Parametrices and hypoellipticity for pseudodifferential operators on spaces of tempered ultradistributions

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Parametrices and hypoellipticity for pseudodifferential operators on spaces of tempered ultraditributions

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Abstract

We construct parametrices for a class of pseudodifferential operators of infinite order acting on spaces of tempered ultradistributions of Beurling and Roumieu type. As a consequence we obtain a result of hypoellipticity in these spaces.

0 Introduction

The main concern in this paper is the study of hypoellipticity for pseudodifferential operators in the setting of tempered ultradistributions of Beurling and Roumieu type on $\mathbb{R}^d$. These distributions represent the global counterpart of the ultradistributions studied by Komatsu, see [12, 13, 16]. We recall that the space of test functions for the ultradistributions of [12, 13, 16] is a natural generalisation of the Gevrey classes. In the same way tempered ultradistributions act on a space which generalises the spaces of type $\mathcal{S}$ introduced by Gelfand and Shilov in [9].

Before presenting our results let us recall some previous results on hypoellipticity in the spaces mentioned above. Hypoellipticity in Gevrey classes has been studied by several authors, see [11, 17, 22, 25] and the references therein. Indeed the functional setting allows to consider very general symbols $a(x, \xi)$ admitting exponential growth at infinity with respect to the covariable ξ. This was first noticed in [25] and generalised in [6, 7] with applications to hyperbolic equations in Gevrey classes. In [25] the hypoellipticity has been obtained by means of the construction of a parametrix. More recently, the results of [25] have been extended by Fernández et al. [8] to the space of ultradistributions of Beurling type and by the first author to the global frame of the Gelfand-Shilov spaces of type $\mathcal{S}$, see [2, 3, 4], allowing exponential growth for the symbols also with respect to the variable x.

It is then natural to study the same problem for pseudodifferential operators acting on tempered ultradistributions. In a recent paper [21], the third author constructed a global calculus for pseudodifferential operators of infinite order of Shubin type in this setting. Here we want to apply this tool to construct parametrices for the class of [21] and to prove a hypoellipticity result.

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Let us first fix some notation and introduce the functional setting where our results are obtained. In the sequel, the sets of integer, non-negative integer, positive integer, real and complex numbers are denoted by $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{Z}_+$, $\mathbb{R}$, $\mathbb{C}$. We denote $\langle x\rangle = (1+|x|^2)^{1/2}$ for $x \in \mathbb{R}^d$, $D^\alpha = D_1^{\alpha_1}\dots D_d^{\alpha_d}$, $D_j^{\alpha_j} = i^{-1}\partial^{\alpha_j}/\partial x^{\alpha_j}$, $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d) \in \mathbb{N}^d$. Finally, fixed $B > 0$ we shall denote by Q_B^c the set of all $(x,\xi) \in \mathbb{R}^{2d}$ for which we have $\langle x\rangle \geq B$ or $\langle \xi\rangle \geq B$.

Following [12], in the sequel we shall consider sequences M_p of positive numbers such that $M_0 = M_1 = 1$ and satisfying all or some of the following conditions:

$(M.1)$ $M_p^2 \leq M_{p-1}M_{p+1}$, $p \in \mathbb{Z}_+$;

$(M.2)$ $M_p \leq c_0 H^p \min\limits_{0\leq q\leq p}\{M_{p-q}M_q\}$, $p, q \in \mathbb{N}$, for some $c_0, H \geq 1$;

$(M.3)$ $\displaystyle\sum_{p=q+1}^{\infty}\frac{M_{p-1}}{M_p} \leq c_0 q\frac{M_q}{M_{q+1}}$, $q \in \mathbb{Z}_+$,

$(M.4)$ $\displaystyle\left(\frac{M_p}{p!}\right)^2 \leq \frac{M_{p-1}}{(p-1)!}\cdot\frac{M_{p+1}}{(p+1)!}$, for all $p \in \mathbb{Z}_+$,

In some assertions in the sequel we could replace $(M.3)$ by the weaker assumption

$(M.3)'$ $\displaystyle\sum_{p=1}^{\infty}\frac{M_{p-1}}{M_p} < \infty$,

cf. [12]. It is important to note that $(M.4)$ implies $(M.1)$.
Note that the Gevrey sequence $M_p = p!^s$, $s > 1$, satisfies all of these conditions.

For a multi-index $\alpha \in \mathbb{N}^d$, M_α will mean $M_{|\alpha|}$, $|\alpha| = \alpha_1 + ... + \alpha_d$. Recall that the associated function for the sequence M_p is defined by

$$M(\rho) = \sup_{p\in\mathbb{N}} \log_+ \frac{\rho^p}{M_p}, \quad \rho > 0.$$

The function $M(\rho)$ is non-negative, continuous, monotonically increasing, it vanishes for sufficiently small $\rho > 0$ and increases more rapidly than $\ln \rho^p$ when ρ tends to infinity, for any $p \in \mathbb{N}$ (see [12]).

For $m > 0$ and a sequence M_p satisfying the conditions $(M.1)-(M.3)$, we shall denote by $\mathcal{S}_\infty^{M_p,m}(\mathbb{R}^d)$ the Banach space of all functions $\varphi \in \mathcal{C}^\infty(\mathbb{R}^d)$ such that

$$\|\varphi\|_m := \sup_{\alpha\in\mathbb{N}^d}\sup_{x\in\mathbb{R}^d}\frac{m^{|\alpha|}|D^\alpha\varphi(x)|e^{M(m|x|)}}{M_\alpha} < \infty, \tag{0.1}$$

endowed with the norm in (0.1) and we denote $\mathcal{S}^{(M_p)}(\mathbb{R}^d) = \varprojlim\limits_{m\to\infty} \mathcal{S}_\infty^{M_p,m}(\mathbb{R}^d)$ and $\mathcal{S}^{\{M_p\}}(\mathbb{R}^d) = \varinjlim\limits_{m\to 0} \mathcal{S}_\infty^{M_p,m}(\mathbb{R}^d)$. In the sequel we shall consider simultaneously the two latter spaces by using the common notation $\mathcal{S}^*(\mathbb{R}^d)$. For each space we will consider a suitable symbol class. Definitions and statements will be formulated first for the (M_p) case and then for the $\{M_p\}$ case, using the notation $*$. We shall denote by $\mathcal{S}^{*\prime}(\mathbb{R}^d)$ the strong dual space of $\mathcal{S}^*(\mathbb{R}^d)$. We refer to [5, 18, 19] for the properties of $\mathcal{S}^*(\mathbb{R}^d)$ and $\mathcal{S}^{*\prime}(\mathbb{R}^d)$. Here we just recall that the Fourier transformation is an automorphism on $\mathcal{S}^*(\mathbb{R}^d)$ and on $\mathcal{S}^{*\prime}(\mathbb{R}^d)$ and that for $M_p = p!^s$, $s > 1$, we have $M(\rho) \sim \rho^{1/s}$. In this case $\mathcal{S}^*(\mathbb{R}^d)$ coincides respectively with the Gelfand-Shilov spaces $\Sigma_s(\mathbb{R}^d)$ (resp. $\mathcal{S}_s(\mathbb{R}^d)$) of all functions $\varphi \in \mathcal{C}^\infty(\mathbb{R}^d)$ such that

$$\sup_{\alpha,\beta\in\mathbb{N}^d} h^{-|\alpha|-|\beta|}(\alpha!\beta!)^{-s} \sup_{x\in\mathbb{R}^d} |x^\beta \partial^\alpha \varphi(x)| < \infty$$

for every $h > 0$ (resp. for some $h > 0$), cf. [9, 18].

