Standardization of a Call-By-Value Lambda-Calculus*

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Abstract

We study an extension of Plotkin's call-by-value lambda-calculus by means of two commutation rules (sigma-reductions). Recently, it has been proved that this extended calculus provides elegant characterizations of many semantic properties, as for example solvability. We prove a standardization theorem for this calculus by generalizing Takahashi's approach of parallel reductions. The standardization property allows us to prove that our calculus is conservative with respect to the Plotkin's one. In particular, we show that the notion of solvability for this calculus coincides with that for Plotkin's call-by-value lambda-calculus.

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1 Introduction

The λ_v-calculus (λ_v for short) has been introduced by Plotkin in [15], in order to give a formal account of the call-by-value evaluation, which is the most commonly used parameter passing policy for programming languages. λ_v shares the syntax with the classical, call-by-name, λ-calculus (λ for short), but its reduction rule, β_v, is a restriction of β, firing only in case the argument is a value (i.e., a variable or an abstraction). While β_v is enough for evaluation, it turned out to be too weak to study operational properties of terms. For example, in λ, the β-reduction is sufficient to characterize solvability and (using extensionality) separability, but, in order to characterize similar properties for λ_v, it has been necessary to introduce different notions of reduction unsuitable for a correct call-by-value evaluation (see [13, 14]): this is disappointing and requires complex reasoning. In this paper we study λ_v^σ, the extension of λ_v proposed in [3]. It keeps the λ_v (and λ) syntax and it adds to the β_v-reduction two commutation rules, called σ_1 and σ_3, which unblock β_v-redexes that are hidden by the "hyper-sequential structure" of terms. It is well-known (see [14, 1]) that in λ_v there are normal forms that are unsolvable, e.g. $(\lambda yx.xx)(zz)(\lambda x.xx)$. The more evident benefit of λ_v^σ

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is that the commutation rules make all normal forms solvable (indeed $(\lambda yx.xx)(zz)(\lambda x.xx)$ is not a λ_v^σ normal form). More generally, the so obtained language, allows us to characterize operational properties, like solvability and potential valuability, in an internal and elegant way (see [3]). In this paper we prove a standardization property in λ_v^σ, and some of its consequences, namely its soundness with respect to the semantics of λ_v.

Let us recall the notion of standardization, which has been first studied in the ordinary λ-calculus (see, for example [5, 8, 2]). A reduction sequence is standard if its redexes are ordered in a given way, and the corresponding standardization theorem establishes that every reduction sequence can constructively be transformed into a standard one. Standardization is a key tool to grasp the way in which reductions work, that sheds some light on redexes relationships and their dependencies. It is useful for characterization of semantic properties through reduction strategies (the proof of operational semantics adequacy is a typical use).

In the λ_v setting standardization theorems have been proved by Plotkin [15], Paolini and Ronchi Della Rocca [14, 12] and Crary [4]. The definition of standard sequence of reductions considered by Plotkin and Crary coincides, and it imposes a partial order on redexes, while Paolini and Ronchi Della Rocca define a total order on them. All these proofs are developed by using the notion of parallel reduction introduced by Tait and Martin-Löf (see Takahashi [17] for details and interesting technical improvements). We emphasize that this method does not involve the notion of residual of a redex, on which many classical proofs for the λ-calculus are based (see for example [8, 2]). As in [15, 17, 14, 4], we use a suitable notion of parallel reduction for developing our standardization theorem for λ_v^σ. In particular we consider two groups of redexes, head β_v-redexes and head σ-redexes (putting together σ_1 and σ_3), and we induce a total order on head redexes of the two groups, without imposing any order on head σ-redexes themselves. More precisely, when σ-redexes are missing, this notion of standardization coincides with that presented in [14]. Moreover, we show that it is not possible to strengthen our standardization by (locally) ordering σ_1-reduction to σ_3-reduction (or viceversa).

As usual, our standardization proof is based on a sequentialization result: inner reductions can always be postponed after the head ones, for a non-standard definition of head reduction. Sequentialization has interesting consequences: it allows us to prove that fundamental operational properties in λ_v^σ, like observational equivalence, potential valuability and solvability, are conservative with respect to the corresponding notions of λ_v. This fully justifies the project in [3] where λ_v^σ has been introduced as a tool for studying the operational behaviour of λ_v.

Other variants of λ_v have been introduced in the literature for modeling the call-by-value computation. We would like to cite here at least the contributions of Moggi [10], Felleisen and Sabry [16], Maraist et al. [9], Herbelin and Zimmerman [7], Accattoli and Paolini [1]. All these proposals are based on the introduction of new constructs to the syntax of λ_v, so the comparison between them is not easy with respect to syntactical properties (some detailed comparison is given in [1]). We point out that the calculi introduced in [10, 16, 9, 7] present some variants of our σ_1 and/or σ_3 rules, often in a setting with explicit substitutions.

Outline. In Section 2 we introduce the language λ_v^σ and its operational behaviour; in Section 3 the sequentialization property is proved; Section 4 contains the main result, i.e., standardization ; in Section 5 some conservativity results with respect to Plotkin's λ_v-calculus are proved. Section 6 concludes the paper, with some hints for future work.

2 The call-by-value lambda calculus with sigma-rules

In this section we present λ_v^σ, a call-by-value λ-calculus introduced in [3] that adds two σ-reduction rules to pure (i.e. without constants) call-by-value λ-calculus defined by Plotkin in [15].

The syntax of terms of λ_v^σ [3] is the same as the one of ordinary λ-calculus and Plotkin's call-by-value λ-calculus λ_v [15] (without constants). Given a countable set $\mathcal{V}$ of *variables* (denoted by $x, y, z, \dots$), the sets Λ of *terms* and Λ_v of *values* are defined by mutual induction:

$$(\Lambda_v) \qquad V, U ::= x \mid \lambda x.M \qquad \textit{values}$$
$$(\Lambda) \qquad M, N, L ::= V \mid MN \qquad \textit{terms}$$

Clearly, $\Lambda_v \subseteq \Lambda$. All terms are considered up to α-conversion. The set of free variables of a term M is denoted by $\mathsf{fv}(M)$. Given $V_1, \dots, V_n \in \Lambda_v$ and pairwise distinct variables $x_1, \dots, x_n$, $M\{V_1/x_1, \dots, V_n/x_n\}$ denotes the term obtained by the *capture-avoiding simultaneous substitution* of V_i for each free occurrence of x_i in the term M (for all $1 \leq i \leq n$). Note that, for all $V, V_1, \dots, V_n \in \Lambda_v$ and pairwise distinct variables $x_1, \dots, x_n$, $V\{V_1/x_1, \dots, V_n/x_n\} \in \Lambda_v$.

Contexts (with exactly one hole $(\!|\cdot|\!)$), denoted by $\mathtt{C}$, are defined as usual via the grammar:

$$\mathtt{C} ::= (\!|\cdot|\!) \mid \lambda x.\mathtt{C} \mid \mathtt{C}M \mid M\mathtt{C}\,.$$

We use $\mathtt{C}(\!|M|\!)$ for the term obtained by the capture-allowing substitution of the term M for the hole $(\!|\cdot|\!)$ in the context $\mathtt{C}$.

▶ **Notation.** From now on, we set $I = \lambda x.x$ and $\Delta = \lambda x.xx$.

The reduction rules of λ_v^σ consist of Plotkin's β_v-reduction rule, introduced in [15], and two simple commutation rules called σ_1 and σ_3, studied in [3].

▶ **Definition 1** (Reduction rules)**.** We define the following binary relations on Λ (for any $M, N, L \in \Lambda$ and any $V \in \Lambda_v$):

$$(\lambda x.M)V \mapsto_{\beta_v} M\{V/x\}$$
$$(\lambda x.M)NL \mapsto_{\sigma_1} (\lambda x.ML)N \quad \text{with } x \notin \mathsf{fv}(L)$$
$$V((\lambda x.L)N) \mapsto_{\sigma_3} (\lambda x.VL)N \quad \text{with } x \notin \mathsf{fv}(V).$$

For any $\mathsf{r} \in \{\beta_v, \sigma_1, \sigma_3\}$, if $M \mapsto_{\mathsf{r}} M'$ then M is a r-*redex* and M' is its r-*contractum*. In this sense, a term of the shape $(\lambda x.M)N$ (for any $M, N \in \Lambda$) is a β-*redex*.

We set $\mapsto_\sigma = \mapsto_{\sigma_1} \cup \mapsto_{\sigma_3}$ and $\mapsto_{\mathsf{v}} = \mapsto_{\beta_v} \cup \mapsto_\sigma$.

The side conditions on $\mapsto_{\sigma_1}$ and $\mapsto_{\sigma_3}$ in Definition 1 can be always fulfilled by α-renaming.

Obviously, any β_v-redex is a β-redex but the converse does not hold: $(\lambda x.z)(yI)$ is a β-redex but not a β_v-redex. Redexes of different kind may overlap: for example, the term $\Delta I \Delta$ is a σ_1-redex and it contains the β_v-redex ΔI; the term $\Delta(I\Delta)(xI)$ is a σ_1-redex and it contains the σ_3-redex $\Delta(I\Delta)$, which contains in turn the β_v-redex $I\Delta$.

According to the Girard's call-by-value "boring" translation $(\cdot)^v$ of terms into Intuitionistic Multiplicative Exponential Linear Logic proof-nets, defined by $(A \Rightarrow B)^v = !A^v \multimap !B^v$ (see [6]), the images under $(\cdot)^v$ of a σ_1-redex (resp. σ_3-redex) and its contractum are equal modulo some "bureaucratic" steps of cut-elimination.

▶ **Notation.** Let R be a binary relation on Λ. We denote by R^* (resp. R^+; $\mathsf{R}^=$) the reflexive-transitive (resp. transitive; reflexive) closure of R.

▶ **Definition 2** (Reductions). Let $\mathsf{r} \in \{\beta_v, \sigma_1, \sigma_3, \sigma, \mathsf{v}\}$.

The r-*reduction* $\to_{\mathsf{r}}$ is the contextual closure of $\mapsto_{\mathsf{r}}$, i.e. $M \to_{\mathsf{r}} M'$ iff there is a context $\mathtt{C}$ and $N, N' \in \Lambda$ such that $M = \mathtt{C}(\!|N|\!)$, $M' = \mathtt{C}(\!|N'|\!)$ and $N \mapsto_{\mathsf{r}} N'$.

The r-*equivalence* $=_{\mathsf{r}}$ is the reflexive-transitive and symmetric closure of $\to_{\mathsf{r}}$.

Let M be a term: M is r-*normal* if there is no term N such that $M \to_{\mathsf{r}} N$; M is r-*normalizable* if there is a r-normal term N such that $M \to_{\mathsf{r}}^* N$; M is *strongly* r-*normalizing* if there is no sequence $(N_i)_{i\in\mathbb{N}}$ such that $M = N_0$ and $N_i \to_{\mathsf{r}} N_{i+1}$ for any $i \in \mathbb{N}$. Finally, $\to_{\mathsf{r}}$ is *strongly normalizing* if every $N \in \Lambda$ is strongly r-normalizing.

