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IBAN: IT90 Y030 6909 2171 0000 0460 182
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RENDICONTI
DEL SEMINARIO
MATEMATICO

Università e Politecnico di Torino

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Volume 71, N. 2 **2013**

RENDICONTI DEL SEMINARIO MATEMATICO 2013

Autorizzazione del Tribunale di Torino N. 2962 del 6.VI.1980

DIRETTORE RESPONSABILE
Marino Badiale

QUESTO FASCICOLO È STAMPATO CON IL CONTRIBUTO DI:
UNIVERSITÀ DEGLI STUDI DI TORINO
POLITECNICO DI TORINO

Preface

The 8th Congress of Isaac (the International Society for Analysis, its Applications and Computation) took place at the People's Friendship University of Russia, Moscow between August 22nd and 27th 2011. There were a large number of parallel sessions at the meeting, and we are grateful to the editors of *Rendiconti del Seminario Matematico di Torino,* particularly Marino Badiale, for agreeing to publish a selection of the talks from the *Stochastic Analysis* session herein.

Just as analysis grew from the need to make fully rigorous sense of the calculus developed by Newton, Leibnitz and their followers in the seventeenth century, so stochastic analysis developed from the equivalent programme for stochastic calculus in the twentieth century. The giant on whose shoulders we all stand was Kyosi Itô (1915-2008), who developed the key ideas of stochastic integral and stochastic differential equation (SDE) in the 1940s, as well as initiating many other important themes in the subject. He was awarded the inaugural Gauss prize in honour of his achievements in 2006. Itô worked primarily, but not exclusively, with *Brownian motion*, and this quickly became the paradigm noise process for workers in the field. Breakthroughs in understanding the more general processes called *martingales* in the 1960s and 1970s by Kunita and Watanabe, and by Meyer (among others) led to the foundations of stochastic analysis being based on the general notion of *semimartingale*. Applications to engineering and science were quickly developed, through the extension of stochastic analysis to stochastic control and stochastic filtering. The remarkable development of mathematical finance since the 1990s stimulated a revival of interest in the rich class of semimartingales with jumps called *Lévy processes*, while interest in modelling with self-similarity led to an extension of the calculus to include *fractional Brownian motion*, which is not a semimartingale.

Meanwhile, from the mid 1970s there were important theoretical advances such as the systematic study of processes on manifolds – *stochastic differential geometry*, and the creation of the *Malliavian calculus*, initially by Paul Malliavin (1925-2010), which enabled the development of an internal differential calculus based on chaotic expansions. In particular, Malliavin obtained a probabilistic proof of Hörmander's theorem that hypoelliptic diffusions have smooth densities. In the last twenty years or so, the study of stochastic partial differential equations (SPDEs) has been a major focus of attention, and this has led to a lot of interest in infinite–dimensional noise. Martin Hairer was awarded a Fields medal in 2014 for a novel theory of regularity structures associated to non-linear SPDEs, and his work made great use of a new way of looking at stochastic equations – the *rough path* technique of Terry Lyons.

Stochastic analysis reaches into many different areas of mathematics. Its roots are in probability theory and mathematical analysis, but it interacts with functional analysis, differential geometry, partial differential equations, potential theory and many areas of applied mathematics, such as numerical analysis. The papers in this volume are good indicators of the breadth and scope of the subject.

Applebaum uses techniques from group representations to find necessary and sufficient conditions for smoothness of densities on compact Lie groups. Belopskaya and Woyczynski present a probabilistic approach to existence and uniqueness of nonlinear, second-order parabolic PDEs, by setting up associated SDEs. Glikikh and Vinokurova obtain existence to an SDE on the total space of a vector bundle, which is nothing but Newton's second law of motion from the point of view of Nelson's theory of "stochastic mechanics"; a theory that provides a probabilistic approach to the problem of quantisation. By investigating conservativity of the associated Markov semigroup, Grigor'yan finds sufficient conditions for stochastic completeness of some processes on metric measure spaces, including Brownian motion and jump processes on manifolds, and random walks on graphs. Melnikova and Alshanskiy apply a theory of random generalised functions, known as the "white noise" calculus, to obtain existence and uniqueness for a class of SPDEs. The COGARCH process is a continuous time version of the GARCH process, which is an important volatility model in mathematical finance (for which Robert Engle won the 2003 Nobel Prize in Economics), and Schnurr shows how to extract probabilistic information from this process by using the pseudo-differential operator representation of its generator. Finally Vives presents a survey of Malliavin calculus for Lévy processes, including the nonanticipating Itô formula.

David Applebaum (October 2014)

Rend. Sem. Mat. Univ. Politec. Torino
Vol. 71, 2 (2013), 197 – 208

D. Applebaum

SMOOTHNESS OF DENSITIES ON COMPACT LIE GROUPS

Abstract. We give necessary and sufficient conditions for both square integrability and smoothness for densities of a probability measure on a compact connected Lie group.

Keywords and Phrases. Lie group, Haar measure, unitary dual, Fourier transform, convolution operator, Lie algebra, weight, Sugiura space, smooth density, central measure, infinitely divisible, deconvolution density estimator.

1. Introduction

The study of probability measures on groups provides a mathematical framework for describing the interaction of chance with symmetry. This subject is broad and interacts with many other areas of mathematics and its applications such as analysis on groups [19], stochastic differential geometry [6], statistics [5] and engineering [4].

In this paper we focus on the important question concerning when a probability measure on a compact group has a regular density with respect to Haar measure. We begin by reviewing work from [1] where Peter-Weyl theory is used to find a necessary and sufficient condition for such a measure to have a square-integrable density. This condition requires the convergence of an infinite series of terms that are formed from the (non-commutative) Fourier transform of the measure in question. We also describe a related result from [2] where it is shown that square-integrability of the measure is a necessary and sufficient condition for the associated convolution operator to be Hilbert-Schmidt (and hence compact) on the L^2-space of Haar measure.

In the second part of our paper we turn our attention to measures with smooth densities. A key element of our approach is the important insight of Hermann Weyl that the unitary dual $\widehat{G}$ of the group G can be parameterised by the space of highest weights. This effectively opens up $\widehat{G}$ to investigation by standard analytical methods. We introduce Suguira's space of rapidly decreasing functions of weights which was shown in [18] to be topologically isomorphic to $C^\infty(G)$. We are then able to prove that a probability measure has a smooth density if and only if its Fourier transform lives in Suguira's space. This improves on results of [3] where the Sobolev embedding theorem was used to find sufficient conditions for such a density to exist.

In the last part of the paper we give a brief application to statistical inference. In [13], Kim and Richards have introduced an estimator for the density of a signal on the group based on i.i.d. (i.e. independent and identically distributed) observations of the signal after it has interacted with an independent noise. To obtain fast rates of convergence to the true density, the noise should be in a suitable "smoothness class" where smoothness is here measured in terms of the decay of the Fourier transform of the measure. We show that the "super-smooth" class is smooth in the usual mathematical sense.

2. Fourier Transforms of Measures on Groups

Throughout this paper G is a compact connected Lie group with neutral element e and dimension d, $\mathcal{B}(G)$ is the Borel σ-algebra of G and $\mathcal{P}(G)$ is the space of probability measures on $(G, \mathcal{B}(G))$, equipped with the topology of weak convergence. The role of the uniform distribution on G is played by *normalised Haar measure* $m \in \mathcal{P}(G)$ and we recall that this is a bi–invariant measure in that

$$m(A\sigma) = m(\sigma A) = m(A),$$

for all $A \in \mathcal{B}(G), \sigma \in G$. We will generally write $m(d\sigma) = d\sigma$ within integrals.

Our main focus in this paper is those $\rho \in \mathcal{P}(G)$ that are absolutely continuous with respect to m and so they have densities $f \in L^1(G,m)$ satisfying

$$\rho(A) = \int_A f(\sigma)d\sigma,$$

for all $A \in \mathcal{B}(G)$.

A key tool which we will use to study these measures is the non-commutative Fourier transform which is defined using representation theory. We recall some key facts that we need. A good reference for the material below about group representations, the Peter-Weyl theorem and Fourier analysis of square-integrable functions is Faraut [7].

If H is a complex separable Hilbert space then $\mathcal{U}(H)$ is the group of all unitary operators on H. A *unitary representation* of G is a strongly continuous homomorphism π from G to $\mathcal{U}(V_\pi)$ for some such Hilbert space V_π. So we have for all $g,h \in G$,:

- $\pi(gh) = \pi(g)\pi(h)$,
- $\pi(e) = I_\pi$ (where I_π is the identity operator on V_π),
- $\pi(g^{-1}) = \pi(g)^{-1} = \pi(g)^*$.

π is *irreducible* if it has no non-trivial invariant closed subspace. Every group has a trivial representation δ acting on $V_\delta = \mathbb{C}$ by $\delta(g) = 1$ for all $g \in G$ and it is clearly irreducible. The *unitary dual* of $G, \widehat{G}$ is defined to be the set of equivalence classes of all irreducible representations of G with respect to unitary conjugation. We will as usual identify each equivalence class with a typical representative element. As G is compact, for all $\pi \in \widehat{G}, d_\pi := \dim(V_\pi) < \infty$ so that each $\pi(g)$ is a unitary matrix. Furthermore in this case $\widehat{G}$ is countable.

For each $\pi \in \widehat{G}$, we define co-ordinate functions $\pi_{ij}(\sigma) = \pi(\sigma)_{ij}$ with respect to some orthonormal basis in V_π.

THEOREM 1 (Peter-Weyl). *The set* $\{\sqrt{d_\pi}\pi_{ij}, 1 \leq i,j \leq d_\pi, \pi \in \widehat{G}\}$ *is a complete orthonormal basis for* $L^2(G,\mathbb{C})$.

The following consequences of Theorem 1 are straightforward to derive using Hilbert space arguments.

COROLLARY 1. *For* $f, g \in L^2(G, \mathbb{C})$

- *Fourier expansion.*

$$f = \sum_{\pi \in \widehat{G}} d_\pi tr(\widehat{f}(\pi)\pi),$$

where $\widehat{f}(\pi) := \int_G f(\sigma^{-1})\pi(\sigma)d\sigma$ *is the Fourier transform of* f*.*

- *The Plancherel theorem.*

$$||f||^2 = \sum_{\pi \in \widehat{G}} d_\pi |||\widehat{f}(\pi)|||^2$$

where $|||\cdot|||$ *is the Hilbert-Schmidt norm* $|||T||| := tr(TT^*)^{\frac{1}{2}}$.

- *The Parseval identity.*

$$\langle f, g \rangle = \sum_{\pi \in \widehat{G}} d_\pi tr(\widehat{f}(\pi)\widehat{g}(\pi)^*).$$

If $\mu \in \mathcal{P}(G)$ we define its *Fourier transform* $\widehat{\mu}$ to be

$$\widehat{\mu}(\pi) = \int_G \pi(\sigma^{-1})\mu(d\sigma),$$

for each $\pi \in \widehat{G}$. For example if ε_e is a Dirac mass at e then $\widehat{\varepsilon_e}(\pi) = I_\pi$ and $\widehat{m}(\pi) = \begin{cases} 0 & \text{if } \pi \neq \delta \\ 1 & \text{if } \pi = \delta \end{cases}$. If μ has a density f then $\widehat{\mu} = \widehat{f}$ as defined in Corollary 1. If we take G to be the d-torus $\mathbb{T}^d$ then $\widehat{G}$ is the dual group $\mathbb{Z}^d$ and the Fourier transform is precisely the usual characteristic function of the measure μ defined by $\widehat{\mu}(n) = \int_{\mathbb{T}^d} e^{-in\cdot x}\mu(dx)$ for $n \in \mathbb{Z}^d$, where $\cdot$ is the scalar product. Note that any non-trivial compact connected abelian Lie group is isomorphic to $\mathbb{T}^d$ for some $d \in \mathbb{N}$.

Fourier transforms of measures on groups have been studied by many authors, see e.g. [12, 10, 9, 16] where proofs of the following basic properties can be found.

For all $\mu, \mu_1, \mu_2 \in \mathcal{P}(G), \pi \in \widehat{G}$,

1. $\widehat{\mu_1 * \mu_2}(\pi) = \widehat{\mu_2}(\pi)\widehat{\mu_1}(\pi)$,
2. $\widehat{\mu}$ determines μ uniquely,
3. $||\widehat{\mu}(\pi)||_\infty \leq 1$, where $||\cdot||_\infty$ denotes the operator norm in V_π.
4. Let $(\mu_n, n \in \mathbb{N})$ be a sequence in $\mathcal{P}(G)$. $\mu_n \to \mu$ (weakly) if and only if $\widehat{\mu_n}(\pi) \to \widehat{\mu}(\pi)$ as $n \to \infty$.

Remark. Most authors define $\widehat{\mu}(\pi) = \int_G \pi(\sigma)\mu(d\sigma)$. This has the advantage that Property 1 above will then read $\widehat{\mu_1 * \mu_2}(\pi) = \widehat{\mu_1}(\pi)\widehat{\mu_2}(\pi)$ but the disadvantage that if μ has density f then $\widehat{\mu}(\pi) = \widehat{f}(\pi)^*$. It is also worth pointing out that the Fourier transform continues to make sense and is a valuable probabilistic tool in the case where G is a general locally compact group (see e.g. [10, 9, 16].)

3. Measures with Square-Integrable Densities

In this section we examine the case where μ has a square-integrable density. The following result can be found in [1] and so we only sketch the proof here.

THEOREM 2. *The probability measure μ has an L^2-density f_μ if and only if*

$$\sum_{\pi \in \widehat{G}} d_\pi |||\widehat{\mu}(\pi)|||^2 < \infty.$$

In this case

$$f_\mu = \sum_{\pi \in \widehat{G}} d_\pi tr(\widehat{\mu}(\pi)\pi(\cdot)).$$

Proof. Necessity is straightforward. For sufficiency define $g := \sum_{\pi \in \widehat{G}} d_\pi \mathrm{tr}(\widehat{\mu}(\pi)\pi)$. Then $g \in L^2(G,\mathbb{C})$ and by uniqueness of Fourier coefficients $\widehat{g}(\pi) = \widehat{\mu}(\pi)$. Using Parseval's identity, Fubini's theorem and Fourier expansion, we find that for each $h \in C(G,\mathbb{C})$:

$$\int_G h(\sigma)\overline{g(\sigma)}d\sigma = \sum_{\pi \in \widehat{G}} d_\pi \mathrm{tr}(\widehat{h}(\pi)\widehat{\mu}(\pi)^*) = \int_G h(\sigma)\mu(d\sigma).$$

This together with the Riesz representation theorem implies that g is real valued and $g(\sigma)d\sigma = \mu(d\sigma)$. The fact that g is non-negative then follows from the Jordan decomposition for signed measures. □

See [1] for specific examples. We will examine some of these in the next section from the finer point of view of smoothness.

To study random walks and Lévy processes in G we need the convolution operator T_μ in $L^2(G,\mathbb{C})$ associated to $\mu \in \mathcal{P}(G)$ by

$$(T_\mu f)(\sigma) := \int_G f(\sigma\tau)\mu(d\tau),$$

for $f \in L^2(G,\mathbb{C}), \sigma \in G$. For example T_μ is the transition operator corresponding to the random walk $(\mu^{*n}, n \in \mathbb{N})$. The following properties are fairly easy to establish.

- T_μ is a contraction.
- T_μ is self-adjoint if and only if μ is symmetric, i.e. $\mu(A) = \mu(A^{-1})$ for all $A \in \mathcal{B}(G)$.

The next result is established in [2].

THEOREM 3. *The operator T_μ is Hilbert-Schmidt if and only if μ has a square-integrable density.*

Proof. Sufficiency follows from the standard representation of Hilbert-Schmidt operators in L^2-spaces. For necessity, suppose that T_μ is Hilbert-Schmidt. Then it has a kernel $k \in L^2(G \times G)$ and

$$(T_\mu f)(\sigma) = \int_G f(\tau) k_\mu(\sigma, \tau) d\tau.$$

In particular for each $A \in \mathcal{B}(G)$,

$$\mu(A) = T_\mu 1_A(e) = \int_A k_\mu(e, \tau) d\tau.$$

It follows that μ is absolutely continuous with respect to m with density $f_\mu = k_\mu(e, \cdot)$.□

Let $(\mu_t, t \geq 0)$ be a weakly continuous convolution semigroup in $\mathcal{P}(G)$ and write $T_t := T_{\mu_t}$. Then $(T_t, t \geq 0)$ is a strongly continuous contraction semigroup on $L^2(G, \mathbb{C})$ (see e.g. [11, 10, 14, 2].)

COROLLARY 2. *The linear operator T_t is trace-class for all $t > 0$ if and only if μ_t has a square-integrable density for all $t > 0$.*

Proof. For each $t > 0$, if μ_t has a square-integrable density then $T_t = T_{\frac{t}{2}} T_{\frac{t}{2}}$ is the product of two Hilbert-Schmidt operators and hence is trace class. The converse follows from the fact that every trace-class operator is Hilbert-Schmidt. □

If for $t > 0, \mu_t$ has a square-integrable density and is symmetric, then by Theorem 3, T_t is a compact self-adjoint operator and so has a discrete spectrum of positive eigenvalues $1 = e^{-t\beta_1} > e^{-t\beta_2} > \cdots > e^{-t\beta_n} \to 0$ as $n \to \infty$. Furthermore by Corollary 2, T_t is trace class and

$$\mathrm{Tr}(T_t) = \sum_{n=1}^{\infty} e^{-t\beta_n} < \infty.$$

Further consequences of these facts including the application to small time asymptotics of densities can be found in [2, 3].

4. Sugiura Space and Smoothness

In this section we will review key results due to Sugiura [18] which we will apply to densities in the next section. In order to do this we need to know about weights on Lie algebras and we will briefly review the necessary theory.

4.1. Weights

Let $\mathfrak{g}$ be the Lie algebra of G and $\exp : \mathfrak{g} \to G$ be the exponential map. For each finite dimensional unitary representation π of G we obtain a Lie algebra representation $d\pi$ by

$$\pi(\exp(tX)) = e^{td\pi(X)} \text{ for all } t \in \mathbb{R}.$$

Each $d\pi(X)$ is a skew-adjoint matrix on V_π and

$$d\pi([X,Y]) = [d\pi(X), d\pi(Y)],$$

for all $X,Y \in \mathfrak{g}$. A maximal torus $\mathbb{T}$ in G is a maximal commutative subgroup of G. Its dimension r is called the rank of G. Here are some key facts about maximal tori.

- Any $\sigma \in G$ lies on some maximal torus.
- Any two maximal tori are conjugate.

Let $\mathfrak{t}$ be the Lie algebra of $\mathbb{T}$. Then it is a maximal abelian subalgebra of $\mathfrak{g}$. The matrices $\{d\pi(X), X \in \mathfrak{t}\}$ are mutually commuting and so simultaneously diagonalisable, i.e. there exists a non-singular matrix Q such that

$$Qd\pi(X)Q^{-1} = \text{diag}(i\lambda_1(X), \ldots, i\lambda_{d_\pi}(X)).$$

The distinct real-valued linear functionals λ_j on $\mathfrak{t}$ are called the *weights* of π.

Let Ad be the adjoint representation of G on $\mathfrak{g}$. We can and will choose an Ad-invariant inner product $(\cdot,\cdot)$ on $\mathfrak{g}$. This induces an inner product on $\mathfrak{t}^*$ the algebraic dual of $\mathfrak{t}$ which we also write as $(\cdot,\cdot)$. We denote the corresponding norm by $|\cdot|$. The weights of the adjoint representation acting on $\mathfrak{g}$ equipped with $(\cdot,\cdot)$ are called the *roots* of G. Let $\mathcal{P}$ be the set of all roots of G. We choose a convention for positivity of roots as follows. Pick $v \in \mathfrak{t}$ such that $\mathcal{P} \cap \{\eta \in \mathfrak{t}^*; \eta(v) = 0\} = \emptyset$. Now define $\mathcal{P}_+ = \{\alpha \in \mathcal{P}; \alpha(v) > 0\}$. We can always find a subset $\mathcal{Q} \subset \mathcal{P}_+$ so that $\mathcal{Q}$ forms a basis for $\mathfrak{t}^*$ and every $\alpha \in \mathcal{P}$ is an linear combination of elements of $\mathcal{Q}$ with integer coefficients, all of which are either nonnegative or nonpositive. The elements of $\mathcal{Q}$ are called *fundamental roots.*

It can be shown that every weight of π is of the form

$$\mu_\pi = \lambda_\pi - \sum_{\alpha \in \mathcal{Q}} n_\alpha \alpha$$

where each n_α is a non-negative integer and λ_π is a weight of π called the *highest weight*. Indeed if μ_π is any other weight of π then $|\mu_\pi| \leq |\lambda_\pi|$. The highest weight of a representation is invariant under unitary conjugation of the latter and so there is a one-to-one correspondence between $\widehat{G}$ and the space of highest weights D of all irreducible representation of G. We can thus parameterise $\widehat{G}$ by D and this is a key step for Fourier analysis on nonabelian compact Lie groups. In fact D can be given a nice geometrical description as the intersection of the weight lattice with the dominant Weyl chamber, but in order to save space we won't pursue that line of reasoning here. From now on we will use the notation d_λ interchangeably with d_π to denote the dimension of the space V_π where $\pi \in \widehat{G}$ has highest weight λ. For a more comprehensive discussion of roots and weights, see e.g. [8] and [17].

4.2. Sugiura Theory

The main result of this subsection is Theorem 4 which is proved in [18].

Let $M_n(\mathbb{C})$ denote the space of all $n \times n$ matrices with complex entries and $\mathcal{M}(G) := \bigcup_{\lambda \in D} M_{d(\lambda)}(\mathbb{C})$. We define the *Sugiura space of rapid decrease* to be $\mathcal{S}(D) := \{F : D \to \mathcal{M}(G)\}$ such that

(i) $F(\lambda) \in M_{d(\lambda)}(\mathbb{C})$ for all $\lambda \in D$,

(ii) $\lim_{|\lambda| \to \infty} |\lambda|^k |||F(\lambda)||| = 0$ for all $k \in \mathbb{N}$.

$\mathcal{S}(D)$ is a locally convex topological vector space with respect to the seminorms $||F||_s = \sup_{\lambda \in D} |\lambda|^s |||F(\lambda)|||$, where $s \geq 0$. We also note that $C^\infty(G)$ is a locally convex topological vector space with respect to the seminorms $||f|_U = \sup_{\sigma \in G} |Uf(\sigma)|$ where $U \in \mathcal{U}(\mathfrak{g})$, which is the universal embedding algebra of g acting on $C^\infty(G)$ as polynomials in left-invariant vector fields on G, as described by the celebrated Poincaré-Birkhoff-Witt theorem.

THEOREM 4. *[Sugiura] There is a topological isomorphism between $C^\infty(G)$ and $\mathcal{S}(D)$ which maps each $f \in C^\infty(G)$ to its Fourier transform $\widehat{f}$.*

We list three useful facts that we will need in the next section. All can be found in [18].

- *Weyl's dimension formula* states that

$$d_\lambda = \frac{\prod_{\alpha \in \mathcal{P}_+} (\lambda + \rho, \alpha)}{\prod_{\alpha \in \mathcal{P}_+} (\rho, \alpha)},$$

  where $\rho := \frac{1}{2} \sum_{\alpha \in \mathcal{P}_+}$ is the celebrated "half-sum of positive roots". From here we can deduce a highly useful inequality. Namely there exists $N > 0$ such that

$$d_\lambda \leq N |\lambda|^m \tag{1}$$

  where $m := \# \mathcal{P}_+ = \frac{1}{2}(d - r)$.

- *Sugiura's zeta function* is defined by

$$\zeta(s) = \sum_{\lambda \in D - \{0\}} \frac{1}{|\lambda|^s}$$

  and it converges if $s > r$.

- Let $(X_1, \ldots, X_d)$ be a basis for $\widehat{G}$ and let $\Delta \in \mathcal{U}(\mathfrak{g})$ be the usual Laplacian on G so that

$$\Delta = \sum_{i,j=1}^{d} g^{ij} X_i X_j$$

where (g^{ij}) is the inverse of the matrix whose (i,j)th component is (X_i, X_j). We may consider Δ as a linear operator on $L^2(G)$ with domain $C^\infty(G)$. It is essentially self-adjoint and

$$\Delta\pi_{ij} = -\kappa_\pi \pi_{ij}$$

for all $1 \leq i, j \leq d_\pi, \pi \in \widehat{G}$, where $\kappa_\delta = 0$ and $\pi \neq \delta \Rightarrow \kappa_\pi > 0$. The numbers $(\kappa_\pi, \pi \in \widehat{G}\}$ are called the *Casimir spectrum* and if λ_π is the highest weight corresponding to $\pi \in \widehat{G}$ then

$$\kappa_\pi = (\lambda_\pi, \lambda_\pi + 2\rho).$$

From here we deduce that there exists $C > 0$ such that

$$|\lambda_\pi|^2 \leq \kappa_\pi \leq C(1 + |\lambda_\pi|^2). \tag{2}$$

4.3. Smoothness of Densities

We can now establish our main theorem.

THEOREM 5. *$\mu \in \mathcal{P}(G)$ has a C^∞ density if and only if $\widehat{\mu} \in \mathcal{S}(D)$.*

Proof. Necessity is obvious. For sufficiency its enough to show μ has an L^2-density. Choose $s > r$ so that Suguira's zeta function converges. Then using Theorem 2 and (1) we have

$$\begin{aligned}
\sum_{\lambda \in D - \{0\}} d_\lambda |||\widehat{\mu}_\lambda|||^2 &\leq N \sum_{\lambda \in D - \{0\}} |\lambda|^m |||\widehat{\mu}_\lambda|||^2 \\
&\leq N \sup_{\lambda \in D - \{0\}} |\lambda|^{m+s} |||\widehat{\mu}_\lambda|||^2 \sum_{\lambda \in D - \{0\}} \frac{1}{|\lambda|^s} \\
&< \infty. \qquad \square
\end{aligned}$$

We now investigate some classes of examples. We say that $\mu \in \mathcal{P}(G)$ is *central* if for all $\sigma \in G$,

$$\mu(\sigma A \sigma^{-1}) = \mu(A).$$

By Schur's lemma μ is central if and only if for each $\pi \in \widehat{G}$ there exists $c_\pi \in \mathbb{C}$ such that

$$\widehat{\mu}(\pi) = c_\pi I_\pi.$$

Clearly m is a central measure. A standard Gaussian measure on G is central where we say that a measure μ on G is a *standard Gaussian* if it can be realised as $\mu_1^{(B)}$ in the convolution semigroup $(\mu_t^{(B)}, t \geq 0)$ corresponding to Brownian motion on G (i.e. the associated Markov semigroup of operators is generated by $\frac{1}{2}\sigma^2\Delta$ where $\sigma > 0$.) For a more general notion of Gaussianity see e.g. [10], section 6.2. To verify centrality, take Fourier transforms of the heat equation to obtain $\widehat{\mu}(\pi) = e^{-\frac{1}{2}\sigma^2\kappa_\pi} I_\pi$ for each $\pi \in \widehat{G}$.

