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LAGRANGIAN SUBMANIFOLDS AT INFINITY AND THEIR PARAMETRIZATION

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Abstract. In this paper, we study a class of Lagrangian submanifolds which may be viewed as intersecting at infinity. They are objects naturally associated with a class of tempered oscillatory integrals. In this context, we prove the adapted versions of the classical theorems, such as parametrization results, as well as equivalence of phase functions.

1. Introduction

The study of Lagrangian submanifolds is an important branch in symplectic geometry. One of the main motivations for their study is due to the fundamental role they play as carriers of singularities in the global theory of Fourier integral operators on manifolds, see [10, 16, 19, 20]. The fundamental connection is that the kernels of Fourier integral operators are Lagrangian distributions associated with a Lagrangian submanifold given by a canonical relation.

The resulting calculus is especially well-suited for working on compact, boundaryless manifolds, while a global theory of Fourier integral operators on unbounded manifolds, even on $\mathbb{R}^d$, is far from being complete. A natural choice of a class of pseudodifferential operators that such operators should contain are those defined through the so-called SG-symbols, see [2, 32, 34]. There are many contributions to the long-standing problem of introducing a suitable global calculus of SG-Fourier integral operators, see for instance [1, 3, 4, 8]. As a key ingredient, it is desirable to understand the suitable class of associated Lagrangian submanifolds that should be considered.

In [25, 26, 30], a geometric approach to the SG-calculus on general asymptotically conic manifolds, the so-called scattering geometry, has been developed. Unbounded geometries are therein viewed as manifolds with boundary and the cotangent bundle is replaced by a rescaled and compactified version, the scattering cotangent bundle. Melrose and Zworski subsequently introduced the so-called Legendrian distributions, see [30], which are smoooth functions with a prescribed singularity at infinity, associated with Legendrian submanifolds "at infinity" (see also [14, 15, 39]). On a vector space, these distributions correspond to Fourier transforms of compactly supported Lagrangian distributions.

In [9] we discussed SG-type tempered oscillatory integrals on $\mathbb{R}^d$, which are Lagrangian distributions with a suitable behaviour at infinity. It turned out that their singularities, encoded by their SG-wave front set, may be decomposed into two sets: one which admits an interpretation as a Lagran-gian submanifold, and one that corresponds to a Legendrian. These sets may thus be used as the starting point of a global theory of SG-Fourier integral operators, and a clear understanding of their geometric properties and local parametrization is then a necessary prerequisite.

Here we provide the details needed to start such analysis. In particular, we introduce a class of pairs of Lagrangian-Legendrian submanifolds and show how they can be parametrized by a class of SG-phase functions. We then review in which sense the resulting objects are suitable to formulate the singularities of SG-Lagrangian distributions. In future publications, we will then establish a calculus of the associated SG-Lagrangian distributions.

The paper is organized as follows. In Section 2, we revisit some features of the scattering geometry and outline the geometric setting in which our analysis takes place. Here we give our main definition of SG-Lagrangian submanifold, and emphasize in which sense the components of it are Lagrangian "at infinity" and "at co-infinity". In Section 3 we introduce the class of phase functions that may be used to parametrize SG-Lagrangians. We check that such (non-degenerate) phase functions indeed give rise to SG-Lagrangian submanifolds. In the main Parametrization Theorem 3.24 we state that the converse is also true. That is, we may always parametrize SG-Lagrangians by a phase function - in a suitable sense of locality. The subsequent Section 4 is devoted to the proof of the Parametrization Theorem. In Section 5 we outline when two phase functions parametrizing the same Lagrangian may be considered equivalent. Section 6 is devoted to reviewing some elements of the theory of tempered oscillatory integrals from [9] and give an example of how SG-Lagrangian submanifolds arise. Finally, for the benefit of the reader, we collected some results on the analysis of manifolds with corners in Appendix A.

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2. Asymptotically Euclidean manifolds and their symplectic structure

2.1. **Scattering manifolds.** There are many definitions available to formulate aysmptotic flatness of manifolds. In this subsection we give a short description of the scattering geometry, introduced in [25, 26, 30]. Therein, an asymptotically Euclidean manifold X is one that admits a specific compactification, in the sense that X may be viewed as a compact manifold with boundary $X^o \cup \partial X$, equipped with a Riemannian metric g defined in the interior. In a neighbourhood of the boundary, of the form $[0,\epsilon)\times\partial X \ni (\tilde{y}, y)$, where $\tilde{y}$ is a *boundary defining function*, it is required that g can be written in the form

$$g(\tilde{y}, y) = \frac{\mathrm{d}\tilde{y}^{\otimes 2}}{\tilde{y}^4} + \frac{h}{\tilde{y}^2}, \tag{1}$$

with h being a smooth symmetric 2-tensor which restricts to a metric on the boundary. The geometry "at infinity", identified with the boundary, is then modelled by the scattering vector fields, that is, vector fields that are tangent to the boundary and of bounded length w.r.t. g. These are the sections of a vector bundle, denoted by ${}^{\mathrm{sc}}TX$, and are spanned by vector fields of the form $\tilde{y}^2\partial_{\tilde{y}}$ and $\tilde{y}\partial_{y_j}$. In fact, these vector fields, the scattering vector fields ${}^{\mathrm{sc}}V$, may be obtained as $\tilde{y}V_b$ where V_b are the vector fields tangent to the boundary. Consequently, there is a natural map ${}^{\mathrm{sc}}TX \hookrightarrow TX$, under which g restricts to a well-defined tensor of ${}^{\mathrm{sc}}TX$, and a corresponding dual bundle $T^*X \hookrightarrow {}^{\mathrm{sc}}T^*X$.

In the next subsection we recall one way of viewing $\mathbb{R}^d$ as a scattering manifold.[1]

2.2. **Radial compactification.** We set $\mathbb{B}^d = \{x \in \mathbb{R}^d \colon |x| \le 1\}$, and denote $\partial\mathbb{B}^d = \mathbb{S}^{d-1}$, $(\mathbb{B}^d)^o = \{x \in \mathbb{R}^d \colon |x| < 1\}$, and $\mathbb{R}_+ = (0,\infty)$. Pick any diffeomorphism $\iota : \mathbb{R}^d \to (\mathbb{B}^d)^o$ that for $|x| > 3$ is given by

$$\iota : x \mapsto \frac{x}{|x|}\left(1 - \frac{1}{|x|}\right).$$

[1] The choice of compactification is motivated by that of [12], here reformulated in terms of scattering geometry.

Then its inverse is given, for $|y| \geq \frac{2}{3}$, by

$$\iota^{-1} : y \mapsto \frac{y}{|y|}(1 - |y|)^{-1}.$$

The map ι is called the *radial compactification map*. The associated polar coordinates equip $\mathbb{R}^d$ with a differential structure "at infinity". Indeed, introducing polar coordinates $(r, \varphi) \in \mathbb{R}^d$ we see that ι is simply given (for large r) by

$$r \mapsto 1 - \frac{1}{r} \qquad \text{and} \qquad \varphi \mapsto \varphi. \tag{2}$$

Denote by $x \mapsto [x]$ any smooth function $\mathbb{R}^d \to \mathbb{R}_+$ that coincides with $|x|$ for $|x| > 3$. Then, the map $\mathbb{B}^d \to [0, \infty)$ given by $y \mapsto \frac{1}{[\iota^{-1}(y)]} =: \tilde{y}$ is a *boundary defining function*, that is, a non-negative smooth function that vanishes on and only on the boundary of $\mathbb{B}^d$, and whose differential is non-vanishing at $\partial\mathbb{B}^d$. Notice that, for $|y| > 2/3$, the map $y \mapsto \tilde{y}$ is simply given by $y \mapsto (1-|y|)$. In a collar neighbourhood of the boundary, $0 \leq \tilde{y} < 1/3$, the metric induced by these coordinates from the standard Euclidean metric on $\mathbb{R}^d$ is given by $g = \frac{\mathrm{d}\tilde{y}^{\otimes 2}}{\tilde{y}^2} + \frac{h}{\tilde{y}^4}$, where h is the (lifted) standard metric on the $(d-1)$-sphere.

2.3. **Scattering geometry at infinity.** We now return to the case of a general scattering manifold. Since ${}^{\mathrm{sc}}T^*X$ is a vector bundle, it is possible to apply radial compactification to its fibres, resulting in an object denoted by ${}^{\mathrm{sc}}\overline{T}^*X$. We denote coordinates therein by $(\tilde{y}, y, \tilde{\eta}, \eta)$, where $\tilde{\eta}$ is the fibre-boundary defining function. The resulting set ${}^{\mathrm{sc}}\overline{T}^*X$ carries the structure of a manifold with corners. Various elements of the theory of manifolds with corners are recalled in Appendix A, based on [23]. In the sequel, we will refer to its contents whenever needed.

The boundary of ${}^{\mathrm{sc}}\overline{T}^*X$ consists of two components,

$$\partial({}^{\mathrm{sc}}\overline{T}^*X) = {}^{\mathrm{sc}}S^*X \cup {}^{\mathrm{sc}}\overline{T}^*_{\partial X}X,$$

which intersect in the corner (of codimension 2) ${}^{\mathrm{sc}}S^*_{\partial X}X$. Here, ${}^{\mathrm{sc}}S^*X$ is a (co-) sphere bundle on X, where the (co-)sphere is interpreted as the boundary of $\mathbb{B}^d$.

It is important to note that by the *rescaling isomorphism* $\frac{\mathrm{d}y}{\tilde{y}} \leftrightarrow \mathrm{d}y$, using the boundary defining function, we have

$${}^{\mathrm{sc}}\overline{T}^*_{\partial X}X \cong T^*_{\partial X}X.$$

This corresponds to the fact that the (dual) *rescaling isomorphism* ${}^{\mathrm{sc}}V \ni v \mapsto \tilde{y}^{-1}v$ induces an isomorphism ${}^{\mathrm{sc}}T_{\partial X}X \cong T_{\partial X}X$. In the same way, using $\tilde{\eta}$, we may identify ${}^{\mathrm{sc}}S^*X$ with the usual co-sphere bundle.

Following [2], we call $\widetilde{\mathcal{W}} := \partial({}^{\mathrm{sc}}\overline{T}^*X)$ the *wave front space* and the boundary components its faces. In our model case $\mathbb{R}^d$, the resulting space is $\widetilde{\mathcal{W}} := \partial(\mathbb{B}^d \times \mathbb{B}^d)$. The one-dimensional case is depicted in Figure 1. Note that it is exceptional, since $\partial\mathbb{B}^1 = \mathbb{S}^0 = \pm 1$, which is not

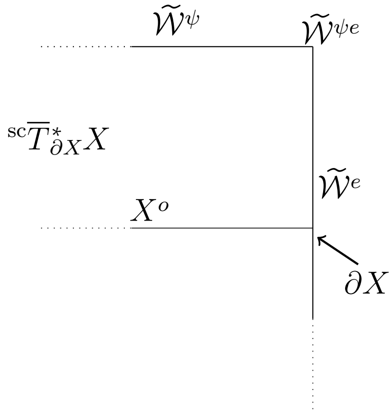

FIGURE 1. The boundary faces and corner of ${}^{\mathrm{sc}}\overline{T}^*_{\partial X}X$

connected. We depict the situation near the top right corner $(+1,+1)$.

Notation. The faces of $\widetilde{\mathcal{W}}$ behave in a lot of ways "symmetrically". In order to reflect this in a more symmetric notation, following [12], we attach to any object defined on ${}^{\mathrm{sc}}S^*(X^o) =: \widetilde{\mathcal{W}}^{\psi}$ an index "ψ" and to the corresponding one in ${}^{\mathrm{sc}}T^*_{\partial X}X =: \widetilde{\mathcal{W}}^{e}$ an index "e". To the corner, ${}^{\mathrm{sc}}S^*_{\partial X}X =: \widetilde{\mathcal{W}}^{\psi e}$, we attach the label "$\psi e$". In the model case this becomes

$$\begin{aligned}\partial(\mathbb{B}^d \times \mathbb{B}^d) &= (\mathbb{S}^{d-1} \times (\mathbb{B}^d)^o) \cup ((\mathbb{B}^d)^o \times \mathbb{S}^{d-1}) \cup (\mathbb{S}^{d-1} \times \mathbb{S}^{d-1}) \\ &=: \widetilde{\mathcal{W}}^{e} \cup \widetilde{\mathcal{W}}^{\psi} \cup \widetilde{\mathcal{W}}^{\psi e}.\end{aligned}$$

We will need to interpret conic subsets of $T^*X \setminus \{0\}$ as subsets of $\widetilde{\mathcal{W}}^{\psi}$ and viceversa. We then form

$$\mathcal{W}^{\psi} := \widetilde{\mathcal{W}}^{\psi} \times \mathbb{R}_+ \cong T^*X \setminus \{0\}, \qquad \mathcal{W}^{e} := \mathbb{R}_+ \times \widetilde{\mathcal{W}}^{e} \cong T^*(\Gamma(\partial X)),$$

where $\Gamma(\partial X)$ is the *cone with base* ∂X, that is $\mathbb{R}_+ \times \partial X$ with metric $\mathrm{d}r + r^2 h$.

2.4. **Symplectic structure at infinity.** In the sequel, we will view $\partial({}^{\mathrm{sc}}\overline{T}^*X)$ from two viewpoints: as the correct space for (scattering) microlocalization, and as the carrier of a natural symplectic structure, induced from the interior bundle $T^*(X^o)$. That is, the symplectic structure on $\widetilde{\mathcal{W}}$ should be obtained from the canonical symplectic 2-form ω defined in the interior. However, this 2-form blows up near the boundary ∂X, see [30]. Since we aim at connecting microlocal phenomena to the symplectic geometry of the underlying manifold, we need a suitable extension of the symplectic structure of $T^*(X^o)$ to $\partial({}^{\mathrm{sc}}\overline{T}^*X)$. This can be achieved by rescaling as follows, cf. [25, 39] and

[19, Sect. 21.1]. We set

$$\tilde{\alpha}^{\psi} := (\tilde{\eta}^2 \partial_{\tilde{\eta}}) \lrcorner\, \omega \text{ restricted to } \widetilde{\mathcal{W}}^{\psi},$$

$$\tilde{\alpha}^{e} := (\tilde{y}^2 \partial_{\tilde{y}}) \lrcorner\, \omega \text{ restricted to } \widetilde{\mathcal{W}}^{e}.$$

The 1-forms $\tilde{\alpha}^{\psi}$ and $\tilde{\alpha}^{e}$ turn out to be two contact forms. Before proceeding, we will establish a different viewpoint for these 1-forms.

Recall that T^*X is a vector bundle and, as such, a conical manifold. This yields a distinguished radial vector field ρ^{ψ}, corresponding to fibrewise dilation, given by $\rho^{\psi}(f)(x,\xi) = \partial_t f(x, t\xi)|_{t=1}$. In local coordinates (x,ξ), ρ^{ψ} is then given by $\sum \xi_j \cdot \partial_{\xi_j}$, which, under radial compactification in the fibre with $\tilde{\eta} = \frac{1}{|\xi|}$, becomes $\tilde{\eta}\partial_{\tilde{\eta}}$. It is well-known that insertion of ρ^{ψ} into ω yields the canonical 1-form α^{ψ}, which restricts to the canonical contact form on the co-sphere bundle. Thus, $\tilde{\alpha}^{\psi}$ corresponds to a rescaling of the canonical 1-form under the rescaling isomorphism ${}^{\mathrm{sc}}S^*X \cong S^*X$. We may obtain $\tilde{\alpha}^{e}$ by an analogous construction. Indeed, any choice of boundary defining function, and corresponding collar decomposition $X = [0,\epsilon) \times \partial X$, introduces a conical structure near the boundary of X. The associated *radial* vector field yields again a 1-form, of which $\tilde{\alpha}^{e}$ is the rescaling. In our model example $T^*\mathbb{R}^d$ with standard symplectic coordinates, these 1-forms correspond to a rescaling of the 1-forms

$$\alpha^{e} = -x \cdot \mathrm{d}\xi, \qquad \alpha^{\psi} = \xi \cdot \mathrm{d}x.$$

Lemma 2.1. *The differential* 1*-forms* $\tilde{\alpha}^{e}$ *and* $\tilde{\alpha}^{\psi}$ *do not depend on the choice of coordinates.*

Proof. We check the statement for $\tilde{\alpha}^{e}$, since it here represents the main new element of the symplectic structure at infinity, and the result for $\tilde{\alpha}^{\psi}$ can be obtained in a completely similar way (cfr. also [30]). Let $(\tilde{y}', y')$ be new coordinates inducing the same smooth structure. Then $(\tilde{y}', y')$ are related to $(\tilde{y}, y)$ by a diffeomorphism. Since $\tilde{y}$ and $\tilde{y}'$ are both boundary defining functions we have $\tilde{y}' = \tilde{y} f(\tilde{y}, y)$ for some smooth $f > 0$. Moreover, the fact that the metric has to take the form (1) for both of them implies that near the boundary $\tilde{y} = \tilde{y}' = 0$ we necessarily have

$$f^{-2}\left(f + \tilde{y}\left(\frac{\partial f}{\partial \tilde{y}}\right)\right) = 1 + \mathcal{O}(\tilde{y}).$$

We compute

$$\tilde{y}^2 \partial_{\tilde{y}} = f^{-2}\left(f + \tilde{y}\left(\frac{\partial f}{\partial \tilde{y}}\right)\right)(\tilde{y}')^2 \partial_{\tilde{y}'} + \frac{(\tilde{y}')^2}{f^2} \sum_{j=1}^{n} \left(\frac{\partial y'}{\partial \tilde{y}}\right) \partial_{y'}.$$

Therefore, $\tilde{y}^2 \partial_{\tilde{y}} = (\tilde{y}')^2 \partial_{\tilde{y}'}$, up to contributions from $\tilde{y}\,{}^{\mathrm{sc}}V$. As sections of ${}^{\mathrm{sc}}\overline{T}X$, the bundle obtained by ${}^{\mathrm{sc}}TX$ through radial compactification of the fibers, the latter vanish under restriction to the boundary $\tilde{y} = 0$. Hence, $\tilde{\alpha}^{e}$ does not depend on the choice of coordinates associated with the same scattering structure, as claimed. $\square$

We have then obtained two well-defined 1-forms, that describe the symplectic structure "at infinity" and at "co-infinity" in the wave front space, induced by the symplectic and scattering structures in the interior. In terms of contact geometry, this corresponds to the freedom of passing from a contact manifold to the associated symplectic cone and going back by contracting with the Liouville vector field ρ^e or ρ^ψ.