Patently, $\to_\sigma \subsetneq \to_{\mathsf{v}}$ and $\to_{\beta_v} \subsetneq \to_{\mathsf{v}}$.

▶ **Remark 3.** For any $\mathsf{r} \in \{\beta_v, \sigma_1, \sigma_3, \sigma, \mathsf{v}\}$ (resp. $\mathsf{r} \in \{\sigma_1, \sigma_3, \sigma\}$), values are closed under r-reduction (resp. r-expansion): for any $V \in \Lambda_v$, if $V \to_{\mathsf{r}} M$ (resp. $M \to_{\mathsf{r}} V$) then $M \in \Lambda_v$; more precisely, $V = \lambda x.N$ and $M = \lambda x.N'$ for some $N, N' \in \Lambda$ with $N \to_{\mathsf{r}} N'$ (resp. $N' \to_{\mathsf{r}} N$).

▶ **Proposition 4** (See [3]). *The σ-reduction is confluent and strongly normalizing. The* v*-reduction is confluent.*

The λ_v^σ-*calculus*, λ_v^σ for short, is the set Λ of terms endowed with the v-reduction $\to_{\mathsf{v}}$. The set Λ endowed with the β_v-reduction $\to_{\beta_v}$ is the λ_v-calculus (λ_v for short), i.e. the Plotkin's call-by-value λ-calculus [15] (without constants), which is thus a sub-calculus of λ_v^σ.

▶ **Example 5.** $M = (\lambda y.\Delta)(xI)\Delta \to_{\sigma_1} (\lambda y.\Delta\Delta)(xI) \to_{\beta_v} (\lambda y.\Delta\Delta)(xI) \to_{\beta_v} \dots$ and $N = \Delta((\lambda y.\Delta)(xI)) \to_{\sigma_3} (\lambda y.\Delta\Delta)(xI) \to_{\beta_v} (\lambda y.\Delta\Delta)(xI) \to_{\beta_v} \dots$ are the only possible v-reduction paths from M and N respectively: M and N are not v-normalizable, and $M =_{\mathsf{v}} N$. Meanwhile, M and N are β_v-normal and different, hence $M \neq_{\beta_v} N$ (by confluence of $\to_{\beta_v}$, see [15]).

Informally, σ-rules unblock β_v-redexes which are hidden by the "hyper-sequential structure" of terms. This approach is alternative to the one in [1] where hidden β_v-redexes are reduced using rules acting at a distance (through explicit substitutions). It can be shown that the call-by-value λ-calculus with explicit substitution introduced in [1] can be embedded in λ_v^σ.

3 Sequentialization

In this section we aim to prove a sequentialization theorem (Theorem 22) for the λ_v^σ-calculus by adapting Takahashi's method [17, 4] based on parallel reductions.

▶ **Notation.** From now on, we always assume that $V, V' \in \Lambda_v$.

Note that the generic form of a term is $VM_1 \dots M_m$ for some $m \in \mathbb{N}$ (in particular, values are obtained when $m = 0$). The sequentialization result is based on a partitioning of v-reduction between head and internal reduction.

▶ **Definition 6** (Head β_v-reduction). We define inductively the *head β_v-reduction* $\xrightarrow{h}_{\beta_v}$ by the following rules ($m \in \mathbb{N}$ in both rules):

$$\frac{}{(\lambda x.M)VM_1 \dots M_m \xrightarrow{h}_{\beta_v} M\{V/x\}M_1 \dots M_m}\,\beta_v \qquad \frac{N \xrightarrow{h}_{\beta_v} N'}{VNM_1 \dots M_m \xrightarrow{h}_{\beta_v} VN'M_1 \dots M_m}\,\mathit{right}$$

The head β_v-reduction $\xrightarrow{h}_{\beta_v}$ reduces exactly the same redexes (see also [13]) as the "left reduction" defined in [15, p. 136] for λ_v and called "evaluation" in [16, 4]. If $N \xrightarrow{h}_{\beta_v} N'$ then N' is obtained from N by reducing the leftmost-outermost β_v-redex, not in the scope of a λ: thus, the head β_v-reduction is deterministic (i.e., it is a partial function from Λ to Λ) and does not reduce values.

▶ **Definition 7** (Head σ-reduction). We define inductively the *head σ-reduction* $\xrightarrow{h}_\sigma$ by the following rules ($m \in \mathbb{N}$ in all the rules, $x \notin \mathsf{fv}(L)$ in the rule σ_1, $x \notin \mathsf{fv}(V)$ in the rule σ_3):

$$\frac{}{(\lambda x.M)NLM_1\dots M_m \xrightarrow{h}_\sigma (\lambda x.ML)NM_1\dots M_m}\,\sigma_1 \qquad \frac{N \xrightarrow{h}_\sigma N'}{VNM_1\dots M_m \xrightarrow{h}_\sigma VN'M_1\dots M_m}\,\mathit{right}$$

$$\frac{}{V((\lambda x.L)N)M_1\dots M_m \xrightarrow{h}_\sigma (\lambda x.VL)NM_1\dots M_m}\,\sigma_3$$

The *head* (v-)*reduction* is $\xrightarrow{h}_{\mathsf{v}} = \xrightarrow{h}_{\beta_v} \cup \xrightarrow{h}_\sigma$. The *internal* ($\mathsf{v}$-)*reduction* is $\xrightarrow{int}_{\mathsf{v}} = \to_{\mathsf{v}} \smallsetminus \xrightarrow{h}_{\mathsf{v}}$.

Notice that $\mapsto_{\beta_v} \subsetneq \xrightarrow{h}_{\beta_v} \subsetneq \to_{\beta_v}$ and $\mapsto_\sigma \subsetneq \xrightarrow{h}_\sigma \subsetneq \to_\sigma$ and $\mapsto_{\mathsf{v}} \subsetneq \xrightarrow{h}_{\mathsf{v}} \subsetneq \to_{\mathsf{v}}$. Values are normal forms for the head reduction, but the converse does not hold: $xI \notin \Lambda_v$ is head-normal.

Informally, if $N \xrightarrow{h}_\sigma N'$ then N' is obtained from N by reducing "one of the leftmost" σ_1- or σ_3-redexes, not in the scope of a λ: in general, a term may contain several head σ_1- and σ_3-redexes. Indeed, differently from $\xrightarrow{h}_{\beta_v}$, the head σ-reduction $\xrightarrow{h}_\sigma$ is not deterministic, for example the leftmost-outermost σ_1- and σ_3-redexes may overlap: if $M = (\lambda y.y')(\Delta(xI))I$ then $M \xrightarrow{h}_\sigma (\lambda y.y'I)(\Delta(xI)) = N_1$ by applying the rule σ_1 and $M \xrightarrow{h}_\sigma (\lambda z.(\lambda y.y')(zz))(xI)I = N_2$ by applying the rule σ_3. Note that N_1 contains only a head σ_3-redex and $N_1 \xrightarrow{h}_\sigma (\lambda z.(\lambda y.y'I)(zz))(xI) = N$ which is normal for $\xrightarrow{h}_{\mathsf{v}}$; meanwhile N_2 contains only a head σ_1-redex and $N_2 \xrightarrow{h}_\sigma (\lambda z.(\lambda y.y')(zz)I)(xI) = N'$ which is normal for $\xrightarrow{h}_{\mathsf{v}}$: $N \neq N'$, hence the head reduction $\xrightarrow{h}_{\mathsf{v}}$ is not confluent and a term may have several head-normal forms (this example does not contradict the confluence of σ-reduction because $N' \to_\sigma N$ but by performing an internal reduction step). Later, in Corollary 26.2 we show that if a term M has a head normal form $N \in \Lambda_v$ then N is the unique head normal form of M.

▶ **Definition 8** (Parallel reduction). We define inductively the *parallel reduction* $\Rightarrow$ by the following rules ($x \notin \mathsf{fv}(L)$ in the rule σ_1, $x \notin \mathsf{fv}(V)$ in the rule σ_3):

$$\frac{V \Rightarrow V' \quad M_i \Rightarrow M_i' \quad (m \in \mathbb{N},\ 0 \le i \le m)}{(\lambda x.M_0)VM_1\dots M_m \Rightarrow M_0'\{V'/x\}M_1'\dots M_m'}\,\beta_v \qquad \frac{N \Rightarrow N' \quad L \Rightarrow L' \quad M_i \Rightarrow M_i' \quad (m\in\mathbb{N},\ 0\le i\le m)}{(\lambda x.M_0)NLM_1\dots M_m \Rightarrow (\lambda x.M_0'L')N'M_1'\dots M_m'}\,\sigma_1$$

$$\frac{V \Rightarrow V' \qquad N \Rightarrow N' \qquad L \Rightarrow L' \qquad M_i \Rightarrow M_i' \quad (m \in \mathbb{N},\ 1 \le i \le m)}{V((\lambda x.L)N)M_1\dots M_m \Rightarrow (\lambda x.V'L')N'M_1'\dots M_m'}\,\sigma_3$$

$$\frac{M_i \Rightarrow M_i' \quad (m \in \mathbb{N},\ 0 \le i \le m)}{(\lambda x.M_0)M_1\dots M_m \Rightarrow (\lambda x.M_0')M_1'\dots M_m'}\,\lambda \qquad \frac{M_i \Rightarrow M_i' \quad (m \in \mathbb{N},\ 1 \le i \le m)}{x\,M_1\dots M_m \Rightarrow x\,M_1'\dots M_m'}\,\mathit{var}$$

In Definition 8 the rule *var* has no premises when $m = 0$: this is the base case of the inductive definition of $\Rightarrow$. The rules σ_1 and σ_3 have exactly three premises when $m = 0$.

Intuitively, $M \Rightarrow M'$ means that M' is obtained from M by reducing a number of β_v-, σ_1- and σ_3-redexes (existing in M) simultaneously.

▶ **Definition 9** (Internal and strong parallel reduction). We define inductively the *internal parallel reduction* $\overset{int}{\Rightarrow}$ by the following rules:

$$\frac{N \Rightarrow N'}{\lambda x.N \overset{int}{\Rightarrow} \lambda x.N'}\,\lambda \qquad \frac{}{x \overset{int}{\Rightarrow} x}\,\mathit{var} \qquad \frac{V \Rightarrow V' \quad N \overset{int}{\Rightarrow} N' \quad M_i \Rightarrow M_i' \quad (m \in \mathbb{N},\ 1 \le i \le m)}{VNM_1\dots M_m \overset{int}{\Rightarrow} V'N'M_1'\dots M_m'}\,\mathit{right}$$

The *strong parallel reduction* $\Rrightarrow$ is defined by: $M \Rrightarrow N$ iff $M \Rightarrow N$ and there exist $M', M'' \in \Lambda$ such that $M \xrightarrow{h}{}^*_{\beta_v} M' \xrightarrow{h}{}^*_\sigma M'' \overset{int}{\Rightarrow} N$.