Following [3] we introduce a class of central probability measures on G which we call the $CID_{\mathbb{R}}(G)$ *class* as they are *central* and are induced by *infinitely divisible* measures on $\mathbb{R}$. Let ρ be a symmetric infinitely divisible probability measure on $\mathbb{R}$ so we have the Lévy-Khintchine formula

$$\int_{\mathbb{R}} e^{iux}\rho(dx) = e^{-\eta(u)} \text{ for all } u \in \mathbb{R}$$

$$\text{where } \eta(u) = \frac{1}{2}\sigma^2 u^2 + \int_{\mathbb{R}-\{0\}} (1-\cos(u))\nu(du),$$

with $\sigma \geq 0$ and ν a Lévy measure, i.e. $\int_{\mathbb{R}-\{0\}}(1\wedge u^2)\nu(du) < \infty$ (see e.g. [15].) We say $\mu \in CID_{\mathbb{R}}(G)$ if there exists η as above such that

$$\widehat{\mu}(\pi) = e^{-\eta(\kappa_\pi^{\frac{1}{2}})} I_\pi \text{ for each } \pi \in \widehat{G}.$$

Examples of such measures are obtained by *subordination* [15]. So let $(\gamma_t^f, t \geq 0)$ be a subordinator with Bernstein function f so that for all $u \geq 0$

$$\int_0^\infty e^{-us}\gamma_t^f(ds) = e^{-tf(u)}.$$

Let $(\mu_t^{(B)}, t \geq 0)$ be a Brownian convolution semigroup on G (with $\sigma = \sqrt{2}$) so that for each $\pi \in \widehat{G}$ $\widehat{\mu_t}(\pi) = e^{-t\kappa_\pi} I_\pi$. then we obtain a convolution semigroup of measures $(\mu_t^f, t \geq 0)$ in $CIG_{\mathbb{R}}(G)$ by

$$\mu_t^f(A) = \int_0^\infty \mu_s^{(B)}(A)\gamma_t^f(ds)$$

for each $A \in \mathcal{B}(G)$ and we have

$$\widehat{\mu_t^f}(\pi) = e^{-tf(\kappa(\pi))} I_\pi.$$

Examples (where we have taken $t = 1$):

- Laplace Distribution $f(u) = \log(1+\beta^2 u)$,

$$\widehat{\mu}(\pi) = (1+\beta^2\kappa_\pi)^{-1} I_\pi.$$

- Stable-like distribution $f(u) = b^\alpha u^{\frac{\alpha}{2}} (0 < \alpha < 2)$,

$$\widehat{\mu}(\pi) = e^{-b^\alpha \kappa_\pi^{\frac{\alpha}{2}}} I_\pi.$$

We now apply Theorem 5 to present some examples of measures in the $CIG_{\mathbb{R}}$ class which have smooth densities (and one that doesn't).

Example 1. η general with $\sigma \neq 0$ (i.e. non-vanishing Gaussian part)

Using (1) and (2) we obtain

$$\begin{aligned}\lim_{|\lambda|\to\infty}|\lambda|^k|||\widehat{\mu}(\lambda)||| &= \lim_{|\lambda|\to\infty}|\lambda|^k e^{-\eta\kappa_\pi^{\frac{1}{2}}} d_\lambda^{\frac{1}{2}} \\ &\leq \lim_{|\lambda|\to\infty}|\lambda|^k e^{-\frac{\sigma^2}{2}\kappa_\lambda} d_\lambda^{\frac{1}{2}} \\ &\leq N^{\frac{1}{2}}\lim_{|\lambda|\to\infty}|\lambda|^{k+\frac{m}{2}} e^{-\frac{\sigma^2}{2}|\lambda|^2} = 0.\end{aligned}$$

Example 2. Stable like laws are all C^∞ by a similar argument.

Example 3. The Laplace distribution is not C^∞. But it is L^2 if $r = 1$ (e.g. $SO(3), SU(2), Sp(1)$.)

5. Deconvolution Density Estimation

We begin by reviewing the work of Kim and Richards in [13]. Let X, Y and ε be G-valued random variables with $Y = X\varepsilon$. Here we interpret X as a signal, Y as the observations and ε as the noise which is independent of X. If all three random variables have densities, then with an obvious notation we have $f_Y = f_X * f_\varepsilon$. The statistical problem of interest is to estimate f_X based on i.i.d. observations $Y_1, \ldots, Y_n$ of the random variable Y. We assume that the matrix $\widehat{f_\varepsilon}(\pi)$ is invertible for all $\pi \in \widehat{G}$. Our key tool is the *empirical characteristic function* $\widehat{f_Y}^{(n)}(\pi) := \frac{1}{n}\sum_{i=1}^n \pi(Y_i^{-1})$. We then define the *non-parametric density estimator* (with smoothing parameters $T_n \to \infty$ as $n \to \infty$) for $\sigma \in G, n \in \mathbb{N}$:

$$f_X^{(n)}(\sigma) := \sum_{\pi\in\widehat{G}:\kappa_\pi<T_n} d_\pi \mathrm{tr}(\pi(\sigma)\widehat{f_Y}^{(n)}(\pi)\widehat{f_\varepsilon}(\pi)^{-1}).$$

The noise ε is said to be *super-smooth* of order $\beta > 0$ if there exists $\gamma > 0$ and $a_1, a_2 \geq 0$ such that

$$||\widehat{f_\varepsilon}(\pi))^{-1}||_\infty = O(\kappa_\pi^{-a_1}\exp(\gamma\kappa_\pi^\beta)) \text{ and } ||\widehat{f_\varepsilon}(\pi)||_\infty = O(\kappa_\pi^{a_2}\exp(-\gamma\kappa_\pi^\beta))$$

as $\kappa_\pi \to \infty$. For example a standard Gaussian is super-smooth with $a_i = 0$ $(i = 1, 2)$.

For $p > 0$, the Sobolev space $\mathcal{H}_p(G) := \{f \in L^2(G); ||f||_p < \infty\}$ where $||f||_p^2 = \sum_{\pi\in\widehat{G}} d_\pi(1+\kappa_\pi)^p|||\widehat{f}(\pi)|||^2$.

THEOREM 6 (Kim, Richards). *If f_ε super-smooth of order β and $||f_X||_{H_s(G)} \leq K$ for some $s > \frac{d}{2}$ where $K > 1$ then the optimal rate of convergence of $f_X^{(n)}$ to f_X is $(\log(n))^{-\frac{s}{2\beta}}$.*

A natural question to ask is "how smooth is super-smooth?" and we answer this as follows:

PROPOSITION 1. *If f is super-smooth then it is smooth.*

Proof. For sufficiently large κ_π and using (1) and (2) we find that there exists $C > 0$ such that

$$\begin{aligned} |||\widehat{f}(\pi)||| &\leq ||\widehat{f}(\pi)||_\infty |||I_\pi||| \\ &= d_\pi^{\frac{1}{2}} ||\widehat{f}(\pi)||_\infty \\ &\leq N^{\frac{1}{2}} |\lambda_\pi|^{\frac{m}{2}} C \kappa_\pi^{a_2} \exp(-\gamma \kappa_\pi^\beta) \\ &\leq K |\lambda_\pi|^{\frac{m}{2}} (1 + |\lambda_\pi|^2)^{a_2} \exp(-\gamma |\lambda_\pi|^{2\beta}) \end{aligned}$$

from which it follows that $\widehat{f} \in \mathcal{S}(D)$ and the result follows by Theorem 5. □.

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AMS Subject Classification: 60B15, 43A05, 43A25, 43A77, 60E07

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Lavoro pervenuto in redazione il 22.05.2013

Rend. Sem. Mat. Univ. Politec. Torino
Vol. 71, 2 (2013), 209 – 217

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SDES, FBSDES AND FULLY NONLINEAR PARABOLIC SYSTEMS

Abstract. In this article we describe two probabilistic approaches to construction of the Cauchy problem solution for a class of nonlinear parabolic systems. Namely, we describe probabilistic models associated with classical and viscosity solutions and use them to state conditions on the problem data that ensure the existence and uniqueness of the required solution of the PDE system.

Introduction

Systems of nonlinear second order parabolic equations appear in various fields of control theory, differential geometry, financial mathematics and others. Here we consider a class of nonlinear PDEs of the form

(1) $$\frac{\partial u_l}{\partial s} + [\mathcal{B}u]_l + g = 0, \quad u_l(T,x) = u_{0l}(x), \quad l = 1,\dots,d_1, \text{ where}$$

$$[\mathcal{B}u]_l = a_i \nabla_i u_l + \frac{1}{2} Tr A^* \nabla^2 u_l A + B^i_{lm} \nabla_i u_m + c_{lm} u_m$$

and all coefficients a, A, B, c and a scalar function g depend on $x, u, \nabla u$ and $\nabla^2 u^l$. A is an invertible operator and * denotes the transposition.

Here and below we assume a common convention about summation over all repeating indices if the contrary is not mentioned.

We call a system (1) semilinear when a, A, B, c and g depend on x, u, quasilinear when these parameters depend on $x, u, \nabla u$ and fully nonlinear when they depend on $x, u, \nabla u$ and $\nabla^2 u$.

A construction of a stochastic problem associated with (1) strictly depends on our understanding of a solution to a system, namely, on our intention to construct a strong, weak or viscosity solution. In this paper we give some new results concerning strong and viscosity solutions of (1) constructed via stochastic approaches.

1. Probabilistic approach to a strong solution of the Cauchy problem for a PDE system

Let $(\Omega, \mathcal{F}, P)$ be a probability space, $w(t) \in \mathbb{R}^d$ be a Wiener process defined on it and $\mathcal{F}_t$ be a flow generated by $w(t)$.

*The first author gratefully acknowledges the financial support of RRBR Grant 12-01-00457-a and project 1.370.2011 Minobrnauki.

To construct a strong solution to (1) in a semilinear case we consider a stochastic problem of the form

$$d\xi(t) = a(\xi(t), u(t,\xi(t))dt + A(\xi(t), u(t,\xi(t))dw(t), \quad \xi(s) = x \in \mathbb{R}^d, \tag{2}$$

$$d\eta(t) = c(\xi(t), u(t,\xi(t))\eta(t)dt + C(\xi(t), u(t,\xi(t))(\eta(t), dw(t)), \quad \eta(s) = h \in \mathbb{R}^{d_1}, \tag{3}$$

$$\langle h, u(s,x))\rangle = E[\langle \eta(T), u_0(\xi(T))\rangle + \int_s^T \langle \eta(\theta), g(\xi(\theta), u(\theta,\xi(\theta)))\rangle], \tag{4}$$

where $B^l_{im} = C^l_{km} A^k_i$ and $\langle h,u\rangle = \sum_{k=1}^{d_1} h_k u_k$ denotes the inner product in $\mathbb{R}^{d_1}$.

Actually, we can set $\gamma(t) = (\xi(t), \eta(t))$ and present (2),(3) in the form

$$d\gamma(t) = n_u(\gamma(t))dt + N_u(\gamma(t))dW(t)), \quad \gamma(s) = \gamma, \tag{5}$$

where $W(t) = (w(t), w(t))^*$, $n_u(\gamma(t)) = n(\gamma(t), u(t,\gamma(t)), N_u(\gamma(t)) = N(\gamma(t), u(t,\gamma(t)),$ and

$$N_u(x,h) = \begin{pmatrix} A(x,u) & 0 \\ 0 & C(x,u)h \end{pmatrix}, \quad n_u(x,h) = \begin{pmatrix} a(x,u) & 0 \\ 0 & c(x,u)h \end{pmatrix}.$$

Let $X = \mathbb{R}^d$, $Y = \mathbb{R}^{d_1}$, $M^d = \mathbb{R}^d \otimes \mathbb{R}^d$, $a(x,u) \in \mathbb{R}^d$, $A(x,u) \in M^d$, $c(x,u) \in M^{d_1}$, $C(x,u) \in M^{d_1} \otimes \mathbb{R}^d$ provided $x \in X, u \in Y$. We say $\mathbf{C}^{\mathbf{1,k}}$ holds if

i) a, A have sublinear growth in $x \in \mathbb{R}^d$, c, C g and u_0 are bounded in x and all of them but u_0 have polynomial growth in $u \in \mathbb{R}^{d_1}$;

ii) a, A, c, C and u_0, g are C^k-smooth in all arguments in correspondent norms.

THEOREM 1. *Assume* $\mathbf{C}^{\mathbf{1,1}}$ *holds. Then there exists a unique solution of (2)-(4).*

Let $\mathbf{C}^{\mathbf{1,k}}$ hold, $k \geq 1$. Along with (2),(3) we consider

$$d\zeta(t) = \nabla a_u(\xi(t))\zeta(t)dt + \nabla A_u(\xi(t))(\zeta(t), dw(t)), \quad \zeta(s) = I, \tag{6}$$

$$d\kappa(t) = c_u(\xi(t)\kappa(t)dt + C_u(\xi(t)(\kappa(t), dw(t)) + \tag{7}$$
$$\nabla c_u(\xi(t))(\zeta(t), \eta(t))dt + \nabla C_u(\xi(t))(\zeta(t), \eta(t), dw(t)), \quad \kappa(s) = 0,$$

$$\langle h, \nabla_y u(s,x)\rangle = E[\langle \eta(T), \nabla_{\zeta(T)} u_0(\xi(T))\rangle + \langle \kappa(T), u_0(\xi(T))\rangle | \mathcal{F}_s] + \tag{8}$$
$$E\left[\int_s^T [\langle \eta(\theta), \nabla_{\zeta(\theta)} g(\xi_{s,x}(\theta), u(\theta, \xi_{s,x}(\theta)))\rangle + \langle \kappa(\theta), g(\xi(\theta), u(\theta,\xi(\theta)))\rangle] d\theta | \mathcal{F}_s\right].$$

THEOREM 2. *Let* $\mathbf{C}^{\mathbf{1,3}}$ *hold. Then there exists an interval* $[T_1, T]$ *on which the Cauchy problem (1) has a unique solution* $u(s,x)$ *which admits a probabilistic representation of the form (4).*

In addition, if $\mathcal{N}^{1,2}$ is the set of scalar functions of the form

$$\Psi(s,x,h) = \langle h, u(s,x)\rangle$$

defined on $[0,T]\times X\times Y$, differentiable in $s\in[0,T]$, and twice differentiable in $z\in X\times Y$, then a solution to (2) belongs to $\mathcal{N}^{1,2}$ if $\Psi_0\in\mathcal{N}^{1,2}$.

Provided $u(s,x)$ is a classical solution of (1), we can check that $\Phi(s,x,h) = \langle h, u(s,x)\rangle$ given by (4) satisfies the scalar Cauchy problem,

$$(9)\qquad \frac{\partial\Phi}{\partial s} + \langle n(\gamma),\nabla\Phi\rangle + \frac{1}{2}TrN^*(\gamma)\nabla^2\Phi N(\gamma) + G(\gamma,u) = 0, \quad \Phi(T,\gamma) = \langle h, u_0(x)\rangle$$

w.r.t. $\Phi(t,\gamma) = \langle h, u(s,x)\rangle, \gamma = (x,h)$, where $G = \langle h, g\rangle$.

Moreover, we can show that systems (2)-(4), (6)-(8), and (2)-(4) have a similar structure. To this end we set $\Theta = Y\oplus X\diamond Y$, and let $\gamma = (\gamma_1,\gamma_2)\in\Theta$ have the form $\gamma_1 = \kappa, \gamma_2 = y\diamond h$. Then one can treat (2)-(4), (6)-(8) as a system that consists of (2) and

$$(10)\qquad d\lambda(t) = m(\xi(t))\lambda(t)dt + M(\xi(t))(\lambda(t), d\tilde{W}(t)), \quad \lambda(s) = \lambda\in\Theta.$$

Here $\tilde{W}(t) = (w(t), w(t)\diamond w(t))^*$ and the coefficients have the form

$$m(x)\begin{pmatrix}\kappa\\ \zeta\diamond\eta\end{pmatrix} = \begin{pmatrix} c(x) & \hat{\nabla}c(x)\\ 0 & \nabla a(x)\oplus c(x)\end{pmatrix}\begin{pmatrix}\kappa\\ \zeta\diamond\eta\end{pmatrix},$$

$$M(x)\begin{pmatrix}\kappa\\ \zeta\diamond\eta\end{pmatrix} = \begin{pmatrix} C(x) & \hat{\nabla}C(x)\\ 0 & \nabla\mathcal{M}(x)\oplus C(x)\end{pmatrix}\begin{pmatrix}\kappa\\ \zeta\diamond\eta\end{pmatrix},$$

where $\hat{\nabla}c(x)\zeta\diamond\eta = \nabla c(x)(\zeta,\eta), \quad \hat{\nabla}C(x)\zeta\diamond\eta = \nabla C(x)(\zeta,\eta)$.

Along with (1) we will consider the Cauchy problem

$$(11)\qquad \frac{\partial v_i^l}{\partial s} + [\mathcal{B}(x)v]_i^l + [[\nabla_i\mathcal{B}]u]^l + \nabla g^l(x,u) = 0, \quad v_i^l(T,x) = \nabla_i u_0^l(x)$$

w.r.t. $v_i^l = \nabla_i u^l$, where $[[\nabla_i\mathcal{B}]u]^l = v_k^l\nabla_i a^k(x) + Tr\nabla_i A(x)\nabla v^l A(x) + \nabla_i B_m^{lk} v_k^m + \nabla_i c_m^l u^m$. Here $\nabla_i g(x,u) = g_i^1(x,u) + g_m^2(x,u)\nabla_i u^m$ and, given $\alpha = (\alpha_1,\dots,\alpha_k)$, we use notation $g_m^j(\alpha) = \frac{\partial g_m(\alpha)}{\partial\alpha_j}, \quad j = 1,\dots,k$. Actually, the system (1),(11) is a semilinear system w.r.t. $V(t,x) = (u(t,x),\nabla u(t,x))$. This together with (10) allows us to apply the above theorems to construct solutions both to SDEs (2), (10) and to the Cauchy problem (1), (11).

Note that another useful way to view (2),(3), (6),(7) is to consider them as an SDE system w.r.t. components of a process $\beta(t) = (\chi(t),\nu(t))$, where the processes $\chi(t)$ and $\nu(t)$ satisfy SDEs

$$(12)\qquad d\chi(t) = b(\chi(t))dt + B(\chi(t))dW(t), \quad \chi(s) = \chi = (x,y)\in H_1,$$

$$(13)\qquad d\nu(t) = q(\chi(t))\nu(t)dt + Q(\chi(t))\nu(t)dW(t), \quad \nu(s) = \nu = (0,h)\in H_2,$$

where

$$q(\chi)\begin{pmatrix}\kappa\\ h\end{pmatrix}=\begin{pmatrix}c(x)\kappa\\ \nabla_y c(x)h\end{pmatrix},\quad Q(\chi)\begin{pmatrix}\kappa\\ h\end{pmatrix}=\begin{pmatrix}C(x)\kappa\\ \nabla_y C(x)h\end{pmatrix},$$

and $b(\chi)=(a(x),\nabla_y A)^*, B(\chi)=(A(x),\nabla_y A(x))^*$.

All the above constructions can be extended to the case when coefficients a, A, c, C, and the function g, depend on $x,u,\nabla u$, and even $\nabla^2 u$. This allows us to include a quasilinear or fully nonlinear system of the form (1) into a semilinear system with a similar structure w.r.t. a function $U=(u,\nabla u,\nabla^2 u)$ or $U=(u,\nabla u,\nabla^2 u,\nabla^3 u)$, respectively, and to prove the existence and uniqueness of its solution on a small interval $[\tau,T]$ depending on coefficients and functions u_0 and g, when they satisfy $\mathbf{C}^{\mathbf{1,k}}$ with $k=5$ or $k=6$. One can see the detailed proof of the above results in [4], [5].

2. Probabilistic approach to a viscosity solution of the Cauchy problem for a nonlinear PDE system

In this section we construct a viscosity solution of a fully nonlinear version of the Cauchy problem (1) based on the BSDE theory developed in [6], [7] in combination with the constructions described in the previous section.

To be more precise we first develop a modification of the approach of [7] that allows us to construct a viscosity solution of a system of quasilinear parabolic equations of the form (1) with coefficients depending on $x,u,\nabla u$ and $g=g(x,u,A\nabla u)$, and then apply a differential prolongation procedure to a system of fully nonlinear parabolic equations to include it into a system of quasilinear parabolic equations. This makes it possible for us to apply the BSDE technique to construct a viscosity solution to a system of fully nonlinear parabolic equations. The details of the corresponding construction can be found in [4], [5].

Let us consider the Cauchy problem (1) in a larger system consisting of (1) and

$$(14)\qquad \frac{\partial v_i^l}{\partial s}+[\mathcal{B}(x)v]_i^l+[[\nabla_i\mathcal{B}]u]^l+\nabla g^l(x,u,A\nabla u,\nabla^2 u_l)=0,\quad v_i^l(T,x)=\nabla_i u_0^l(x)$$

w.r.t. $v_i^l=\nabla_i u^l$, where $[[\nabla_i\mathcal{B}]u]_l=v_k^l\nabla_i a^k\rangle+Tr\nabla_i A\nabla v_l A+\nabla_i B_m^{lk}v_{mk}+\nabla_i c_m^l u_m$,

$$\nabla_i g(x,u,A\nabla u,\nabla^2 u_l)=g_i^1(x,u,A\nabla u,\nabla^2 u_l)+g_m^2(x,u,A\nabla u,\nabla^2 u_l)\nabla_i u_m+$$

$$g_{jm}^3(x,u,A\nabla u,\nabla^2 u_l)\nabla_i(A\nabla u)_{jm}+g_{lkj}^4(x,u,\nabla u,\nabla^2 u_l)\nabla_{kj}^2 u_{li}.$$

At this point we need to examine a fully coupled system of forward-backward SDEs (FBSDEs) associated with (1), (14), state conditions on their coefficients and functions g and u_0 to ensure the existence and uniqueness of a solution to the resulting FBSDE system and, finally, check that our results lead to construction of a viscosity solution of (1).

Let $V(s,x,y)=(u(s,x),p(s,x,y))$, $p(s,x,y)=\langle y,\nabla u(s,x)\rangle$. Then (1), (14) may be rewritten as

(15) $$\frac{\partial V_m}{\partial s}+\mathcal{G}V_m+\hat{C}^i_{lm}\mathcal{M}^k_i\nabla_k V_l+\hat{c}_{lm}V_l+G_m(x,y,V,\nabla V)=0, \quad \text{where}$$

(16) $$\mathcal{G}V_m=\frac{1}{2}Tr\mathcal{M}^*(x,V,\nabla V)\nabla^2 V_m\mathcal{M}(x,V,\nabla V)+\langle m(x,V,\nabla V),\nabla V_m\rangle,$$

$$m=\begin{pmatrix} a\\ a\end{pmatrix},\quad \frac{1}{2}[\mathcal{M}^*\mathcal{M}]_{jk}=\begin{pmatrix} A_{jk}+\frac{\partial g}{\partial q_{jk}} & 0\\ 0 & A_{jk}+\frac{\partial g}{\partial q_{jk}}\end{pmatrix},$$

$$G(x,y,V,\nabla V_l)=\begin{pmatrix} g(x,u,p,\nabla p_l)\\ g^1(x,y,u,p,\nabla p^l)\end{pmatrix},$$

and $\hat{C}$, $\hat{c}$ depend on C,c,a,A and their derivatives.

Assume that $\mathbf{C}^{\mathbf{1,2}}$ holds. Then (1) and (15) have similar structures.

Consider the Cauchy problem for (15) with the Cauchy data

(17) $$V(T,x,y)=V_0(x,y)=(u_0(x),\nabla_y u_0(x)).$$

Set $H_1=X\times X, H_2=Y\times Y, H_3=M\times M_X$ and $\chi(t)=(\xi(t),\zeta(t))\in H_1$, $\beta(t)=(\eta(t),\kappa(t))\in H_2, \quad Y(t)=(y(t),p(t))\in H_2, \quad Z(t)=(p(t),q(t))\in H_3.$

Assume that $\lambda(t)$ satisfies an equation of the form (10) associated with (15), (17). Consider a stochastic process $\Pi(t)=\Phi(t,\beta(t))$, where

$$\Phi(t,\beta(t))=\Phi^h_1(t,\xi(t))+\Phi^h_2(t,\chi(t))=$$

$$[\langle\eta(t),u_0(\xi(t))\rangle]+[\langle\kappa(t),u_0(\xi(t))\rangle+\langle\eta(t),\nabla_{\zeta(t)}u_0(\xi(t))\rangle]$$

and notice that $\Phi^h_2(t,\chi(t))$ is linear in h and y. From Ito's formula, and (12), we deduce that the stochastic differential of the process $\tilde{Y}(t)=\Phi^h_2(t,\chi(t))=\langle\nu(t),V(t,\chi(t))\rangle$ has the form

(18) $$d\tilde{Y}(t)=-\tilde{G}(\chi(t),V(t,\chi(t)),\nabla V(t,\chi(t))\zeta(t))dt+$$
$$\langle\nabla\Phi^h_2(t,\chi(t)),\mathcal{M}(\chi(t))dW(t)\rangle,$$

where $G(\chi,V,\nabla V)=(g(x,u,\nabla u,\nabla^2 u),\nabla_y g(x,u,\nabla u,\nabla^2 u),$

$$\tilde{G}(\chi(t),V(t,\chi(t)),\nabla_{\zeta(t)}V(t,\chi(t))=\langle\beta,\Xi^*(s,t)G(\chi(t),V(t,\chi(t)),\nabla_{\zeta(t)}V(t,\chi(t)))\rangle,$$

and

(19) $$\langle\nabla\Phi_2(t,\chi(t)),\mathcal{M}(t,\chi(t))dW(t)\rangle=\langle\nabla C(\xi(t))(\zeta(t),\eta(t),dw(t)),u(t,\xi(t))\rangle+$$
$$\langle C(\xi(t))(\kappa(t),dw(t)),u(t,\xi(t))\rangle+\langle\eta(t),\nabla u(t,\xi(t))\nabla\mathcal{M}(\xi(t))(\zeta(t),dw(t))\rangle.$$

We deduce from (18), (19) that the process $\tilde{Y}(t)$ satisfies

(20) $d\tilde{Y}(t) = -\tilde{G}(\chi(t), \tilde{Y}(t), \tilde{Z}(t))dt + \langle \tilde{Z}(t), dW(t)\rangle, \quad \tilde{Y}(T) = \langle \beta(T), V_0(\chi(T))\rangle,$

and the processes $Y(t) = (Y^1(t), Y^2(t)), Z(t) = (Z^1(t), Z^2(t))$ defined by

$$\tilde{Y}(t) = \langle \beta(t), Y(t)\rangle = \langle \beta, \Xi^*(s,t)Y(t)\rangle = \langle \kappa(t), Y^1(t)\rangle + \langle \eta(t), Y^2(t)\rangle =$$

$$\langle \kappa, \Xi_1^*(s,t)Y_1(t)\rangle + \langle h, \Xi_2^*(s,t)Y_2(t)\rangle, \quad \tilde{Z}(t) = \langle \beta, Z(t)\rangle$$

satisfy the BSDE

(21) $dY(t) = -G(\chi(t), Y(t), Z(t))dt + ZdW(t), \quad Y(T) = \Xi^*(s,T)V_0(\chi(T)).$

Finally, we deduce that one can associate with (15), (17) the following FBSDEs w.r.t. $\mathcal{F}_t$-measurable stochastic processes $\chi(t) = (\xi(t), \zeta(t)) \in H_1$, $Y(t) = (y(t), p(t)) \in H_2$, $\quad Z(t) = (p(t), q(t)) \in H_3 = M \times M_X$,

(22) $d\chi(t) = b(\chi(t), Y(t), Z(t))dt + B(\chi(t), Y(t), Z(t))dW(t), \quad \chi(s) = \chi \in H_1,$

(23) $dY(t) = -G(\chi(t), Y(t), Z(t))dt + Z(t)dW, \quad \alpha = Y(T) = (\alpha_1, \alpha_2) \in H_2.$

Here $b = (a, \nabla A), B = (A, \nabla A)$, $G = (g, g^1) \in H_2$, and $Y(T)$ is $\mathcal{F}_T$-measurable.

Let $\mathcal{M}^2([0,T];\mathbb{R}^d)$ denote the set of progressively measurable square integrable stochastic processes $\xi(t) \in \mathbb{R}^d$, $\quad E\left[\int_0^T \|\xi(\tau)\|^2 d\tau\right] < \infty$, and $\mathcal{S}^2([0,T],X)$ denote the set of semimartingales $\eta(t) \in \mathbb{R}^d$, such that $E\left[\sup_{0\le t\le T} \|\eta(t)\|^2\right] < \infty$.

A solution to FBSDE (22),(23) is a triple of progressively measurable processes $(\chi(t), Y(t), Z(t))$ in $\mathcal{S}^2([0,T];H_1) \times \mathcal{S}^2([0,T];H_2) \times \mathcal{M}^2([0,T];H_3)$ such that

(24) $\chi(t) = \chi + \int_s^t b(\chi(\tau), Y(\tau), Z(\tau))d\tau + \int_s^t B(\chi(\tau), Y(\tau), Z(\tau))dW(\tau),$

and

(25) $Y(t) = \alpha + \int_s^t G(\chi(\tau), Y(\tau), Z(\tau))d\tau - \int_s^t Z(\tau)dW(\tau), \quad 0 \le t \le T,$

with probability 1.