Following [21] we now introduce the class of pseudodifferential operators to which our results apply. Let M_p, A_p be two sequences of positive numbers. We assume that M_p satisfies $(M.1), (M.2)$ and $(M.3)$ and that A_p satisfies $A_0 = A_1 = 1$, $(M.1), (M.2), (M.3)'$ and $(M.4)$. Moreover we suppose that $A_p \subset M_p$ i.e. there exist $c_0 > 0, L > 0$ such that $A_p \leq c_0 L^p M_p$ for all $p \in \mathbb{N}$. Let $\rho_0 = \inf\{\rho \in \mathbb{R}_+ |\, A_p \subset M_p^\rho\}$. Obviously $0 < \rho_0 \leq 1$. Let $\rho \in \mathbb{R}_+$ be arbitrary but fixed such that $\rho_0 \leq \rho \leq 1$ if the infimum can be reached, or otherwise $\rho_0 < \rho \leq 1$. For any fixed $h > 0, m > 0$ we denote by $\Gamma^{M_p,\infty}_{A_p,\rho}(\mathbb{R}^{2d}; h, m)$ the space of all functions $a(x,\xi) \in \mathcal{C}^\infty(\mathbb{R}^{2d})$ such that

$$\sup_{\alpha,\beta\in\mathbb{Z}^d_+} \sup_{(x,\xi)\in\mathbb{R}^{2d}} \frac{|D^\alpha_\xi D^\beta_x a(x,\xi)| \langle (x,\xi)\rangle^{\rho|\alpha+\beta|} e^{-(M(m|x|)+M(m|\xi|))}}{h^{|\alpha+\beta|} A_\alpha A_\beta} < \infty, \tag{0.2}$$

where $M(\cdot)$ is the associated function for the sequence M_p. Then we define

$$\Gamma^{(M_p),\infty}_{A_p,\rho}(\mathbb{R}^{2d}) = \varinjlim_{m\to\infty} \varprojlim_{h\to 0} \Gamma^{M_p,\infty}_{A_p,\rho}(\mathbb{R}^{2d}; h, m);$$
$$\Gamma^{\{M_p\},\infty}_{A_p,\rho}(\mathbb{R}^{2d}) = \varinjlim_{h\to\infty} \varprojlim_{m\to 0} \Gamma^{M_p,\infty}_{A_p,\rho}(\mathbb{R}^{2d}; h, m).$$

Remark 1. *We notice that in the case $M_p = p!^s, s > 1$, we can replace $M(m|x|) + M(m|\xi|)$ by $M(m|x||\xi|)$ in (0.2). In particular, in the case of non-quasi-analytic Gelfand-Shilov spaces, we can include symbols of the form $e^{\pm\langle (x,\xi)\rangle^{1/s}}$ in our class, cf. [20].*

We associate to any symbol $a \in \Gamma^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d})$ a pseudodifferential operator $a(x, D)$ defined, as it is usual, by

$$a(x,D)f(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{i\langle x,\xi\rangle} a(x,\xi) \hat{f}(\xi) d\xi, \qquad f \in \mathcal{S}^*(\mathbb{R}^d), \tag{0.3}$$

where $\hat{f}$ denotes the Fourier transform of f. In [21] it was proved that operators of the form (0.3) act continuously on $\mathcal{S}^*(\mathbb{R}^d)$ and on $\mathcal{S}^{*\prime}(\mathbb{R}^d)$. Moreover, a symbolic calculus for $\Gamma^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d})$ (denoted there by $\Gamma^{*,\infty}_{A_p,A_p,\rho}(\mathbb{R}^{2d})$) has been constructed. As a consequence it was proved that the class of pseudodifferential operators with symbols in $\Gamma^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d})$ is closed with respect to composition and adjoints. Here we introduce a notion of hypoellipticity for this class.

Definition 0.1. *Let $a \in \Gamma^{*,\infty}_{A_p,\rho}\left(\mathbb{R}^{2d}\right)$. We say that a is $\Gamma^{*,\infty}_{A_p,\rho}$-hypoelliptic if*

i) there exists $B > 0$ such that there exist $c, m > 0$ (resp. for every $m > 0$ there exists $c > 0$) such that

$$|a(x,\xi)| \geq c e^{-M(m|x|)-M(m|\xi|)}, \quad (x,\xi) \in Q^c_B \tag{0.4}$$

ii) there exists $B > 0$ such that for every $h > 0$ there exists $C > 0$ (resp. there exist $h, C > 0$) such that

$$\left| D_\xi^\alpha D_x^\beta a(x,\xi) \right| \leq C \frac{h^{|\alpha|+|\beta|}|a(x,\xi)|A_\alpha A_\beta}{\langle (x,\xi)\rangle^{\rho(|\alpha|+|\beta|)}}, \ \alpha,\beta \in \mathbb{N}^d, \ (x,\xi) \in Q_B^c. \tag{0.5}$$

The main result of the paper is the following

Theorem 0.2. *Let $a \in \Gamma^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d})$ be $\Gamma^{*,\infty}_{A_p,\rho}$-hypoelliptic and let $v \in \mathcal{S}^*(\mathbb{R}^d)$. Then every solution $u \in \mathcal{S}^{*\prime}(\mathbb{R}^d)$ to the equation $a(x,D)u = v$ belongs to $\mathcal{S}^*(\mathbb{R}^d)$.*

Remark 2. *In the case $M_p = p!^s, s > 1$, symbols of the form $e^{\langle (x,\xi)\rangle^{1/s}}$ satisfy the conditions (0.4), (0.5), cf. [20, Section 5] for details and other examples of hypoelliptic operators. Moreover, using the results obtained in [10] for Gelfand-Shilov spaces, it is easy to verify that the lower bound assumption (0.4) is sharp if we consider operators of the form $\exp(-P^{1/ms})u := \sum_{j=1}^\infty e^{-\lambda_j^{1/ms}} u_j \varphi_j$, where P is a positive globally elliptic Shubin differential operator of order m, cf. [24], λ_j are its eigenvalues, $\{\varphi_j\}_{j\in\mathbb{N}}$ is an orthonormal basis of eigenfunctions of P and u_j are the Fourier coefficients of u.*

The proof of Theorem 0.2 is based on the construction of a parametrix for a $\Gamma^{*,\infty}_{A_p,\rho}$-hypoelliptic operator. To perform this step we use the global calculus developed in [21]. In Section 1 we recall some facts about this calculus. Section 2 is devoted to the construction of the parametrix and to the proof of Theorem 0.2.

