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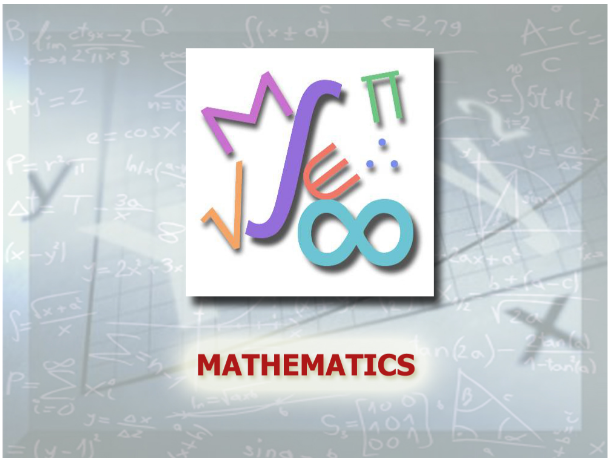

Non-integrable special geometric structures in dimensions six and seven

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NON-INTEGRABLE SPECIAL GEOMETRIC STRUCTURES IN DIMENSIONS SIX AND SEVEN

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XXVIII ciclo – March 4th, 2016

Ai miei genitori.
A Ginevra.

Abstract

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Six-dimensional manifolds admitting an SU(3)-structure and seven-dimensional manifolds endowed with a G_2-structure are the main object of study in this thesis.

In the six-dimensional case, we consider SU(3)-structures (ω, ψ_+) satisfying the condition $d\omega = c\,\psi_+$, $c \in \mathbb{R} - \{0\}$, known in literature as coupled. They are half-flat and generalize the class of nearly Kähler SU(3)-structures. We study their properties in the general case and in relation with the rôle they play in supersymmetric string theory, the conditions under which the associated metric is Einstein, their behaviour with respect to the Hitchin flow equations and various classes of examples.

In the seven-dimensional case, we focus on G_2-structures defined by a stable 3-form φ which is locally conformal equivalent to a closed one. We study the restrictions arising when the underlying metric is Einstein, we use warped products and the mapping torus construction to provide noncompact and compact examples of 7-manifolds endowed with such a structure starting from 6-manifolds with a coupled SU(3)-structure and, finally, we prove a structure result for compact 7-manifolds.

We conclude studying a generalization of the Hitchin flow equations and a geometric flow of spinors on 6-manifolds. The latter gives rise to a flow of SU(3)-structures.

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Contents

Introduction

"If I have seen further,
it is by standing on the shoulders of Giants"
I. Newton

In Riemannian geometry, the study of special geometric structures is closely related to holonomy theory. Let (M, g) be an m-dimensional Riemannian manifold, it is well-known that the Riemannian holonomy group $\mathrm{Hol}(g)$, namely the holonomy group of the Levi Civita connection ∇^g associated with the Riemannian metric g, is a subgroup of the orthogonal group $\mathrm{O}(m)$.

The classification of the possible Riemannian holonomy groups when (M, g) is simply connected and complete began with Cartan's classification of simply connected Riemannian symmetric spaces [36, 37] in the twenties of the last century and was achieved with the results of de Rham [58] and Berger [21] in the 1950s. In [58], it was shown that (M, g) is isometric to a product $(M_0 \times M_1 \times \cdots \times M_k, g_0 \times g_1 \times \cdots \times g_k)$ of simply connected and complete Riemannian manifolds such that (M_0, g_0) is flat, $\mathrm{Hol}(g_i)$ acts irreducibly on the tangent spaces of M_i for every $i = 1, \ldots, k$, and $\mathrm{Hol}(g)$ is isomorphic to $\mathrm{Hol}(g_1) \times \cdots \times \mathrm{Hol}(g_k)$, while in [21] the list of all possible holonomy groups for irreducible and non-symmetric Riemannian manifolds was obtained. In detail, $\mathrm{Hol}(g)$ must be one of $\mathrm{SO}(m), \mathrm{U}(n), \mathrm{SU}(n)$ when $m = 2n \geq 4$, $\mathrm{Sp}(n)\mathrm{Sp}(1), \mathrm{Sp}(n)$ when $m = 4n \geq 8$, G_2 when $m = 7$ and $\mathrm{Spin}(7)$ when $m = 8$. The proof that all of the groups appearing in Berger's list actually occur as Riemannian holonomy groups was completed in the 1980s with Bryant and Salamon's first examples of (complete) metrics with holonomy G_2 and $\mathrm{Spin}(7)$ [29, 32].

Given a Lie subgroup G of $\mathrm{GL}(m,\mathbb{R})$, a G-*structure* on M is by definition a reduction of the structure group of the frame bundle FM from $\mathrm{GL}(m,\mathbb{R})$ to G, that is, a principal subbundle Q of FM with typical fiber G. If G is one of the groups appearing in Berger's list except $\mathrm{SO}(m)$, the corresponding G-structures are called *special geometric structures*.

Whenever $\mathrm{G} \subseteq \mathrm{O}(m)$, a G-structure Q on M gives rise to a Riemannian metric g on it and some extra geometric data, which are usually represented by certain tensor fields on M whose common stabilizer at each point of the manifold is G. Actually, the existence of such tensor fields is equivalent to a reduction of the structure group of FM to G. This is exactly what happens for every special geometric structure.

The obstruction for the Riemannian holonomy group $\mathrm{Hol}(g)$ to reduce to G is represented by the so-called *intrinsic torsion* τ of the G-structure, which is a section of a vector bundle over M with typical fiber the G-module $(\mathbb{R}^m)^* \otimes \mathfrak{g}^\perp$, being $\mathfrak{g}^\perp$ the orthogonal complement of the Lie algebra $\mathfrak{g}$ of G in $\mathfrak{so}(m)$ with respect to the Killing form. More precisely, such a reduction is characterized by the vanishing of τ. Furthermore, when the G-structure is defined by some tensor fields, τ can be identified with their covariant derivatives with respect to the Levi Civita connection ∇^g and, consequently, the holonomy reduction holds when they are all ∇^g-parallel. A G-structure with identically vanishing intrinsic torsion is called *torsion-free*, while it is said to be *non-integrable* otherwise.

In this thesis, we are mainly interested in 6-manifolds endowed with an SU(3)-structure, evolution equations of SU(3)-structures and 7-manifolds admitting a G_2-structure. In the following, we briefly review these topics explaining the motivations of our study.

An SU(3)-*structure* on a six-dimensional manifold M can be defined by the data of a 2-form ω and a 3-form ψ_+ which are stable in the sense of [101, 161]: at each point p of M their orbit under the action of the general linear group $\mathrm{GL}(T_pM)$ is open and there exists a basis $(e^1,\ldots,e^6)$ of the cotangent space T_p^*M such that

$$\omega = e^{12} + e^{34} + e^{56}, \quad \psi_+ = e^{135} - e^{146} - e^{236} - e^{245},$$

where $e^{ijk\cdots}$ is a shorthand for the wedge product $e^i \wedge e^j \wedge e^k \wedge \cdots$ of 1-forms.

On the manifold M, the pair (ω, ψ_+) gives rise to an almost complex structure J, to a complex volume form $\Psi = \psi_+ + i\psi_-$ of nonzero constant length, where $\psi_- := J\psi_+$, and to a Riemannian metric defined by $g(\cdot,\cdot) = \omega(\cdot, J\cdot)$.

The intrinsic torsion τ of an SU(3)-structure (ω, ψ_+) is a section of a rank 42 vector bundle $\mathcal{W}$ over M with typical fiber $(\mathbb{R}^6)^* \otimes \mathfrak{su}(3)^\perp$ and can be identified with the covariant derivatives $\nabla^g\omega$, $\nabla^g\Psi$. The decomposition of the SU(3)-module $(\mathbb{R}^6)^* \otimes \mathfrak{su}(3)^\perp$ into SU(3)-irreducible summands induces a splitting

$$\mathcal{W} = \mathcal{W}_1^+ \oplus \mathcal{W}_1^- \oplus \mathcal{W}_2^+ \oplus \mathcal{W}_2^- \oplus \mathcal{W}_3 \oplus \mathcal{W}_4 \oplus \mathcal{W}_5$$

and the intrinsic torsion decomposes accordingly. This gives rise to 128 classes of SU(3)-structures, which are defined in terms of the identically vanishing components of τ and can be characterized by the expressions of $d\omega$, $d\psi_+$ and $d\psi_-$, as shown by Chiossi and Salamon in [40]. When ω, ψ_+ and ψ_- are all closed, the SU(3)-structure is torsion-free, the Riemannian holonomy group $\mathrm{Hol}(g)$ is a subgroup of SU(3) and g is Ricci-flat, i.e., its Ricci tensor $\mathrm{Ric}(g)$ vanishes identically.

One of the most remarkable classes of SU(3)-structures is defined by the equations

$$d\omega \wedge \omega = 0, \quad d\psi_+ = 0,$$

which constrain the intrinsic torsion to lie in the rank 21 vector bundle $\mathcal{W}_1^- \oplus \mathcal{W}_2^- \oplus \mathcal{W}_3$. Since the rank is exactly half of the rank of $\mathcal{W}$, such structures have been named *half-flat* [40] or *half-integrable* [42]. By [40, 102], every oriented hypersurface of a Riemannian 7-manifold with holonomy in G_2 is naturally endowed with a half-flat SU(3)-structure and, conversely, a six-dimensional manifold with a real analytic half-flat SU(3)-structure can be realized as a hypersurface of a 7-manifold with holonomy in G_2 [31, 102]. The latter result is proved studying the following system of evolution equations for an SU(3)-structure $(\omega(t), \psi_+(t))$ depending on a parameter $t \in \mathrm{I} \subseteq \mathbb{R}$

$$\begin{cases} \frac{\partial}{\partial t}\psi_+(t) = d\omega(t) \\ \frac{\partial}{\partial t}\omega(t) \wedge \omega(t) = -d\psi_-(t) \end{cases}.$$

Such evolution equations, introduced by Hitchin in [102] and now commonly known as *Hitchin flow equations*, are not a geometric flow in the usual sense, as the evolution of $\omega(t)$ and $\psi_+(t)$ is not described by partial differential equations of the form

$$\frac{\partial}{\partial t}\omega(t) = L_t(\omega(t)), \quad \frac{\partial}{\partial t}\psi_+(t) = K_t(\psi_+(t)),$$

where L_t, K_t are certain differential operators depending smoothly on t. In fact, to our knowledge, up to now in literature there are no known examples of geometric flows evolving SU(3)-structures.

An almost Hermitian manifold (M, g, J) is said to be *strict nearly Kähler* if

$$(\nabla^g_X J)X = 0,$$

for every $X \in \mathfrak{X}(M)$, and $\nabla^g_X J \neq 0$ for all non-vanishing $X \in \mathfrak{X}(M)$. By [162], in dimension six the structure group of FM admits a natural reduction to SU(3) and the corresponding SU(3)-structure (ω, ψ_+) can be characterized by the conditions

$$d\omega = 3\,\psi_+, \quad d\psi_- = -2\,\omega^2.$$

Since ψ_+ is exact and the compatibility condition $\omega \wedge \psi_+ = 0$ always holds, nearly Kähler SU(3)-structures are half-flat and their intrinsic torsion belongs to $\mathcal{W}_1^-$. Moreover, the associated Riemannian metric g is Einstein [90, 144], that is, $\mathrm{Ric}(g) = \mu g$ for some real number μ.

Nearly Kähler manifolds have been widely studied in literature, for instance in [88, 90], and the relevance of the six-dimensional case is evident from the results of [149, 150], where the author proved that a complete and simply connected nearly Kähler manifold is locally a Riemannian product of Kähler manifolds, twistor spaces over Kähler manifolds and six-dimensional nearly Kähler manifolds.

Up to now, very few examples of manifolds endowed with a nearly Kähler SU(3)-structure are known. In the homogeneous case there are only finitely many of them by [34], namely the 6-sphere S^6, the product of 3-spheres $S^3 \times S^3$, the complex projective space $\mathbb{CP}^3$ and the flag manifold $\mathbb{F}(1, 2)$, and each one carries a unique invariant nearly Kähler SU(3)-structure up to homothety. Moreover, the existence of new non-homogeneous examples was recently proved on S^6 and $S^3 \times S^3$ in [73].

Because of the rareness of examples, one may weaken the defining conditions of the class $\mathcal{W}_1^-$ and study the resulting SU(3)-structures in relation to the properties of nearly Kähler SU(3)-structures. In the half-flat class there are two natural subclasses allowing to do this: the class $\mathcal{W}_1^- \oplus \mathcal{W}_3$ of *double half-flat* and the class $\mathcal{W}_1^- \oplus \mathcal{W}_2^-$ of *coupled* SU(3)-structures.

Properties and explicit examples of double half-flat SU(3)-structures were studied for instance in [42, 137, 165, 167]. By [42, 167], they are defined by the conditions

$$d\psi_+ = 0, \quad d\psi_- = k\,\omega^2,$$

for some nonzero real number k, and can be characterized as the half-flat SU(3)-structures having totally skew-symmetric Nijenhuis tensor.

Natural spaces motivating the study of coupled SU(3)-structures, named in this way in [164], are $S^3 \times S^3$ and the twistor spaces over self-dual Einstein 4-manifolds of positive scalar curvature, where also nearly Kähler SU(3)-structures exist [147]. Moreover, a further motivation comes from supersymmetric string theory, since a necessary and sufficient condition for $\mathcal{N} = 1$ compactifications of Type IIA string theory on spaces of the form $\mathrm{AdS}_4 \times M$, where AdS_4 is the four-dimensional anti-de Sitter space, is that the internal compact 6-manifold M is endowed with a coupled SU(3)-structure satisfying some additional properties [118, 136]. We will see later that coupled SU(3)-structures are also useful to construct examples and to study the properties of manifolds endowed with a certain class of G_2-structures.

A G_2*-structure* on a seven-dimensional manifold M is defined by a stable 3-form φ which can be pointwise written as

$$\varphi = e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245},$$

with respect to a basis $(e^1, \ldots, e^7)$ of T_p^*M. Such a 3-form induces a Riemannian metric g_φ and a volume form dV_φ on the manifold via the identity

$$g_\varphi(X,Y)dV_\varphi = \frac{1}{6}\,(\iota_X\varphi) \wedge (\iota_Y\varphi) \wedge \varphi,$$

for every pair of vector fields X, Y on M.

The intrinsic torsion of a G_2-structure φ can be identified with the covariant derivative $\nabla^{g_\varphi}\varphi$. In [67], Fernández and Gray showed that the G_2-module $\mathcal{X}$ of tensors satisfying the same symmetries as $\nabla^{g_\varphi}\varphi$ decomposes into a direct sum of four G_2-irreducible submodules

$$\mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \mathcal{X}_3 \oplus \mathcal{X}_4,$$

allowing to divide G_2-structures into 16 classes characterized by the identically vanishing components of $\nabla^{g_\varphi}\varphi$. Denoted by $*_\varphi$ the Hodge operator determined by g_φ and dV_φ, it is also possible to show that each class can be completely characterized by the expressions of $d\varphi$ and $d*_\varphi\varphi$, as they contain the same informations on the intrinsic torsion as $\nabla^{g_\varphi}\varphi$ [30, 40, 141]. Torsion-free G_2-structures φ are then equally defined by the condition $\nabla^{g_\varphi}\varphi = 0$ or by $d\varphi = 0, d*_\varphi\varphi = 0$, their underlying Riemannian metric g_φ has holonomy contained in G_2 and is Ricci flat [26].

Noncompact examples of 7-manifolds endowed with a G_2-structure can be constructed starting from a 6-manifold M endowed with an $\mathrm{SU}(3)$-structure (ω, ψ_+) and considering the *warped product* $M \times \mathrm{I}$ endowed with the metric $f(t)g + dt^2$, where g is the metric induced by (ω, ψ_+), t is the coordinate on the interval $\mathrm{I} \subseteq \mathbb{R}$ and f is a positive real-valued function defined on I. For instance, $M \times \mathbb{R}$ admits the G_2-structure

$$\varphi = \omega \wedge dt + \psi_+$$

inducing the cylindrical metric $g_\varphi = g + dt^2$, while $M \times (0, +\infty)$ is endowed with the G_2-structure

$$\varphi = t^2\,\omega \wedge dt + t^3\,\psi_+,$$

whose underlying metric is the conical one, namely $g_\varphi = t^2\,g + dt^2$.

Compact examples can be obtained considering the *mapping torus* of a diffeomorphism ν of a compact 6-manifold M, that is, the compact 7-manifold M_ν defined as the quotient of $M \times \mathbb{R}$ by the infinite cyclic group of diffeomorphisms generated by

$$\begin{array}{rccc} \widetilde{\nu}: & M \times \mathbb{R} & \longrightarrow & M \times \mathbb{R} \\ & (p,t) & \longmapsto & (\nu(p), t+1) \end{array}.$$

In detail, when ν is an automorphism of an $\mathrm{SU}(3)$-structure (ω, ψ_+) on M, i.e., when $\nu^*\omega = \omega$ and $\nu^*\psi_+ = \psi_+$, then a 2-form $\widetilde{\omega}$ and a 3-form $\widetilde{\psi}_+$ are naturally induced by ω and ψ_+ on the mapping torus M_ν and the 3-form

$$\varphi = \widetilde{\omega} \wedge \eta + \widetilde{\psi}_+$$

defines a G_2-structure on it, where η is the closed 1-form on M_ν induced by the 1-form dt on $\mathbb{R}$. In this case, M_ν is the total space of a fibration over the circle S^1 and each fiber is endowed with an SU(3)-structure. Observe that if ν is the identity diffeomorphism, then M_ν is none other than the product $M \times S^1$.

The intrinsic torsion of a G_2-structure defined by a closed 3-form φ lies in $\mathcal{X}_2$. In this case, the G_2-structure is called *closed* or *calibrated*, as it defines a calibration on the manifold by [95], and can be seen as the G_2-analogue of an *almost Kähler* structure, i.e., an almost Hermitian structure (g, J) whose fundamental form ω is closed. The first example of compact 7-manifold endowed with a calibrated G_2-structure was given by Fernández in [64], while, more recently, examples of invariant calibrated G_2-structures were obtained in [50] on compact nilmanifolds, i.e., compact quotients of simply connected nilpotent Lie groups by a lattice. Examples of calibrated G_2-structures on mapping tori of 6-manifolds endowed with an SU(3)-structure whose defining forms ω and ψ_+ are both closed, known as *symplectic half-flat*, were constructed in [140].

The geometry of calibrated G_2-structures was studied in [47] by Cleyton and Ivanov. Furthermore, Bryant proved in [30] that the scalar curvature of the metric underlying a calibrated G_2-structure is nonpositive and vanishes identically if and only if the G_2-structure is torsion-free.

A G_2-structure φ is said to be Einstein if the underlying Riemannian metric g_φ is Einstein. As an analogous of Goldberg conjecture for almost-Kähler manifolds [83], in [30, 47] it was proved that on a compact manifold every Einstein calibrated G_2-structure is necessarily torsion-free. In the noncompact homogeneous case, it was recently shown that a seven-dimensional solvmanifold cannot admit any left-invariant Einstein calibrated G_2-structure unless g_φ is flat [65].

A G_2-structure whose defining 3-form φ is locally conformal equivalent to a closed stable 3-form is called *locally conformal calibrated* and is characterized by the condition

$$d\varphi = -\theta \wedge \varphi,$$

for a unique closed 1-form θ, known as the Lee form of φ. In this case, the intrinsic torsion belongs to $\mathcal{X}_2 \oplus \mathcal{X}_4$ and the G_2-structure generalizes both calibrated and *locally conformal parallel* G_2-structures, namely those satisfying

$$d\varphi = -\theta \wedge \varphi, \quad d *_\varphi \varphi = -\frac{4}{3}\,\theta \wedge *_\varphi\varphi$$

and whose intrinsic torsion lies in $\mathcal{X}_4$. Seven-dimensional manifolds endowed with a locally conformal calibrated G_2-structure are called *locally conformal calibrated* G_2*-manifolds* and represent the G_2-analogue of *locally conformal symplectic manifolds* [176], that is, even-dimensional manifolds endowed with a non-degenerate 2-form ω which is locally conformal equivalent to a symplectic 2-form. In literature, properties of locally conformal calibrated G_2-manifolds were studied in [69].

Because of the local conformal equivalence between 3-forms defining a calibrated G_2-structure and those defining a locally conformal calibrated one, it is natural to ask whether the results of [30, 47, 65] previously recalled extend to manifolds endowed with an Einstein locally conformal calibrated G_2-structure. It is also interesting to study the conditions for which the mapping torus of a 6-manifold endowed with an $\mathrm{SU}(3)$-structure is a locally conformal calibrated G_2-manifold and to describe the geometry of a compact locally conformal calibrated G_2-manifold. Known results motivating this are for instance those of [104], where a characterization of compact locally conformal parallel G_2-manifolds as fiber bundles over S^1 with compact nearly Kähler fiber was obtained, and those of [13, 176], where fibration results for compact locally conformal symplectic manifolds were established.