Now we are in the framework of the FBSDE theory and have to consider a fully coupled system of forward-backward stochastic equations. To prove the existence and uniqueness of a solution to (22), (23) we need some additional conditions that allow us to apply the technique of homotopy prolongation [8].

We say that **C 2** holds when **C 1,1** holds and the random function $F(t,Y,Z) = G(\chi(t),Y,Z) \in H_2$ satisfies the standard conditions of the BSDE theory [6] which ensure the existence and uniqueness of a solution to a BSDE equation

$$dY(t) = -F(t, Y(t), Z(t))dt + Z(t)dW, \quad \alpha = Y(T) \in H_2.$$

Let

$$\mathcal{H}_1 = \{\chi(t) \in H_1 : E \sup_{t\in[0,T]} \|\chi(t)\|^2 < \infty\}, \quad \mathcal{H}_2 = \{Y(t) \in H_2 : E \sup_{t\in[0,T]} \|Y(t)\|^2 < \infty\},$$

$$\mathcal{H}_3 = \{Z(t) \in H_3 : E \int_0^T |Z(t)|^2 dt < \infty\}, \quad \mathcal{H} = \mathcal{H}_1 \times \mathcal{H}_2 \times \mathcal{H}_3$$

and $\|\cdot\|_{\mathcal{H}}$ denote the norm in $\mathcal{H}$, that is, if $\Theta = (\chi, Y, Z) \in \mathcal{H}$, then

$$\|\Theta\|_{\mathcal{H}}^2 = E\left[\sup_{[0,T]} \|\chi(t)\|^2 + \sup_{[0,T]} \|Y(t)\|^2 + \int_0^T |Z(t)|^2 dt\right].$$

Denote by $D = H_1 \times H_2 \times H_3$, $\mathcal{D} = \mathcal{M}^2(0,T;D) \cap \mathcal{H}$ and, for $\Theta = (\chi, \kappa, \upsilon) \in D$, let $\Upsilon(\Theta) = (-F(\Theta), b(\Theta), B(\Theta))$. We say that $\mathbf{C}^{\,\mathbf{3}}$ holds if there exists a constant $C > 0$ such that functions $\Upsilon : D \to D$ and V_0 satisfy the estimates

$$\|\Upsilon(\Theta) - \Upsilon(\Theta_1)\|_D \le C\|\Theta - \Theta_1\|_D, \quad \forall \Theta, \Theta_1 \in D, \quad P-\text{a.s.}$$

$$\|V_0(\chi) - V_0(\chi_1)\| \le C\|\chi - \chi_1\|, \quad \forall \chi, \chi_1 \in H_1 \quad P-\text{a.s.}.$$

We say that $\mathbf{C}^{\,\mathbf{4}}$ holds if there exists a constant $C_1 > 0$ such that

$$\langle\langle \Upsilon(\Theta) - \Upsilon(\Theta_1), \Theta - \Theta_1 \rangle\rangle \le -C_1\|\chi - \chi_1\|^2, \quad \forall \chi, \chi_1 \in H_1, P-\text{a.s.},$$

where $\langle\langle \cdot, \cdot \rangle\rangle$ is an inner product in $\tilde{D}$ and

$$\langle V_0(\chi) - V_0(\chi_1), N[\chi - \chi_1] \rangle \ge C_1\|\chi - \chi_1\|^2 \quad \chi, \chi_1 \in H_1, P-\text{a.s.}$$

Let us start with a simple case as the starting point in the homotopy construction.

LEMMA 1. *Let* $(b^0, F^0, B^0) \in \mathcal{D}$, $\kappa^0 \in L^2(\Omega, \mathcal{F}_T, P)$*. Then there exists a unique solution* $(\chi, Y, Z) \in \mathcal{D}$ *of FBSDE*

$$d\chi(t) = [Y(t) - b^0(t)]dt + [Z(t) - B^0(t)]dw(t), \quad \chi(0) = \chi, \tag{26}$$

$$dY(t) = -[F^0(t) - \chi(t)]dt + Z(t)dw(t), \quad Y(T) = \chi(T) + \alpha, \quad 0 \le t \le T. \tag{27}$$

Next, for a given $\mu \in [0,1]$, denote by

$$b^\mu(\chi, Y, Z) = (1-\mu)Y - \mu b(\chi, Y, Z), \quad B^\mu(\chi, Y, Z) = (1-\mu)z - \mu B(\chi, Y, Z),$$

$$F^\mu(\chi, Y, Z) = (1-\mu)\chi - \mu F(\chi, Y, Z), \quad V_0^\mu(\chi) = \mu V_0(\chi) + (1-\mu)\chi.$$

From general results of BSDE theory and Lemma 1 we can deduce that at least for $\mu = 0$ there exists a unique solution of the FBSDE

$$\chi(t) = \chi + \int_0^t [b^\mu(\Theta(\tau)) - b^0(\tau)]d\tau + \int_0^t [B^\mu(\Theta(\tau)) - B^0(\tau)]dW(\tau), \tag{28}$$

$$Y(t) = (V_0^\mu(\chi(T)) + \kappa^0) - \int_0^t [F^\mu(\Theta(\tau)) - F^0(\tau)]d\tau - \int_0^t Z(\tau)dW(\tau). \tag{29}$$

LEMMA 2. *Assume that* $\mathbf{C}^{\mathbf{2}}$-$\mathbf{C}^{\mathbf{4}}$ *hold,* $(b^\mu, B^\mu, F^\mu) \in \mathcal{D}$ *and, for* $\mu = \mu_0 \in [0,1]$, *there exists a unique solution* $\Theta^{\mu_0}(t) = (\chi^{\mu_0}(t), Y^{\mu_0}(t), Z^{\mu_0}(t)) \in \mathcal{D}$ *of (28), (29). Then there exists a constant* $\delta_0 \in [0,1)$, *depending on* C_1, C_2 *and* T *such that there exists a unique solution* $(\chi^\mu(t), Y^\mu(t), Z^\mu(t)) \in \mathcal{D}$ *of (28), (29) for* $\mu = \mu_0 + \delta$, *where* $\delta \in [0, \delta_0]$.

As a result we can deduce the following statement.

THEOREM 3. *Assume that* $\mathbf{C}^{\mathbf{1,1}} - \mathbf{C}^{\mathbf{3}}$ *hold. Then there exists a unique solution* (χ, Y, Z) *of (24)-(25). In addition, the function* $V(s,\chi) = Y(s)$ *is a continuous viscosity solution of (15),* $V(s,\chi) = (u(s,x), \nabla_y u(s,x))$ *and its first component* $u(s,x)$ *is a viscosity solution of (1).*

To verify that $V(s,\chi)$ is a viscosity solution to (15) and hence $u(s,x)$ is a viscosity solution to (1) one needs comparison theorems which are well known for scalar equations but are much less known for the case of nondiagonal systems. Actually, we can overcome this difficulty due to the special structure of the systems under consideration, and our ability to reduce them to scalar equations in a new phase space (described in Section 1).

Finally, applying Ito's formula, it is not difficult to check that the following inequalities hold

$$E\left(\int_s^\tau \Lambda^l(\theta, \chi(\theta), Y(\theta), Z^l(\theta))d\theta\right) \geq 0 (\leq 0),$$

where

$$\Lambda^l(s,\chi,Y,Z) = [\frac{\partial \Phi^l}{\partial \theta} + \mathcal{A}\Phi^l](s,\chi) - F^l(\chi,Y,Z),$$

$\Phi = (\phi, \nabla_y \phi)$, and $\phi(s,x) \in \mathbb{R}^d$ is a C^3- smooth function such that (s,χ) is a point, where a local maximum (minimum) of $V^l(s,\chi) - \Phi^l(s,\chi), l = 1,\ldots,d_2 = 2d_1$ is attained. Combining this with the comparison results we can prove the last statement of the theorem.

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AMS Subject Classification: 60H10, 60J70

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Lavoro pervenuto in redazione il 22.05.2013

Rend. Sem. Mat. Univ. Politec. Torino
Vol. 71, 2 (2013), 219 – 226

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ON THE NEWTON-NELSON TYPE EQUATIONS ON VECTOR BUNDLES WITH CONNECTIONS

Abstract. An equation of Newton-Nelson type on the total space of vector bundle with a connection, whose right-hand side is generated by the curvature form, is described and investigated. An existence of solution theorem is obtained.

Introduction

In [5] (see also [6]) a certain second order differential equation on the total space of vector bundle with a connection was constructed and investigated. In some cases it was interpreted as an equation of motion of a classical particle in the classical gauge field. The form of this equation allowed one to apply the quantization procedure in the language of Nelson's Stochastic Mechanics (see, e.g., [8, 9]). In [7] this procedure was realized for the vector bundles over Lorentz manifolds with complex fibers. The corresponding relativistic-type Newton-Nelson equation (the equation of motion in Stochastic Mechanics) was constructed and the existence of its solutions under some natural conditions was proved. The results of [7] were interpreted as the description of motion of a quantum particle in the gauge field.

In this paper we consider the analogous non-relativistic Newton-Nelson equation in the situation where the base of the bundle is a Riemannian manifold and the fiber is a real linear space. In this case some deeper results are obtained under some less restrictive conditions with respect to the case of [7].

We refer the reader to [2, 6] for the main facts of the geometry of manifolds and to [4, 6] for general facts of Stochastic Analysis on Manifolds.

1. Necessary facts from the Geometry of Manifolds

Recall that for every bundle E over a manifold M, in each tangent space $T_{(m,x)}E$ to the total space E there is a special sub-space $\mathsf{V}_{(m,x)}$, called *vertical*, that consists of the vectors tangent to the fiber E_m (called also vertical). In the case of principal or vector bundle, a connection H on E is a collection of sub-spaces in tangent spaces to E such that $T_{(m,x)}E = \mathsf{H}_{(m,x)} \oplus \mathsf{V}_{(m,x)}$ at each $(m,x) \in E$ and this collection possesses some properties of smoothness and invariance (see, e.g., [6]).

Denote by $\mathcal{M}$ a Riemannian manifold with metric tensor $g(\cdot,\cdot)$. Let $\Pi : \mathcal{E} \to \mathcal{M}$ be a principal bundle over $\mathcal{M}$ with a structure group G. By $\mathfrak{g}$ we denote the Lie algebra of G. Let a connection H with connection form θ and curvature form $\Phi = D\theta$ be given

*The research is supported in part by the RFBR Grants 10-01-00143 and 12-01-00183

on $\mathcal{E}$. Here D is the covariant differential (see, e.g., [2]). Recall that the 1-form θ and the 2-form Φ are equivariant and take values in the algebra $\mathfrak{g}$ of G and that Φ is horizontal (equals zero on vertical vectors).

We suppose G to be a subgroup of $GL(k,\mathbb{R})$ for a certain k. Let $\mathcal{F}$ be a k-dimensional real vector space, on which G acts from the left, and let on $\mathcal{F}$ an inner product $h(\cdot,\cdot)$, invariant with respect to the action of G, be given. We suppose that a mapping $e:\mathcal{F}\to\mathfrak{g}^*$ (where $\mathfrak{g}^*$ is the co-algebra) having constant values on the orbits of G, is given. This mapping is called *charge*.

Consider the vector bundle $\pi : Q\to\mathcal{M}$ with standard fiber $\mathcal{F}$, associated to $\mathcal{E}$. We denote by Q_m the fiber at $m\in\mathcal{M}$. Consider the factorization $\lambda:\mathcal{E}\times\mathcal{F}\to Q$ that yields the bundle Q (see [2]). The tangent mapping $T\lambda$ translates the connection H from the tangent spaces to $\mathcal{E}$ to tangent spaces to Q. This connection on Q is denoted by H^π. Recall that the spaces of connection are the kernels of operator $K^\pi : TQ\to Q$ called *connector*, that is constructed as follows. Consider the natural expansion of the tangent vector $X\in T_{(m,q)}Q$ at $(m,q)\in Q$ into horizontal and vertical components $X=\mathsf{H}X+\mathsf{V}X$, where $\mathsf{H}X\in\mathsf{H}^\pi_{(m,q)}$ and $\mathsf{V}X\in\mathsf{V}_{(m,q)}$. Introduce the operator $\mathbf{p}:\mathsf{V}_{(m,q)}\to Q_m$, the natural isomorphism of the linear tangent space $\mathsf{V}_{(m,q)}=T_qQ_m$ to the fiber Q_m of Q onto the fiber (linear space) Q_m. Then $K^\pi X=\mathbf{p}\mathsf{V}X$.

On the manifold Q (the total space of bundle) we construct the Riemannian metric g^Q as follows: in the horizontal subspaces H^π we introduce it as the pull-back $T\pi^*g$, in the vertical subspaces V – as h and define that H^π are V orthogonal to each other.

We denote the projection of tangent bundle $T\mathcal{M}$ to $\mathcal{M}$ by $\tau : T\mathcal{M}\to\mathcal{M}$ and by H^τ the Levi-Civita connection of metric g on $\mathcal{M}$. Its connector is denoted by $K^\tau : T^2\mathcal{M}\to T\mathcal{M}$. The construction of K^τ is quite analogous to that of K^π where Q is replaced by $T\mathcal{M}$ and TQ by $T^2\mathcal{M}=TT\mathcal{M}$.

Recall the standard construction of a connection on the total space of bundle Q, based on the connections H^π and H^τ (see, e.g., [3, 6]). The connector $K^Q : T^2Q\to TQ$ of this connection has the form: $K^Q=K^H+K^V$ where $K^H : T^2Q\to\mathsf{H}^\pi$ and $K^V : T^2Q\to\mathsf{V}$, and the latter connectors are introduced as: $K^H=T\pi^{-1}\circ K^\tau\circ T^2\pi$ where $T^2\pi=T(T\pi):T^2Q\to T^2\mathcal{M}$ and $T\pi^{-1}$ is the linear isomorphism of tangent spaces to $\mathcal{M}$ onto the spaces of connection H^π; $K^V=\mathbf{p}^{-1}\circ K^\pi\circ TK^\pi$.

Recall that λ is a one-to-one mapping of the standard fiber $\mathcal{F}$ onto the fibers of bundle Q, hence the charge e is well-defined on the entire Q. Since $T\lambda$ is also a one-to-one mapping of the connections and Φ is equivariant, we can introduce the differential form $\tilde\Phi$ on Q with values in $\mathfrak{g}$ as follows. Consider $(m,q)=\lambda((m,p),f)$ for $(m,p)\in\mathcal{E}$ and $f\in\mathcal{F}$. For X, $Y\in T_{(m,q)}Q$ we denote by $\mathsf{H}X$ and $\mathsf{H}Y$ their horizontal components. Then we define $\tilde\Phi_{(m,q)}(X,Y)=\Phi_{(m,p)}(T\lambda^{-1}\mathsf{H}X,T\lambda^{-1}\mathsf{H}Y)$.

Denote by $\bullet$ the coupling of elements of $\mathfrak{g}$ and $\mathfrak{g}^*$.Consider the vector $((m,q),X)$ tangent to Q at (m,q). It is clear that $e((m,q))\bullet\tilde\Phi_{(m,q)}(\cdot,X)$ is an ordinary 1-form (i.e., differential form with values in real line). Denote by $\overline{e((m,q))\bullet\tilde\Phi_{(m,q)}(\cdot,X)}$ the tangent vector to the total space of Q physically equivalent to the form $e((m,q))\bullet\tilde\Phi_{(m,q)}(\cdot,X)$ (i.e., obtained by lifting the indices with the use of Riemannian metric g^Q).

LEMMA 1 ([5]). *The vector field $\overline{e((m,q))\bullet\tilde{\Phi}_{(m,q)}(\cdot,X)}$ is horizontal, i.e., it belongs to the spaces of connection H^{π}.*

THEOREM 1 ([7]). *Let $(m(t),q(t))$ be a smooth curve in Q. Let $X(t)$ be the parallel translation of the vector $X\in T_{(m(t_0),q(t_0))}Q$ along $(m(t),q(t))$ with respect to H^Q. (i) Both the horizontal $\mathsf{H}X(t)$ and vertical $\mathsf{V}X(t)$ components of $X(t)$ are parallel along $(m(t),q(t))$ with respect to H^Q. (ii) The parallel translation of horizontal vectors preserves constant the norms and scalar products with respect to g^Q. (iii) The vector field $T\pi X(t)$ is parallel along $m(t)$ on $\mathcal{M}$ with respect to H^{τ}.*

2. Mean derivatives on manifolds and vector bundles

Consider a stochastic process $\xi(t)$ with values in $\mathcal{M}$, given on a certain probability space $(\Omega,\mathfrak{F},\mathsf{P})$. By $\mathfrak{N}_t^{\xi}$ we denote the minimal σ-sub-algebra of σ-algebra $\mathfrak{F}$ generated by the pre-images of Borel sets in $\mathcal{M}$ under the mapping $\xi(t):\Omega\to\mathcal{M}$ (the "present" or "now" of $\xi(t)$) and by $E(\cdot\mid\mathfrak{N}_t^{\xi})$ the conditional expectation with respect to $\mathfrak{N}_t^{\xi}$. Recall that the conditional expectation of a random element ϑ with respect to $\mathfrak{N}_t^{\xi}$ can be represented as $\Theta(\xi(t))$ where Θ is the so-called *regression* introduced by the formula $\Theta(m)=E(\theta\mid\xi(t)=m)$ (see, e.g., [10]).

Specify a point in $\mathcal{M}$ and consider the normal chart U_m at this point with respect to the exponential mapping of Levi-Civita connection on $\mathcal{M}$. In U_m construct the following regressions

(1) $$Y^{U_m}(t,m')=\lim_{\Delta t\downarrow 0}E\left(\frac{\xi(t+\Delta t)-\xi(t)}{\Delta t}\mid\xi(t)=m'\right);$$

(2) $${}_*^{U_m}(t,m')=\lim_{\Delta t\downarrow 0}E\left(\frac{\xi(t)-\xi(t-\Delta t)}{\Delta t}\mid\xi(t)=m'\right).$$

Introduce $X^0(t,m)=Y^{U_m}(t,m)$ and $X_*^0(t,m)=Y_*^{U_m}(t,m)$. Note that $X^0(t,m)$ and $X_*^0(t,m)$ are vector fields on $\mathcal{M}$, i.e., under the coordinate changes they transform like cross-sections of the tangent bundle $T\mathcal{M}$.

Forward and backward mean derivatives of $\xi(t)$ are defined by the formulae $D\xi(t)=X^0(t,\xi(t))$ and $D_*\xi(t)=X_*^0(t,\xi(t))$.

The vector $v^{\xi}(t)=\frac{1}{2}(D+D_*)\xi(t)$ is called the *current velocity* of $\xi(t)$. From the properties of conditional expectation it follows that there exists a Borel measurable vector field (regression) $v^{\xi}(t,m)$ on $\mathcal{M}$ such that $v^{\xi}(t)=v^{\xi}(t,\xi(t))$.

Introduce the increment $\Delta\xi(t)$ of process $\xi(t)$: $\Delta\xi(t)=\xi(t+\Delta t)-\xi(t)$ and the so called quadratic mean derivative D_2 (see [1, 6]) $D_2\xi(t)=\lim\limits_{\Delta\tau\downarrow 0}E(\frac{\Delta\xi(t)\otimes\Delta\xi(t)}{\Delta t}\mid\mathfrak{N}_t^{\xi})$. If $D_2\xi(t)$ exists, it takes values in $(2,0)$-tensors.

Everywhere below we are dealing with processes, along which the parallel translation with respect to an appropriate connection is well-posed. Here we use $\xi(\cdot)$ and parallel translation with respect to the connection H^{τ} and such an assumption is true, for example, if $\xi(t)$ is an Itô process on $\mathcal{M}$, i.e., an Itô development of an Itô process in a certain tangent space to $\mathcal{M}$ as it is defined in [6]. Denote by $\Gamma_{t,s}$ the operator of such parallel translation along $\xi(\cdot)$ of tangent vectors from the (random) point $\xi(s)$ of the process to the (random) point $\xi(t)$.

For a vector field $Z(t,m)$ on $\mathcal{M}$ the covariant forward and backward mean derivatives $\mathbf{D}Z(t,\xi(t))$ and $\mathbf{D}_*Z(t,\xi(t))$ are constructed by the formulae

(3) $$\mathbf{D}Z(t,\xi(t)) = \lim_{\Delta t\downarrow 0} E\left(\frac{\Gamma_{t,t+\Delta t}Z(t+\Delta t,\xi(t+\Delta t)) - Z(t,\xi(t))}{\Delta t} \mid \mathfrak{N}_t^\xi\right);$$

(4) $$\mathbf{D}_*Z(t,\xi(t)) = \lim_{\Delta t\downarrow 0} E_t^\xi\left(\frac{Z(t,\xi(t)) - \Gamma_{t,t-\Delta t}Z(t-\Delta t,\xi(t-\Delta t))}{\Delta t} \mid \mathfrak{N}_t^\xi\right).$$

From formulae (1), (2), (3) and (4) it evidently follows that $T\pi\mathbf{D}Z(t,\xi(t)) = D\xi(t)$ and $T\pi\mathbf{D}_*Z(t,\xi(t)) = D_*\xi(t)$.

Now consider a stochastic process $\eta(t)$ in the total space of bundle Q and introduce the process $\xi(t) = \pi\eta(t)$ on $\mathcal{M}$. Denote by $\Gamma^\pi_{t,s}$ the parallel translation of random vectors from the fiber $Q_{\xi(s)}$ to the fiber $Q_{\xi(t)}$ along $\xi(\cdot)$ with respect to connection H^π. For $\eta(t)$ we introduce the covariant mean derivatives by formulae

(5) $$\mathbf{D}\eta(t) = \lim_{\Delta t\downarrow 0} E\left(\frac{\Gamma^\pi_{t,t+\Delta t}\eta(t+\Delta t) - \eta(t)}{\Delta t} \mid \mathfrak{N}_t^\xi\right);$$

(6) $$\mathbf{D}_*\eta(t) = \lim_{\Delta t\downarrow 0} E\left(\frac{\eta(t) - \Gamma^\pi_{t,t-\Delta t}\eta(t-\Delta t)}{\Delta t} \mid \mathfrak{N}_t^\xi\right).$$

(analogs of (3) and (4)). As above, $v^\eta(t) = \frac{1}{2}(\mathbf{D}+\mathbf{D}_*)\eta(t)$ is called the *current velocity* of $\eta(t)$.

In order to define the mean derivatives of a vector field along $\eta(t)$ on Q we use the parallel translation $\Gamma^Q_{t,s}$ of vectors tangent to Q at $\eta(s)$, to vectors tangent to Q at $\eta(t)$ along $\eta(\cdot)$ with respect to connection H^Q. By analogy with formulae (3) and (4) for a vector field $Z(t,(m,q))$ on Q we introduce the covariant mean derivatives by formulae

(7) $$\mathbf{D}^Q Z(t,\eta(t)) = \lim_{\Delta t\downarrow 0} E\left(\frac{\Gamma^Q_{t,t+\Delta t}Z(t+\Delta t,\eta(t+\Delta t)) - Z(t,\eta(t))}{\Delta t} \mid \mathfrak{N}_t^\xi\right);$$

(8) $$\mathbf{D}^Q_* Z(t,\eta(t)) = \lim_{\Delta t\downarrow 0} E\left(\frac{Z(t,\eta(t)) - \Gamma^Q_{t,t-\Delta t}Z(t-\Delta t,\eta(t-\Delta t))}{\Delta t} \mid \mathfrak{N}_t^\xi\right).$$

LEMMA 2. *$\Gamma^Q_{t,s}$ translates $\mathsf{H}^\pi_{\eta(s)}$ onto $\mathsf{H}^\pi_{\eta(t)}$ and $\mathsf{V}_{\eta(s)}$ onto $\mathsf{V}_{\eta(t)}$; the parallel translation of horizontal components preserves the norms and inner products with respect to g^Q.*

The assertion of Lemma 2 follows from Theorem 1 and from the fact that (see [3, 6]) that the parallel translation along random processes can be described as the limit of parallel translations along the processes whose sample paths are piece-wise geodesic approximations of the sample paths of the process under consideration.

By symbols $\mathbf{D}^H$ and $\mathbf{D}^H_*$ we denote the derivatives introduced by formulae (7) and (8), respectively, for the horizontal components of vectors (i.e., taking values in H^π). By symbols $\mathbf{D}^V$ and $\mathbf{D}^V_*$ we denote the derivatives for vertical components (i.e., taking values in V). Thus, $\mathbf{D}^Q = \mathbf{D}^H + \mathbf{D}^V$ and $\mathbf{D}^Q_* = \mathbf{D}^H_* + \mathbf{D}^V_*$.

3. The Newton-Nelson equation on the total space of vector bundle

In the problem under consideration the Newton-Nelson equation takes the form

$$\left\{ \begin{array}{l} \frac{1}{2}(\mathbf{D}^Q\mathbf{D}_* + \mathbf{D}^Q_*\mathbf{D})\eta(t) = \overline{e(\eta(t)) \bullet \tilde{\Phi}_{\eta(t)}(\cdot, v^\eta(t))} \\ D_2\xi(t) = \frac{\hbar}{\mathrm{m}} I \end{array} \right. , \tag{9}$$

where $\xi(t) = \pi\eta(t)$ (cf. [8, 9]).

Expand the current velocity v^η in the right-hand side of (9) into the sum of vertical and horizontal components: $v^\eta = v^H_\eta + v^V_\eta$, where $v^H_\eta \in \mathsf{H}^\pi$ and $v^V_\eta \in \mathsf{V}$. Since $\tilde{\Phi}_{\eta(t)}(\cdot,\cdot)$ is linear in both arguments, $\tilde{\Phi}_{\eta(t)}(\cdot, v^\eta) = \tilde{\Phi}_{\eta(t)}(\cdot, v^H_\eta) + \tilde{\Phi}_{\eta(t)}(\cdot, v^V_\eta)$. Then, since the form $\tilde{\Phi}$ is horizontal (see Lemma 1) we obtain that $\tilde{\Phi}_{\eta(t)}(\cdot, v^V_\eta) = 0$. Thus, the first equation of system (9) is equivalent to the following system:

$$\frac{1}{2}(\mathbf{D}^H\mathbf{D}_* + \mathbf{D}^H_*\mathbf{D})\eta(t) = \overline{e(\eta(t)) \bullet \tilde{\Phi}_{\eta(t)}(\cdot, v^H_\eta(t))}, \tag{10}$$

$$\frac{1}{2}(\mathbf{D}^V\mathbf{D}_* + \mathbf{D}^V_*\mathbf{D})\eta(t) = 0. \tag{11}$$

For simplicity of presentation we denote $\overline{e(\eta(t)) \bullet \tilde{\Phi}_{\eta(t)}(\cdot, v^H_\eta(t))}$ by $\alpha_{(t,\eta(t))} v^H_\eta$ where, by construction, $\alpha_{(t,(m',q'))}(\cdot)$ is a linear operator in $\mathsf{H}^\pi_{(m',q')}$ ($(1,1)$-tensor).

Introduce the horizontal $(1,2)$-tensor field $\nabla^H\alpha(\cdot,\cdot) = K^H T\alpha(\cdot)$ on Q. The vector $\mathrm{tr}\,\nabla^H\alpha(\alpha\cdot,\cdot)$ is horizontal by construction.