1 Pseudodifferential operators on $\mathcal{S}^*(\mathbb{R}^d), \mathcal{S}^{*\prime}(\mathbb{R}^d)$

In this section we recall some facts about the pseudodifferential calculus for operators with symbols in $\Gamma^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d})$ which will be used in the proofs of the next section. Since the statements below are proved in [21] for slightly more general classes of symbols, we prefer to report here the same results as they should be read for the class $\Gamma^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d})$ in order to make the paper self-contained. For proofs and further details we refer to [21]. First we recall the notion of asymptotic expansion, cf. [21, Definition 2].

Definition 1.1. *Let M_p and A_p be as in the definition of $\Gamma^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d})$ and let $m_0 = 0, m_p = M_p/M_{p-1}, p \in \mathbb{Z}_+$. We denote by $FS^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d})$ the space of all formal sums $\sum_{j\in\mathbb{N}} a_j$ such that for some $B > 0$, $a_j \in \mathcal{C}^\infty(int Q^c_{Bm_j})$ and satisfy the following condition: there exists $m > 0$ such that for every $h > 0$ (resp. there exists $h > 0$ such that for every $m > 0$) we have*

$$\sup_{j\in\mathbb{N}} \sup_{\alpha,\beta\in\mathbb{N}^d} \sup_{(x,\xi)\in Q^c_{Bm_j}} \frac{|D_\xi^\alpha D_x^\beta a_j(x,\xi)|\langle (x,\xi)\rangle^{\rho(|\alpha+\beta|+2j)} e^{-M(m|x|)-M(m|\xi|)}}{h^{|\alpha+\beta|+2j} A_\alpha A_\beta A_j^2} < \infty.$$

Notice that any symbol $a \in \Gamma^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d})$ can be regarded as an element $\sum\limits_{j\in\mathbb{N}} a_j$ of $FS^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d})$ with $a_0 = a, a_j = 0$ for $j \geq 1$.

Definition 1.2. *A symbol* $a \in \Gamma^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d})$ *is equivalent to* $\sum_{j\in\mathbb{N}} a_j \in FS^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d})$ *(we write* $a \sim \sum_{j\in\mathbb{N}} a_j$ *in this case) if there exist* $m, B > 0$ *such that for every* $h > 0$ *(resp. there exist* $h, B > 0$ *such that for every* $m > 0$*) the following condition holds:*

$$\sup_{N\in\mathbb{Z}_+}\sup_{\alpha,\beta\in\mathbb{N}^d}\sup_{(x,\xi)\in Q^c_{Bm_N}} \frac{\left|D^\alpha_\xi D^\beta_x\big(a(x,\xi)-\sum_{j<N} a_j(x,\xi)\big)\right| e^{-M(m|x|)-M(m|\xi|)}}{h^{|\alpha+\beta|+2N}A_\alpha A_\beta A_N^2\langle(x,\xi)\rangle^{-\rho(|\alpha+\beta|+2N)}} < \infty.$$

In [21] it was proved that if $a \sim 0$, then the operator $a(x,D)$ is $*$-regularizing, i.e. it extends to a continuous map from $\mathcal{S}^{*\prime}(\mathbb{R}^d)$ to $\mathcal{S}^*(\mathbb{R}^d)$. Moreover we have the following result, cf. [21, Theorem 4].

Proposition 1.3. *Let* $\sum_{j\in\mathbb{N}} a_j \in FS^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d})$*. Then there exists a symbol* $a \in \Gamma^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d})$ *such that* $a \sim \sum_{j\in\mathbb{N}} a_j$*.*

Finally we recall the following composition theorem, cf. [21, Corollary 1].

Theorem 1.4. *Let* $a, b \in \Gamma^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d})$ *with asymptotic expansions* $a \sim \sum_{j\in\mathbb{N}} a_j$ *and* $b \sim \sum_{j\in\mathbb{N}} b_j$*. Then there exists* $c \in \Gamma^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d})$ *and a* $*$*-regularizing operator* T *such that* $a(x,D)b(x,D) = c(x,D)+T$*. Moreover* c *has the following asymptotic expansion*

$$c(x,\xi) \sim \sum_{j\in\mathbb{N}}\sum_{s+k+l=j}\sum_{|\alpha|=l}\frac{1}{\alpha!}\partial^\alpha_\xi a_s(x,\xi)D^\alpha_x b_k(x,\xi).$$

2 Hypoellipticity and parametrix

In this section we construct the symbol of a left (and right) parametrix for a $\Gamma^{*,\infty}_{A_p,\rho}$-hypoelliptic operator starting from the asymptotic expansion of the symbol and using the symbolic calculus developed in [21]. To do this we need some preliminary results.

Lemma 2.1. *Let* M_p *be a sequence of positive numbers satisfying* $(M.4)$ *and* $M_0 = M_1 = 1$*. Then for all* $2 \le q \le p$, $\left(\frac{M_q}{q!}\right)^{1/(q-1)} \le \left(\frac{M_p}{p!}\right)^{1/(p-1)}$.

Proof. For brevity in notation put $N_p = M_p/p!$. Then $N_0 = N_1 = 1$ and N_p satisfies $(M.1)$. Morever the sequence N_{p-1}/N_p is monotonically decreasing. It is enough to prove that $N_p^{1/(p-1)} \le N_{p+1}^{1/p}$ for $p \ge 2$, $p \in \mathbb{N}$. The proof goes by induction. For $p = 2$ one easily verifies this. Assume that it holds for some $p \ge 2$. Then we have

$$\begin{aligned} N^{2p+2}_{p+1} &\le N^{p+1}_p N^{p+1}_{p+2} \le N_p N^{p-1}_{p+1} N^{p+1}_{p+2} = N^{2p}_{p+2} N_p \left(\frac{N_{p+1}}{N_{p+2}}\right)^{p-1} \\ &\le N^{2p}_{p+2} N_p \frac{N_{p-1}}{N_p}\cdot \ldots \cdot \frac{N_1}{N_2} = N^{2p}_{p+2}, \end{aligned}$$

from which the desired inequality follows. □

Lemma 2.2. *Let M_p satisfy $(M.4)$ and $M_0 = M_1 = 1$. Then for all $\alpha, \beta \in \mathbb{N}^d$ such that $\beta \leq \alpha$ and $1 \leq |\beta| \leq |\alpha| - 1$ the inequality $\binom{\alpha}{\beta} M_{\alpha-\beta} M_\beta \leq |\alpha| M_{|\alpha|-1}$ holds.*

Proof. We will consider two cases.

Case 1. $2 \leq |\beta| \leq |\alpha| - 2$.
If we use Lemma 2.1 and the inequality $\binom{\kappa}{\nu} \leq \binom{|\kappa|}{|\nu|}$ for $\nu \leq \kappa$, $\kappa, \nu \in \mathbb{N}^d$, we have

$$\begin{aligned}\binom{\alpha}{\beta} M_{\alpha-\beta} M_\beta &\leq |\alpha|! \cdot \frac{M_{\alpha-\beta}}{(|\alpha| - |\beta|)!} \cdot \frac{M_\beta}{|\beta|!} \\ &\leq |\alpha|! \cdot \left(\frac{M_{|\alpha|-1}}{(|\alpha| - 1)!}\right)^{\frac{|\alpha|-|\beta|-1}{|\alpha|-2}} \cdot \left(\frac{M_{|\alpha|-1}}{(|\alpha| - 1)!}\right)^{\frac{|\beta|-1}{|\alpha|-2}} = |\alpha| M_{|\alpha|-1}.\end{aligned}$$

Case 2. $|\beta| = 1$ or $|\beta| = |\alpha| - 1$.
Then obviously $\binom{\alpha}{\beta} M_{\alpha-\beta} M_\beta \leq |\alpha| M_{|\alpha|-1}$. □

In the following we assume that A_p satisfies the conditions $(M.1)$, $(M.2)$, $(M.3)'$ and $(M.4)$. Furthermore we suppose that $A_0 = A_1 = 1$. Because of $(M.3)'$, $A_p/(pA_{p-1}) \to \infty$, when $p \to \infty$, see [12]. Under these assumptions we can prove the following result.