Notice that the rule *right* for $\overset{int}{\Rightarrow}$ has exactly two premises when $m = 0$.

▶ **Remark 10.** The relations $\Rightarrow$, $\Rrightarrow$ and $\overset{int}{\Rightarrow}$ are reflexive. The reflexivity of $\Rrightarrow$ follows immediately from the reflexivity of $\Rightarrow$ and $\overset{int}{\Rightarrow}$. The proofs of reflexivity of $\Rightarrow$ and $\overset{int}{\Rightarrow}$ are both by structural induction on a term: in the case of $\Rightarrow$, recall that every term is of the form $(\lambda x.N)M_1 \dots M_m$ or $x\, M_1 \dots M_m$ for some $m \in \mathbb{N}$ and then apply the rule λ or *var* respectively, together with the inductive hypothesis; in the case of $\overset{int}{\Rightarrow}$, recall that every term is of the form $\lambda x.M$ or x or $VNM_1 \dots M_m$ for some $m \in \mathbb{N}$ and then apply the rule λ (together with the reflexivity of $\Rightarrow$) or *var* or *right* (together with the reflexivity of $\Rightarrow$ and the inductive hypothesis) respectively.

One has $\overset{int}{\Rightarrow} \subsetneq \Rrightarrow \subseteq \Rightarrow$ (first, prove that $\overset{int}{\Rightarrow} \subseteq \Rightarrow$ by induction on the derivation of $M \overset{int}{\Rightarrow} M'$, the other inclusions follow from the definition of $\Rrightarrow$) and, since $\Rightarrow$ is reflexive (Remark 10), $\overset{h}{\to}_{\beta_v} \subsetneq \Rightarrow$ and $\overset{h}{\to}_{\sigma} \subsetneq \Rightarrow$. Observe that $\Delta\Delta \mathsf{R} \Delta\Delta$ for any $\mathsf{R} \in \{\mapsto_{\beta_v}, \overset{h}{\to}_{\beta_v}, \Rightarrow, \overset{int}{\Rightarrow}, \Rrightarrow\}$, even if for different reasons: for example, $\Delta\Delta \overset{int}{\Rightarrow} \Delta\Delta$ by reflexivity of $\overset{int}{\Rightarrow}$ (Remark 10), whereas $\Delta\Delta \overset{h}{\to}_{\beta_v} \Delta\Delta$ by reducing the (leftmost-outermost) β_v-redex.

Next two further remarks collect many minor properties that can be easily proved.

▶ **Remark 11.**
1. The head β_v-reduction $\overset{h}{\to}_{\beta_v}$ does not reduce a value (in particular, does not reduce under λ's), i.e., for any $M \in \Lambda$ and any $V \in \Lambda_v$, one has $V \overset{h}{\not\to}_{\beta_v} M$.
2. The head σ-reduction $\overset{h}{\to}_{\sigma}$ does neither reduce a value nor reduce to a value, i.e., for any $M \in \Lambda$ and any $V \in \Lambda_v$, one has $V \overset{h}{\not\to}_{\sigma} M$ and $M \overset{h}{\not\to}_{\sigma} V$.
3. Variables and abstractions are preserved under $\overset{int}{\Leftarrow}$ ($\overset{int}{\Rightarrow}$-expansion), i.e., if $M \overset{int}{\Rightarrow} x$ (resp. $M \overset{int}{\Rightarrow} \lambda x.N'$) then $M = x$ (resp. $M = \lambda x.N$ for some $N \in \Lambda$ such that $N \Rightarrow N'$).
4. If $M \Rightarrow M'$ then $\lambda x.M \mathsf{R} \lambda x.M'$ for any $\mathsf{R} \in \{\Rightarrow, \overset{int}{\Rightarrow}, \Rrightarrow\}$. Indeed, for $\mathsf{R} \in \{\Rightarrow, \overset{int}{\Rightarrow}\}$ apply the rule λ to conclude, then $\lambda x.M \Rrightarrow \lambda x.M'$ according to the definition of $\Rrightarrow$.
5. For any $V, V' \in \Lambda_v$, $V \overset{int}{\Rightarrow} V'$ iff $V \Rightarrow V'$. The left-to-right direction holds because $\overset{int}{\Rightarrow} \subseteq \Rightarrow$; conversely, assume $V \Rightarrow V'$: if V is a variable then necessarily $V = V'$ and hence $V \overset{int}{\Rightarrow} V'$ by applying the rule *var* for $\overset{int}{\Rightarrow}$; otherwise $V = \lambda x.N$ for some $N \in \Lambda$, and then necessarily $V' = \lambda x.N'$ with $N \Rightarrow N'$, so $V \overset{int}{\Rightarrow} V'$ by applying the rule λ for $\overset{int}{\Rightarrow}$.

▶ **Remark 12.**
1. If $M \Rightarrow M'$ and $N \Rightarrow N'$ then $MN \Rightarrow M'N'$. For the proof, it is sufficient to consider the last rule of the derivation of $M \Rightarrow M'$.
2. If $\mathsf{R} \in \{\overset{h}{\to}_{\beta_v}, \overset{h}{\to}_{\sigma}\}$ and $M \mathsf{R} M'$, then $MN \mathsf{R} M'N$ for any $N \in \Lambda$. For the proof, it is sufficient to consider the last rule of the derivation of $M \mathsf{R} M'$, for any $\mathsf{R} \in \{\overset{h}{\to}_{\beta_v}, \overset{h}{\to}_{\sigma}\}$.
3. If $M \overset{int}{\Rightarrow} M'$ and $N \Rightarrow N'$ where $M' \notin \Lambda_v$, then $MN \overset{int}{\Rightarrow} M'N'$: indeed, the last rule in the derivation of $M \overset{int}{\Rightarrow} M'$ can be neither λ nor *var* because $M' \notin \Lambda_v$. The hypothesis $M' \notin \Lambda_v$ is crucial: for example, $x \overset{int}{\Rightarrow} x$ and $I\Delta \Rightarrow \Delta$ but $I\Delta \overset{int}{\not\Rightarrow} \Delta$ and thus $x(I\Delta) \overset{int}{\not\Rightarrow} x\Delta$.
4. $\to_{\mathsf{v}} \subseteq \Rightarrow \subseteq \to_{\mathsf{v}}^*$. As a consequence, $\Rightarrow^* = \to_{\mathsf{v}}^*$ and (by Proposition 4) $\Rightarrow$ is confluent.
5. $\overset{int}{\to}_{\mathsf{v}} \subseteq \overset{int}{\Rightarrow} \subseteq \overset{int}{\to}_{\mathsf{v}}^*$, so $\overset{int}{\Rightarrow}^* = \overset{int}{\to}_{\mathsf{v}}^*$. Thus, by Remark 11.3, variables and abstractions are preserved under $\overset{int}{\to}_{\mathsf{v}}^*$-expansion, i.e., if $M \overset{int}{\to}_{\mathsf{v}}^* x$ (resp. $M \overset{int}{\to}_{\mathsf{v}}^* \lambda x.N'$) then $M = x$ (resp. $M = \lambda x.N$ with $N \to_{\mathsf{v}}^* N'$).
6. For any $\mathsf{R} \in \{\overset{h}{\to}_{\beta_v}, \overset{h}{\to}_{\sigma}\}$, if $M \mathsf{R} M'$ then $M\{V/x\} \mathsf{R} M'\{V/x\}$ for any $V \in \Lambda_v$. The proof is by straightforward induction on the derivation of $M \mathsf{R} M'$ for any $\mathsf{R} \in \{\overset{h}{\to}_{\beta_v}, \overset{h}{\to}_{\sigma}\}$.

As expected, a basic property of parallel reduction $\Rightarrow$ is the following:

▶ **Lemma 13** (Substitution lemma for $\Rightarrow$)**.** *If $M \Rightarrow M'$ and $V \Rightarrow V'$ then $M\{V/x\} \Rightarrow M'\{V'/x\}$.*

Proof. By straightforward induction on the derivation of $M \Rightarrow M'$. ◀

The following lemma will play a crucial role in the proof of Lemmas 18-19 and shows that the head σ-reduction $\overset{h}{\to}_{\sigma}$ can be postponed after the head β_v-reduction $\overset{h}{\to}_{\beta_v}$.

▶ **Lemma 14** (Commutation of head reductions)**.**
1. *If* $M \xrightarrow{h}_\sigma L \xrightarrow{h}_{\beta_v} N$ *then there exists* $L' \in \Lambda$ *such that* $M \xrightarrow{h}_{\beta_v} L' \xrightarrow{h}{}^{=}_\sigma N$*.*
2. *If* $M \xrightarrow{h}{}^{*}_\sigma L \xrightarrow{h}{}^{*}_{\beta_v} N$ *then there exists* $L' \in \Lambda$ *such that* $M \xrightarrow{h}{}^{*}_{\beta_v} L' \xrightarrow{h}{}^{*}_\sigma N$*.*
3. *If* $M \xrightarrow{h}{}^{*}_{\mathsf{v}} M'$ *then there exists* $N \in \Lambda$ *such that* $M \xrightarrow{h}{}^{*}_{\beta_v} N \xrightarrow{h}{}^{*}_\sigma M'$*.*

Proof. 1. By induction on the derivation of $M \xrightarrow{h}_\sigma L$. Let us consider its last rule r.
If $\mathsf{r} = \sigma_1$ then $M = (\lambda x.M_0)N_0L_0M_1 \dots M_m$ and $L = (\lambda x.M_0L_0)N_0M_1 \dots M_m$ where $m \in \mathbb{N}$ and $x \notin \mathsf{fv}(L_0)$. Since $L \xrightarrow{h}_{\beta_v} N$, there are only two cases:
- either $N_0 \xrightarrow{h}_{\beta_v} N_0'$ and $N = (\lambda x.M_0L_0)N_0'M_1 \dots M_m$ (according to the rule *right* for $\xrightarrow{h}_{\beta_v}$), then $M \xrightarrow{h}_{\beta_v} (\lambda x.M_0)N_0'L_0M_1 \dots M_m \xrightarrow{h}_\sigma N$;
- or $N_0 \in \Lambda_v$ and $N = M_0\{N_0/x\}L_0M_1 \dots M_m$ (by the rule β_v, as $x \notin \mathsf{fv}(L_0)$), so $M \xrightarrow{h}_{\beta_v} N$.

