, we would find it difficult to abstract or generalize, to use "compressed" versions of processes as inputs in building new mathematical entities and patterns (Fauconnier & Turner, 2002). We would argue that the fundamental metaphor of "mathematical entities as physical objects" makes possible the intellectual work that allows us to transform mathematical processes into objects.

Symbols and Inscriptions

Algebra, and its use of non-numerical symbols, is based on a common linguistic mechanism, which Lakoff and Núñez (2000) call the Fundamental Metonymy of Algebra. Metonymy refers to the ability to use a specific part or aspect of a referent to stand for the whole, as when we say, "the White House" when we actually mean the U.S. president. The Fundamental Metonymy of Algebra is defined as a "metonymic mechanism that makes the discipline of algebra possible, by allowing us to reason about numbers or other entities without knowing which particular entities we are talking about" (p. 75). Utilizing this linguistic mechanism, a letter or phrase can stand for an unknown or a range of possible values (e.g., the values that make the equation "$x + y = 12$" true; the set of natural numbers). Building on the fundamental metonymy that letters can stand for numerical values, the metaphor "mathematical symbols are physical objects" makes it possible to manipulate and "move" mathematical symbols as if they were physical objects. Of course, there are more constraints on the manipulation of mathematical symbols (and what they stand for) than there are on physical objects, because of the formal definitional nature of mathematical entities. However, this metaphor can help us understand a relatively common type of error in learning algebra, which occurs when the definitional constraints are ignored in favor of the more flexible affordances of physical objects, leading to "symbol-pushing" (e.g., Arcavi, 1994; Kieran, 1992; Matz, 1980).

Written (or displayed) mathematical symbols are an example of external inscriptions, without which mathematical actions and results would be ephemeral and easily forgotten. We use the term "inscription" to refer to an external "representation," whether symbolic or imagistic, which is non-ephemeral and therefore amenable to reflection, review and revision. The invention of mathematical symbols is one example of the development of external representational scaffolding, which includes the range of methods humans have created to keep track of thoughts and prior actions (including language, imagery, and concrete artifacts). Yet the use of external structure to keep track of or "represent" thought is not a simple unidirectional process. That is, representations do not simply "carry" meanings from one person to another, or from the mind to the external world, as if they were "conduits" of information (Edwards, 1995; Lakoff, 1992; Reddy, 1979). Instead, inscriptions are important elements in a process of feedback and feedforward, in which the very act of creating external representations can change what it is one is trying to represent (e.g., Clark, 2001).

Mathematics is clearly a domain in which the use of external scaffolding, in the form of conventional symbols as well as both conventional and idiosyncratic graphical inscriptions, has contributed in vital ways to the evolution of the domain (Cajori, 1993). Hutchins would call these inscriptions "material anchors" (Hutchins, 2005), and, again, from an embodied perspective, inscriptions do not simply "represent" an internal collection of pre-existing thoughts. Instead, mathematical arguments and ideas are developed through the iterative practice of recording, manipulating, considering, erasing and re-writing symbols, while using the record created with symbols to test and refine the ideas themselves. Clark (2001) describes this affordance of writing in the more general case of creating "text:"

> By writing down our ideas, we generate a trace in a format that opens up a range of new possibilities. We can then inspect and reinspect the same ideas […]. We can hold the original ideas steady so that we may judge them, and safely experiment with subtle alterations. We can store them in ways that allow us to

> compare and combine them with other complexes of ideas in ways that would quickly defeat the unaugmented imagination. In these ways […], the real properties of physical text transform the space of possible thoughts. (p. 208)

Clark (2001) goes on to discuss how written language and symbols make possible what he calls "second-order cognitive dynamics," the ability to think about our own thinking, including, "coming to see why we reached a particular conclusion by appreciating the logical transitions in our own thought" (pp. 208-9), an essential aspect of mathematical reasoning. Clark describes the way that viewing thoughts as objects makes possible metacognitive processes:

> As soon as we formulate a thought in words (or on paper), it becomes an object for ourselves and others. As an object, it is the kind of thing that we can have thoughts about […] The process of linguistic formulation thus creates the stable structure to which subsequent thinkings attach. (Clark, 2001, p. 209)

Thus, the fact that human minds are, fundamentally, embodied controllers of action, and that, at the same time, we exist in a cultural world that offers scaffolding for actions and thoughts, provides a starting point in considering the particular modalities utilized in thinking and doing mathematics.