THEOREM 2. *Let for the tensor field $\alpha_{(t,(m,q))}(\cdot)$ there exist a constant $C > 0$ such that $\int_0^T (\|\alpha_{(t,x(t))}(\cdot)\|^2 + \|\mathrm{tr}\,\nabla^H\alpha_{(t,x(t))}(\alpha\cdot,\cdot)\|^2)dt < C$ for a certain $T > 0$ and every continuous curve $x(t)$ in Q given on $t \in [0,T]$. Here $\|\alpha_{(t,x)}(\cdot)\|$ is the operator norm (all the norms are generated by g^Q). Let also the connections H^τ and H^π be stochastically complete (see [6]). Then for every point $(m,q) \in Q$, every vector $\beta_0 \in \mathsf{H}^\pi_{(m,q)}$ and every time instant $t_0 \in (0,T)$ there exists a stochastic process $\eta(t)$ in Q such that: (i) it is well-defined on $[0,T]$; (ii) $\eta(0) = (m,q)$ and $D\eta(0) = \beta_0$; (iii) for all $t \in (t_0,T)$ the processes $\eta(t)$ and $\xi(t) = \pi\eta(t)$ satisfy* (9)*; (iv) along $\eta(t)$ the charge $e(\eta(t))$ is constant.*

Proof. For simplicity and without loss of generality we suppose that $\frac{\hbar}{\mathrm{m}} = 1$.

Consider on the space of continuous curves $C^0([0,T], T_mM)$ the filtration $\mathcal{P}_t$ where for every $t \in [0,T]$ the σ-algebra $\mathcal{P}_t$ is generated by cylinder sets with bases over $[0,t]$. Consider the Wiener measure ν on the measure space $(C^0([0,T],T_mM),\mathcal{P}_T)$ and so the standard Wiener process $W_m(t)$ in T_mM as the coordinate process on the probability space $(C^0([0,T],T_mM),\mathcal{P}_T,\nu)$. Since H^τ is stochastically complete, the Itô development $W^M(t)$ of $W_m(t)$ with respect to H^τ on M is well-posed. Since H^π is also stochastically complete, the horizontal lift $W^Q(t)$ of $W^M(t)$ onto Q with respect to H^π with initial condition (m,q) is also well-posed. A detailed description of the construction of processes $W^M(t)$ and $W^Q(t)$ can be found in [6].

Since $T\pi : \mathsf{H}^\pi_{(m,q)} \to T_mM$ is a linear isomorphism that defines the metric tensor g^Q in $\mathsf{H}^\pi_{(m,q)}$ by the pull back of g from T_mM, we can translate the Wiener measure and the Wiener process from T_mM to $\mathsf{H}^\pi_{(m,q)}$. Denote by $W(t)$ the Wiener process obtained by this construction. This is a coordinate process on the space of continuous curves in $\mathsf{H}^\pi_{(m,q)}$ with σ-algebra $\mathcal{P}_T$ and Wiener measure.

For $t_0 \geq 0$ we introduce the real-valued function $t_0(t)$ that equals $\frac{1}{t_0}$ for $t < t_0$ and $\frac{1}{t}$ for $t \geq t_0$. Its derivative $t_0'(t)$ is equal to 0 for $t < t_0$ and to $-\frac{1}{t^2}$ for $t \geq t_0$.

Now consider the following Itô equation in $\mathsf{H}^\pi_{(m,q)}$:

$$
\begin{aligned}
\beta(t) = \beta_0 + \frac{1}{2}\int_0^t \Gamma^Q_{0,s}\, tr\, \nabla^H \alpha_{(s,W^Q(s))}(\alpha\cdot,\cdot)ds + \int_0^t \Gamma^Q_{0,s}\alpha_{(s,W^Q(s))}dW(s) \\
-\frac{1}{2}\int_0^t t_0(s)\beta(s)ds - \frac{1}{2}\int_0^t t_0'(s)W(s)ds.
\end{aligned} \tag{12}
$$

Since equation (12) is linear in β, it has a strong and strongly unique solution $\beta(t)$. Since this solution is strong, it can be given on the space of continuous curves in $\mathsf{H}^\pi_{(m,q)}$ equipped with Wiener measure. Consider the following density on the latter space of curves $\theta(l) = \exp\left(-\frac{1}{2}\int_0^T \beta(s)^2 ds + \int_0^T (\beta(s)\cdot dW(s))\right)$. From the hypothesis and from Lemma 2 it follows that it is well-posed. Introduce the measure that has this density with respect to the Wiener measure. It is well-known that with the new measure the coordinate process takes the form $\zeta(t) = \int_0^t \beta(s)ds + w(t)$ where $w(t)$ is a certain Wiener process adapted to $\mathcal{P}_t$. Denote $W^Q(t)$, considered with respect to the new measure, by the symbol $\eta(t)$ and introduce the process $\xi(t) = \pi\eta(t)$; $\xi(t)$ is obtained from $W^M(t)$ by the change of measure. Equation (12) turns into

$$
\begin{aligned}
\beta(t) = \beta_0 + \frac{1}{2}\int_0^t \Gamma^Q_{0,s}\, tr\, \nabla^H \alpha_{(s,\eta(t))}(\alpha\cdot,\cdot)ds + \int_0^t \Gamma^Q_{0,s}\alpha_{(s,\eta(s))}\beta(s)ds \\
+\int_0^t \left(\Gamma^Q_{0,s}\alpha_{(s,\eta(s))}(\cdot) + \frac{1}{2}t_0(s)\right)dw(s) - \frac{1}{2}\int_0^t t_0(s)\beta(s)ds - \frac{1}{2}\int_0^t t_0'(t)\zeta(t)ds.
\end{aligned}
$$

By construction, $\eta(0) = (m,q)$ and $D\eta(t) = \beta_0$. The process $\eta(t)$ satisfies (11) also by construction. The fact that for $t \in (t_0,T)$ the processes $\eta(t)$ and $\xi(t) = \pi\eta(t)$ satisfy (10) and that $D_2\xi(t) = I$ follows from the formulae for mean derivatives obtained in [6, Chapters 12 and 18].

Evidently $\eta(t)$ is the horizontal lift of the process $\xi(t)$ with respect to connection H^π with the initial condition (m,q). Recall that the horizontal lift $\eta(t)$ of $\xi(t)$ is a parallel translation of (m,q) along $\xi(\cdot)$ with respect to H^{π}. Hence, it can be presented in the form $(\xi(t), b_t(f))$ where b_t is the horizontal lift of $\xi(t)$ to $\mathcal{E}$ with respect to connection H and f is a certain vector in the standard fiber $\mathcal{F}$. Thus, the sample paths of $\eta(t)$ belong to an orbit of G and so the charge e is constant along $\eta(t)$. □

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AMS Subject Classification: 58J65; 60H30; 60H10

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Lavoro pervenuto in redazione il 22.05.2013

Rend. Sem. Mat. Univ. Politec. Torino
Vol. 71, 2 (2013), 227 – 237

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STOCHASTIC COMPLETENESS OF SYMMETRIC MARKOV PROCESSES AND VOLUME GROWTH

Abstract. We discuss sufficient conditions for stochastic completeness of various types of Markov processes (diffusions on Riemannian manifolds, jump processes, random walks on graphs) in terms of the volume growth function of the underlying metric measure space.

1. Brownian motion on Riemannian manifolds

Let (M,g) be a Riemannian manifold and μ be the Riemannian measure on M. The Laplace operator (or Laplace-Beltrami operator) Δ is defined to satisfy the Green formula: for all $u,v \in C_0^\infty(M)$

$$\int_M \Delta u \, v d\mu = -\int_M \langle \nabla u, \nabla v \rangle d\mu, \tag{1}$$

where ∇ is the Riemannian gradient and $\langle \cdot,\cdot \rangle$ is the Riemannian inner product (see [2], [6], [10]).

The symmetry of the operator Δ with respect to μ (that follows from (1)) allows to extend it to a self-adjoint operator in $L^2(M,\mu)$. In general, this extension may not be unique, but if M is geodesically complete (which will be assumed throughout) then this extension is unique, that is, Δ is essentially self-adjoint. With some abuse of notation, the self-adjoint extension of Δ will be denoted by the same letter.

As one can see from (1), the operator Δ is non-positive definite, which implies that the operator $P_t := e^{t\Delta}$ is a bounded self-adjoint operator for any $t \geq 0$. The family $\{P_t\}_{t\geq 0}$ is called the *heat semigroup* of Δ for the reason that it resolves the heat equation. More precisely, the following is true:

- for any $f \in L^2$, the function $u(t,x) = P_t f(x)$ is C^∞ smooth in $(t,x) \in (0,+\infty) \times M$, satisfies the heat equation $\frac{\partial u}{\partial t} = \Delta u$ and the initial condition $u(t,\cdot) \overset{L^2}{\to} f$ as $t \to 0+$.
- If $f \geq 0$ then $P_t f \geq 0$; if $f \leq 1$ then $P_t f \leq 1$.
- The semigroup property: $P_t P_s = P_{t+s}$.

Furthermore, the operator P_t is in fact an integral operator with a kernel $p_t(x,y)$ that is a smooth positive function of $t > 0$ and $x,y \in M$ such that

$$P_t f(x) = \int_M p_t(x,y) f(y) d\mu(y) \tag{2}$$

for all $f \in L^2$. The function $p_t(x,y)$ is called the *heat kernel* of Δ (or of M). It is also the minimal positive fundamental solution of the heat equation and the transition density of Brownian motion on M. For example, if $M = \mathbb{R}^n$ then

$$p_t(x,y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x-y|^2}{4t}\right).$$

For general manifolds there is no explicit formula for the heat kernel.

The existence of the heat kernel allows to extend the domain of the operator P_t from L^2 to other spaces. For that, let us use now the identity (2) as the definition of P_t where f is any function such that the integral converges. In particular, P_t extends to a bounded operator on L^∞.

DEFINITION 1. A manifold (M,g) is called *stochastically complete* if $P_t 1 \equiv 1$.

Note that in general we have $0 \le P_t 1 \le 1$. If $P_t 1 \not\equiv 1$ then the manifold M is called *stochastically incomplete.*

Easy examples of stochastically incomplete processes are given by diffusions in bounded domains with the Dirichlet boundary condition. A by far less trivial example was discovered by R.Azencott [1] in 1974: he showed that Brownian motion on a geodesically complete non-compact manifold can be stochastically incomplete. In his example, the manifold has negative sectional curvature that grows to $-\infty$ very fast with the distance to an origin. The stochastic incompleteness occurs because negative curvature plays the role of a drift towards infinity, and a very high negative curvature produces an extremely fast drift that sweeps the Brownian particle to infinity in a finite time.

The first sufficient condition for stochastic completeness of geodesically complete manifolds in terms of lower bound of Ricci curvature was proved by S.-T. Yau [15]. Below we present a condition in terms of the volume growth function.

Let us first state various equivalent conditions for the stochastic completeness. Fix $0 < T \le \infty$, set $I = (0,T)$ and consider the Cauchy problem in $I \times M$

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u & \text{in } I \times M, \\ u|_{t=0} = 0. \end{cases} \tag{3}$$

The problem (3) is understood in the classical sense, that is, $u \in C^\infty(I \times M)$ and $u(t,x) \to 0$ locally uniformly in $x \in M$ as $t \to 0$. We are interested in the uniqueness of the trivial solution $u \equiv 0$ of (3).

THEOREM 1. (Khas'minskii [9]) *For any* $\alpha > 0$ *and* $T \in (0,\infty]$, *the following conditions are equivalent.*

(a) *M is stochastically complete.*

(b) *The equation* $\Delta v = \alpha v$ *in M has the only bounded non-negative solution* $v \equiv 0$.

(c) *The Cauchy problem in* $(0,T)\times M$ *has a unique bounded solution* $u\equiv 0$.

DEFINITION 2. Define the *volume function* $V(x,r)$ of a manifold (M,g) by $V(x,r):=\mu(B(x,r))$, where $B(x,r)$ is the geodesic ball of radius r centered at x.

Note that $0<V(x,r)<\infty$ for all $x\in M$ and $r>0$ provided M is geodesically complete.

THEOREM 2. *Let* (M,g) *be a geodesically complete connected Riemannian manifold. If, for some point* $x_0\in M$,

$$\int^{\infty}\frac{rdr}{\log V(x_0,r)}=\infty, \tag{4}$$

then M *is stochastically complete.*

Condition (4) holds, in particular, if

$$V(x_0,r)\le \exp\left(Cr^2\right) \tag{5}$$

for all r large enough or even if (5) holds for a sequence $\{r_k\}$ of values r that goes to ∞ as $k\to\infty$.

Theorem 2 follows from the equivalence $(a)\Leftrightarrow(c)$ of Theorem 1 and the following more general result.

THEOREM 3. *Let* (M,g) *be a complete connected Riemannian manifold, and let* $u(x,t)$ *be a solution to the Cauchy problem* (3)*. Assume that, for some* $x_0\in M$ *and for all* $R>0$,

$$\int_0^T\int_{B(x_0,R)}u^2(x,t)\,d\mu(x)dt\le \exp\left(f(R)\right), \tag{6}$$

where $f(r)$ *is a positive increasing function on* $(0,+\infty)$ *such that*

$$\int^{\infty}\frac{rdr}{f(r)}=\infty. \tag{7}$$

Then $u\equiv 0$ *in* $I\times M$.

Condition (6) determines hence a uniqueness class for the Cauchy problem. Clearly, (7) holds for $f(r)=Cr^2$, but fails for $f(r)=Cr^{2+\varepsilon}$ with $\varepsilon>0$.

Theorems 2 and 3 were proved in [4] (see also [5] and [6]). Without going into details, let us emphasize, that the argument repeatedly uses the following property of the geodesic distance function d on the manifold: $|\nabla d|\le 1$.

Let us mention the following consequence for $\mathbb{R}^n$.

COROLLARY 1. *If* $M=\mathbb{R}^n$ *and* $u(t,x)$ *be a solution to* (3) *satisfying the condition*

$$|u(t,x)|\le C\exp\left(C|x|^2\right)\quad \text{for all } t\in I,\ x\in\mathbb{R}^n, \tag{8}$$

then $u \equiv 0$. *Moreover, the same is true if u satisfies instead of* (8) *the condition*

$$|u(t,x)| \leq C \exp\left(f\left(|x|\right)\right) \quad \textit{for all } t \in I,\ x \in \mathbb{R}^n, \tag{9}$$

where $f(r)$ *is a convex increasing function on* $(0,+\infty)$ *satisfying* (7).

The class of functions u satisfying (8) is called the *Tikhonov class*, and the conditions (9) and (7) define the *Täcklind class*. The uniqueness of the Cauchy problem in $\mathbb{R}^n$ in each of these classes is a classical result of Tikhonov [13] and Täcklind [12], respectively.

The hypothesis (4) of Theorem 2 is sufficient for the stochastic completeness of M but not necessary. Moreover, there are examples of stochastically complete manifolds with arbitrarily large volume function.

Nevertheless, the condition (4) is sharp in the following sense: if $f(r)$ is a smooth positive convex function on $(0,+\infty)$ with $f'(r) > 0$ and such that

$$\int^{\infty} \frac{rdr}{f(r)} < \infty,$$

then there exists a geodesically complete but stochastically incomplete manifold M such that $\log V(x_0,r) = f(r)$, for some $x_0 \in M$ and large enough r (see [5]).

2. Jump processes

Let (M,d) be a metric space such that all closed metric balls

$$B(x,r) = \{y \in M : d(x,y) \leq r\}$$

are compact. In particular, (M,d) is locally compact and separable. Let μ be a Radon measure on M with a full support.

Recall that a *Dirichlet form* $(\mathcal{E},\mathcal{F})$ in $L^2(M,\mu)$ is a symmetric, non-negative definite, bilinear form $\mathcal{E} : \mathcal{F} \times \mathcal{F} \to \mathbb{R}$ defined on a dense subspace $\mathcal{F}$ of $L^2(M,\mu)$, which satisfies in addition the following properties:

- Closedness: $\mathcal{F}$ is a Hilbert space with respect to the following inner product:

$$\mathcal{E}_1(f,g) := \mathcal{E}(f,g) + (f,g). \tag{10}$$

- The Markov property: if $f \in \mathcal{F}$ then also $\widetilde{f} := (f \wedge 1)_+$ belongs to $\mathcal{F}$ and $\mathcal{E}(\widetilde{f}) \leq \mathcal{E}(f)$, where $\mathcal{E}(f) := \mathcal{E}(f,f)$.

Then $(\mathcal{E},\mathcal{F})$ has the *generator* $\mathcal{L}$ that is a non-positive definite, self-adjoint operator on $L^2(M,\mu)$ with domain $\mathcal{D} \subset \mathcal{F}$ such that $\mathcal{E}(f,g) = (-\mathcal{L}f,g)$ for all $f \in \mathcal{D}$ and $g \in \mathcal{F}$. The generator $\mathcal{L}$ determines the *heat semigroup* $\{P_t\}_{t\geq 0}$ by $P_t = e^{t\mathcal{L}}$ in the sense of functional calculus of self-adjoint operators. It is known that $\{P_t\}_{t\geq 0}$ is strongly continuous, contractive, symmetric semigroup in L^2, and is *Markovian*, that is, $0 \leq P_t f \leq 1$ for any $t > 0$ if $0 \leq f \leq 1$.

The Markovian property of the heat semigroup implies that the operator P_t preserves the inequalities between functions, which allows to use monotone limits to extend P_t from L^2 to L^∞ (in fact, P_t extends to any L^q, $1 \leq q \leq \infty$ as a contraction). In particular, $P_t 1$ is defined.

DEFINITION 3. The form $(\mathcal{E}, \mathcal{F})$ is called *conservative* or *stochastically complete* if $P_t 1 = 1$ for every $t > 0$.

Assume in addition that $(\mathcal{E}, \mathcal{F})$ is *regular*, that is, the set $\mathcal{F} \cap C_0(M)$ is dense both in $\mathcal{F}$ with respect to the norm (10) and in $C_0(M)$ with respect to the sup-norm. By a theory of Fukushima [3], for any regular Dirichlet form there exists a Hunt process $\{X_t\}_{t \geq 0}$ such that, for all bounded Borel functions f on M,

$$\mathbb{E}_x f(X_t) = P_t f(x) \tag{11}$$

for all $t > 0$ and almost all $x \in M$, where $\mathbb{E}_x$ is expectation associated with the law of $\{X_t\}$ started at x. Using the identity (11), one can show that the lifetime of X_t is almost surely ∞ if and only if $P_t 1 = 1$ for all $t > 0$, which motivates the term "stochastic completeness" in the above definition.

One distinguishes local and non-local Dirichlet forms. The Dirichlet form $(\mathcal{E}, \mathcal{F})$ is called *local* if $\mathcal{E}(f, g) = 0$ for all functions $f, g \in \mathcal{F}$ with disjoint compact support. It is called *strongly local* if the same is true under a milder assumption that $f = \text{const}$ on a neighborhood of $\operatorname{supp} g$.

For example, the classical Dirichlet form on a Riemannian manifold

$$\mathcal{E}(f, g) = \int_M \nabla f \cdot \nabla g \, d\mu$$

is strongly local. The domain of this form is the Sobolev space H^1, the generator is the self-adjoint Laplace-Beltrami operator Δ, and the Hunt process is Brownian motion on M.

A well-studied non-local Dirichlet form in $\mathbb{R}^n$ is given by

$$\mathcal{E}(f, g) = \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{(f(x) - f(y))(g(x) - g(y))}{|x - y|^{n+\alpha}} dx dy \tag{12}$$

where $0 < \alpha < 2$. The domain of this form is the Besov space $B_{2,2}^{\alpha/2}$, the generator is (up to a constant multiple) the operator $-(-\Delta)^{\alpha/2}$, where Δ is the Laplace operator in $\mathbb{R}^n$, and the Hunt process is the the symmetric stable process of index α.

By a theorem of Beurling and Deny (cf. [3]), any regular Dirichlet form can be represented in the form

$$\mathcal{E} = \mathcal{E}^{(c)} + \mathcal{E}^{(j)} + \mathcal{E}^{(k)},$$

where $\mathcal{E}^{(c)}$ is a strongly local part that has the form

$$\mathcal{E}^{(c)}(f, g) = \int_M \Gamma(f, g) \, d\mu,$$

where $\Gamma(f,g)$ is a so called *energy density* (generalizing $\nabla f \cdot \nabla g$ on manifolds); $\mathcal{E}^{(j)}$ is a jump part that has the form

$$\mathcal{E}^{(j)}(f,g) = \frac{1}{2}\int\int_{M\times M} (f(x) - f(y))(g(x) - g(y))\, dJ(x,y)$$

with some measure J on $M \times M$ that is called a *jump measure*; and $\mathcal{E}^{(k)}$ is a killing part that has the form

$$\mathcal{E}^{(k)}(f,g) = \int_M fg\,dk$$

where k is a measure on M that is called a *killing measure*.

In terms of the associated process this means that X_t is in some sense a mixture of a diffusion process, jump process and a killing condition.

The log-volume test of Theorem 2 can be extended to strongly local Dirichlet forms, provided the distance function satisfies the condition

$$\Gamma(d(\cdot,x_0), d(\cdot,x_0)) \le C, \tag{13}$$

for some point $x_0 \in M$ and constant C, and the volume function $V(x,r) := \mu(B(x,r))$ satisfies (4). The method of the proof is basically the same as in Theorem 2 because for strongly local forms the same chain rule and product rules are available, and the condition (13) replaces the condition $|\nabla d| \le 1$ (see [11]).

Now let us turn to jump processes. For simplicity let us assume that the jump measure J has a density $j(x,y)$. Namely, let $j(x,y)$ be is a non-negative Borel function on $M \times M$ that satisfies the following two conditions:

(a) $j(x,y)$ is symmetric: $j(x,y) = j(y,x)$;

(b) there is a positive constant C such that

$$\int_M (1 \wedge d(x,y)^2)\, j(x,y)\, d\mu(y) \le C \ \text{ for all } x \in M. \tag{14}$$

DEFINITION 4. We say that a distance function d is *adapted* to a kernel $j(x,y)$ (or j is adapted to d) if (b) is satisfied.

For the purpose of investigation of stochastic completeness the condition (b) plays the same role as (13) does for diffusion.

Consider the following bilinear functional

$$\mathcal{E}(f,g) = \frac{1}{2}\int\int_{M\times M} (f(x) - f(y))(g(x) - g(y))\, j(x,y)\, d\mu(x)\, d\mu(y) \tag{15}$$

that is defined on Borel functions f and g whenever the integral makes sense. Define the maximal domain of $\mathcal{E}$ by

$$\mathcal{F}_{\max} = \left\{ f \in L^2 : \mathcal{E}(f,f) < \infty \right\},$$

where $L^2 = L^2(M,\mu)$. By the polarization identity, $\mathcal{E}(f,g)$ is finite for all $f,g \in \mathcal{F}_{\max}$. Moreover, $\mathcal{F}_{\max}$ is a Hilbert space with the following norm:

$$\|f\|^2_{\mathcal{F}_{\max}} = \mathcal{E}_1(f,f) := \|f\|^2_{L^2} + \mathcal{E}(f,f).$$

Denote by $\mathrm{Lip}_0(M)$ the class of Lipschitz functions on M with compact support. It follows from (14) that $\mathrm{Lip}_0(M) \subset \mathcal{F}_{\max}$. Define the space $\mathcal{F}$ as the closure of $\mathrm{Lip}_0(M)$ in $(\mathcal{F}_{\max}, \|\cdot\|_{\mathcal{F}_{\max}})$. Under the above hypothesis, $(\mathcal{E},\mathcal{F})$ is a regular Dirichlet form in $L^2(M,\mu)$. The associated Hunt process $\{X_t\}$ is a pure jump process with the jump density $j(x,y)$.

Many examples of jump processes are provided by Lévy-Khintchine theorem where the Lévy measure corresponds to $j(x,y)\,d\mu(y)$. The condition (14) appears also in Lévy-Khintchine theorem, so that the Euclidean distance in $\mathbb{R}^n$ is adapted to any Lévy measure. An explicit example of a jump density in $\mathbb{R}^n$ is

$$j(x,y) = \frac{\mathrm{const}}{|x-y|^{n+\alpha}},$$

where $\alpha \in (0,2)$, which defines the Dirichlet form (12).

Sufficient condition for stochastic completeness of the Dirichlet form of jump type is given in the following theorem that was proved in [7].

THEOREM 4. *Assume that j satisfies (a) and (b) and let $(\mathcal{E},\mathcal{F})$ be the jump form defined as above. Fix a constant $b < \frac{1}{2}$. If, for some $x_0 \in M$ and for all large enough* r,*

$$V(x_0,r) \le \exp(br\log r), \tag{16}$$

then the Dirichlet form $(\mathcal{E},\mathcal{F})$ is stochastically complete.

It is not known if the borderline value $\frac{1}{2}$ for b is sharp.

For example, (16) is satisfied if, for some constant C and all large r,

$$V(x_0,r) \le \exp(Cr)$$

For the proof of Theorem 4 we split the jump kernel $j(x,y)$ into the sum of two parts:

$$j'(x,y) = j(x,y)\mathbf{1}_{\{d(x,y)\le 1\}} \text{ and } j''(x,y) = j(x,y)\mathbf{1}_{\{d(x,y)>1\}}$$

and show first the stochastic completeness of the Dirichlet form $(\mathcal{E}',\mathcal{F})$ associated with j'. For that we adapt the methods used for stochastic completeness for the local form. The bounded range of j' allows to treat the Dirichlet form $(\mathcal{E}',\mathcal{F})$ as "almost" local: if f,g are two functions from $\mathcal{F}$ such that $d(\operatorname{supp} f, \operatorname{supp} g) > 1$ then $\mathcal{E}(f,g) = 0$. The condition (14) plays in the proof the same role as the condition (13) in the local case. However, the lack of locality brings up in the estimates various additional terms that

*In fact it suffices to have (16) for $r = r_k$ where $\{r_k\}$ is any sequence such that $r_k \to \infty$ as $k \to \infty$.

have to be compensated by a stronger hypothesis of the volume growth (16), instead of the quadratic exponential growth in Theorem 2.

The tail j'' can regarded as a small perturbation of j' in the following sense: $(\mathcal{E},\mathcal{F})$ is stochastically complete if and only if $(\mathcal{E}',\mathcal{F})$ is so. The proof is based on the fact that the integral operator with the kernel j'' is a bounded operator in $L^2(M,\mu)$, because by (14)

$$\int_M j''(x,y)\,d\mu(y) \le C.$$

It is not clear if the volume growth condition (16) in Theorem 4 is sharp.

Let us briefly mention a recent result of Xueping Huang [8], that is analogous of Theorem 3 about the uniqueness class for the Cauchy problem on a geodesically complete manifold. X.Huang proved a similar theorem for the heat equation associated with the jump Dirichlet form on graphs satisfying (a) and (b): namely, the associated heat equation has the following uniqueness class

$$\int_0^T \int_{B(x,R)} u^2(t,x)\,d\mu(x)\,dt \le \exp(br\log r)$$

where b is as above any constant smaller than $\frac{1}{2}$. Moreover, he has shown that for $b > 2\sqrt{2}$ this statement fails. The optimal value of b remains unknown. Note that the function u in that example is unbounded, so that it cannot serve to show the sharpness of the condition (16) in Theorem 4.

3. Random walks on graphs

Let us now turn to random walks on graphs. Let (X,E) be a locally finite, infinite, connected graph, where X is the set of vertices and E is the set of edges. We assume that the graph is undirected, simple, without loops. Let μ be the counting measure on X. Define the jump kernel by $j(x,y) = 1_{\{x\sim y\}}$, where $x \sim y$ means that x,y are neighbors, that is, $(x,y) \in E$. The corresponding Dirichlet form is

$$\mathcal{E}(f) = \frac{1}{2}\sum_{\{x,y:x\sim y\}} (f(x) - f(y))^2,$$

and its generator is

$$\Delta f(x) = \sum_{y\sim x} (f(y) - f(x)).$$

The operator Δ is called *unnormalized* (or *physical)* Laplace operator on (X,E). This is to distinguish from the *normalized* or *combinatorial* Laplace operator

$$\hat{\Delta} f(x) = \frac{1}{\deg(x)}\sum_{y\sim x} (f(y) - f(x)),$$

where $\deg(x)$ is the number of neighbors of x. The normalized Laplacian $\hat{\Delta}$ is the generator of the same Dirichlet form but with respect to the degree measure $\deg(x)$.

Both Δ and $\hat{\Delta}$ generate the heat semigroups $e^{t\Delta}$ and $e^{t\hat{\Delta}}$ and, hence, associated continuous time random walks on X. It is easy to prove that $\hat{\Delta}$ is a bounded operator in $L^2(X,\deg)$, which then implies that the associated random walk is always stochastically complete. On the contrary, the random walk associated with the unnormalized Laplace operator can be stochastically incomplete.