We now summarize the content of the thesis, describing the main results.

The first chapter is mainly an overview of well-known topics on which the content of the thesis is based. We begin recalling basic definitions and properties about manifolds, vector bundles, principal bundles and holonomy theory, in order to fix the notations and clarify some conventions we use. We then review G-structures, explaining more in detail the results outlined in the first part of this introduction and describing some explicit examples. After doing this, we consider homogeneous Riemannian manifolds, focusing our attention on the properties of (compact) nilmanifolds and solvmanifolds. Moreover, we introduce the notations used for real Lie algebras and we make some observations on metric Lie algebras. We conclude the chapter with a review of Einstein and Ricci soliton metrics. Related results by Heber

[98] and Lauret [127] on Einstein solvmanifolds and by Lauret [124, 125] on Ricci nilsolitons, i.e., left-invariant Ricci soliton metrics on simply connected nilpotent Lie groups, are also recalled.

In the second chapter, we consider 6-manifolds endowed with an SU(3)-structure. General results on this topic are reviewed in detail in the first part, while in the second part we focus on special half-flat SU(3)-structures, namely SU(3)-structures whose class is contained in $\mathcal{W}_1^- \oplus \mathcal{W}_2^- \oplus \mathcal{W}_3$. In Section 2.3, we discuss the results on nearly Kähler manifolds previously sketched and we add some details for those on double half-flat. Then, in Section 2.4, we consider the class $\mathcal{W}_1^- \oplus \mathcal{W}_2^-$ of coupled SU(3)-structures, proving that on a connected manifold M it is characterized by the condition

$$d\omega = c\,\psi_+$$

for some real constant c (Proposition 2.4.2). It is then evident that coupled SU(3)-structures with nonzero *coupled constant* c are a natural generalization of nearly Kähler SU(3)-structures and, as it happens for nearly Kähler, they are completely determined by the 2-form ω. Consequently, a diffeomorphism ν of M such that $\nu^*\omega = \omega$ is an automorphism of the coupled SU(3)-structure (Corollary 2.4.3). Moreover, the almost Hermitian structure (g, J) underlying a coupled structure is quasi Kähler, i.e., its fundamental form ω is $\overline{\partial}$-closed (Proposition 2.4.6). In Section 2.4.1, we look for examples of manifolds endowed with a coupled SU(3)-structure. We classify six-dimensional non-Abelian nilpotent Lie algebras admitting a coupled SU(3)-structure, showing that up to isomorphism only two cases occur (Theorem 2.4.12). One of them, namely the real Lie algebra underlying the complex Heisenberg group of complex dimension three, is endowed with a coupled SU(3)-structure whose associated metric is a Ricci nilsoliton. We prove that this is the unique example of this kind (Proposition 2.4.14). We conclude giving an example of left-invariant coupled SU(3)-structure on the Lie group $\mathrm{SU}(2) \times \mathrm{SU}(2)$. As the half-flat condition is preserved by the Hitchin flow equations, in Section 2.4.2 we examine the behaviour of coupled SU(3)-structures with respect to this flow. We characterize solutions of the Hitchin flow equations starting from a coupled SU(3)-structure and remaining coupled as long as they exist (Proposition 2.4.17), we review a known example in the light of this characterization and we prove that such solutions need not to exist in general (Proposition 2.4.19). In Section 2.4.3, we study the properties of coupled SU(3)-structures which are of interest in supersymmetric string theory, discussing also explicit examples. The chapter ends with Section 2.5, where we consider the problem of finding (special) half-flat SU(3)-structures whose associated metric is Einstein. We begin with the homogeneous manifold $S^3 \times S^3$ identified with the Lie group $\mathrm{SU}(2) \times \mathrm{SU}(2)$, proving that it does not admit any left-invariant coupled SU(3)-structure inducing one of the currently known Einstein metrics existing on it (Theorem 2.5.5) and giving an example of a left-invariant half-flat SU(3)-structure with intrinsic torsion in $\mathcal{W}_1^- \oplus \mathcal{W}_2^- \oplus \mathcal{W}_3$ and whose associated metric is Einstein. Then, we consider twistor spaces over oriented Riemannian 4-manifolds, reviewing some known results which allow to prove the existence of coupled SU(3)-structures whose underlying metric is Einstein. Finally, we move to the noncompact homogeneous case, where we prove that there are no coupled SU(3)-structures inducing the Einstein metric on Einstein solvmanifolds (Theorem 2.5.16) and on homogeneous Einstein manifolds of nonpositive sectional curvature (Corollary 2.5.19). In some cases, the result is stronger and holds for all half-flat SU(3)-structures.

The third chapter is devoted to the study of locally conformal calibrated G_2-manifolds. After reviewing the general properties of G_2-structures, their classification and their relation with SU(3)-structures in Sections 3.1 and 3.2, we focus our attention on the class $\mathcal{X}_2 \oplus \mathcal{X}_4$. In Section 3.3, we discuss the main properties of locally conformal calibrated G_2-structures and we look for new examples. We show that noncompact examples of locally conformal calibrated G_2-manifolds can be constructed on the cylinder over a 6-manifold endowed with a coupled SU(3)-structure and also on the cone when the coupled constant c is not 3, generalizing known results for nearly Kähler SU(3)-structures (Proposition 3.3.8). We then study the conditions under which the mapping torus of an automorphism of an SU(3)-structure is a locally conformal calibrated G_2-manifold, proving that it suffices to consider a 6-manifold M endowed with a coupled SU(3)-structure (ω, ψ_+) and a diffeomorphism ν of M such that $\nu^*\omega = \omega$ to obtain the result (Proposition 3.3.11). When the coupled SU(3)-structure is nearly Kähler, we prove that the same hypothesis gives rise to a mapping torus endowed with a locally conformal parallel G_2-structure (Proposition 3.3.14). Finally, we state necessary and sufficient conditions guaranteeing the existence of a locally conformal calibrated G_2-structure on seven-dimensional Lie algebras obtained as rank-one extensions of six-dimensional Lie algebras endowed with a coupled SU(3)-structure (Propositions 3.3.17 and 3.3.20). Using these results, we are able to give two new examples of invariant locally conformal calibrated G_2-structures on compact solvmanifolds, namely compact quotients of simply connected solvable Lie groups by a lattice. In Section 3.4, we study the restrictions imposed by requiring that the Riemannian metric underlying a locally conformal calibrated G_2-structure φ is Einstein. In the compact case, we show that the scalar curvature of g_φ is nonpositive (Theorem 3.4.4) and, as a consequence, we prove that a seven-dimensional compact homogeneous manifold cannot admit any invariant Einstein locally conformal calibrated G_2-structure unless the underlying metric is flat (Corollary 3.4.5). In contrast to the compact case, we construct a noncompact example of left-invariant Einstein (non-flat) locally conformal calibrated G_2-structure on a seven-dimensional solvmanifold and we give a noncompact example of a locally conformal calibrated G_2-structure whose associated metric is Ricci-flat. The geometry of compact locally conformal calibrated G_2-manifolds is studied in Section 3.5. Here, we first discuss the conditions under which the 3-form φ defining a locally conformal calibrated G_2-structure with Lee form θ can be expressed as $\varphi = d\beta + \theta \wedge \beta$ for some 2-form β. This requirement, which implies in particular that $d\varphi = -\theta \wedge \varphi$, is motivated by the results of [13, 176] mentioned earlier, as they are proved for locally conformal symplectic manifolds whose non-degenerate 2-form ω satisfies a similar identity. Then, we prove that a compact locally conformal calibrated G_2-manifold (M, φ) with non-vanishing Lee form θ such that $\mathcal{L}_{\theta^\sharp}\varphi = 0$ is fibered over S^1 and each fiber is endowed with a coupled SU(3)-structure (Theorem 3.5.17). This is a partial converse of the mapping torus construction stated above.

In the last chapter, we study evolution equations of SU(3)-structures. In Section 4.1, we consider a generalization of the Hitchin flow equations introduced in the physical paper [57] and we show that it can be used to define a system of evolution equations for an SU(3)-structure preserving the class $\mathcal{W}_1^- \oplus \mathcal{W}_2^-$ of coupled SU(3)-structures (Proposition 4.1.7). Geometric flows are reviewed in Section 4.2, where we recall the fundamental definitions and a result guaranteeing the short-time existence and uniqueness of solutions of an initial value problem. In Section 4.3, we explain some ideas which could be useful to study the open problem regarding the existence of geometric flows evolving SU(3)-structures, we summarize them here. It is well-known that a Riemannian 6-manifold (M, g) is endowed with an SU(3)-structure if and only if it is orientable and admits a spin structure [129]. As a consequence, it is possible to consider the *spinor bundle* ΣM over M, which is a complex vector bundle with typical fiber $\mathbb{C}^8$, and show that there is a correspondence between the differential forms (ω, ψ_+) defining an SU(3)-structure and unit real spinor fields $\phi \in \Gamma(\Sigma M)$. Since the correspondence is one-to-one up to a sign in the definition of ϕ, instead of studying evolution equations for the differential forms ω and ψ_+, we look for flows evolving ϕ. The advantage of this approach is that we have to control only one object instead of two objects and the compatibility conditions they have to satisfy. The evolution equation we consider is the following

$$\frac{\partial}{\partial t}\phi(t) = -D^2\phi(t),$$

where D is the Dirac operator of the Riemannian spin manifold (M, g). As $-D^2$ is a strongly elliptic second-order differential operator, the previous equation is strictly parabolic and the short-time existence and uniqueness of solutions for a given unit real spinor field $\phi_0 \in \Gamma(\Sigma M)$ is guaranteed on compact manifolds (Theorem 4.3.16). The solution $\phi(t)$ is non-vanishing as long as it exists, as it depends smoothly on t and being non-vanishing is an open condition. It is then possible to normalize it using the real metric of ΣM and get an SU(3)-structure on M depending on t. Therefore, the flow at the spinor level translates into a flow of SU(3)-structures on M leaving the metric g fixed. This argument fails when ϕ_0 is an eigenspinor of the Dirac operator with constant eigenfunction. Indeed, in this case the solution $\phi(t)$ is just a rescaling of ϕ_0 and the normalization is ϕ_0 itself (Proposition 4.3.18). This happens, for instance, when ϕ_0 corresponds to a coupled SU(3)-structure with coupled constant c, as it satisfies

$$D\phi_0 = -c\,\phi_0.$$

We conclude the section discussing two examples of solutions on Lie algebras.

The computations on Lie algebras have been done with the aid of the software Maple 18, its packages *difforms*, *LinearAlgebra*, *PolynomialIdeals* (only for the proof of Theorem 2.5.5), and some Maple procedures written by the author.

The original results collected in this thesis are contained in the papers [66, 70, 71, 160] and in the work-in-progress [72].

Chapter 1

Preliminaries

In this chapter, we review the main topics on which the content of this thesis is based. After recalling some basic facts about manifolds and fiber bundles, we describe the foremost properties of G-structures and of homogeneous Riemannian manifolds, paying particular attention to the case of nilmanifolds and solvmanifolds. Finally, we consider two classes of Riemannian metrics satisfying remarkable properties, namely Einstein and Ricci soliton metrics, and we discuss some related results.

Since most of the results appearing here are well-known, instead of proving every assertion, we will suggest one or more references where the reader can find the proofs and more details on the topics.

1.1 Basics, notations and conventions

This section is mainly a summary of fundamental definitions and properties in differential and Riemannian geometry. Several notations and conventions used in this thesis are introduced here.

1.1.1 Smooth manifolds, vector bundles and tensor fields

A *smooth manifold* M of dimension m is an m-dimensional topological manifold admitting a (maximal) differentiable atlas of class C^∞. The manifolds considered in this thesis will be always assumed to be smooth.

A *vector bundle* E of *rank* k over M is a fiber bundle $\pi : E \to M$ whose fibers are vector spaces of the same dimension k. In particular, each point p of M has an open neighborhood $\mathcal{U} \subseteq M$ such that $\pi^{-1}(\mathcal{U})$ is diffeomorphic to $\mathcal{U} \times V$, where the k-dimensional vector space V is the *typical fiber* of E. The space of (*smooth*) *sections* of E, i.e., the smooth maps $\sigma : M \to E$ such that $\sigma_p := \sigma(p)$ belongs to the fiber $E_p := \pi^{-1}(p)$ of E over p, is denoted by $\Gamma(E)$. A section $\sigma \in \Gamma(E)$ is said to be *vanishing* at a point p of M if $\sigma(p) = 0$, *identically vanishing* if it is vanishing at each point of the manifold and *non-vanishing* if it is nowhere vanishing.

T_pM denotes the *tangent space* to M at a point p, while T_p^*M denotes the *cotangent space* at p, namely the dual vector space of T_pM. The bundle T_s^rM of *r-contravariant s-covariant tensors*, or (r,s)-*tensors* for short, is the vector bundle over M whose fiber over each point p is the vector space $T_s^r(T_pM) = (T_pM)^{\otimes r} \otimes (T_p^*M)^{\otimes s}$ of (r,s)-tensors on T_pM. The *tangent bundle* is $TM = T_0^1M$ and the *cotangent bundle* is $T^*M = T_1^0M$, while $T_0^0M = M \times \mathbb{R}$.

The space $\Gamma(T_s^rM)$ of smooth sections of T_s^rM is alternatively denoted by $\mathcal{T}_s^r(M)$ and its elements are called (r,s)-*tensor fields* on M. In particular, $\mathfrak{X}(M) := \mathcal{T}_0^1(M)$ is the space of *vector fields* on M and $\Omega^1(M) := \mathcal{T}_1^0(M)$ is the space of *covector fields* (or 1-*forms*) on M. More in general, it is possible to consider the vector bundle $\Lambda^k(T^*M)$ of antisymmetric $(0,k)$-tensors over M for each $0 \le k \le m$, where $\Lambda^0(T^*M) = M \times \mathbb{R}$ and $\Lambda^1(T^*M) = T^*M$, and define the space of *differential k-forms* on M as $\Omega^k(M) := \Gamma(\Lambda^k(T^*M))$. Clearly, a 0-form is just a smooth real valued function defined on M, thus $\Omega^0(M) = C^\infty(M)$. The *wedge product* of two differential forms $\omega \in \Omega^k(M)$ and $\eta \in \Omega^l(M)$ is a $(k+l)$-form $\omega \wedge \eta$ on M defined by

$$(\omega \wedge \eta)_p = \omega_p \wedge \eta_p = \frac{(k+l)!}{k!\, l!} \operatorname{Alt}(\omega_p \otimes \eta_p),$$

being $\operatorname{Alt} : T_k^0(T_pM) \to \Lambda^k(T_p^*M)$ the *alternating projection* sending a $(0,k)$-tensor σ_p on T_pM to its antisymmetric part

$$\operatorname{Alt}(\sigma_p) = \frac{1}{k!} \sum_{\zeta \in \mathfrak{S}_k} \operatorname{sgn}(\zeta)\, {}^{\zeta}\sigma_p,$$

where $\operatorname{sgn}(\zeta)$ is the sign of the permutation $\zeta \in \mathfrak{S}_k$ and, for any k vectors $X_{i_1}, \dots, X_{i_k}$

of T_pM, ${}^{\zeta}\sigma_p\left(X_{i_1},\ldots,X_{i_k}\right)=\sigma_p\left(X_{i_{\zeta(1)}},\ldots,X_{i_{\zeta(k)}}\right)$. We use the notation

$$\omega^n=\underbrace{\omega\wedge\cdots\wedge\omega}_{n}$$

as a shortening for the wedge product of a k-form ω by itself for n-times.

The vector bundle of symmetric $(0,k)$-tensors is denoted by $\mathcal{S}^k(T^*M)$ and the space of its sections by $\mathcal{S}^k(M)$. Given two *symmetric tensor fields* $\alpha\in\mathcal{S}^k(M)$ and $\beta\in\mathcal{S}^l(M)$, their *symmetric product* is the symmetric tensor field $\alpha\beta\in\mathcal{S}^{k+l}(M)$ defined as

$$(\alpha\beta)_p=\alpha_p\beta_p=\mathrm{Sym}(\alpha_p\otimes\beta_p),$$

where $\mathrm{Sym}:T^0_k(T_pM)\to\mathcal{S}^k(T^*_pM)$ is the *symmetrization*

$$\mathrm{Sym}(\sigma_p)=\frac{1}{k!}\sum_{\zeta\in\mathfrak{S}_k}{}^{\zeta}\sigma_p.$$

The shortening

$$(\alpha)^n=\underbrace{\alpha\cdots\alpha}_{n}$$

is used to denote the symmetric product of a symmetric tensor by itself for n-times.

The *differential* at $p\in M$ of a smooth map $F:M\to N$ between two manifolds M and N is denoted by $F_{*p}:T_pM\to T_{F(p)}N$, while the *pullback* is denoted by $F^*:T^*_{F(p)}N\to T^*_pM$. If ω is a $(0,s)$-tensor field on N, then its pullback by F is a $(0,s)$-tensor field on M defined in the following way

$$(F^*\omega)_p(X_1,\ldots,X_s)=\omega_{F(p)}(F_{*p}X_1,\ldots,F_{*p}X_s),\quad p\in M,\quad X_i\in T_pM.$$

A (*linear*) *connection* on a vector bundle E over M is an $\mathbb{R}$-linear map $\nabla:\Gamma(E)\to\Gamma(T^*M\otimes E)$ such that for all $f\in C^\infty(M)$ and $\sigma\in\Gamma(E)$

$$\nabla(f\sigma)=f\nabla\sigma+df\otimes\sigma.$$

Given $X\in\mathfrak{X}(M)$, the *covariant derivative* of $\sigma\in\Gamma(E)$ in the direction of X is

$$\nabla_X\sigma:=\nabla\sigma(X)\in\Gamma(E).$$

A connection ∇ on the tangent bundle TM over M induces a connection on every vector bundle $T^r_s M$, which is denoted by the same symbol. In particular, it is possible to compute the covariant derivative of any tensor field on M.

The *torsion* $T(\nabla)$ of a connection ∇ on TM is the $(1,2)$-tensor field on M defined by

$$T(\nabla)(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y],$$

for all $X, Y \in \mathfrak{X}(M)$. Since $T(\nabla)(X,Y) = -T(\nabla)(Y,X)$, it is actually a section of the bundle $\Lambda^2(T^*M) \otimes TM$. A connection ∇ is said to be *torsion-free* if $T(\nabla)$ vanishes identically.

A *Riemannian metric* on M is a symmetric tensor field $g \in \mathcal{S}^2(M)$ such that g_p is an inner product on the vector space T_pM for each point p of M. The pair (M,g) is called *Riemannian manifold*. On its tangent bundle there always exists a unique connection ∇^g, the *Levi Civita connection*, which is *metric* (or *compatible* with g), i.e.,

$$\left(\nabla^g_X g\right)(Y,Z) = \nabla^g_X(g(Y,Z)) - g\left(\nabla^g_X Y, Z\right) - g\left(Y, \nabla^g_X Z\right) = 0,$$

for all $X, Y, Z \in \mathfrak{X}(M)$, and has identically vanishing torsion $T(\nabla^g)$.

The *Riemannian curvature endomorphism* of (M,g) is the (1,3)-tensor field defined by

$$R^g(X,Y)Z := \nabla^g_X\left(\nabla^g_Y Z\right) - \nabla^g_Y\left(\nabla^g_X Z\right) - \nabla^g_{[X,Y]} Z,$$

for all $X, Y, Z \in \mathfrak{X}(M)$. The manifold (M,g) is *flat*, i.e., locally isometric to the Euclidean space $\mathbb{R}^m$, if and only if R^g vanishes identically.

The *Ricci curvature tensor* of g is the symmetric $(0,2)$-tensor field on M

$$\mathrm{Ric}(g)(Y,Z) := \mathrm{tr}(X \mapsto R^g(X,Y)Z).$$

Moreover, it is possible to define a $(1,1)$-tensor field $\mathrm{Rc}(g)$ from $\mathrm{Ric}(g)$, called *Ricci operator*, via the identity

$$\mathrm{Ric}(g)(Y,Z) = g(\mathrm{Rc}(g)(Y), Z).$$

Finally, the *scalar curvature* of g is a smooth function obtained taking the trace of $\mathrm{Ric}(g)$ with respect to the metric g

$$\mathrm{Scal}(g) := \mathrm{tr}_g(\mathrm{Ric}(g)).$$

A Riemannian metric g can be extended to a fiber metric on each tensor bundle $T^r_s M$ over M. It is defined in any local coordinate system by

$$g(\sigma,\beta) = g^{i_1 j_1} \cdots g^{i_s j_s} \sigma^{k_1 \cdots k_r}_{i_1 \cdots i_s} \beta^{l_1 \cdots l_r}_{j_1 \cdots j_s} g_{k_1 l_1} \cdots g_{k_r l_r},$$

for all $\sigma, \beta \in \mathcal{T}^r_s(M)$, where the Einstein summation convention is used. The *norm* induced by this metric is denoted by

$$|\sigma| := g(\sigma,\sigma)^{\frac{1}{2}}.$$

Remark 1.1.1. Unless specified otherwise, we always use the Einstein summation convention over repeated indices.