Recall, see e.g. [10, Section 3.7], that a conic submanifold L of $T^*X \subset \{0\}$ is called Lagrangian if $\alpha^\psi|_L \equiv 0$. Since α^e and α^ψ are defined by contraction with radial vector fields, they vanish on that radial vector field by antisymmetry of ω. Consequently, by applying the rescaling isomorphism for the tangential vector fields, we have also proved:

Lemma 2.2. *A compact submanifold $\tilde{L}$ of $\widetilde{\mathcal{W}}^\psi$ is Legendrian with respect to $\tilde{\alpha}^\psi$, that is, it satisfies $\tilde{\alpha}^\psi|_{{}^{\mathrm{sc}}T_{\tilde{L}}X} \equiv 0$, if and only if on the associated cone $L = \tilde{L} \times (0,\infty)$ we have $\alpha^\psi|_L = 0$. Correspondingly, a compact submanifold $\tilde{L}$ of $\widetilde{\mathcal{W}}^e$ is Legendrian with respect to $\tilde{\alpha}^e$ if and only if on the associated cone $L = (0,\infty) \times \tilde{L}$ we have $\alpha^e|_L = 0$.*

It follows that we may view such submanifolds either as *conic and Lagrangian* or *Legendrian* for the corresponding contact forms. In the next step we will generalize this notion to pairs of ψ- and e-Legendrian submanifolds *with boundary* that intersect in the corner. That is, we want to have a *Lagrangian structure across the corner*, as shown schematically[2] in Figure 2. We are now in the position to introduce

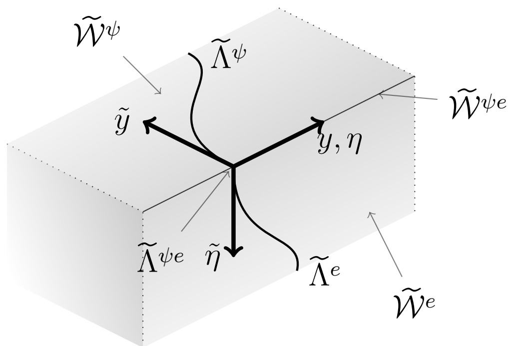

FIGURE 2. Intersection of $\widetilde{\Lambda}^\psi \subset \widetilde{\mathcal{W}}^\psi$ and $\widetilde{\Lambda}^e \subset \widetilde{\mathcal{W}}^e$ at the corner $\widetilde{\mathcal{W}}^{\psi e}$

such Legendrian submanifolds.

[2]The simplest non-trivial situation, in which this arises, is in 4 dimensions. In Figure 2, the boundary coordinates (y, η) were combined and drawn as one-dimensional.

Definition 2.3. A closed, embedded *Legendrian* (corner-crossing) *submanifold* $\widetilde{\Lambda} = (\widetilde{\Lambda}^e, \widetilde{\Lambda}^\psi)$ is a pair of closed, embedded submanifolds with boundary of $\partial({}^{\mathrm{sc}}\overline{T}^*X)$, such that

1. $(\widetilde{\Lambda}^e)^o \subset \widetilde{\mathcal{W}}^e$, $(\widetilde{\Lambda}^\psi)^o \subset \widetilde{\mathcal{W}}^\psi$,
2. $\dim(\widetilde{\Lambda}^e) = \dim(\widetilde{\Lambda}^\psi) = d-1$,
3. $(\widetilde{\Lambda}^e \cap \widetilde{\Lambda}^\psi) = \partial\widetilde{\Lambda}^e = \partial\widetilde{\Lambda}^\psi =: \widetilde{\Lambda}^{\psi e} \subset \widetilde{\mathcal{W}}^{\psi e}$ (with $\dim(\widetilde{\Lambda}^{\psi e}) = d-2$), with the intersection being clean,
4. The contact forms $\tilde{\alpha}^e$ and $\tilde{\alpha}^\psi$ satisfy
$$\tilde{\alpha}^e|_{\widetilde{\Lambda}^e} \equiv 0, \qquad \tilde{\alpha}^\psi|_{\widetilde{\Lambda}^\psi} \equiv 0.$$

Furthermore, we make the following assumptions, which will become clear in the later parts of the document:

5. $(\widetilde{\Lambda}^e)^o \cap (\partial X \times \{0\}) = \varnothing$,
6. $\widetilde{\Lambda}^{\psi e}$ is the conormal sphere to some submanifold $\tilde{S}$ of ∂X, namely, $\widetilde{\Lambda}^{\psi e} = \partial({}^{\mathrm{sc}}\overline{N}^*\tilde{S})$.

We call the associated pair $(\Lambda^e, \Lambda^\psi)$ of submanifolds, of $\mathcal{W}^e$ and $\mathcal{W}^\psi$ respectively, an SG-Lagrangian submanifold.

Remark 2.4. We remark that Definition 2.3 only covers the case where there is actually an intersection in the corner. The case where Λ is a compact Legendrian submanifold (w.r.t. one of the contact forms) without boundary in the corresponding face - or a finite collection thereof - is straightforward to formulate. Since this case is already well-studied in [19] and [30], respectively, we focus on the corner-crossing case in the remainder of the paper.

3. Phase functions and associated submanifolds

Having formulated what a Legendrian submanifold/pair of Lagrangian submanifolds is in our context, we now turn towards its parametrization. In this section, we will discuss the class of phase functions that may be used to parametrize the previously defined objects. We start by recalling elements from the classical theory.

3.1. **Introduction and motivation.** Assume for now that X is a compact manifold without boundary, and $\Lambda \subset T^*X \smallsetminus \{0\}$ is a Lagrangian submanifold conic in the fiber. Introduce local coordinates around a given point $x_0 \in X$ such that $(x_0, \xi_0) \in \Lambda$. Then, it is always possible to find a (real) phase function $\varphi(x,\theta) \in \mathscr{C}^\infty(\mathbb{R}^d \times (\mathbb{R}^s \smallsetminus \{0\}))$, that is, a smooth function, positively 1-homogeneous w.r.t. θ, that has no critical points, i.e. $d_x\varphi + d_\theta\varphi \neq 0$, which locally parametrizes Λ. This means that, in a suitable neighbourhood of (x_0, ξ_0), conic in ξ, we have

$$\Lambda \equiv \Lambda_\varphi = \{(x, \nabla_x\varphi(x,\theta)) \,|\, \nabla_\theta\varphi(x,\theta) = 0\}. \tag{3}$$

In fact, φ may be assumed non-degenerate, meaning that the map $\lambda_\varphi : (x,\theta) \mapsto (x, \nabla_x \varphi)$ is a local diffeomorphism

$$\{(x,\theta) \colon \nabla_\theta \varphi(x,\theta) = 0\} =: C_\varphi \to \Lambda.$$

In the following, we will reinterpret this analysis and establish a suitable analogue in the non-compact setting. The guiding idea is that the phase function φ, in the previous setting of compact manifolds without boundary, is actually determined, by homogeneity, by its restriction to $X \times \mathbb{S}^{s-1}$. Moreover, Λ is determined by its (Legendrian) intersection with the co-sphere bundle, $\Lambda \cap S^*X$. In local coordinates, we thus have a (local) correspondence of smooth, 1-homogenous, non-critical functions on $\mathbb{R}^d \times \mathbb{S}^{s-1}$ and Legendrian submanifolds in $\widetilde{\mathcal{W}}^\psi$. This motivates that a Legendrian corner-crossing submanifold should be locally parametrized by a smooth function $\tilde{\varphi}$ on a "suitable model corner" $\partial(\mathbb{B}^d \times \mathbb{B}^s)$. Revisiting Figure 2 the situation is schematically modelled in Figure 3.

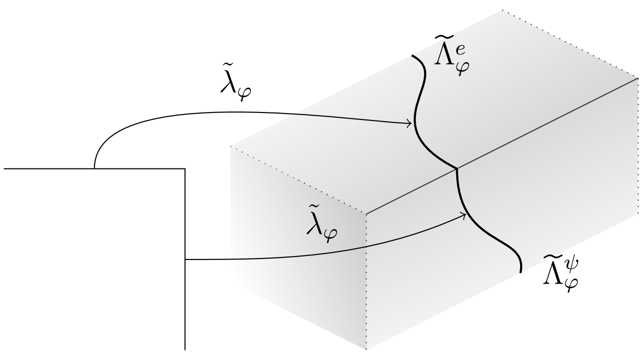

FIGURE 3. Parametrization of $\widetilde{\Lambda}_\varphi$ from a model corner

Smooth functions on such a model corner are given by pairs $\tilde{\varphi} = (\tilde{\varphi}^\psi, \tilde{\varphi}^e)$ of smooth function on the respective faces that are compatible in the corner. Actually, such a smooth function is the restriction of a smooth function on $\mathbb{B}^d \times \mathbb{B}^s$ to the boundary. This gives rise, under inverse radial compactification, to the classical SG symbols, also called scattering symbols.

3.2. **Classical** SG**-symbols and phase functions.**

Notation. In the sequel, we need to introduce some notation in order to distinguish the different faces and the corner in the (compactified) cotangent bundle and the space used for parametrization by a phase function. These different faces of manifolds and corners behave somewhat symmetrically. To avoid confusion when different spaces are involved, we make systematic use of the following notation:

- y denotes "variable-type" elements of $\mathbb{B}^d$, η denotes "co-variable-type" elements of $\mathbb{B}^d$, γ denotes "co-variable-type" elements of

$\mathbb{B}^s$; if we distinguish one variable as a boundary defining function, we mark it by adding a tilde to it;
- the corresponding elements of $\mathbb{R}^d$ are denoted by x and ξ and elements of $\mathbb{R}^s$ are named θ.

Subsets of $\mathbb{B}^d$ and $\mathbb{B}^s$ that correspond to subsets of $\mathbb{R}^d \sqcup (\mathbb{R}^d \setminus \{0\})$ or $\mathbb{R}^s \sqcup (\mathbb{R}^s \setminus \{0\})$ are usually denoted by the same symbol equipped with a tilde. We recall that the SG-*wave front space* is defined as $\widetilde{\mathcal{W}} := \partial(\mathbb{B}^d \times \mathbb{B}^d) = \widetilde{\mathcal{W}}^e \sqcup \widetilde{\mathcal{W}}^\psi \sqcup \widetilde{\mathcal{W}}^{\psi e}$, where

$$\widetilde{\mathcal{W}}^e := \mathbb{S}^{d-1} \times \left(\mathbb{B}^d\right)^o, \quad \widetilde{\mathcal{W}}^\psi := \left(\mathbb{B}^d\right)^o \times \mathbb{S}^{d-1}, \quad \widetilde{\mathcal{W}}^{\psi e} := \mathbb{S}^{d-1} \times \mathbb{S}^{d-1}. \tag{4}$$

In a completely similar fashion to the faces of $\widetilde{\mathcal{W}}$, substituting s in place of d in the dimensions of the second factors in (4), we define $\widetilde{\mathcal{B}} := \partial(\mathbb{B}^d \times \mathbb{B}^s) = \widetilde{\mathcal{B}}^e \sqcup \widetilde{\mathcal{B}}^\psi \sqcup \widetilde{\mathcal{B}}^{\psi e}$. We also set $\mathcal{W} = \mathcal{W}^e \sqcup \mathcal{W}^\psi \sqcup \mathcal{W}^{\psi e}$, with

$$\mathcal{W}^e := \mathbb{R}_+ \times \widetilde{\mathcal{W}}^e, \quad \mathcal{W}^\psi := \widetilde{\mathcal{W}}^\psi \times \mathbb{R}_+, \quad \mathcal{W}^{\psi e} := \mathbb{R}_+ \times \widetilde{\mathcal{W}}^\psi \times \mathbb{R}_+, \tag{5}$$

and, again with s in place of d in the dimensions of the second factors of (5), $\mathcal{B} := \mathcal{B}^e \sqcup \mathcal{B}^\psi \sqcup \mathcal{B}^{\psi e}$. Finally, we set $\mathcal{S} = \mathcal{S}^e \sqcup \mathcal{S}^\psi \sqcup \mathcal{S}^{\psi e}$, with

$$\mathcal{S}^e = \mathbb{S}^{d-1} \times \mathbb{R}^s, \quad \mathcal{S}^\psi = \mathbb{R}^d \times \mathbb{S}^{s-1}, \quad \mathcal{S}^{\psi e} = \mathbb{S}^{d-1} \times \mathbb{S}^{s-1}.$$

Moreover, we will use the symbol χ for any excision function, that is, a smooth function $\chi : \mathbb{R}^d \to [0,1]$ (or defined on $\mathbb{R}^s$) that equals 0 in a neighbourhood of the origin and is identically equal to 1 outside some compact set. When appropriate, we attach a label "e" or "ψ" to it, to emphasize the variables on which χ acts.

SG-*symbol classes.*The class of SG-symbols on $\mathbb{R}^d \times \mathbb{R}^s$ of order $(m_e, m_\psi) \in \mathbb{R}^2$ consists of those $a \in \mathscr{C}^\infty(\mathbb{R}^d \times \mathbb{R}^s)$ such that, for all $\alpha, \beta \in \mathbb{N}_0^d$, there exist $C_{\alpha\beta} > 0$ such that, for all $(x,\theta) \in \mathbb{R}^d \times \mathbb{R}^s$,

$$|\partial_x^\alpha \partial_\theta^\beta a(x,\theta)| \le C_{\alpha\beta} \langle x \rangle^{m_e - |\alpha|} \langle \theta \rangle^{m_\psi - |\beta|}.$$

Such symbol classes will be denoted by $\mathrm{SG}^{m_e,m_\psi}(\mathbb{R}^d \times \mathbb{R}^s)$. On a general (SG-admissible) manifold X, these are introduced by covering X with local coordinate patches that make X look like $\mathbb{R}^d$, see [25].

There is an important subclass of SG-symbols, denoted by $\mathrm{SG}_{\mathrm{cl}}^{m_e,m_\psi}(\mathbb{R}^d \times \mathbb{R}^s)$, which consists of those elements of $\mathrm{SG}^{m_e,m_\psi}(\mathbb{R}^d \times \mathbb{R}^s)$ that admit polyhomogeneous expansions (in both x and θ separately), see [12, 31, 41]. The important feature for our analysis is that these are precisely given, by radial compactification, as weighted smooth functions on the corresponding compactifications, see [12, 25, 41]. From here on, we will denote coordinates on $\mathbb{B}^s$, viewed as the radial compactification of $\mathbb{R}^s$, by $(\tilde{\gamma}, \gamma)$. We use the symbol ι for both the radial compactification map in $\mathbb{R}^d$ as well as $\mathbb{R}^s$.