If $\mathsf{r} = \sigma_3$ then $M = V((\lambda x.L_0)N_0)M_1 \dots M_m$ and $L = (\lambda x.VL_0)N_0M_1 \dots M_m$ with $m \in \mathbb{N}$ and $x \notin \mathsf{fv}(V)$. Since $L \xrightarrow{h}_{\beta_v} N$, there are only two cases:
- either $N_0 \xrightarrow{h}_{\beta_v} N_0'$ and $N = (\lambda x.VL_0)N_0'M_1 \dots M_m$ (according to the rule *right* for $\xrightarrow{h}_{\beta_v}$), then $M \xrightarrow{h}_{\beta_v} V((\lambda x.L_0)N_0')M_1 \dots M_m \xrightarrow{h}_\sigma N$;
- or $N_0 \in \Lambda_v$ and $N = VL_0\{N_0/x\}M_1 \dots M_m$ (by the rule β_v, as $x \notin \mathsf{fv}(V)$), so $M \xrightarrow{h}_{\beta_v} N$.

Finally, if $\mathsf{r} = \textit{right}$ then $M = VN_0M_1 \dots M_m$ and $L = VN_0'M_1 \dots M_m$ with $m \in \mathbb{N}$ and $N_0 \xrightarrow{h}_\sigma N_0'$. By Remark 11.2, $N_0' \notin \Lambda_v$ and thus, since $L \xrightarrow{h}_{\beta_v} N$, the only possibility is that $N_0' \xrightarrow{h}_{\beta_v} N_0''$ and $N = VN_0''M_1 \dots M_m$ (according to the rule *right* for $\xrightarrow{h}_{\beta_v}$). By induction hypothesis, there exists $N_0''' \in \Lambda$ such that $N_0 \xrightarrow{h}_{\beta_v} N_0''' \xrightarrow{h}{}^{=}_\sigma N_0''$. Therefore, $M \xrightarrow{h}_{\beta_v} VN_0'''M_1 \dots M_m \xrightarrow{h}{}^{=}_\sigma N$.

2. Immediate consequence of Lemma 14.1, using standard techniques of rewriting theory.
3. Immediate consequence of Lemma 14.2, using standard techniques of rewriting theory. ◀

We are now able to travel over again Takahashi's method [17, 4] in our setting with β_v- and σ-reduction. The next four lemmas govern the strong parallel reduction and will be used to prove Lemma 19.

▶ **Lemma 15.** *If* $M \Rrightarrow M'$ *and* $N \Rightarrow N'$ *and* $M' \notin \Lambda_v$ *, then* $MN \Rrightarrow M'N'$*.*

Proof. One has $MN \Rightarrow M'N'$ by Remark 12.1 and since $M \Rightarrow M'$. By hypothesis, there exist $m, n \in \mathbb{N}$ and $M_0, \dots, M_m, N_0, \dots, N_n$ such that $M = M_0$, $M_m = N_0$, $N_n \overset{int}{\Rrightarrow} M'$, $M_i \xrightarrow{h}_{\beta_v} M_{i+1}$ for any $0 \le i < m$ and $N_j \xrightarrow{h}_\sigma N_{j+1}$ for any $0 \le j < n$; by Remark 12.2, $M_iN \xrightarrow{h}_{\beta_v} M_{i+1}N$ for any $0 \le i < m$ and $N_jN \xrightarrow{h}_\sigma N_{j+1}N$ for any $0 \le j < n$. As $M' \notin \Lambda_v$, one has $N_nN \overset{int}{\Rrightarrow} M'N'$ by Remark 12.3. Therefore, $MN \Rrightarrow M'N'$. ◀

▶ **Lemma 16.** *If* $M \Rrightarrow M'$ *and* $N \Rrightarrow N'$ *then* $MN \Rrightarrow M'N'$*.*

Proof. If $M' \notin \Lambda_v$ then $MN \Rrightarrow M'N'$ by Lemma 15 and since $N \Rightarrow N'$.
Assume $M' \in \Lambda_v$: $MN \Rightarrow M'N'$ by Remark 12.1, as $M \Rightarrow M'$ and $N \Rightarrow N'$. By hypothesis, there are $m, m', n, n' \in \mathbb{N}$ and $M_0, \dots, M_m, M_0', \dots, M_{m'}', N_0, \dots, N_n, N_0', \dots, N_{n'}'$ such that:
- $M = M_0$, $M_m = M_0'$, $M_{m'}' \overset{int}{\Rrightarrow} M'$, $M_i \xrightarrow{h}_{\beta_v} M_{i+1}$ for any $0 \le i < m$, and $M_{i'}' \xrightarrow{h}_\sigma M_{i'+1}'$ for any $0 \le i' < m'$,
- $N = N_0$, $N_n = N_0'$, $N_{n'}' \overset{int}{\Rrightarrow} N'$, $N_j \xrightarrow{h}_{\beta_v} N_{j+1}$ for any $0 \le j < n$ and $N_{j'}' \xrightarrow{h}_\sigma N_{j'+1}'$ for any $0 \le j' < n'$.

By Remark 11.3, $M_{m'}' \in \Lambda_v$ since $M' \in \Lambda_v$, therefore $m' = 0$ by Remark 11.2, and thus $M_m = M_0' \overset{int}{\Rrightarrow} M'$ (and $M_m \Rightarrow M'$ since $\overset{int}{\Rrightarrow} \subseteq \Rightarrow$) and $M_m \in \Lambda_v$. Using the rules *right* for $\xrightarrow{h}_{\beta_v}$ and $\xrightarrow{h}_\sigma$, one has $M_mN_j \xrightarrow{h}_{\beta_v} M_mN_{j+1}$ for any $0 \le j < n$, and $M_mN_{j'}' \xrightarrow{h}_\sigma M_mN_{j'+1}'$ for any $0 \le j' < n'$. By Remark 12.2, $M_iN_0 \xrightarrow{h}_{\beta_v} M_{i+1}N_0$ for any $0 \le i < m$. By applying the rule *right* for $\stackrel{int}{\Rrightarrow}$, one has $M_m N'_{n'} \stackrel{int}{\Rrightarrow} M'N'$. Therefore, $MN = M_0 N_0 \stackrel{h}{\to}{}^*_{\beta_v} M_m N_0 \stackrel{h}{\to}{}^*_{\beta_v} M_m N_n = M_m N'_0 \stackrel{h}{\to}{}^*_{\sigma} M_m N'_{n'} \stackrel{int}{\Rrightarrow} M'N'$ and hence $MN \Rrightarrow M'N'$. ◀

▶ **Lemma 17.** *If $M \stackrel{int}{\Rrightarrow} M'$ and $V \Rrightarrow V'$, then $M\{V/x\} \Rrightarrow M'\{V'/x\}$.*

Proof. By Lemma 13, one has $M\{V/x\} \Rightarrow M'\{V'/x\}$ since $M \Rightarrow M'$ and $V \Rightarrow V'$. We proceed by induction on $M \in \Lambda$. Let us consider the last rule r of the derivation of $M \stackrel{int}{\Rrightarrow} M'$.

If $\mathsf{r} = var$ then there are two cases: either $M = x$ and then $M\{V/x\} = V \Rrightarrow V' = M'\{V'/x\}$; or $M = y \neq x$ and then $M\{V/x\} = y = M'\{V'/x\}$, so $M\{V/x\} \Rrightarrow M'\{V'/x\}$ by Remark 10.

If $\mathsf{r} = \lambda$ then $M = \lambda y.N$ and $M' = \lambda y.N'$ with $N \Rightarrow N'$; we can suppose without loss of generality that $y \notin \mathsf{fv}(V) \cup \{x\}$. One has $N\{V/x\} \Rightarrow N'\{V'/x\}$ according to Lemma 13. By applying the rule λ for $\stackrel{int}{\Rrightarrow}$, one has $M\{V/x\} = \lambda y.N\{V/x\} \stackrel{int}{\Rrightarrow} \lambda y.N'\{V'/x\} = M'\{V'/x\}$ and thus $M\{V/x\} \Rrightarrow M'\{V'/x\}$.

Finally, if $\mathsf{r} = right$ then $M = UNM_1 \dots M_m$ and $M' = U'N'M'_1 \dots M'_m$ for some $m \in \mathbb{N}$ with $U, U' \in \Lambda_v$, $U \Rightarrow U'$, $N \stackrel{int}{\Rrightarrow} N'$ and $M_i \Rightarrow M'_i$ for any $1 \leq i \leq m$. By induction hypothesis, $U\{V/x\} \Rrightarrow U'\{V'/x\}$ (indeed $U \stackrel{int}{\Rrightarrow} U'$ according to Remark 11.5) and $N\{V/x\} \Rrightarrow N'\{V'/x\}$. By Lemma 13, $M_i\{V/x\} \Rightarrow M'_i\{V'/x\}$ for any $1 \leq i \leq m$. By Lemma 16, $U\{V/x\}N\{V/x\} \Rrightarrow U'\{V'/x\}N'\{V/x\}$ and hence, by applying Lemma 15 m times since $U'\{V'/x\}N'\{V/x\} \notin \Lambda_v$, one has $M\{V/x\} = U\{V/x\}N\{V/x\}M_1\{V/x\} \dots M_m\{V/x\} \Rrightarrow U'\{V'/x\}N'\{V'/x\}M'_1\{V'/x\} \dots M'_m\{V'/x\} = M'\{V'/x\}$. ◀

In the proof of the two next lemmas, as well as in the proof of Corollary 21 and Theorem 22, our Lemma 14 plays a crucial role: indeed, since the head σ-reduction well interact with the head β_v-reduction, Takahashi's method [17, 4] is still working when adding the reduction rules σ_1 and σ_3 to Plotkin's β_v-reduction.

▶ **Lemma 18** (Substitution lemma for $\Rrightarrow$)**.** *If $M \Rrightarrow M'$ and $V \Rrightarrow V'$ then $M\{V/x\} \Rrightarrow M'\{V'/x\}$.*

Proof. By Lemma 13, one has $M\{V/x\} \Rightarrow M'\{V'/x\}$ since $M \Rightarrow M'$ and $V \Rightarrow V'$. By hypothesis, there exist $m, n \in \mathbb{N}$ and $M_0, \dots, M_m, N_0, \dots, N_n$ such that $M = M_0$, $M_m = N_0$, $N_n \stackrel{int}{\Rrightarrow} M'$, $M_i \stackrel{h}{\to}_{\beta_v} M_{i+1}$ for any $0 \leq i < m$ and $N_j \stackrel{h}{\to}_{\sigma} N_{j+1}$ for any $0 \leq j < n$; by Remark 12.6, $M_i\{V/x\} \stackrel{h}{\to}_{\beta_v} M_{i+1}\{V/x\}$ for any $0 \leq i < m$, and $N_j\{V/x\} \stackrel{h}{\to}_{\sigma} N_{j+1}\{V/x\}$ for any $0 \leq j < n$. By Lemma 17, one has $N_n\{V/x\} \Rrightarrow M'\{V'/x\}$, thus there exist $L, N \in \Lambda$ such that $M\{V/x\} \stackrel{h}{\to}{}^*_{\beta_v} N_0\{V/x\} \stackrel{h}{\to}{}^*_{\sigma} N_n\{V/x\} \stackrel{h}{\to}{}^*_{\beta_v} N \stackrel{h}{\to}{}^*_{\sigma} L \stackrel{int}{\Rrightarrow} M'\{V'/x\}$. By Lemma 14.2, there exists $N' \in \Lambda$ such that $M\{V/x\} \stackrel{h}{\to}{}^*_{\beta_v} N_0\{V/x\} \stackrel{h}{\to}{}^*_{\beta_v} N' \stackrel{h}{\to}{}^*_{\sigma} N \stackrel{h}{\to}{}^*_{\sigma} L \stackrel{int}{\Rrightarrow} M'\{V'/x\}$, therefore $M\{V/x\} \Rrightarrow M'\{V'/x\}$. ◀

Now we are ready to prove a key lemma, which states that parallel reduction $\Rightarrow$ coincides with strong parallel reduction $\Rrightarrow$ (the inclusion $\Rrightarrow \subseteq \Rightarrow$ is trivial).