If the Riemannian manifold (M,g) is oriented and dV_g denotes its *Riemannian volume form*, that is, the unique m-form on M satisfying $dV_g(e_1,\dots,e_m) = 1$ whenever $(e_1,\dots,e_m)$ is an oriented orthonormal basis of T_pM, then it is possible to introduce the *Hodge operator*

$$* : \Omega^k(M) \to \Omega^{m-k}(M), \quad k = 0,1,\dots,m,$$

uniquely defined in such a way that for each pair of forms $\omega, \eta \in \Omega^k(M)$

$$\omega \wedge *\eta = g(\omega,\eta) dV_g.$$

The Hodge operator is an $\mathbb{R}$-linear map satisfying $*(*\omega) = (-1)^{k(m-k)}\omega$. As a consequence, it is also an isometry with respect to the fiber metric induced by g on $\Lambda^k(T^*M)$.

Let $d : \Omega^k(M) \to \Omega^{k+1}(M)$ denote the *exterior derivative* on M, the *coderivative* $d^* : \Omega^k(M) \to \Omega^{k-1}(M)$ is $d^* := (-1)^{km+m+1} * d *$. It obviously satisfies $(d^*)^2 = 0$.

For more details on the topics of this section, the reader may refer for example to [110, 130, 131].

1.1.2 Principal bundles and connections

Let G be a Lie group, a *principal* G-*bundle* over a manifold M is the data of a manifold P, on which G acts smoothly and freely on the right, and a smooth projection

$\pi : P \to M$ such that (P, π, M) is a locally trivial fibration with fibers the orbits of the G-action. Each fiber is then diffeomorphic to the group G, which is called the *structure group* of the principal bundle, and the base manifold M is diffeomorphic to the orbit space P/G of the G-action on P. As customary, we refer equally to (P, π, M), $\pi : P \to M$ or, simply, to P as the principal G-bundle over M.

A *reduction* of the structure group G of P to a closed subgroup $\mathrm{K} \subseteq \mathrm{G}$ is a principal subbundle Q of P with structure group K, that is, a submanifold Q of P which is invariant under the restriction to K of the G-action on P and such that $(Q, \pi|_Q, M)$ is a principal K-bundle over M.

Consider a principal G-bundle P over M and suppose that G also acts smoothly on the left on a manifold N. Then, G acts smoothly on the right on the product manifold $P \times N$ as

$$(u, p) \cdot a = (u \cdot a, a^{-1} \cdot p),$$

for all $a \in \mathrm{G}$, $u \in P$ and $p \in N$, where the symbol $\cdot$ denotes indifferently the various actions. The quotient space of $P \times N$ by this action is denoted by $P \times_{\mathrm{G}} N$, it is the total space of a fiber bundle over M with standard fiber N and structure group G. $P \times_{\mathrm{G}} N$ is called the fiber bundle associated with P with standard fiber N. If π denotes the projection of this bundle, a *cross section* of $P \times_{\mathrm{G}} N$ is a smooth map $\sigma : M \to P \times_{\mathrm{G}} N$ such that $\pi \circ \sigma$ is the identity map of M.

As a particular case of the previous construction, we can consider a vector space V and a representation $\rho : \mathrm{G} \to \mathrm{GL}(V)$, that is, a Lie group homomorphism between G and the Lie group $\mathrm{GL}(V)$ of invertible linear transformations of V. G acts on the left on V via the representation as $a \cdot v = \rho(a)(v)$, for all $a \in \mathrm{G}$ and $v \in V$. We can then construct the fiber bundle $\rho(P) := P \times_{\mathrm{G}} V$, which turns out to be a vector bundle over M with fiber V. If Q is a reduction of the structure group G of P to K, then the vector bundle $\rho|_{\mathrm{K}}(Q)$ associated with Q using the restriction to K of the representation ρ is isomorphic to $\rho(P)$ (see for instance [17, Thm. 2.14]).

The vector bundle with fiber the Lie algebra $\mathfrak{g}$ of G obtained starting from the adjoint representation $\mathrm{Ad} : \mathrm{G} \to \mathrm{GL}(\mathfrak{g})$ is called *adjoint bundle* of P over M and is usually denoted by $\mathrm{Ad}(P)$ or by $\mathfrak{g}(P)$.

If $\pi : P \to M$ is a principal G-bundle, it is possible to define the *vertical subbundle*

C of TP as the vector subbundle of TP obtained taking the union of the vector spaces $C_u := \ker(\pi_{*u}) = T_u(\pi^{-1}(p)) \subset T_uP$, where $u \in P$ and $\pi(u) = p$. A G-*principal connection* on P is a vector subbundle H of TP, called the *horizontal subbundle*, which is invariant under the action of G on P and satisfies $T_uP = C_u \oplus H_u$ for each $u \in P$. The map $\pi_{*u} : T_uP \to T_pM$ induces an isomorphism between H_u and T_pM, while the Lie algebra $\mathfrak{g}$ of G is isomorphic to C_u via the map sending $A \in \mathfrak{g}$ to $A^*_u := \frac{d}{dt}\big|_{t=0} (u \cdot \exp(tA))$. If $(Q, \pi|_Q, M)$ is a principal K-subbundle of P, a connection H on P *reduces* to Q if H is a subbundle of TQ.

Let X be a vector field on M, there is a unique vector field $\widetilde{X}$ on P which is *horizontal*, i.e., $\widetilde{X}_u \in H_u$, and satisfies $\pi_{*u}(\widetilde{X}_u) = X_{\pi(u)}$ for each $u \in P$. $\widetilde{X}$, called *horizontal lift* of X, is invariant under the action of G (see for example [116, Ch. II, Prop. 1.2]).

Given a principal G-bundle $\pi : P \to M$ and a representation ρ of G on a vector space V, there exists a correspondence between G-principal connections on P and connections on the vector bundle $\rho(P)$. In order to make this explicit, we first observe that there exists a $C^\infty(M)$-module isomorphism identifying every $\sigma \in \Gamma(\rho(P))$ with the G-equivariant smooth map $\sigma^P : P \to V$ defined uniquely by $\sigma(\pi(u)) = [(u, \sigma^P(u))] \in \rho(P)$ (see e.g. [17, Thm. 2.8]). Then, if H is a connection on P, the corresponding connection ∇^H on $\rho(P)$ is defined in the following way for every $u \in P$ with $\pi(u) = p$

$$\left(\nabla^H_X \sigma\right)_p = \sigma^P_{*u}\left(\widetilde{X}_u\right).$$

Therefore, a map $H \mapsto \nabla^H$ from the set of G-principal connections on the principal G-bundle P to the set of connections on the vector bundle $\rho(P)$ is defined. In general, this map is not a bijection.

We refer the reader to [17, 110, 116] for more informations on principal bundles and principal connections.

1.1.3 The holonomy group of a connection

Let E be a vector bundle of rank k over M with a connection ∇. The set of parallel transport maps $P_\gamma : E_p \to E_p$ along loops $\gamma : [0, 1] \to M$ based at a point p of M

is a subgroup of $\mathrm{GL}(E_p)$ called the *holonomy group of* ∇ *based at* p and denoted by $\mathrm{Hol}_p(\nabla)$. If M is connected, $\mathrm{Hol}_p(\nabla)$ can be viewed as a subgroup of $\mathrm{GL}(k,\mathbb{R}) = \mathrm{GL}(\mathbb{R}^k)$ defined up to conjugation. In this sense, it does not depend on the base point and we can refer to it as the *holonomy group* $\mathrm{Hol}(\nabla)$ of the connection ∇. If M is also simply connected, $\mathrm{Hol}(\nabla)$ is a connected Lie subgroup of $\mathrm{GL}(k,\mathbb{R})$. More in general, we can consider the *restricted holonomy group* $\mathrm{Hol}^0(\nabla)$, first defined at each point p considering only the contractible loops at p, and then viewed as a subgroup of $\mathrm{GL}(k,\mathbb{R})$ defined up to conjugation. $\mathrm{Hol}^0(\nabla)$ is always a connected Lie subgroup of $\mathrm{GL}(k,\mathbb{R})$. In particular, it is the connected component of $\mathrm{Hol}(\nabla)$ containing the identity and coincides with it when M is simply connected.

If we consider a principal G-bundle $\pi : P \to M$ with a connection H, we can define an equivalence relation $\sim$ on P by declaring two points $u_1, u_2 \in P$ to be equivalent if and only if there exists a piecewise smooth horizontal curve $\gamma : [0,1] \to P$ such that $\gamma(0) = u_1$ and $\gamma(1) = u_2$, where *horizontal* means that $\dot{\gamma}(t) := \frac{d}{dt}\gamma(t) \in H_{\gamma(t)}$ for each t belonging to the open and dense subset of $[0,1]$ where γ is smooth. The *holonomy group of* (P,H) *based at* $u \in P$ is then defined as the subgroup $\mathrm{Hol}_u(P,H)$ of G whose elements are the $a \in \mathrm{G}$ such that $u \sim u \cdot a$. When M is connected, the holonomy groups of (P,H) based at two different points of P are conjugated by an element of G. Thus, we can define the *holonomy group* $\mathrm{Hol}(P,H)$ of (P,H) as the equivalence class of these subgroups under conjugation. The *restricted holonomy group* $\mathrm{Hol}^0_u(P,H)$ of (P,H) at a point $u \in P$ is defined in a similar way, requiring in addition that the piecewise smooth horizontal curve γ appearing in the definition of $\sim$ is such that $\pi \circ \gamma$ is contractible in M. The restricted holonomy group $\mathrm{Hol}^0(P,H)$ of (P,H) is then defined as the equivalence class of subgroups $\mathrm{Hol}^0_u(P,H)$ of G under conjugation. It is a connected Lie subgroup of G and the connected component of $\mathrm{Hol}(P,H)$ containing the identity. The groups $\mathrm{Hol}^0(P,H)$ and $\mathrm{Hol}(P,H)$ coincide when M is simply connected.

The two definitions of holonomy group of a connection are closely related. Indeed, if P is a principal G-bundle over M, ρ is a representation of G on a vector space V and $\rho(P)$ is the vector bundle over M with fiber V, then given a connection H on P and considered the corresponding connection ∇^H on $\rho(P)$, the holonomy groups $\mathrm{Hol}(P,H)$ and $\mathrm{Hol}(\nabla^H)$ are subgroups of G and $\mathrm{GL}(V)$, respectively, both defined up to conjugation, and $\rho(\mathrm{Hol}(P,H)) = \mathrm{Hol}(\nabla^H)$.

More details on the holonomy group of a (principal) connection and the proofs of the properties we have just described can be found for instance in [110, 116, 163].

1.2 G-structures

1.2.1 Definition, properties and examples

Let M be a connected manifold of dimension m. A *linear frame* at $p \in M$ is an ordered basis $(E_1, \dots, E_m)$ of the tangent space T_pM or, equivalently, an isomorphism $u : \mathbb{R}^m \to T_pM$ sending each vector e_i of the canonical basis $(e_1, \dots, e_m)$ of $\mathbb{R}^m$ to the vector E_i. Let FM_p denote the set of all these isomorphisms, there is a free and transitive right action of the Lie group $\mathrm{GL}(m, \mathbb{R})$ on FM_p given by

$$u \cdot a = u \circ a, \tag{1.1}$$

for all $u \in FM_p$ and $a \in \mathrm{GL}(m, \mathbb{R})$. If we consider the matrix associated with $a \in \mathrm{GL}(m, \mathbb{R})$ with respect to the canonical basis of $\mathbb{R}^m$ and we denote by ${a^i}_k$ its entries, this right action on any ordered basis $(E_1, \dots, E_m)$ of T_pM reads

$$(E_1, \dots, E_m) \cdot a = \left(E_i {a^i}_1, \dots, E_i {a^i}_m\right). \tag{1.2}$$

As a consequence of the previous equivalence, there exists a bijection between the $\mathrm{GL}(m, \mathbb{R})$-orbit of the action (1.1) on FM_p and the $\mathrm{GL}(m, \mathbb{R})$-orbit of the action (1.2) on the set of ordered basis of the tangent space T_pM. The smooth structure of M induces a smooth structure on the set

$$FM := \coprod_{p \in M} FM_p,$$

which is therefore a smooth manifold. Together with the smooth projection $\pi : FM \to M$ sending each $u \in FM_p$ to the point p, (FM, π, M) becomes a principal $\mathrm{GL}(m, \mathbb{R})$-bundle over M, called the *frame bundle* of M.

Definition 1.2.1. Let G be a closed subgroup of $\mathrm{GL}(m, \mathbb{R})$. A G-*structure* on M is a reduction of the structure group $\mathrm{GL}(m, \mathbb{R})$ of FM to G.

Observe that if Q is a G-structure on M, then each fiber $Q_p := (\pi|_Q)^{-1}(p)$ is a G-orbit in FM_p with respect to the restriction of the action (1.1) to G. Equivalently, Q_p can be thought as a G-orbit in the space of ordered basis of T_pM with respect to the restriction of the action (1.2) to G.

It follows from a general result of principal bundles theory (see for example [116, Ch. I, Prop. 5.6]) that G-structures on M are in one-to-one correspondence with cross sections of the bundle $FM \times_{\mathrm{GL}(m,\mathbb{R})} (\mathrm{GL}(m,\mathbb{R})/\mathrm{G}) \cong FM/\mathrm{G}$ associated with FM with standard fiber $\mathrm{GL}(m,\mathbb{R})/\mathrm{G}$. Moreover, the existence of such cross sections is usually a topological matter. It is guaranteed, for instance, whenever $\mathrm{GL}(m,\mathbb{R})/\mathrm{G}$ is diffeomorphic to some Euclidean space $\mathbb{R}^k$ (cf. [116, Ch. I, Thm. 5.7]).

Before giving the statement of the next result, we first need to recall some definitions. Let us consider the vector space $\mathbb{R}^m$ with canonical basis $(e_1, \ldots, e_m)$ and let us denote by $(e^1, \ldots, e^m)$ its dual basis. The group $\mathrm{GL}(m,\mathbb{R})$ acts linearly on the left on the space $T^r_s(\mathbb{R}^m)$ of (r,s)-tensors on $\mathbb{R}^m$ via the action uniquely defined on any basis tensor $\beta = e_{i_1} \otimes \cdots \otimes e_{i_r} \otimes e^{j_1} \otimes \cdots \otimes e^{j_s}$ by

$$a \cdot \beta = a(e_{i_1}) \otimes \cdots \otimes a(e_{i_r}) \otimes \left(e^{j_1} \circ a^{-1}\right) \otimes \cdots \otimes \left(e^{j_s} \circ a^{-1}\right),$$

for every $a \in \mathrm{GL}(m,\mathbb{R})$. The usual linear maps a_* and a^* from $T^r_s(\mathbb{R}^m)$ to itself induced by a can be defined uniquely in terms of this action in the following way

$$a \cdot \beta = a_*(\beta) = \left(a^{-1}\right)^* (\beta).$$

The *stabilizer* of a tensor $\sigma_0 \in T^r_s(\mathbb{R}^m)$ in $\mathrm{GL}(m,\mathbb{R})$ is the subgroup of $\mathrm{GL}(m,\mathbb{R})$ defined equivalently as

$$\{a \in \mathrm{GL}(m.\mathbb{R}) \mid a \cdot \sigma_0 = \sigma_0\} = \{a \in \mathrm{GL}(m,\mathbb{R}) \mid a^*\sigma_0 = \sigma_0\}.$$

When G is the stabilizer of one or more tensors defined on $\mathbb{R}^m$, the existence of a G-structure is related to the existence of certain global tensor fields defined on M.

Proposition 1.2.2. *Let a closed subgroup* $\mathrm{G} \subseteq \mathrm{GL}(m,\mathbb{R})$ *be the stabilizer of a tensor* $\sigma_0 \in T^r_s(\mathbb{R}^m)$. *Then,* σ_0 *gives rise to a one-to-one correspondence between* G*-structures on* M *and tensor fields* $\sigma \in \mathcal{T}^r_s(M)$ *such that at each point* p *of* M *there exists* $u \in FM_p$ *satisfying* $u^*(\sigma_p) = \sigma_0$.

Proof. First, suppose that Q is a G-structure on M, where G is the stabilizer of σ_0. Then, we can define the (r, s)-tensor field σ on M as

$$\sigma : p \mapsto \sigma_p = \left(u^{-1}\right)^* \sigma_0,$$

where $u \in Q_p$. The definition does not depend on the choice of u. Indeed, any other element of the fiber Q_p is of the form $u \circ a$, for some $a \in \mathrm{G}$, and $\left((u \circ a)^{-1}\right)^* \sigma_0 = \left(u^{-1}\right)^* (a \cdot \sigma_0) = \left(u^{-1}\right)^* \sigma_0$.

Conversely, we can define the principal G-bundle Q as the disjoint union of the G-invariant sets

$$Q_p = \{u \in FM_p \mid u^*(\sigma_p) = \sigma_0\}, \quad p \in M.$$

With this choice, $(Q, \pi|_Q, M)$ becomes a G-structure on M (we omit the details). □

When a G-structure is defined by a tensor field as in the previous proposition, each tangent space to M has a distinguished basis.

Definition 1.2.3. Let $\sigma \in \mathcal{T}^r_s(M)$ be a tensor field on M and let $u : \mathbb{R}^m \to T_pM$ be an isomorphism such that $u^*(\sigma_p) = \sigma_0$, where $\sigma_0 \in T^r_s(\mathbb{R}^m)$ has stabilizer G. Then, the basis $(u(e_1), \ldots, u(e_m))$ of T_pM is called *adapted basis* for σ or G-*basis*.

The previous results can be easily extended to the case where G is the common stabilizer of a finite number of tensors on $\mathbb{R}^m$. We can then refer to the corresponding family of tensor fields $\sigma_1, \ldots, \sigma_k$ on M as a G-structure and to the tensors on $\mathbb{R}^m$ from which they are defined as their *model tensors*. This motivates the

Definition 1.2.4. Let Q be a G-structure on M which can be defined by a family of tensor fields $\sigma_1, \ldots, \sigma_k$ in the sense described above. An *automorphism* of the G-structure is an automorphism of the principal fiber bundle Q or, equivalently, a diffeomorphism $\nu : M \to M$ such that $\nu^*\sigma_i = \sigma_i$, for every $i = 1, \ldots, k$.

We describe now some examples of G-structures for $\mathrm{G} = \mathrm{O}(m)$, $\mathrm{SO}(m)$ and $\mathrm{G} = \mathrm{GL}(n, \mathbb{C})$, $\mathrm{Sp}(2n, \mathbb{R})$, when $m = 2n$, while in the next chapters we study more in depth the cases $\mathrm{G} = \mathrm{U}(n), \mathrm{SU}(n)$ and $\mathrm{G} = \mathrm{G}_2$. The description is based essentially on the result of Proposition 1.2.2 and the details can be worked out following its proof. In what follows, $(e_1, \ldots, e_m)$ still denotes the canonical basis of $\mathbb{R}^m$ and $(e^1, \ldots, e^m)$ denotes its dual basis.

Example 1.2.5. Consider the inner product on $\mathbb{R}^m$ given by

$$g_0 = \sum_{i=1}^{m} (e^i)^2,$$

where $(e^i)^2 = e^i e^i$ is the symmetric product of covectors. The basis $(e_1, \dots, e_m)$ is orthonormal with respect to g_0 and the stabilizer of g_0 is the *orthogonal group*

$$\mathrm{O}(m) = \{a \in \mathrm{GL}(m, \mathbb{R}) \mid g_0(a\cdot, a\cdot) = g_0(\cdot, \cdot)\}.$$

$\mathrm{O}(m)$-structures on a manifold M are then in one-to-one correspondence with Riemannian metrics defined on it. If Q denotes an $\mathrm{O}(m)$-structure and g is the corresponding metric, then for each $p \in M$ the fiber Q_p consists of all isomorphisms sending the canonical basis of $\mathbb{R}^m$ to a g-orthonormal basis of T_pM and the model tensor of g is g_0.

Remark 1.2.6. Since $\mathrm{GL}(m, \mathbb{R})/\mathrm{O}(m)$ is diffeomorphic to the space $\mathbb{R}^k$, with $k = \frac{m(m+1)}{2}$, it follows from a result previously recalled that a manifold M always admits an $\mathrm{O}(m)$-structure, that is, a Riemannian metric.

Example 1.2.7. The common stabilizer of the inner product g_0 on $\mathbb{R}^m$ introduced in the previous example and of the volume form

$$dV_0 = e^1 \wedge \cdots \wedge e^m$$

is the *special orthogonal group*

$$\mathrm{SO}(m) = \{a \in \mathrm{O}(m) \mid \det(a) = 1\} = \mathrm{O}(m) \cap \mathrm{SL}(m, \mathbb{R}).$$

Therefore, a manifold M admits an $\mathrm{SO}(m)$-structure if and only if it is an oriented Riemannian manifold, that is, if and only if there exist on it a Riemannian metric g and a nowhere vanishing m-form dV whose model tensors are g_0 and dV_0, respectively.