Theorem 3.1 (Realization as smooth functions)**.** *For $(m_e, m_\psi) \in \mathbb{R} \times \mathbb{R}$, consider the map $\iota_{\mathrm{SG}}^{m_e,m_\psi}$ on $\mathrm{SG}_{\mathrm{cl}}^{m_e,m_\psi}(\mathbb{R}^d \times \mathbb{R}^s)$ given by*

$$a(x,\theta) \mapsto b(y,\gamma) := \tilde{y}^{m_e}\, \tilde{\gamma}^{m_\psi}\, [(\iota^{-1} \times \iota^{-1})^* a](\tilde{y}, y, \tilde{\gamma}, \gamma) = \tilde{y}^{m_e}\, \tilde{\gamma}^{m_\psi}\, \tilde{a}(\tilde{y}, y, \tilde{\gamma}, \gamma). \tag{6}$$

Then, (6) *extends to an isomorphism*

$$\iota_{\mathrm{SG}}^{m_e,m_\psi} : \mathrm{SG}_{\mathrm{cl}}^{m_e,m_\psi}(\mathbb{R}^d \times \mathbb{R}^s) \to \mathscr{C}^\infty(\mathbb{B}^d \times \mathbb{B}^s).$$

Remark 3.2. This means that $\tilde{a} = (\iota^{-1} \times \iota^{-1})^* a \in \tilde{y}^{-m_e} \tilde{\gamma}^{-m_\psi} \mathscr{C}^\infty(\mathbb{B}^d \times \mathbb{B}^s)$, and that under radial compactification we have a filtration-preserving isomorphism[3]. Namely,

$$\mathrm{SG}_{\mathrm{cl}}(\mathbb{R}^d \times \mathbb{R}^s) = \bigcup_{m_e,m_\psi} \mathrm{SG}_{\mathrm{cl}}^{m_e,m_\psi}(\mathbb{R}^d \times \mathbb{R}^s) \cong \bigcup_{m_e,m_\psi} \tilde{y}^{-m_e} \tilde{\gamma}^{-m_\psi} \mathscr{C}^\infty(\mathbb{B}^d \times \mathbb{B}^s).$$

Under this isomorphism, the polyhomogeneous expansions correspond to Taylor series in polar coordinates. In particular we may recover the *principal symbols.*

Definition 3.3. Let $a \in \mathrm{SG}_{\mathrm{cl}}^{m_e,m_\psi}(\mathbb{R}^d \times \mathbb{R}^s)$. The *principal symbol* of a is the triple of smooth functions $\sigma(a) = (\sigma^\psi(a), \sigma^e(a), \sigma^{\psi e}(a))$ on $\mathcal{B}^\psi$, $\mathcal{B}^e$ and on $\mathcal{B}^{\psi e}$, respectively, obtained by

$$\begin{aligned} \sigma^\psi(a)(x,\theta) &= \lim_{\lambda\to\infty} \lambda^{-m_\psi} a(x,\lambda\theta), \\ \sigma^e(a)(x,\theta) &= \lim_{\mu\to\infty} \mu^{-m_e} a(\mu x,\theta), \\ \sigma^{\psi e}(a)(x,\theta) &= \lim_{\mu\to\infty}\lim_{\lambda\to\infty} \lambda^{-m_\psi}\mu^{-m_e} a(\mu x,\lambda\theta). \end{aligned}$$

Lemma 3.4 (Properties of the principal symbol)**.** *Let* $a \in \mathrm{SG}_{\mathrm{cl}}^{m_e,m_\psi}(\mathbb{R}^d \times \mathbb{R}^s)$. *Then*

a) for any two excision functions χ^e *and* χ^ψ *we have*

$$a(x,\theta) - \chi^\psi \sigma^\psi(a) - \chi^e \sigma^e(a) + \chi^e \chi^\psi \sigma^{\psi e}(a) \in \mathrm{SG}_{\mathrm{cl}}^{m_e-1,m_\psi-1}(\mathbb{R}^d \times \mathbb{R}^s); \quad (7)$$

b) for all $(\alpha,\beta) \in \mathbb{N}_0^d \times \mathbb{N}_0^s$, $\bullet \in \{e,\psi,\psi e\}$ *we have*

$$\sigma^\bullet\left(\partial_x^\alpha \partial_\theta^\beta a\right)(x,\theta) = \partial_x^\alpha \partial_\theta^\beta (\sigma^\bullet a)(x,\theta);$$

c) the symbol a *is said to be* $\bullet$*-elliptic at a given point* (x,θ) *in the respective domain of* $\sigma^\bullet(a)$, $\bullet \in \{\psi,e,\psi e\}$, *if we have* $\sigma^\bullet(a)(x,\theta) \neq 0$*;*

d) under the isomorphism in Theorem 3.1, the principal symbol may be computed as the restriction of $\tilde{a}$ *to the respective boundary face, homogeneously continued into the interior, that is, we*

[3]We remark that this isomorphism may be used to equip $\mathrm{SG}_{\mathrm{cl}}$ with a Fréchet topology, see [12, 41].

have $\sigma^\bullet(a)|_{\mathcal{S}^\bullet}(x,\theta) = \tilde{a}|_{\widetilde{\mathcal{B}}^\bullet}(y,\gamma)$*; explicity, using polar coordinates,*

$$\sigma^\psi(a)(x,\theta) = |\theta|^{m_\psi}\tilde{a}\left(\iota(x);0,\frac{\theta}{|\theta|}\right),$$

$$\sigma^e(a)(x,\theta) = |x|^{m_e}\tilde{a}\left(0,\frac{x}{|x|};\iota(\theta)\right),$$

$$\sigma^{\psi e}(a)(x,\theta) = |x|^{m_e}|\theta|^{m_\psi}\tilde{a}\left(0,\frac{x}{|x|};0,\frac{\theta}{|\theta|}\right).$$

Remark 3.5. Notice that, in view of point d) in Lemma 3.4, equation (7) is nothing else than the fact that $\tilde{y}^{m_e}\tilde{\gamma}^{m_\psi}\tilde{a}$, subtracted by (any smooth continuation to the interior of) its restriction to the boundary, vanishes there. Similarly, ellipticity at a point in $\mathbb{R}^d\times\mathbb{R}^s$ is simply the non-vanishing of $\tilde{y}^{m_e}\tilde{\gamma}^{m_\psi}\tilde{a}$ at the corresponding point on $\widetilde{\mathcal{B}}$.

SG-Phase functions. We will now introduce the class of SG-symbols that may be used to parametrize Lagrangian submanifolds, that is, SG-phase functions. Such phase functions were introduced in [9], see also Section 6 below. We restrict our attention to phase functions of order $(1,1)$.[4]

Definition 3.6. An element of $\mathrm{SG}^{1,1}(\mathbb{R}^d\times\mathbb{R}^s)$ is called an (admissible, classical) SG-*phase function* near $(x_0,\theta_0)\in\mathcal{B}$ if it is real-valued and the associated function

$$\Phi(x,\theta) := \langle x\rangle^2\,|\nabla_x\varphi(x,\theta)|^2 + \langle\theta\rangle^2\,|\nabla_\theta\varphi(x,\theta)|^2 \tag{8}$$

is elliptic at (x_0,θ_0). We associate with a phase function the critical set

$$C_\varphi = \big\{(x_0,\theta_0)\in\mathcal{B} : |\nabla_\theta\varphi| \text{ is not elliptic at } (x_0,\theta_0)\big\}.$$

More precisely, we write $\varphi^\bullet = \sigma^\bullet(\varphi)$ and split C_φ into

$$\mathcal{C}^e_\varphi := \{(x_0,\theta_0)\in\mathcal{B}^e : \nabla_\theta\varphi^e(x_0,\theta_0) = 0\},$$

$$\mathcal{C}^\psi_\varphi := \{(x_0,\theta_0)\in\mathcal{B}^\psi : \nabla_\theta\varphi^\psi(x_0,\theta_0) = 0\},$$

$$\mathcal{C}^{\psi e}_\varphi := \{(x_0,\theta_0)\in\mathcal{B}^{\psi e} : \nabla_\theta\varphi^{\psi e}(x_0,\theta_0) = 0\}.$$

Remark 3.7. The principal symbol of a $\mathrm{SG}^{1,1}$-symbol φ is a triple of functions $\varphi^e(x,\theta)$, $\varphi^\psi(x,\theta)$ and $\varphi^{\psi e}(x,\theta)$, each 1-homogeneous in x, θ and both separately, respectively. It follows that φ^ψ is then a phase function in the ordinary sense, and the $\mathcal{C}^\psi_\varphi$-component coincides with the standard notion of critical set $\mathcal{C}_\varphi$ for a homogenous phase function φ^ψ.

[4] In view of Theorem 3.1, this is no real restriction, since any space $\mathrm{SG}^{(m_e,m_\psi)}_{\mathrm{cl}}$ is isomorphic to the smooth functions, and this isomorphism actually contains the geometric information.

We will now characterize $\mathcal{C}_\varphi$ in the boundary components on the compactified space $\widetilde{\mathcal{B}}$ by the compactification map ι.

Lemma 3.8. *The condition that the associated function* $\Phi = |\pi_{1,0} \cdot \nabla_x \varphi|^2 + |\pi_{0,1} \cdot \nabla_\theta \varphi|^2$ *is* SG*-elliptic of order* $(2,2)$ *is equivalent to the condition that* $\big(\tilde{\gamma}\widetilde{\nabla_x\varphi}, \tilde{y}\widetilde{\nabla_\theta\varphi}\big)$ *is not vanishing on the corresponding point* $\widetilde{\mathcal{B}}$. *We can write*

$$\widetilde{\mathcal{C}}_\varphi := \{(y_0,\gamma_0) \in \widetilde{\mathcal{B}} \colon \tilde{y}\widetilde{\nabla_\theta\varphi}(y_0,\gamma_0) = 0\},$$

for which we have

$$\widetilde{\mathcal{C}}^e_\varphi = (\iota \times \mathrm{id})\big(\mathcal{C}^e_\varphi \cap \mathcal{S}^e\big), \quad \widetilde{\mathcal{C}}^\psi_\varphi = (\mathrm{id} \times \iota)\big(\mathcal{C}^\psi_\varphi \cap \mathcal{S}^\psi\big), \quad \widetilde{\mathcal{C}}^{\psi e}_\varphi = \mathcal{C}^{\psi e}_\varphi \cap \mathcal{S}^{\psi e}.$$

Proof. By Lemma 3.4, Φ is elliptic if and only if $\iota^{2,2}_{\mathrm{SG}}(\Phi)$ is not vanishing at the corresponding point in $\widetilde{\mathcal{B}}$. We can also write,

$$\begin{aligned}\iota^{2,2}_{\mathrm{SG}}(\Phi)(y,\gamma) &= \tilde{\gamma}^2\tilde{y}^2\left(\langle x\rangle^2|\nabla_x\varphi(x,\theta)|^2 + \langle\theta\rangle^2|\nabla_\theta\varphi(x,\theta)|^2\right)\big|_{(x,\theta)=(\iota^{-1}(y),\iota^{-1}(\gamma))}\\ &= \left[|(\iota^{1,0}_{\mathrm{SG}}(\langle x\rangle)\cdot\iota^{0,1}_{\mathrm{SG}}(\nabla_x\varphi)|^2 + |(\iota^{0,1}_{\mathrm{SG}}(\langle\theta\rangle)\cdot\iota^{1,0}_{\mathrm{SG}}(\nabla_\theta\varphi)|^2\right](y,\gamma)\end{aligned}$$

Since $\langle x\rangle$ and $\langle\theta\rangle$ are elliptic, their images under $\iota^{1,0}_{\mathrm{SG}}$ and $\iota^{0,1}_{\mathrm{SG}}$ are nowhere vanishing, which proves the first assertion. The characterization of $\widetilde{\mathcal{C}}_\varphi$ follows by repeating the same argument for $|\langle\theta\rangle\cdot\nabla_\theta\varphi(x,\theta)|^2$, in view of Definition 3.6. □

Lemma 3.8 allows for us to write the image of C_φ as the null set of a smooth function. We will now show that $\widetilde{\mathcal{C}}_\varphi$ may be regarded as a pair of smooth manifolds in the boundary faces which intersect cleanly in the corner.

Definition 3.9 (Non-degenerate classical SG-phase functions)**.** Let $\varphi \in \mathrm{SG}^{1,1}_{\mathrm{cl}}(\mathbb{R}^d \times \mathbb{R}^s)$ be a classical SG-phase function. Then φ is called *non-degenerate* if the differentials $\left\{d\left(\widetilde{\tilde{y}\partial_{\theta_j}\varphi}|_X\right)\right\}_{j=1,\dots,s}$ form, for every $(y_0,\gamma_0) \in \widetilde{\mathcal{C}}_\varphi$, a set of linearly independent vectors in $T^*_{(y_0,\gamma_0)}(\widetilde{\mathcal{B}}^\bullet)$, for any choice of $\bullet \in \{e,\, \psi,\, \psi e\}$.

Each of the boundary faces $\widetilde{\mathcal{B}}^e$ and $\widetilde{\mathcal{B}}^\psi$ are submanifolds (with boundary) of the manifold with corners $\mathbb{B}^d \times \mathbb{B}^s$, that intersect cleanly at their joint boundary $\widetilde{\mathcal{B}}^{\psi e}$. That is, for every $(y_0,\gamma_0) \in \mathbb{S}^{d-1} \times \mathbb{S}^{s-1}$ we have

$$T_{(y_0,\gamma_0)}\widetilde{\mathcal{B}}^{\psi e} = T_{(y_0,\gamma_0)}\widetilde{\mathcal{B}}^e \cap T_{(y_0,\gamma_0)}\widetilde{\mathcal{B}}^\psi.$$

We recall that, by Lemma 3.8, $\widetilde{\mathcal{C}}_\varphi$ is the set of boundary elements (y_0,γ_0) jointly annihilated by $\tilde{y}\widetilde{\nabla_\theta\varphi}$, $j = 1,\dots,s$. From that we are able to obtain a similar set-up for the different components of $\widetilde{\mathcal{C}}_\varphi$, detailed in the next Proposition 3.10.

Proposition 3.10. *Let* $\varphi \in \mathrm{SG}^{1,1}_{\mathrm{cl}}(\mathbb{R}^d \times \mathbb{R}^s)$ *be a non-degenerate* SG*-phase function. Then, the following properties hold true.*

(1) *The different components of* $\widetilde{\mathcal{C}}_\varphi$ *are totally neat submanifolds of the corresponding boundary component in* $\mathbb{B}^d \times \mathbb{B}^s$. *That is, we have*

$$\widetilde{\mathcal{C}}_\varphi = \widetilde{\mathcal{C}}^e_\varphi \cup \widetilde{\mathcal{C}}^\psi_\varphi \subset \widetilde{\mathcal{B}}^e \cup \widetilde{\mathcal{B}}^\psi,$$

and their possible boundaries form a subset $\widetilde{\mathcal{C}}^{\psi e}_\varphi$ *of* $\widetilde{\mathcal{B}}^{\psi e}$.

(2) *The codimension of the respective component is (if non-empty) always* s, *meaning* $\dim(\widetilde{\mathcal{C}}^e_\varphi) = \dim(\widetilde{\mathcal{C}}^\psi_\varphi) = d-1$ *and* $\dim(\widetilde{\mathcal{C}}^{\psi e}_\varphi) = d-2$.

(3) *The tangent space to each face of* $\widetilde{\mathcal{C}}^\bullet_\varphi$ *in* $\widetilde{\mathcal{B}}^\bullet$ *may be calculated as*

$$\left\{v \in T_{(y_0,\gamma_0)}(\widetilde{\mathcal{B}}^\bullet) \,\Big|\, \left(d_{y,\gamma}\left(\tilde{y}\widetilde{\partial_{\theta_j}\varphi}\big|_{\widetilde{\mathcal{B}}^\bullet}\right)\right)v = 0 \;\; \forall j \in \{1,\dots,s\}\right\}.$$

(4) *The intersection* $\widetilde{\mathcal{C}}^\psi_\varphi \cap \widetilde{\mathcal{C}}^e_\varphi = \widetilde{\mathcal{C}}^{\psi e}_\varphi$ *is clean.*

Proof. Statements *(1)-(3)* are consequences of the regular value theorem for manifolds with corners, see Theorem A.17. Then, also the cleanness of the intersection follows. □

We now show how a non-degenerate SG-phase functions φ parametrizes a pair of associated submanifolds $\Lambda^\bullet_\varphi$ over its critical set $\mathcal{C}^\bullet_\varphi$. We seek to generalize (3). To that end, we set

Definition 3.11. Let $\varphi \in \mathrm{SG}^{1,1}_{\mathrm{cl}}(\mathbb{R}^d \times \mathbb{R}^s)$ be a classical SG-phase function. We define

$$\begin{aligned}
\Lambda^e_\varphi &:= \left\{((x, \nabla_x\varphi^e(x,\theta)) \,\big|\, \exists\, (x,\theta) \in \mathcal{B}^e : \; \nabla_\theta\varphi^e(x,\theta) = 0\right\},\\
\Lambda^\psi_\varphi &:= \left\{\left((x, \nabla_x\varphi^\psi(x,\theta)\right) \,\big|\, \exists\, (x,\theta) \in \mathcal{B}^\psi : \; \nabla_\theta\varphi^\psi(x,\theta) = 0\right\},\\
\Lambda^{\psi e}_\varphi &:= \left\{\left((x, \nabla_x\varphi^{\psi e}(x,\theta)\right) \,\big|\, \exists\, (x,\theta) \in \mathcal{B}^{\psi e} : \; \nabla_\theta\varphi^{\psi e}(x,\theta) = 0\right\}.
\end{aligned}$$

The problem of Definition 3.11 is that it is hard to extract geometric insight "at infinity". In this "limit", $\Lambda^e_\varphi = \Lambda^\psi_\varphi = \Lambda^{\psi e}_\varphi$, but this is hard to define for manifolds that are not even submanifolds of the same space. In order to overcome this difficulty, we pass again to the compactified space.