Meanings for Multimodality

The term "multimodality" has been used in many different fields and analytic contexts, ranging from the study of communication to examinations of neurological processes. From the discussion below, it should be apparent that the meanings used in these different fields of study are not mutually exclusive, but intersect and complement each other. We will briefly discuss these usages and conclude by stating how we use the term in analyzing mathematical thinking.

Sensory Modalities

Aristotle is held to have distinguished five senses, and these are often considered to be the primary sensory modalities (or channels): sight, hearing, touch, taste, and smell. According to contemporary physiologists, there are five characteristics that define a sensory modality (Kling & Riggs, 1971):

> They have (1) *markedly different receptive organs* that (2) respond to *characteristic stimuli.* Each set of receptive organs has its (3) *own nerve* that goes to a (4) *different part of the brain,* and the (5) *sensations are different.* (p. 118, emphasis in original)

Based on the criteria above, four additional sensory modalities have been identified beyond those specified by Aristotle; they are: kinesthesia (joint sense), vestibular sense (balance as signaled by the inner ear), temperature sense, and pain. These four are called "somatosensory" modalities.

Clearly, sensory modalities make up an important element of learning, whether of mathematics or other subjects, for it is only via the senses that a learner has access to either direct experience or culturally transmitted knowledge. Although sight and hearing are the most important modalities used in formal schooling, other sensory modalities may provide unconscious grounding for understanding fields like mathematics. For example, the sense of balance provided by one's vestibular sense can provide the foundation for understanding the algebraic process of "balancing" an equation, among other mathematical concepts (Johnson, 1987; Núñez, Edwards, & Matos, 1999), and the experience of touch underlies the comprehension of the “behavior” of a function near an asymptote.

Neural Multimodality

Vittorio Gallese, a neuroscientist, and George Lakoff, a linguist, utilize the term “multimodality” in specific way in their model of how concepts are created in the brain. This model offers an alternative to the information-processing stance toward cognition, in which it is held that perception, thought, and motor action are three separate brain processes (Barsalou, 2008). In the information-processing model, the perceptual system first takes in outside stimuli, which are then processed in an “association area” in the cortex. The cortex subsequently directs action through the premotor and motor cortices, resulting in a possible action in response to the stimulus. In contrast, Gallese and Lakoff (2005) propose an interactionist theory built on recent discoveries that, in addition to action-only or perception-only neurons, there are neuron assemblages in the premotor and parietal areas that do two things at once: respond to sensory input and initiate or simulate action. One particular neuron of this kind is called a “mirror neuron,” which act in the following way (Gallese & Lakoff, 2005):

> [M]irror neurons […] discharge when the subject (a monkey in the classical experiments) performs various types of hand actions that are goal-related and also when the subject observes another individual performing similar kinds of actions. (p. 460)

In other words, certain neurons are activated not only by particular actions, but also by seeing such actions performed by others.

It is this linkage of perception and action that Gallese and Lakoff (2005) characterize as "multimodality" at the neuronal level. They also note that the entire sensory-motor system, as well as language itself, is multimodal:

> …circuitry across brain regions links modalities, infusing each with properties of others. The sensory-motor system of the brain is thus "multimodal" rather than modular. Accordingly, language is inherently multimodal in this sense, that is, it uses many modalities linked together—sight, hearing, touch, motor actions, and so on. Language exploits the pre-existing multimodal character of the sensory-motor system. If this is true, it follows that there is no single "module" for language—and that human language makes use of mechanisms also present in nonhuman primates. (p. 456)