Example 1.2.8. We recall that a 2-form ω on a vector space V of dimension $2n$ is *non-degenerate* if $\omega(v, w) = 0$ for all $v \in V$ implies $w = 0$ or, equivalently, if $\omega^n \neq 0$. A 2-form $\omega \in \Omega^2(M)$ is non-degenerate if ω_p is non-degenerate at each point p of M.

Consider on $\mathbb{R}^{2n}$ the non-degenerate 2-form

$$\omega_0 = \sum_{k=1}^{n} e^{2k-1} \wedge e^{2k},$$

its stabilizer is the *symplectic group*

$$\mathrm{Sp}(2n, \mathbb{R}) = \{a \in \mathrm{GL}(2n, \mathbb{R}) \mid \omega(a\cdot, a\cdot) = \omega(\cdot, \cdot)\}.$$

Moreover, every $a \in \mathrm{Sp}(2n, \mathbb{R})$ fixes the volume form $\frac{\omega_0^n}{n!} = e^1 \wedge \cdots \wedge e^{2n}$ on which acts as multiplication by $\det(a)$, realizing in this way the inclusion $\mathrm{Sp}(2n, \mathbb{R}) \subset \mathrm{SL}(2n, \mathbb{R})$.

An $\mathrm{Sp}(2n, \mathbb{R})$-structure on a $2n$-dimensional manifold M is then a non-degenerate differential form $\omega \in \Omega^2(M)$ with model tensor ω_0.

Example 1.2.9. A *complex structure* on $\mathbb{R}^m$ is an endomorphism $J : \mathbb{R}^m \to \mathbb{R}^m$ such that $J^2 = -I$, where I is the identity isomorphism. From the equation $(\det(J))^2 = (-1)^m$, we get that $m = 2n$ is necessarily even.

The real vector space $\mathbb{R}^{2n}$ endowed with J admits a natural structure of complex vector space obtained by defining the multiplication of a vector $v \in \mathbb{R}^{2n}$ by a complex number $x + iy \in \mathbb{C}$ as $(x + iy)\, v = x\, v + y\, J(v)$. An isomorphism $a \in \mathrm{GL}(2n, \mathbb{R})$ is then a complex linear isomorphism of the complex vector space $(\mathbb{R}^{2n}, J)$ if and only if it belongs to the group

$$\mathrm{GL}(n, \mathbb{C}) = \left\{a \in \mathrm{GL}(2n, \mathbb{R}) \mid a\, J\left(a^{-1}\cdot\right) = J(\cdot)\right\}.$$

Observe that J can be thought as a $(1,1)$-tensor on $\mathbb{R}^{2n}$ whose stabilizer is exactly $\mathrm{GL}(n, \mathbb{C})$. From this follows that a $\mathrm{GL}(n, \mathbb{C})$-structure on a manifold M of dimension $2n$ is none other than an *almost complex structure* J on M, that is, an endomorphism $J : TM \to TM$ such that $J^2 = -\mathrm{Id}$.

If $(e_1, \ldots, e_{2n})$ is the canonical basis of the vector space $\mathbb{R}^{2n}$, we can choose as model tensor for J the complex structure J_0 on $\mathbb{R}^{2n}$ defined on the basis vectors with odd index by

$$J_0(e_{2k-1}) = e_{2k}, \quad 1 \leq k \leq n,$$

and on the remaining basis vectors in such a way that $(J_0)^2 = -I$.

Remark 1.2.10. With the choice of J_0 just described, at the matrix level the inclusion $\mathrm{GL}(n,\mathbb{C}) \subset \mathrm{GL}(2n,\mathbb{R})$ can be realized associating to a nonsingular complex $n \times n$ matrix a with complex entries ${a^j}_k$ the real $2n \times 2n$ matrix $\hat{a}$ obtained by replacing each entry of the first one with the 2×2 matrix

$$\begin{pmatrix} \Re({a^j}_k) & -\Im({a^j}_k) \\ \Im({a^j}_k) & \Re({a^j}_k) \end{pmatrix}.$$

Their determinants satisfy the identity $\det(\hat{a}) = |\det(a)|^2 = \det(a)\overline{\det(a)}$.

1.2.2 The intrinsic torsion of a G-structure

Consider the frame bundle FM over M, if ρ is the standard representation of $\mathrm{GL}(m,\mathbb{R})$ on $\mathbb{R}^m$, then $\rho(FM)$ is a vector bundle over M isomorphic to the tangent bundle TM. In this case, the correspondence between $\mathrm{GL}(m,\mathbb{R})$-principal connections on FM and connections on $\rho(FM) = TM$ described in Section 1.1.2 is one-to-one and the holonomy group of a $\mathrm{GL}(m,\mathbb{R})$-principal connection on FM coincides, as subgroup of $\mathrm{GL}(m,\mathbb{R})$ defined up to conjugation, with the holonomy group of the corresponding connection on TM.

Since the correspondence between connections on TM and connections on FM is one-to-one, it makes sense to introduce the

Definition 1.2.11. Let Q be a G-structure on M. A connection ∇ on TM is called a G-*connection* (or *compatible* with the G-structure) if the corresponding connection on FM reduces to Q.

If H is a G-principal connection on Q, then there exists a unique connection on FM which reduces to H on Q (see for instance [17, Thm. 4.1]) and the set of G-principal connections on Q is an affine space modeled on $\Gamma(T^*M \otimes \mathfrak{g}(Q))$ (cf. [163, p. 16]), where $\mathfrak{g}(Q)$ is the adjoint bundle of Q over M. Thus, G-connections on TM always exist.

Remark 1.2.12. From the general theory of principal bundles, we know that if ρ is the standard representation of $\mathrm{GL}(m,\mathbb{R})$ on $\mathbb{R}^m$ and Q is a G-structure on M, then the vector bundle with fiber $\mathbb{R}^m$ associated with Q with respect to $\rho|_{\mathrm{G}}$ is isomorphic to $\rho(FM) = TM$ (see also Section 1.1.2).

It follows from [163, Lemma 1.3] that when a G-structure is defined by one or more tensor fields $\sigma_1, \ldots, \sigma_k$ on M as described in Proposition 1.2.2, then a connection ∇ on TM is a G-connection if and only if the σ_i are *parallel* with respect to ∇, that is, $\nabla \sigma_i = 0$ for $i = 1, \ldots, k$.

Example 1.2.13. Let Q be an $\mathrm{O}(m)$-structure on M. Then, M is endowed with a Riemannian metric g (see Example 1.2.5) and a connection ∇ on TM is an $\mathrm{O}(m)$-connection if and only if it is metric.

The tensor fields which are parallel with respect to a given connection ∇ on TM can be characterized in terms of the holonomy group $\mathrm{Hol}(\nabla)$ in the following way (cf. [110, Prop. 2.5.2]).

Proposition 1.2.14. *Let M be a connected manifold and let ∇ be a connection on TM. Fix a point p of M, the holonomy group $\mathrm{Hol}_p(\nabla)$ is a subgroup of $\mathrm{GL}(T_pM)$ and there exists a natural representation of it on each fiber $T^r_s(T_pM)$ of the bundle of (r,s)-tensors over M. If a tensor field $\sigma \in \mathcal{T}^r_s(M)$ is parallel with respect to ∇, then $\sigma(p)$ is fixed by the action of $\mathrm{Hol}_p(\nabla)$ on $T^r_s(T_pM)$. Conversely, if $\sigma_p \in T^r_s(T_pM)$ is fixed by the action of $\mathrm{Hol}_p(\nabla)$, then there exists a unique tensor field $\sigma \in \mathcal{T}^r_s(M)$ which is parallel with respect to ∇ and whose value at p is exactly σ_p.*

The previous result, known in literature as the *holonomy principle*, has the following important consequence

Corollary 1.2.15. *Let $p \in M$ be a given point and let G^∇ be the subgroup of $\mathrm{GL}(T_pM)$ that fixes σ_p for all tensor fields σ on M which are parallel with respect to ∇. Then, $\mathrm{Hol}_p(\nabla)$ is a subgroup of G^∇.*

Thus, if a G-structure on M is defined by certain tensor fields $\sigma_1, \ldots, \sigma_k$ and ∇ is a connection on TM, the non-vanishing of the covariant derivatives $\nabla \sigma_i$ constitutes an obstruction for ∇ to be a G-connection and for $\mathrm{Hol}(\nabla)$ to be a subgroup of G.

When ∇ is torsion-free, the latter obstruction can be also expressed in terms of the so-called *intrinsic torsion* of a G-structure. To introduce this object, we begin considering the adjoint representation of the group $\mathrm{GL}(m,\mathbb{R})$ on its Lie algebra $\mathfrak{gl}(m,\mathbb{R}) \cong (\mathbb{R}^m)^* \otimes \mathbb{R}^m$. We can construct the adjoint bundle $\mathfrak{gl}(m,\mathbb{R})(FM)$, which is isomorphic to the space $\mathrm{End}(TM) \cong T^*M \otimes TM$. Furthermore, if Q is a G-structure on M, we can also consider the adjoint bundle $\mathfrak{g}(Q)$, which is clearly a subbundle of $T^*M \otimes TM$. It then makes sense to define the map $\delta : T^*M \otimes \mathfrak{g}(Q) \to \Lambda^2(T^*M) \otimes TM$ which acts as antisymmetrization in the first two arguments.

Definition 1.2.16. Let Q be a G-structure on M and let ∇ be a G-connection on TM. Denoted by $T(\nabla)$ its torsion tensor, the *intrinsic torsion* $\tau(Q)$ of Q is defined by

$$\tau(Q) := [T(\nabla)] \in \Gamma\left(\Lambda^2(T^*M) \otimes TM/\mathrm{Im}(\delta)\right).$$

Q is said to be *torsion-free* if $\tau(Q)$ vanishes identically.

Observe that the previous definition is well-posed. Indeed, if $\nabla, \widetilde{\nabla}$ are two G-connections on TM, then $\nabla - \widetilde{\nabla}$ is a smooth section of $T^*M \otimes \mathfrak{g}(Q) \subseteq T^*M \otimes T^*M \otimes TM$ and, using this fact, the difference of their torsion tensors $T(\nabla) - T(\widetilde{\nabla})$ is easily seen to belong to $\mathrm{Im}(\delta)$, since for every $X, Y \in \mathfrak{X}(M)$ it holds

$$\begin{aligned}(T(\nabla) - T(\widetilde{\nabla}))(X,Y) &= (\nabla_X - \widetilde{\nabla}_X)Y - (\nabla_Y - \widetilde{\nabla}_Y)X \\ &= -(T(\nabla) - T(\widetilde{\nabla}))(Y,X).\end{aligned}$$

By [163, Prop. 1.6], the existence of a G-connection with identically vanishing torsion is guaranteed when the G-structure Q is *integrable*, that is, when around each point of M there exists a local coordinate frame $\left\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^m}\right\}$ which is also a local section of Q. This motivates the

Definition 1.2.17. A G-structure Q on M is called *non-integrable* if $\tau(Q) \neq 0$.

We can now give the statement of the aforementioned result, for the proof we refer the reader to [110, Prop. 2.6.5].

Proposition 1.2.18. *M admits a torsion-free* G*-structure if and only if there exists a connection ∇ on TM with identically vanishing torsion such that* $\mathrm{Hol}(\nabla)$ *is a subgroup of* G*.*

Example 1.2.19. Consider an $\mathrm{O}(m)$-structure on M. As we saw in Example 1.2.13, ∇ is an $\mathrm{O}(m)$-connection on TM if and only if it is compatible with the Riemannian metric g on M. The existence of a unique $\mathrm{O}(m)$-connection with identically vanishing torsion on M, the Levi Civita connection ∇^g, implies that every $\mathrm{O}(m)$-structure is torsion-free.

1.2.3 Riemannian holonomy groups

Let (M, g) be a connected Riemannian manifold with Levi Civita connection ∇^g. By definition, $\nabla^g g = 0$, therefore the holonomy group of ∇^g is a subgroup of $\mathrm{O}(m)$ uniquely defined up to conjugation by Proposition 1.2.14.

Definition 1.2.20. The (*Riemannian*) *holonomy group* $\mathrm{Hol}(g)$ of g is the holonomy group of the Levi Civita connection ∇^g. It is a subgroup of $\mathrm{O}(m)$ defined up to conjugation in $\mathrm{O}(m)$. The *restricted holonomy group* $\mathrm{Hol}^0(g)$ of g is the restricted holonomy group $\mathrm{Hol}^0(\nabla^g)$, it is a connected Lie subgroup of $\mathrm{SO}(m)$ defined up to conjugation in $\mathrm{O}(m)$.

Suppose now that G is a closed subgroup of $\mathrm{O}(m)$. Then, a G-structure Q on M gives rise to a Riemannian metric g on M and some extra geometric data (think for instance about Example 1.2.7, where we saw that an $\mathrm{SO}(m)$-structure is equivalent to the existence of a Riemannian metric g and a volume form dV on the manifold). In particular, each point $u \in Q$, which is (in correspondence with) a linear frame of T_pM for some $p \in M$, is g-orthonormal and the principal $\mathrm{O}(m)$-bundle corresponding to g can be reconstructed from Q as $Q \cdot \mathrm{O}(m)$. For sake of simplicity, let us denote it by $\mathrm{O}(M)$.

In this situation, the Lie algebra $\mathfrak{so}(m)$ of $\mathrm{O}(m)$ can be decomposed as $\mathfrak{so}(m) = \mathfrak{g} \oplus \mathfrak{g}^\perp$, where $\mathfrak{g}^\perp$ is the subspace of $\mathfrak{so}(m)$ orthogonal to $\mathfrak{g} = \mathrm{Lie}(\mathrm{G})$ with respect to the Killing form. The restriction to G of the adjoint representation of $\mathrm{O}(m)$ on $\mathfrak{so}(m)$ induces an action of G on the spaces $\mathfrak{g}$ and $\mathfrak{g}^\perp$ and, as a consequence, we can construct the vector bundle associated with Q with fiber $\mathfrak{g}^\perp$, which we denote by $\mathfrak{g}^\perp(Q)$. The bundle $\mathfrak{so}(m)(\mathrm{O}(M))$ associated with $\mathrm{O}(M)$ with respect to the adjoint representation of $\mathrm{O}(m)$ on $\mathfrak{so}(m)$ splits according to the decomposition of $\mathfrak{so}(m)$ as $\mathfrak{g}(Q) \oplus \mathfrak{g}^\perp(Q)$. Moreover, it follows from the isomorphism $\mathfrak{so}(m) \cong \Lambda^2((\mathbb{R}^m)^*)$ that the map δ introduced in the previous section is an isomorphism between the vector bundles $T^*M\otimes\mathfrak{so}(m)(\mathrm{O}(M))$ and $\Lambda^2(T^*M)\otimes TM$. We then get the following bundle isomorphisms

$$T^*M\otimes\mathfrak{g}^{\perp}(Q)\cong\frac{T^*M\otimes\mathfrak{so}(m)(\mathrm{O}(M))}{T^*M\otimes\mathfrak{g}(Q)}\cong\frac{\Lambda^2(T^*M)\otimes TM}{\delta(T^*M\otimes\mathfrak{g}(Q))}.$$

Consequently, the intrinsic torsion $\tau(Q)$ of Q can be seen as a section of $T^*M\otimes\mathfrak{g}^{\perp}(Q)$, the vector bundle over M associated with Q with respect to the action of G on $(\mathbb{R}^m)^*\otimes\mathfrak{g}^{\perp}$. Moreover, there exists a unique metric G-connection $\overline{\nabla}$, called the *minimal connection* of Q, such that

$$\tau(Q)=\overline{\nabla}-\nabla^g,$$

where ∇^g is the Levi Civita connection of the Riemannian metric g induced by the G-structure Q. Using this result, it is possible to prove the

Proposition 1.2.21. *Let a closed subgroup* $\mathrm{G}\subseteq\mathrm{O}(m)$ *be the stabilizer of a tensor* $\sigma_0\in T^r_s(\mathbb{R}^m)$*, let* Q *be a* G*-structure on* M *and denote by* $\sigma\in\mathcal{T}^r_s(M)$ *the corresponding tensor field with model tensor* σ_0*. Then, there exists an injective vector bundle homomorphism*

$$F:T^*M\otimes\mathfrak{g}^{\perp}(Q)\to T^*M\otimes T^r_sM$$

mapping the intrinsic torsion $\tau(Q)$ *of* Q *to* $-\nabla^g\sigma$*, where* ∇^g *is the Levi Civita connection of the Riemannian metric* g *induced by the* G*-structure.*

Proof. Consider the map

$$f:\mathrm{O}(m)\to T^r_s(\mathbb{R}^m),\quad f(a)=a\cdot\sigma_0,$$

its differential is a linear map $f_*:\mathfrak{so}(m)\to T^r_s(\mathbb{R}^m)$ with kernel $\mathfrak{g}$. Thus, it induces an injective map $f_*|_{\mathfrak{g}^{\perp}}:\mathfrak{g}^{\perp}\to T^r_s(\mathbb{R}^m)$ which can be used to construct an injective vector bundle homomorphism

$$F:T^*M\otimes\mathfrak{g}^{\perp}(Q)\to T^*M\otimes T^r_sM.$$

Now,

$$F(\tau(Q))=(\overline{\nabla}-\nabla^g)\sigma=\overline{\nabla}\sigma-\nabla^g\sigma=-\nabla^g\sigma,$$

since the minimal connection $\overline{\nabla}$ of Q is a G-connection. □

A first consequence of the previous proposition is the possibility to classify the G-structures on a manifold M in two standard ways when $\mathrm{G} \subseteq \mathrm{O}(m)$. The first one consists in decomposing the G-module $(\mathbb{R}^m)^* \otimes \mathfrak{g}^\perp$ into the direct sum of G-irreducible submodules. This induces a decomposition of the bundle $T^*M \otimes \mathfrak{g}^\perp(Q)$ and the intrinsic torsion $\tau(Q)$ can be decomposed accordingly. The G-structures can then be divided into classes according to the vanishing components of $\tau(Q)$. The second way works in a similar manner, starting with the decomposition into G-irreducible summands of the G-module of tensors satisfying the same identities as $\nabla^g\sigma$ and then defining the classes of G-structures according to the vanishing of the components of $\nabla^g\sigma$. This result extends in the obvious way to the case where the G-structure is defined by a finite number of tensor fields $\sigma_1, \ldots, \sigma_k$ on M.

Furthermore, an immediate consequence of propositions 1.2.14 and 1.2.21 is the following

Proposition 1.2.22. *Let* $\mathrm{G} \subseteq \mathrm{O}(m)$ *be a closed subgroup and let* Q *be a* G*-structure on* M *defined by the tensor fields* $\sigma_1, \ldots, \sigma_k$ *and inducing a Riemannian metric* g*. Then,* $\tau(Q) = 0$ *if and only if* $\nabla^g\sigma_i = 0$ *for all* $i = 1, \ldots, k$*. Whenever this happens,* $\mathrm{Hol}(g)$ *is a subgroup of* G*.*

Remark 1.2.23. When a G-structure is defined by certain tensor fields $\sigma_1, \ldots, \sigma_k$, we denote the intrinsic torsion simply by τ and the corresponding bundle by $T^*M \otimes \mathfrak{g}^\perp$, being understood that there exists a reduction Q of $\mathrm{GL}(m, \mathbb{R})$ to G such that $\tau = \tau(Q)$ and $T^*M \otimes \mathfrak{g}^\perp = T^*M \otimes \mathfrak{g}^\perp(Q)$.

A natural question arising for Riemannian manifolds is which subgroups of $\mathrm{O}(m)$ can occur as holonomy groups of a Riemannian metric. A classification of the possible holonomy groups for simply connected and complete Riemannian manifolds was achieved with the results of Cartan [36, 37], de Rham [58] and Berger [21]. We recall it here.

First of all, observe that $\mathrm{Hol}(g) = \mathrm{Hol}^0(g) \subseteq \mathrm{SO}(m)$ when M is simply connected.

A Riemannian manifold (M, g) is said to be *irreducible* if it is not locally isometric to a Riemannian product $(M_1 \times M_2, g_1 \times g_2)$, where (M_i, g_i) are Riemannian manifolds of dimension at least one. In this case, the natural representations of $\mathrm{Hol}(g)$ and $\mathrm{Hol}^0(g)$ on $\mathbb{R}^m$ are irreducible. By [58], a simply connected, complete Riemannian manifold (M, g) is isometric to a product $(M_0 \times M_1 \times \cdots \times M_k, g_0 \times g_1 \times \cdots \times g_k)$ of simply connected and complete Riemannian manifolds such that (M_0, g_0) is flat, thus $\mathrm{Hol}(g_0) = \{1\}$, the representation of $\mathrm{Hol}(g_i)$ on the fiber of TM_i is irreducible for every $i = 1, \ldots, k$, and $\mathrm{Hol}(g)$ is isomorphic to $\mathrm{Hol}(g_1) \times \cdots \times \mathrm{Hol}(g_k)$.