We say that the graph (X,E) is stochastically complete if the heat semigroup $e^{t\Delta}$ is stochastically complete.

Denote by $\rho(x,y)$ the graph distance on X, that is the minimal number of edges in an edge chain connecting x and y. Let $B_\rho(x,r)$ be closed metric balls with respect to this distance ρ and set $V_\rho(x,r) = \left|B_\rho(x,r)\right|$ where $|\cdot| := \mu(\cdot)$ denotes the number of vertices in a given set.

The stochastic completeness can be determined in terms of the function V_ρ as follows.

THEOREM 5. *If there is a point $x_0 \in X$ and a constant $c > 0$ such that*

$$V_\rho(x_0,r) \le cr^3 \tag{17}$$

for all large enough r, then the graph (X,E) is stochastically complete.

Note that the cubic rate of the volume growth is sharp here. Indeed, Wojciechowski [14] has shown that, for any $\varepsilon > 0$ there is a stochastically incomplete graph that satisfies $V_\rho(x_0,r) \le cr^{3+\varepsilon}$. For any non-negative integer r, set

$$S_r = \{x \in X : \rho(x_0,x) = r\}.$$

In the example of Wojciechowski every vertex on S_r is connected to all vertices on S_{r-1} and S_r.

For this type of graphs, that are called *anti-trees*, the stochastic incompleteness is equivalent to the following condition ([14]):

$$\sum_{r=1}^{\infty} \frac{V_\rho(x_0,r)}{|S_{r+1}|\,|S_r|} < \infty. \tag{18}$$

Indeed, assuming (18), one constructs a non-trivial bounded solution to the equation $\Delta u - u = 0$, which is enough to ensure the stochastic incompleteness (cf. Theorem 1). For a radial function $u = u(r)$ this equation acquires the form

$$u(r+1) = u(r) + \frac{1}{|S_{r+1}|\,|S_r|}\sum_{i=0}^{r} |S_i|\,u(i).$$

Setting $u(0) = 1$ and solving this equation inductively in r, we obtain a positive solution $u(r)$ that increases in r. It follows that

$$u(r+1) \le \left(1 + \frac{1}{|S_{r+1}|\,|S_r|}\sum_{i=0}^{r} |S_i|\right) u(r)$$

whence by induction

$$u(R) \le \prod_{r=0}^{R-1} \left(1 + \frac{V_\rho(x_0, r)}{|S_{r+1}|\,|S_r|}\right).$$

The condition (18) implies that the product in the right hand side is bounded so that u is a bounded function.

If $|S_r| \simeq r^{2+\varepsilon}$ then $V_\rho(x_0, r) \simeq r^{3+\varepsilon}$ and the condition (18) is satisfied so that the graph is stochastically incomplete.

The proof of Theorem 5 is based on the following ideas. First observe that the graph distance ρ is in general not adapted. More precisely, ρ is adapted if and only if the graph has uniformly bounded degree, which is not an interesting case.

Let us construct an adapted distance as follows. For any edge $x \sim y$ define first its length $\sigma(x, y)$ by

$$\sigma(x,y) = \frac{1}{\sqrt{\deg(x)}} \wedge \frac{1}{\sqrt{\deg(y)}}.$$

Then, for all $x, y \in X$ define $d(x, y)$ as the smallest total length of all edges in an edge chain connecting x and y. It is easy to verify that d satisfies (14) with $C = 1$.

Next one proves that (17) for ρ-balls implies that the d-balls have at most exponential volume growth, so that the stochastic completeness follows by Theorem 4.

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AMS Subject Classification: 47D07, 60J75

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Lavoro pervenuto in redazione il 22.05.2013

Rend. Sem. Mat. Univ. Politec. Torino
Vol. 71, 2 (2013), 239 – 249

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GENERALIZED SOLUTIONS TO EQUATIONS WITH MULTIPLICATIVE NOISE IN HILBERT SPACES

Abstract. We suggest a framework that allows to introduce multiplicative stochastic perturbation of the Gaussian white noise type into a linear differential equation in a Hilbert space and prove existence of the unique solution for the obtained stochastic problem in a certain space of generalized functions.

1. Introduction

Our model problem is

$$\frac{\partial u(t,s)}{\partial t} = -\frac{\partial u(t,s)}{\partial s} + \eta(s)u(t,s),\ 0 < s < 1,\ t > 0,\ u(t,0) = 0,\ u(0,s) = \varphi(s),$$

where $\eta \in L_\infty[0;1]$. It can be written as the Cauchy problem for an operator-differential equation in Hilbert space $H = L^2[0,1]$ in the following way:

$$\frac{du(t)}{dt} = Au(t)\,,\ \ t > 0\,,\ u(0) = \varphi\,, \tag{1}$$

where

$$A = A_0 + B_0 = -\frac{d}{ds} + \eta(s),\ \ \mathrm{dom}A = \{x \in L^2[0,1]\,, \frac{dx}{ds} \in L^2[0,1]\,, x(0) = 0\}. \tag{2}$$

Operator A_0 is the generator of the right shift semigroup, which is a C_0-semigroup in H. Its perturbation by B_0, which is bounded in H, gives A which is also the generator of a C_0-semigroup. Such problems arise for example in population dynamics. In this case u represents population density with respect to a certain numerical characteristic, say age, or size of an individual, A_0 is usually the generator of a shift-type semigroup, B_0 is a multiplication operator (or a sum of multiplication operators) that reflects the influence of such phenomena as death and birth. We will be concerned with the situation when B_0 is subject to random fluctuations, so that instead of $\eta(s)$ we have $\eta(s) + \nu(t,s)$, where ν is a random process taking values in a certain space of functions on $[0,1]$. If we want multiplication by $\eta + \nu$ to be a bounded operator in H, ν must be a sufficiently smooth function of s. We use multiplication by smoothed values of an H-valued white noise. Namely, consider $B(\cdot) \in \mathcal{L}\big(H;\mathcal{L}(H)\big)$ defined by

$$[B(x)y](s) := \varepsilon \cdot x(s) \int_0^1 \psi(s-\tau)y(\tau)\,d\tau\,, \tag{3}$$

*This work was supported by the Ministry of Education and Science of Russian Federation (Program 1.1016.2011) and by RFBR, project 13-01-00090

where $\varepsilon > 0$, $\psi \in C_0^\infty(\mathbb{R})$. Thus we come to the following stochastic problem:

$$dX(t) = AX(t)dt + B(X(t))dW(t)\,,\ t \geq 0\,,\quad X(0) = \Phi\,. \tag{4}$$

where $W(t)$ is a cylindrical H-valued Wiener process on a probability space $(\Omega, \mathcal{F}, \mathrm{P})$ with normal filtration $\{\mathcal{F}_t\}$, Φ is an $\mathcal{F}_0$-measurable random variable.

In our work we introduce spaces of H-valued generalized random variables $(\mathcal{S})_{-\rho}(H)$, $0 \leq \rho \leq 1$, so that (4) can be written as

$$\frac{dX(t)}{dt} = AX(t) + B\big(X(t)\big) \diamond \mathbb{W}(t)\,,\ t \geq 0\,,\quad X(0) = \Phi\,, \tag{5}$$

where $\mathbb{W}(t)$ is H-valued cylindrical singular white noise and "$\diamond$" is the Wick product. Using S-transform we reduce the problem (5) to a deterministic one and thus prove the existence and uniqueness of its solution in $(\mathcal{S})_{-0}(H)$.

2. Framework

Let $(\mathcal{S}', \mathcal{B}(\mathcal{S}'), \mu)$ be the white noise probability space, where $\mathcal{S}'$ is the space of tempered distributions over the space of rapidly decreasing functions $\mathcal{S}$, $\mathcal{B}(\mathcal{S}')$ is the σ-algebra of Borel subsets of $\mathcal{S}'$ and μ is the white noise probability measure on $\mathcal{B}(\mathcal{S}')$ (Minlos – Sasonov measure) with

$$\int_{\mathcal{S}'} e^{i\langle\omega,\theta\rangle} d\mu(\omega) = e^{-\frac{1}{2}|\theta|_0^2}\,,\quad \theta \in \mathcal{S}\,. \tag{6}$$

We denote by $|\cdot|_0 = \sqrt{(\cdot,\cdot)_0}$ the norm of $L^2(\mathbb{R})$. Let (L^2) be the space of μ-square integrable $\mathbb{R}$-valued functions (random variables) on $\mathcal{S}'$ with norm $\|\cdot\|_0$. It follows from (6) that for any $\theta, \eta \in \mathcal{S}$ we have $\big(\langle\cdot,\theta\rangle, \langle\cdot,\eta\rangle\big)_{(L^2)} = (\theta,\eta)_{L^2(\mathbb{R})}$, $\|\langle\cdot,\theta\rangle\|_0^2 = E\langle\cdot,\theta\rangle^2 = |\theta|_0^2$. It follows from here that the mapping $\theta \mapsto \langle\cdot,\theta\rangle$ can be extended by continuity from $\mathcal{S}$ to the whole $L^2(\mathbb{R})$, so that $\langle\cdot,\phi\rangle \in (L^2)$ is well defined for all $\phi \in L^2(\mathbb{R})$ and (6) is still valid for $\theta \in L^2(\mathbb{R})$.

Let $\{\xi_k\}_{k=1}^\infty$ be the orthonormal basis of $L^2(\mathbb{R})$, consisting of Hermite functions $\xi_k(x) = \dfrac{e^{-\frac{x^2}{2}} h_{k-1}(x)}{\pi^{\frac{1}{4}}((k-1)!)^{\frac{1}{2}}}$, where $h_k(x) = (-1)^k e^{\frac{x^2}{2}} \frac{d^k}{dx^k} e^{-\frac{x^2}{2}}$ are Hermite polynomials.

Let $\mathcal{T} \subset \big(\mathbb{N} \cup \{0\}\big)^{\mathbb{N}}$ be the set of all finite multi-indices. Stochastic Hermite polynomials, defined by $\mathbf{h}_\alpha(\omega) := \prod_k h_{\alpha_k}(\langle\omega, \xi_k\rangle)\,, \omega \in \mathcal{S}'\,, \alpha \in \mathcal{T}$, form an orthogonal basis of (L^2) with $(\mathbf{h}_\alpha\,, \mathbf{h}_\beta)_{(L^2)} = \delta_{\alpha,\beta}\alpha!$, where $\alpha! := \prod_k \alpha_k!$.

The Gelfand triple

$$(\mathcal{S})_\rho \subset (L^2) \subset (\mathcal{S})_{-\rho}\,,\quad (0 \leq \rho \leq 1) \tag{7}$$

is widely used in white noise analysis (see [1, 3]). Here $(\mathcal{S})_\rho = \cap_{p\in\mathbb{N}}(\mathcal{S}_p)_\rho$ with projective limit topology, where

$$(\mathcal{S}_p)_\rho = \Big\{\varphi = \sum_{\alpha\in\mathcal{T}} \varphi_\alpha \mathbf{h}_\alpha \in (L^2) : \sum_{\alpha\in\mathcal{T}} (\alpha!)^{1+\rho} |\varphi_\alpha|^2 (2\mathbb{N})^{2p\alpha} < \infty\Big\}$$

and the norm $|\cdot|_{p,\rho}$, generated by the scalar product

$$(\varphi,\psi)_{p,\rho}=\sum_{\alpha\in\mathcal{T}}(\alpha!)^{1+\rho}\varphi_\alpha\psi_\alpha(2\mathbb{N})^{2p\alpha}, \quad (2\mathbb{N})^{p\alpha}:=\prod_{i\in\mathbb{N}}(2i)^{p\alpha_i};$$

$(\mathcal{S})_{-\rho}=\cup_{p\in\mathbb{N}}(\mathcal{S}_{-p})_{-\rho}$ with inductive limit topology, where $(\mathcal{S}_{-p})_{-\rho}$ is the adjoint to $(\mathcal{S}_p)_\rho$. The space $(\mathcal{S}_{-p})_{-\rho}$ can be identified with the Hilbert space of all formal expansions $\Phi=\sum_{\alpha\in\mathcal{T}}\Phi_\alpha\mathbf{h}_\alpha$ such that $\sum_{\alpha\in\mathcal{T}}(\alpha!)^{1-\rho}\frac{|\Phi_\alpha|^2}{(2\mathbb{N})^{2p\alpha}}<\infty$ with scalar product $(\Phi,\Psi)_{-p,-\rho}=\sum_{\alpha\in\mathcal{T}}(\alpha!)^{1-\rho}\frac{\Phi_\alpha\Psi_\alpha}{(2\mathbb{N})^{2p\alpha}}$. We will denote $|\cdot|^2_{-p,-\rho}=(\cdot,\cdot)_{-p,-\rho}$. For $\Phi=\sum_{\alpha\in\mathcal{T}}\Phi_\alpha\mathbf{h}_\alpha\in(\mathcal{S})_{-\rho}$, $\varphi=\sum_{\alpha\in\mathcal{T}}\varphi_\alpha\mathbf{h}_\alpha\in(\mathcal{S})_\rho$ we have $\langle\Phi,\varphi\rangle=\sum_{\alpha\in\mathcal{T}}\alpha!\Phi_\alpha\varphi_\alpha$.

A set $M\subseteq(\mathcal{S})_\rho$ is called bounded if for any $\{\varphi_n\}\subseteq M$ and for any $\{\varepsilon_n\}\subset\mathbb{R}$ converging to 0, the sequence $\{\varepsilon_n\varphi_n\}$ converges to zero in $(\mathcal{S})_\rho$. It is easy to see that boundedness of a set in $(\mathcal{S})_\rho$ is equivalent to its boundedness in any $(\mathcal{S}_p)_\rho$.

Let H be a separable Hilbert space over $\mathbb{C}$ with scalar product $(\cdot,\cdot)$ and corresponding norm $\|\cdot\|$. Denote by $(L^2)(H)$ the space of H-valued functions on $\mathcal{S}'$, square Bochner integrable with respect to μ. Let $\{e_j\}_{j=1}^\infty$ be an orthonormal basis in H. The family $\{\mathbf{h}_\alpha e_j\}_{\alpha\in\mathcal{T},j\in\mathbb{N}}$ is an orthogonal basis in $(L^2)(H)$. Any $f\in(L^2)(H)$ can be expanded into Fourier series $f=\sum_{\alpha\in\mathcal{T},j\in\mathbb{N}}f_{\alpha,j}\mathbf{h}_\alpha e_j=\sum_{\alpha\in\mathcal{T}}f_\alpha\mathbf{h}_\alpha=\sum_{j=1}^\infty f_je_j$, where $f_{\alpha,j}\in\mathbb{R}$, $f_\alpha=\sum_j f_{\alpha,j}e_j\in H$, $f_j=\sum_{\alpha\in\mathcal{T}}f_{\alpha,j}\mathbf{h}_\alpha\in(L^2)$, and we have $\|f\|^2_{(L^2)(H)}=\sum_{\alpha\in\mathcal{T},j\in\mathbb{N}}\alpha!|f_{\alpha,j}|^2=\sum_{\alpha\in\mathcal{T}}\alpha!\|f_\alpha\|^2_H=\sum_{j=1}^\infty\|f_j\|^2_{(L^2)}$.

Define the space $(\mathcal{S})_{-\rho}(H)$ of H-valued generalized functions over the space $(\mathcal{S})_\rho$ of test functions as the space of all linear continuous operators $\Phi:(\mathcal{S})_\rho\to H$ with the topology of uniform convergence on bounded subsets of $(\mathcal{S})_\rho$. We will denote by $\Phi[\varphi]$ the action of $\Phi\in(\mathcal{S})_{-\rho}(H)$ on a test function $\varphi\in(\mathcal{S})_\rho$.

Now we describe the structure of $(\mathcal{S})_{-\rho}(H)$. It is easy to prove the following proposition:

Proposition 1 *Any $\Phi\in(\mathcal{S})_{-\rho}(H)$ is bounded as an operator from $(\mathcal{S}_p)_\rho$ to H for some $p\in\mathbb{N}$.*

Since $(\mathcal{S})_\rho$ is a countably Hilbert nuclear space, it follows from Proposition 1:

Corollary 1 *Any $\Phi\in(\mathcal{S})_{-\rho}(H)$ is a Hilbert–Schmidt operator from $(\mathcal{S}_p)_\rho$ to H for some $p\in\mathbb{N}$.*

For any $\Phi\in(\mathcal{S})_{-\rho}(H)$ denote by Φ_j the linear functional, defined on $(\mathcal{S})_\rho$ by $\langle\Phi_j,\varphi\rangle:=(\Phi[\varphi],e_j)$. Let Φ be Hilbert–Schmidt from $(\mathcal{S}_p)_\rho$ to H, then all Φ_j, $j\in\mathbb{N}$, belong to the corresponding $(\mathcal{S}_{-p})_{-\rho}$ and thus we have

$$\Phi_j=\sum_{\alpha\in\mathcal{T}}\Phi_{\alpha,j}\mathbf{h}_\alpha, \quad \sum_{\alpha\in\mathcal{T}}(\alpha!)^{1-\rho}\frac{|\Phi_{\alpha,j}|^2}{(2\mathbb{N})^{2p\alpha}}<\infty.$$

For the Hilbert–Schmidt norm of Φ as an operator from $(\mathcal{S}_p)_\rho$ to H we have:

$$\|\Phi\|^2_{\mathrm{HS},p,\rho}=\sum_{\alpha\in\mathcal{T}}\left\|\Phi\left[\frac{\mathbf{h}_\alpha}{(\alpha!)^{\frac{1+\rho}{2}}(2\mathbb{N})^{p\alpha}}\right]\right\|^2=\sum_{\alpha\in\mathcal{T},j\in\mathbb{N}}(\alpha!)^{1-\rho}\frac{|\Phi_{\alpha,j}|^2}{(2\mathbb{N})^{2p\alpha}}.$$

Denote by $\mathrm{HS}\big((\mathcal{S}_p)_\rho;H\big)$ the space of Hilbert–Schmidt operators from $(\mathcal{S}_p)_\rho$ to H.It is a separable Hilbert space. The family of operators $\{\mathbf{h}_\alpha\otimes e_j\}_{\alpha\in\mathcal{T},j\in\mathbb{N}}$, defined by $(\mathbf{h}_\alpha\otimes e_j)\varphi := \big(\mathbf{h}_\alpha,\varphi\big)_{(L^2)}e_j$, $\varphi\in(\mathcal{S}_p)_\rho$ is an orthogonal basis of $\mathrm{HS}\big((\mathcal{S}_p)_\rho;H\big)$. It follows from Proposition 1 that $(\mathcal{S})_{-\rho}(H)=\bigcup_{p\in\mathbb{N}}\mathrm{HS}\big((\mathcal{S}_p)_\rho;H\big)$. Any $\Phi\in(\mathcal{S})_{-\rho}(H)$ has the following decomposition:

$$\Phi[\cdot]=\sum_{j\in\mathbb{N}}\langle\Phi_j,\cdot\rangle e_j=\sum_{\alpha\in\mathcal{T},j\in\mathbb{N}}\Phi_{\alpha,j}(\mathbf{h}_\alpha\otimes e_j)=\sum_{\alpha\in\mathcal{T}}\Phi_\alpha(\mathbf{h}_\alpha,\cdot)_{(L^2)},$$

where $\Phi_j=(\Phi[\cdot],e_j)\in(\mathcal{S}_{-p})_{-\rho}$ for some $p\in\mathbb{N}$, $\Phi_\alpha=\sum_{j\in\mathbb{N}}\Phi_{\alpha,j}e_j\in H$. We have

$$\|\Phi\|^2_{\mathrm{HS},p,\rho}=\sum_{j\in\mathbb{N}}|\Phi_j|^2_{-p,-\rho}=\sum_{\alpha\in\mathcal{T},j\in\mathbb{N}}(\alpha!)^{1-\rho}\frac{|\Phi_{\alpha,j}|^2}{(2\mathbb{N})^{2p\alpha}}=\sum_{\alpha\in\mathcal{T}}(\alpha!)^{1-\rho}\frac{\|\Phi_\alpha\|^2}{(2\mathbb{N})^{2p\alpha}}<\infty.$$

For all $p_1<p_2$ and $\Phi\in\mathrm{HS}\big((\mathcal{S}_{p_1})_\rho;H\big)$ we evidently have

$$\mathrm{HS}\big((\mathcal{S}_{p_1})_\rho;H\big)\subseteq\mathrm{HS}\big((\mathcal{S}_{p_2})_\rho;H\big),\quad \|\Phi\|_{HS,p_1,\rho}\geq\|\Phi\|_{HS,p_2,\rho}.$$

A set $\mathcal{M}\subseteq(\mathcal{S})_{-\rho}(H)$ is called bounded if for any sequence $\{\Phi_n\}\subseteq\mathcal{M}$ and any $\{\varepsilon_n\}\subset\mathbb{R}$ convergent to zero, $\{\varepsilon_n\Phi_n\}$ converges to zero in $(\mathcal{S})_{-\rho}(H)$. It is easy to prove the following propositions:

Proposition 2 *A set $\mathcal{M}$ is bounded in $(\mathcal{S})_{-\rho}(H)$ if and only if for any bounded $M\subset(\mathcal{S})_\rho$ there exists $K>0$ such that $\|\Phi[\varphi]\|\leq K$ for any $\varphi\in M$, $\Phi\in\mathcal{M}$.*

Proposition 3 *If $\mathcal{M}$ is bounded in $(\mathcal{S})_{-\rho}(H)$, then there exist $p\in\mathbb{N}$ and $K>0$ such that $\|\Phi[\varphi]\|\leq K|\varphi|_{p,\rho}$ for all $\Phi\in\mathcal{M}$, $\varphi\in(\mathcal{S})_\rho$.*

Thus, if a set $\mathcal{M}$ is bounded in $(\mathcal{S})_{-\rho}(H)$, then all elements of $\mathcal{M}$ are bounded operators from $(\mathcal{S}_p)_\rho$ to H for some $p\in\mathbb{N}$ and $\mathcal{M}$ is bounded in $\mathcal{L}\big((\mathcal{S}_p)_\rho,H\big)$. Consequently we have

Proposition 4 *If $\mathcal{M}$ is bounded in $(\mathcal{S})_{-\rho}(H)$, then $\mathcal{M}\subset\mathrm{HS}\big((\mathcal{S}_p)_\rho;H\big)$ for some $p\in\mathbb{N}$, and $\mathcal{M}$ is bounded in $\mathrm{HS}\big((\mathcal{S}_p)_\rho;H\big)$.*

The next proposition, which we state omitting the proof, gives characterization of convergence in $(\mathcal{S})_{-\rho}(H)$.

Proposition 5 *Let $\Phi_n=\sum_\alpha\Phi^{(n)}_\alpha\mathbf{h}_\alpha$, $\Phi=\sum_\alpha\Phi_\alpha\mathbf{h}_\alpha\in(\mathcal{S})_{-\rho}(H)$. The following assertions are equivalent:*

(i) $\{\Phi_n\}$ *converges to* Φ *in* $(\mathcal{S})_{-\rho}(H)$;

(ii) *All elements of the sequence* $\{\Phi_n\}$ *and* Φ *belong to* $\mathrm{HS}\big((\mathcal{S}_p)_\rho;H\big)$ *for some* $p\in\mathbb{N}$ *and* $\lim_{n\to\infty}\|\Phi_n-\Phi\|_{\mathrm{HS},p,\rho}=0$.

Let $\Phi(\cdot):\mathbb{R}\to(\mathcal{S})_{-\rho}(H)$. We will write $\Psi=\lim_{t\to t_0}\Phi(t)$ if $\Phi(t_n)\to\Psi$ uniformly on any bounded subset of $(\mathcal{S})_\rho$ for any sequence $t_n\to t_0$. The derivative $\Phi'(t_0)$ will be understood in the same way. It is easy to derive from Proposition 5 the following

Corollary 2 *Let* $\Phi(t) = \sum_\alpha \Phi_\alpha(t)\mathbf{h}_\alpha \in (\mathcal{S})_{-\rho}(H)$ *for* $t \in [a,b]$ *and let* $t_0 \in [a,b]$.

1. $\lim_{t\to t_0} \Phi(t) = \Phi(t_0)$ *in* $(\mathcal{S})_{-\rho}(H)$ *if and only if all* $\Phi(t), t \in [a,b]$, *belong to* $\mathrm{HS}\big((\mathcal{S}_p)_\rho; H\big)$ *for some* $p \in \mathbb{N}$ *and* $\lim_{n\to\infty} \|\Phi(t) - \Phi(t_0)\|_{\mathrm{HS},p,\rho} = 0$;

2. $\Phi(t)$ *is differentiable at* $t_0 \in [a,b]$ *if and only if* $\dfrac{d\Phi}{dt} := \lim_{t\to t_0} \dfrac{\Phi(t) - \Phi(t_0)}{t - t_0}$ *exists in* $\mathrm{HS}\big((\mathcal{S}_p)_\rho; H\big)$ *for some* p.

Example. (H-valued cylinder Wiener process and white noise).

Let $n(\cdot,\cdot) : \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ be a bijection with

$$n(i,j) \geq ij\,, \quad i,j \in \mathbb{N}\,. \tag{8}$$

Denote $\varepsilon_n := (0,0,\ldots,\underset{n}{1},0,\ldots)$. The sequence $\beta_j(t) = \sum_{i=1}^\infty \int_0^t \xi_i(s)\,ds\,\mathbf{h}_{\varepsilon_{n(i,j)}}$ is a sequence of independent Brownian motions. Then the H-valued random process

$$W(t) = \sum_{j\in\mathbb{N}} \beta_j(t)e_j = \sum_{n\in\mathbb{N}} W_{\varepsilon_n}(t)\,\mathbf{h}_{\varepsilon_n}\,, \quad W_{\varepsilon_n}(t) = \int_0^t e_{j(n)}\,\xi_{i(n)}(s)\,ds \in H\,,$$

is a cylindrical Wiener process (here $i(n), j(n) \in \mathbb{N}$ are such that $n(i(n), j(n)) = n$).

It is easy to show that $W(t) \notin (L^2)(H)$ for all $t \in \mathbb{R}$. At the same time it follows from the well known estimate $\int_0^t \xi_i(s)\,ds = O(i^{-\frac{3}{4}})$ and (8) that $\|W(t)\|^2_{\mathrm{HS},1,\rho} < \infty$. So we have $W(t) \in \mathrm{HS}\big((\mathcal{S}_1)_\rho; H\big) \subset (\mathcal{S})_{-\rho}(H)$.

Define the H-valued cylindrical white noise by

$$\mathbb{W}(t) := \sum_{i,j\in\mathbb{N}} \xi_i(t)\,(\mathbf{h}_{\varepsilon_{n(i,j)}} e_j) = \sum_{n\in\mathbb{N}} \mathbb{W}_{\varepsilon_n}(t)\,\mathbf{h}_{\varepsilon_n}\,, \quad \mathbb{W}_{\varepsilon_n}(t) = \xi_{i(n)}(t)\,e_{j(n)} \in H\,.$$

Since $\xi_i(t) = O\big(i^{-\frac{1}{4}}\big)$, we have $\|\mathbb{W}(t)\|^2_{\mathrm{HS},1,\rho} < \infty$, thus

$$\mathbb{W}(t) \in \mathrm{HS}\big((\mathcal{S}_1)_\rho; H\big) \subset (\mathcal{S})_{-\rho}(H).$$

Note that for all $t \in \mathbb{R}$ we have $\dfrac{d}{dt}W(t) = \mathbb{W}(t)$.