▶ **Lemma 19** (Key Lemma)**.** *If $M \Rightarrow M'$ then $M \Rrightarrow M'$.*

Proof. By induction on the derivation of $M \Rightarrow M'$. Let us consider its last rule r.

If $\mathsf{r} = var$ then $M = x\,M_1 \dots M_m$ and $M' = x\,M'_1 \dots M'_m$ where $m \in \mathbb{N}$ and $M_i \Rightarrow M'_i$ for any $1 \leq i \leq m$. By reflexivity of $\Rrightarrow$ (Remark 10), $x \Rrightarrow x$. By induction hypothesis, $M_i \Rrightarrow M'_i$ for any $1 \leq i \leq m$. Therefore, $M \Rrightarrow M'$ by applying Lemma 16 m times.

If $\mathsf{r} = \lambda$ then $M = (\lambda x.M_0)M_1 \dots M_m$ and $M' = (\lambda x.M'_0)M'_1 \dots M'_m$ where $m \in \mathbb{N}$ and $M_i \Rightarrow M'_i$ for any $0 \leq i \leq m$. By induction hypothesis, $M_i \Rrightarrow M'_i$ for any $1 \leq i \leq m$. According to Remark 11.4, $\lambda x.M_0 \Rrightarrow \lambda x.M'_0$. So $M \Rrightarrow M'$ by applying Lemma 16 m times.

If $\mathsf{r} = \beta_v$ then $M = (\lambda x.M_0)VM_1 \dots M_m$ and $M' = M'_0\{V'/x\}M'_1 \dots M'_m$ where $m \in \mathbb{N}$, $V \Rightarrow V'$ and $M_i \Rightarrow M'_i$ for any $0 \leq i \leq m$. By induction hypothesis, $V \Rrightarrow V'$ and $M_i \Rrightarrow M'_i$

for any $0 \leq i \leq m$. By applying the rule β_v for $\xrightarrow{h}_{\beta_v}$, one has $M \xrightarrow{h}_{\beta_v} M_0\{V/x\}M_1 \ldots M_m$; moreover $M_0\{V/x\}M_1 \ldots M_m \Rrightarrow M'$ by Lemma 18 and by applying Lemma 16 m times, thus there are $L, N \in \Lambda$ such that $M \xrightarrow{h}_{\beta_v} M_0\{V/x\}M_1 \ldots M_m \xrightarrow{h}{}^{*}_{\beta_v} L \xrightarrow{h}{}^{*}_{\sigma} N \stackrel{int}{\Rrightarrow} M'$. So $M \Rrightarrow M'$.

If $\mathsf{r} = \sigma_1$ then $M = (\lambda x.M_0)N_0L_0M_1 \ldots M_m$ and $M' = (\lambda x.M_0'L_0')N_0'M_1' \ldots M_m'$ where $m \in \mathbb{N}$, $L_0 \Rightarrow L_0'$, $N_0 \Rightarrow N_0'$ and $M_i \Rightarrow M_i'$ for any $0 \leq i \leq m$. By induction hypothesis, $N_0 \Rrightarrow N_0'$ and $M_i \Rrightarrow M_i'$ for any $1 \leq i \leq m$. By applying the rule σ_1 for $\xrightarrow{h}_{\sigma}$, one has $M \xrightarrow{h}_{\sigma} (\lambda x.M_0L_0)N_0M_1 \ldots M_m$. By Remark 12.1, $M_0L_0 \Rightarrow M_0'L_0'$ and thus $\lambda x.M_0L_0 \Rrightarrow \lambda x.M_0'L_0'$ according to Remark 11.4. So $(\lambda x.M_0L_0)N_0M_1 \ldots M_m \Rrightarrow M'$ by applying Lemma 16 $m+1$ times, hence there are $L, N \in \Lambda$ such that $M \xrightarrow{h}_{\sigma} (\lambda x.M_0L_0)N_0M_1 \ldots M_m \xrightarrow{h}{}^{*}_{\beta_v} L \xrightarrow{h}{}^{*}_{\sigma} N \stackrel{int}{\Rrightarrow} M'$. By Lemma 14.2, there is $L' \in \Lambda$ such that $M \xrightarrow{h}{}^{*}_{\beta_v} L' \xrightarrow{h}{}^{*}_{\sigma} L \xrightarrow{h}{}^{*}_{\sigma} N \stackrel{int}{\Rrightarrow} M'$ and thus $M \Rrightarrow M'$.

Finally, if $\mathsf{r} = \sigma_3$ then $M = V((\lambda x.L_0)N_0)M_1 \ldots M_m$ and $M' = (\lambda x.V'L_0')N_0'M_1' \ldots M_m'$ with $m \in \mathbb{N}$, $V \Rightarrow V'$, $L_0 \Rightarrow L_0'$, $N_0 \Rightarrow N_0'$ and $M_i \Rightarrow M_i'$ for any $1 \leq i \leq m$. By induction hypothesis, $N_0 \Rrightarrow N_0'$ and $M_i \Rrightarrow M_i'$ for any $1 \leq i \leq m$. By the rule σ_3 for $\xrightarrow{h}_{\sigma}$, one has $M \xrightarrow{h}_{\sigma} (\lambda x.VL_0)N_0M_1 \ldots M_m$. By Remark 12.1, $VL_0 \Rightarrow V'L_0'$ and thus $\lambda x.VL_0 \Rrightarrow \lambda x.V'L_0'$ according to Remark 11.4. So $(\lambda x.VL_0)N_0M_1 \ldots M_m \Rrightarrow M'$ by applying Lemma 16 $m+1$ times, hence there are $L, N \in \Lambda$ such that $M \xrightarrow{h}_{\sigma} (\lambda x.VL_0)N_0M_1 \ldots M_m \xrightarrow{h}{}^{*}_{\beta_v} L \xrightarrow{h}{}^{*}_{\sigma} N \stackrel{int}{\Rrightarrow} M'$. By Lemma 14.2, there is $L' \in \Lambda$ such that $M \xrightarrow{h}{}^{*}_{\beta_v} L' \xrightarrow{h}{}^{*}_{\sigma} L \xrightarrow{h}{}^{*}_{\sigma} N \stackrel{int}{\Rrightarrow} M'$, so $M \Rrightarrow M'$. ◀

Next Lemma 20 and Corollary 21 show that internal parallel reduction can be shifted after head reductions.

▶ **Lemma 20** (Postponement). *If $M \stackrel{int}{\Rrightarrow} L$ and $L \xrightarrow{h}_{\beta_v} N$ (resp. $L \xrightarrow{h}_{\sigma} N$) then there exists $L' \in \Lambda$ such that $M \xrightarrow{h}_{\beta_v} L'$ (resp. $M \xrightarrow{h}_{\sigma} L'$) and $L' \Rightarrow N$.*

Proof. By induction on the derivation of $M \stackrel{int}{\Rrightarrow} L$. Let us consider its last rule r.

If $\mathsf{r} = \mathit{var}$, then $M = x = L$ which contradicts $L \xrightarrow{h}_{\beta_v} N$ and $L \xrightarrow{h}_{\sigma} N$ by Remarks 11.1-2.

If $\mathsf{r} = \lambda$ then $L = \lambda x.L'$ for some $L' \in \Lambda$, which contradicts $L \xrightarrow{h}_{\beta_v} N$ and $L \xrightarrow{h}_{\sigma} N$ by Remarks 11.1-2.

Finally, if $\mathsf{r} = \mathit{right}$ then $M = VM_0M_1 \ldots M_m$ and $L = V'L_0L_1 \ldots L_m$ where $m \in \mathbb{N}$, $V \Rightarrow V'$ (so $V \stackrel{int}{\Rrightarrow} V'$ by Remark 11.5), $M_0 \stackrel{int}{\Rrightarrow} L_0$ (thus $M_0 \Rightarrow L_0$ since $\stackrel{int}{\Rrightarrow} \subseteq \Rightarrow$) and $M_i \Rightarrow L_i$ for any $1 \leq i \leq m$.

- If $L \xrightarrow{h}_{\beta_v} N$ then there are two cases, depending on the last rule r' of the derivation of $L \xrightarrow{h}_{\beta_v} N$.
  - If $\mathsf{r}' = \beta_v$ then $V' = \lambda x.N_0'$, $L_0 \in \Lambda_v$ and $N = N_0'\{L_0/x\}L_1 \ldots L_m$, thus $M_0 \in \Lambda_v$ and $V = \lambda x.N_0$ with $N_0 \Rightarrow N_0'$ by Remark 11.3. By Lemma 13, one has $N_0\{M_0/x\} \Rightarrow N_0'\{L_0/x\}$. Let $L' = N_0\{M_0/x\}M_1 \ldots M_m$: so $M = (\lambda x.N_0)M_0M_1 \ldots M_m \xrightarrow{h}_{\beta_v} L'$ (apply the rule β_v for $\xrightarrow{h}_{\beta_v}$) and $L' \Rightarrow N$ by applying Remark 12.1 m times.
  - If $\mathsf{r}' = \mathit{right}$ then $N = V'N_0L_1 \ldots L_m$ with $L_0 \xrightarrow{h}_{\beta_v} N_0$. By induction hypothesis, there exists $L_0' \in \Lambda$ such that $M_0 \xrightarrow{h}_{\beta_v} L_0' \Rightarrow N_0$. Let $L' = VL_0'M_1 \ldots M_m$: so $M \xrightarrow{h}_{\beta_v} L'$ (apply the rule *right* for $\xrightarrow{h}_{\beta_v}$) and $L' \Rightarrow N$ by applying Remark 12.1 $m+1$ times.
- If $L \xrightarrow{h}_{\sigma} N$ then there are three cases, depending on the last rule r' of the derivation of $L \xrightarrow{h}_{\sigma} N$.
  - If $\mathsf{r}' = \sigma_1$ then $m > 0$, $V' = \lambda x.N_0'$ and $N = (\lambda x.N_0'L_1)L_0L_2 \ldots L_m$, thus $V = \lambda x.N_0$ with $N_0 \Rightarrow N_0'$ by Remark 11.3. Using Remarks 12.1 and 11.4, one has $\lambda x.N_0M_1 \Rightarrow \lambda x.N_0'L_1$. Let $L' = (\lambda x.N_0M_1)M_0M_2 \ldots M_m$: so $M = (\lambda x.N_0)M_0M_1 \ldots M_m \xrightarrow{h}_{\sigma} L'$ (apply the rule σ_1 for $\xrightarrow{h}_{\sigma}$) and $L' \Rightarrow N$ by applying Remark 12.1 m times.
  - If $\mathsf{r}' = \sigma_3$ then $L_0 = (\lambda x.L_{01})L_{02}$ and $N = (\lambda x.V'L_{01})L_{02}L_1 \ldots L_m$. Since $M_0 \stackrel{int}{\Rrightarrow} (\lambda x.L_{01})L_{02}$, necessarily $M_0 = (\lambda x.M_{01})M_{02}$ with $M_{01} \Rightarrow L_{01}$ and $M_{02} \stackrel{int}{\Rrightarrow} L_{02}$ (so $M_{02} \Rightarrow L_{02}$). Using Remarks 12.1 and 11.4, one has $\lambda x.VM_{01} \Rightarrow \lambda x.V'L_{01}$. Let

$L' = (\lambda x.VM_{01})M_{02}M_1 \dots M_m$: therefore $M = V((\lambda x.M_{01})M_{02})M_1 \dots M_m \xrightarrow{h}_\sigma L'$ (apply the rule σ_3 for $\xrightarrow{h}_\sigma$) and $L' \Rightarrow N$ by applying Remark 12.1 $m+1$ times.