We may first look at the map $\lambda_\varphi : \mathbb{R}^d \times \mathbb{R}^s \to \mathbb{R}^d \times \mathbb{R}^d$ given by $(x,\theta) \mapsto (x, \nabla_x\varphi(x,\theta))$. This is a map whose components are $\mathrm{SG}^{1,0}$ and $\mathrm{SG}^{0,1}$-symbols, respectively. We want to find an analogue to this function on $(\mathbb{B}^d)^o \times (\mathbb{B}^s)^o \to (\mathbb{B}^d)^o \times (\mathbb{B}^d)^o$ that extends it to (parts of) the boundary that becomes an isomorphism suitably close to $\widetilde{\mathcal{C}}_\varphi$. We start by considering the map

$$(y,\gamma) \mapsto \left(\iota^{-1}(y),\, \widetilde{\nabla_x\varphi}(y,\gamma)\right) \hat{=} (x, \widetilde{\nabla_x\varphi}(x,\theta)),$$

defined on $(\mathbb{B}^d)^o \times (\mathbb{B}^s)^o$. We may compactify the image space to $\mathbb{B}^d \times \mathbb{B}^d$, by means of the map $\iota \times \iota$, to look at the extension of

$$\tilde{\lambda}_\varphi\big|_{(\mathbb{B}^d)^o \times (\mathbb{B}^s)^o} = (\iota \times \iota) \circ \left((\iota^{-1} \times \iota^{-1})^* \lambda_\varphi\right) \tag{9}$$

to the subset

$$
\begin{aligned}
&\widetilde{\mathcal{E}} = ((\mathbb{B}^d)^o \times (\mathbb{B}^s)^o) \sqcup \widetilde{\mathcal{B}}^e \sqcup \widetilde{\mathcal{B}}_{\text{ell}},\\
&\widetilde{\mathcal{B}}_{\text{ell}} = \{(y_0,\gamma_0) \in \widetilde{\mathcal{B}}^\psi \cup \widetilde{\mathcal{B}}^{\psi e} \colon |\nabla_x \varphi|^2 \text{ is elliptic at } (y_0,\gamma_0)\}.
\end{aligned}
\tag{10}
$$

Remark 3.12. This construction may be visualized through the following commuting diagram:

$$
\begin{array}{ccc}
\widetilde{\mathcal{E}} & \xrightarrow{\quad \tilde{\lambda}_\varphi \quad} & \mathbb{B}^d \times \mathbb{B}^d \\
\uparrow & & \uparrow \\
(\mathbb{B}^d)^o \times (\mathbb{B}^s)^o & \xrightarrow{\quad \tilde{\lambda}_\varphi \quad} & (\mathbb{B}^d)^o \times (\mathbb{B}^d)^o \\
{\scriptstyle \iota^{-1} \times \iota^{-1}} \downarrow & & \uparrow {\scriptstyle \iota \times \iota} \\
\mathbb{R}^d \times \mathbb{R}^s & \xrightarrow{\quad \lambda_\varphi \quad} & \mathbb{R}^d \times \mathbb{R}^s
\end{array}
$$

Indeed, we know by Theorem 3.1 that the map $(\iota^{1,0}_{\text{SG}} \times \iota^{0,1}_{\text{SG}})\lambda_\varphi : \mathbb{B}^d \times \mathbb{B}^s \to \mathbb{B}^d \times \mathbb{R}^d$ given by

$$
(y,\gamma) \mapsto \big(y, \tilde{\gamma}\, \widetilde{\nabla_x \varphi}(y,\gamma)\big)
\tag{11}
$$

is smooth up to the boundary. We will show that, close to the boundary components of $\widetilde{\mathcal{E}}$, this property yields the desired extension of $\tilde{\lambda}_\varphi$.

Proposition 3.13. *$\tilde{\lambda}_\varphi$ defined on $(\mathbb{B}^d)^o \times (\mathbb{B}^s)^o$ by* (9)*, can be extended as a smooth map to the subset $\widetilde{\mathcal{E}} \subset \mathbb{B}^d \times \mathbb{B}^s$ defined in* (10)*.*

Proof. Since ι is a diffeomorphism, it is clear that $\tilde{\lambda}_\varphi$ is smooth in the interior, i.e. on $(\mathbb{B}^d)^o \times (\mathbb{B}^s)^o$. So, it is enough that we look at (9) for $|y|, |\gamma| > 2/3$. It is also clear that we have to prove the existence of the extension only for the second component of $\tilde{\lambda}_\varphi$, since the first one coincides with pr_1, the projection on the first set of variables, which is of course smoothly extendable from the interior to the whole of $\mathbb{B}^d \times \mathbb{B}^s$.

By Theorem 3.1 and Lemma 3.4, we have, for a vector-valued symbol $p \in \text{SG}^{-1,1}$,

$$
\begin{aligned}
\iota(\widetilde{\nabla_x \varphi}(y,\gamma)) &= \iota\left(\nabla_x \varphi^e\left(\frac{y}{|y|}(1-|y|)^{-1}, \frac{\gamma}{|\gamma|}(1-|\gamma|)^{-1}\right) + \tilde{p}(y,\gamma)\right)\\
&= \iota\left(\nabla_x \varphi^e\left(\frac{y}{|y|}, \frac{\gamma}{|\gamma|}(1-|\gamma|)^{-1}\right) + \tilde{p}(y,\gamma)\right).
\end{aligned}
\tag{12}
$$

Then, $\tilde{\lambda}_\varphi$ can be extended smoothly to

$$
A_1 = \{y \in \mathbb{B}^d \colon 2/3 < |y| \le 1\} \times \{\gamma \in \mathbb{B}^s \colon |\gamma| < r'\},
$$

with arbitrary r', $1 > r' > 2/3$. In fact, this is clearly true for the first term appearing in the argument of ι in the right hand side of (12).

For the second term, it is enough to observe that, by Theorem 3.1, for any $p \in \mathrm{SG}^{-1,1}$, $\tilde{p} \in \tilde{y}\tilde{\gamma}^{-1}\mathscr{C}^\infty(\mathbb{B}^d \times \mathbb{B}^s)$, that is, also $\tilde{p}$ is smooth on A_1. Moreover, the values of both such extensions to A_1 remain bounded, and ι is smooth on $\mathbb{R}^d$. This implies that $\tilde{\lambda}_\varphi$ can be smoothly extended to any point in $\widetilde{\mathcal{B}}^e$.

We now consider the subset of $\mathbb{B}^d \times \mathbb{B}^s$ given by

$$A_2 = \{y \in \mathbb{B}^d : 2/3 < |y| < 1\} \times \{\gamma \in \mathbb{B}^s : |\gamma| > r\},$$

$r' > r > 2/3$, so that, of course, $\widetilde{\mathcal{B}}_{\mathrm{ell}} \subset A_2$. Observe that, by Lemma 3.4 (in fact by its analogue for vector-valued symbols),

$$\begin{aligned}\iota(\widetilde{\nabla_x\varphi}(y,\gamma)) &= \iota\left(\nabla_x\varphi^\psi\left(\frac{y}{|y|}(1-|y|)^{-1}, \frac{\gamma}{|\gamma|}(1-|\gamma|)^{-1}\right) + \tilde{q}(y,\gamma)\right) \\ &= \iota\left(\nabla_x\varphi^\psi\left(\frac{y}{|y|}(1-|y|)^{-1}, \frac{\gamma}{|\gamma|}\right)(1-|\gamma|)^{-1} + \tilde{q}(y,\gamma)\right).\end{aligned} \tag{13}$$

$\tilde{q}$ can be extended smoothly to $\mathbb{B}^d \times \mathbb{B}^s$, since, by Theorem 3.1, for any $q \in \mathrm{SG}^{0,0}_{\mathrm{cl}}$, $i^{0,0}_{\mathrm{SG}}(q) = \tilde{q} \in \mathscr{C}^\infty(\mathbb{B}^d \times \mathbb{B}^s)$. At points $(y_0,\gamma_0) \in \widetilde{\mathcal{B}}_{\mathrm{ell}}$, we have

$$\begin{aligned}\text{either}\quad & (y_0,\gamma_0) \in \widetilde{\mathcal{B}}^\psi \text{ and } \nabla_x\varphi^\psi(\iota^{-1}(y_0),\gamma_0) \neq 0,\\ \text{or}\quad & (y_0,\gamma_0) \in \widetilde{\mathcal{B}}^{\psi e} \text{ and } \nabla_x\varphi^{\psi e}(y_0,\gamma_0) \neq 0.\end{aligned}$$

In the former case, the norm of the first term in the argument of ι in the right hand side of (13) tends to $+\infty$ when $|\gamma| \nearrow 1$. Then, sufficiently close to (y_0,γ_0) we have

$$\iota(\widetilde{\nabla_x\varphi}) = \frac{\widetilde{\nabla_x\varphi}}{|\widetilde{\nabla_x\varphi}|}\left(1 - \frac{1}{|\widetilde{\nabla_x\varphi}|}\right) = \frac{\tilde{\gamma}\widetilde{\nabla_x\varphi}}{|\tilde{\gamma}\widetilde{\nabla_x\varphi}|}\left(1 - \frac{\tilde{\gamma}}{|\tilde{\gamma}\widetilde{\nabla_x\varphi}|}\right), \tag{14}$$

where $\tilde{\gamma}\widetilde{\nabla_x\varphi} = \iota^{0,1}_{\mathrm{SG}}(\nabla_x\varphi)$ is smooth up to the boundary. Moreover,

$$\tilde{\gamma}\widetilde{\nabla_x\varphi}(y,\gamma) = \tilde{\gamma}(\iota^{-1}\times\iota^{-1})^*\nabla_x\varphi(y,\gamma) = \nabla_x\varphi^\psi\left(\frac{y}{|y|}(1-|y|)^{-1}, \frac{\gamma}{|\gamma|}\right) + \tilde{\gamma}\cdot\tilde{q}(y,\gamma),$$

so such an expression cannot vanish close to (y_0,γ_0), since $|\nabla_x\varphi^\psi(\iota^{-1}(y_0),\gamma_0)| = k > 0$ and $|\tilde{\gamma}\cdot\tilde{q}(y,\gamma)| < k/2$ for $(y,\gamma) \in V$, suitably small neighborhood of (y_0,γ_0), by $|\tilde{\gamma}(\gamma_0)\cdot\tilde{q}(y_0,\gamma_0)| = 0$. Then the smooth extendability of (14) to points in $\widetilde{\mathcal{B}}_{\mathrm{ell}} \cap \widetilde{\mathcal{B}}^\psi$ follows.

The remaining case, that is, the result for $(y_0,\gamma_0) \in \widetilde{\mathcal{B}}_{\mathrm{ell}} \cap \widetilde{\mathcal{B}}^{\psi e}$, follows in a similar way, writing

$$\begin{aligned}\iota(\widetilde{\nabla_x\varphi}(y,\gamma)) &= \iota((\iota^{-1}\times\iota^{-1})^*\nabla_x\varphi(y,\gamma)) \\ &= \iota\left(\nabla_x\varphi^{\psi e}\left(\frac{y}{|y|}(1-|y|)^{-1}, \frac{\gamma}{|\gamma|}(1-|\gamma|)^{-1}\right) + \tilde{p}(y,\gamma) + \tilde{q}(y,\gamma)\right) \\ &= \iota\left(\nabla_x\varphi^{\psi e}\left(\frac{y}{|y|}, \frac{\gamma}{|\gamma|}\right)(1-|\gamma|)^{-1} + \tilde{p}(y,\gamma) + \tilde{q}(y,\gamma)\right),\end{aligned}$$

with $p \in \mathrm{SG}^{-1,1}$, $q \in \mathrm{SG}^{0,0}$ and $\nabla_x \varphi(y_0, \gamma_0) \neq 0$, so that

$$\tilde{\gamma}(\widetilde{\nabla_x \varphi}(y,\gamma) = \nabla_x \varphi^{\psi e}\left(\frac{y}{|y|}, \frac{\gamma}{|\gamma|}\right) + \tilde{\gamma} \cdot \tilde{p}(y,\gamma) + \tilde{\gamma} \cdot \tilde{q}(y,\gamma),$$

with the last two terms smoothly extendable to (y_0, γ_0) and vanishing there.

The proof is complete. □

Remark 3.14. Observe that, in view of the assumption (8) on Φ, $\tilde{\lambda}_\varphi$ is well defined in a neighborhood of $\widetilde{\mathcal{C}}_\varphi$. In fact, at points $(y_0, \gamma_0) \in \widetilde{\mathcal{C}}_\varphi$ we necessarily have $\tilde{\gamma} \nabla_y \tilde{\varphi}(y_0, \gamma_0) \neq 0 \Leftrightarrow |\nabla_x \varphi|^2$ is elliptic at (y_0, γ_0). Since this is equivalent to the fact that $\iota^{0,1}_{\mathrm{SG}}(\nabla_x \varphi)$ does not vanish at (y_0, γ_0), the same holds, by continuity, in a neighborhood of (y_0, γ_0) in $\widetilde{\mathcal{B}}$.

Finally, we can state in which sense a non-degenerate phase function may parametrize a pair of Lagrangian submanifolds.

Definition 3.15. Let $\varphi \in \mathrm{SG}^{1,1}_{\mathrm{cl}}(\mathbb{R}^d \times \mathbb{R}^s)$ be a classical SG-phase function. Then we set $\widetilde{\Lambda}_\varphi := \tilde{\lambda}_\varphi(\widetilde{\mathcal{C}}_\varphi)$.
For a given Legendrian submanifold $\widetilde{\Lambda}$ we say that φ parametrizes $\widetilde{\Lambda}$ near some $p \in \widetilde{\Lambda}$ if we have, in a neighbourhood $\widetilde{U}$ of p in $\widetilde{\mathcal{W}}$, that $\widetilde{\Lambda} = \widetilde{\Lambda}_\varphi$ or, equivalently, if $\Lambda = (\Lambda^e, \Lambda^\psi)$ is the corresponding pair of Lagrangian submanifolds, we have $\Lambda^\bullet \cap U^\bullet = \Lambda^\bullet_\varphi \cap U^\bullet$, $\bullet \in \{e, \psi\}$, in the associated[5] neighbourhoods U^e and U^ψ.

Remark 3.16. Notice that if p is a corner point, U^e and U^ψ will necessarily be unbounded (asymptotically conic) in both variables, that is we have a local parametrization "up to infinity".

From the smoothness of $\tilde{\lambda}_\varphi$ up to the boundary in a neighborhood of $\widetilde{\mathcal{C}}_\varphi$, we now obtain a statement similar to Lemma 3.8 for $\widetilde{\Lambda}_\varphi$.

[5]Associated under inverse radial compactification.

Proposition 3.17. *Let* $\varphi \in \mathrm{SG}^{1,1}_{\mathrm{cl}}(\mathbb{R}^d \times \mathbb{R}^s)$ *be a non-degenerate* SG-*phase function. Then, the following properties hold true.*

(1) *The different components of* $\widetilde{\Lambda}_\varphi$ *are each totally neat, immersed submanifolds of the corresponding boundary component* $\mathbb{B}^d \times \mathbb{B}^s$. *That is, we have*

$$\widetilde{\Lambda}_\varphi = \underbrace{\widetilde{\Lambda}^e_\varphi}_{\subset \widetilde{\mathcal{W}}^e} \cup \underbrace{\widetilde{\Lambda}^\psi_\varphi}_{\subset \widetilde{\mathcal{W}}^\psi},$$

and their possible boundaries form a subset $\widetilde{\Lambda}^{\psi e}_\varphi$ *of* $\widetilde{\mathcal{W}}^{\psi e}$.

(2) *The codimension of the respective component is always* s, *meaning* $\dim(\widetilde{\Lambda}^e_\varphi) = \dim(\widetilde{\Lambda}^\psi_\varphi) = d-1$ *and (if non-empty)* $\dim(\widetilde{\Lambda}^{\psi e}_\varphi) = d-2$.

(3) *The tangent space to each face of* $\widetilde{\Lambda}^\bullet_\varphi$ *in* $\widetilde{\mathcal{W}}$ *may be calculated by means of the differential of* $\tilde{\lambda}_\varphi$, *that is, via*

$$T\widetilde{\Lambda}^\bullet_\varphi = \left(d\left(\tilde{\lambda}_\varphi|_{\widetilde{\mathcal{C}}^\bullet_\varphi}\right)\right) T\widetilde{\mathcal{C}}^\bullet_\varphi, \quad \bullet \in \{e, \psi\}.$$

(4) *The intersection* $\widetilde{\Lambda}^\psi_\varphi \cap \widetilde{\Lambda}^e_\varphi = \widetilde{\Lambda}^{\psi e}_\varphi$ *is clean.*

Proof. By the non-degeneracy of φ, $\tilde{\lambda}_\varphi$ is an immersion near $\widetilde{\mathcal{C}}_\varphi$, and we may use Theorem A.14. □

Again, Figure 2 provides a schematic visualization of the geometric situation. We now check that $\widetilde{\Lambda}^\psi$ is truly the analogue of Λ_φ under radial compactification.