Based on this conception of multimodality, Gallese and Lakoff (2005) propose a redefinition of the notion of "concept," one quite different from that found in classical cognitive science. In "first generation" cognitive science, the definition of a concept is based on a set of necessary and sufficient conditions, and concepts are seen as "modality-neutral and symbolic" (p. 466). However, according to Gallese and Lakoff, concepts are embodied: they arise as a consequence of human action or internal simulation of such action, through the formation of clusters of functional neurons within larger structures they call schemas. For Gallese and Lakoff (2005), these schemas are unlike the purely internal schemas described by Piaget or information processing psychology:

> Schemas are *interactional*, arising from (1) the nature of our bodies, (2) the nature of our brains, and (3) the nature of our social and physical *interactions* in the

> world. Schemas are therefore not purely internal, nor are they purely representations of external reality. (p. 466)

This description of schemas as interactional, as well as the work on neural multimodality, offers a biologically grounded basis for a theory of embodied mathematics. Although it may not be on the immediate horizon, it is reasonable to foresee the eventual identification of neurally based schemas and concepts for specific mathematical ideas.

Semiotic Multimodality

In recent decades, linguists, semioticians, and other scholars interested in discourse have drawn attention to the fact that communication occurs in ways that go beyond oral speech and written language, introducing the notion of semiotic multimodality. For example, Kress (2001b) describes multimodality as “the idea that communication and representation always draw on a multiplicity of semiotic modes of which language may be one” (p. 67-68). Researchers in mathematics education have also fruitfully utilized a semiotic approach in the examination of the multiple means of expression found in mathematical practice, including spoken words, mathematical symbols, and various kinds of imagery, including gesture (e.g., Arzarello, Paola, Robutti, & Sabena, 2009; Arzarello & Robutti, 2010; Radford, 2009, 2011).

Among the primary semiotic modes discussed by Kress and others (e.g., Bateman, 2009; Norris, 2004) are language, imagery and sound. With the evolution of the discipline, semioticians now also look at more complex and broad-ranging modes, including music, theater, color, clothing, and even furniture layout (see Kress & Van Leeuwen, 2002). From this perspective, virtually any means that humans use to express or organize themselves can be seen as a semiotic mode:

> I use the term “mode” for the culturally and socially produced resources for representation and “medium” as the term for the culturally produced means for distribution of these representations-as-meanings, that is, as messages. These technologies—those of representation, the modes and those of dissemination, the media—are always both independent of and interdependent with each other. (Kress, 2005, p. 6-7)

Two aspects of the definitions above should be noted: Kress restricts modes (which we take as a synonym for modalities) to resources that are “culturally and socially produced.” He also distinguishes between modes (for representation) and media (as methods of dissemination). This distinction is further clarified when he states:

> Media are the material resources used in the production of semiotic products and events, including both the tools and the materials used (e.g., the musical instrument and air; the chisel and the block of wood). They usually are specially produced for this purpose, not only in culture (ink, paint, cameras, computers), but also in nature (our vocal apparatus). (Kress, 2001a, p. 22)

Within this framework, Kress does not explicitly discuss the sources of the meanings or messages that are being represented or disseminated; yet his language suggests a conduit metaphor of representation (Lakoff, 1992; Reddy, 1979). In such a metaphor, an idea or message originates (presumably in abstract form) in the subject’s mind, and then is expressed or transferred via one or more semiotic modes (language, imagery, sound, etc.). These modes, in turn, are made concrete via particular material media. For example, the (abstract) mode of language can be delivered through the spoken word, in writing or type on paper, via characters on a computer screen, and so on.

The metaphor of communication or representation as conduit and the conceptualization of modes as purely social or cultural resources reveal important differences with the theory of embodied cognition. According to this theory, ideas do not originate as abstractions that are made concrete through particular media. Instead, the generation of ideas and concepts is intimately linked to motor action, as well as simulated or imagined motor action, as discussed by Gallese and Lakoff (2005). Ideas are embodied from the start, based on individual experience and human physical capabilities. In addition, from an embodied perspective, modes are not restricted to "culturally and socially produced resources" (Kress, 2005, p. 6) – instead, the body itself offers numerous resources for creating and expressing meanings, including gesture, bodily stance and movement, gaze, rhythm, and prosody in speech.