Let $\mathcal{E}_\theta := e^{\langle\cdot,\theta\rangle - \frac{1}{2}|\theta|_0^2}$. For any $\theta \in \mathcal{S}$ it is a random variable on $\mathcal{S}'$ belonging to $(\mathcal{S})_\rho$ for $0 \leq \rho < 1$ with $|\mathcal{E}_\theta|_{p,\rho} \leq 2^{\rho/2} \exp\Big[(1-\rho)^{\frac{2\rho-1}{1-\rho}} |\theta|_p^{\frac{2}{1-\rho}}\Big]$ (see [1]). The following expansion holds:

$$\mathcal{E}_\theta = \sum_{\alpha\in\mathcal{T}} e_\alpha \mathbf{h}_\alpha\,, \quad e_\alpha = \frac{1}{\alpha!}\prod_{i=1}^\infty (\theta,\xi_i)_0^{\alpha_i}\,.$$

Let $\Phi \in (\mathcal{S})_{-\rho}(H)$, $0 \leq \rho < 1$. Define *the S-transform* of Φ by

$$(S\Phi)(\theta) = \Phi[\mathcal{E}_\theta]\,, \quad \theta \in \mathcal{S}\,.$$

The proof of the following characteristic theorem almost completely repeats the proof of the corresponding theorem for the $\mathbb{C}$-valued case (see, for example, [1]), and is thus omitted.

Theorem 1 *Let $\Phi \in (\mathcal{S})_{-\rho}(H)$, $0 \le \rho < 1$. Then $F = S\Phi$ satisfies the following conditions:*

(i) for any $\theta, \nu \in \mathcal{S}$ the function $F(\theta + z\nu)$ is entire analytic function of $z \in \mathbb{C}$.

(ii) There exist $K > 0, a > 0$, $p \in \mathbb{N}$, such that

$$\|F(\theta)\| \le K \exp\left[a|\theta|_p^{\frac{2}{1-\rho}}\right], \quad \theta \in \mathcal{S}. \tag{9}$$

If $F : \mathcal{S} \to H$ satisfies (i) and (ii), then there exists a unique $\Phi \in (\mathcal{S})_{-\rho}(H)$ such that $F = S\Phi$ and for any q such that $e^2\left(\frac{2a}{1-\rho}\right)^{1-\rho}\sum_{i=1}^{\infty}(2i)^{-2(q-p)} < 1$, it holds

$$\|\Phi\|_{\mathrm{HS},q,\rho} \le K\left(1 - e^2\left(\frac{2a}{1-\rho}\right)^{1-\rho}\sum_{i=1}^{\infty}(2i)^{-2(q-p)}\right)^{-1/2}.$$

Example. For the above defined cylinder white noise we have:

$$\big(S\mathbb{W}(t)\big)(\theta) = \mathbb{W}(t)\left[\mathcal{E}_\theta\right] = \sum_{i,j\in\mathbb{N}} \xi_i(t) e_j(\theta, \xi_{n(i,j)})_0 .$$

Let H_1 and H_2 be separable Hilbert spaces. Since the space $\mathrm{HS}(H_1;H_2)$ of Hilbert–Schmidt operators acting from H_1 to H_2 is a separable Hilbert space, we can consider the space $(\mathcal{S})_{-\rho}\big(\mathrm{HS}(H_1;H_2)\big)$ of $\mathrm{HS}(H_1;H_2)$-valued generalized random variables over $(\mathcal{S})_\rho$. For S-transforms of any $\Psi \in (\mathcal{S})_{-\rho}\big(\mathrm{HS}(H_1;H_2)\big)$ and $\Phi \in (\mathcal{S})_{-\rho}(H_1)$, $F(\theta) = S\Psi(\theta)S\Phi(\theta) \in H_2$ is well defined for any $\theta \in \mathcal{S}$. Since $S\Psi(\theta)$ and $S\Phi(\theta)$ satisfy conditions (i) and (ii) of Theorem 1, for any $\theta, \nu \in \mathcal{S}$ the function $F(\theta + z\nu)$ is an entire analytic function of $z \in \mathbb{C}$ and

$$\|S\Psi(\theta)S\Phi(\theta)\|_{H_2} \le \|S\Psi(\theta)\|_{\mathrm{HS}(H_1;H_2)}\|S\Phi(\theta)\|_{H_1} \le K_1K_2\exp\left[(a_1+a_2)|\theta|_p^{\frac{2}{1-\rho}}\right],$$

where K_1, K_2, a_1, a_2 are the constants from condition (ii) of Theorem 1 for Ψ and Φ correspondingly (we can obviously suppose the constant p in these conditions to be the same). It follows that F is an S-transform of a unique generalized random variable $\Theta \in (\mathcal{S})_{-\rho}(H_2)$. This justifies the following definition.

Let $\Psi \in (\mathcal{S})_{-\rho}\big(\mathrm{HS}(H_1;H_2)\big)$, $\Phi \in (\mathcal{S})_{-\rho}(H_1)$. We will call $\Theta \in (\mathcal{S})_{-\rho}(H_2)$ such that $S\Theta = S\Psi S\Phi$ the *Wick product* of Ψ and Φ and denote it $\Psi \diamond \Phi$.

Let $Q \in \mathrm{HS}(H)$, $H_Q = Q^{\frac{1}{2}}(H)$ with scalar product $(u,v)_{H_Q} = (Q^{-\frac{1}{2}}u, Q^{-\frac{1}{2}}v)$. For the H-valued cylindrical white noise, from the estimate

$$\|\mathbb{W}_{\varepsilon_{n(i,j)}}\|_{H_Q}^2 \big(2\mathbb{N}\big)^{-2p\varepsilon_{n(i,j)}} = \frac{|\xi_i(t)|^2}{\sigma_j^2\big(2n(i,j)\big)^{2p}} \le \frac{|\xi_i(t)|^2}{\sigma_j^2\big(2ij\big)^{2p}} = O\big(\sigma_j^{-2} i^{-2p-\frac{1}{2}} j^{-2p}\big),$$

it follows

Proposition 6 *For any* $Q = \sum_{j=1}^{\infty} \sigma_j^2 (e_j \otimes e_j) \in \mathrm{HS}(H;H)^*$, *(i.e.* $\sum_{j=1}^{\infty} \sigma_j^2 < \infty$*), if*

$$\sum_{j=1}^{\infty} \sigma_j^{-2} j^{-2p} < \infty \text{ for some } p \in \mathbb{N}, \tag{10}$$

then $\mathbb{W}(t) \in (\mathcal{S})_{-\rho}(H_Q)$ *for all* $t \in \mathbb{R}$ *and any* $\rho \in [0;1]$.

It follows from proposition 6 that if Q satisfies (10), then for any stochastic process $\Psi(t)$ with values in $(\mathcal{S})_{-\rho}\big(\mathrm{HS}(H_Q;H)\big)$ the $(\mathcal{S})_{-1}(H)$-valued random process $\Psi(t) \diamond \mathbb{W}(t)$ is well defined.

We will call an $(\mathcal{S})_{-\rho}\big(\mathrm{HS}(H_Q;H)\big)$-valued random process $\Psi(t)$ *Hitsuda–Skorohod integrable* on $[0;T]$, if $\Psi(t) \diamond \mathbb{W}(t)$ is integrable on $[0;T]$ as an $(\mathcal{S})_{-1}(H)$-valued function and will call $\int_0^T \Psi(t) \diamond \mathbb{W}(t)\, dt$ the Hitsuda–Skorohod integral of $\Psi(t)$.

The Hitsuda–Skorohod integral is a generalization of the Ito integral $\int_0^T \Psi(t)\, dW(t)$ with respect to the cylindrical Wiener process. Namely, if $\Psi(t) \in (L^2)\big(\mathrm{HS}(H_Q;H)\big)$ for all $t \in [0;T]$, $\Psi(t)$ is adapted to the filtration generated by $W(t)$ and

$$\int_0^T \|\Psi(t)\|^2_{(L^2)(\mathrm{HS}(H_Q;H))} dt < \infty,$$

then

$$\int_0^T \Psi(t) \diamond \mathbb{W}(t)\, dt = \int_0^T \Psi(t)\, dW(t)$$

Let H_1 and H_2 be separable Hilbert spaces. For $A \in \mathcal{L}(H_1, H_2)$ define

$$A\Phi := \sum_{\alpha \in \mathcal{T}} A\Phi_\alpha \mathbf{h}_\alpha, \text{ for } \Phi = \sum_{\alpha \in \mathcal{T}} \Phi_\alpha \mathbf{h}_\alpha \in (\mathcal{S})_{-\rho}(H_1). \tag{11}$$

(See the proof in [5]). Defined in such a way A is a linear continuous operator with values in $(\mathcal{S})_{-\rho}(H_2)$. If A is not bounded, define $(\mathrm{dom}A)$ as the set of all $\sum_{\alpha \in \mathcal{T}} \Phi_\alpha \mathbf{h}_\alpha \in (\mathcal{S})_{-\rho}(H_1)$ such that $\Phi_\alpha \in \mathrm{dom}A$ for any $\alpha \in \mathcal{T}$ and $\sum_{\alpha \in \mathcal{T}} (\alpha!)^{1-\rho} \frac{\|A\Phi_\alpha\|^2_{H_2}}{(2\mathbb{N})^{2p\alpha}} < \infty$ for some $p \in \mathbb{N}$. Then (11) defines a linear operator on $(\mathrm{dom}A)$ with values in $(\mathcal{S})_{-\rho}(H_2)$. It is easy to verify that it is closed if A is a closed operator from H_1 to H_2, and to prove the next proposition.

Proposition 7 *Let* $A : H_1 \to H_2$ *be linear and closed. For any* $\Phi \in (\mathrm{dom}A) \subseteq (\mathcal{S})_{-\rho}(H_1)$ *we have* $\big[S\Phi\big](h) \in \mathrm{dom}A \subseteq H_1$ *and* $\big[SA\Phi\big](h) = A\big[S\Phi\big](h),\ h \in \mathcal{S}$.

3. The Cauchy problem for a linear operator-differential equation with multiplicative noise

Consider the Cauchy problem (5) with a linear closed operator A acting in H, $B(\cdot) \in \mathcal{L}\big(H, \mathcal{L}(H)\big)$, $\Phi \in (\mathrm{dom}A) \subseteq (\mathcal{S})_{-\rho}(H)$. We obtain it by substituting the Hitsuda–Skorohod integral for the Ito one in equation (4) and differentiating both sides of the

[*] For $v \in V$, $u \in U$, where V and U are Hilbert spaces, we denote by $v \otimes u$ the operator from U to V, defined by $(v \otimes u)h := v(u, h)_U$.

equation with respect to t. Note that if Q is a nuclear operator in H satisfying (10), then since $B\big(X(t)\big) \in (\mathcal{S})_{-\rho}\big(\mathrm{HS}(H_Q;H)\big)$ for any $X(t) \in (\mathcal{S})_{-\rho}(H)$, the Wick product in (5) is well defined. Our main result is the following theorem.

Theorem 2 *Let A be the generator of a C_0-semigroup in H, B be such that for each $y \in H$*

- *(BI)* $\ker B(\cdot)y = \{0\}$*;*
- *(BII)* $B(\mathrm{dom}A)y \subseteq \mathrm{dom}A$*;*
- *(BIII) The operator $C(\cdot)y : H \to \mathcal{L}(H)$, defined by $C(x)y := AB(x)y - B(Ax)y$ for $x \in \mathrm{dom}A$, is bounded.*

Then for any $\Phi \in (\mathrm{dom}A) \subseteq (\mathcal{S})_{-0}(H)$ the problem (5) has a unique solution in the space $(\mathcal{S})_{-0}(H)$.

Proof. Note that by the uniform boundedness principle it follows from (BIII) that there exists $M_{AB} > 0$ such that

$$\|C(x)y\| \le M_{AB}\|x\|\,\|y\|, \quad x \in \mathrm{dom}A,\ y \in H. \tag{12}$$

Applying S-transform to (5) we obtain the next Cauchy problem:

$$\frac{d}{dt}\hat{X}(t,\theta) = A\hat{X}(t,\theta) + B\big(\hat{X}(t,\theta)\big)\hat{\mathbb{W}}(t,\theta),\ t \ge 0, \quad \hat{X}(0,\theta) = \hat{\Phi}(\theta), \quad \theta \in \mathcal{S}, \tag{13}$$

where $\hat{X}(t,\theta) = S[X(t)](\theta)$, $\hat{\mathbb{W}}(t,\theta) = S[\mathbb{W}(t)](\theta)$, $\hat{\Phi}(\theta) = S\Phi(\theta)$.

We first prove the uniqueness of solution. Note that if $\hat{X}(\cdot,\theta)$ is a solution of (13) for some $\theta \in \mathcal{S}$, it satisfies the equation

$$\hat{X}(t,\theta) = U(t)\hat{\Phi}(\theta) + \int_0^t U(t-s)B(\hat{X}(s,\theta))\hat{\mathbb{W}}(s,\theta)\,ds, \quad t \ge 0.$$

Thus it is sufficient to prove that equation

$$\hat{X}(t,\theta) - \int_0^t U(t-s)B(\hat{X}(s,\theta))\hat{\mathbb{W}}(s,\theta)\,ds = 0, \quad t \ge 0 \tag{14}$$

has the only solution $\hat{X}(t,\theta) \equiv 0$ for any $\theta \in \mathcal{S}$, where $\{U(t),\, t \ge 0\}$ is the C_0-semigroup generated by A with $M > 0$, $a \in \mathbb{R}$ such that

$$\|U(t)\| \le Me^{at}, \quad t \ge 0. \tag{15}$$

This can be proved using the Volterra equations technique and the fact that $\hat{\mathbb{W}}(s,\theta)$ is an infinitely differentiable $\mathbb{H}$-valued function of s and thus is bounded on any segment of $\mathbb{R}$.

To prove existence of solution consider the series

$$T(t,\theta) = \sum_{k=0}^{\infty} T_k(t,\theta), \quad \theta \in \mathcal{S}, \tag{16}$$

where operators $T_k(t,\theta), t \geq 0,\ k = 0,1,2,\dots$ are defined as follows:

$$T_0(t,\theta) = U(t), \quad T_k(t,\theta)x = \int_0^t U(t-s)B\big(T_{k-1}(s,\theta)x\big)\hat{\mathbb{W}}(s,\theta)\,ds,\ x \in H.$$

Proving first for $t \geq 0$, $\theta \in \mathcal{S}$, $k \in \mathbb{N} \cup \{0\}$ and $\Phi \in (\mathrm{dom}A)$ the estimates

$$\|T_k(t,\theta)\|_{\mathcal{L}(H)} \leq M^{k+1}\|B\|^k e^{at}|\theta|_0^k \sqrt{\frac{t^k}{k!}}\,, \tag{17}$$

$$\|AT_k(t,\theta)\hat{\Phi}(\theta)\| \leq M^{k+1}\|B\|^{k-1}|\theta|_0^k e^{at}\sqrt{\frac{t^k}{k!}}\left(\|B\|\|A\hat{\Phi}(\theta)\| + kM_{AB}\|\hat{\Phi}(\theta)\|\right), \tag{18}$$

where $M > 0$ and $a \in \mathbb{R}$ are constants from (15), $\|B\| = \|B\|_{\mathcal{L}(H,\mathcal{L}(H))}$, M_{AB} is from (12), we obtain by (17) for any $n,m \in \mathbb{N}$

$$\begin{aligned}\sum_{k=n}^{n+m}\|T_k(t,\theta)\| &\leq Me^{at}\sum_{k=n}^{n+m}\frac{\big(M\sqrt{2}\|B\||\theta|_0\sqrt{t}\big)^k}{\sqrt{k!}}\cdot\frac{1}{\sqrt{2^k}} \leq \\ &\leq Me^{at}\left(\sum_{k=n}^{n+m}\frac{\big(2M^2\|B\|^2|\theta|_0^2 t\big)^k}{k!}\right)^{1/2}\left(\sum_{k=n}^{n+m}\frac{1}{2^k}\right)^{1/2}.\end{aligned} \tag{19}$$

Hence (16) is absolutely convergent to $T(t,\theta)$ in $\mathcal{L}(H)$ for any $t \geq 0$, $\theta \in \mathcal{S}$.

For any $\Phi \in (\mathrm{dom}A)$, by Proposition 7 and properties of C_0-semigroups we obtain: $T_0(t,\theta)\hat{\Phi}(\theta) \in \mathrm{dom}A$ for all $t \geq 0$ and $\theta \in \mathcal{S}$. It follows from (BII) that $B\big(\mathrm{dom}A\big)\hat{W}(t,\theta) \subseteq \mathrm{dom}A$ for all $t \geq 0$ and $\theta \in \mathcal{S}$ and by induction we obtain that $T_k(t,\theta)\hat{\Phi}(\theta) \in \mathrm{dom}A$ for all $\Phi \in (\mathrm{dom}A)$, $k \in \mathbb{N}$, $t \geq 0$ and $\theta \in \mathcal{S}$. It also follows from (BII) that $B\big(T_k(s,\theta)\hat{\Phi}(\theta)\big)\hat{\mathbb{W}}(t,\theta) \in \mathrm{dom}A$. Moreover, we have

$$\frac{d}{dt}U(t-s)B\big(T_k(s,\theta)\hat{\Phi}(\theta)\big)\hat{\mathbb{W}}(t,\theta) = AU(t-s)B\big(T_k(s,\theta)\hat{\Phi}(\theta)\big)\hat{\mathbb{W}}(t,\theta),\ t \geq 0, \theta \in \mathcal{S}.$$

Thus for all $\Phi \in (\mathrm{dom}A)$ we have

$$\frac{d}{dt}T_0(t,\theta)\hat{\Phi}(\theta) = AT_0(t,\theta)\hat{\Phi}(\theta)\,, \tag{20}$$

$$\begin{aligned}\frac{d}{dt}T_k(t,\theta)\hat{\Phi}(\theta) = &\int_0^t AU(t-s)B\big(T_{k-1}(s,\theta)\hat{\Phi}(\theta)\big)\hat{\mathbb{W}}(s,\theta)\,ds + \\ &+ B\big(T_{k-1}(t,\theta)\hat{\Phi}(\theta)\big)\hat{\mathbb{W}}(t,\theta)\,.\end{aligned} \tag{21}$$

Since A is closed we can rewrite (21) as

$$\frac{d}{dt}T_k(t,\theta)\hat{\Phi}(\theta) = AT_k(t,\theta)\hat{\Phi}(\theta) + B\big(T_{k-1}(t,\theta)\hat{\Phi}(\theta)\big)\hat{\mathbb{W}}(t,\theta)\,. \tag{22}$$

Using (18) we obtain

$$\sum_{k=n+1}^{m} \|AT_k(t,\theta)\hat{\Phi}(\theta)\| \le Me^{at}\left(\sum_{k=n+1}^{m} \frac{(\sqrt{2}M\|B\||\theta|_{L^2(\mathbb{R})}\sqrt{t})^k}{\sqrt{k!}}\cdot\frac{1}{\sqrt{2^k}}\right)\|A\hat{\Phi}(\theta)\|+$$

$$+\frac{M}{\|B\|}e^{at}\left(\sum_{k=n+1}^{m} \frac{(\sqrt{2}M\|B\||\theta|_{L^2(\mathbb{R})}\sqrt{t})^k}{\sqrt{k!}}\cdot\frac{k}{\sqrt{2^k}}\right)M_{AB}\|\hat{\Phi}(\theta)\| \le$$

$$\le Me^{at}\left(\sum_{k=n+1}^{m} \frac{\left(2M^2\|B\|^2|\theta|_0^2 t\right)^k}{k!}\right)^{1/2}\cdot\left(\sum_{k=n+1}^{m}\frac{1}{2^k}\right)^{1/2}\|A\hat{\Phi}(\theta)\|+$$

$$+\frac{M}{\|B\|}e^{at}\left(\sum_{k=n+1}^{m} \frac{\left(2M^2\|B\|^2|\theta|_0^2 t\right)^k}{k!}\right)^{1/2}\cdot\left(\sum_{k=n+1}^{m}\frac{k^2}{2^k}\right)^{1/2}M_{AB}\|\hat{\Phi}(\theta)\|.$$

it follows from here that the series $\sum_{k=0}^{\infty} AT_k(t,\theta)\hat{\Phi}(\theta)$ converges in H for all $\theta \in \mathcal{S}$, $\Phi \in (\mathrm{dom}A)$. Taking sum of equalities (20) and (22) with respect to all $k \in \mathbb{N}$ we obtain in the right hand side a series converging in H for all $t \ge 0$, $\theta \in \mathcal{S}$. This proves that $\hat{X}(t,\theta) = T(t,\theta)\hat{\Phi}(\theta)$ is a solution of (13).

It follows from (19) that

$$\|T(t,\theta)\| \le \sum_{k=0}^{\infty}\|T_k(t,\theta)\| \le Me^{at}\sum_{k=0}^{\infty}\frac{\left(M\sqrt{2}\|B\||\theta|_{L^2(\mathbb{R})}\sqrt{t}\right)^k}{\sqrt{k!}}\cdot\frac{1}{\sqrt{2^k}} \le$$

$$\le Me^{at}\left(\sum_{k=0}^{\infty}\frac{\left(2M^2\|B\|^2|\theta|_0^2 t\right)^k}{k!}\right)^{1/2}\left(\sum_{k=0}^{\infty}\frac{1}{2^k}\right)^{1/2} = M\sqrt{2}\,e^{at}\exp\left(M^2\|B\|^2|\theta|_0^2 t\right).$$

By (9) we have $\|\hat{\Phi}(\theta)\| \le \|\Phi\|_{HS,p,0}\exp\left(|\theta|_p^2\right)$, $\theta \in \mathcal{S}$, for some $p \in \mathbb{N}$. It follows that for $t \ge 0$ we have

$$\|\hat{X}(t,\theta)\| \le M\sqrt{2}\,e^{at}\exp\left(\left(M^2\|B\|^2 t+1\right)|\theta|_p^2\right)\|\Phi\|_{HS,p,0}\,,\quad \theta \in \mathcal{S}.$$

It follows from here that for each $t \ge 0$ $\hat{X}(t,\theta)$ is an S-transform of a unique $X(t) \in (\mathcal{S})_{-0}(H)$, which is a unique solution of problem (13). □

It is easy to see that A and B defined by (2) and (3) respectively satisfy the conditions of Theorem 2. Thus the stochastic perturbation of our model problem described in introduction has a unique solution in $(\mathcal{S})_{-0}(L^2[0;1])$.

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AMS Subject Classification: 46F25; 47D06; 34K30; 60H40

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Lavoro pervenuto in redazione il 22.05.2013

Rend. Sem. Mat. Univ. Politec. Torino
Vol. 71, 2 (2013), 251 – 260

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COGARCH: SYMBOL, GENERATOR AND CHARACTERISTICS

Abstract. We describe the technique how to use the symbol in order to calculate the generator and the characteristics of an Itô process. As an example we analyze the COGARCH process which is used to model financial data.

1. Introduction

The COGARCH process was introduced by Klüppelberg et al. in [13] in order to model financial data. It is a continuous time analog of the classic GARCH process (in discrete time) and it is based on a single background driving Lévy process in contrast to the well known model by Barndorff-Nielsen and Shephard [1]. Lévy processes are càdlàg universal Markov processes which are homogeneous in time *and* space. Our main reference for this class of processes is [16]. For the Lévy triplet we write (ℓ, Q, N).

In the present paper we calculate the so called *symbol* of the COGARCH process (and its volatility process). The origins of the symbols are in the theory of partial differential equations, namely they appear in the Fourier representation of certain operators. The symbol found its way into probability theory for the following reason: suppose we are given a Feller process X with associated semigroup $(T_t)_{t\geq 0}$ and generator $(A, D(A))$. Suppose further that the test functions $C_c^\infty(\mathbb{R}^d)$ are contained in the domain $D(A)$. In this case A is a pseudo-differential operator with symbol $-q(x,\xi)$. For every $x \in \mathbb{R}^d$ $q(x,\cdot)$ is a continuous negative definite function in the sense of Schoenberg (cf. [2] Chapter 2).

For a detailed, self contained treatment on the interplay between the process and its symbol cf. the monograph [9]. In this context the following four questions are of interest:
I) Given a process, (say as the solution of an SDE) what is its symbol? (E.g. [19])
II) Given a symbol, does there exist a corresponding process? ([6, 7, 11])
III) Which properties of the process can be characterized via the symbol? ([17, 18])
IV) For which bigger classes of processes is it possible (and useful) to define a symbol? ([20, 21])
All four questions are a vital part of ongoing research. In the present paper we emphasize, how one can calculate the symbol of a given process using a probabilistic formula and derive directly the generator as well as the semimartingale characteristics.

The notation we are using is (more or less) standard. Vectors are meant to be

*Acknowledgments: Most of this work was done as a part of my PhD thesis, written under the guidance of René L. Schilling to whom I am deeply grateful. Financial Support by the DFG-SFB 823 is gratefully acknowledged. Furthermore I would like to thank an anonymous referee for carefully reading the manuscript and offering useful suggestions which helped to improve the paper.

column vectors and the transposed of a vector v or a matrix Q is denoted by v' respective Q'.

Let us recall how the COGARCH process is defined:
we start with a Lévy process $Z = (Z_t)_t$ with triplet (ℓ, Q, N). Fix $0 < \delta < 1$, $\beta > 0$, $\lambda \geq 0$. Then the volatility process $(\sigma_t)_{t\geq 0}$ is the solution of the SDE

$$\begin{aligned} d\sigma_t^2 &= \beta\, dt + \sigma_t^2 \left(\log\delta\, dt + \frac{\lambda}{\delta}\, d[Z,Z]_t^{disc} \right) \\ \sigma_0 &= S \end{aligned}$$

where $S > 0$ and

$$[Z,Z]_t^{disc} = \sum_{0<s\leq t} (\Delta Z_s)^2.$$

It turns out, that $(\sigma_t)_{t\geq 0}$ is a time homogeneous Markov process.

Definition: The process

$$G_t := g + \int_0^t \sigma_{s-}\, dZ_t, \qquad g \in \mathbb{R},$$

is called **COGARCH process** (starting in g).

We allow the process to start everywhere in order to bring our methods into account. The pair (G_t, σ_t^2) is a (normal) Markov process which is homogeneous in time. It is homogeneous in space in the first component. Furthermore (G_t, σ_t^2) is an Itô process, which follows from Theorem 3.33 of [4] which characterizes Itô processes as solutions of certain stochastic differential equations and Proposition IX.5.2. of [10] giving a representation of the semimartingale characteristics of a stochastic integral.

To avoid problems which might arise for processes defined on $\mathbb{R} \times \mathbb{R}_+$ we consider in the following: $(G_t, V_t) = (G_t, \log(\sigma_t^2))$, i.e., V is the logarithmic squared volatility.

2. The Symbol of a Stochastic Process

Definition: Let X be an $\mathbb{R}^d$-valued universal Markov process, which is conservative and normal. Fix a starting point x and define $T = T_R^x$ to be the first exit time from the ball of radius $R > 0$:

(1) $$T := T_R^x := \inf\{t \geq 0 : \|X_t - x\| > R\} \text{ under } \mathbb{P}^x (x \in \mathbb{R}^d).$$

We call the function $p : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$, given by

(2) $$p(x,\xi) := -\lim_{t\downarrow 0} \mathbb{E}^x \frac{e^{i(X_t^T - x)'\xi} - 1}{t},$$

the **(probabilistic) symbol** of the process, if the limit exists for every x, ξ and R and is independent of the choice of R.

In [21] Theorem 4.4. we have shown that for Itô processes in the sense of Cinlar, Jacod, Protter and Sharpe (cf. [5]) having differential characteristics which are finely continuous (cf. [3]) and locally bounded the above limit exists and coincides for every choice of R. For the reader's convenience we recall the the definition of Itô processes, as it is used here:

Definition: A Markov semimartingale $X = (X_t)_{t\geq 0}$, i.e., a universal Markov process which is a semimartingale with respect to every initial probability $\mathbb{P}^x$ ($x \in \mathbb{R}$), is called **Itô process** if it has characteristics of the form:

$$\begin{aligned} B_t^j(\omega) = & \int_0^t \ell^j(X_s(\omega))\, ds \quad j = 1, \ldots, d \\ C_t^{jk}(\omega) = & \int_0^t Q^{jk}(X_s(\omega))\, ds \quad j,k = 1, \ldots, d \\ \nu(\omega; ds, dy) = & \; N(X_s(\omega), dy)\, ds \end{aligned}$$

where $\ell^j, Q^{jk} : \mathbb{R}^d \to \mathbb{R}$ are measurable functions, $Q(x) = (Q^{jk}(x))_{1\leq j,k\leq d}$ is a positive semidefinite matrix for every $x \in \mathbb{R}^d$, and $N(x,\cdot)$ is a Borel transition kernel on $\mathbb{R}^d \times \mathcal{B}(\mathbb{R}^d \setminus \{0\})$. ℓ, Q and $\int_{y\neq 0}(1 \wedge y^2)N(\cdot, dy)$ are called **differential characteristics**.