Lemma 3.18. *Let* $\varphi \in \mathrm{SG}^{1,1}_{\mathrm{cl}}(\mathbb{R}^d \times \mathbb{R}^s)$ *be a classical* SG-*phase function. We have*

$$\widetilde{\Lambda}^e_\varphi = (\mathrm{id} \times \iota)\big(\Lambda^e_\varphi \cap \mathcal{S}^e\big), \quad \widetilde{\Lambda}^\psi_\varphi = (\iota \times \mathrm{id})\big(\Lambda^\psi_\varphi \cap \mathcal{S}^\psi\big), \quad \widetilde{\Lambda}^{\psi e}_\varphi = \Lambda^{\psi e}_\varphi \cap \mathcal{S}^{\psi e}.$$

Proof. We start with the proof for Λ^ψ_φ, which coincides with the classical definition of the manifold of stationary points for a classical homogeneous phase function. We have $\widetilde{\Lambda}^\psi_\varphi = \tilde{\lambda}_\varphi(\widetilde{\mathcal{C}}^\psi_\varphi)$. By Lemma 3.8 we have $\tilde{y}\widetilde{\nabla_\theta\varphi}(y,\gamma) = 0$ on $\widetilde{\mathcal{C}}_\varphi$. Thus, in view of the same Lemma, $\tilde{\gamma}\widetilde{\nabla_x\varphi}(y,\gamma) \neq 0$. Recalling (14) from the proof of Proposition 3.13 and using the fact that $\tilde{\gamma}$ vanishes on $\widetilde{\mathcal{C}}^\psi_\varphi$ and Lemma 3.4, we can write

$$(\iota^{-1} \times \mathrm{id})(\widetilde{\Lambda}^\psi_\varphi) = \left\{ \left((\iota^{-1}(y), \frac{\tilde{\gamma}\widetilde{\nabla_x\varphi}(y,\gamma)}{|\tilde{\gamma}\widetilde{\nabla_x\varphi}(y,\gamma)|} \right) \middle| (y,\gamma) \in \widetilde{\mathcal{C}}^\psi_\varphi \right\},$$

and the cone over it as

$$(\iota^{-1} \times \mathrm{id})(\widetilde{\Lambda}^\psi_\varphi) \times \mathbb{R}_+ = \left\{ \left((x, \mu \frac{\nabla_x\varphi^\psi(x,\theta)}{|\nabla_x\varphi^\psi(x,\theta)|} \right) \middle| (x,\theta) \in \mathcal{C}^\psi_\varphi, \mu > 0 \right\}.$$

Making use of the homogeneity of φ^ψ, we may write this simply as

$$(\iota^{-1} \times \Gamma)(\widetilde{\Lambda}^\psi_\varphi) =$$
$$= \left\{ \big((x, \nabla_x\varphi^\psi(x,\theta) \big) : (x,\theta) \in \mathbb{R}^d \times (\mathbb{R}^s \smallsetminus 0) \mathrm{and} \nabla_\theta\varphi^\psi(x,\theta) = 0 \right\},$$

which is the definition of Λ^{ψ}_{φ}, as claimed. In the same way we can write

$$\begin{aligned}\mathbb{R}_+ \times (\mathrm{id} \times \iota^{-1})(\widetilde{\Lambda}^e_{\varphi}) &= \mathbb{R}_+ \times \left[(\mathrm{id} \times \iota^{-1})\tilde{\lambda}_{\varphi}\right](\widetilde{\mathcal{C}}^e_{\varphi}) \\ &= \left\{\left(\mu y, \widetilde{\nabla_x \varphi}(y,\gamma)\right) : (y,\gamma) \in \widetilde{\mathcal{C}}^e_{\varphi}\right\} \\ &= \left\{(x, \nabla_x \varphi^e(x,\theta)) : (x,\theta) \in \mathcal{C}^e_{\varphi}\right\},\end{aligned}$$

where we have again made use of Lemma 3.4. The characterization of the corner component $\Lambda^{\psi e}_{\varphi}$ follows in exactly the same way. □

Remark 3.19. Note that in the classical theory, also clean phase functions are permitted to parametrize Lagrangian submanifolds, see [19], in which case λ_{φ} is locally a fibration of a fixed dimension, called the excess e. In our case, this would give rise to complicated geometric structures, such as (compactified) fibrations over manifolds with corners. While there are tools available to also treat these, see [24, 28], we omit such complications here, and will address the question of excess phase parameters and the elimination thereof in future works on the calculus of Lagrangian distributions.

3.3. **Lagrangian properties of the components and their parametrization.** So far, we have only stated how phase function parametrize a submanifold, but have not actually discussed its Lagrangian properties. We will now prove:

Theorem 3.20. *Let $\varphi \in \mathrm{SG}^{1,1}_{\mathrm{cl}}(\mathbb{R}^d \times \mathbb{R}^s)$ be a non-degenerate* SG*-phase function. Then $\widetilde{\Lambda}_{\varphi} = (\widetilde{\Lambda}^e_{\varphi}, \widetilde{\Lambda}^{\psi}_{\varphi})$ is an* SG*-Legendrian submanifold of $\widetilde{\mathcal{W}}$ in the sense of Definition 2.3.*

We start by checking the symplectic properties. For a classical phase function, Λ^{ψ} is well-known to be Lagrangian. We will now obtain an analogous statement for Λ^e.

Lemma 3.21. *Let φ be a non-degenerate classical* SG*-phase function. Then α^e vanishes on Λ^e_{φ}. As a consequence, $\tilde{\alpha}^e$ vanishes on $\widetilde{\Lambda}^{\psi e}$.*

Remark 3.22. We remark that, to our best knowledge, Lemma 3.21 indeed requires its own proof, and cannot be simply "deduced by symmetry" from the classical theory, due to the "asymmetrical definition" of Λ_{φ} with respect to x and θ.

Proof. We adopt here the notation in [10], and denote the induced coordinates on $T_x M$ by δx. We first notice that Λ^e_{φ} is, by definition, the image of

$$\mathcal{C}^e_{\varphi} = \{(x_0,\theta_0) \in \mathbb{R}^d \setminus \{0\} \times \mathbb{R}^s \colon \nabla_{\xi} \varphi^e(x_0,\theta_0) = 0\},$$

which, by the non-degeneracy of φ, is a smooth manifold, under the map $\lambda^e_{\varphi} = (\mathrm{pr}_1, \nabla_x \varphi^e)$. We can thus calculate its tangent space in terms of that of the preimage.[6] $T_{(x,\theta)}\mathcal{C}^e_\varphi$ is given by

$$(\delta x \cdot \nabla_x)\nabla_\theta\varphi + (\delta\theta \cdot \nabla_\theta)\nabla_\theta\varphi = 0, \tag{15}$$

and we thus have

$$T_{(x,\nabla_x\varphi^e(x,\theta))}\Lambda^e_\varphi = J(\mathrm{pr}_1, \nabla_x\varphi^e) \cdot T_{(x,\theta)}\mathcal{C}^e_\varphi,$$

where $J(\mathrm{pr}_1, \nabla_x\varphi^e)$ denotes the Jacobian matrix of the map $(\mathrm{pr}_1, \nabla_x\varphi^e)$. Furthermore,

$$J_{(x,\theta)}(\mathrm{pr}_1, \nabla_x\varphi^e)(\delta x, \delta\theta) = (\delta x, (\delta x \cdot \nabla_x)\nabla_x\varphi^e + (\delta\theta \cdot \nabla_\theta)\nabla_x\varphi^e). \tag{16}$$

Computing $\alpha^e = x \cdot d\xi$ on such a vector, we see that

$$\begin{aligned} x \cdot (\delta x \cdot \nabla_x)\nabla_x\varphi^e &+ x \cdot (\delta\theta \cdot \nabla_\theta)\nabla_x\varphi^e \\ &= \sum_{j,k} x_j(\delta x_k \partial_{x_k})\partial_{x_j}\varphi^e + \sum_j (\delta\theta \cdot \nabla_\theta)x_j\partial_{x_j}\varphi^e \\ &= \sum_{j,k}(\delta x_k\partial_{x_k})x_j\partial_{x_j}\varphi^e - \sum_k \delta x_k\partial_{x_k}\varphi^e + \sum_j(\delta\theta\cdot\nabla_\theta)x_j\partial_{x_j}\varphi^e. \end{aligned} \tag{17}$$

Since φ^e is 1-homogeneous in the first set of variables, by Euler's theorem for homogeneous functions (17) is equal to

$$\begin{aligned} &\sum_k(\delta x_k\partial_{x_k})\varphi^e - \sum_k \delta x_k\partial_{x_k}\varphi^e + (\delta\theta\cdot\nabla_\theta)\varphi^e, \\ &= \delta\theta \cdot (\nabla_\theta\varphi^e) \overset{(x,\theta)\in\mathcal{C}^e_\varphi}{=} 0 \end{aligned}$$

This proves the assertion. □

Finally, we observe the additional properties that these kind of submanifolds, arising from SG-classical phase functions, possess, which limit their behaviour at infinity.

Lemma 3.23. *Let* $\varphi \in \mathrm{SG}^{1,1}_{\mathrm{cl}}(\mathbb{R}^d \times \mathbb{R}^s)$ *be a non-degenerate classical* SG*-phase function. Then,*

(1) the pairing $\langle x, \xi\rangle$ *vanishes on* $\Lambda^{\psi e}_\varphi$*, meaning that* $\widetilde{\Lambda}_\varphi$ *is contained in the conormal to its base* $\mathrm{pr}_1(\widetilde{\Lambda}^\psi)$*;*
(2) Λ^e_φ *does not intersect* $(\mathbb{R}^d \setminus \{0\}) \times \{0\}$.

Proof. On $\Lambda^{\psi e}$ we have, by Euler's theorem for homogeneous functions applied twice,

$$\langle x, \xi\rangle = \langle x, \nabla_x\varphi^{\psi e}(x,\theta)\rangle = \varphi^{\psi e}(x,\theta) = \theta \cdot \nabla_\theta\varphi^{\psi e}(x,\theta) = 0.$$

The second assertion follows from the characterization of Λ^e_φ in Lemma 3.18, since the assumption on Φ in Definition 3.6 implies that if $\nabla_\theta\varphi^e(x,\theta) = 0$ we have $\nabla_x\varphi^e(x,\theta) \neq 0$. □

[6] As in Lemma 2.3.2 of [10], we can conclude from (15) and (16) that $(\mathrm{pr}_1, \nabla_x\varphi^e)$ is an immersion, and thus its image is an immersed d-dimensional conic submanifold.

We have then proved Theorem 3.20, meaning that every non-degenerate classical SG-phase function gives rise to an associated SG-Lagrangian submanifold. We are now ready to prove our main result, namely, that it is always possible to find a SG-classical phase function to locally[7] parametrize a given SG-Lagrangian.

Theorem 3.24 (Parametrization Theorem)**.** *Let $\Lambda = (\Lambda^e, \Lambda^\psi)$ be an* SG-*Lagrangian submanifold. Then, Λ is locally parametrizable by a non-degenerate* SG-*classical phase function. That is, $\forall (y_0, \eta_0) \in \widetilde{\Lambda}$ there exist*

(1) a neighbourhood $\widetilde{U}$ of (y_0, η_0) in $\mathbb{B}^d \times \mathbb{B}^d$,
(2) an open set $\widetilde{V} \subset \mathbb{B}^d \times \mathbb{B}^s$,
(3) a function $\tilde{\varphi} \in \tilde{\gamma}^{-1}\tilde{y}^{-1}\mathscr{C}^\infty(\tilde{V})$ such that the corresponding (locally defined) phase function $\varphi = (\iota \times \iota)^* \tilde{\varphi}$ is non-degenerate,

such that

$$\widetilde{\Lambda} \cap \widetilde{U} = \tilde{\lambda}_\varphi \left(\left\{ (y_0, \gamma_0) \in \tilde{V} \cap \widetilde{\mathcal{B}} \colon (y_0, \gamma_0) \in \widetilde{\mathcal{C}}_\varphi \right\} \right).$$

Remark 3.25. We will say, for short, that both the functions $\tilde{\varphi}$ and $\varphi = (\iota \times \iota)^* \tilde{\varphi}$, satisfying the properties stated in Theorem 3.24, are *(local) non-degenerate* SG-*phase functions*, associated with the Legendrian $\widetilde{\Lambda}$ and/or the corresponding SG-Lagrangian Λ.

4. Proof of the Parametrization Theorem

We will only consider the case where $(y_0, \eta_0) \in \widetilde{\Lambda}^{\psi e}$, since the other possible situations are far simpler and will be covered by the same argument. The outline of the proof is classical, cf. [15] and [19], but here some tools from the theory of manifolds with corners are essential to achieve the result, as well as the extension of $\tilde{\lambda}_\varphi$ and the symplectic structure "at infinity" discussed in Section 2.

Let $(y_0, \eta_0) \in \widetilde{\Lambda}^{\psi e}$. $\widetilde{\Lambda}^{\psi e}$ is a $(d-2)$-dimensional embedded submanifold of $\mathbb{S}^{d-1} \times \mathbb{S}^{d-1}$ and we may assume, possibly after a rearrangement of variables in a neighbourhood $\tilde{U}$ of (y_0, η_0), that $\widetilde{\Lambda}^{\psi e}$ is parametrized as

$$\tilde{U} \cap \widetilde{\Lambda}^{\psi e} = \left\{ y', y'', \sqrt{1 - (y')^2 - (y'')^2}, \sqrt{1 - (\eta')^2 - (\eta'')^2}, \eta', \eta'' \right\},$$

where, for some $s \le d-1$, we have that $\eta' = (\eta_2, \dots, \eta_s)$ and $y'' = (y_{s+1}, \dots, y_{d-1})$ are independent variables and the remaining variables,

$$y' = \tilde{Y}^{\psi e}(y'', \eta'), \qquad \eta'' = \tilde{H}^{\psi e}(y'', \eta'),$$

are smoothly dependent on (y'', η'). We may further assume that y_d and η_1 do not vanish in the chosen coordinate neighbourhood, that is we have, for some $1 \ge c > 0$, $y_d > c$ and $\eta_1 > c$.

Due to the clean intersection at the corner $\widetilde{\Lambda}^{\psi e} = \widetilde{\Lambda}^e \cap \widetilde{\Lambda}^\psi = \partial \widetilde{\Lambda}^e = \partial \widetilde{\Lambda}^\psi$,

[7] Notice that "locally" near the corner component means "up to infinity", which is where the difficulty of the theory lies.

that is $T_{\widetilde{\Lambda}^{\psi e}}\widetilde{\Lambda}^e \cap T_{\widetilde{\Lambda}^{\psi e}}\widetilde{\Lambda}^\psi = T\widetilde{\Lambda}^{\psi e}$, we may find, accordingly, parametrizations of $\widetilde{\Lambda}^e$ and $\widetilde{\Lambda}^\psi$ near the corner point (y_0, η_0), namely

$$\tilde{U} \cap \widetilde{\Lambda}^e = \left\{ y', y'', \sqrt{1-(y')^2-(y'')^2}, \eta_1, \eta', \eta'' \right\},$$
$$\tilde{U} \cap \widetilde{\Lambda}^\psi = \left\{ y', y'', y_d, \sqrt{1-(\eta')^2-(\eta'')^2}, \eta', \eta'' \right\}.$$

Here we have the independent coordinates (y'', η_1, η') on $\widetilde{\Lambda}^e$ and (y'', y_d, η') on $\widetilde{\Lambda}^\psi$. The remaining variables on $\tilde{U} \cap \widetilde{\Lambda}^\psi$ may be written as functions smooth up to the boundary,

$$y' = \tilde{Y}^e(y'', \eta_1, \eta'), \quad \eta'' = \tilde{H}^e(y'', \eta_1, \eta'),$$

and on $\tilde{U} \cap \widetilde{\Lambda}^\psi$ as

$$y' = \tilde{Y}^\psi(y'', y_d, \eta'), \quad \eta'' = \tilde{H}^\psi(y'', y_d, \eta').$$

By $\widetilde{\Lambda}^e \cap \widetilde{\Lambda}^\psi = \partial\widetilde{\Lambda}^e = \partial\widetilde{\Lambda}^\psi = \widetilde{\Lambda}^{\psi e}$ we conclude that, if

$$\left(\eta_1, \eta', \tilde{H}^e(y'', \eta_1, \eta')\right) \in \mathbb{S}^{d-1} \text{ and } \left(\tilde{Y}^\psi(y'', y_d, \eta'), y'', y_d\right) \in \mathbb{S}^{d-1},$$

we have

$$\tilde{Y}^e(y'', \eta_1, \eta') = \tilde{Y}^\psi(y'', y_d, \eta') = \tilde{Y}^{\psi e}(y'', \eta'), \tag{18}$$

$$\tilde{H}^e(y'', \eta_1, \eta') = \tilde{H}^\psi(y'', y_d, \eta') = \tilde{H}^{\psi e}(y'', \eta'). \tag{19}$$

This choice of coordinates induces coordinates on the associated conic manifolds $\Lambda^e = \mathbb{R}_+ \times \widetilde{\Lambda}^e$ and $\Lambda^\psi = \widetilde{\Lambda}^\psi \times \mathbb{R}_+$. That is, we may take, as independent variables on Λ^e,

$$x'' = (\mu y'', \mu\sqrt{1-(y')^2-(y'')^2}), \qquad \xi' = \iota^{-1}(\eta_1, \eta').$$

In particular, x'' may be defined implicitly in terms of the map

$$(y'', \mu) \mapsto \left(\mu(\mathrm{id}\times\iota)^*\tilde{Y}^e(y'', \xi'), \mu y'', \mu\sqrt{1-((\mathrm{id}\times\iota)^*\tilde{Y}^e(y'', \xi'))^2-(y'')^2}\right).$$

We obtain that $x' = \mu(\mathrm{id}\times\iota)^*\tilde{Y}^e(y'', \xi') =: X^e(x'', \xi')$ is a smooth function of x'' and ξ' and polyhomogeneous in ξ', of maximal degree 0. By $|(x', x'')| = \mu$ it is further 1-homogeneous in x''. Similarly we have that

$$\xi'' = \iota^{-1}\left((\mathrm{id}\times\iota)^*\tilde{H}^e(y'', \xi')\right) =: \Xi^e(x'', \xi')$$

is 0-homogeneous in x'' and polyhomogeneous in ξ'. We can thus write, locally near $(x_0, \xi_0) = (\mathrm{id}\times\iota^{-1})(y_0, \eta_0)$,

$$\Lambda^e = \left\{ \left(X^e(x'', \xi'), x''; \xi', \Xi^e(x'', \xi')\right) \right\}.$$

In the same way we may write, in coordinates

$$x'' = \iota^{-1}(y'', y_d), \qquad \xi' = (\mu\eta_1, \mu\eta'),$$

that

$$\Lambda^\psi = \left\{ \left(X^\psi(x'', \xi'), x''; \xi', \Xi^\psi(x'', \xi')\right) \right\}.$$

We now define phase functions parametrizing these conic submanifolds in the given neighbourhoods. We set

$$\phi^e(x,\xi) = \langle x',\xi'\rangle + \langle x'', \Xi^e(x'',\xi')\rangle, \tag{20}$$

$$\phi^\psi(x,\xi) = \langle x',\xi'\rangle - \langle X^\psi(x'',\xi'),\xi'\rangle. \tag{21}$$