Lemma 2.3. *Let $a \in \Gamma^{*,\infty}_{A_p,\rho}\left(\mathbb{R}^{2d}\right)$ be $\Gamma^{*,\infty}_{A_p,\rho}$-hypoelliptic. Then, the function $p_0(x,\xi) = a(x,\xi)^{-1}$ satisfies the following condition: for every $h > 0$ there exists $C > 0$ (resp. there exist $h, C > 0$) such that*

$$\left|D^\alpha_\xi D^\beta_x p_0(x,\xi)\right| \leq C \frac{h^{|\alpha|+|\beta|} |p_0(x,\xi)| A_{\alpha+\beta}}{\langle (x,\xi) \rangle^{\rho(|\alpha|+|\beta|)}}, \ \alpha, \beta \in \mathbb{N}^d, \ (x,\xi) \in Q^c_B. \tag{2.1}$$

Proof. We observe preliminary that $(M.1)$ and $(M.2)$ on A_p imply that (0.5) is equivalent to saying that there exists $B > 0$ such that for every $h > 0$ there exists $C > 0$ (resp. there exist $h, C > 0$) such that

$$\left|D^\alpha_\xi D^\beta_x a(x,\xi)\right| \leq C \frac{h^{|\alpha+\beta|} |a(x,\xi)| A_{\alpha+\beta}}{\langle (x,\xi) \rangle^{\rho(|\alpha+\beta|)}}, \ \alpha, \beta \in \mathbb{N}^d, \ (x,\xi) \in Q^c_B. \tag{2.2}$$

Then, to simplify the notation, we set $w = (x,\xi)$. First we will consider the (M_p) case. Let $h > 0$ be arbitrary but fixed and take $h_1 > 0$ such that $2^{4d+2} h_1 \leq h$. Then there exists $C_{h_1} \geq 1$ such that

$$|D^\alpha_w a(w)| \leq C_{h_1} \frac{h_1^{|\alpha|} |a(w)| A_\alpha}{\langle w \rangle^{\rho|\alpha|}}, \ \alpha \in \mathbb{N}^{2d}, \ w \in Q^c_B. \tag{2.3}$$

Now, there exists $t \in \mathbb{Z}_+$ such that $C_{h_1} \leq 2^t$. Then, for $|\alpha| \geq t$,

$$|D^\alpha_w a(w)| \leq \frac{(2h_1)^{|\alpha|} |a(w)| A_\alpha}{\langle w \rangle^{\rho|\alpha|}}, \ w \in Q^c_B. \tag{2.4}$$

Choose $s\in\mathbb{N}$, $s>t+1$, such that

$$C_{h_1}s'A_{s'-1}\leq A_{s'},\ \text{ for all } s'\geq s. \tag{2.5}$$

We will prove that

$$|D^{\alpha}_w p_0(w)|\leq C_{h_1}^{\min\{s,|\alpha|\}}\frac{h^{|\alpha|}|p_0(w)|A_\alpha}{\langle w\rangle^{\rho|\alpha|}},\ \alpha\in\mathbb{N}^{2d},\ w\in Q^c_B, \tag{2.6}$$

which will complete the proof in the (M_p) case.

For $|\alpha|=0$, (2.6) is obviously true. Suppose that it is true for $|\alpha|\leq k$, for some $0\leq k\leq s-1$. We will prove that it holds for $|\alpha|=k+1$. If we differentiate the equality $a(w)p_0(w)=1$ on Q^c_B, we have

$$|a(w)||D^{\alpha}_w p_0(w)|\ \leq\ \sum_{\substack{\beta\leq\alpha\\ \beta\neq 0}}\binom{\alpha}{\beta}|D^{\alpha-\beta}_w p_0(w)|\cdot|D^{\beta}_w a(w)|.$$

We can use the inductive hypothesis for the terms $|D^{\alpha-\beta}_w p_0(w)|$, Lemma 2.2 and the fact that $qA_{q-1}\leq A_q$, $\forall q\in\mathbb{Z}_+$, (which follows from $(M.4)$) to obtain

$$\begin{aligned}|D^{\alpha}_w p_0(w)|\ &\leq\ \frac{C^{k+1}_{h_1}|p_0(w)|}{\langle w\rangle^{\rho|\alpha|}}\sum_{\substack{\beta\leq\alpha\\ \beta\neq 0}}\binom{\alpha}{\beta}h^{|\alpha|-|\beta|}h_1^{|\beta|}A_{\alpha-\beta}A_\beta\\ &\leq\ \frac{C^{k+1}_{h_1}|p_0(w)|h^{|\alpha|}A_\alpha}{\langle w\rangle^{\rho|\alpha|}}\sum_{\substack{\beta\leq\alpha\\ \beta\neq 0}}\left(\frac{h_1}{h}\right)^{|\beta|}\\ &\leq\ \frac{C^{k+1}_{h_1}|p_0(w)|h^{|\alpha|}A_\alpha}{\langle w\rangle^{\rho|\alpha|}}\sum_{r=1}^{\infty}\left(\frac{h_1}{h}\right)^{r}\sum_{|\beta|=r}1.\end{aligned}$$

Since

$$\sum_{r=1}^{\infty}\left(\frac{h_1}{h}\right)^{r}\sum_{|\beta|=r}1\leq\sum_{r=1}^{\infty}\binom{r+2d-1}{2d-1}\left(\frac{h_1}{h}\right)^{r}\leq\sum_{r=1}^{\infty}\left(\frac{2^{4d}h_1}{h}\right)^{r}\leq 1,$$

(2.6) is true for $0\leq|\alpha|\leq s$. To continue the induction, assume that it is true for $|\alpha|\leq k$, with $k\geq s$. To prove it for $|\alpha|=k+1$, differentiate the equality $a(w)p_0(w)=1$ for $w\in Q^c_B$. We obtain

$$|a(w)|\,|D^{\alpha}_w p_0(w)|\leq\sum_{\substack{\beta\leq\alpha\\ \beta\neq 0,\,\beta\neq\alpha}}\binom{\alpha}{\beta}\left|D^{\alpha-\beta}_w p_0(w)\right|\left|D^{\beta}_w a(w)\right|+|p_0(w)|\,|D^{\alpha}_w a(w)|\,.$$