Example 1: Let X be a d-dimensional Lévy process. It is a well known fact that the characteristic function of X_t ($t \geq 0$) can be written as

$$\mathbb{E}^0 \exp(iX_t'\xi) = \exp(-t\psi(\xi)).$$

The function $\psi : \mathbb{R}^d \to \mathbb{C}$ is called characteristic exponent. By an elementary calculation one obtains $p(x,\cdot) = \psi(\cdot)$ for every $x \in \mathbb{R}^d$.

Example 2: Let X be a rich Feller process, i.e., the test functions $C_c^\infty(\mathbb{R}^d)$ are contained in the domain $D(A)$ of the generator A. In this case the generator restricted to $C_c^\infty(\mathbb{R}^d)$ is a pseudo-differential operator with (functional analytic) symbol $-q(x,\xi)$. In [21] we have shown that X is an Itô process and $p(x,\xi) = q(x,\xi)$ for every $x,\xi \in \mathbb{R}^d$.

Example 3: Let $(Z_t)_{t\geq 0}$ be an $\mathbb{R}^n$-valued Lévy process. The solution of the stochastic differential equation ($x \in \mathbb{R}^d$),

$$\begin{aligned} dX_t^x &= \Phi(X_{t-}^x)\, dZ_t \\ X_0^x &= x, \end{aligned}$$

where $\Phi : \mathbb{R}^d \to \mathbb{R}^{d\times n}$ is Lipschitz continuous admits the symbol

$$p(x,\xi) = \psi(\Phi(x)'\xi).$$

This was shown in [19].

3. Symbol, Generator and Characteristics

In the present section we calculate the symbol of the COGARCH process. Using the close relationship between the symbol, the extended generator and the semimartingale characteristics we are able to write down the latter two objects directly. Let us emphasize that the symbol does *not* depend on g, since the process is homogeneous in the first component.

Theorem: The stochastic process $(G_t, V_t) = (G_t, \log(\sigma_t^2))$ admits the symbol $p : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{C}$ given by

$$\begin{aligned}
&p\left(\begin{pmatrix} g \\ v \end{pmatrix}, \xi\right) = \\
&-i\xi_1 \left(\ell e^{v/2} + e^{v/2} \int_{\mathbb{R}\setminus\{0\}} y \cdot (1_{\{|e^{v/2} y|<1\}} \cdot 1_{\{|\log(1+(\lambda/\delta)\, y^2)|<1\}} - 1_{\{|y|<1\}})\, N(dy)\right) \\
&-i\xi_2 \left(\frac{\beta}{e^v} + \log\delta + \int_{\mathbb{R}\setminus\{0\}} \log(1+\frac{\lambda}{\delta} y^2) \cdot (1_{\{|e^{v/2} y|<1\}} \cdot 1_{\{|\log(1+(\lambda/\delta)\, y^2)|<1\}})\, N(dy)\right) \\
&+\frac{1}{2}\xi_1^2 e^v Q \\
&-\int_{\mathbb{R}^2\setminus\{0\}} \left(e^{i(z_1,z_2)\xi} - 1 - iz'\xi \cdot (1_{\{|z_1|<1\}} \cdot 1_{\{|z_2|<1\}})\right) \tilde{N}\left(\begin{pmatrix} g \\ v \end{pmatrix}, dz\right),
\end{aligned}$$

where $\tilde{N}$ is the image measure

$$\tilde{N}\left(\begin{pmatrix} g \\ v \end{pmatrix}, dz\right) = N(f_v \in dz)$$

under $f : \mathbb{R} \to \mathbb{R}^2$ given by

$$f_v(w) = \begin{pmatrix} e^{v/2} w \\ \log(1+(\lambda/\delta)\, w^2) \end{pmatrix}.$$

Remark: It is not surprising, that the transformation of the jump measure depends only on v since the process is space homogeneous in the first component.

Proof: Let T be the stopping time defined in (1). At first we use Itô's formula:

$$\frac{\mathbb{E}^{g,v} e^{i(G_t^T - g, V_t^T - v)\xi} - 1}{t} = \frac{\mathbb{E}^{0,v} e^{i(G_t^T, V_t^T - v)\xi} - 1}{t}$$

$$\text{(I)} \qquad = \frac{1}{t}\mathbb{E}^{0,v} \int_{0+}^t i\xi_1 e^{i(G_{s-}^T, V_{s-}^T - v)\xi}\, dG_s^T$$

$$\text{(II)} \qquad + \frac{1}{t}\mathbb{E}^{0,v} \int_{0+}^t i\xi_2 e^{i(G_{s-}^T, V_{s-}^T - v)\xi}\, dV_s^T$$

$$\text{(III)} \qquad - \frac{1}{2t}\mathbb{E}^{0,v} \int_{0+}^t \xi_1^2 e^{i(G_{s-}^T, V_{s-}^T - v)\xi}\, d[G^T, G^T]_s^c$$

$$\text{(IV)} \qquad - \frac{1}{t}\mathbb{E}^{0,v} \int_{0+}^t \xi_1\xi_2 e^{i(G_{s-}^T, V_{s-}^T - v)\xi}\, d[G^T, V^T]_s^c$$

$$\text{(V)} \qquad - \frac{1}{2t}\mathbb{E}^{0,v} \int_{0+}^t \xi_2^2 e^{i(G_{s-}^T, V_{s-}^T - v)\xi}\, d[V^T, V^T]_s^c$$

$$\text{(VI)} \qquad + \frac{1}{t}\mathbb{E}^{0,v} \sum_{0<s\le t} e^{(G_{s-}^T, V_{s-}^T - v)\xi} \left(e^{i\Delta(G_s^T, V_s^T)\xi} - 1 - (i\xi_1 \Delta G_s^T + i\xi_2 \Delta V_s^T)\right).$$

We deal with this formula term-by-term. In the calculation of the first term we use

$$dG_s^T = \sigma_{s-} 1_{\{s\in[[0,T]]\}}\, dZ_s.$$

Recall that the integrand is bounded and for the Lévy process Z we have the Lévy-Itô-decomposition:

$$Z_t = \ell t + \sqrt{Q} W_t + \int_{[0,t]\times\{|y|<1\}} y\,(\mu^Z(ds,dy) - ds N(dy)) \\ + \sum_{0<s\le t} \Delta Z_s 1_{\{|\Delta Z_s|\ge 1\}},$$

where μ^Z denotes the jump measure of the process (cf. [10] Proposition II.1.16). The integrals with respect to the martingale parts are again L^2-martingales and the respective terms disappear. What remains from the first term is:

$$\frac{1}{t}\mathbb{E}^{0,v}\int_{0+}^t i\xi_1 e^{i(G^T_{s-},V^T_{s-}-v)\xi}\sigma_{s-}1_{\{s\in[[0,T]]\}}\, d\left(\ell s + \sum_{0<r\le s}\Delta Z_r \cdot 1_{\{|\Delta Z_r|\ge 1\}}\right). \tag{3}$$

For the first part of this integrand we get:

$$\begin{aligned}
&\frac{1}{t}\mathbb{E}^{0,v}\int_{0+}^t i\xi_1 e^{i(G^T_{s-},V^T_{s-}-v)\xi}\sigma_{s-}1_{\{s\in[[0,T]]\}}\, d(\ell s)\\
&\quad = \mathbb{E}^{0,v}\frac{1}{t}\int_0^t i\xi_1 \ell e^{i(G^T_s,V^T_s-v)\xi}1_{\{s\in[[0,T[[\}}\sigma_s\, ds\\
&\quad = i\xi_1\ell\,\mathbb{E}^{0,v}\underbrace{\int_0^1 e^{i(G^T_{st},V^T_{st}-v)\xi}1_{\{st\in[[0,T[[\}}}_{\to 1}\underbrace{\sigma_{st}}_{\to S}\, ds\\
&\underset{t\downarrow 0}{\longrightarrow} i\xi_1\ell S.
\end{aligned}$$

In the first equation we used the fact that we are integrating with respect to Lebesgue measure. For this the countable number of jump times is a nullset. In the last step we used Lebesgue's theorem twice. A similar argumentation is used in the consideration of the second and the third term. The jump term of (3) above will be compared to the sixth term.

Using Itô's formula we obtain for the second term

$$\frac{1}{t}\mathbb{E}^{0,v}\int_{0+}^t i\xi_2 e^{i(G^T_{s-},V^T_{s-}-v)\xi}\left\{\frac{1}{\sigma^2_{s-}}\,d(\sigma^T_s)^2 + d\left(\sum_{0<r\le s}\log\sigma^2_r - \log\sigma^2_{r-} - \frac{1}{\sigma^2_{r-}}\Delta(\sigma^2_r)\right)\right\}$$

and by plugging in the defining SDE for (σ^2):

$$\begin{aligned}
&\frac{1}{t}\mathbb{E}^{0,v}\int_{0+}^t i\xi_2 e^{i(G^T_{s-},V^T_{s-}-v)\xi}1_{\{s\in[[0,T]]\}}\left\{\left(\frac{\beta}{\sigma^2_{s-}}\,ds + \frac{\sigma^2_{s-}}{\sigma^2_{s-}}\log\delta\, ds\right)\right.\\
&\left. + \frac{\lambda}{\delta}\, d\left(\sum_{0<r\le s}(\Delta Z_r)^2)\right) + d\left(\sum_{0<r\le s}\Delta(\log\sigma^2_r) - \frac{1}{\sigma^2_{r-}}\Delta(\sigma^2_r)\right)\right\}.
\end{aligned}$$

We postpone the jump parts and for the remainder term we get in the limit, using a similar argumentation as for the first term,

$$\underset{t\downarrow 0}{\longrightarrow} i\xi_2\beta/S^2 + i\xi_2\log\delta.$$

For the third term we obtain in an analogous manner to the first one

$$
\begin{aligned}
-\frac{1}{2t}\mathbb{E}^{0,v}&\int_{0+}^{t}\xi_1^2 e^{i(G_{s-}^T,V_{s-}^T-v)\xi}\,d[G^T,G^T]_s^c\\
&=-\frac{1}{2t}\mathbb{E}^{0,v}\int_{0+}^{t}\xi_1^2 e^{i(G_{s-}^T,V_{s-}^T-v)\xi}\,1_{\{s\in[[0,T]]\}}\sigma_{s-}^2\,d[Z,Z]_s^c\\
&=-\frac{1}{2t}\mathbb{E}^{0,v}\int_{0}^{t}\xi_1^2 e^{i(G_{s-}^T,V_{s-}^T-v)\xi}\,1_{\{s\in[[0,T[[\}}\sigma_{s-}^2\,d(Qs)\\
\xrightarrow[t\downarrow 0]{}&-\frac{1}{2}\xi_1^2 S^2 Q.
\end{aligned}
$$

The terms four and five are constant zero: since $(t)_t$ and $([Z,Z]_t)_t$ are both of finite variation on compacts, the process $(\sigma_t^2)_t$ has this property as well, by its very definition. Therefore it is a quadratic pure jump process (see [14] Section II.6). Using Itô's formula we obtain that $V=\log(\sigma^2)$ is again a quadratic pure jump process and therefore

$$
[V^T,V^T]_s^c=0 \text{ and } [V^T,G^T]_s^c=0.
$$

The only thing that remains to do is dealing with the various 'jump parts'. From the first term we left the following behind

$$
\begin{aligned}
\frac{1}{t}\mathbb{E}^{0,v}&\int_{0+}^{t} i\xi_1 e^{i(G_{s-}^T,V_{s-}^T-v)\xi}\sigma_{s-}1_{\{s\in[[0,T]]\}}\,d\left(\sum_{0<r\le s}\Delta Z_r\cdot 1_{\{|\Delta Z_r|\ge 1\}}\right)\\
&=\frac{1}{t}\mathbb{E}^{0,v}\sum_{0<s\le t} i\xi_1 e^{i(G_{s-}^T,V_{s-}^T-v)\xi}\sigma_{s-}1_{\{s\in[[0,T]]\}}\Delta Z_s\cdot 1_{\{|\Delta Z_s|\ge 1\}}
\end{aligned}
$$

and from the second one

$$
\begin{aligned}
&\frac{1}{t}\mathbb{E}^{0,v}\int_{0+}^{t} i\xi_2 e^{i(G_{s-}^T,V_{s-}^T-v)\xi}1_{\{s\in[[0,T]]\}}\frac{\lambda}{\delta}\,d\left(\sum_{0<r\le s}(\Delta Z_r)^2\right)\\
&\qquad+\frac{1}{t}\mathbb{E}^{0,v}\int_{0+}^{t} i\xi_2 e^{i(G_{s-}^T,V_{s-}^T-v)\xi}1_{\{s\in[[0,T]]\}}\,d\left(\sum_{0<r\le s}\Delta V_r-\frac{1}{\sigma_{r-}^2}\Delta(\sigma_r^2)\right)\\
&=\frac{1}{t}\mathbb{E}^{0,v}\sum_{0<s\le t} i\xi_2 e^{i(G_{s-}^T,V_{s-}^T-v)\xi}1_{\{s\in[[0,T]]\}}\frac{\lambda}{\delta}(\Delta Z_s)^2\\
&\qquad+\frac{1}{t}\mathbb{E}^{0,v}\sum_{0<s\le t} i\xi_2 e^{i(G_{s-}^T,V_{s-}^T-v)\xi}1_{\{s\in[[0,T]]\}}\left(\Delta V_s-\frac{1}{\sigma_{s-}^2}\Delta(\sigma_s^2)\right).
\end{aligned}
$$

Adding these terms to term number six and using the equalities

$$
\Delta G_s^T=(\sigma_{s-}1_{\{s\in[[0,T]]\}})\Delta Z_s \text{ and } (\Delta\sigma_s^T)^2=\frac{\lambda}{\delta}(\sigma_{s-}^2 1_{\{s\in[[0,T]]\}})(\Delta Z_s)^2
$$

as well as

$$
\Delta\log(\sigma_s^2)^T=\log\left(\frac{(\sigma_{s-}^2)^T+\Delta(\sigma_s^2)^T}{(\sigma_{s-}^2)^T}\right)=\log\left(1+\frac{\Delta(\sigma_s^2)^T}{(\sigma_{s-}^2)^T}\right)
$$

we obtain

$$\frac{1}{t}\mathbb{E}^{0,v}\sum_{0<s\le t} e^{i(G^T_{s-},V^T_{s-}-v)\xi}1_{\{s\in[[0,T]]\}}\times \left(e^{i\sigma_{s-}\Delta Z_s\xi_1+i\log(1+(\lambda/\delta)\Delta(Z_s)^2)\xi_2}-1-i\xi_1\sigma_{s-}\Delta Z_s\cdot 1_{\{|\Delta Z_s|<1\}}\right)$$

$$=\frac{1}{t}\mathbb{E}^{0,v}\int_{]0,t]\times\{y\neq 0\}} e^{i(G^T_{s-},V^T_{s-}-v)\xi}1_{\{s\in[[0,T]]\}}\times \left(e^{i\sigma_{s-}y\xi_1+i\log(1+(\lambda/\delta)y^2)\xi_2}-1-i\xi_1\sigma_{s-}y\cdot 1_{\{|y|<1\}}\right)\mu^Z(\cdot;ds,dy)$$

$$=\frac{1}{t}\mathbb{E}^{0,v}\int_{]0,t]\times\{y\neq 0\}} e^{i(G^T_{s-},V^T_{s-}-v)\xi}1_{\{s\in[[0,T]]\}}\times \left(\left(e^{i\sigma_{s-}y\xi_1+i\log(1+(\lambda/\delta)y^2)\xi_2}-1-i\begin{pmatrix}\sigma_{s-}y\\ \log(1+\frac{\lambda}{\delta}y^2)\end{pmatrix}'\xi\cdot 1_{\{|Sy|<1\}}\cdot 1_{\{|\log(1+\frac{\lambda}{\delta}y^2)|<1\}}\right)\right.$$
$$+\left(i\xi_1\sigma_{s-}y\cdot(1_{\{|Sy|<1\}}\cdot 1_{\{|\log(1+\frac{\lambda}{\delta}y^2)|<1\}})-1_{\{|y|<1\}})\right)$$
$$\left.+\left(i\xi_2\log(1+\frac{\lambda}{\delta}y^2)\cdot 1_{\{|Sy|<1\}}\cdot 1_{\{|\log(1+\frac{\lambda}{\delta}y^2)|<1\}}\right)\right)\mu^Z(\cdot;ds,dy).$$

It is possible to calculate the integral with respect to the compensator $\nu(\cdot;ds,dy)=N(dy)\,ds$ instead of the measure itself 'under the expectation', since the integrands are of class F_p^2 of Ikeda-Watanabe ([8]):

$$F_p^2=\left\{f(s,y,\omega): f\text{ is predictable, }\mathbb{E}\int_0^t\int_{\mathbb{R}}|f(s,y,\cdot)|^2N(dy)ds\text{ for every }t>0\right\}.$$

One obtains this, because $1_{\{|Sy|<1\}}\cdot 1_{\{|\log(1+(\lambda/\delta)\,y^2)|<1\}}-1_{\{|y|<1\}}$ is zero near the origin and bounded and $\log(1+\frac{\lambda}{\delta}y^2)\le(\lambda/\delta)\cdot y^2$ for $|(\lambda/\delta)\cdot y^2|<1$.
For t tending to zero (and multiplying with -1) we obtain by using Lebesgue's theorem again twice

$$p\left(\begin{pmatrix}g\\ v\end{pmatrix},\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}\right)=$$
$$-i\xi_1\left(\ell S+S\int_{\mathbb{R}\setminus\{0\}}y\cdot(1_{\{|Sy|<1\}}\cdot 1_{\{|\log(1+(\lambda/\delta)\,y^2)|<1\}}-1_{\{|y|<1\}})\,N(dy)\right)$$
$$-i\xi_2\left(\frac{\beta}{S^2}+\log\delta+\int_{\mathbb{R}\setminus\{0\}}\log(1+\frac{\lambda}{\delta}y^2)\cdot(1_{\{|Sy|<1\}}\cdot 1_{\{|\log(1+(\lambda/\delta)\,y^2)|<1\}})\,N(dy)\right)$$
$$+\frac{1}{2}\xi_1^2S^2Q$$
$$-\int_{\mathbb{R}^2\setminus\{0\}}\left(e^{i(z_1,z_2)\xi}-1-iz'\xi\cdot(1_{\{|z_1|<1\}}\cdot 1_{\{|z_2|<1\}})\right)\tilde{N}\left(\begin{pmatrix}g\\ S\end{pmatrix},dz\right),$$

where $\tilde{N}$ is the image measure

$$\tilde{N}\left(\binom{g}{S}, dz\right) = N\left(\begin{pmatrix} S\cdot \\ \log(1+(\lambda/\delta)\ \cdot^2) \end{pmatrix} \in dz\right).$$

And by writing the starting point as $S = \exp(v/2)$ we obtain the result. □

It is an advantage of our approach that, having calculated the symbol, one can write down the (extended) generator and the semimartingale characteristics at once. For the reader's convenience we recall the definition of the extended generator (cf. Definition (7.1) of [5]):

Definition: An operator G with domain $\mathcal{D}_G$ is called **extended generator** of a Markov semimartingale X if $\mathcal{D}_G$ consists of those functions $f \in \mathcal{B}(\mathbb{R}^d)$ for which there exists a function $Gf \in \mathcal{B}(\mathbb{R}^d)$ such that the process

$$C_t^f := f(X_t) - f(X_0) - \int_0^t Gf(X_s)\, ds$$

is well defined and a local martingale.

Combining Theorem 4.4 of [21] and Theorem 7.16 of [5] we obtain:

Corollary 1: The extended generator G on $C_b^2(\mathbb{R}^2)$ of the process $(X^{(1)}, X^{(2)})' = (G, \log(\sigma^2))'$ can be written as

$$\begin{aligned}
&Gu(x) = \\
&\partial_1 u(x)\left(\ell e^{x_2/2} + e^{x_2/2}\int_{\mathbb{R}\setminus\{0\}} y\cdot(1_{\{|e^{x_2/2}y|<1\}}\cdot 1_{\{|\log(1+(\lambda/\delta)\ y^2)|<1\}} - 1_{\{|y|<1\}})\ N(dy)\right) \\
&+\partial_2 u(x)\left(\frac{\beta}{e^{x_2}} + \log\delta + \int_{\mathbb{R}\setminus\{0\}} \log(1+\frac{\lambda}{\delta}y^2)\cdot(1_{\{|e^{x_2/2}y|<1\}}\cdot 1_{\{|\log(1+(\lambda/\delta)\ y^2)|<1\}})\ N(dy)\right) \\
&+\partial_1\partial_1 u(x) e^{x_2} Q \\
&+\int_{\mathbb{R}^2\setminus\{0\}}\Big(u(x-y) - u(x) + y'\nabla u(x)\cdot(1_{\{|y_1|<1\}}\cdot 1_{\{|y_2|<1\}})\Big)\tilde{N}(x, dy)
\end{aligned}$$

with the $\tilde{N}$ from above.

Writing $D(A)$ for the domain of the generator A of the process we have $D(A) \subseteq \mathcal{D}_G$ and the operators A and G coincide on $D(A)$.

Corollary 2: The semimartingale characteristics (B, C, ν) of the process $(X^{(1)}, X^{(2)})' = (G, \log(\sigma^2))'$ are

$$\begin{aligned}
B_t^{(1)} &= \int_0^t \left(\ell e^{\frac{X^{(2)}}{2}} + e^{\frac{X^{(2)}}{2}}\int_{\mathbb{R}\setminus\{0\}} y\cdot(1_{\left\{\left|e^{\frac{X^{(2)}}{2}} y\right|<1\right\}}\cdot 1_{\{|\log(1+(\frac{\lambda}{\delta})\ y^2)|<1\}} - 1_{\{|y|<1\}})\ N(dy)\right) ds \\
B_t^{(2)} &= \int_0^t \left(\frac{\beta}{e^{X^{(2)}}} + \log\delta + \int_{\mathbb{R}\setminus\{0\}} \log(1+\frac{\lambda}{\delta}y^2)\cdot(1_{\left\{|e^{X^{(2)}/2}y|<1\right\}}\cdot 1_{\{|\log(1+(\frac{\lambda}{\delta})\ y^2)|<1\}})\ N(dy)\right) ds \\
C_t &= \int_0^t \begin{pmatrix} e^{X^{(2)}}Q & 0 \\ 0 & 0 \end{pmatrix} ds \\
\nu(\cdot; ds, dy) &= \tilde{N}(X_s(\cdot), dy)\ ds
\end{aligned}$$

with the $\tilde{N}$ from above.

Remark: A different approach to calculate the characteristics of the COGARCH process is described in [12]. Furthermore our results are related to earlier work of B. Rajput and J. Rosinski. In their interesting article [15] they derive under certain restrictions a representation of the characteristic function of processes of the form $X_t = \int_0^t f(t,s)\, dZ_s$ where f is a deterministic function and Z is a Lévy process.

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AMS Subject Classification: 60J75 (primary), 60G51, 60J25 (secondary)

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Lavoro pervenuto in redazione il 22.05.2013

Rend. Sem. Mat. Univ. Politec. Torino
Vol. 71, 2 (2013), 261 – 272

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MALLIAVIN CALCULUS FOR LÉVY PROCESSES: A SURVEY

Abstract. Since Itô (1956) it is known that Lévy processes enjoy the chaotic representation property in a certain generalized form. In other words, the space of square integrable functionals of a certain independent random measure associated to a Lévy process has Fock space structure. The Fock space structure gives the possibility to develop a formal calculus where a gradient and a divergence operators, that are dual between them, are the main tools. On every space of random functionals with Fock space structure we can interpret probabilistically these operators and develop an stochastic calculus of Malliavin - Skorohod type. In this survey I present, first of all, a probabilistic interpretation of these operators in the case of functionals of a Lévy process. This interpretation generalizes the well-known interpretation for the standard Poisson process presented in Nualart and Vives (1990 and 1995) and, of course, the genuine Malliavin - Skorohod calculus for the Wiener process. As an application I obtain an anticipating Itô formula that extends both the usual adapted formula for Lévy processes and the anticipative version of the Itô formula on the Wiener space.

1. Introduction

This paper is a survey of Malliavin Calculus for Lévy processes since the point of view developed mainly in Solé, Utzet and Vives [15], that is strongly based on Itô [7], where the fact that square integrable functionals adapted to the filtration of a certain independent random measure associated to a Lévy process enjoy the chaotic representation property is proved. Of course, being Wiener process a particular example of Lévy process, Malliavin calculus for Lévy processes is an extension of Malliavin calculus for the Wiener process. Good references of Malliavin calculus for the Wiener process and for Gaussian processes in general are Sanz-Solé [13] and Nualart [8].

The fact that a process enjoys the chaotic representation property can be described also saying that the space of square integrable functionals has Fock space structure. This structure gives the possibility to develop a formal calculus where a gradient and a divergence operators (dual between them) are the main tools. On every space of random functionals with Fock space structure we can interpret probabilistically these operators and develop an stochastic calculus of Malliavin - Skorohod type. See Nualart-Vives [9] and Applebaum [5] for details.

In this paper, the probabilistic interpretation of these operators in the case of functionals of a Lévy process is presented following Solé, Utzet and Vives [15]. Previously, a canonical space for Lévy processes is constructed following the ideas developed by Neveu [11] for the standard Poisson case. This interpretation of the operators generalizes the interpretation given by Nualart and Vives in [9] and [10] for the standard Poisson case.

As an application I present an anticipating Itô formula, based on Alòs, León and Vives [1], that extends both the usual adapted formula for Lévy processes (see for

*This work has been financed by grants MEC FEDER MTM 2009, 08869 and 07203.

example Cont and Tankov [6]) and the anticipative version of the Itô formula on the Wiener space developed in Alòs and Nualart [3]. Another recent application that can be found in Alòs, León, Pontier and Vives [2], is a Hull and White formula (pricing formula) for plain vanilla options based on an stochastic volatility jump diffusion price model. We have no space here to present this nice financial application.

Section 2 is devoted to Fock space structure. In section 3 we give the construction of the canonical space for a Lévy process. In section 4 we present the probabilistic interpretation of the operators. Finally, Section 5 is devoted to the anticipative Itô formula.

2. Formal calculus based on the Fock space structure

Let H be a real separable Hilbert space. For any $n \geq 0$ we consider the tensor products $H^{\otimes n}$. Recall that $H^{\otimes 0} = \mathbb{R}$ and $H^{\otimes 1} = H$. We define the Hilbert subspaces $H^{\odot n} \subseteq H^{\otimes n}$ given by the symmetric elements with the scalar product

$$\langle f_n, g_n \rangle_{\odot n} := n! \langle f_n, g_n \rangle_{\otimes n}.$$

The Fock space associated to H is defined by the Hilbert space

$$\Phi(H) := \bigoplus_{n=0}^{\infty} H^{\odot n}$$

with the scalar product $\langle f, g \rangle = \sum_{n=0}^{\infty} \langle f_n, g_n \rangle_{H^{\odot n}}$, where $f = \sum_{n=0}^{\infty} f_n$ and $g = \sum_{n=0}^{\infty} g_n$.

If $(S, \mathcal{B}(S), \mu_S)$ is a certain measure space we can consider $H = L^2(S)$. In this case we have $H^{\odot n} = L_s^2(S^n)$, that is the space of $n-$dimensional and symmetric square integrable functions, with the modified scalar product. So, if $F \in \Phi(H)$, we have $F = \sum_{n=0}^{\infty} f_n$ with $f_n \in L_s^2(S^n)$.