A Riemannian manifold (M, g) is called *locally symmetric* if $\nabla^g R^g = 0$ and *non-symmetric* otherwise. When a simply connected and complete (M, g) is locally symmetric and irreducible, $\mathrm{Hol}(g)$ is isomorphic to its isotropy group [22, Prop. 10.79] and the classification follows from Cartan's classification of simply connected Riemannian symmetric spaces [36, 37] (see e.g. [110, Sect. 3.3] for more details).

In view of the previous results, to complete the classification it is sufficient to study the problem when (M, g) is irreducible and non-symmetric.

Theorem 1.2.24 ([21])**.** *Let (M, g) be a complete, simply connected, irreducible, non-symmetric Riemannian manifold of dimension m. Then,* $\mathrm{Hol}(g)$ *is one of the following groups:*

i) $\mathrm{SO}(m)$*;*

ii) $\mathrm{U}(n)$*, with $m = 2n \geq 4$;*

iii) $\mathrm{SU}(n)$*, with $m = 2n \geq 4$;*

iv) $\mathrm{Sp}(n)\mathrm{Sp}(1)$*, with $m = 4n \geq 8$;*

v) $\mathrm{Sp}(n)$*, with $m = 4n \geq 8$;*

vi) G_2*, with $m = 7$;*

vii) $\mathrm{Spin}(7)$*, with $m = 8$.*

Remark 1.2.25. The list of groups in the previous theorem originally contained also $\mathrm{Spin}(9)$ for $m = 16$. However, it was proved later that a Riemannian manifold with $\mathrm{Hol}(g) = \mathrm{Spin}(9)$ is symmetric (see [3, 28]).

Remark 1.2.26. After the publication of [21], it took about thirty years to complete the proof that all of the groups appearing in Berger's Theorem actually occur as holonomy group of a Riemannian metric g. For an exhaustive list of references on this topic, the reader may refer to [110, Sect. 3.4.1].

1.3 Homogeneous Riemannian manifolds

Consider a connected m-dimensional Riemannian manifold (M, g), it is a classical result (see for instance [115, 148]) that the group of isometries of M

$$\mathrm{Isom}(M, g) = \{\nu \in \mathrm{Diff}(M) \mid \nu^* g = g\}$$

is a Lie group of dimension at most $\frac{m(m+1)}{2}$ which acts smoothly on M and is compact if M is compact as well. Moreover, the *isotropy subgroup* (or *stabilizer*) at a point p of M

$$\mathrm{I}_p(M, g) = \{\nu \in \mathrm{Isom}(M, g) \mid \nu(p) = p\}$$

is a closed, compact subgroup of $\mathrm{Isom}(M, g)$.

Definition 1.3.1. A Riemannian manifold (M, g) is *homogeneous* if the group of isometries $\mathrm{Isom}(M, g)$ acts transitively on M, that is, for each pair of points p, q of M there exists an isometry $\nu \in \mathrm{Isom}(M, g)$ such that $\nu(p) = q$.

If (M, g) is homogeneous, then all isotropy subgroups are isomorphic via the map sending $\gamma \in \mathrm{I}_p(M, g)$ to $\nu \circ \gamma \circ \nu^{-1} \in \mathrm{I}_q(M, g)$, where $\nu \in \mathrm{Isom}(M, g)$ satisfies $\nu(p) = q$.

In general, the group $\mathrm{Isom}(M, g)$ may contain proper subgroups acting transitively on M, this motivates the following

Definition 1.3.2. If G is a closed subgroup of $\mathrm{Isom}(M, g)$ acting transitively on M, then (M, g) is said to be G-homogeneous.

If (M, g) is G-homogeneous, then it is diffeomorphic to the quotient G/G_p, where $\mathrm{G}_p = \{\nu \in \mathrm{G} \mid \nu(p) = p\}$ is a compact subgroup of $\mathrm{I}_p(M, g)$.

The homogeneous manifolds we will be mostly interested in are the (compact) nilmanifolds and solvmanifolds. To introduce them, we first need to recall the definitions of nilpotent and solvable Lie groups.

Definition 1.3.3. Consider a Lie group G with Lie algebra $\mathfrak{g}$ and define

$$\begin{aligned} \mathcal{C}^1(\mathfrak{g}) &= [\mathfrak{g}, \mathfrak{g}] \\ \mathcal{C}^i(\mathfrak{g}) &= [\mathfrak{g}, \mathcal{C}^{i-1}(\mathfrak{g})], \quad i \geq 2. \end{aligned}$$

Each $\mathcal{C}^i(\mathfrak{g})$ is an ideal in $\mathcal{C}^{i-1}(\mathfrak{g})$. G is called (*k-step*) *nilpotent* if there exists an integer k such that $\mathcal{C}^k(\mathfrak{g}) = \{0\}$ and $\mathcal{C}^i(\mathfrak{g}) \neq \{0\}$ if $i < k$.

Define now

$$\begin{aligned} \mathcal{D}^1(\mathfrak{g}) &= [\mathfrak{g}, \mathfrak{g}], \\ \mathcal{D}^i(\mathfrak{g}) &= \left[\mathcal{D}^{i-1}(\mathfrak{g}), \mathcal{D}^{i-1}(\mathfrak{g})\right], \ i \geq 2. \end{aligned}$$

Also in this case each $\mathcal{D}^i(\mathfrak{g})$ is an ideal in $\mathcal{D}^{i-1}(\mathfrak{g})$. G is called (*k-step*) *solvable* if there exists an integer k such that $\mathcal{D}^k(\mathfrak{g}) = \{0\}$ and $\mathcal{D}^i(\mathfrak{g}) \neq \{0\}$ for $i < k$.

Clearly, since for each i it holds $\mathcal{D}^i(\mathfrak{g}) \subseteq \mathcal{C}^i(\mathfrak{g})$, every nilpotent Lie group is also solvable, but the converse is not true in general.

If $\mathfrak{g}$ is a real Lie algebra of dimension m with Lie bracket $[\cdot, \cdot]$, we can consider a basis $(e_1, \ldots, e_m)$ of it and define its *structure equations* with respect to this basis by

$$[e_i, e_j] = c_{ij}^r e_r.$$

The real numbers $c_{ij}^r = -c_{ji}^r$ are called the *structure constants* of $\mathfrak{g}$. If we consider the dual basis $(e^1, \ldots, e^m)$ of $\mathfrak{g}^*$ and compute the exterior derivative of each basis 1-form (thought as a left-invariant 1-form on G), we get

$$de^r(e_i, e_j) = e_i(e^r(e_j)) - e_j(e^r(e_i)) - e^r([e_i, e_j]) = -e^r([e_i, e_j]) = -c_{ij}^r.$$

As a consequence, the structure equations can also be written in the following way

$$de^r = -\frac{1}{2} c_{ij}^r \, e^i \wedge e^j = \sum_{1 \leq i < j \leq m} (-c_{ij}^r) e^i \wedge e^j.$$

This defines a linear map $d : \Lambda^1(\mathfrak{g}^*) \to \Lambda^2(\mathfrak{g}^*)$, which can be extended to linear maps $d : \Lambda^k(\mathfrak{g}^*) \to \Lambda^{k+1}(\mathfrak{g}^*)$, $2 \leq k \leq m-1$, by requiring that

$$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^k \alpha \wedge (d\beta), \quad \alpha \in \Lambda^k(\mathfrak{g}^*), \ \beta \in \Lambda^s(\mathfrak{g}^*).$$

From the Jacobi identity for the Lie bracket $[\cdot, \cdot]$ and the previous rule, it follows that $d \circ d = 0$ always holds. Therefore, $\{\Lambda^{\cdot}(\mathfrak{g}^*), d\}$ is a differential complex which can be naturally identified with the complex of left-invariant forms on G. $\{\Lambda^{\cdot}(\mathfrak{g}^*), d\}$ is usually called the *Chevalley-Eilenberg complex* of $\mathfrak{g}$ and d is known as the *Chevalley-Eilenberg differential.*

If $(e_1, \dots, e_m)$ is a basis of $\mathfrak{g}$, we denote the structure equations with respect to its dual basis $(e^1, \dots, e^m)$ by

$$\left(-\frac{1}{2}\, c^1_{ij}\, e^i \wedge e^j, \dots, -\frac{1}{2}\, c^m_{ij}\, e^i \wedge e^j\right).$$

For example, $(0, \dots, 0)$ are the structure equations of the m-dimensional Abelian Lie algebra, while $(0, \dots, 0, e^{12})$ means that $de^i = 0$ for $i = 1, \dots, m-1$, and $de^m = e^{12}$, where $e^{12} = e^1 \wedge e^2$.

Remark 1.3.4. From now on, we use the notation $e^{i_1 \cdots i_k}$ as a shortening for the wedge product $e^{i_1} \wedge \cdots \wedge e^{i_k}$ of the covectors $e^{i_1}, \dots, e^{i_k}$.

A connected Riemannian manifold (M, g) is called homogeneous nilmanifold if the group $\mathrm{Isom}(M, g)$ contains a nilpotent Lie subgroup acting transitively on M. It follows from the proof of [180, Thm. 3] that (M, g) can be identified with a simply connected nilpotent Lie group endowed with a left-invariant metric. This motivates the following

Definition 1.3.5. A (*homogeneous*) *nilmanifold* is a simply connected, nilpotent Lie group N endowed with a left-invariant Riemannian metric g.

Solvmanifolds are defined in a similar way

Definition 1.3.6. A *solvmanifold* is a simply connected, solvable Lie group S endowed with a left-invariant Riemannian metric g.

Observe that when G is a Lie group endowed with a left-invariant Riemannian metric g, by definition for every $a \in \mathrm{G}$ and any pair of vectors $X, Y \in T_a\mathrm{G}$ it holds

$$g_a(X, Y) = g_e((L_{a^{-1}})_* X, (L_{a^{-1}})_* Y),$$

where $L_a : \mathrm{G} \to \mathrm{G}$ is the left-multiplication map $L_a(b) = ab$ and $e \in \mathrm{G}$ is the identity element. Then, a left-invariant Riemannian metric is completely determined by the inner product g_e on the Lie algebra $\mathfrak{g} \cong T_e\mathrm{G}$ of G and, if G is simply connected, the pair (G, g) can be identified with the *metric Lie algebra* $(\mathfrak{g}, g_e)$. The same result holds more in general for left-invariant tensor fields on G, which can then be identified with tensors of the same type defined on $\mathfrak{g}$. For brevity, we may use the same symbol to denote a tensor on $\mathfrak{g}$ and the left-invariant tensor it defines on G. That being so, a nilmanifold (N, g) can be identified with its *metric nilpotent Lie algebra* $(\mathfrak{n}, g)$ and a solvmanifold (S, g) can be identified with its *metric solvable Lie algebra* $(\mathfrak{s}, g)$.

A classical problem consists in classifying nilpotent and solvable Lie algebras up to isomorphism. For instance, the solution is known in dimension seven and lower in the real nilpotent case (see [84, 138]), and up to dimension six in the real solvable case, while in higher dimensions the problem is still open and only some partial results are known. Since the result will be useful later, we recall that in dimension six there are 34 non-isomorphic real nilpotent Lie algebras overall (including the Abelian one). They are listed with their structure equations with respect to a given basis in Table 1.1.

Besides the definition of nilmanifolds, we can introduce the compact nilmanifolds as follows

Definition 1.3.7. Let N be a simply connected, nilpotent Lie group and Γ a cocompact discrete subgroup (lattice) of N. The compact quotient manifold N/Γ is called *compact nilmanifold.*

In this case, every left-invariant tensor on N passes to the quotient defining an *invariant* tensor on the compact nilmanifold N/Γ. In particular, if (N, g) is a nilmanifold and we denote by $\pi : \mathrm{N} \to \mathrm{N}/\Gamma$ the projection (universal covering) to the quotient, then the left-invariant metric g on N induces an invariant metric on N/Γ whose pullback by π is exactly g.

Example 1.3.8. Consider the *Heisenberg group*

$$\mathrm{H} = \left\{ \begin{pmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix},\ z_k \in \mathbb{C},\ k = 1, 2, 3 \right\},$$

it is a complex Lie group of complex dimension 3. H can be seen as a real Lie group: a real basis $(e^1, \dots, e^6)$ of $\mathfrak{h}^*$, the dual space of the Lie algebra $\mathfrak{h}$ of H, can be obtained by setting

$$e^1 + ie^2 = dz_1, \quad e^3 + ie^4 = dz_2, \quad e^5 + ie^6 = -dz_3 + z_1 dz_2,$$

where the forms appearing in the right-hand side of the identities are all left-invariant on H. The structure equations of $\mathfrak{h}$ can then be computed from this definition, obtaining

$$\left(0, 0, 0, 0, e^{13} - e^{24}, e^{14} + e^{23}\right).$$

In particular, $\mathfrak{h}$ is a 2-step nilpotent Lie algebra (it is exactly the algebra $\mathfrak{n}_{28}$ of Table 1.1). Moreover, H admits a cocompact discrete subgroup Γ, defined as the subgroup for which the z_k are Gaussian integers, that is, $z_k = x_k + i\, y_k$ with $x_k, y_k \in \mathbb{Z}$. The quotient space H/Γ is then a compact nilmanifold, known in literature as the *Iwasawa manifold.* The left-invariant 1-forms e^k on H pass to the quotient defining a frame of invariant 1-forms on H/Γ. A tensor on H/Γ is then invariant if it can be expressed in terms of this frame using constant coefficients.

The following result of Malčev gives a necessary and sufficient condition for the existence of a lattice of a nilpotent Lie group.

Proposition 1.3.9 ([139])**.** *Let* N *be a simply connected, nilpotent Lie group with Lie algebra* $\mathfrak{n}$*. Then, there exists a basis of* $\mathfrak{n}$ *such that the structure constants are rational numbers if and only if there exists a lattice* Γ *of* N *such that* N/Γ *is a compact nilmanifold.*

For instance, six-dimensional compact nilmanifolds can be constructed from all of the non-isomorphic six-dimensional nilpotent Lie algebras, since they all satisfy the hypothesis of the previous proposition, as one can check directly in Table 1.1.

A further result, proved by Nomizu in [155], states that the de Rham cohomology of a compact nilmanifold N/Γ is completely determined by the cohomology of the Chevalley-Eilenberg complex $\{\Lambda^\cdot(\mathfrak{n}^*), d\}$. More in detail

Theorem 1.3.10 ([155])**.** *Let* N/Γ *be a compact nilmanifold. Then, the natural inclusion* $\{\Lambda^\cdot(\mathfrak{n}^*), d\} \subseteq \{\Omega^\cdot(\mathrm{N}/\Gamma), d\}$ *induces an isomorphism between every de Rham cohomology group* $H^k_{\mathrm{dR}}(\mathrm{N}/\Gamma)$ *of the compact nilmanifold and the cohomology group* $H^k(\mathfrak{n}^*)$ *of the Chevalley-Eilenberg complex of* $\mathfrak{n}$*.*

$\mathfrak{n}$.	$(de^1, de^2, de^3, de^4, de^5, de^6)$	$\mathfrak{n}$.	$(de^1, de^2, de^3, de^4, de^5, de^6)$
$\mathfrak{n}_1$	$(0,0,e^{12},e^{13},e^{14}+e^{23},e^{34}-e^{25})$	$\mathfrak{n}_{18}$	$(0,0,0,e^{12},e^{13}-e^{24},e^{14}+e^{23})$
$\mathfrak{n}_2$	$(0,0,e^{12},e^{13},e^{14},e^{34}-e^{25})$	$\mathfrak{n}_{19}$	$(0,0,0,e^{12},e^{14},e^{13}-e^{24})$
$\mathfrak{n}_3$	$(0,0,e^{12},e^{13},e^{14},e^{15})$	$\mathfrak{n}_{20}$	$(0,0,0,e^{12},e^{13}+e^{14},e^{24})$
$\mathfrak{n}_4$	$(0,0,e^{12},e^{13},e^{14}+e^{23},e^{24}+e^{15})$	$\mathfrak{n}_{21}$	$(0,0,0,e^{12},e^{13},e^{14}+e^{23})$
$\mathfrak{n}_5$	$(0,0,e^{12},e^{13},e^{14},e^{23}+e^{15})$	$\mathfrak{n}_{22}$	$(0,0,0,e^{12},e^{13},e^{24})$
$\mathfrak{n}_6$	$(0,0,e^{12},e^{13},e^{23},e^{14})$	$\mathfrak{n}_{23}$	$(0,0,0,e^{12},e^{13},e^{14})$
$\mathfrak{n}_7$	$(0,0,e^{12},e^{13},e^{23},e^{14}-e^{25})$	$\mathfrak{n}_{24}$	$(0,0,0,e^{12},e^{13},e^{23})$
$\mathfrak{n}_8$	$(0,0,e^{12},e^{13},e^{23},e^{14}+e^{25})$	$\mathfrak{n}_{25}$	$(0,0,0,0,e^{12},e^{15}+e^{34})$
$\mathfrak{n}_9$	$(0,0,0,e^{12},e^{14}-e^{23},e^{15}+e^{34})$	$\mathfrak{n}_{26}$	$(0,0,0,0,e^{12},e^{15})$
$\mathfrak{n}_{10}$	$(0,0,0,e^{12},e^{14},e^{15}+e^{23})$	$\mathfrak{n}_{27}$	$(0,0,0,0,e^{12},e^{14}+e^{25})$
$\mathfrak{n}_{11}$	$(0,0,0,e^{12},e^{14},e^{15}+e^{23}+e^{24})$	$\mathfrak{n}_{28}$	$(0,0,0,0,e^{13}-e^{24},e^{14}+e^{23})$
$\mathfrak{n}_{12}$	$(0,0,0,e^{12},e^{14},e^{15}+e^{24})$	$\mathfrak{n}_{29}$	$(0,0,0,0,e^{12},e^{14}+e^{23})$
$\mathfrak{n}_{13}$	$(0,0,0,e^{12},e^{14},e^{15})$	$\mathfrak{n}_{30}$	$(0,0,0,0,e^{12},e^{34})$
$\mathfrak{n}_{14}$	$(0,0,0,e^{12},e^{13},e^{14}+e^{35})$	$\mathfrak{n}_{31}$	$(0,0,0,0,e^{12},e^{13})$
$\mathfrak{n}_{15}$	$(0,0,0,e^{12},e^{23},e^{14}+e^{35})$	$\mathfrak{n}_{32}$	$(0,0,0,0,0,e^{12}+e^{34})$
$\mathfrak{n}_{16}$	$(0,0,0,e^{12},e^{23},e^{14}-e^{35})$	$\mathfrak{n}_{33}$	$(0,0,0,0,0,e^{12})$
$\mathfrak{n}_{17}$	$(0,0,0,e^{12},e^{14},e^{24})$	$\mathfrak{n}_{34}$	$(0,0,0,0,0,0)$

Table 1.1: Non-isomorphic, real nilpotent Lie algebras of dimension six.

Similarly to the definition of compact nilmanifolds, we have the

Definition 1.3.11. Let S be a simply connected, solvable Lie group and Γ a cocompact discrete subgroup (lattice) of S. The compact quotient manifold S/Γ is called *compact solvmanifold.*

The results previously recalled for compact nilmanifolds do not extend in general to the case of compact solvmanifolds. For instance, there are no sufficient conditions guaranteeing the existence of a lattice of a simply connected solvable Lie group.

Anyway, a general result [145, Lemma 6.2] states that if a Lie group G admits a lattice, then it must be *unimodular*, i.e., the trace of the adjoint operator

$$\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g}, \quad \mathrm{ad}_X(Y) = [X, Y]$$

has to be zero for every $X \in \mathfrak{g}$.

Moreover, as shown by Hattori in [96], if a Lie group S is *completely solvable*, that is, if ad_X has only real eigenvalues for every $X \in \mathfrak{s} = \mathrm{Lie(S)}$, then a result similar to Nomizu's Theorem can be proved for compact solvmanifolds obtained as the quotient of simply connected, completely solvable Lie groups by a lattice.

We conclude this section with some observations on metric Lie algebras. Consider two metric Lie algebras $(\mathfrak{g}_1, g_1)$ and $(\mathfrak{g}_2, g_2)$, they are said to be *isometric* if the corresponding simply connected Lie groups are isometric as Riemannian manifolds endowed with the left-invariant Riemannian metrics induced by g_1 and g_2, while they are called *isomorphic* if there exists a Lie algebra isomorphism $f : \mathfrak{g}_1 \to \mathfrak{g}_2$ satisfying $f^* g_2 = g_1$. Clearly, isomorphic metric Lie algebras are isometric, but the converse is not true in general (cf. [5]). However, by [4], if $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are completely solvable, then $(\mathfrak{g}_1, g_1)$ and $(\mathfrak{g}_2, g_2)$ are isometric if and only if they are isomorphic. This is the case, for instance, of metric nilpotent Lie algebras, since nilpotent Lie algebras are in particular completely solvable.