We can use the inductive hypothesis for the terms $\left|D^{\alpha-\beta}_w p_0(w)\right|$, Lemma 2.2 and (2.5) to obtain

$$|D^{\alpha}_w p_0(w)|\ \leq\ \frac{C^{s}_{h_1}|p_0(w)|}{\langle w\rangle^{\rho|\alpha|}}\left((2h_1)^{|\alpha|}A_\alpha+\sum_{\substack{\beta\leq\alpha\\ \beta\neq 0,\,\beta\neq\alpha}}\binom{\alpha}{\beta}C_{h_1}h^{|\alpha|-|\beta|}h_1^{|\beta|}A_{\alpha-\beta}A_\beta\right)$$

$$
\begin{aligned}
&\leq \frac{C_{h_1}^s |p_0(w)|}{\langle w\rangle^{\rho|\alpha|}} \left((2h_1)^{|\alpha|} A_\alpha + \sum_{\substack{\beta\leq\alpha \\ \beta\neq 0, \beta\neq\alpha}} h^{|\alpha|-|\beta|} h_1^{|\beta|} C_{h_1} |\alpha| A_{|\alpha|-1} \right) \\
&\leq \frac{C_{h_1}^s |p_0(w)|}{\langle w\rangle^{\rho|\alpha|}} \left((2h_1)^{|\alpha|} A_\alpha + A_\alpha h^{|\alpha|} \sum_{\substack{\beta\leq\alpha \\ \beta\neq 0, \beta\neq\alpha}} \left(\frac{h_1}{h}\right)^{|\beta|} \right) \\
&\leq \frac{C_{h_1}^s h^{|\alpha|} |p_0(w)| A_\alpha}{\langle w\rangle^{\rho|\alpha|}} \sum_{r=1}^{\infty} \left(\frac{2h_1}{h}\right)^r \sum_{|\beta|=r} 1 \\
&= \frac{C_{h_1}^s h^{|\alpha|} |p_0(w)| A_\alpha}{\langle w\rangle^{\rho|\alpha|}} \sum_{r=1}^{\infty} \binom{r+2d-1}{2d-1} \left(\frac{2h_1}{h}\right)^r .
\end{aligned}
$$

Finally, we observe that

$$
\sum_{r=1}^{\infty} \binom{r+2d-1}{2d-1} \left(\frac{2h_1}{h}\right)^r \leq \sum_{r=1}^{\infty} \left(\frac{2^{4d+1} h_1}{h}\right)^r \leq 1.
$$

This completes the induction.

In the $\{M_p\}$ case, there exist $h_1, C_{h_1} > 0$ such that (2.3) holds. Take h such that $2^{4d+2} h_1 \leq h$. Choose t and s as in (2.4) and (2.5). Then we can prove (2.6) in the same way as for the (M_p) case. □

Remark 3. *We observe that to prove Lemma 2.3 we can replace the assumption* $(M.4)$ *on* A_p *by a weaker asssumption. Namely we can assume that there exists* $K > 0$ *such that* $\left(\frac{M_q}{q!}\right)^{1/q} \leq K \left(\frac{M_p}{p!}\right)^{1/p}$, *for all* $1 \leq q \leq p$. *In fact, the latter condition is the same adopted to prove that* $1/f \in \mathcal{E}^*(\mathbb{R})$ *when* $f \in \mathcal{E}^*(\mathbb{R})$ *and* $\inf |f(x)| \neq 0$ *(cf. [1] for the Beurling case and [23] for the Roumieu case). The proof in [1], [23] relies on careful considerations of the coefficients in the Faà di Bruno formula applied to the composition of the mapping* $t \mapsto 1/t$ *with* $a(x,\xi)$. *On the contrary* $(M.4)$ *is needed to prove the next Lemma 2.4.*

Lemma 2.4. *Let* $a \in \Gamma^{*,\infty}_{A_p,\rho}\left(\mathbb{R}^{2d}\right)$ *be* $\Gamma^{*,\infty}_{A_p,\rho}$*-hypoelliptic. Define* $p_0(x,\xi) = a(x,\xi)^{-1}$ *and inductively*

$$
p_j(x,\xi) = -p_0(x,\xi) \sum_{0<|\nu|\leq j} \frac{1}{\nu!} \partial_\xi^\nu p_{j-|\nu|}(x,\xi) D_x^\nu a(x,\xi), j \in \mathbb{Z}_+.
$$

Then, the functions p_j *satisfy the following conditions:*

there exist $B > 0$ *such that for every* $h > 0$ *there exists* $C > 0$ *(resp. there exist* $h, C > 0$*) such that*

$$
\left| D_\xi^\alpha D_x^\beta p_j(x,\xi) \right| \leq C \frac{h^{|\alpha|+|\beta|+2j} A_{|\alpha|+|\beta|+2j} |p_0(x,\xi)|}{\langle (x,\xi)\rangle^{\rho(|\alpha|+|\beta|+2j)}}, \tag{2.7}
$$

for all $\alpha, \beta \in \mathbb{N}^d$, $(x,\xi) \in Q_B^c$, $j \in \mathbb{Z}_+$;

there exist $m, B > 0$ *such that for every* $h > 0$ *there exists* $C > 0$ *(resp. there exist* $h, B > 0$ *such that for every* $m > 0$ *there exists* $C > 0$*) such that*

$$\left|D_\xi^\alpha D_x^\beta p_j(x,\xi)\right| \leq C\frac{h^{|\alpha|+|\beta|+2j}A_{|\alpha|+|\beta|+2j}e^{M(m|x|)}e^{M(m|\xi|)}}{\langle(x,\xi)\rangle^{\rho(|\alpha|+|\beta|+2j)}}, \tag{2.8}$$

for all $\alpha, \beta \in \mathbb{N}^d$, $(x,\xi) \in Q_B^c$, $j \in \mathbb{Z}_+$.

Proof. First, observe that it is enough to prove (2.7) since (2.8) follows from (2.7) by (0.4) (possibly with different constants). As before, we put $w = (x,\xi)$. We will consider first the (M_p) case. Let $h > 0$ be fixed. Choose $h_1 > 0$ so small such that $2^{9d+1}h_1 \leq h$ and $e^{4^d dh_1/h} - 1 \leq 1/2$. Then by assumption and Lemma 2.3, there exists $C_{h_1} \geq 1$ such that

$$|D_w^\alpha a(w)| \leq C_{h_1}\frac{h_1^{|\alpha|}|a(w)|A_\alpha}{\langle w\rangle^{\rho|\alpha|}},\ \alpha \in \mathbb{N}^{2d},\ w \in Q_B^c, \tag{2.9}$$

$$|D_w^\alpha p_0(w)| \leq C_{h_1}\frac{h_1^{|\alpha|}|p_0(w)|A_\alpha}{\langle w\rangle^{\rho|\alpha|}},\ \alpha \in \mathbb{N}^{2d},\ w \in Q_B^c, \tag{2.10}$$