In order to incorporate these additional means for making and expressing ideas, we would like to propose a theoretical framework for multimodality that integrates the body and that looks more closely at how concepts originate in the embodied mind.

An Expanded View of Modalities

From the perspective of embodied cognition, bodily resources are vital in the production of meanings, not just in communicating them (Barsalou, 2008; Clark, 2001; Gallese & Lakoff, 2005; Goodwin, 2003; Johnson, 2007; Lakoff & Johnson, 1999; McNeill, 1992). This is true within mathematics no less than within any other domain. A student who painstakingly plots the points of a function for the first time and connects them into a (more or less) smooth curve is not simply expressing concepts that already exist, conveyed via the medium of pencil and paper. His or her physical engagement with the graph paper and pencil, and the iterative action of consulting a table of values, locating and plotting those values, we would argue, are essential aspects of the construction of the concept of a graph of a function. Later work with graphs may include other modalities, perhaps entering equations into a computer, or the intentional production of gestures (see Gerofsky, 2010). The body is thus not simply a medium, but an important resource in the construction and communication of meaning. It is also, clearly, a vital element in receiving meanings generated by others, via the sensory modalities.

Thus, we propose a broader definition for modality that encompasses and goes beyond the traditional notion of semiotic mode. Within this expanded perspective, we see modalities as the entire range of cultural, social and bodily resources available for receiving, creating, and expressing meaning. In addition to sensory modalities, which receive information, this category would also include motor modalities, such as gesture, bodily stance, touch and so on – essentially anything that humans can do with their bodies to communicate or construct ideas. Along with Kress, we also see the body as a medium for the expression of ideas; however, we see it not simply a medium but also a primary modality for thought.

Our framework also includes a category for the expressive products created by humans through the use of language, imagery, bodily motion or any other modality. By "expressive product," we refer to the physical "traces", whether permanent or ephemeral, of people's actions. These may take the form of writing and other inscriptions, utterances, song, dance, computer imagery, physical constructions or any other observable production. Expressive products are sometimes called "representations;" however, this term has often been used to imply the existence of an abstract internal meaning that is simply mapped onto an external representational system. Since we do not adopt this perspective, we have chosen to use the term "expressive product" rather than "representation."

Table 1 presents an outline of this four-category framework for understanding multimodality. The categories include bodily modalities (both sensory channels and motor actions), semiotic modes, material media, and concrete expressive products.

Table 1.

A Framework for Multimodality

Bodily modalities	**Semiotic modes**
Sensory - Sight, hearing, touch, vestibular, etc. *Motor* - Motor actions in general - Gesture - Gaze - Head movement - Full body movement - Bodily stance - Manipulation of artifacts - Prosody, rhythm, etc.	*Sensory and motor* - Language - Mathematical symbols - Musical notation - Other formal notation systems - Visual imagery (external) - Sound - Clothing, architecture, dance, colors, etc. - Any cultural system for making meaning
Material media	**Expressive products**
Bodily based - Voice - Hands - Body *External to body* - Paper & pencil - Blackboard - Computer screen - Other electronic devices - Paint, clay, stone, etc. - Math manipulatives, blocks, etc. - Musical instruments	*Bodily based* - Speech, song, chant - Sign language, gestures - Dance, marching, posing, etc. *External to body* Inscriptions: - Written words, books, musical scores, etc. - Written math symbols, graphs, visuals - Text messages, web pages, computer games Other products: - Paintings, pottery, sculpture, buildings, etc. - A configuration of cubes, rods, blocks, etc. - Instrumental music

Often, we find that research in mathematics education attends to only one or perhaps two of these modalities, which is understandable given the richness of each mode of expression. However, all four of the categories in Table 1 tend to be involved in living acts of communication or cognition. If we examine the semiotic mode of language, for example, it might take the form of oral speaking. Speech, seen both as a motor action and an expressive product, is mediated via air moving through the lungs, larynx and mouth. It requires the motor activity of these bodily parts, as well as the use of the sensory channel of hearing. Although the paradigmatic situation for speech involves a speaker and interlocutor in physical proximity, oral speech might also be transmitted via a different medium, such as a telephone or radio.