1.4 Einstein and Ricci soliton metrics

In Riemannian geometry there exist some types of Riemannian metrics whose properties distinguish them from others. In this section, we consider two cases, namely Einstein and Ricci soliton metrics, reviewing also related results on solvmanifolds and nilmanifolds.

1.4.1 Einstein metrics

Definition 1.4.1. A Riemannian metric g on a manifold M is said to be an *Einstein metric* if it is proportional to its Ricci tensor at each point of M. In this case, (M, g) is called *Einstein manifold*.

Thus, if g is an Einstein metric, there exists a smooth function $f \in C^\infty(M)$ such that

$$\mathrm{Ric}(g) = fg.$$

Taking the trace of both sides of the previous identity with respect to the metric, one easily gets that g is Einstein if and only if

$$\mathrm{Ric}(g) = \frac{1}{m}\,\mathrm{Scal}(g)\,g. \tag{1.3}$$

The property of being Einstein for a Riemannian metric is relevant in dimension $m \geq 3$. Indeed, in dimension $m = 1$ the Ricci curvature is zero, while in dimension $m = 2$ the identity (1.3) always holds. Moreover, when $m \geq 3$, taking the covariant derivative of both sides of (1.3) and tracing it with respect to g in a proper way gives $d(\mathrm{Scal}(g)) = 0$. This proves the

Proposition 1.4.2. *Let (M, g) be a connected Einstein manifold of dimension $m \geq 3$. Then,* $\mathrm{Scal}(g)$ *is constant.*

On a connected manifold of dimension $m \geq 3$ it is then possible to write the Einstein condition (1.3) as

$$\mathrm{Ric}(g) = \mu g, \tag{1.4}$$

for the real constant $\mu := \frac{1}{m}\mathrm{Scal}(g)$, usually called *Einstein constant.* In particular, when $\mu = 0$ the metric g is Ricci-flat.

The rôle of Einstein metrics was widely discussed in [22], where the reader can find more details on the subject and some good motivations explaining why they can be considered as "best" or "distinguished" metrics on a Riemannian manifold.

In the general case, Einstein metrics may not exist. The following result by Milnor describes a typical example where this happens (see also [108]).

Theorem 1.4.3 ([145])**.** *Let* G *be a Lie group with nilpotent, non-Abelian Lie algebra. Then, for every left-invariant metric on* G *there exists a direction of strictly negative Ricci curvature and a direction of strictly positive Ricci curvature.*

Remark 1.4.4. We stress that the non-existence of Einstein metrics in the setting of the previous theorem follows from the obvious fact that for any Einstein metric g and any non-vanishing vector field X, the quantity $\mathrm{Ric}(g)(X, X) = \mu|X|^2$ has a definite sign.

1.4.2 Einstein solvmanifolds

Milnor's result does not extend to the solvable case and, moreover, the simply connected solvable Lie groups endowed with a left-invariant Einstein metric, or *Einstein solvmanifolds* for short, are of particular interest. Indeed, they constitute the only currently known examples of noncompact homogeneous Einstein manifolds and a conjecture attributed to D.V. Alekseevskiĭ states that every noncompact homogeneous Einstein manifold might be of this kind when the group acting transitively on it is linear (cf. [22, 7.57]). As reviewed in [107], this conjecture is known to be true in several cases, like for instance the case of homogeneous Einstein spaces of negative sectional curvature, the case of four- and five-dimensional noncompact homogeneous Einstein manifolds and also in dimension up to ten if the Einstein manifold is G-homogeneous with G non-semisimple, as shown in [8, 9]. For more details on this problem and the most recent results concerning it, we refer the reader to the works just cited and the references therein.

Consider now an Einstein solvmanifold (S, g) and identify it with its Einstein metric solvable Lie algebra $(\mathfrak{s}, g)$. In [127], Lauret showed that every Einstein solvmanifold is *standard*, i.e., the orthogonal complement $\mathfrak{a}$ to $\mathfrak{n} := [\mathfrak{s}, \mathfrak{s}]$ is always an Abelian subalgebra of $\mathfrak{s} = \mathfrak{n} \oplus \mathfrak{a}$, whose dimension is called the (*algebraic*) *rank* of $(\mathfrak{s}, g)$.

The properties of standard Einstein solvmanifolds were studied earlier by Heber in [98], who proved many structural and uniqueness results for them. By the aforementioned result of Lauret, they are valid for every Einstein solvmanifold.

Unlike what happens in the compact homogeneous case, if a simply connected solvable Lie group admits a left-invariant Einstein metric, then this is unique up to isometry and scaling [98, Thm. 5.1]. Moreover, since any unimodular Einstein solvmanifold is flat by [60], we can restrict our attention to the nonunimodular case, where it is possible to prove that any nonunimodular standard Einstein metric solvable Lie algebra $(\mathfrak{s} = \mathfrak{n} \oplus \mathfrak{a}, g)$ is an Iwasawa-type algebra up to isometry [98, Thm. 4.10]. This means that every ad_A is symmetric with respect to g and nonzero for each $A \in \mathfrak{a} - \{0\}$ and that there exists some $A^0 \in \mathfrak{a}$ such that the restriction $\mathrm{ad}_{A^0}|_{\mathfrak{n}}$ is positive definite. As a consequence, $\mathfrak{n} = [\mathfrak{s}, \mathfrak{s}]$ is the maximal nilpotent ideal of $\mathfrak{s}$. A canonical choice of A^0 is the vector $H \in \mathfrak{a}$ defined by requiring that $\mathrm{tr}(\mathrm{ad}_X) = g(X, H)$ holds for all $X \in \mathfrak{s}$, since there exists a positive number k such that the eigenvalues of $\mathrm{ad}_{kH}|_{\mathfrak{n}}$ are all positive integers $n_1, \ldots, n_r$ with multiplicities $d_1, \ldots, d_r$, respectively, and without common divisors (cf. [98, Thm. 4.14]). The collection $(n_1 < \cdots < n_r; d_1, \ldots, d_r)$ is called the *eigenvalue type* of (S, g), it is invariant under isometries and scaling and in each dimension only finitely many eigenvalue types occur. Finally, by [98, Thm. 4.18], the study of standard Einstein metric solvable Lie algebras can be reduced to the rank-one case. Indeed, if $(\mathfrak{s} = \mathfrak{n} \oplus \mathfrak{a}, g)$ is the metric solvable Lie algebra of an Einstein solvmanifold (S, g), then the solvable metric Lie algebra $(\mathfrak{s}_0 := \mathfrak{n} \oplus \mathbb{R}H, g)$ is Einstein of Iwasawa-type and S can be reconstructed from it.

Remark 1.4.5. Observe that an Iwasawa-type algebra $(\mathfrak{s}, g)$ is nonunimodular, since $\mathrm{tr}(\mathrm{ad}_{A^0}) \neq 0$, and completely solvable. Consequently, two metric solvable Lie algebras of Iwasawa-type are isometric if and only if they are isomorphic.

1.4.3 Ricci soliton metrics

A natural generalization of Einstein metrics is given by Ricci soliton metrics. To define them, we first need to recall briefly some results about the Ricci flow. The reader can find more details on this topic for example in the introductory book [46], in its sequels [43–45] and in the references cited in this section.

Let g_0 be a fixed Riemannian metric on a manifold M, the *Ricci flow* is the second order, non-parabolic flow

$$\begin{cases} \frac{\partial}{\partial t} g(t) = -2\,\mathrm{Ric}(g(t)) \\ g(0) = g_0 \end{cases} . \tag{1.5}$$

A solution of the Ricci flow is a family of Riemannian metrics $g(t)$ defined on M and depending on a real parameter t, the *time*, which satisfies the PDE in (1.5) with initial condition $g(0) = g_0$. The Ricci flow was introduced in [94] by Hamilton, who developed a program, later completed by Perelman's works [157–159], aiming to solve Thurston's Geometrization Conjecture for compact 3-manifolds using it.

Remark 1.4.6. Unless stated otherwise, compact manifolds are always assumed to have empty boundary.

Although the flow equation is not strictly parabolic, it is possible to prove local existence and uniqueness for solutions of (1.5) on compact manifolds.

Theorem 1.4.7 ([94])**.** *Let (M, g_0) be a compact Riemannian manifold. Then, there exists a unique solution for the Ricci flow defined on some interval $[0, \varepsilon)$ and such that $g(0) = g_0$.*

This theorem was first proved by Hamilton in [94] using the complex machinery of Nash-Moser inverse function Theorem, while an alternative proof of it was given shortly after by DeTurck in [59]. The latter consists in modifying the flow equation by adding on the right-hand side the Lie derivative of the metric with respect to a suitable vector field. The new equation is a strictly parabolic PDE for which the local existence and uniqueness of solutions is guaranteed by standard PDEs theory (see also Section 4.2); the unique local solution of the Ricci flow is then obtained by pulling back the solution of the modified flow by an appropriate family of diffeomorphisms depending on t. Furthermore, the solution can be extended to a unique one, called *singular solution*, defined on a maximal time interval $[0, T)$, where $T \leq +\infty$ is the *singularity time* (see [43, Thm. 6.45]).

Even if short-time existence and uniqueness of solutions of (1.5) on an arbitrary Riemannian manifold cannot be proved, they still hold in the case of complete noncompact manifolds with bounded curvature (see [39, 170]). Moreover, in the Riemannian homogeneous case there always exists a homogeneous solution of (1.5) starting at a given homogeneous metric and which is unique among homogeneous metrics. For a proof the reader may refer to [106, 128].

A distinguished family of solutions of the Ricci flow is given by the so called self-similar solutions, which are obtained by rescaling and pulling back the initial metric g_0 by a family of diffeomorphisms of M depending on t. More formally

Definition 1.4.8. A solution $g(t)$ of the Ricci flow with initial metric g_0 is said to be *self-similar* if there exist a positive real valued smooth function $\sigma(t)$ and a

1-parameter family of diffeomorphisms $\nu_t : M \to M$ such that

$$g(t) = \sigma(t)\nu_t^*(g_0). \tag{1.6}$$

Observe that differentiating a solution of the form (1.6) with respect to t and evaluating the result in $t = 0$, we get

$$\mathrm{Ric}(g_0) = \mu g_0 + \mathcal{L}_X g_0, \tag{1.7}$$

where $\mu = -\frac{1}{2}\dot{\sigma}(0)$ and $X = -\frac{1}{2}\sigma(0)\widehat{X}(0)$, being $\widehat{X}(t)$ the time-dependent vector field such that $\widehat{X}_{\nu_t(p)} = \frac{d}{dt}(\nu_t(p))$. Conversely, if we consider a Riemannian metric g_0 satisfying (1.7), define $\sigma(t) = 1 - 2\mu t$, $Y(t) = -\frac{2}{\sigma(t)}X$, and let ν_t denote the 1-parameter family of diffeomorphisms generated by $Y(t)$ with $\nu_0 = \mathrm{Id}_M$, then

$$g(t) = \sigma(t)\nu_t^*(g_0)$$

is a solution of the Ricci flow. These results can be summarized as follows.

Proposition 1.4.9. *$g(t)$ is a self-similar solution of the Ricci flow with initial condition g_0 if and only if g_0 satisfies*

$$\mathrm{Ric}(g_0) = \mu g_0 + \mathcal{L}_X g_0,$$

for some real constant μ and some vector field $X \in \mathfrak{X}(M)$.

In light of the previous proposition, it is possible to introduce the

Definition 1.4.10. A Riemannian metric g on a manifold M is said to be a *Ricci soliton* if there exist a real number μ and a vector field $X \in \mathfrak{X}(M)$ such that

$$\mathrm{Ric}(g) = \mu g + \mathcal{L}_X g.$$

If the vector field X appearing in the definition of a Ricci soliton g is everywhere zero or if it is a Killing vector field for it, i.e., $\mathcal{L}_X g = 0$, then g is actually an Einstein metric. Therefore, Ricci soliton metrics are a generalization of Einstein metrics, which can be considered as *trivial* Ricci solitons.

Example 1.4.11. Let g_0 be an Einstein metric with $\mathrm{Ric}(g_0) = \mu g_0$. It is a trivial Ricci soliton and the self-similar solution of (1.5) associated with it starting at g_0 is $g(t) = (1 - 2\mu t)g_0$.

The interest for Ricci solitons is motivated not only by the fact that they are (in correspondence with) self-similar solutions of the Ricci flow and they generalize the Einstein condition (1.4), but also because they are singularity models for the Ricci flow, namely complete non-flat solutions of (1.5) which occur as limits of dilations of a singular solution. The reader can find further details and properties of Ricci solitons for example in [43, Ch. 1].

1.4.4 Ricci soliton metrics on nilpotent Lie groups

We now focus our attention on left-invariant metrics on nilpotent Lie groups. As we already recalled, if the nilpotent Lie algebra is not Abelian, then there are no left-invariant Einstein metrics on the Lie group. Since Ricci solitons are a natural generalization of Einstein metrics, it makes sense to consider left-invariant Ricci solitons as distinguished left-invariant metrics on nilpotent Lie groups. Lauret studied the properties of these metrics in [124], we recall here some of his results.

Consider a simply connected nilpotent Lie group N endowed with a left-invariant Riemannian metric g, which we identify with $(\mathfrak{n}, g)$ as explained in Section 1.3. Observe that the Ricci operator $\mathrm{Rc}(g)$ of the inner product g on $\mathfrak{n}$ is the restriction to $\mathfrak{n} \cong T_e\mathrm{N}$ of the Ricci operator on N. If g is a left-invariant Ricci soliton on N, Lauret proved that its Ricci operator on $\mathfrak{n}$ differs from a constant multiple of the identity automorphism $I : \mathfrak{n} \to \mathfrak{n}$ only by a *derivation* of $\mathfrak{n}$, that is, an element belonging to the Lie algebra $\mathrm{Der}(\mathfrak{n})$ of the Lie group $\mathrm{Aut}(\mathfrak{n})$ of automorphisms of $\mathfrak{n}$, where

$$\begin{aligned} \mathrm{Aut}(\mathfrak{n}) &= \{A \in \mathrm{GL}(\mathfrak{n}) \mid A[\cdot,\cdot] = [A\cdot, A\cdot]\}, \\ \mathrm{Der}(\mathfrak{n}) &= \{D \in \mathrm{End}(\mathfrak{n}) \mid D[\cdot,\cdot] = [D\cdot,\cdot] + [\cdot, D\cdot]\}. \end{aligned}$$

More in detail

Proposition 1.4.12 ([124])**.** *A left-invariant Riemannian metric g on a simply connected nilpotent Lie group* N *is a Ricci soliton if and only if the Ricci operator on* $\mathfrak{n}$

has the following form

$$\mathrm{Rc}(g) = \mu I + D, \tag{1.8}$$

for some $\mu \in \mathbb{R}$ *and some derivation* $D \in \mathrm{Der}(\mathfrak{n})$.

Proof. First, suppose that $\mathrm{Rc}(g) = \mu I + D$. Then, for every pair $X, Y \in \mathfrak{n}$ we have

$$\mathrm{Ric}(g)(X,Y) = g(\mathrm{Rc}(g)X, Y) = \mu g(X,Y) + g(DX, Y).$$

Now, if $\nu_t \in \mathrm{Aut}(\mathrm{N})$ denotes the unique t-depending automorphism of N such that $(\nu_t)_{*e} = \exp\left(-\frac{t}{2}D\right)$ for each t, and $X \in \mathfrak{X}(\mathrm{N})$ is the vector field defined by $X_p = \frac{d}{dt}\Big|_{t=0} \nu_t(p)$, $p \in \mathrm{N}$, then it follows from the definition of Lie derivative that

$$\mathcal{L}_X g(\cdot,\cdot) = \frac{d}{dt}\bigg|_{t=0} \nu_t^* g = g(D\cdot,\cdot).$$

Therefore, g is a Ricci soliton.

Conversely, if g is a Ricci soliton, then there exist a family ν_t of diffeomorphisms of N and a real valued function $\sigma(t)$ such that $g(t) = \sigma(t)\nu_t^*(g)$ is a self similar solution of the Ricci flow with $g(0) = g$. By the uniqueness of solutions of (1.5), $\nu_t^*(g)$ has to be left-invariant for all t. Using this fact and the results of [180, Thm. 2], it follows that there exists a family of automorphisms ϱ_t of N such that $\nu_t^*(g) = \varrho_t^*(g)$ for all t. In particular, $\varrho_{*e} = \exp\left(-\frac{t}{2}D\right)$ for some derivation $D \in \mathrm{Der}(\mathfrak{n})$. Now, a computation similar to the previous one proves the assertion. ☐

Remark 1.4.13. It follows from the proof that an inner product satisfying (1.8) on a Lie algebra $\mathfrak{g}$ always induces a left-invariant Ricci soliton metric on the simply connected Lie group G with Lie(G) $= \mathfrak{g}$, while the converse is not true in general. In literature, a detailed study of Ricci solitons in the more general setting of homogeneous Riemannian manifolds was carried out by many authors, the interested reader may refer to the works [97, 105, 106, 121, 122] and the references therein for more details on this topic.

Motivated by the previous result, it is possible to give the

Definition 1.4.14. A left-invariant Riemannian metric g on a nilmanifold N is said to be a *nilsoliton* if its Ricci operator on $\mathfrak{n}$ belongs to the space $\mathbb{R}I \oplus \mathrm{Der}(\mathfrak{n})$.

The existence of a nilsoliton metric on a simply connected nilpotent Lie group N clearly implies the existence of a nonzero symmetric derivation of its Lie algebra $\mathfrak{n}$. This provides an obstruction to the existence of nilsoliton metrics. For instance, if N is *characteristically nilpotent*, i.e., $\mathrm{Der}(\mathfrak{n})$ consists only of nilpotent elements, then there are no symmetric derivations of $\mathfrak{n}$ and N cannot admit nilsoliton metrics. Therefore, in the general case nilsolitons may not exist. However, when they exist, they are unique up to isometry and scaling among left-invariant metrics [124, Thm. 3.5].

We conclude this section recalling a characterization of Einstein solvmanifolds in terms of nilsolitons proved in [125]. Consider an Einstein solvmanifold of dimension $m+1$, it is standard and we can always suppose that it has rank one (cf. Section 1.4.2). Let $\mathfrak{n}$ be a vector space of dimension m and consider an inner product vector space $(\mathfrak{s} = \mathfrak{n} \oplus \mathbb{R}H, \widetilde{g})$ with $\widetilde{g}(H, \mathfrak{n}) = 0$ and $\widetilde{g}(H,H) = 1$. Then, the metric Lie algebra of any $(m+1)$-dimensional rank-one solvmanifold S can be modeled on $(\mathfrak{s} = \mathfrak{n} \oplus \mathbb{R}H, \widetilde{g})$ for some nilpotent Lie bracket $[\cdot,\cdot]_{\mathfrak{n}}$ on $\mathfrak{n}$ and some symmetric derivation $D \in \mathrm{Der}(\mathfrak{n})$. Indeed, using them it is possible to define a solvable Lie bracket $[\cdot,\cdot]_{\mathfrak{s}}$ on $\mathfrak{s}$ by

$$[H,X]_{\mathfrak{s}} = DX, \quad [X,Y]_{\mathfrak{s}} = [X,Y]_{\mathfrak{n}},$$

for all $X, Y, \in \mathfrak{n}$. The rank-one solvmanifold S is then obtained as the simply connected solvable Lie group with solvable Lie algebra $(\mathfrak{s}, [\cdot,\cdot]_{\mathfrak{s}})$ endowed with the left-invariant Riemannian metric determined by $\widetilde{g}$.

Proposition 1.4.15 ([125])**.** *Let D be a symmetric derivation of $(\mathfrak{n}, [\cdot,\cdot]_{\mathfrak{n}})$, let $(\mathrm{S}, \widetilde{g})$ be the solvmanifold with metric solvable Lie algebra $(\mathfrak{s} = \mathfrak{n} \oplus \mathbb{R}H, \widetilde{g})$ and denote by g the restriction of the inner product $\widetilde{g}$ on $\mathfrak{n}$. Then, $(\mathrm{S}, \widetilde{g})$ is Einstein if and only if the Ricci operator of g on $\mathfrak{n}$ satisfies*

$$\mathrm{Rc}(g) = \mu I + \mathrm{tr}(D)D,$$

where $\mu = \frac{\mathrm{tr}(\mathrm{Rc}(g)^2)}{\mathrm{tr}(\mathrm{Rc}(g))}$. In that case, the Ricci operator of $\widetilde{g}$ on $\mathfrak{s}$ equals μI.

Remark 1.4.16. When the solvmanifold corresponding to the metric solvable Lie algebra $(\mathfrak{s} = \mathfrak{n} \oplus \mathbb{R}H, \widetilde{g})$ is Einstein, its eigenvalue type is determined by the eigenvalues of the operator $\mathrm{ad}_{kH}|_{\mathfrak{n}} = kD$, for some $k > 0$.