Take $s \in \mathbb{Z}_+$, such that

$$C_{h_1}^2 s' A_{s'-1} \leq A_{s'},\ \text{ for all } s' \geq s. \tag{2.11}$$

We will prove that, for $j \geq 1$,

$$|D_w^\alpha p_j(w)| \leq C_{h_1}^{2\min\{s,j\}+1}\frac{h^{|\alpha|+2j}A_{|\alpha|+2j}|p_0(w)|}{\langle w\rangle^{\rho(|\alpha|+2j)}}, \tag{2.12}$$

for all $\alpha \in \mathbb{N}^{2d}$, $w \in Q_B^c$, $j \in \mathbb{Z}_+$, which will prove the lemma in the (M_p) case. We can argue by induction on j. For $j = 1$, we have

$$\begin{aligned}|D_w^\alpha p_1(w)| &\leq \sum_{\beta+\gamma+\delta=\alpha}\sum_{|\nu|=1}\frac{\alpha!}{\beta!\gamma!\delta!}\left|D_w^\beta p_0(w)\right|\left|D_w^\gamma D_\xi^\nu p_0(w)\right|\left|D_w^\delta D_x^\nu a(w)\right| \\ &\leq \frac{C_{h_1}^3|p_0(w)|}{\langle w\rangle^{\rho(|\alpha|+2)}}\sum_{\beta+\gamma+\delta=\alpha}\frac{d\cdot\alpha!}{\beta!\gamma!\delta!}h_1^{|\beta|}A_{|\beta|}h^{|\gamma|+1}A_{|\gamma|+1}h_1^{|\delta|+1}A_{|\delta|+1}.\end{aligned}$$

For $|\gamma| \geq 1$, by using Lemma 2.1, we obtain

$$A_{|\gamma|+1} \leq (|\gamma|+1)!\left(\frac{A_{|\alpha|+2}}{(|\alpha|+2)!}\right)^{\frac{|\gamma|}{|\alpha|+1}}.$$

For $|\gamma| = 0$ this trivially holds. Also, if $|\beta| \geq 2$,

$$A_\beta \leq |\beta|!\left(\frac{A_{|\alpha|+2}}{(|\alpha|+2)!}\right)^{\frac{|\beta|-1}{|\alpha|+1}} \leq |\beta|!\left(\frac{A_{|\alpha|+2}}{(|\alpha|+2)!}\right)^{\frac{|\beta|}{|\alpha|+1}}$$

and this obviously holds if $|\beta| = 1$ or $|\beta| = 0$ (note that $(M.4)$ implies that $A_p \geq p!$ for all $p \in \mathbb{N}$). Moreover for $|\delta| \geq 1$, by Lemma 2.1, we have

$$A_{|\delta|+1} \leq (|\delta| + 1)! \left(\frac{A_{|\alpha|+2}}{(|\alpha| + 2)!} \right)^{\frac{|\delta|}{|\alpha|+1}}.$$

If $|\delta| = 0$ this inequality obviously holds. Insert these inequalities in the estimate for $|D_w^\alpha p_1(w)|$ to obtain

$$\begin{aligned} |D_w^\alpha p_1(w)| &\leq \frac{C_{h_1}^3 h^{|\alpha|+2} A_{|\alpha|+2} |p_0(w)|}{\langle w \rangle^{\rho(|\alpha|+2)}} \sum_{\beta+\gamma+\delta=\alpha} \frac{d \cdot \alpha!}{\beta!\gamma!\delta!} \left(\frac{h_1}{h} \right)^{|\beta|+|\delta|+1} \\ &\quad \cdot \frac{(|\gamma| + 1)! |\beta|! (|\delta| + 1)!}{(|\alpha| + 2)!}. \end{aligned}$$

Observe that

$$\begin{aligned} \frac{\alpha!}{\beta!\gamma!\delta!} &= \binom{\alpha}{\beta+\gamma} \binom{\beta+\gamma}{\beta} \leq \binom{|\alpha|}{|\beta+\gamma|} \binom{|\beta+\gamma|}{|\beta|} \\ &= \frac{|\alpha|!}{|\beta|!|\gamma|!|\delta|!} \leq \frac{(|\alpha| + 1)!}{|\beta|!(|\gamma| + 1)!|\delta|!} \leq \frac{(|\alpha| + 2)!}{|\beta|!(|\gamma| + 1)!(|\delta| + 1)!}. \end{aligned}$$

We obtain

$$|D_w^\alpha p_1(w)| \leq \frac{C_{h_1}^3 h^{|\alpha|+2} A_{|\alpha|+2} |p_0(w)|}{\langle w \rangle^{\rho(|\alpha|+2)}} \sum_{\beta+\gamma+\delta=\alpha} \left(\frac{2^d h_1}{h} \right)^{|\beta|+|\delta|+1}.$$

Note that

$$\begin{aligned} \sum_{\beta+\gamma+\delta=\alpha} \left(\frac{2^d h_1}{h} \right)^{|\beta|+|\delta|+1} &\leq \sum_{l=0}^{\infty} \sum_{|\beta|+|\delta|=l} \left(\frac{2^d h_1}{h} \right)^{l+1} \\ &\leq \sum_{l=0}^{\infty} \binom{l+4d-1}{4d-1} \left(\frac{2^d h_1}{h} \right)^{l+1} \\ &\leq \sum_{l=0}^{\infty} \left(\frac{2^{9d} h_1}{h} \right)^{l+1} \leq 1, \end{aligned}$$

which completes the proof for $j = 1$. Suppose that it holds for all $j \leq k$, $k \leq s - 1$, $k \in \mathbb{Z}_+$. We will prove it for $j = k + 1$.

$$\begin{aligned} |D_w^\alpha p_j(w)| &\leq \sum_{\beta+\gamma+\delta=\alpha} \sum_{0<|\nu|\leq j} \frac{\alpha!}{\beta!\gamma!\delta!} \cdot \frac{1}{\nu!} |D_w^\beta p_0(w)| \cdot |D_w^\gamma D_\xi^\nu p_{j-|\nu|}(w)| \cdot |D_w^\delta D_x^\nu a(w)| \\ &\leq \frac{C_{h_1}^{2j+1} |p_0(w)|}{\langle w \rangle^{\rho(|\alpha|+2j)}} \sum_{\beta+\gamma+\delta=\alpha} \sum_{0<|\nu|\leq j} \frac{\alpha!}{\beta!\gamma!\delta!\nu!} \cdot h_1^{|\beta|} A_{|\beta|} h^{|\gamma|+2j-|\nu|} A_{|\gamma|+2j-|\nu|} h_1^{|\delta|+|\nu|} A_{|\delta|+|\nu|}, \end{aligned}$$

where we used the inductive hypothesis for the derivatives of the terms $p_{j-|\nu|}(w)$. By using Lemma 2.1, we obtain (note that $2j-|\nu|\geq 2$)

$$\begin{aligned}A_{|\gamma|+2j-|\nu|} &\leq (|\gamma|+2j-|\nu|)!\left(\frac{A_{|\alpha|+2j}}{(|\alpha|+2j)!}\right)^{\frac{|\gamma|+2j-|\nu|-1}{|\alpha|+2j-1}}\\ &\leq (|\gamma|+2j-|\nu|)!\left(\frac{A_{|\alpha|+2j}}{(|\alpha|+2j)!}\right)^{\frac{|\gamma|+2j-|\nu|}{|\alpha|+2j-1}},\end{aligned}$$

where the last inequality follows from $A_p\geq p!$, $p\in\mathbb{N}$, which in turn follows from $(M.4)$. Also, if $|\beta|\geq 2$,

$$A_\beta\leq|\beta|!\left(\frac{A_{|\alpha|+2j}}{(|\alpha|+2j)!}\right)^{\frac{|\beta|-1}{|\alpha|+2j-1}}\leq|\beta|!\left(\frac{A_{|\alpha|+2j}}{(|\alpha|+2j)!}\right)^{\frac{|\beta|}{|\alpha|+2j-1}}$$

and this obviously holds if $|\beta|=1$ or $|\beta|=0$. Moreover for $|\delta|\geq 1$, by Lemma 2.1 (because $|\nu|\geq 1$), we have

$$A_{|\delta|+|\nu|}\leq(|\delta|+|\nu|)!\left(\frac{A_{|\alpha|+2j}}{(|\alpha|+2j)!}\right)^{\frac{|\delta|+|\nu|-1}{|\alpha|+2j-1}}.$$