- If $\mathsf{r}' = \mathit{right}$ then $N = V'N_0L_1 \dots L_m$ with $L_0 \xrightarrow{h}_\sigma N_0$. By induction hypothesis, there exists $L'_0 \in \Lambda$ such that $M_0 \xrightarrow{h}_\sigma L'_0 \Rightarrow N_0$. Let $L' = VL'_0M_1 \dots M_m$: so $M \xrightarrow{h}_\sigma L'$ (apply the rule *right* for $\xrightarrow{h}_\sigma$) and $L' \Rightarrow N$ by applying Remark 12.1 $m+1$ times.

◀

▶ **Corollary 21.** *If $M \overset{int}{\Rightarrow} L$ and $L \xrightarrow{h}_{\beta_v} N$ (resp. $L \xrightarrow{h}_\sigma N$), then there exist $L', L'' \in \Lambda$ such that $M \xrightarrow{h}{}^{+}_{\beta_v} L' \xrightarrow{h}{}^{*}_{\sigma} L'' \overset{int}{\Rightarrow} N$ (resp. $M \xrightarrow{h}{}^{*}_{\beta_v} L' \xrightarrow{h}{}^{*}_{\sigma} L'' \overset{int}{\Rightarrow} N$).*

Proof. Immediate by Lemma 20 and Lemma 19, applying Lemma 14.2 if $L \xrightarrow{h}_\sigma N$. ◀

Now we obtain our first main result (Theorem 22): any v-reduction sequence can be sequentialized into a head β_v-reduction sequence followed by a head σ-reduction sequence, followed by an internal reduction sequence. In ordinary λ-calculus, the well-known result corresponding to our Theorem 22 says that a β-reduction sequence can be factorized in a head reduction sequence followed by an internal reduction sequence (see for example [17, Corollary 2.6]).

▶ **Theorem 22** (Sequentialization). *If $M \to^*_{\mathsf{v}} M'$ then there exist $L, N \in \Lambda$ such that $M \xrightarrow{h}{}^{*}_{\beta_v} L \xrightarrow{h}{}^{*}_{\sigma} N \xrightarrow{int}{}^{*}_{\mathsf{v}} M'$.*

Proof. By Remark 12.4, $M \Rightarrow^* M'$ and thus there are $m \in \mathbb{N}$ and $M_0, \dots, M_m \in \Lambda$ such that $M = M_0$, $M_m = M'$ and $M_i \Rightarrow M_{i+1}$ for any $0 \le i < m$. We prove by induction on $m \in \mathbb{N}$ that there are $L, N \in \Lambda$ such that $M \xrightarrow{h}{}^{*}_{\beta_v} L \xrightarrow{h}{}^{*}_{\sigma} N \overset{int}{\Rightarrow}{}^{*} M'$, so $N \xrightarrow{int}{}^{*}_{\mathsf{v}} M'$ by Remark 12.5.

If $m = 0$ then $M = M_0 = M'$ and hence we conclude by taking $L = M' = N$.

Finally, suppose $m > 0$. By induction hypothesis applied to $M_1 \Rightarrow^* M'$, there exist $L', N' \in \Lambda$ such that $M_1 \xrightarrow{h}{}^{*}_{\beta_v} L' \xrightarrow{h}{}^{*}_{\sigma} N' \overset{int}{\Rightarrow}{}^{*} M'$. By applying Lemma 19 to M, there exist $L_0, N_0 \in \Lambda$ such that $M \xrightarrow{h}{}^{*}_{\beta_v} L_0 \xrightarrow{h}{}^{*}_{\sigma} N_0 \overset{int}{\Rightarrow} M_1$. By applying Corollary 21 repeatedly, there exists $N \in \Lambda$ such that $N_0 \xrightarrow{h}{}^{*}_{\mathsf{v}} N \overset{int}{\Rightarrow} N'$ and hence $M \xrightarrow{h}{}^{*}_{\mathsf{v}} N \overset{int}{\Rightarrow}{}^{*} M'$. According to Lemma 14.3, there exists $L \in \Lambda$ such that $M \xrightarrow{h}{}^{*}_{\beta_v} L \xrightarrow{h}{}^{*}_{\sigma} N \overset{int}{\Rightarrow}{}^{*} M'$. ◀

It is worth noticing that in Definition 7 there is no distinction between head σ_1- and head σ_3-reduction steps, and, according to it, the sequentialization of Theorem 22 imposes no order between head σ-reductions. We denote by $\xrightarrow{h}_{\sigma_1}$ and $\xrightarrow{h}_{\sigma_3}$ respectively the reduction relations $\to_{\sigma_1} \cap \xrightarrow{h}_\sigma$ and $\to_{\sigma_3} \cap \xrightarrow{h}_\sigma$. So, a natural question arises: is it possible to sequentialize them? The answer is negative, as proved by the next two counterexamples.

- Let $M = x((\lambda y.z')(zI))\Delta$ and $N = (\lambda y.xz'\Delta)(zI)$: $M \xrightarrow{h}_{\sigma_3} (\lambda y.xz')(zI)\Delta \xrightarrow{h}_{\sigma_1} N$, but there exists no L such that $M \xrightarrow{h}{}^{*}_{\sigma_1} L \xrightarrow{h}{}^{*}_{\sigma_3} N$. In fact M contains only a head σ_3-redex and $(\lambda y.xz')(zI)\Delta$ contains only a head σ_1-redex
- Let $M = x((\lambda y.z')(zI)\Delta)$ and $N = (\lambda y.x(z'\Delta))(zI)$: $M \xrightarrow{h}_{\sigma_1} x((\lambda y.z'\Delta)(zI)) \xrightarrow{h}_{\sigma_3} N$ but there is no L such that $M \xrightarrow{h}{}^{*}_{\sigma_3} L \xrightarrow{h}{}^{*}_{\sigma_1} N$. In fact M contains only a head σ_1-redex and $x((\lambda y.z'\Delta)(zI))$ contains only a head σ_3-redex.

So, the impossibility of sequentializing a head σ-reduction sequence is due to the fact that a head σ_1-reduction step can create a head σ_3-redex, and viceversa. This is not a problem, since head σ-reduction is strongly normalizing (by Proposition 4 and since $\xrightarrow{h}_\sigma \subseteq \to_\sigma$). Our approach does not force a strict order of head σ-reductions.

4 Standardization

Now we are able to prove the main result of this paper, i.e., a standardization theorem for λ_v^σ (Theorem 25). In particular we provide a notion of standard reduction sequence that avoids any auxiliary notion of residual redexes, by closely following the definition given in [15].

▶ **Notation.** For any $k, m \in \mathbb{N}$ with $k \leq m$, we denote by $\lceil M_0, \cdots, M_k, \cdots M_m \rceil^{head}$ a sequence of terms such that $M_i \xrightarrow{h}_{\beta_v} M_{i+1}$ when $0 \leq i < k$, and $M_i \xrightarrow{h}_{\sigma} M_{i+1}$ when $k \leq i < m$.

It is easy to check that $\lceil M \rceil^{head}$ for any $M \in \Lambda$. The notion of standard sequence of terms is defined by using the previous notion of *head*-sequence. Our notion of standard reduction sequence is mutually defined together with the notion of inner-sequence of terms (Definition 23). This definition allows us to avoid non-deterministic cases remarked in [7] (we provide more details at the end of this section). We denote by $\lceil M_0, \cdots, M_m \rceil^{std}$ (resp. $\lceil M_0, \cdots, M_m \rceil^{in}$) a standard (resp. inner) sequence of terms.

▶ **Definition 23** (Standard and inner sequences). *Standard* and *inner sequences* of terms are defined by mutual induction as follows:

1. if $\lceil M_0, \cdots, M_m \rceil^{head}$ and $\lceil M_m, \cdots, M_{m+n} \rceil^{in}$ then $\lceil M_0, \cdots, M_{m+n} \rceil^{std}$, where $m, n \in \mathbb{N}$;
2. $\lceil M \rceil^{in}$, for any $M \in \Lambda$;
3. if $\lceil M_0, \cdots, M_m \rceil^{std}$ then $\lceil \lambda z.M_0, \cdots, \lambda z.M_m \rceil^{in}$, where $m \in \mathbb{N}$;
4. if $\lceil V_0, \cdots, V_h \rceil^{std}$ and $\lceil N_0, \cdots, N_n \rceil^{in}$ then $\lceil V_0 N_0, \cdots, V_0 N_n, \cdots, V_h N_n \rceil^{in}$, where $h, n \in \mathbb{N}$;
5. if $\lceil N_0, \cdots, N_n \rceil^{in}$, $\lceil M_0, \cdots, M_m \rceil^{std}$ and $N_0 \notin \Lambda_v$, then $\lceil N_0 M_0, \cdots, N_n M_0, \cdots, N_n M_m \rceil^{in}$, where $m, n \in \mathbb{N}$.

For instance, let $M = (\lambda y.Ix)(z(\Delta I))(II)$: $M \to_{\mathsf{v}} (\lambda y.Ix)(z(\Delta I))I \to_{\mathsf{v}} (\lambda y.x)(z(\Delta I))I$ and $M \to_{\mathsf{v}} (\lambda y.Ix(II))(z(\Delta I)) \to_{\mathsf{v}} (\lambda y.Ix(II))(z(II))$ are not standard sequences; $M \to_{\mathsf{v}} (\lambda y.Ix)(z(\Delta I))I$ and $M \to_{\mathsf{v}} (\lambda y.Ix)(z(II))(II) \to_{\mathsf{v}} (\lambda y.Ix)(zI)(II) \to_{\mathsf{v}} (\lambda y.Ix(II))(zI) \to_{\mathsf{v}} (\lambda y.x(II))(zI) \to_{\mathsf{v}} (\lambda y.xI)(zI)$ are standard sequences.