By the above definitions of Ξ^e and X^ψ we observe that ϕ^e is 1-homogeneous in x and 1-polyhomogeneous in ξ, whereas ϕ^ψ is 1-homogeneous in ξ and polyhomogeneous in x. In fact these functions, restricted to (suitable neighbourhoods in) $\mathbb{S}^{d-1}\times\mathbb{R}^d$ and $\mathbb{R}^d\times\mathbb{S}^{d-1}$, respectively, may be written as

$$\phi^e(x,\xi)|_{\mathbb{S}^{d-1}\times\mathbb{R}^d} = (\mathrm{id}\times\iota)^*\underbrace{\left(\left\langle(y',y'',y_d)\,,\,\iota^{-1}\left(\eta_1,\eta',\tilde H^e(y'',\eta_1,\eta')\right)\right\rangle\right)}_{=:\tilde y\cdot\tilde\phi^e|_{\widetilde{\mathcal{W}}^e}}, \tag{22}$$

$$\phi^\psi(x,\xi)|_{\mathbb{R}^d\times\mathbb{S}^{d-1}} = (\iota\times\mathrm{id})^*\underbrace{\left(\left\langle\iota^{-1}(y') - \iota^{-1}\left(\tilde Y^\psi(y'',y_d,\eta')\right),(\eta_1,\eta')\right\rangle\right)}_{=:\tilde\eta\cdot\tilde\phi^\psi|_{\widetilde{\mathcal{W}}^\psi}}. \tag{23}$$

Using $\iota^{-1}(y) = \frac{y}{|y|}(1-|y|)^{-1} = \tilde y^{-1}\frac{y}{|y|}$ for large arguments and Theorem 3.1, we obtain the desired symbol properties.
We now show that ϕ^e and ϕ^ψ may be obtained as the respective principal symbol components of a single SG-phase function. To this aim, we calculate the principal symbols of ϕ^e and ϕ^ψ by means of the proof of Lemma 3.4. Using $\lim_{n\to\infty}\tilde{y_n}\,\iota^{-1}(y_n) = \frac{y}{|y|}$ in case $y_n\to y$ with $y_n\in(\mathbb{B}^d)^o$ and $y\in\mathbb{S}^{d-1}$ as well as (18) and (19) in (22) and (23) we obtain in the corner component

$$\sigma_\psi(\phi^e)|_{\mathbb{S}^{d-1}\times\mathbb{S}^{d-1}} = (\mathrm{id}\times\mathrm{id})^*\left\langle(y',y'',y_d),\left(\eta_1,\eta',\tilde H^{\psi e}(y'',\eta')\right)\right\rangle,$$
$$\sigma_e(\phi^\psi)|_{\mathbb{S}^{d-1}\times\mathbb{S}^{d-1}} = (\mathrm{id}\times\mathrm{id})^*\left\langle y' - \tilde Y^{\psi e}(y'',\eta'),(\eta_1,\eta')\right\rangle,$$

and thus we have

$$\sigma_\psi(\phi^e)|_{\mathbb{S}^{d-1}\times\mathbb{S}^{d-1}} - \sigma_e(\phi^\psi)|_{\mathbb{S}^{d-1}\times\mathbb{S}^{d-1}} = \\ (\mathrm{id}\times\mathrm{id})^*\left(\left\langle\tilde Y^{\psi e}(y'',\eta'),(\eta_1,\eta')\right\rangle + \left\langle(y'',y_d),\tilde H^{\psi e}(y'',\eta')\right\rangle\right),$$

which is nothing else than $\langle x,\,\xi\rangle$ restricted to $\mathbb{S}^{d-1}\times\mathbb{S}^{d-1}$ in $\Lambda^{\psi e}$ and thus vanishes by the conormality assumption. We are then able, using (18) and (19), Lemma 3.4 and Remark 3.5, to continue (ϕ^e,ϕ^ψ) to a single SG-symbol with principal symbol $(\phi^e,\phi^\psi,\phi^{\psi e})$.
To have a chance of non-degeneracy, we first reduce the number of phase variables since, so far, the resulting phase function is constant in the ξ''-variables. Getting rid of these redundant variables, we may define $\varphi:\mathbb{R}^d\times\mathbb{R}^s\to\mathbb{R}$ by $((x',x'');\theta)\mapsto\phi((x',x'');(\theta,\xi''_0))$ for some arbitrary ξ''_0. We then obtain the components of the principal symbol $\varphi^\bullet = \sigma^\bullet(\varphi)$ for $\bullet\in\{e,\psi,\psi e\}$ and may define $\tilde\varphi\in\tilde\gamma^{-1}\tilde y^{-1}\mathscr{C}^\infty(\tilde U)$ via

$(\iota^{-1} \times \iota^{-1})^* \varphi$.

We now have to see that the functions $\varphi^\bullet$ indeed parametrize Λ_φ. For that we gather, by $\alpha^\bullet|_{\Lambda^\bullet} = 0$, the identities

$$\begin{aligned} X^e(x'', \xi') + \nabla_{\xi'} \left(x'' \cdot \Xi^e(x'', \xi')\right) &= 0, \\ x'' \cdot \partial_{x''_j} \Xi^e(x'', \xi') &= 0, \qquad j \in \{s+1, \dots, d\}, \\ \theta \cdot \partial_{\xi'_k} X^\psi(x'', \xi') &= 0, \qquad k \in \{1, \dots, s\}, \\ \nabla_{x''} \big(\theta \cdot X^\psi(x'', \xi')\big) + \Xi^\psi(x'', \xi') &= 0. \end{aligned}$$

We may then use these to compute, using (20) and (21),

$$\begin{aligned} \nabla_\theta \varphi^e(x, \theta) &= x' + \underbrace{x'' \cdot \nabla_\theta \Xi^e(x'', \theta)}_{= -X^e(x'', \theta)}, \\ \partial_{\theta_k} \varphi^\psi(x, \theta) &= (x'_k - X^\psi_k(x'', \theta)) - \underbrace{\big(\partial_{\theta_k} X^\psi(x'', \theta)\big) \cdot \theta}_{=0}. \end{aligned}$$

We therefore have $\nabla_\theta \varphi^\bullet = 0$ if and only if $x' = X^\bullet(x'', \theta)$, and we have obtained

$$\mathcal{C}^\bullet_\varphi = \{\big(X^\bullet(x'', \theta), x''; \theta\big)\}, \quad \bullet \in \{e, \psi\}.$$

In a similar fashion, using the remaining two identities,

$$\Lambda^\bullet_\varphi = \Big\{\big(X^\bullet(x'', \theta), x''; \theta, \Xi^\bullet(x'', \theta)\big)\Big\} = \Lambda^\bullet, \quad \bullet \in \{e, \psi\}.$$

We can thus (locally) parametrize $\Lambda^\bullet$ by $\varphi^\bullet$, $\bullet \in \{e, \psi\}$. Finally, we have to check that the symbol φ actually defines a phase function in the sense of Definition 3.6, which means $\sigma^\bullet(\Phi) \neq 0$ on $\mathcal{B}^\bullet$, $\bullet \in \{e, \psi\}$. By assumption, $\nabla_\theta \varphi^\bullet$ vanishes only on $\mathcal{C}^\bullet_\varphi$, $\bullet \in \{e, \psi\}$. There, however, we always have $\nabla_x \varphi^\bullet \neq 0$, $\bullet \in \{e, \psi\}$, since, by assumption, none of the faces of Λ_φ contains a point of the form $(x, 0)$.

This concludes the proof of Theorem 3.24. □

5. Equivalence of phase functions

Having established that we can always find a (local) non-degenerate SG-phase function parametrizing any SG-Lagrangian, we now investigate when two such phase functions may be considered *equivalent*. Here we rely again on the identification provided in Theorem 3.1

Theorem 5.1. *Let $\tilde\varphi_1$, $\tilde\varphi_2 \in \tilde\gamma^{-1}\tilde y^{-1}\mathscr{C}^\infty(\mathbb{B}^d \times \mathbb{B}^s)$ be two non-degenerate* SG*-phase functions that parametrize the same* SG*-Legendrian $\widetilde\Lambda \subset \widetilde{\mathcal{W}}$ in a neighbourhood of $(y_0, \eta_0) \in \widetilde\Lambda$ such that*

(1) there exists $(y_0, \gamma_{0,1}) \in \widetilde{\mathcal{C}}_{\varphi_1}$ and $(y_0, \gamma_{0,2}) \in \widetilde{\mathcal{C}}_{\varphi_2}$ such that[8]

$$(y_0, \eta_0) = \tilde\lambda_{\varphi_i}(y_0, \gamma_{0,i}) \text{ and } \tilde\varphi_1(y_0, \gamma_{0,1}) = \tilde\varphi_2(y_0, \gamma_{0,2}),$$

[8] We note that this is always fulfilled in the classical case since, by homogeneity, φ_i vanishes on $\mathcal{C}_{\varphi_i}$, $i = 1, 2$.

(2) The matrices

$$\left(\tilde\gamma^{-1}\tilde y\,\widetilde{\partial^2_{\theta_j\theta_k}\varphi_1|_X}\right)_{j,k=1,\dots,s} \quad \textit{and} \quad \left(\tilde\gamma^{-1}\tilde y\,\widetilde{\partial^2_{\theta_j\theta_k}\varphi_2|_X}\right)_{j,k=1,\dots,s}$$

have the same signature at $(y_0,\gamma_{0,i}) \in \widetilde{\mathcal{C}}_{\varphi_i}$, *where* $\varphi_i := (\iota\times\iota)^*\tilde\varphi_i$ *are the (locally defined) phase functions associated with* $\tilde\varphi_i$, $i = 1,2$.

Then, there exists a local diffeomorphism[9] $\tilde\kappa$ *of the boundary* $\widetilde{\mathcal{B}} \mapsto \widetilde{\mathcal{B}}$ *that is defined in a neighbourhood of* $(y_0,\gamma_{0,2})$ *in the corresponding faces, which is smooth on each face and such that* $\tilde\varphi_2\circ\tilde\kappa = \tilde\varphi_1|_{\widetilde{\mathcal{B}}}$. *In this case,* $\tilde\varphi_1$ *and* $\tilde\varphi_2$ *are called equivalent phase-functions.*

Remark 5.2. Note that the statement only ensures that the principal symbols of the corresponding phase functions φ_i may be arranged to agree, that is, the triples $(\varphi_i^e,\varphi_i^\psi,\varphi_i^{\psi e})$, $i=1,2$. This is, however, not a drawback, since only the principal symbols of φ_i, $i=1,2$, are used in the definition of $\widetilde\Lambda_\varphi$ and $\widetilde{\mathcal{C}}_\varphi$.

Proof of Theorem 5.1. We assume $(y_0,\eta_0)\in\widetilde\Lambda^{\psi e}$ since again this case (with slight adaptations) includes the others. Indeed, the case of $\widetilde\Lambda^\psi_\varphi$ is known from the classical theory and our proof follows the classical outline of [16] and [10]. We begin by arranging $\tilde\varphi_1$ and $\tilde\varphi_2$ such that they agree "up to second order" on $\widetilde{\mathcal{C}}_{\varphi_1}$. Consider the maps $\tilde\Phi_1$, $\tilde\Phi_2$ given by

$$(y,\gamma)\mapsto\tilde\Phi_i(y,\gamma) := (\tilde\lambda_{\varphi_i},\tilde y\widetilde{\nabla_\theta\varphi_i})\in\mathbb{B}^d\times\mathbb{B}^d\times\mathbb{R}^d.$$

By Theorem 3.1 and Proposition 3.13, these maps are well-defined and smooth up to the boundary in a neighbourhood of $\widetilde{\mathcal{C}}_{\varphi_i}$. By Lemma 3.8 and Lemma 3.15 we have, for $(y,\gamma)\in\widetilde{\mathcal{B}}$,

$$(\mathrm{pr}_3\circ\tilde\Phi_i)(y,\gamma)=0 \iff (y,\gamma)\in\widetilde{\mathcal{C}}_{\varphi_i} \iff \tilde\Phi_i(y,\gamma)\in\widetilde\Lambda\times\{0\},\ i=1,2.$$

By the implicit function theorem on manifolds with corners, that is, Theorem A.10, and the non-degeneracy assumption of $\tilde\varphi_i$, $i=1,2$, we may thus locally invert, in each closed face $\widetilde{\mathcal{B}}^\psi\cup\widetilde{\mathcal{B}}^{\psi e}=\mathbb{B}^d\times\mathbb{S}^{s-1}$ and $\widetilde{\mathcal{B}}^e\cup\widetilde{\mathcal{B}}^{\psi e}=\mathbb{S}^{d-1}\times\mathbb{B}^s$ separately, to obtain two maps defined in neighbourhoods of $(y_0,\eta_{0,i},0)$, namely,

$$\widetilde\Psi_i^\psi : (\widetilde{\mathcal{W}}^\psi\cup\widetilde{\mathcal{W}}^{\psi e})\times\mathbb{R}^d\to\mathbb{B}^d\times\mathbb{S}^{s-1},$$
$$\widetilde\Psi_i^e : (\widetilde{\mathcal{W}}^e\cup\widetilde{\mathcal{W}}^{\psi e})\times\mathbb{R}^d\to\mathbb{S}^{d-1}\times\mathbb{B}^s,$$

such that

$$\widetilde\Psi_i^\bullet\circ\left(\tilde\Phi_i|_{\widetilde{\mathcal{B}}^\bullet}\right)=\mathrm{id}_{\widetilde{\mathcal{B}}^\bullet},\quad \bullet\in\{e,\psi\}, i=1,2.$$

That is, we have the diagrams, for $\bullet\in\{e,\psi\}$, $i=1,2$,

[9] In the sense of manifolds with boundary, meaning it is the restriction of a diffeomorphism of surrounding extensions, see [24].

$$\begin{array}{ccccc}
(y_0,\gamma_{0,i}) & \in & \widetilde{\mathcal{C}}^{\bullet}_{\varphi_i} \cup \widetilde{\mathcal{C}}^{\psi e}_{\varphi_i} & \subset & \widetilde{\mathcal{B}}^{\bullet} \cup \widetilde{\mathcal{B}}^{\psi e} \\
\widetilde{\Psi}^{\bullet}_i \Big\uparrow \Big\downarrow \widetilde{\Phi}^{\bullet}_i & & \widetilde{\Psi}^{\bullet}_i \Big\uparrow \Big\downarrow \widetilde{\Phi}^{\bullet}_i & & \widetilde{\Psi}^{\bullet}_i \Big\uparrow \Big\downarrow \widetilde{\Phi}^{\bullet}_i \\
(y_0,\eta_0,0) & \in & \widetilde{\Lambda}^{\bullet} \times \{0\} & \subset & (\widetilde{\mathcal{W}}^{\bullet} \cup \widetilde{\mathcal{W}}^{\psi e}) \times \mathbb{R}^d
\end{array}$$

Notice that the last column is only meant locally, since, in general, we cannot achieve a global definition of $\widetilde{\Phi}^{\bullet}_i$ and $\widetilde{\Psi}^{\bullet}_i$, $\bullet \in \{e,\psi\}$, $i = 1,2$. However, in a neighbourhood of $(y_0,\eta_{0,i},0)$ in $\widetilde{\mathcal{W}}^{\psi e} \times \mathbb{R}^d$, we have

$$\widetilde{\Psi}^{\psi}_i|_{\widetilde{\mathcal{W}}^{\psi e}\times\mathbb{R}^d} = \widetilde{\Psi}^{e}_i|_{\widetilde{\mathcal{W}}^{\psi e}\times\mathbb{R}^d},\ i = 1,2.$$

We also note that $\mathrm{pr}_1 \circ \tilde{\lambda}_{\varphi_i} = \mathrm{id}$, $i = 1,2$. Therefore, the compositions $\widetilde{\Psi}^{\bullet}_1 \circ \big(\tilde{\Phi}_2|_{\widetilde{\mathcal{S}}^{\bullet}}\big)$ induce a diffeomorphism $\tilde{\kappa}$, which on each face is given by

$$\tilde{\kappa}^{\bullet}\colon \widetilde{W}^{\bullet} \subseteq \widetilde{\mathcal{C}}^{\bullet}_{\varphi_2} \longrightarrow \widetilde{\mathcal{C}}^{\bullet}_{\varphi_1}\colon (y,\gamma_2) \mapsto \big(y,\gamma_1(y,\gamma_2)\big),$$

where $\widetilde{W}^{\bullet}$ is a neighbourhood of (y_0,γ_0) in $\widetilde{\mathcal{C}}^{\bullet}_{\varphi_2}$, $\bullet \in \{e,\psi\}$. We then define the new (local) phase function

$$\tilde{f} := \begin{cases} \tilde{\varphi}_2 \circ \tilde{\kappa}^{e} & (y,\gamma) \in \widetilde{\mathcal{B}}^{e} \\ \tilde{\varphi}_2 \circ \tilde{\kappa}^{\psi} & (y,\gamma) \in \widetilde{\mathcal{B}}^{\psi}. \end{cases}$$

This yields a continuous function on the boundary $\widetilde{\mathcal{B}}$ that is smooth in the interior of each boundary face up to the corner. Since $\tilde{f}$ and $\tilde{\varphi}_2$ are related by the diffeomorphism $\tilde{\kappa}$, we may continue our analysis by replacing $\tilde{\varphi}_2$ with $\tilde{f}$. If we thus look at the principal symbol of this phase function, by means of Lemma 3.4, we see that $\tilde{f}$ agrees (at the boundary) up to second order with $\tilde{\varphi}_1$ on $\widetilde{\mathcal{C}}_{\varphi_1}$. In fact, their differentials vanish there, and both functions agree at the point $(y_0,\gamma_{0,1})$.

We may now essentially argue as in [16] on each of the two faces. Indeed, since all the involved objects are smooth up to the boundary of each face, Seeley's Extension Theorem allows us to extend them smoothly to a *mirror copy* of $\widetilde{\mathcal{B}}^{\bullet}$, $\bullet \in \{e,\psi\}$, across $\widetilde{S}^{\psi e}$. It is then possible to consider Taylor expansions around points in $\widetilde{\mathcal{B}}^{\psi e}$.