If $|\delta|=0$ and $|\nu|\geq 2$ Lemma 2.1 implies the same inequality and if $|\delta|=0$ and $|\nu|=1$ this inequality obviously holds. If we insert these inequalities in the estimate for $|D_w^\alpha p_j(w)|$, we obtain

$$\begin{aligned}&|D_w^\alpha p_j(w)|\\ &\leq \frac{C_{h_1}^{2j+1}|p_0(w)|}{\langle w\rangle^{\rho(|\alpha|+2j)}}\sum_{\beta+\gamma+\delta=\alpha}\sum_{0<|\nu|\leq j}\frac{\alpha!}{\beta!\gamma!\delta!\nu!}h_1^{|\beta|}h^{|\gamma|+2j-|\nu|}h_1^{|\delta|+|\nu|}\\ &\quad\cdot(|\gamma|+2j-|\nu|)!\left(\frac{A_{|\alpha|+2j}}{(|\alpha|+2j)!}\right)^{\frac{|\gamma|+2j-|\nu|}{|\alpha|+2j-1}}|\beta|!\left(\frac{A_{|\alpha|+2j}}{(|\alpha|+2j)!}\right)^{\frac{|\beta|}{|\alpha|+2j-1}}\\ &\quad(|\delta|+|\nu|)!\left(\frac{A_{|\alpha|+2j}}{(|\alpha|+2j)!}\right)^{\frac{|\delta|+|\nu|-1}{|\alpha|+2j-1}}\\ &= \frac{C_{h_1}^{2j+1}h^{|\alpha|+2j}A_{|\alpha|+2j}|p_0(w)|}{\langle w\rangle^{\rho(|\alpha|+2j)}}\sum_{\beta+\gamma+\delta=\alpha}\sum_{0<|\nu|\leq j}\frac{\alpha!}{\beta!\gamma!\delta!\nu!}\left(\frac{h_1}{h}\right)^{|\beta|+|\delta|+|\nu|}\\ &\quad\cdot\frac{(|\gamma|+2j-|\nu|)!|\beta|!(|\delta|+|\nu|)!}{(|\alpha|+2j)!}.\end{aligned}$$

Similarly as above, we have

$$\begin{aligned}\frac{\alpha!}{\beta!\gamma!\delta!} &\leq \frac{|\alpha|!}{|\beta|!|\gamma|!|\delta|!}\leq\frac{(|\alpha|+2j-|\nu|)!}{|\beta|!(|\gamma|+2j-|\nu|)!|\delta|!}\\ &\leq \frac{(|\alpha|+2j)!}{|\beta|!(|\gamma|+2j-|\nu|)!(|\delta|+|\nu|)!}.\end{aligned}$$

We obtain

$$|D^\alpha_w p_j(w)| \le \frac{C_{h_1}^{2j+1} h^{|\alpha|+2j} A_{|\alpha|+2j} |p_0(w)|}{\langle w\rangle^{\rho(|\alpha|+2j)}} \sum_{\beta+\gamma+\delta=\alpha} \sum_{r=1}^{\infty} \sum_{|\nu|=r} \frac{1}{\nu!} \left(\frac{h_1}{h}\right)^{|\beta|+|\delta|+r}.$$

We have the estimate

$$\begin{aligned}
&\sum_{\beta+\gamma+\delta=\alpha} \sum_{r=1}^{\infty} \sum_{|\nu|=r} \frac{1}{\nu!} \left(\frac{h_1}{h}\right)^{|\beta|+|\delta|+r} \\
&\le \sum_{\beta+\gamma+\delta=\alpha} \sum_{r=1}^{\infty} \binom{r+d-1}{d-1} \frac{d^r}{r!} \left(\frac{h_1}{h}\right)^{|\beta|+|\delta|+r} \\
&\le \sum_{\beta+\gamma+\delta=\alpha} \left(\frac{h_1}{h}\right)^{|\beta|+|\delta|} \sum_{r=1}^{\infty} \frac{1}{r!} \left(\frac{2^{2d} d h_1}{h}\right)^{r} \\
&= \left(e^{4^d d h_1/h} - 1\right) \sum_{\beta+\gamma+\delta=\alpha} \left(\frac{h_1}{h}\right)^{|\beta|+|\delta|} = \left(e^{4^d d h_1/h} - 1\right) \sum_{\beta+\delta\le\alpha} \left(\frac{h_1}{h}\right)^{|\beta|+|\delta|} \\
&\le \left(e^{4^d d h_1/h} - 1\right) \sum_{l=0}^{\infty} \left(\frac{h_1}{h}\right)^{l} \sum_{|\beta|+|\delta|=l} 1 \\
&= \left(e^{4^d d h_1/h} - 1\right) \sum_{l=0}^{\infty} \left(\frac{h_1}{h}\right)^{l} \binom{l+4d-1}{4d-1} \\
&\le \left(e^{4^d d h_1/h} - 1\right) \sum_{l=0}^{\infty} \left(\frac{2^{8d} h_1}{h}\right)^{l} \le 1.
\end{aligned}$$

Hence, we proved (2.12) for $1 \le j \le s$. Suppose that it holds for all $j \le k$, $k \ge s$. For $j = k+1$, similarly as above, we obtain

$$\begin{aligned}
|D^\alpha_w p_j(w)| \le\ & \frac{C_{h_1}^{2s+1} |p_0(w)|}{\langle w\rangle^{\rho(|\alpha|+2j)}} \sum_{\beta+\gamma+\delta=\alpha} \sum_{0<|\nu|\le j} \frac{\alpha!}{\beta!\gamma!\delta!\nu!} \\
& \cdot C_{h_1}^2 h_1^{|\beta|} A_{|\beta|} h^{|\gamma|+2j-|\nu|} A_{|\gamma|+2j-|\nu|} h_1^{|\delta|+|\nu|} A_{|\delta|+|\nu|}.
\end{aligned}$$

Note that $|\gamma| + 2j - |\nu| \ge s$, so, by (2.11), we have

$$C_{h_1}^2 A_{|\gamma|+2j-|\nu|} \le A_{|\gamma|+2j-|\nu|+1}/(|\gamma| + 2j - |\nu| + 1).$$