We define the gradient or annihilation operator D as an application that maps an element $F \in \Phi(H)$ to an element $DF \in \Phi(H) \times H \cong L^2(S, \Phi(H))$ such that

$$D_t F = \sum_{n=1}^{\infty} n f_n(\cdot, t), t - a.e.,$$

of course provided that $DF \in L^2(S, \Phi(H))$, that is equivalent to

$$\sum_{n=1}^{\infty} n n! ||f_n||_{L^2(S^n)}^2 < \infty.$$

It is easy to see that this operator is densely defined and closed. Its domain is denoted by Dom$D$.

Let $u \in L^2(S, \Phi(H))$. Of course we have $u_t = \sum_{n=0}^{\infty} u_n(t, \cdot)$, $\mu_S - a.e.$ where $u_n \in L^2(S^{n+1})$ is symmetric with respect to the n last variables. Denote by $\widetilde{u}_n$ be the symmetrization in all $n+1$ variables. Then we define the divergence or creation operator of u by

$$\delta(u) = \sum_{n=0}^{\infty} \widetilde{u}_n,$$

provided this series is in $\Phi(H)$, that is equivalent to assume

$$\sum_{n=0}^{\infty} (n+1)! \, \|\widetilde{u}_n\|^2_{L^2(S^{n+1})} < \infty.$$

We denote by Domδ its domain. This operator is also densely defined and closed.

Operators D and δ are dual. Concretely we have that if $F \in DomD$ and $u \in Dom\delta$ then

$$\langle u, DF \rangle_{L^2(S,\Phi(H))} = \langle F, \delta(u) \rangle_{\Phi(H)}.$$

This is the basis of a calculus on the Fock space, that we can name Malliavin-Skorohod calculus without probability, and that can be largely developed, obtaining abstract formulas such as a Clark-Ocone type one (see Nualart and Vives [9]).

3. Lévy processes

In all the paper X will be a Lévy process with triplet (γ, σ^2, ν) where $\gamma \in \mathbb{R}$, $\sigma^2 > 0$ and ν is a Lévy measure. Good references for Lévy processes are Sato [14] and Cont and Tankov [6]. Recall that Lévy processes can be usefully represented by the so called Lévy-Itô representation $X_t = \gamma t + \sigma W_t + J_t$, where W is the standard Wiener process and J is a *pure jump* Lévy process, independent of W, such that

$$J_t := \int_0^t \int_{\{|x|>1\}} x \, dN(s,x) + \lim_{\varepsilon \downarrow 0} \int_0^t \int_{\{\varepsilon<|x|\leq 1\}} x \, d\widetilde{N}(s,x),$$

where $N(B) = \#\{t : (t, \Delta X_t) \in B\}$, for $B \in \mathcal{B}((0,\infty) \times \mathbb{R}_0)$, is the jump measure of the process, $d\widetilde{N}(t,x) := dN(t,x) - dt \, d\nu(x)$ is the compensated jump measure and the limit is $a.s.$ uniform in t on every bounded interval. Recall also that for every $t \geq 0$, $\mathcal{F}_t^X = \mathcal{F}_t^W \vee \mathcal{F}_t^J$.

From Itô [7], a Lévy process X can be associated to a centered and independent random measure M on $\mathbb{R}_+ \times \mathbb{R}$. We consider the continuous measure $\mu(dt,dx) = \eta(dx)dt$, where $\eta(dx) := \sigma^2 \delta_0(dx) + x^2 \nu(dx)$. More explicitly, we have, for any $E \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R})$,

$$\mu(E) = \sigma^2 \int_{E(0)} dt + \iint_{E'} x^2 d\nu(x) dt,$$

where $E(0) = \{t \in \mathbb{R}_+ : (t,0) \in E\}$ and $E' = E - \{(t,0) \in E\}$. Then, for $E \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R})$ with $\mu(E) < \infty$, we define the measure

$$M(dt,dx) = \sigma W(dt) \delta_0(dx) + x \widetilde{N}(dt,dx),$$

that is,

$$M(E) = \sigma \int_{E(0)} dW_t + \iint_{E'} x \, d\widetilde{N}(t,x),$$

and it is a centered independent random measure such that $E\left[M(E_1)M(E_2)\right] = \mu(E_1 \cap E_2)$, for $E_1, E_2 \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R})$ with $\mu(E_1) < \infty$ and $\mu(E_2) < \infty$.

Let $S := [0,\infty) \times \mathbb{R}$ endowed with the Borel $\sigma-$algebra and the measure μ defined above. Then we can consider

$$H^{\otimes n} = L_n^2 := L^2\Big((\mathbb{R}_+ \times \mathbb{R})^n, \mathcal{B}(\mathbb{R}_+ \times \mathbb{R})^n, \mu^{\otimes n}\Big).$$

For $f_n \in L_n^2$, following Itô [7], we can define a multiple stochastic integral $I_n(f_n)$ with respect M, through the same steps as in the Wiener case, and prove that $L^2(\Omega, \mathcal{F}^X)$ has Fock space structure, that is,

$$L^2(\Omega, \mathcal{F}^X) = \bigoplus_{n=0}^{\infty} I_n(L_n^2).$$

Then, we can represent any functional $F \in L^2(\Omega, \mathcal{F}^X)$ via the expansion

$$F = \sum_{n=0}^{\infty} I_n(f_n), \quad f_n \in L_n^2.$$

This expansion is unique if we take every f_n symmetric.

This fact makes possible to apply the machinery of annihilation and creation operators in a Fock space as presented before.

If $F \in L^2(\Omega)$, with chaotic representation $F = \sum_{n=0}^{\infty} I_n(f_n)$, ($f_n$ symmetric) and such that $\sum_{n=1}^{\infty} n \, n! \| f_n\|^2_{L_n^2} < \infty$, we define its gradient as

$$D_z F = \sum_{n=1}^{\infty} n I_{n-1}\Big(f_n\big(z,\cdot\big)\Big), \;\; z \in \mathbb{R}_+ \times \mathbb{R},$$

Recall that $D_z F$ is an element of $L^2(\mathbb{R}_+ \times \mathbb{R} \times \Omega, \mu \otimes \mathbb{P})$.

In particular we can consider the two particular cases

$$D_{t,0} F = \sum_{n=1}^{\infty} n I_{n-1}\Big(f_n\big((t,0),\cdot\big)\Big), \quad t \in \mathbb{R}_+,$$

in $L^2(\mathbb{R}_+ \times \Omega, dt \otimes \mathbb{P})$ and

$$D_{t,x} F = \sum_{n=1}^{\infty} n I_{n-1}\Big(f_n\big((t,x),\cdot\big)\Big), \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}_0,$$

in $L^2(\mathbb{R}_+ \times \mathbb{R}_0 \times \Omega, dt \, x^2 d\nu(x) \otimes \mathbb{P})$.

If we define its domains analogously to previous cases and denote them by $\mathrm{Dom}D^0$ and $\mathrm{Dom}D^J$ respectively, we have that if $\sigma > 0$ and $\nu \neq 0$, $\mathrm{Dom}D = \mathrm{Dom}D^0 \cap \mathrm{Dom}D^J$.

On other hand, let $u \in L^2\big(\mathbb{R}_+ \times \mathbb{R} \times \Omega, \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}) \otimes \mathcal{F}^X, \mu \otimes \mathbb{P}\big)$. As before, we have the chaotic decomposition

$$u(t,x) = \sum_{n=0}^{\infty} I_n(u_n((t,x),\cdot))$$

where $u_n \in L^2_{n+1}$ is symmetric in the n last variables. Then, if $\widetilde{u}_n$ denotes the symmetrization in all $n+1$ variables we have

$$\delta(u) = \sum_{n=0}^{\infty} I_{n+1}(\widetilde{u}_n),$$

in $L^2(\Omega)$, provided $u \in \mathrm{Dom}\,\delta$, that means $\sum_{n=0}^{\infty}(n+1)!\,\|\widetilde{u}_n\|^2_{L^2_{n+1}} < \infty$.

The duality property, in this case can be written in the following way: If $u \in \mathrm{Dom}\,\delta$ and $F \in \mathrm{Dom}\,D$ we have

$$E[\delta(u)F] = E\iint_{\mathbb{R}_+ \times \mathbb{R}} u(t,x)\, D_{t,x}F\, \mu(dt,dx).$$

4. Probabilistic interpretation of gradient and divergence operators

4.1. A canonical space for Lévy processes

The usual canonical Lévy process is built on the space of measurable functions from $\mathbb{R}_+$ to $\mathbb{R}$ or on the space of *càdlàg* functions, in both cases with the σ-field generated by the cylinders and using the Kolmogorov extension theorem. In order to have a probabilistic interpretation of the operator D, in Solé, Utzet and Vives [15] a different canonical Lévy process is constructed. This construction is an extension of the canonical Poisson process defined by Neveu [11] and is done in several steps. First of all we construct a canonical space for a compound Poisson process in a finite time interval, then we extend it to $\mathbb{R}$ and after this, we construct the canonical space for a pure jump Lévy case. In fact, in this last case, the probability space is the set of all finite or infinite sequences of pairs (t_i,x_i) such that for every $T > 0$, there is only a finite number of $t_i \leq T$, including the empty sequence. Finally, for a general Lévy process we consider the canonical Wiener space $(\Omega_W, \mathcal{F}_W, \mathbb{P}_W, \{\overline{W}_t,\, t \geq 0\})$ and the canonical pure jump Lévy space $(\Omega_J, \mathcal{F}_J, \mathbb{P}_J, \{\overline{J}_t, t \in \mathbb{R}_+\})$. Then we define

$$(\Omega_W \times \Omega_J, \mathcal{F}_W \otimes \mathcal{F}_J, \mathbb{P}_W \otimes \mathbb{P}_J)$$

with $W_t(\omega,\omega') := \overline{W}_t(\omega)$ and $J_t(\omega,\omega') := \overline{J}_t(\omega')$. The process $X_t = \gamma t + \sigma W_t + J_t$ is the canonical Lévy process.

4.2. Probabilistic interpretation of the operator $D_{t,0}$

We are going to see that $D_{t,0}$ turns to be the derivative with respect to the Wiener part of X and that the usual rules of classical Malliavin Calculus apply

Recall that we have the isometry $L^2(\Omega_W \times \Omega_J) \simeq L^2(\Omega_W; L^2(\Omega_J))$ and then we can apply the theory of Malliavin calculus for Hilbert space valued random variables as it is developed for example in Nualart [8].

Let be D^W the classical Malliavin derivative and denote by $\mathrm{Dom}\, D^W$ its domain. Given a real separable Hilbert space $\mathcal{H}$, we can extend this notion to $\mathcal{H}$-valued random variables. We write D^{W^*} to denote the extended notion and $\mathrm{Dom}\, D^{W^*}$ to denote its domain. In this case we have $\mathrm{Dom}\, D^{W^*} \simeq \mathrm{Dom}\, D^W \otimes \mathcal{H}$. In the particular case of $\mathcal{H} = L^2(\Omega')$, for a certain probability space $(\Omega', \mathcal{F}', \mathbb{P}')$, such that $L^2(\Omega')$ is separable, we have,

$$\mathrm{Dom}\, D^{W^*} \simeq \mathrm{Dom}\, D^W \otimes L^2(\Omega') \simeq L^2(\Omega'; \mathrm{Dom}\, D^W).$$

As a consequence, if $F \in L^2(\Omega \times \Omega')$ such that for all $\omega' \in \Omega'$, $\mathbb{P}'$-a.s., $F(\cdot, \omega') \in \mathrm{Dom}\, D^W$, then $F \in \mathrm{Dom}\, D^{W^*}$ and

$$D_t^{W^*} F(\omega, \omega') = D_t^W F(\cdot, \omega')(\omega), \ell \otimes \mathbb{P} \otimes \mathbb{P}' - \text{a.e.}$$

In our particular case we have $L^2(\Omega') = L^2(\Omega_J)$, which is a separable Hilbert space, and so $L^2(\Omega_W \times \Omega_J) \simeq L^2(\Omega_W; L^2(\Omega_J))$. Therefore we can compute both $D_{t,0}F$ and $D_t^{W^*}F$, and to obtain $\mathrm{Dom}\, D^{W^*} \subset \mathrm{Dom}\, D^0$, and for $F \in \mathrm{Dom}\, D^{W^*}$, we have $D_{t,0}F = \frac{1}{\sigma} D_t^{W^*} F$. This gives the probabilistic interpretation of $D_{t,0}$.

The most general chain rule is proved in Petrou [12]: If $F = f(Z)$ with $Z \in \mathrm{Dom}\, D^{W*}$ and f in $C_b^1(\mathbb{R})$, then $F \in \mathrm{Dom}\, D^{W*}$ and $D_t^{W*} F = f'(Z) D_t^{W*} Z$.

4.3. Probabilistic interpretation of $D_{t,x}$ for $x \neq 0$.

Consider now a pure jump Lévy process J with Lévy measure ν. Given $\omega \in \Omega^J$ and $z = (t,x) \in \mathbb{R}_+ \times \mathbb{R}_0$, we introduce in ω a jump of size x at instant t, and call the new element $\omega_z = \big((t_1,x_1),(t_2,x_2),\dots,(t,x),\dots\big)$.

For a $\mathcal{F}^J$-random variable F, we define the transformation $(T_z F)(\omega) := F(\omega_z)$, and the application $TF \colon \mathbb{R}_+ \times \mathbb{R}_0 \times \Omega \longrightarrow \mathbb{R}$, that applies (z,ω) to $F(\omega_z)$ is $\mathcal{B}(\mathbb{R}_+ \times \mathbb{R}_0) \otimes \mathcal{F}^J$ measurable and if $F = 0$, $P-$almost surely, then $TF = 0$, $\ell \otimes \nu \otimes P$ a.e.

Now we can define the increment quotient operator

$$\Psi_{t,x} F(\omega) := \frac{(T_{t,x}F)(\omega) - F(\omega)}{x}.$$

Thanks to the results given above, $\Psi_{t,x}$ is a measurable operator from $L^0(\Omega^J)$ to $L^0(\mathbb{R}_+ \times \mathbb{R}_0 \times \Omega^J)$. It is linear, closed and if $F, G \in L^0(\Omega^J)$,

$$\Psi_{t,x}(F\,G) = G\,\Psi_{t,x}F + F\,\Psi_{t,x}G + x\,\Psi_{t,x}(F)\,\Psi_{t,x}(G).$$

Using the same ideas as in Nualart and Vives [10], given $F \in L^2(\Omega^J)$, we have

$$F \in \operatorname{Dom} D^J \Longleftrightarrow \Psi F \in L^2(\mathbb{R}_+ \times \mathbb{R}_0 \times \Omega^J),$$

and in this case $D_{t,x}F = \Psi_{t,x}F,\ \mu \otimes P-$a.e. This gives the probabilistic interpretation of $D_{t,x}$ for $x \neq 0$.

In the general case, given $z = (t,x) \in \mathbb{R}_+ \times \mathbb{R}_0$, for $\omega = (\omega^W, \omega^J) \in \Omega_W \times \Omega_J$ define $\omega_z = (\omega^W, \omega_z^J)$, and for a random variable $F \in L^0(\Omega_W \times \Omega_J)$ let $(T_z^* F)(\omega) := F(\omega_z)$. Define also the operator

$$\Psi^*_{t,x} F := \frac{F(\omega_{t,x}) - F(\omega)}{x}.$$

Then, for $F \in L^2(\Omega)$ we have that $F \in \operatorname{Dom} D$ if and only if $F \in \operatorname{Dom} D^{W*}$ and $\Psi^* F \in L^2(\Omega \times [0,\infty) \times \mathbb{R}_0)$, and in this case,

$$D_{t,x}F = \mathbb{1}_{\{\sigma>0\}} \mathbb{1}_{\{0\}}(x) \frac{1}{\sigma} D_t^{W*} F + \mathbb{1}_{\mathbb{R}_0}(x) \Psi^*_{t,x} F.$$

4.4. Probabilistic interpretation of δ

From now on, fix a finite time $T > 0$ and consider the process $\{X_t, t \in [0,T]\}$. Consider the independent random measure M restricted to $[0,T] \times \mathbb{R}$. Assume also $\int_{\mathbb{R}} x^2 d\nu(x) < \infty$.

Following Applebaum [4], the random measure M, with the filtration $\left\{\mathcal{F}_t^X, t \in [0,T]\right\}$, induces a martingale-valued measure and allows to define an stochastic integral.

Let u be a predictable process such that $E \iint_{[0,T]\times\mathbb{R}} u^2(z)\, \mu(dz) < \infty$. We can define a stochastic integral $\iint_{[0,T]\times\mathbb{R}} u(z)\, dM_z$ such that for u and v square integrable predictable processes we have

$$E\Big[\iint_{[0,T]\times\mathbb{R}} u\, dM \cdot \iint_{[0,T]\times\mathbb{R}} v\, dM\Big] = E\Big[\iint_{[0,T]\times\mathbb{R}} u v\, d\mu\Big].$$

An explicit expression for the integral $\iint_{[0,T]\times\mathbb{R}} u(z)\, dM_z$ is given by

$$\iint_{[0,T]\times\mathbb{R}} u(z)\, dM_z = \sigma \int_0^T u(t,0)\, dW_t + \iint_{[0,T]\times\mathbb{R}_0} x u(t,x)\, d\widetilde{N}(t,x).$$

As in the Wiener case, the Skorohod integral restricted to predictable processes coincides with the integral with respect to the random measure M.

In fact, if δ^0 is the dual operator of $D_{t,0}$ and δ^J is the dual operator of $D_{t,x}$ for $x \neq 0$, we have

$$\delta(u) = \delta^0(u_{\cdot,0}) + \delta^J(u \mathbb{1}_{\mathbb{R}_0}(x)).$$

In particular δ^0 coincides with $\sigma\delta^W$ and δ^J coincides with the path by path integral with respect to $x\widetilde{N}$ over predictable processes.

Next result will play a key role in the application:

LEMMA 1. *Let $F \in \mathrm{Dom}\, D$ be a bounded random variable and $u \in \mathrm{Dom}\,\delta$ such that*

$$E\int_{[0,T]\times\mathbb{R}} (u(t,x)(F + xD_{t,x}F))^2 \mu(dt,dx) < \infty.$$

Then $u(t,x)(F + xD_{t,x}F) \in \mathrm{Dom}\,\delta$ if and only if

$$F\delta(u) - \int_{[0,T]\times\mathbb{R}} u(t,x)D_{t,x}F\mu(dt,dx) \in L^2(\Omega)$$

and in this case $\delta(Fu) = F\delta(u) - \delta(xuDF) - \int_{[0,T]\times\mathbb{R}} u(t,x)D_{t,x}F\mu(dt,dx)$.

4.5. The space $\mathbb{L}^F$

To go further we need some structure into the space Domδ. We follow Alòs and Nualart [3]. It is known that $L^2_a([0,T]\times\mathbb{R}\times\Omega)$, the space of square integrable and adapted processes, is included in Domδ. So, we search for a Hilbert space included in the domain of δ but that inludes adapted and square integrable processes.

We define $\mathbb{L}^{1,2,f}$ as the space of processes $u \in L^2([0,T]\times\mathbb{R}\times\Omega)$ such that $D_{s,x}u_{t,y}$ exists a.e. for $s \geq t$. and belongs to $L^2(([0,T]\times\mathbb{R})^2\times\Omega)$. Observe that $\mathbb{L}^{1,2,f}$ is a Hilbert space with the norm

$$\|u\|^2_{\mathbb{L}^{1,2,f}} := \|u\|^2_{L^2([0,T]\times\mathbb{R}\times\Omega)} + \|D_{s,x}u_{t,y}1\!\!1_{\{s\geq t\}}\|^2_{L^2(([0,T]\times\mathbb{R})^2\times\Omega)}$$

and $L^2_a([0,T]\times\mathbb{R}\times\Omega) \subseteq \mathbb{L}^{1,2,f} \subseteq L^2([0,T]\times\mathbb{R}\times\Omega)$.

Then we consider the space $\mathbb{L}^F$ that it is defined in the following way: $u \in \mathbb{L}^F$ if and only if $u \in L^{1,2,f}$ and $D_{r,w}D_{s,x}u_{t,y}$ exists a.e. for $r \vee s \geq t$ and belongs to $L^2([0,T]^3\times\mathbb{R}^3\times\Omega)$. $\mathbb{L}^F$ is a Hilbert space with the norm

$$\|u\|^2_{\mathbb{L}^F} := \|u\|^2_{\mathbb{L}^{1,2,f}} + \|D_{r,w}D_{s,x}u_{t,y}1\!\!1_{\{r\vee s\geq t\}}\|^2_{L^2(([0,T]\times\mathbb{R})^3\times\Omega)}.$$

and $L^2_a([0,T]\times\mathbb{R}\times\Omega) \subseteq \mathbb{L}^F \subseteq \mathbb{L}^{1,2,f} \cap \mathrm{Dom}\delta \subseteq L^2([0,T]\times\mathbb{R}\times\Omega)$. Moreover,

$$\mathbb{E}(\delta(u)^2) \leq 2\|u\|^2_{\mathbb{L}^F}.$$

Observe that this inequality allow to control convergence of $\delta(u)$ by convergence with respect the norm of $\mathbb{L}^F$ and apply, when necessary, dominated convergence theorem.

5. An anticipating Itô formula

In Alòs, León and Vives [1] we use the techniques presented before to obtain an anticipative version of the Itô formula for Lévy processes, where the coefficients are assumed to be in $\mathbb{L}^F$. Our Itô formula is not only an extension of the usual adapted formula for Lévy processes, but also an extension of the anticipative version of the Itô formula on the Wiener space, obtained by Alòs and Nualart (2008).

Consider the semimartingale

$$\begin{aligned} X_t &= X_0 + \int_0^t u_s dW_s + \int_0^t v_s ds \\ &+ \int_0^t \int_{|y|>1} z_1(s-,y) y N(ds,dy) + \int_0^t \int_{|y|\leq 1} z_2(s-,y) y \tilde{N}(ds,dy) \end{aligned}$$

where u and $z_2(s-,y)y$ are adapted and have L^2 trajectories a.s. and v is adapted and has L^1 trajectories a.s. This is in fact a generalization of a generic Lévy process.

In this case (see Cont and Tankov [6] for example) it is well known that

$$\begin{aligned} F(X_t) &= F(X_0) + \int_0^t F'(X_{s-}) u_s dW_s \\ &+ \int_0^t F'(X_{s-}) v_s ds + \frac{1}{2} \int_0^t F''(X_{s-}) u_s^2 ds \\ &+ \int_0^t \int_{|y|>1} [F(X_s) - F(X_{s-})] N(ds,dy) \\ &+ \int_0^t \int_{|y|\leq 1} [F(X_s) - F(X_{s-}) - F'(X_{s-}) z_2(s-,y) y] N(ds,dy) \\ &+ \int_0^t \int_{|y|\leq 1} F'(X_{s-}) z_2(s-,y) y \tilde{N}(ds,dy). \end{aligned}$$

Our purpose is to obtain an analogous formula changing Itô stochastic integrals by Skorohod versions, that is, an anticipating version of this formula. Recall that if u, v, z_1 and z_2 are anticipating processes, the Itô integral with respect to W is not defined, so we need the Skorohod extension. Moreover, the integrals with respect $\tilde{N}$ are well defined path by path, but they are not zero expectation integrals, so we are also interested in an Skorohod type version for this case. Coefficients will be assumed to be in the domain of the gradient operator in the future sense. So, this application includes also the Lévy extension of the corresponding domains in the Wiener case as presented in Alòs and Nualart [3].

We introduce the space $\mathbb{L}_-^{1,2,f}$. A random field $u = \{u(s,y) : (s,y) \in [0,T] \times \mathbb{R}\}$ in $\mathbb{L}^{1,2,f}$ belongs to the space $\mathbb{L}_-^{1,2,f}$ if there exists D^-u in $L^2(\Omega \times [0,T] \times \mathbb{R})$ such that

$$\int_0^T \int_{\mathbb{R}} \sup_{(s-\frac{1}{n})\vee 0 \leq r < s, y \leq x \leq y+\frac{1}{n}} E[|D_{s,y}u(r,x) - D^-u(s,y)|^2] \mu(ds,dy)$$

converges to zero as n goes to infinity.

We need also to precise the relationship between Skorohod and path by path integrals. Let $z = \{z(s,x) : (s,x) \in [0,T] \times \mathbb{R}\}$ be a measurable random field such that:

- If $s_n \uparrow s$ in $[0,T]$ and $y_m \to y$, $y \neq 0$, the limit $z(s-,y) = \lim_{n,m\to\infty} z(s_n, y_m)$ is well–defined and belongs to $\mathbb{L}_{-}^{1,2,f}$.
- The random fields $z(s-,y)$ and $yD^{-}z(s-,y)$ belongs to $\mathbb{L}^F$.
- The random field $z(s-,y)y$ is pathwise integrable with respect to $\tilde{N}$.

Then we have that for any interval $(a,b]$ or (a,∞) in $(0,\infty)$,

$$\int_0^t \int_{\{a<|y|\leq b\}} z(s-,y)y\tilde{N}(ds,dy)$$

$$= \delta((z(s-,y) + yD^{-}z(s-,y))1\!\!1_{\{a<|y|\leq b\}} 1\!\!1_{[0,t]}(s))$$

$$+ \int_0^t \int_{\{a<|y|\leq b\}} D^{-}z(s-,y)\mu(ds,dy), \quad t \in [0,T].$$

Finally, consider the process

$$\begin{aligned} X_t &= X_0 + \delta^W(u1\!\!1_{[0,t]}) + \int_0^t v_s ds + \int_0^t \int_{\{|x|>1\}} z_1(s-,x)xN(ds,dx) \\ &+ \int_0^t \int_{\{0<|x|\leq 1\}} z_2(s-,x)x\tilde{N}(ds,dx), \quad t \in [0,T]. \end{aligned}$$

with the hypotheses

- $X_0 \in \mathrm{DomD}$.
- $u \in \mathbb{L}^F$, $\delta^W(u1\!\!1_{[0,t]})$ has continuous paths and $\int_0^T u_s^2 ds$ is a.s. bounded by a constant.
- $v \in \mathbb{L}^{1,2,f}$ and $\int_0^T v_s^2 ds$ is a.s. bounded by a constant.
- z_1 and z_2 are bounded and satisfies the conditions of Theorem 1 on $(1,\infty)$ and $(0,1]$ respectively. Moreover, $D^{-}z_2 \in \mathbb{L}^{1,2,f}$

Then, if $F \in C^2(\mathbb{R})$, we have that

$$F'(X_{s-})(u_s 1\!\!1_{\{y=0\}} + z_2(s-,y)1\!\!1_{\{0<|y|\leq 1\}})1\!\!1_{[0,t]}(s)$$

and

$$D^-(z_2(s-,y)F'(X_{s-}))(s,y)y1\!\!1_{\{0<|y|\leq 1\}}1\!\!1_{[0,t]}(s)$$

belong to *Dom*δ and

$$\begin{aligned} & F(X_t)-F(X_0) \\ = & \delta((F'(X_{s-})(u_s 1\!\!1_{\{y=0\}}+z_2(s-,y)1\!\!1_{\{0<|y|\leq 1\}}) \\ + & y1\!\!1_{\{0<|y|\leq 1\}}D^-_{(s,y)}(z_2(s-,y)F'(X_{s-})))1\!\!1_{[0,t]}(s)) \\ + & \frac{1}{2}\int_0^t F''(X_s)u_s^2 ds+\int_0^t F'(X_s)v_s ds+\int_0^t F''(X_s)D^-_{(s,0)}X_s u_s ds \\ + & \int_0^t\int_{\{0<|y|\leq 1\}} D^-_{(s,y)}F'(X_{s-})z_2(s,y)\mu(ds,dy) \\ + & \int_0^t\int_{0<|y|\leq 1}[F(X_s)-F(X_{s-})-F'(X_{s-})z_2(s-,y)y]N(ds,dy) \\ + & \int_0^t\int_{\{|y|>1\}}(F(X_s)-F(X_{s-}))N(ds,dy). \end{aligned}$$

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AMS Subject Classification: 60H07, 60H30, 91B70

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Lavoro pervenuto in redazione il 22.05.2013

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dell'Università e del Politecnico di Torino

Volume 71, N. 2 (2013)

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ISSN 0373-1243

UNIVERSITÀ DEGLI STUDI DI TORINO
DIPARTIMENTO DI MATEMATICA