▶ **Remark 24.** For any $n \in \mathbb{N}$, if $\lceil N_0, \cdots, N_n \rceil^{in}$ (resp. $\lceil N_0, \cdots, N_n \rceil^{head}$) then $\lceil N_0, \cdots, N_n \rceil^{std}$. Indeed, $\lceil N_0 \rceil^{head}$ (resp. $\lceil N_n \rceil^{in}$ by Definition 23.2), so $\lceil N_0, \cdots, N_n \rceil^{std}$ by Definition 23.1.

In particular, $\lceil N \rceil^{std}$ for any $N \in \Lambda$: apply Definition 23.2 and Remark 24 for $n = 0$.

▶ **Theorem 25** (Standardization).

1. *If $M \to_{\mathsf{v}}^* M'$ then there is a sequence $\lceil M, \cdots, M' \rceil^{std}$.*
2. *If $M \xrightarrow{int}_{\mathsf{v}}^* M'$ then there is a sequence $\lceil M, \cdots, M' \rceil^{in}$.*

Proof. Both statements are proved simultaneously by induction on $M' \in \Lambda$.

1. - If $M' = z$ then, by Theorem 22, $M \xrightarrow{h}_{\beta_v}^* L \xrightarrow{h}_{\sigma}^* N \xrightarrow{int}_{\mathsf{v}}^* z$ for some $L, N \in \Lambda$. By Remarks 12.5 and 11.2, $L = N = z$; therefore $M \xrightarrow{h}_{\beta_v}^* z$ and hence there is a sequence $\lceil M, \cdots, z \rceil^{head}$. Thus, $\lceil M, \cdots, z \rceil^{std}$ by Remark 24.
   - If $M' = \lambda z.N'$ then, by Theorem 22, $M \xrightarrow{h}_{\beta_v}^* L \xrightarrow{h}_{\sigma}^* L' \xrightarrow{int}_{\mathsf{v}}^* \lambda z.N'$ for some $L, L' \in \Lambda$. By Remarks 12.5 and 11.2, $L = L' = \lambda z.N$ with $N \to_{\mathsf{v}}^* N'$. So $M \xrightarrow{h}_{\beta_v}^* \lambda z.N$ and hence there is a sequence $\lceil M, \cdots, \lambda z.N \rceil^{head}$. By induction on (1), there is a sequence $\lceil N, \cdots, N' \rceil^{std}$, thus $\lceil \lambda z.N, \cdots, \lambda z.N' \rceil^{in}$ by Definition 23.3. Therefore $\lceil M, \cdots, \lambda z.N, \cdots, \lambda z.N' \rceil^{std}$ by Definition 23.1.
   - If $M' = N'L'$ then, by Theorem 22, $M \xrightarrow{h}_{\beta_v}^* M'' \xrightarrow{h}_{\sigma}^* M_0 \xrightarrow{int}_{\mathsf{v}}^* N'L'$ for some $M'', M_0 \in \Lambda$. By Remark 3, $M_0 = NL$ for some $N, L \in \Lambda$, since $\xrightarrow{int}_{\mathsf{v}}^* \subseteq \to_{\mathsf{v}}^*$ and $M' \notin \Lambda_v$. Thus there is a sequence $\lceil M, \cdots, M'', \cdots, NL \rceil^{head}$. By Remark 12.5, $NL \overset{int}{\Rightarrow}^* N'L'$; clearly, each step of $\overset{int}{\Rightarrow}$ is an instance of the rule *right* of Definition 9. There are two sub-cases.

- If $N \in \Lambda_v$ then $N \Rightarrow^* N'$ and $L \overset{int}{\Rightarrow}{}^* L'$, so $N \to_{\mathsf{v}}^* N'$ and $L \overset{int}{\to}{}_{\mathsf{v}}^* L'$ by Remarks 12.4-5. By induction respectively on (1) and (2), there are sequences $\lceil N, \cdots, N' \rceil^{std}$ and $\lceil L, \cdots, L' \rceil^{in}$, thus $\lceil NL, \cdots, NL', \cdots, N'L' \rceil^{in}$ by Definition 23.4. Therefore $\lceil M, \cdots, M'', \cdots, NL, \cdots, NL', \cdots, N'L' \rceil^{std}$ by Definition 23.1.
- If $N \notin \Lambda_v$ (i.e., $N = VM_1 \dots M_m$ with $m > 0$) then $N \overset{int}{\Rightarrow}{}^* N'$ and $L \Rightarrow^* L'$, so $N \overset{int}{\to}{}_{\mathsf{v}}^* N'$ and $L \to_{\mathsf{v}}^* L'$ by Remarks 12.4-5. By induction respectively on (2) and (1), there are sequences $\lceil N, \cdots, N' \rceil^{in}$ and $\lceil L, \cdots, L' \rceil^{std}$. Hence $\lceil NL, \cdots, N'L, \cdots, N'L' \rceil^{in}$ by Definition 23.5 . Thus $\lceil M, \cdots, M'', \cdots, NL, \cdots, N'L, \cdots, N'L' \rceil^{std}$ by Definition 23.1.

2. - If $M' = z$ then $M = z$ by Remark 12.5, hence $\lceil z \rceil^{in}$ by Definition 23.2.
   - If $M' = \lambda z.L'$ then $M = \lambda z.L$ and $L \to_{\mathsf{v}}^* L'$ by Remark 12.5. Hence there is a sequence $\lceil L, \cdots, L' \rceil^{std}$ by induction on (1). By Definition 23.3, $\lceil \lambda z.L, \cdots, \lambda z.L' \rceil^{in}$.
   - If $M' = N'L'$ then $M = NL$ for some $N, L \in \Lambda$ by Remark 3, since $\overset{int}{\to}{}_{\mathsf{v}}^* \subseteq \to_{\mathsf{v}}^*$ and $M' \notin \Lambda_v$. By Remark 12.5, $NL \overset{int}{\Rightarrow}{}^* N'L'$; clearly, each step of $\overset{int}{\Rightarrow}$ is an instance of the rule *right* of Definition 9. There are two sub-cases.
     - If $N \in \Lambda_v$ then $N \Rightarrow^* N'$ and $L \overset{int}{\Rightarrow}{}^* L'$, so $N \to_{\mathsf{v}}^* N'$ and $L \overset{int}{\to}{}_{\mathsf{v}}^* L'$ by Remarks 12.4-5. Thus there are sequences $\lceil N, \cdots, N' \rceil^{std}$ and $\lceil L, \cdots, L' \rceil^{in}$ by induction respectively on (1) and (2). Therefore, by Definition 23.4, $\lceil NL, \cdots, NL', \cdots, N'L' \rceil^{in}$.
     - If $N \notin \Lambda_v$ (i.e. $N = VM_1 \dots M_m$ with $m > 0$) then $N \overset{int}{\Rightarrow}{}^* N'$ and $L \Rightarrow^* L'$, thus $N \overset{int}{\to}{}_{\mathsf{v}}^* N'$ and $L \to_{\mathsf{v}}^* L'$ by Remarks 12.4-5. By induction respectively on (2) and (1), there are sequences $\lceil N, \cdots, N' \rceil^{in}$ and $\lceil L, \cdots, L' \rceil^{std}$. So $\lceil NL, \cdots, N'L, \cdots, N'L' \rceil^{in}$ by Definition 23.5. ◀

Due to non-sequentialization of head σ_1- and head σ_3-reductions, several standard sequences may have the same starting term and ending term: for instance, if $M = I(\Delta I)I$ and $N = (\lambda z.(\lambda x.xI)(zz))I$ then $M \to_{\mathsf{v}} (\lambda x.xI)(\Delta I) \to_{\mathsf{v}} N$ and $M \to_{\mathsf{v}} (\lambda z.I(zz))II \to_{\mathsf{v}} (\lambda z.I(zz)I)I \to_{\mathsf{v}} N$ are both standard sequences from M to N.

Finally, we can compare our notion of standardization with that given in [15]. To make the comparison possible we avoid σ-reductions and we recall that $\overset{h}{\to}_{\beta_v}$ is exactly the Plotkin's left-reduction [15, p. 136]. As remarked in [7, §1.5 p. 149], both sequences $(\lambda z.II)(II) \to_{\mathsf{v}} (\lambda z.I)(II) \to_{\mathsf{v}} (\lambda z.I)I$ and $(\lambda z.II)(II) \to_{\mathsf{v}} (\lambda z.II)I \to_{\mathsf{v}} (\lambda z.I)I$ are standard according to [15]. On the other hand, only the second sequence is standard in our sense. It is easy to check that collapsing the two notions of inner and standard sequence given in Definition 23, we get a notion of standard sequence that accept both the above sequences.

5 Some conservativity results

The sequentialization result (Theorem 22) has some interesting semantic consequences. It allows us to prove that (Corollary 29) the λ_v^σ-calculus is sound with respect to the call-by-value observational equivalence introduced by Plotkin in [15] for λ_v. Moreover we can prove that some notions, like that of potential valuability and solvability, introduced in [13] for λ_v, coincide with the respective notions for λ_v^σ (Theorem 31). This justifies the idea that λ_v^σ is a useful tool for studying the properties of λ_v. Our starting point is the following corollary.

▶ **Corollary 26.**

1. *If $M \to_{\mathsf{v}}^* V \in \Lambda_v$ then there exists $V' \in \Lambda_v$ such that $M \overset{h}{\to}{}_{\beta_v}^* V' \overset{int}{\to}{}_{\mathsf{v}}^* V$.*
2. *For every $V \in \Lambda_v$, $M \overset{h}{\to}{}_{\beta_v}^* V$ if and only if $M \overset{h}{\to}{}_{\mathsf{v}}^* V$.*

Proof. The first point is proved by observing that, by Theorem 22, there are $N, L \in \Lambda$ such that $M \overset{h}{\to}{}_{\beta_v}^* L \overset{h}{\to}{}_{\sigma}^* N \overset{int}{\to}{}_{\mathsf{v}}^* V$. By Remark 12.5 , $N \in \Lambda_v$ and thus $L = N$ according to

Remark 11.2. Concerning the second point, the right-to-left direction is a consequence of Lemma 14.3 and Remark 11.2; the left-to-right direction follows from $\xrightarrow{h}_{\beta_v} \subseteq \xrightarrow{h}_{\mathsf{v}}$. ◀

Let us recall the notion of observational equivalence defined by Plotkin [15] for λ_v.

▶ **Definition 27** (Halting, observational equivalence). Let $M \in \Lambda$. We say that (*the evaluation of*) M *halts* if there exists $V \in \Lambda_v$ such that $M \xrightarrow{h}{}^{*}_{\beta_v} V$.