Also $|\gamma| + 2j - |\nu| + 1 \le |\alpha| + 2j$, hence Lemma 2.1 implies

$$C_{h_1}^2 A_{|\gamma|+2j-|\nu|} \le \frac{A_{|\gamma|+2j-|\nu|+1}}{|\gamma| + 2j - |\nu| + 1} \le (|\gamma| + 2j - |\nu|)! \left(\frac{A_{|\alpha|+2j}}{(|\alpha|+2j)!}\right)^{\frac{|\gamma|+2j-|\nu|}{|\alpha|+2j-1}}.$$

In the same manner as above we obtain

$$A_\beta \le |\beta|! \left(\frac{A_{|\alpha|+2j}}{(|\alpha|+2j)!}\right)^{\frac{|\beta|}{|\alpha|+2j-1}} \text{ and } A_{|\delta|+|\nu|} \le (|\delta| + |\nu|)! \left(\frac{A_{|\alpha|+2j}}{(|\alpha|+2j)!}\right)^{\frac{|\delta|+|\nu|-1}{|\alpha|+2j-1}}.$$

If we insert these inequalities in the estimate for $|D^\alpha_w p_j(w)|$ and use the above inequality for $\dfrac{\alpha!}{\beta!\gamma!\delta!}$ we obtain

$$|D^\alpha_w p_j(w)| \leq \frac{C^{2s+1}_{h_1} h^{|\alpha|+2j} A_{|\alpha|+2j} |p_0(w)|}{\langle w\rangle^{\rho(|\alpha|+2j)}} \sum_{\beta+\gamma+\delta=\alpha} \sum_{r=1}^{\infty} \sum_{|\nu|=r} \frac{1}{\nu!} \left(\frac{h_1}{h}\right)^{|\beta|+|\delta|+r}.$$

We already proved that $\displaystyle\sum_{\beta+\gamma+\delta=\alpha} \sum_{r=1}^{\infty} \sum_{|\nu|=r} \frac{1}{\nu!} \left(\frac{h_1}{h}\right)^{|\beta|+|\delta|+r} \leq 1$, hence the proof for the (M_p) case is complete.

Next, we consider the $\{M_p\}$ case. By assumption and Lemma 2.3, there exist $h_1, C_{h_1} \geq 1$ such that (2.9) and (2.10) hold. Take h so large such that $2^{9d+1}h_1 \leq h$ and $e^{4^d dh_1/h} - 1 \leq 1/2$. There exists $s \in \mathbb{Z}_+$ such that $C^2_{h_1} s' A_{s'-1} \leq A_{s'}$, for all $s' \geq s$. One proves that

$$|D^\alpha_w p_j(w)| \leq C^{2\min\{s,j\}+1}_{h_1} \frac{h^{|\alpha|+2j} A_{|\alpha|+2j} |p_0(w)|}{\langle w\rangle^{\rho(|\alpha|+2j)}},$$

for all $\alpha \in \mathbb{N}^{2d}$, $w \in Q^c_B$, $j \in \mathbb{Z}_+$, by induction on j in the same manner as for (2.12) in the (M_p) case. This completes the proof in the $\{M_p\}$ case. □

Theorem 2.5. *Let $a \in \Gamma^{*,\infty}_{A_p,\rho}\left(\mathbb{R}^{2d}\right)$ be $\Gamma^{*,\infty}_{A_p,\rho}$-hypoelliptic. Then there exist *-regularizing operators T and T' and $b, b' \in \Gamma^{*,\infty}_{A_p,\rho}\left(\mathbb{R}^{2d}\right)$ such that $b(x,D)a(x,D) = \mathrm{Id} + T$ and $a(x,D)b'(x,D) = \mathrm{Id} + T'$.*

Proof. Let p_j, $j \in \mathbb{N}$, be as in Lemma 2.4. Then the functions p_0 and p_j, $j \in \mathbb{Z}_+$, satisfy the estimates given in Lemmas 2.3 and 2.4. Since A_p satisfies $(M.1)$ and $(M.2)$, these estimates are equivalent to the following:

there exist $m, B > 0$ such that for every $h > 0$ there exists $C > 0$ (resp. there exist $h, B > 0$ such that for every $m > 0$ there exists $C > 0$) such that

$$\left|D^\alpha_\xi D^\beta_x p_j(x,\xi)\right| \leq C \frac{h^{|\alpha|+|\beta|+2j} A_\alpha A_\beta A^2_j e^{M(m|x|)} e^{M(m|\xi|)}}{\langle (x,\xi)\rangle^{\rho(|\alpha|+|\beta|+2j)}}, \tag{2.13}$$

for all $\alpha, \beta \in \mathbb{N}^d$, $(x,\xi) \in Q^c_B$, $j \in \mathbb{N}$. One can modify p_0 near the boundary of Q^c_B so that it can be extended to $\mathcal{C}^\infty$ function on $\mathbb{R}^{2d}$ and satisfy (2.13) on the whole $\mathbb{R}^{2d}$. Hence, (2.13) remains true for all $j \in \mathbb{Z}_+$ with larger B. We obtain $\sum_{j=0}^\infty p_j \in FS^{\infty,*}_{A_p,\rho}\left(\mathbb{R}^{2d}\right)$. Let $b \sim \sum_j p_j$, $b \in \Gamma^{*,\infty}_{A_p,\rho}\left(\mathbb{R}^{2d}\right)$. By Theorem 1.4 there exist $c \in \Gamma^{*,\infty}_{A_p,\rho}\left(\mathbb{R}^{2d}\right)$ and a *-regularizing operator $\widetilde{T}'_1$ such that $b(x,D)a(x,D) = c(x,D) + \widetilde{T}$ and c has the asymptotic expansion $c \sim \sum_j c_j$, where

$$c_j(x,\xi) = \sum_{s+l=j} \sum_{|\nu|=l} \frac{1}{\nu!} \partial^\nu_\xi p_s(x,\xi) D^\nu_x a(x,\xi).$$

One easily verifies that $c_0(x,\xi) = 1$ on Q^c_B. Also, for $j \in \mathbb{Z}_+$,

$$c_j = p_j a + \sum_{l=1}^{j} \sum_{|\nu|=l} \frac{1}{\nu!} \partial^\nu_\xi p_{j-l} \cdot D^\nu_x a = p_j a + \sum_{0<|\nu|\leq j} \frac{1}{\nu!} \partial^\nu_\xi p_{j-|\nu|} \cdot D^\nu_x a = 0,$$

on Q_B^c, by the definition of p_j. Hence, $b(x,D)a(x,D) = \mathrm{Id}+T$ for some *-regularizing operator T. With similar constructions one obtains b' such that $a(x,D)b'(x,D) = \mathrm{Id} + T'$, where T' is a *-regularizing operator. □

Proof of Theorem 0.2. Let $u \in \mathcal{S}^{*\prime}(\mathbb{R}^d)$ be a solution of $a(x,D)u = v \in \mathcal{S}^*(\mathbb{R}^d)$. Then, applying the left parametrix $b(x,D)$ of $a(x,D)$, we obtain $u = b(x,D)v - Tu$ for some *-regularizing operator T. Hence $u \in \mathcal{S}^*(\mathbb{R}^d)$. The theorem is proved. □

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