In the setting of the previous theorem, $\mathfrak{s}$ is said to be a *rank-one solvable extension* of the nilpotent Lie algebra $\mathfrak{n}$. Using a variational method developed in [125], the existence of a rank-one solvable extension for every nilpotent Lie algebra of dimension less or equal than five was proved in [126], while in the six-dimensional case it was proved in [179]. Thus, all non-isomorphic nilpotent Lie algebras of dimension up to six admit a unique nilsoliton metric and their rank-one solvable extensions admit an Einstein metric. As an example, we write the details in the case of the Lie algebra of the Iwasawa manifold examined in Example 1.3.8.

Example 1.4.17. Consider the Lie algebra

$$\mathfrak{h} = \mathfrak{n}_{28} = \left(0, 0, 0, 0, e^{13} - e^{24}, e^{14} + e^{23}\right)$$

and endow it with the inner product g for which the basis $(e^1, \ldots, e^6)$ is orthonormal. A simple computation shows that with respect to the basis $(e_1, \ldots, e_6)$ we have

$$\mathrm{Rc}(g) = -3I + 4\,\mathrm{diag}\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1, 1\right),$$

and $D = \mathrm{diag}\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1, 1\right)$ is a symmetric derivation of $\mathfrak{h}$ with $\mathrm{tr}(D) = 4$. Thus, the rank-one solvable extension $\mathfrak{s} := \mathfrak{h} \oplus \mathbb{R}e_7$ endowed with the inner product $\widetilde{g} = g + (e^7)^2$, where e^7 is the covector of $\mathfrak{s}^*$ dual to e_7, is Einstein with $\mathrm{Rc}(\widetilde{g}) = -3I$. In particular, its structure equations with respect to the basis $(e^1, \ldots, e^6, e^7)$ are

$$\left(\frac{1}{2}e^{17}, \frac{1}{2}e^{27}, \frac{1}{2}e^{37}, \frac{1}{2}e^{47}, e^{13} - e^{24} + e^{57}, e^{14} + e^{23} + e^{67}, 0\right)$$

and the eigenvalue type of the corresponding solvmanifold is $(1 < 2; 4, 2)$.

Chapter 2

Special half-flat SU(3)-structures

Six-dimensional manifolds endowed with an SU(3)-structure are the subject of this chapter. We begin in a more general setting, recalling the definition of U(n)- and SU(n)-structures on manifolds of dimension $2n$ and some related results, which will be useful in the sequel. Then, we focus on the six-dimensional case, describing equivalent definitions of SU(3)-structures and the classification in terms of the intrinsic torsion. Among all of the classes, we consider the one of half-flat SU(3)-structures and certain special subclasses, namely nearly Kähler, double half-flat and coupled. For the first two we recall some known results, while for the third we describe the results obtained in the papers [70, 71, 160] and we discuss further properties.

2.1 SU(n)-structures

2.1.1 Almost complex manifolds

In Example 1.2.9 of Chapter 1, we introduced almost complex structures on manifolds as GL($n,\mathbb{C}$)-structures. We recall the definition here.

Definition 2.1.1. An *almost complex structure* on an even-dimensional manifold M is a vector bundle endomorphism $J : TM \to TM$ such that $J^2 = -\mathrm{Id}$, where Id is the identity map. The pair (M, J) is called *almost complex manifold.*

As we observed in the same example, the structure group of the frame bundle of a $2n$-dimensional almost complex manifold (M, J) reduces to $\mathrm{GL}(n, \mathbb{C})$. Moreover, at each point p of M it is always possible to find a real basis $(e_1, \ldots, e_{2n})$ of T_pM which is adapted for J, i.e., such that at p

$$\begin{array}{rcl} J(e_{2k-1}) & = & e_{2k} \\ J(e_{2k}) & = & -e_{2k-1} \end{array}, \quad 1 \leq k \leq n.$$

A manifold M of real dimension $2n$ is said to be *complex* if it admits a *holomorphic atlas*, that is, an atlas $\{(\mathcal{U}_k, \phi_k)\}$ whose charts are of the form $\phi_k : \mathcal{U}_k \to \phi_k(\mathcal{U}_k) \subseteq \mathbb{C}^n$ and whose transition functions are holomorphic maps between open sets of $\mathbb{C}^n$. A complex manifold always admits an almost complex structure (see for example [103, Prop. 2.6.2]), while the presence of an almost complex structure is not sufficient to guarantee the existence of a holomorphic atlas. Indeed, by Newlander-Nirenberg Theorem [152], an almost complex manifold (M, J) is complex if and only if the *Nijenhuis tensor* $N_J \in \mathcal{T}^1_2(M)$, defined for every $X, Y \in \mathfrak{X}(M)$ as

$$N_J(X,Y) \quad = \quad [X,Y] + J[JX,Y] + J[X,JY] - [JX,JY], \tag{2.1}$$

vanishes identically. In this case, the almost complex structure J is said to be *integrable* or *complex*.

For every almost complex manifold (M, J), the complexification of the tangent bundle $T_{\mathbb{C}}M := TM \otimes_{\mathbb{R}} \mathbb{C}$ splits into a direct sum $T^{1,0}M \oplus T^{0,1}M$, where the two summands are complex vector bundles over M whose fibers over each point p of M are the eigenspaces of the $\mathbb{C}$-linear extension of J :

$$\begin{array}{rcl} T^{1,0}_pM & := & \{X \in T_pM \otimes \mathbb{C} \mid JX = iX\}, \\ T^{0,1}_pM & := & \{X \in T_pM \otimes \mathbb{C} \mid JX = -iX\} = \overline{T^{1,0}_pM}. \end{array}$$

As a consequence, the vector bundles $\Lambda^k(T_{\mathbb{C}}M^*)$ of complex k-forms admit the natural decomposition

$$\Lambda^k(T_{\mathbb{C}}M^*) = \bigoplus_{r+s=k} \Lambda^{r,s}(T^*M), \tag{2.2}$$

where $\Lambda^{r,s}(T^*M) := \Lambda^r\left(\left(T^{1,0}M\right)^*\right) \otimes_{\mathbb{C}} \Lambda^s\left(\left(T^{0,1}M\right)^*\right)$. The sections of $\Lambda^{r,s}(T^*M)$ are called (r,s)-*forms* or *forms of type* (r,s) (with respect to J) and the space of (r,s)-forms is denoted by $\Omega^{r,s}(M)$. According to (2.2), the space $\Omega^k_{\mathbb{C}}(M) := \Gamma(\Lambda^k(T_{\mathbb{C}}M^*))$

splits as

$$\Omega^k_{\mathbb{C}}(M) = \bigoplus_{r+s=k} \Omega^{r,s}(M)$$

and $\overline{\Omega^{r,s}(M)} = \Omega^{s,r}(M)$. Every complex form $\alpha \in \Omega^k_{\mathbb{C}}(M)$ decomposes accordingly and the symbol $(\alpha)^{r,s}$ is used to indicate its component in $\Omega^{r,s}(M)$.

The bundles $\Lambda^{r,s}(T^*M) \oplus \Lambda^{s,r}(T^*M)$, for $r \neq s$, and $\Lambda^{r,r}(T^*M)$ are complexifications of real vector bundles, denoted by $[\![\Lambda^{r,s}(T^*M)]\!]$ and $[\Lambda^{r,r}(T^*M)]$, respectively. A real differential form is said to be *of type* $(r,s)+(s,r)$ if it belongs to $[\![\Omega^{r,s}(M)]\!] := \Gamma([\![\Lambda^{r,s}(T^*M)]\!])$, while it is called *of type* (r,r) if it belongs to $[\Omega^{r,r}(M)] := \Gamma([\Lambda^{r,r}(T^*M)])$. In particular, the spaces of real differential forms on M can be written as

$$\begin{aligned} \Omega^{2r}(M) &= \bigoplus_{j=0}^{r-1} [\![\Omega^{2r-j,j}(M)]\!] \oplus [\Omega^{r,r}(M)]\,, \quad r = 1,\dots,n, \\ \Omega^{2r+1}(M) &= \bigoplus_{j=0}^{r} [\![\Omega^{2r+1-j,j}(M)]\!]\,, \quad r = 0,\dots,n-1. \end{aligned}$$

The almost complex structure J extends to an operator on real k-forms as

$$(J\alpha)(X_1,\dots,X_k) = \alpha(JX_1,\dots,JX_k),$$

for $\alpha \in \Omega^k(M)$ and $X_j \in \mathfrak{X}(M)$. Furthermore, it is possible to intoduce the operators $J_{(l)} : \Omega^k(M) \to \Omega^k(M)$, for $l = 1,\dots,k$, defined by

$$\left(J_{(l)}\alpha\right)(X_1,\dots,X_k) = \alpha(X_1,\dots,JX_l,\dots,X_k).$$

Using them, a characterization of real forms of type $(r,s)+(s,r)$ and (r,r) can be given. For instance,

$$\left[\Omega^{1,1}(M)\right] = \left\{\alpha \in \Omega^2(M) \mid J\alpha = \alpha\right\}$$

and

$$[\![\Omega^{r,0}(M)]\!] = \left\{\alpha \in \Omega^r(M) \mid J_{(k)}\left(J_{(l)}\alpha\right) = -\alpha, \text{ for all } k \neq l\right\}.$$

The $\mathbb{C}$-linear extension of the exterior derivative $d : \Omega^k_{\mathbb{C}}(M) \to \Omega^{k+1}_{\mathbb{C}}(M)$ satisfies

$$d : \Omega^{r,s}(M) \to \Omega^{r+2,s-1}(M) \oplus \Omega^{r+1,s}(M) \oplus \Omega^{r,s+1}(M) \oplus \Omega^{r-1,s+2}(M).$$

Consequently, it decomposes in the obvious way as $d = A + \partial + \overline{\partial} + \overline{A}$ and the following result can be proved (see for instance [103, Prop. 2.6.15] and subsequent results).

Proposition 2.1.2. *Let (M, J) be an almost complex manifold. Then, the following are equivalent:*

i) *J is integrable;*

ii) *$d\alpha = \partial\alpha + \overline{\partial}\alpha$ for all $\alpha \in \Omega^{r,s}(M)$;*

iii) *$\overline{A}\alpha = (d\alpha)^{0,2} = 0$ for all $\alpha \in \Omega^{1,0}(M)$.*

When one of the previous conditions holds, the identity $d^2 = 0$ gives $\partial^2 = 0$, $\overline{\partial}^2 = 0$ and $\partial\overline{\partial} = -\overline{\partial}\partial$.

2.1.2 The group $\mathrm{SU}(n)$ as stabilizer of tensors on $\mathbb{R}^{2n}$

As matrix group, the *special unitary group* $\mathrm{SU}(n)$ is defined as the subgroup of the *unitary group* $\mathrm{U}(n)$ whose elements are $n \times n$ unitary matrices having determinant equal to 1. $\mathrm{SU}(n)$ is a compact, connected, simply connected, real Lie group of real dimension $n^2 - 1$, subgroup of $\mathrm{SO}(2n)$. Here, following the philosophy of Section 1.2.1, we describe $\mathrm{SU}(n)$ as the stabilizer of certain tensors defined on $\mathbb{R}^{2n}$.

We recall that an inner product g on the vector space $\mathbb{R}^{2n}$ is said to be *compatible* with a complex structure $J : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ if J is g-orthogonal, i.e., $g(J\cdot, J\cdot) = g(\cdot, \cdot)$. In this case, the tensor

$$\omega(\cdot, \cdot) := g(J\cdot, \cdot) \tag{2.3}$$

is skew-symmetric and, thus, $\omega \in \Lambda^2((\mathbb{R}^{2n})^*)$.

Let us consider the canonical basis $(e_1, \ldots, e_{2n})$ of $\mathbb{R}^{2n}$ with dual basis $(e^1, \ldots, e^{2n})$. It is orthonormal with respect to the inner product

$$g_0 = \sum_{i=1}^{2n} (e^i)^2.$$

As we did in the first chapter, we can choose

$$\begin{array}{rcl} J_0(e_{2k-1}) & = & e_{2k} \\ J_0(e_{2k}) & = & -e_{2k-1} \end{array}, \quad 1 \leq k \leq n.$$

and it is easy to check that g_0 is compatible with J_0. From (2.3), we then get the 2-form

$$\omega_0 = \sum_{k=1}^{n} e^{2k-1} \wedge e^{2k},$$

which is exactly the non-degenerate 2-form of Example 1.2.8 whose stabilizer is the symplectic group $\mathrm{Sp}(2n, \mathbb{R})$.

On the complex vector space $(\mathbb{R}^{2n}, J_0)$, we can define a positive Hermitian inner product

$$h_0(\cdot, \cdot) := g_0(\cdot, \cdot) - i\omega_0(\cdot, \cdot).$$

Since its stabilizer is the unitary group $\mathrm{U}(n)$, it follows from this description that

$$\mathrm{U}(n) = \mathrm{O}(2n) \cap \mathrm{Sp}(2n, \mathbb{R})$$

is the stabilizer of the pair (g_0, ω_0). Moreover, from the fact that any two objects in the triple (g_0, J_0, ω_0) determine the third via (2.3), it is possible to obtain these further equivalent descriptions of the unitary group

$$\mathrm{U}(n) = \mathrm{O}(2n) \cap \mathrm{GL}(n, \mathbb{C}) = \mathrm{GL}(n, \mathbb{C}) \cap \mathrm{Sp}(2n, \mathbb{R})$$

and, from the inclusion $\mathrm{Sp}(2n, \mathbb{R}) \subset \mathrm{SL}(2n, \mathbb{R})$, it is clear that $\mathrm{U}(n)$ can be embedded into $\mathrm{SO}(2n)$ as

$$\mathrm{U}(n) = \{a \in \mathrm{SO}(2n) \mid \omega_0(a\cdot, a\cdot) = \omega_0(\cdot, \cdot)\}.$$

Summarizing, the stabilizer of the triple (g_0, J_0, ω_0) constituted by the inner product g_0, the g_0-orthogonal complex structure J_0 and the non-degenerate 2-form ω_0 is the unitary group $\mathrm{U}(n)$.

Allowing complex coefficients via the usual identification $\mathbb{R}^{2n} \cong \mathbb{C}^n$, the form

$$\Psi_0 = (e^1 + ie^2) \wedge (e^3 + ie^4) \wedge \cdots \wedge (e^{2n-1} + ie^{2n})$$

is a complex form of type $(n, 0)$ with respect to J_0 and an element $a \in \mathrm{GL}(n, \mathbb{C})$ acts on it by multiplication with $\det(a)$, making evident that the stabilizer of Ψ_0 in $\mathrm{GL}(n, \mathbb{C})$ is the group $\mathrm{SL}(n, \mathbb{C})$. The common stabilizer of the tensors $(g_0, J_0, \omega_0, \Psi_0)$ is then the special unitary group

$$\mathrm{SU}(n) = \mathrm{U}(n) \cap \mathrm{SL}(n, \mathbb{C}),$$

which is clearly a subgroup of $\mathrm{SO}(2n)$.

Remark 2.1.3. When $\mathbb{R}^{2n}$ is identified with $\mathbb{C}^n$, using the common notation $dz_k = e^{2k-1} + ie^{2k}$, $k = 1, \dots, n$, for the standard basis of complex linear 1-forms on $\mathbb{C}^n$, it is also possible to write

$$\begin{aligned} \omega_0 &= \frac{i}{2}\sum_{k=1}^{n}\left(dz_k \wedge \overline{dz_k}\right), \\ \Psi_0 &= dz_1 \wedge \ldots \wedge dz_n, \\ g_0 &= \sum_{k=1}^{n}\left(dz_k \overline{dz_k}\right). \end{aligned} \tag{2.4}$$

2.1.3 Special almost Hermitian manifolds

We are now ready to define U(n)- and SU(n)-structures.

Definition 2.1.4. An *almost Hermitian structure* or U(n)-*structure* on a real manifold M of dimension $2n$ is the data of a Riemannian metric g and an almost complex structure J satisfying

$$g(JX, JY) = g(X, Y), \tag{2.5}$$

for any pair of vector fields X, Y on M. A manifold M endowed with an almost Hermitian structure (g, J) is called *almost Hermitian manifold* and is denoted by (M, g, J).

If follows from (2.5) that the tensor

$$\omega(\cdot, \cdot) := g(J\cdot, \cdot) \tag{2.6}$$

is a real 2-form of type $(1,1)$ with respect to J, that is, $\omega \in \left[\Omega^{1,1}(M)\right]$. The 2-form ω is moreover non-degenerate, since g is, and it is called *fundamental form* or *Kähler form* of the almost complex structure (g, J). Its exterior power ω^n is proportional to the Riemannian volume form dV_g of g

$$dV_g = \frac{1}{n!}\,\omega^n.$$

Clearly, any two of the three tensors g, J, ω determine the remaining one via the relation (2.6). Thus, an almost Hermitian structure can be alternatively defined as the data of any two of them.

The minimal connection $\overline{\nabla}$ of an almost Hermitian structure (g, J) on M is given by [63, 82]

$$\overline{\nabla}_X Y = \nabla^g_X Y - \frac{1}{2} J \left(\nabla^g_X J\right) Y, \tag{2.7}$$

for every pair of vector fields X, Y on M. It is metric, satisfies $\overline{\nabla} J = 0$ (cf. Section 1.2.2) and, consequently, $\overline{\nabla}\omega = 0$. The intrinsic torsion of (g, J) is then $\tau = \overline{\nabla} - \nabla^g \in \Gamma(T^*M \otimes \mathfrak{u}(n)^\perp)$ and it can be identified with the covariant derivative of the fundamental form ω with respect to the Levi Civita connection of g by the general result recalled in Proposition 1.2.21. As a consequence, it is possible to classify almost Hermitian manifolds in terms of $\nabla^g\omega$, which is exactly what Gray and Hervella did in [91]. In detail, starting from a real vector space V of dimension $2n$ endowed with an inner product g compatible with an almost complex structure J, they considered the subspace of $(V^*)^{\otimes 3}$

$$W := \{\alpha \in (V^*)^{\otimes 3} \mid \alpha(X, Y, Z) = -\alpha(X, Z, Y) = -\alpha(X, JY, JZ)\}, \tag{2.8}$$

whose dimension as subspace of $(V^*)^{\otimes 3}$ is $2n^2(n-1) = \dim((\mathbb{R}^{2n})^* \otimes \mathfrak{u}(n)^\perp)$, and showed that it decomposes under the action of $\mathrm{U}(n)$ into the direct sum of irreducible $\mathrm{U}(n)$-representations

$$W = W_1 \oplus W_2 \oplus W_3 \oplus W_4,$$

where the summands W_1, W_3 are trivial if $n = 2$ and $W = \{0\}$ for $n = 1$. Then, on an almost Hermitian manifold (M, g, J), they considered the space $\mathcal{W}$ of tensors satisfying the same identities as $\nabla^g\omega$. This space is pointwise given by the subspace W_p of $(T_p^*M)^{\otimes 3}$, defined as in (2.8) with (V, g, J) replaced by (T_pM, g_p, J_p), and splits according to the decomposition of W_p as

$$\mathcal{W} = \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4.$$

When $n \geq 3$, this decomposition allows to divide almost Hermitian manifolds into sixteen classes. For instance, $\mathcal{W}_1$ denotes the set of all almost Hermitian manifolds such that $(\nabla^g\omega)_p \in W_{p1}$ for all $p \in M$, and so on. Moreover, the classes can be all alternatively described in terms of the exterior derivative $d\omega$ and of the Nijenhuis tensor N_J, since they contain the same informations on the intrinsic torsion as $\nabla^g\omega$.

This result follows from the identity (see e.g. [151, Prop. 2.2])

$$2(\nabla^g_X\omega)(Y,Z) = d\omega(X,Y,Z) - d\omega(X,JY,JZ) - g(JX, N_J(Y,Z)). \tag{2.9}$$

For our purposes, we recall the definition of only six classes in Table 2.1. Usually, the name appearing in the table is used to indicate both the manifold and the almost Hermitian structure.

Class	Name	Defining conditions
$\{0\}$	Kähler	$\nabla^g\omega = 0$ or $\nabla^g J = 0$ or $d\omega = 0$ and $N_J = 0$
$\mathcal{W}_1$	nearly Kähler	$(\nabla^g_X J)X = 0$ or $d\omega = 3\nabla^g\omega$
$\mathcal{W}_2$	almost Kähler	$d\omega = 0$
$\mathcal{W}_1 \oplus \mathcal{W}_2$	quasi Kähler	$\overline{\partial}\omega = 0$ or $(\nabla^g_X J)(Y) = -(\nabla^g_{JX} J)(JY)$
$\mathcal{W}_3 \oplus \mathcal{W}_4$	Hermitian	$N_J = 0$
$\mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$	$\mathcal{G}_1$	$g(N_J(X,Y),X) = 0$

Table 2.1: Some classes of almost Hermitian manifolds for $n \geq 3$.