As a second case of the use of language as a semiotic mode, we can consider signed communication by the Deaf. A deaf person might not use oral speech for communication; instead, she might express herself, for example, in American Sign Language. This form of language uses a different medium from oral speech, specifically, the hands, head, face, and body. It also utilizes a different sensory channel, sight. Thus, sign is a different expressive product from oral speech, employing different modalities. Yet, at a higher level of abstraction, both products (oral speech and signing) are seen as language, a culturally based semiotic mode, developed and shared within specific communities.

In the realm of mathematics, an analysis utilizing the categories in Table 1 would suggest, for example, that “doing geometry” is a very different experience, conceptually, for a learner who is working with pencil and paper versus a dynamic geometry tool instantiated on a computer. Indeed, a robust line of research has investigated the nature of students' experiences with computer-based dynamic geometry systems, examining how such systems afford different experiences from those afforded by paper and pencil (e.g., Lehrer & Chazan, 1998; Laborde, 1995; Mariotti, 2000; Moreno-Armella, Hegedus, & Kaput, 2008). Further examples of the impact of modality are found in this volume. For example, the experiences of blind learners exploring the geometric concepts of area and volume would, of necessity, differ from those of learners with sight (Healy & Fernandes, this volume), and people interacting verbally while imagining a 3-dimensional geometric situation would enact still other modalities of expression (Moore-Russo & Viglietti, this volume).

We have created this framework for multimodality not solely for the sake of systematicity, but also to bring attention to elements of the process of making meaning that may be overlooked during research on mathematical practice. The field of education has moved beyond a simplistic model of teaching as the transmission of information, but it is still in the process of elaborating the complex means through which knowledge is constructed. We would argue that this process involves not just "interior" cogitation and "external" representations, but a nuanced interaction of the body with shared social and cultural resources.

Affordances of Modalities

This brings us to the question of what these different modalities make possible; that is, what are their affordances (Gibson, 1979)? Norman, following Gibson, defines affordances as "the actionable properties between the world and an actor" (Norman, 1999, p. 39). A more detailed definition is offered by Rizzo (2006):

> Affordances are opportunities for actions available in the environments for individuals with proper sensory-motor abilities. They do not belong to the environment neither to the individual, but to their relationships. Affordances are emergent phenomena between distribution of energy in the environment and potential agents' behavior … (p. 239)

For example, a horizontal surface affords sitting (or eating, or writing, depending on the height and surface smoothness), and a teapot with a handle affords easy pouring of liquids, assuming an agent capable of the given motor actions. A computer-based dynamic geometry package offers different affordances to the learner than do graph paper and pencil, among which is the capability to "drag" and continuously transform an image of a geometric object (e.g., Laborde, 1995).

It is reasonable to assume that different modalities have developed at least in part *because* they have different affordances; in other words, the characteristics of each modality bring into play different possibilities for action and communication. In this section, we will examine the affordances, constraints, and complementarities among different modalities and expressive products, particularly those involved in doing mathematics.

Gesture and Speech

A particularly clear case of the complementary affordances of modalities can be found in examining oral speech and gesture. Kendon, citing the Oxford English Dictionary, says gesture "refers to 'a movement of the body or of any part of it' that is 'expressive of thought or feeling'" (1997, p. 109). McNeill (1992) was one of the first researchers to point out the complementary nature of speech and gesture, proposing that,

> Speech and gesture are elements of a single integrated process of utterance formation in which there is a synthesis of opposite modes of thought — global-synthetic and instantaneous imagery with linearly-segmented temporally extended verbalization. Utterances and thoughts realized in them are both imagery and language. (p. 35)

He also points out the way in which "each modality performs its own functions, the two modalities mutually supporting one another" (p 6). This is in contrast to a view of gesture as epiphenomenal, a simple illustration of what is being expressed through speech. A growing body of research has shown that gesturing has a much more significant role in reasoning, problem solving, and cognitive development than merely reinforcing speech (e.g., Alibali, Kita, & Young, 2000; Alibali, Spencer, Knox, & Kita, 2011; Arzarello, Paola, Robutti, & Sabena, 2009; Edwards, 2009; Gerofsky, 2010; Goldin-Meadow, 2003; Robutti, 2006; Roth, 2001).