The (*call-by-value*) *observational equivalence* is an equivalence relation $\cong$ on Λ defined by: $M \cong N$ if, for every context $\mathtt{C}$, one has that $\mathtt{C}(\!|M|\!)$ halts iff $\mathtt{C}(\!|N|\!)$ halts.[1]

Clearly, similar notions can be defined for λ_v^σ using $\xrightarrow{h}_{\mathsf{v}}$ instead of $\xrightarrow{h}_{\beta_v}$. Head σ-reduction plays no role neither in deciding the halting problem for evaluation (Corollary 26.1), nor in reaching a particular value (Corollary 26.2). So, we can conclude that the notions of halting and observational equivalence in λ_v^σ coincide with the ones in λ_v, respectively.

Now we compare the equational theory of λ_v^σ with Plotkin's observational equivalence.

▶ **Theorem 28** (Adequacy of v-reduction). *If $M \to_{\mathsf{v}}^* M'$ then: M halts iff M' halts.*

Proof. If $M' \xrightarrow{h}{}^{*}_{\beta_v} V \in \Lambda_v$ then $M \to_{\mathsf{v}}^* M' \to_{\mathsf{v}}^* V$ since $\xrightarrow{h}_{\beta_v} \subseteq \to_{\mathsf{v}}$. By Corollary 26.1, there exists $V' \in \Lambda_v$ such that $M \xrightarrow{h}{}^{*}_{\beta_v} V'$. Thus M halts.

Conversely, if $M \xrightarrow{h}{}^{*}_{\beta_v} V \in \Lambda_v$ then $M \to_{\mathsf{v}}^* V$ since $\xrightarrow{h}_{\beta_v} \subseteq \to_{\mathsf{v}}$. By confluence of $\to_{\mathsf{v}}$ (Proposition 4, since $M \to_{\mathsf{v}}^* M'$) and Remark 3 (since $V \in \Lambda_v$), there is $V' \in \Lambda_v$ such that $V \to_{\mathsf{v}}^* V'$ and $M' \to_{\mathsf{v}}^* V'$. By Corollary 26.1, there is $V'' \in \Lambda_v$ such that $M' \xrightarrow{h}{}^{*}_{\beta_v} V''$. So M' halts. ◀

▶ **Corollary 29** (Soundness). *If $M =_{\mathsf{v}} N$ then $M \cong N$.*

Proof. Let $\mathtt{C}$ be a context. By confluence of $\to_{\mathsf{v}}$ (Proposition 4), $M =_{\mathsf{v}} N$ implies that there exists $L \in \Lambda$ such that $M \to_{\mathsf{v}}^* L$ and $N \to_{\mathsf{v}}^* L$, hence $\mathtt{C}(\!|M|\!) \to_{\mathsf{v}}^* \mathtt{C}(\!|L|\!)$ and $\mathtt{C}(\!|N|\!) \to_{\mathsf{v}}^* \mathtt{C}(\!|L|\!)$. By Theorem 28, $\mathtt{C}(\!|M|\!)$ halts iff $\mathtt{C}(\!|L|\!)$ halts iff $\mathtt{C}(\!|N|\!)$ halts. Therefore, $M \cong N$. ◀

Plotkin [15, Theorem 5] has already proved that $M =_{\beta_v} N$ implies $M \cong N$, but our Corollary 29 is not obvious since our λ_v^σ-calculus equates more than Plotkin's λ_v-calculus ($=_{\beta_v} \subseteq =_{\mathsf{v}}$ since $\to_{\beta_v} \subseteq \to_{\mathsf{v}}$, and Example 5 shows that this inclusion is strict).

The converse of Corollary 29 does not hold since $\lambda x.x(\lambda y.xy) \cong \Delta$ but $\lambda x.x(\lambda y.xy)$ and Δ are different v-normal forms and hence $\lambda x.x(\lambda y.xy) \neq_{\mathsf{v}} \Delta$ by confluence of $\to_{\mathsf{v}}$ (Proposition 4).

A further remarkable consequence of Corollary 26.1 is that the notions of potential valuability and solvability for λ_v^σ-calculus (studied in [3]) can be shown to coincide with the ones for Plotkin's λ_v-calculus (studied in [13, 14]), respectively. Let us recall their definition.

▶ **Definition 30** (Potential valuability, solvability). Let M be a term:

- M is v-*potentially valuable* (resp. β_v-*potentially valuable*) if there are $m \in \mathbb{N}$, pairwise distinct variables $x_1, \ldots, x_m$ and $V, V_1, \ldots, V_m \in \Lambda_v$ such that $M\{V_1/x_1, \ldots, V_m/x_m\} \to_{\mathsf{v}}^* V$ (resp. $M\{V_1/x_1, \ldots, V_m/x_m\} \to_{\beta_v}^* V$);
- M is v-*solvable* (resp. β_v-*solvable*) if there are $n,m \in \mathbb{N}$, variables $x_1, \ldots, x_m$ and $N_1, \ldots, N_n \in \Lambda$ such that $(\lambda x_1 \ldots x_m.M)N_1 \cdots N_n \to_{\mathsf{v}}^* I$ (resp. $(\lambda x_1 \ldots x_m.M)N_1 \cdots N_n \to_{\beta_v}^* I$).

▶ **Theorem 31.** *Let M be a term:*

1. *M is v-potentially valuable if and only if M is β_v-potentially valuable;*
2. *M is v-solvable if and only if M is β_v-solvable.*

[1] Original Plotkin's definition of call-by-value observational equivalence (see [15]) also requires that $\mathtt{C}(\!|M|\!)$ and $\mathtt{C}(\!|N|\!)$ are closed terms, according to the tradition identifying programs with closed terms.

Proof. In both points, the implication from right to left is trivial since $\to_{\beta_v} \subseteq \to_{\mathsf{v}}$. Let us prove the other direction.

1. Since M is v-potentially valuable, there are variables $x_1, \ldots, x_m$ and $V, V_1, \ldots, V_m \in \Lambda_v$ (with $m \geq 0$) such that $M\{V_1/x_1, \ldots, V_m/x_m\} \to_{\mathsf{v}}^* V$; then, there exists $V' \in \Lambda_v$ such that $M\{V_1/x_1, \ldots, V_m/x_m\} \to_{\beta_v}^* V'$ by Corollary 26.1 and because $\xrightarrow{h}_{\beta_v} \subseteq \to_{\beta_v}$, therefore M is β_v-potentially valuable.
2. Since M is v-solvable, there exist variables $x_1, \ldots, x_m$ and terms $N_1, \ldots, N_n$ (for some $n, m \geq 0$) such that $(\lambda x_1 \ldots x_m.M)N_1 \cdots N_n \to_{\mathsf{v}}^* I$; then, there exists $V \in \Lambda_v$ such that $(\lambda x_1 \ldots x_m.M)N_1 \cdots N_n \to_{\beta_v}^* V \xrightarrow{int}{}^*_{\mathsf{v}} I$ by Corollary 26.1 and because $\xrightarrow{h}_{\beta_v} \subseteq \to_{\beta_v}$. According to Remark 12.5 , $V = \lambda x.N$ for some $N \in \Lambda$ such that $N \to_{\mathsf{v}}^* x$. By Corollary 26.1, there is $V' \in \Lambda_v$ such that $N \xrightarrow{h}{}^*_{\beta_v} V' \xrightarrow{int}{}^*_{\mathsf{v}} x$, hence $V' = x$ by Remark 12.5 again. Since $\xrightarrow{h}_{\beta_v} \subseteq \to_{\beta_v}$, $N \to_{\beta_v}^* x$ and thus $V = \lambda x.N \to_{\beta_v}^* I$, therefore M is β_v-solvable. ◀

So, due to Theorem 31, the semantic (via a relational model) and operational (via two sub-reductions of $\to_{\mathsf{v}}$) characterization of v-potential valuability and v-solvability given in [3, Theorems 24-25] is also a semantic and operational characterization of β_v-potential valuability and β_v-solvability. The difference is that in λ_v^σ these notions can be studied operationally inside the calculus, while it has been proved in [13, 14] that the β_v-reduction is too weak to characterize them: an operational characterization of β_v-potential valuability and β_v-solvability cannot be given inside λ_v. Hence, λ_v^σ is a useful, conservative and "complete" tool for studying semantic properties of λ_v.

6 Conclusions

In this paper we have proved a standardization theorem for the λ_v^σ-calculus introduced in [3]. The used technique is a notion of parallel reduction. Let us recall that parallel reduction in λ-calculus has been defined by Tait and Martin-Löf in order to prove confluence of the β-reduction, without referring to the difficult notion of residuals. Takahashi in [17] has simplified this technique and showed that it can be successfully applied to standardization. We would like to remark that our parallel reduction cannot be used to prove confluence of $\to_{\mathsf{v}}$. Indeed, take $M = (\lambda x.L)\big((\lambda y.N)((\lambda z.N')N'')\big)L'$, $M_1 = (\lambda x.LL')\big((\lambda y.N)((\lambda z.N')N'')\big)$ and $M_2 = \big((\lambda y.(\lambda x.L)N)((\lambda z.N')N'')\big)L'$: then $M \Rightarrow M_1$ and $M \Rightarrow M_2$ but there is no term M' such that $M_1 \Rightarrow M'$ and $M_2 \Rightarrow M'$. To sum up, $\Rightarrow$ does not enjoy the Diamond Property.

The standardization result allows us to formally verify the correctness of λ_v^σ with respect to the semantics of λ_v, so we can use λ_v^σ as a tool for studying properties of λ_v. This is a remarkable result: in fact some properties, like potential valuability and solvability, cannot be characterized in λ_v by means of β_v-reduction (as proved in [13, 14]), but they have a natural operational characterization in λ_v^σ (via two sub-reductions of $\to_{\mathsf{v}}$).

We plan to continue to explore the call-by-value computation, using λ_v^σ. As a first step, we would like to revisit and improve the Separability Theorem given in [11] for λ_v. Still the issue is more complex than in the call-by-name, indeed in ordinary λ-calculus different $\beta\eta$-normal forms can be separated (by the Böhm Theorem), while in λ_v there are different normal forms that cannot be separated, but which are only semi-separable (e.g. I and $\lambda z.(\lambda u.z)(zz)$). We hope to completely characterize separable and semi-separable normal forms in λ_v^σ. This should be a first step aimed to define a semantically meaningful notion of approximants. Then, we should be able to provide a new insight on the denotational analysis of the call-by-value, maybe overcoming limitations as that of the absence of fully abstract filter models [14, Theorem 12.1.25]. Last but not least, an unexplored but challenging research direction is the use of commutation rules to improve the call-by-value evaluation. We do not have concrete evidence supporting such possibility, but since λ_v^σ is strongly related to the calculi presented in [7, 1], which are endowed with explicit substitutions, we are confident that a sharp use of commutations can have a relevant impact in the evaluation.

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