Let now $\tilde{\varphi}$ and $\tilde{\chi}$ be two non-degenerate SG-phase functions parametrizing the same Legendrian and agreeing up to second order on $\widetilde{\mathcal{C}}_{\varphi} = \widetilde{\mathcal{C}}_{\psi}$, up to the boundary, in the sense above. Using the non-degeneracy of $\tilde{\varphi}$, setting $\tilde{h}_j = \tilde{y}\widetilde{\partial_{\theta_j}\varphi}(y,\gamma)$, $j = 1,\dots,s$, we can write, at any given point in $\widetilde{\mathcal{C}}_{\varphi}$,

$$\tilde{y}\tilde{\gamma}\tilde{\chi}(y,\gamma) = \tilde{y}\tilde{\gamma}\tilde{\varphi}(y,\gamma) + \sum_{j,k=1}^{s} \tilde{b}_{jk}(y,\gamma)\tilde{h}_j\tilde{h}_k,$$

for a symmetric matrix $\tilde{B} = (\tilde{b}_{jk}(y,\gamma))$. The non-degeneracy of $\tilde{\chi}$ is then equivalent to

$$\det(I + \tilde{B}\tilde{A}) \neq 0 \text{ at } (y_0,\gamma_0),$$

where we have set $\tilde{A} = \left(\tilde{\gamma}^{-1}\tilde{y}\,\widetilde{\partial^2_{\theta_j\theta_k}\varphi}(y,\gamma)\right)_{j,k=1,\dots,s}$. When $\tilde{B}$ is sufficiently small, we can show the equivalence between $\tilde{\chi}$ and $\tilde{\varphi}$. In fact, by Taylor expansion,

$$\tilde{y}\tilde{\gamma}\tilde{\varphi}(y,\delta) = \tilde{y}\tilde{\gamma}\tilde{\varphi}(y,\gamma) + \sum_{j=1}^{s}(\delta_j - \gamma_j)\tilde{\gamma}\widetilde{\partial_{\theta_j}\varphi}(y,\gamma) + \sum_{j,k=1}^{s}\tilde{c}_{jk}(y,\gamma,\delta)(\delta_j-\gamma_j)\,(\delta_k-\gamma_k),$$

with a symmetric matrix $\tilde{C} = (\tilde{c}_{jk})_{j,k=1,\dots,s}$. Setting

$$\delta_j = \gamma_j + \sum_{k=1}^{s}\tilde{w}_{jk}(y,\gamma)\,h_k,$$

we prove the assertion if we show that there exist a matrix $\tilde{W} = (\tilde{w}_{j,k})_{j,k=1,\dots,s}$ such that

$$\tilde{W} + {}^t\tilde{W}\,\tilde{C}\,\tilde{W} = \tilde{B}.$$

It is well known that, under the condition that the signatures of $\tilde{A}$ and $\tilde{C}$ agree, this equation has a solution for small $\tilde{B}$, which holds true in our cases, in view of the hypothesis *(2)* and the fact that the two phase functions agree on $\widetilde{\mathcal{C}_\varphi}$. The statement then follows, by determining a continuous family of non-degenerate phase functions $\tilde{\chi}_t$, $t \in [0,1]$, such that $\tilde{\chi}_0 = \tilde{\varphi}$ and $\tilde{\chi}_1 = \tilde{\chi}$. In fact, two elements $\tilde{\chi}_s$ and $\tilde{\chi}_t$ of such a family will be equivalent for $|s-t|$ sufficiently small. Since the procedure can be performed separately on the two faces, and $\tilde{\chi}$ and $\tilde{\varphi}$ agree to second order up to the boundary including the corner, they are equivalent also there. The remaining details of this analysis, with reference to [16], are left to the reader. □

6. Tempered oscillatory integrals

In this section we give a brief summary of the results we obtained in [9], to provide an example of how the previously introduced geometric structures arise in the study of tempered distributions. In [9] we associated with a given (inhomogeneous) SG-phase function φ a family of tempered distributions, denoted by $I_\varphi(a)$, parametrized by amplitudes that are SG-symbols.

Theorem 6.1. *With any fixed admissible* SG*-phase function* φ *of order* $(1,1)$ *we may associate a map*

$$I_\varphi : \mathrm{SG}(\mathbb{R}^d \times \mathbb{R}^s) \to \mathscr{S}'(\mathbb{R}^d),$$

uniquely determined by the the following properties:

(1) $a \mapsto I_\varphi(a)$ *is a linear map,*

(2) If $a \in \mathscr{S}(\mathbb{R}^d \times \mathbb{R}^s)$, then $I_\varphi(a)$ coincides with the (absolutely convergent) integral

$$I_\varphi(a) = \int_{\mathbb{R}^s} e^{i\varphi(x,\theta)}\, a(x,\theta)\, d\theta, \tag{24}$$

(3) the restriction of I_φ to $\mathrm{SG}^{m_e,m_\psi}(\mathbb{R}^d \times \mathbb{R}^s)$ is a continuous map

$$\mathrm{SG}^{m_e,m_\psi}(\mathbb{R}^d \times \mathbb{R}^s) \to \mathscr{S}'(\mathbb{R}^d).$$

We call the resulting distribution $I_\varphi(a)$ a SG*-oscillatory integral.*

For these families of tempered oscillatory integrals we proved an inclusion for their so-called SG-*wave front set*, which generalizes the corresponding statement valid for Hörmander's wave front set $\mathrm{WF}_{\mathrm{cl}}(u)$ and the usual class of oscillatory integrals, see [16].[10] In order to state our result in the SG setting, we first recall the definition of the SG-wave front set.

Definition 6.2. Let $u \in \mathscr{S}'(\mathbb{R}^d)$. Then $\mathrm{WF}_{\mathrm{SG}}(u) \subset \mathcal{W}$ is defined in terms of its complement as follows: $(x_0, \xi_0) \notin \mathrm{WF}_{\mathrm{SG}}(u)$ if and only if there exists a pseudo-differential operator with symbol in $\mathrm{SG}^{0,0}_{\mathrm{cl}}(\mathbb{R}^d \times \mathbb{R}^d)$ elliptic at (x_0, ξ_0) such that $Au \in \mathscr{S}(\mathbb{R}^d)$.

For a broader exposition and description of the properties of this notion of wave front set, we refer to [2, 7, 9, 25, 26]. In [9] we proved the following bounds for the singularities of the temperate oscillatory integral $I_\varphi(a)$ defined in Theorem 6.1.

Theorem 6.3. *Let φ be an admissible* SG*-phase function. Then, for any amplitude $a \in \mathrm{SG}^{m_e,m_\psi}(\mathbb{R}^d \times \mathbb{R}^s)$ we have the inclusions*

$$\mathrm{pr}_1(\mathrm{WF}_{\mathrm{SG}}(I_\varphi(a))) \subset \mathrm{pr}_1(\mathcal{C}_\varphi) \text{ and } \mathrm{WF}_{\mathrm{SG}}(I_\varphi(a)) \subset \Lambda_\varphi\,.$$

This theorem establishes a connection between the singularities of oscillatory integrals and the geometric structures established above.

Remark 6.4. Recalling the existence of a canonical principal part for classical SG-symbols, we can write

$$\varphi(x,\theta) = \chi^e(x)\,\varphi^e(x,\theta) + \chi^\psi(\theta)\,\varphi^\psi(x,\theta) - \chi^e(x)\,\chi^\psi(\theta)\,\varphi^{\psi e}(x,\theta) + r_\varphi(x,\theta).$$

Since $e^{ir_\varphi} \in \mathrm{SG}^{0,0}$, we may absorb the r_φ part of the phase function in an oscillatory integral into the amplitude. We are thus reduced to the case of studying phase functions of the form

$$\varphi(x,\theta) = \chi^e(x)\,\varphi^e(x,\theta) + \chi^\psi(\theta)\,\varphi^\psi(x,\theta) - \chi^e(x)\,\chi^\psi(\theta)\,\varphi^{\psi e}(x,\theta).$$

and thus we have found that only the behaviour at infinity, i.e. the principal symbol of φ, enters in the study of the SG-singularities of such oscillatory integrals. It is by this logic that only the boundary

[10]In the present paper we follow a notation close to the one used in [19], different from the one we adopted in [9]. In particular, in the original statement of Theorem 6.3 proved there, $\widetilde{\mathcal{C}}_\varphi$ was denoted by M_φ, and $\widetilde{\Lambda}_\varphi$ by SP_φ, respectively.

components Λ_φ play a role in the study of Lagrangian distributions, and this is why we do not ask for SG-Lagrangians Λ to arise as the boundary of a manifold in the interior, which would be simpler.

6.1. **An example.** In this subsection, we revisit the example of [9], see also [35, 42], and study a distribution associated with an SG-Lagrangian that has a non-trivial (ψe)-component, and hence it is neither Legendrian, nor a Lagrangian in the classical sense.

We consider the two-point function of a free, scalar, bosonic quantum field theory on a flat space-time, that is, Minkowski space $\mathbb{R}\times\mathbb{R}^d$, wherein we denote points by (x_0,x) for $x_0\in\mathbb{R}$ and $x\in\mathbb{R}^d$. Let $m>0$, $\omega_m(x)=\sqrt{m^2+|x|^2}$. The *two-point function*[11] is the distribution given by the formal oscillatory integral (see [33, Sect. IX.8])

$$\Delta_+(x_0,x) := \frac{i}{2(2\pi)^d}\int_{\mathbb{R}^d}\frac{e^{i(-x_0\omega_m(\theta)+x\theta)}}{\omega_m(\theta)}\,d\theta \tag{25}$$

It also arises in the study of the fundamental solutions of the Klein-Gordon equation, and therein its microlocal properties play a significant role, see [21].

We observe that (25) is expressed as a formal oscillatory integral like the one in (24). In fact, we have[12] $\omega_m(\theta)\in \mathrm{SG}^{0,1}_{\mathrm{cl}}(\mathbb{R}^{d+1}\times\mathbb{R}^d)$, and we conclude

$$\varphi(x_0,x;\theta) := -x_0\omega_m(\theta)+x\theta\in \mathrm{SG}^{1,1}_{\mathrm{cl}}(\mathbb{R}^{d+1}\times\mathbb{R}^d).$$

We may then compute the principal symbols and their gradients at an arbitrary point $(x_0,x;\theta)$ on their respective domains of definition:

$$\begin{aligned}
\varphi^\psi &= -x_0|\theta|+x\theta\ , & \nabla_\theta\varphi^\psi &= -x_0\tfrac{\theta}{|\theta|}+x\ , & \nabla_x\varphi^\psi &= (-|\theta|,\theta)^t;\\
\varphi^e &= -x_0\omega_m(\theta)+x\theta\ , & \nabla_\theta\varphi^e &= -x_0\tfrac{\theta}{\omega_m(\theta)}+x\ , & \nabla_x\varphi^e &= (-\omega_m(\theta),\theta)^t;\\
\varphi^{\psi e} &= -x_0|\theta|+x\theta\ , & \nabla_\theta\varphi^{\psi e} &= -x_0\tfrac{\theta}{|\theta|}+x\ , & \nabla_x\varphi^{\psi e} &= (-|\theta|,\theta)^t.
\end{aligned}$$

Since $\nabla_x\varphi^\bullet$ does not vanish on $\mathcal{W}^\bullet$, $\bullet\in\{\psi,e,\psi e\}$, the function Φ associated with φ is SG-elliptic, hence φ is a classical SG-phase function in the sense of Definition 3.6. Theorem 6.1 then defines (25) as a tempered oscillatory SG-integral. From the principal symbols of φ, we may now reproduce the bounds on the singularities of Δ_+ in terms of the

[11]Recall that the two-point function takes this form in difference variables $x=y-y'$.

[12]Indeed,

$$\omega_m(\theta)=\sqrt{m^2+|\theta|^2}=|\theta|\sqrt{1+(|\theta|/m)^{-2}}=|\theta|\sum_{j=0}^{\infty}\frac{(-1)^j(2j)!}{(1-2j)(j!)^2(4^j)}(|\theta|/m)^{-2j},$$

where the series converges for $|\theta|>m$, and therefore we have a polyhomogeneous expansion.

associated geometric sets $\mathcal{C}_\varphi$ and Λ_φ given in [9, 35]. We find

$$\mathcal{C}_\varphi^\psi = \{(0,0;\theta)\colon \theta \in (\mathbb{R}^d \smallsetminus \{0\})\} \cup \{(\pm|x|, x; \pm\lambda x)\colon x \in \mathbb{R}^d \smallsetminus \{0\},\ \lambda > 0\},$$

$$\mathcal{C}_\varphi^e = \left\{\left(\pm x_0, x; \frac{\pm m x}{\sqrt{x_0^2 - |x|^2}}\right)\colon x_0 \in \mathbb{R}_+,\ x \in \mathbb{R}^d,\ |x|^2 < x_0^2\right\},$$

$$\mathcal{C}_\varphi^{\psi e} = \{(\pm|x|, x; \pm\lambda x)\colon x \in \mathbb{R}^d \smallsetminus \{0\},\ \lambda > 0\},$$

and Λ_φ is, by Definition 3.11, the union of

$$\Lambda_\varphi^\psi = \{(0,0;-|\xi|,\xi)\colon \xi \in \mathbb{R}^d\} \cup \{(\pm|x|, x; -\lambda|x|, \pm\lambda x)\colon x \in \mathbb{R}^d \smallsetminus \{0\},\ \lambda > 0\},$$

$$\Lambda_\varphi^e = \left\{\left(\pm x_0, x; \frac{-m|x_0|}{\sqrt{x_0^2 - |x|^2}}, \frac{\pm m x}{\sqrt{x_0^2 - |x|^2}}\right)\colon x_0 \in \mathbb{R}_+,\ x \in \mathbb{R}^d,\ |x|^2 < x_0^2\right\},$$

$$\Lambda_\varphi^{\psi e} = \{(\pm|x|, x; -\lambda|x|, \pm\lambda x)\colon x \in \mathbb{R}^d \smallsetminus \{0\},\ \lambda > 0\}.$$

As in [9], we may parametrize the e-component of Λ_φ also as follows:

$$\Lambda_\varphi^e = \{(\pm\lambda\omega_m(\theta), \pm\lambda\theta; -\omega_m(\theta), \theta)\colon \theta \in \mathbb{R}^d,\ \lambda > 0\}. \tag{26}$$

Thus, Theorem 6.3 yields $\mathrm{WF}_{\mathrm{SG}}(\Delta_+) \subseteq \Lambda_\varphi$, and, in fact, equality holds true (see [9, 35]).

We now turn to a discussion of these sets. $\mathrm{pr}_1(\Lambda_\varphi^\psi)$ yields the *light-cone*, that is $\{(x_0, x)\colon |x_0| = |x|\}$, and $\mathrm{pr}_1(\widetilde{\Lambda}_\varphi^{\psi e})$ is simply the boundary of the light-cone "at infinity". Then, Λ_φ^ψ and $\Lambda_\varphi^{\psi e}$ are formed by attaching those tangential co-vectors to the light-cones that have a negative ξ_0-component. On the other hand, $\mathrm{pr}_1(\Lambda_\varphi^e)$ is formed by all the time-like directions that satisfy $|x|^2 < x_0^2$. Λ_φ^e can also be understood by considering (26) as a bundle over the second set of variables, the (negative) *mass shell* $\{-\omega_m(\theta), \theta\}$, reversing the role of fibre and base space. This information - (schematically, in $1+1$ dimensions) - is visualized in Figure 4, consider also [9, 33, 35].

Obviously, $\Lambda^{\psi e}$ and Λ^e are manifolds. The remaining Λ^ψ has a (bi-

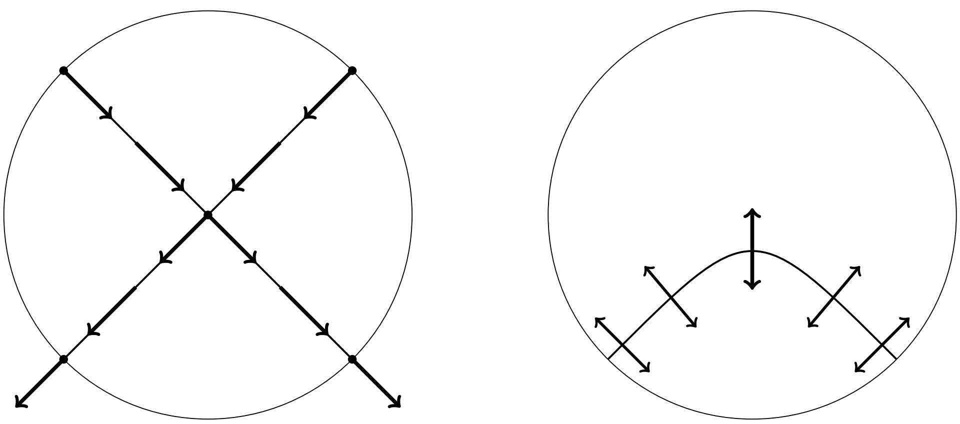

FIGURE 4. The sets $\Lambda_\varphi^\psi \cup \Lambda_\varphi^{\psi e}$ and Λ_φ^e associated with φ

)conical singularity at $(x_0, x) = 0$. This stems from the fact that φ is everywhere non-degenerate, except at all point of $\mathcal{C}_\varphi$ of the form $(0, 0, \theta)$.