Goodwin (2003) also notes the complementarity of speech and gesture, stating that speech is not simply a more "evolved" form of communication than gesture:

> [T]he way in which the structure of gesture differs markedly from language might reflect not the development of a new, more complex system from a simpler one, but instead a process of progressive differentiation within a larger set of interacting systems in which gesture is organized precisely to provide participants with resources that complement, and thus differ significantly from those afforded by language. (p. 23)

The potential for speech and gesture to convey different meanings has been examined experimentally by Goldin-Meadow and her colleagues (e.g., Goldin-Meadow, 2003, 2006; Goldin-Meadow, Kim, & Singer, 1999; Özçaliskan & Goldin-Meadow, 2009). Following McNeill (1992), Goldin-Meadow (2006) describes the different affordances of speech and gesture:

> Speech conveys meaning discretely, relying on codified words and grammatical devices. Gesture that accompanies speech conveys meaning holistically, relying on visual and mimetic imagery. Because gesture and speech employ such different forms of representation, the two modalities rarely contribute identical information to a message. (p. 36)

When different information is conveyed simultaneously by speech and gesture, Goldin-Meadow (2003) refers to this as a gesture-speech "mismatch." Research has shown that such mismatches (or non-redundancies) can signal a readiness to learn or an imminent cognitive transition. In these cases, a learner's gestures can express a change in understanding that has not yet been expressed in his or her speech; this phenomenon has been demonstrated in arithmetic, language-learning, science, and the development of Piagetian conservations (Alibali, Church, Kita, & Hostetter, this volume; Goldin-Meadow, 2003, 2006; Goldin-Meadow, Levine, & Jacobs, this volume; Özçaliskan & Goldin-Meadow, 2009; Roth, 2001).

Affordances of Modalities for Mathematics

The different affordances of gesture and speech in teaching, learning, and doing mathematics have been investigated in this volume and elsewhere (e.g., Alibali, Spencer, Knox, & Kita, 2011; Arzarello, Paola, Robutti, & Sabena, 2009; Edwards, 2008, 2009; Ferrara, 2006; Gerofsky, 2010; Goldin-Meadow, Kim, & Singer, 1999; Nemirovsky, 2003; Núñez, 2009; Radford, 2009; Robutti, 2006; Roth, 2001; Valenzeno, Alibali, &

Klatzky, 2003). We now turn to a range of additional modalities and expressive products that are commonly used in mathematics, with the goal of highlighting their important affordances and constraints. The current analysis considers the following characteristics, drawn from prior research on gesture and speech (McNeill, 1992; Goldin-Meadow, 2006):

- Permanence: Does the modality result in an ephemeral or more permanent expressive product or representation?
- Temporality: Is the modality or expression linear (where the message emerges sequentially in time), as in speech? Or is the expressive product perceived as a global whole (holistically or nonsequentially), as with gesture or inscribed imagery?
- Structure: Is the expression analytic, that is, made up of meaningful sub-units? Or is it a synthetic, non-decomposable whole?

Table 2 presents a set of modalities and expressive products that are commonly utilized in doing, teaching, and learning mathematics, as well as a brief summary of the affordances and other important characteristics of each:

Table 2.

Affordances of Modalities/Expressive Products for Mathematics

Modality or Mode		
	Expressive Product	**Characteristics/Affordances**
Language	• Speech	Ephemeral, linear, analytic (composed of meaningful sub-units). Prosody, rhythm and volume can give emphasis.
	• Written text	Permanent, linear, analytic (composed of meaningful sub-units).
Formal Notations	• Written mathematical symbols	Permanent, generally linear, generally synthetic (although some symbols have meaningful sub-units). Compressions of

		more complex/abstract ideas utilizing metonymy.
Visual Imagery	• Static graphs • e.g., using Cartesian coordinates: an important conventional blend	Permanent. Global/holistic. Analytic – by convention, the parts are meaningful.
	• Static geometric diagrams	Permanent. Global (or holistic). Analytic. Iconic to elements of physical world, but intended to "point to" ideal forms.
	• Static conventional mathematical diagrams (other than graphs and geometric diagrams; e.g., Venn diagrams)	Permanent. Generally global/holistic. Have some characteristics of drawings and some of symbols. Non-arbitrary. Can be synthetic or analytic.
	• Marks drawn to highlight, emphasize or direct attention	Often spontaneous, can be permanent or ephemeral. Global/holistic. Synthetic.
Visual Imagery and/or Formal Notations	• Computer/calculator-based mathematics systems • Dynamic geometry systems, function graphers, etc.	Same characteristics as the components (mathematical symbols, graphs, etc.). However, the system affords instantaneous feedback and iterative exploration. Interaction via mouse & keyboard, or finger & touchscreen.
Motor Actions	• Gestures with empty hands	Ephemeral, global, synthetic.
	• Gestures holding artifact (pen, pointer etc.)	Affords more precise boundaries and point locations when gesturing.
	• Gesture involving an object in environment (table surface or edge, etc.)	The affordances of the object can be incorporated into the meaning of the gesture.
	• Other bodily actions/postures/gaze	Ephemeral, global, synthetic. Each with own particular affordances.

Discussion

As we have seen, the term "multimodality" is itself polysemous: it has been used to refer to the range of sensory channels in the human organism, to the linkage of action and perception at the neural level, and to the use of multiple means of expression or representation, from formal systems such as speech to spontaneous and idiosyncratic bodily gestures. Given the rich collection of modalities and expressive products related to mathematics, we hope that an analysis such as the one presented here will encourage researchers and teachers alike to attend to the various affordances of different modalities. In fact, there is a small but growing body of research that supports the efficacy of appropriate gesturing, by teachers and learners, in mathematics instruction (e.g., Gerofsky, 2010; Goldin-Meadow, Kim, & Singer, 1999; Valenzeno, Alibali, & Klatzky, 2003). Similarly, designers of computer-based learning environments for mathematics have long discussed the power of multiple representations (graphs, symbols, tables, words, etc.); the perspective presented here on complementary affordances has the potential to both explain the synergy among representations and to inspire designs that incorporate modalities that might have been overlooked.

Yet a closer look at embodiment and multimodality in mathematics perhaps raises more questions than it answers, at the level of both beginning and advanced mathematics. A common approach to mathematics teaching at the elementary school level includes the use of concrete physical materials, or manipulatives. These materials, by design, offer certain affordances and constraints to the learner, and it is assumed that using them is an important component in the construction of new mathematical concepts. But how does this happen? How do physical actions with blocks or tiles provide grounding for understanding the mathematical concept and the conventional language and symbols associated with it? Does this embodied interaction with concrete materials persist as part of the student's mathematical knowledge? There are suggestions that such actions are re-externalized and expressed later through gesture (see for example, Edwards, 2009); what other functions do gestures perform? What about more advanced mathematical ideas which are not obviously grounded in physical action? How are the various modalities utilized in learning and doing mathematics at the secondary and university levels? Some of these questions are addressed in the current volume, but a full understanding of embodiment and multimodality in mathematics will require further sustained inquiry.

Embodied experience may be an essential component of knowledge, but it does not always support mathematical understanding (for example, see Nuñéz, Edwards, & Matos, 1999). Additional research is needed into how embodiment may constrain or limit understanding of formal mathematics; after all, the requirements of the discipline for consistency and universality have resulted in structures that do not correspond neatly to everyday experience. Yet we would argue that the entire range of modalities, including those which are bodily-based, are essential to learning, teaching and practicing mathematics. Exploring embodiment and multimodality in diverse mathematical contexts, we believe, can only strengthen our understanding of mathematics education.

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