Therefore, the singular sets associated with the two-point function (in fact, with any of the distinguished fundamental solutions to the Klein-Gordon equation) provide an example of a SG-Lagrangian submanifold, apart from a singularity at the origin. This singularity, however, is expected, since the it lies in the classical ψ-part of the Lagrangian. To allow also such kind of singularity, which arises, for instance, in the construction of parametrices to hyperbolic Cauchy problems, one could pass to an extended version of the calculus of *paired Lagrangian distributions*, see [29].

The previous example shows how SG-Lagrangian submanifolds, which can be decomposed into two suitable submanifolds, one of which is Legendrian, while the other one is Lagrangian, arise.

Appendix A. Manifolds with corners

In this appendix we will present some results from the analysis on manifolds with corners that are employed in the study of SG-Lagrangians. There are different definitions of manifold with corners, see [28], and, e.g. [22, 23]. Since in the main part of this document we only deal with finite-dimensional manifolds with corners, here we shortly recall the approach of [23] in such a case, while in its original formulation it is based on quadrants in general Banach spaces. Therein, the results needed for our purposes (notably, Theorem A.17 below) are explained in full detail, within the complete presentation of this theory.

Definition A.1. With $d \in \mathbb{N}$, let $\Lambda \subseteq \{1, \dots, d\}$. The set

$$E^+_{\Lambda,d} = \begin{cases} \mathbb{R}^d, & \text{if } \Lambda = \varnothing, \\ \{x \in \mathbb{R}^d : x_j \geq 0, j \in \Lambda\}, & \text{otherwise,} \end{cases}$$

is called (Λ-)quadrant of $\mathbb{R}^d$. The notation $E^+_{j,d}$ is used when $\Lambda = \{j\}$. Obviously,

$$E^+_{\Lambda,d} = \bigcap_{j \in \Lambda} E^+_{j,d}.$$

The notion of differentiability on open subsets of a quadrant of $\mathbb{R}^d$ can be introduced exactly as on open subsets of $\mathbb{R}^d$.

Definition A.2. Let U be an open subset of $E^+_{\Lambda,d}$, $f : U \to \mathbb{R}^{d'}$ a map, and $x \in U$. Then, if there exists an element $u \in \mathcal{L}(\mathbb{R}^d, \mathbb{R}^{d'})$ such that

$$\lim_{y \to x} \frac{\|f(y) - f(x) - u(y - x)\|}{\|y - x\|} = 0,$$

$\|.\|$ denoting the standard Euclidean norms on $\mathbb{R}^d$, $\mathbb{R}^{d'}$, f is said to be differentiable at x. In such a case, u is called differential of f at x and is denoted by $Jf(x)$. If f is differentiable at every $x \in U$, f is said to be differentiable on U.

The notion of differentiability and of differential in Definition A.2 is well-defined and coincides with the ordinary one when $\Lambda = \emptyset$. The basic properties and notions of differentiability, such as continuous differentiability and higher order differentiability, carry over to this notion of differentiation on quadrants. In particular, we call f *of class* ∞, or *smooth (up to the boundary)* in a (relatively) open subset $U \subset \mathbb{R}^d$, denoted $f \in \mathscr{C}^\infty(U)$, if for every $p \in \mathbb{N}_0$ the maps $J^p f : (\mathbb{R}^d)^{\otimes p} \to \mathbb{R}^{d'}$ are continuous and differentiable at every $x \in U$.

Equivalent alternative definitions of smooth maps on $E^+_{\Lambda,d}$ can be given in terms of existence of extensions on open sets of $\mathbb{R}^d$ including U, or on neighbourhoods in $\mathbb{R}^d$ of points $x \in U$, which are continuously differentiable of any order with respect to the standard notion, see [23], Sections 1.1 and 2.1, for details.

Definition A.3. Let X be a set. The triple $(U, \nu, E^+_{\Lambda,d})$ is a chart on X if:

(1) $U \subseteq X$;
(2) $\nu: U \to E^+_{\Lambda,d}$ is an injective map and $\nu(U)$ is an open set of $E^+_{\Lambda,d}$.

Let $(U, \nu, E^+_{\Lambda,d})$, $(U', \nu', E^+_{\Lambda',d})$ be charts on X. They are smoothly compatible if $U \cap U' = \emptyset$ or, if $U \cap U' \neq \emptyset$,

(3) $\nu(U \cap U')$ and $\nu'(U \cap U')$ are open subsets of $E^+_{\Lambda,d}$ and $E^+_{\Lambda',d}$, respectively;
(4) $\nu' \circ \nu^{-1}: \nu(U \cap U') \to \nu'(U \cap U')$ and $\nu \circ \nu'^{-1}: \nu'(U \cap U') \to \nu(U \cap U')$ are smooth maps.

A collection $\mathcal{A}$ of smoothly compatible charts that cover X is called a *smooth atlas*. As usual, two atlases $\mathcal{A}$, $\mathcal{A}'$ are called equivalent if $\mathcal{A} \cup \mathcal{A}'$ is an atlas, which yields an equivalence relation. An equivalence class $[\mathcal{A}]_\sim$ is called *smooth differentiable structure on* X and the pair $(X, [\mathcal{A}]_\sim)$ is called smooth manifold or a $\mathscr{C}^\infty$-manifold, denoted simply by X. If Λ cannot be chosen as empty, X is called a smooth manifold with corners.

Given a $\mathscr{C}^\infty$-manifold X, the set

$$\{U \subseteq X \colon U \text{ is the domain of a chart on } X\}$$

is a basis for a topology on X. The space of smooth maps among $\mathscr{C}^\infty$-manifolds X and Y, denoted by $\mathscr{C}^\infty(X, Y)$, is defined in a completely similar fashion to the usual way. In particular the tangent bundle may be defined in a neighbourhood U given by a chart as $U \times \mathbb{R}^d$, and consequently over the full manifold by imposing contravariant transformation behaviour. The differential of a smooth map $f : X \to Y$ in local coordinates then induces a map $df : TX \to TY$.

Definition A.4. Let U be an open set of $E^+_{\Lambda,d}$.

(1) For $x \in E^+_{\Lambda,d}$, $\mathrm{ind}(x) := \mathrm{ind}_\Lambda(x) = \#\{j \in \Lambda : x_j = 0\}$;
(2) The set $\{x \in U : \mathrm{ind}(x) \geq 1\}$ is called boundary of U, and denoted $\partial_\Lambda U = \partial U$;
(3) The set $\{x \in U : \mathrm{ind}(x) = 0\}$ is called interior of U, and denoted $\mathrm{int}_\Lambda U = \mathrm{int}\, U = U^o$.

It can be proved that the value $\mathrm{ind}(x)$ is invariant under smooth diffeomorphisms[13], that is, it has an invariant meaning on a manifold X. This implies that also the notions of boundary and interior are invariantly defined on X. More generally, for any $k \in \mathbb{N}_0$, it is possible to define $\partial^k X$, the k-boundary of X, as the set of all points $x \in X$ such that $\mathrm{ind}(x) \geq k$. We set $\partial X := \partial^1 X$. Moreover, for any $k \in \mathbb{N}_0$, the set $\{x \in X : \mathrm{ind}(x) = k\}$ is denoted by $B_k X$. The set $B_0 X$ is called the interior of X.

Example A.5. Consider $d \in \mathbb{N}$, $\mathbb{B}^d = \{y \in \mathbb{R}^d : \|y\| \leq 1\}$, and, for all $j \in \{1, \dots, d\}$, $(V_j^+, \nu_j^+, E^+_{j,d})$, $(V_j^-, \nu_j^-, E^+_{j,d})$, where

- $V_j^+ = \{y \in \mathbb{B}^d : y_j > 0\}$, $V_j^- = \{y \in \mathbb{B}^d : y_j < 0\}$;
- $\nu_j^+(y) = (\dots, y_{j-1}, \sqrt{1 - (\dots + y_{j-1}^2 + y_{j+1}^2 + \dots)} - y_j, y_{j+1}, \dots)$;
- $\nu_j^-(y) = (\dots, y_{j-1}, \sqrt{1 - (\dots + y_{j-1}^2 + y_{j+1}^2 + \dots)} + y_j, y_{j+1}, \dots)$.

Then, it turns out that

$$\mathcal{A} = \{(V_j^+, \nu_j^+, E^+_{j,d})\}_{j=1,\dots,n} \cup \{(V_j^-, \nu_j^-, E^+_{j,d})\}_{j=1,\dots,n} \cup \{(\mathbb{B}^d)^o, \mathrm{id}, \mathbb{R}^d)\}$$

is a smooth atlas on $\mathbb{B}^n$. Furthermore, the topology of of the manifold $(\mathbb{B}^d, [\mathcal{A}])$ is the usual (subset) topology of $\mathbb{B}^d \subset \mathbb{R}^d$, $\partial \mathbb{B}^d = \mathbb{S}^{n-1}$, $\partial^2 \mathbb{B}^d = \varnothing$.

Proposition A.6. *Let X, X' be $\mathscr{C}^\infty$-manifolds, $f : X \to X'$ a diffeomorphism. Then, for any $k \in \mathbb{N}$, $f(\partial^k X) = \partial^k X'$. In particular, if $\partial^2 X = \varnothing$, f is a diffeomorphism of ∂X onto $\partial X'$.*

It is well known that the finite Cartesian product of manifolds without boundary is a natural, well-defined construction, which yields another manifold without boundary. However, in the category of manifolds with boundary (i.e., $\partial^2 X = \varnothing$), there is no such a natural finite product construction. It turns out that the category of manifolds with corners is the suitable one in which to define finite Cartesian products.

Proposition A.7. *Let X, X' be $\mathscr{C}^\infty$-manifolds. Then, there exists a unique $\mathscr{C}^\infty$-structure $[\mathcal{A}]$ on $X \times Y$ such that, for every chart $(U, \nu, E^+_{\Lambda,d})$ on X and every chart $(U', \nu', E^+_{\Lambda',d'})$ on X', $(U \times U', \nu \times \nu', E^+_{\Lambda \sqcup \Lambda', d+d'})$, $\Lambda \sqcup \Lambda' = \Lambda \cup \{d + j' : j' \in \Lambda'\}$, is a chart of $(X \times X', [\mathcal{A}])$. The pair $(X \times X', [\mathcal{A}])$ is called the product manifold of X and X'.*

[13] A smooth diffeomorphism is a smooth bijective map $X \to X$ whose inverse is also smooth.

Proposition A.8. *Let X, X' be $\mathscr{C}^\infty$-manifolds. Then, the following statements hold true.*

1. *The topology of the product manifold $X \times X'$ is the product topology of those on X and X'.*
2. *For every $(x, x') \in X \times X'$, $\operatorname{ind}(x, x') = \operatorname{ind}(x) + \operatorname{ind}(x')$.*
3. *For all $l \in \mathbb{N}$, $\partial^l(X \times X') = \bigcup_{\substack{j+k=l \\ j,k \geq 0}} \partial^j X \times \partial^k X'$. Moreover, $(X \times X')^o = X^o \times (X')^o$.*

Example A.9. This proposition allows us to construct a differential structure on $\mathbb{B}^d \times \mathbb{B}^s$, $s \in \mathbb{N}$, in terms of that in Example A.5, that turns this set into a manifold with corners of codimension 2 such that

$$B_k(\mathbb{B}^d \times \mathbb{B}^s) = \begin{cases} (\mathbb{B}^d)^o \times (\mathbb{B}^s)^o & k = 0 \\ ((\mathbb{B}^d)^o \times \mathbb{S}^{s-1}) \cup \left(\mathbb{S}^{d-1} \times (\mathbb{B}^s)^o\right) & k = 1 \\ \mathbb{S}^{d-1} \times \mathbb{S}^{s-1} & k = 2 \\ \varnothing & k > 2. \end{cases}$$

It is a remarkable aspect of this theory that the implicit function theorem extends to manifolds with corners, under a rather mild (and natural) additional condition on boundaries. In the next statement, given a map $f: X \times Y \to Z$, for any $(a, b) \in X \times Y$, we write $d_{(a,b)} f = (d^X_{(a,b)} f, d^Y_{(a,b)} f)$ with the linear morphisms $d^X_{(a,b)} f: T_a X \to T_{f(a,b)} Z$ and $d^Y_{(a,b)} f: T_b Y \to T_{f(a,b)} Z$.

Theorem A.10. *Let X, Y, Z be $\mathscr{C}^\infty$-manifolds, $f: X \times Y \to Z$ a smooth map and $(a, b) \in X \times Y$. Suppose that $d^Y_{(a,b)} f: T_b Y \to T_{f(a,b)} Z$ is a linear homeomorphism, and suppose that there are open neighbourhoods V_a of a and V_b of b such that $f(V_a \times (V_b \cap \partial Y)) \subset \partial Z$.*
Then there exist an open neighborhood W_a of a, an open neighbourhood W_b of b and a unique map $g: W_a \to W_b$ such that $f(x, g(x)) = f(a, b)$ for $x \in W_a$. Furthermore:

1. *$g(a) = b$, and g is smooth on W_a;*
2. *for every $x \in W_a$, $d^Y_{(x,g(x))} f$ is a linear homeomorphism and*

$$d_x g = -(d^Y_{(x,g(x))} f)^{-1} \circ d^X_{(x,g(x))} f.$$

We now state the definition of a submanifold (with corners) in this setting.

Definition A.11. Let X be a $\mathscr{C}^\infty$-manifold and $X' \subset X$. Then, X' is a $\mathscr{C}^\infty$-submanifold of X if, for every $x' \in X'$, there exist:

1. a chart $(U, \nu, E^+_{\Lambda,d})$ of X such that $x' \in U$ and $\nu(x') = 0$;
2. an integer $d' \in \mathbb{N}$, $d' \leq d$, and $\Lambda' \subseteq \{1, \ldots, d'\}$, such that $\nu(U \cap X') = \nu(U) \cap E^+_{\Lambda',d'}$, and $\nu(U) \cap E^+_{\Lambda',d'}$ is an open subset of $E^+_{\Lambda',d'}$.

In particular, X^o is an open submanifold of X and if $\partial^2 X = \varnothing$, ∂X is a submanifold of X. In general, there is no relation between the boundary of X and that of a submanifold of X. This leads to the definition of special submanifolds, whose boundaries have "good positions" within the boundary of the ambient manifold.

Definition A.12. Let X' be a submanifold of X. Then:

(1) X' is a *neat submanifold* of X if $\partial X' = (\partial X) \cap X'$;
(2) X' is a *totally neat submanifold* of X if, for all $x' \in X'$, $\mathrm{ind}_{X'}(x') = \mathrm{ind}_X(x')$, that is, $B_k X' = X' \cap B_k X$ for any $k \in \mathbb{N}_0$.

An equivalent condition for X' to be a totally neat submanifold of X is that, for all $x' \in X' \cap B_k X$,

$$\partial X' = (\partial X) \cap X' \text{ and } T_{x'}X = (d_{x'}j')(T_{x'}X') + (d_{x'}j)(T_{x'}B_kX),$$

where $j' : X' \hookrightarrow X$ and $j : B_k X \hookrightarrow X$ are the canonical inclusions. The properties of being a neat or totally neat submanifold are invariant under diffeomorphisms.

Definition A.13. Let $f : X \to X'$ be a $\mathscr{C}^\infty$-map and $x \in X$. f is called (smooth) immersion at x if there is a chart $(U, \nu, E^+_{\Lambda,d})$ on X such that $\nu(x) = 0$, and a chart $(U', \nu', E^+_{\Lambda',d'})$ on X' with $\nu'(f(x)) = 0$, such that $f(U) \subseteq U'$, $\nu(U) \subset \nu'(U')$ and $\nu' \circ f_{|U} \circ \nu^{-1} : \nu(U) \to \nu(U')$ is the inclusion map. If f is an immersion $\forall x \in X$, it is called immersion on X.

Theorem A.14. *Let $f : X \to X'$ be a smooth map and $x \in X$ such that $f(x) \in (X')^o$. Then, the following statements are equivalent:*

(1) f is an immersion at x;
(2) $d_x f$ is an injective map.

We now recall the definition of embeddings in this context, and describe how they can be characterized.

Definition A.15. Let $f : X \to X'$ be a map of class p. Then, f is called embedding if it is an immersion and $f : X \to f(X)$ is a homeomorphism.

We may now give a characterization of embedded submanifolds.

Proposition A.16. *Let X, X' be $\mathscr{C}^\infty$-manifolds and $f : X \to X'$ a map. Then, the following statements are equivalent:*

(1) f is a smooth embedding;
(2) $f(X)$ is a $\mathscr{C}^\infty$-submanifold of X' and $f : X \to f(X)$ is a diffeomorphism.

The next result, [23, Prop. 4.2.10], with which we conclude this appendix, shows that also on manifolds with corners the solutions to systems of equations give rise to submanifolds, provided that the corresponding differentials are linearly independent.

Theorem A.17. *Let X be a smooth manifold and $f_1, \dots, f_s : X \to \mathbb{R}$ be $\mathscr{C}^\infty(X)$-maps. Consider the set $Y = \{x \in X : f_1(x) = \dots = f_s(x) =$*

0}, and suppose that, for every $x \in Y$, $(d_x(f_1|_{B_k X}), \dots, d_x(f_s|_{B_k X}))$ *is a linearly independent system of elements of* $(T_x(B_k X))^*$, *where* $k = \mathrm{ind}(x)$. *Then we have*

(1) *Y is a closed totally neat $\mathscr{C}^\infty$-submanifold of X;*
(2) $T_x(j)(T_x Y) = \{v \in T_x X : T_x f_1(v) = \dots = T_x f_n(v) = 0\}$, *where* $j: Y \to X$ *is the inclusion map and* $x \in Y$;
(3) *For all* $x \in Y$, $\mathrm{codim}_x Y = s$.

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