It is worth observing here that the manifolds (M,g,J) in the class $\mathcal{W} = \{0\}$, known as *Kähler manifolds* in literature, have a torsion-free $\mathrm{U}(n)$-structure and, then, $\mathrm{Hol}(g) \subseteq \mathrm{U}(n)$ by Proposition 1.2.22. The equivalence between the defining conditions can be proved using (2.9), the identities

$$\begin{aligned} \left(\nabla^g_X\omega\right)(Y,Z) &= g\left(\left(\nabla^g_X J\right)Y, Z\right), & (2.10)\\ d\omega(X,Y,Z) &= \mathfrak{S}_{X,Y,Z}(\nabla^g_X\omega)(Y,Z) & (2.11)\\ &= (\nabla^g_X\omega)(Y,Z) + (\nabla^g_Y\omega)(Z,X) + (\nabla^g_Z\omega)(X,Y), \end{aligned}$$

and the expression of the Nijenhuis tensor in terms of the covariant derivative $\nabla^g J$

$$N_J(X,Y) = (\nabla^g_{JY}J)X + J(\nabla^g_X J)Y - (\nabla^g_{JX}J)Y - J(\nabla^g_Y J)X. \tag{2.12}$$

For the complete classification of almost Hermitian manifolds, the description of the summands $\mathcal{W}_i$ and more details on the construction, we refer the reader to the paper [91]. The properties of some classes of manifolds appearing in Table 2.1 were also studied by Gray in [86, 89], where it is possible to find explicit examples.

We can now introduce SU(n)-structures, also known as *special almost Hermitian structures* in literature.

Definition 2.1.5. An SU(n)-*structure* on a real manifold M of dimension $2n$ is the data of an almost Hermitian structure (g, J) and a complex $(n, 0)$-form Ψ of nonzero constant length satisfying the *normalization condition*

$$\Psi \wedge \overline{\Psi} = (-1)^{\frac{n(n+1)}{2}} \frac{(2i)^n}{n!} \omega^n,$$

where ω is the fundamental form of (g, J).

Since ω is of type $(1, 1)$ and Ψ is of type $(n, 0)$, their wedge product is zero. The equation $\omega \wedge \Psi = 0$ is sometimes called the *compatibility condition* between ω and Ψ.

Remark 2.1.6. Observe that given an SU(n)-structure, we can take the tensors $g_0, J_0, \omega_0, \Psi_0$ introduced in the previous section as model tensors for g, J, ω and Ψ, respectively.

The intrinsic torsion of an SU(n)-structure (g, J, Ψ) is a section of the vector bundle $\mathcal{W} = T^*M \otimes \mathfrak{su}(n)^\perp$ and is completely determined by the covariant derivatives $\nabla^g \omega$ and $\nabla^g \Psi$. The decomposition of $\mathcal{W}$ into SU(n)-irreducible components depends on n. In [40], Chiossi and Salamon studied the case $n = 3$, while in [142], Martín Cabrera described the case $n \geq 4$, generalizing some results of [40]. When $n \geq 4$, the SU(n)-irreducible decomposition is

$$\mathcal{W} = \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4 \oplus \mathcal{W}_5,$$

where the first four summands are exactly those appearing in Gray and Hervella's description previously recalled and $\mathcal{W}_5 \cong T^*M$. When $n = 3$, the spaces $\mathcal{W}_1$ and $\mathcal{W}_2$ further decompose into the direct sum of two SU(3)-irreducible components. We describe this situation more in detail in the next section, while we refer the reader to [142] for the case $n \geq 4$.

2.2 SU(3)-structures and their classification

2.2.1 SU(3)-structures revisited

Let M be a six-dimensional manifold endowed with an SU(3)-structure. M admits an almost Hermitian structure (g, J) with fundamental form ω and a complex $(3,0)$-form Ψ of nonzero constant length. We can write

$$\Psi = \psi_+ + i\psi_-,$$

where $\psi_+ := \Re(\Psi)$ and $\psi_- := \Im(\Psi)$ are real forms of type $(3,0)+(0,3)$. The compatibility condition then reads

$$\omega \wedge \psi_\pm = 0,$$

the normalization condition is

$$\psi_+ \wedge \psi_- = \frac{2}{3}\,\omega^3 = 4\,dV_g,$$

and the 3-forms ψ_+ and ψ_- are related by

$$\psi_- = J\psi_+, \quad \psi_+ = -J\psi_-.$$

Moreover, the metric g and the volume form $dV_g = \frac{\omega^3}{6}$ determine the Hodge operator $* : \Omega^k(M) \to \Omega^{6-k}(M)$, $k = 0, \dots, 6$, which is an isometry of the metric induced by g on $\Lambda^k(T^*M)$, satisfies $*^2\alpha = (-1)^k\alpha$ for every $\alpha \in \Omega^k(M)$ and commutes with the almost complex structure J

$$J* = *J.$$

At each point p of M, it is always possible to find a g-orthonormal basis $(e_1, \dots, e_6)$ of T_pM with dual basis $(e^1, \dots, e^6)$ which is adapted for the SU(3)-structure. This means that at p one can always write

$$\begin{array}{rcccl} \omega &=& e^{12}+e^{34}+e^{56} &=& \frac{1}{2}\omega_{jk}e^{jk}, \\ \psi_+ &=& e^{135}-e^{146}-e^{236}-e^{245} &=& \frac{1}{6}\psi_{jkl}e^{jkl}, \\ \psi_- &=& e^{136}+e^{145}+e^{235}-e^{246}, && \end{array} \tag{2.13}$$

where the symbols ω_{jk} and ψ_{jkl} are skew-symmetric in their indices and uniquely defined via the previous identities, and

$$J(e_{2k-1}) = e_{2k}, \quad J(e_{2k}) = -e_{2k-1}, \quad k = 1, 2, 3. \tag{2.14}$$

In particular, $\Psi = \psi_+ + i\psi_- = (e^1 + ie^2) \wedge (e^3 + ie^4) \wedge (e^5 + ie^6)$ at p.

We call equally the basis $(e_1, \dots, e_6)$ and its dual $(e^1, \dots, e^6)$ an SU(3)-*basis* for the SU(3)-structure at the point p.

Using the local expressions (2.13), it is easy to show that the Riemannian metric g can also be described in terms of ω and ψ_+ as

$$g(X, Y)\,\frac{\omega^3}{6} = -\frac{1}{2}(\iota_X\omega) \wedge (\iota_Y\psi_+) \wedge \psi_+,$$

for all $X, Y \in T_pM$, where ι denotes the *contraction* of differential forms by vectors, and that for every $\alpha \in \Omega^1(M)$ and $X \in \mathfrak{X}(M)$

$$\alpha \wedge \omega = 0 \iff \alpha = 0, \tag{2.15}$$

$$\alpha \wedge \omega^2 = 0 \iff \alpha = 0, \tag{2.16}$$

$$\alpha \wedge \psi_\pm = 0 \iff \alpha = 0, \tag{2.17}$$

$$\iota_X\psi_\pm = 0 \iff X = 0. \tag{2.18}$$

Furthermore, the Hodge operator applied to ω and to ψ_+ gives

$$\begin{aligned} *\omega &= \frac{1}{2}\,\omega^2, \\ *\psi_+ &= \psi_-, \end{aligned}$$

from which follows in particular that $|\psi_+|^2 = |\psi_-|^2 = 4$. Indeed, the norms of ψ_+ and ψ_- are the same, since $*$ is an isometry, and

$$|\psi_+|^2\,dV_g = g(\psi_+, \psi_+)dV_g = \psi_+ \wedge *\psi_+ = \psi_+ \wedge \psi_- = 4\,dV_g.$$

The Riemannian metric g is not the only tensor depending on ω and ψ_+. In fact, the whole SU(3)-structure is completely determined by these differential forms. This is a long-standing result, which follows from Reichel's thesis [161] of 1907 and which was later reformulated by Hitchin in [101]. The starting point is the observation that the differential forms ω and ψ_+ are *stable* in the sense of the following

Definition 2.2.1. Let V be a real vector space of dimension m, a k-form $\sigma \in \Lambda^k(V^*)$ is *stable* if its orbit under the action of $\mathrm{GL}(V)$ is open in $\Lambda^k(V^*)$. A k-form $\sigma \in \Omega^k(M)$ on a manifold M is *stable* if the k-form $\sigma(p)$ on T_pM is stable for every $p \in M$.

Stability occurs in very few cases, namely for $k = 2, m-2$ when m is even and $k = 3, m-3$ when $m = 6, 7, 8$. Moreover, given a stable form it is always possible to define a volume form from it. We shall describe the case $m = 6$ in what follows and $m = 7$ in the next chapter. The reader may refer to [53, 102] for a complete picture.

Stable 2-forms on a six-dimensional vector space represent a special case of a more general situation: a 2-form on a $2n$-dimensional vector space V is stable if and only if it is non-degenerate. Indeed, $\Lambda^2(V^*)$ contains only one open orbit, which must coincide with the orbit $\mathrm{GL}(2n,\mathbb{R})/\mathrm{Sp}(2n,\mathbb{R})$ of a non-degenerate 2-form by dimension counting. Given a $2n$-manifold M, this implies that $\omega \in \Omega^2(M)$ is stable if and only if $\omega^n \neq 0$. The volume form defined by ω is the so-called *Liouville volume form* $\frac{1}{n!}\,\omega^n$.

Suppose now that V is an oriented, six-dimensional real vector space with volume form $\Omega \in \Lambda^6(V^*)$. There is a canonical isomorphism

$$A : \Lambda^5(V^*) \to V \otimes \Lambda^6(V^*),$$

defined for every $\alpha \in \Lambda^5(V^*)$ by $A(\alpha) = v \otimes \Omega$, where $v \in V$ is the unique vector such that $\iota_v \Omega = \alpha$. Fix a 3-form $\rho \in \Lambda^3(V^*)$ and define

$$K_\rho : V \to V \otimes \Lambda^6(V^*), \quad K_\rho(v) = A((\iota_v \rho) \wedge \rho)$$

and

$$\lambda : \Lambda^3(V^*) \to \left(\Lambda^6(V^*)\right)^{\otimes 2}, \quad \lambda(\rho) = \frac{1}{6}\left(\mathrm{tr} K_\rho^2\right).$$

$\lambda(\rho)$ is said to be *positive* and is denoted by $\lambda(\rho) > 0$ if there exists $\beta \in \Lambda^6(V^*)$ such that $\lambda(\rho) = \beta \otimes \beta$, while $\lambda(\rho) < 0$ if $-\lambda(\rho)$ is positive. By [101],

$$\rho \text{ is stable} \Longleftrightarrow \lambda(\rho) \neq 0.$$

In this case, the positively oriented squared root $\sqrt{|\lambda(\rho)|} \in \Lambda^6(V^*)$ defines a volume form on V. Moreover, the space $\Lambda^3(V^*)$ contains an invariant quartic hypersurface

$\lambda(\rho) = 0$ which divides it into the open subsets

$$\begin{aligned} \mathcal{O}^+ &= \{\rho \in \Lambda^3(V^*) \mid \lambda(\rho) > 0\}, \\ \mathcal{O}^- &= \{\rho \in \Lambda^3(V^*) \mid \lambda(\rho) < 0\}. \end{aligned}$$

If $(v^1, \dots, v^6)$ is an oriented basis of V^*, $\mathcal{O}^-$ is the open $\mathrm{GL}(V)$-orbit of the 3-form

$$v^{135} - v^{146} - v^{236} - v^{245}$$

and the identity component of the stabilizer of a 3-form lying in it is conjugated to $\mathrm{SL}(3, \mathbb{C})$. As a consequence, $\rho \in \mathcal{O}^-$ defines a complex structure J_ρ on V, which is given by

$$J_\rho = -\frac{1}{\sqrt{|\lambda(\rho)|}}\, K_\rho. \tag{2.19}$$

Moreover, ρ is the real part of the complex (3,0)-form

$$\rho + i(J_\rho \rho).$$

Remark 2.2.2. Observe that the complex structure induced by a stable 3-form $\rho \in \mathcal{O}^-$ does not change if ρ is rescaled by a nonzero real constant, i.e., $J_\rho = J_{r\rho}$ for every $r \in \mathbb{R} - \{0\}$.

Remark 2.2.3. The expression (2.19) for J_ρ differs from that given in the papers [53, 101, 102] by a sign. This is due to the fact that here we are using a convention in the definition of SU(3)-structures which is slightly different from the one used by the authors in the aforementioned papers.

Given a stable 2-form ω and a stable 3-form ρ on V, it is possible to consider the orientation defined by the volume form $\Omega := \frac{1}{6}\,\omega^3$ and define J_ρ in the way previously described. The forms ω and ρ are said to be *compatible* if

$$\omega \wedge \rho = 0 \Longleftrightarrow \omega \in \left[\Lambda^{1,1}(V^*)\right]$$

and *normalized* if

$$\rho \wedge (J_\rho \rho) = \frac{2}{3}\omega^3.$$

When the symmetric tensor $g(\cdot,\cdot) := \omega(\cdot, J_\rho \cdot)$ is positive definite, the pair of compatible and normalized stable forms (ω, ρ) defines an SU(3)-structure on the vector space V.

Referring to the description of the group SU(n) given in Section 2.1.2 and using the previous results, we shall see in the next example that for $n = 3$ the 2-form ω_0 and the 3-form $\Re(\Psi_0)$ on $\mathbb{R}^6$ are sufficient to determine the data $(g_0, J_0, \omega_0, \Psi_0)$. As a consequence, the group SU(3) can be described as

$$\mathrm{SU}(3) = \mathrm{Sp}(6, \mathbb{R}) \cap \mathrm{SL}(3, \mathbb{C}).$$

Example 2.2.4. On the vector space $\mathbb{R}^6$ with canonical basis $(e_1, \ldots, e_6)$ and dual basis $(e^1, \ldots, e^6)$, consider the stable 2-form $\omega_0 = e^{12} + e^{34} + e^{56}$ and the 3-form

$$\rho_0 = \Re(\Psi_0) = e^{135} - e^{146} - e^{236} - e^{245}.$$

The pair (ω_0, ρ_0) is compatible, ω_0 induces the volume form $\Omega = \frac{1}{6}\,\omega_0^3 = e^{123456}$ and a simple computation shows that for $k = 1, 2, 3$

$$K_{\rho_0}(e_{2k-1}) = -2\, e_{2k} \otimes \Omega, \quad K_{\rho_0}(e_{2k}) = 2\, e_{2k-1} \otimes \Omega,$$

from which follows

$$\lambda(\rho_0) = -4\,\Omega \otimes \Omega < 0.$$

Then, the 3-form ρ_0 is stable, it defines the volume form $\sqrt{|\lambda(\rho_0)|} = 2\,\Omega$ and the complex structure J_{ρ_0} given on the basis vectors by

$$J_{\rho_0}(e_{2k-1}) = e_{2k}, \quad J_{\rho_0}(e_{2k}) = -e_{2k-1}, \quad k = 1, 2, 3.$$

Therefore, $g_0(\cdot,\cdot) = \omega_0(\cdot, J_{\rho_0}\cdot)$ is the inner product

$$g_0 = \sum_{k=1}^{6} (e^k)^2,$$

and

$$J_{\rho_0}\rho_0 = e^{136} + e^{145} + e^{235} - e^{246} = \Im(\Psi_0).$$

The previous construction extends in the obvious way to the manifold level with V replaced by the tangent spaces. Thus, if a 6-manifold M is endowed with a pair of stable forms $\omega \in \Omega^2(M)$ and $\rho \in \Omega^3(M)$, with $\lambda(\rho(p)) < 0$ for each $p \in M$, then $J : TM \to TM$, $J_p = J_{\rho(p)}$, defines an almost complex structure on it and the following alternative definition of SU(3)-structures can be given (see also [167, Prop. 3.3])

Definition 2.2.5. Let M be a six-dimensional manifold. An SU(3)-structure on M is a pair of stable forms $(\omega, \rho) \in \Omega^2(M) \times \Omega^3(M)$, with $\lambda(\rho(p)) < 0$ for each point p of M, which are compatible, normalized and induce a Riemannian metric $g(\cdot,\cdot) := \omega(\cdot, J_\rho \cdot)$.

To be consistent with the notations introduced earlier, form now on we use ψ_+ instead of ρ to denote the stable 3-form appearing in the definition of an SU(3)-structure. The almost complex structure associated with (ω, ψ_+) is then $J = J_{\psi_+}$, the complex $(3,0)$-form is $\Psi = \psi_+ + i\psi_-$, where $\psi_- = J\psi_+$, and the Riemannian metric is $g(\cdot,\cdot) = \omega(\cdot, J\cdot)$.

As observed in [165], since the construction of the tensors J, ψ_- and g from the pair (ω, ψ_+) defining an SU(3)-structure is invariant, a diffeomorphism $\nu : M \to M$ preserving the stable forms ω and ψ_+ preserves also J, ψ_- and g. Thus, it is an automorphism of the SU(3)-structure and, in particular, an isometry.

In this thesis, when we consider SU(3)-structures we mainly refer to Definition 2.2.5. Nevertheless, it is worth mentioning here that an alternative description of SU(3)-structures can be given using the spinorial approach. Indeed, any six-dimensional Riemannian manifold (M, g) admits an SU(3)-structure if and only if it is orientable and has a *spin structure* [129]. It is then possible to consider the *spinor bundle* ΣM over (M, g), which is a complex vector bundle with typical fiber $\mathbb{C}^8$, and show that there is a correspondence between SU(3)-structures and unit real spinor fields, that is, global sections $\phi \in \Gamma(\Sigma M)$ of length one satisfying $\overline{\phi} = \phi$. Moreover, up to a sign in the choice of ϕ, the correspondence is one-to-one. We will review this result in detail in Section 4.3.2 of Chapter 4. For more informations on spin structures and related topics, the reader may refer for instance to [16, 18, 76, 129].

2.2.2 The classification of SU(3)-structures

Let M be a 6-manifold endowed with an SU(3)-structure (ω, ψ_+). The intrinsic torsion τ of (ω, ψ_+) is a section of the rank 42 vector bundle $\mathcal{W}$ pointwise modeled on the space $W = (\mathbb{R}^6)^* \otimes \mathfrak{su}(3)^\perp$, which by [40] decomposes into SU(3)-irreducible summands as

$$W = W_1^+ \oplus W_1^- \oplus W_2^+ \oplus W_2^- \oplus W_3 \oplus W_4 \oplus W_5,$$

where $W_1^{\pm} \cong \mathbb{R}$, $W_2^{\pm} \cong \mathfrak{su}(3)$, $W_3 \cong [\![\Lambda_0^{2,1}((\mathbb{R}^6)^*)]\!]$, the space of real forms of type $(2,1)+(1,2)$ whose wedge product with ω_0 is zero, and $W_4, W_5 \cong (\mathbb{R}^6)^*$. Accordingly,

$$\mathcal{W} = \mathcal{W}_1^+ \oplus \mathcal{W}_1^- \oplus \mathcal{W}_2^+ \oplus \mathcal{W}_2^- \oplus \mathcal{W}_3 \oplus \mathcal{W}_4 \oplus \mathcal{W}_5.$$

Moreover, by [40, Thm. 1.1], τ is completely determined by the exterior derivatives of ω, ψ_+ and ψ_-. In detail, the irreducible decompositions of the SU(3)-modules $\Lambda^3((\mathbb{R}^6)^*)$ and $\Lambda^4((\mathbb{R}^6)^*)$ induce on (M, ω, ψ_+) the g-orthogonal decompositions

$$\begin{aligned} \Omega^3(M) &= \underbrace{C^\infty(M)\,\psi_+ \oplus C^\infty(M)\,\psi_-}_{(3,0)+(0,3)} \oplus \underbrace{\left[\!\left[\Omega_0^{2,1}(M)\right]\!\right] \oplus \Omega^1(M) \wedge \omega}_{(2,1)+(1,2)}, \\ \Omega^4(M) &= \underbrace{C^\infty(M)\,\omega^2 \oplus \left[\Omega_0^{1,1}(M)\right] \wedge \omega}_{(2,2)} \oplus \underbrace{\Omega^1(M) \wedge \psi_+}_{(3,1)+(1,3)}, \end{aligned} \tag{2.20}$$

where

$$\left[\!\left[\Omega_0^{2,1}(M)\right]\!\right] = \left\{\alpha \in [\![\Omega^{2,1}(M)]\!] \;\mid \alpha \wedge \omega = 0\right\}$$

is the space of *primitive* real forms of type $(2,1) + (1,2)$ and

$$\left[\Omega_0^{1,1}(M)\right] = \left\{\beta \in \left[\Omega^{1,1}(M)\right] \;\mid \beta \wedge \omega^2 = 0\right\}$$

is the space of *primitive* real forms of type $(1,1)$. Consequently (see also [19]), there exist unique differential forms $w_1^{\pm} \in C^\infty(M)$, $w_2^{\pm} \in \left[\Omega_0^{1,1}(M)\right]$, $w_3 \in \left[\!\left[\Omega_0^{2,1}(M)\right]\!\right]$, $w_4 \in \Omega^1(M)$ and $w_5 \in \Omega^1(M)$, such that

$$\begin{aligned} d\omega &= -\tfrac{3}{2}w_1^-\psi_+ + \tfrac{3}{2}w_1^+\psi_- + w_3 + w_4 \wedge \omega, \\ d\psi_+ &= w_1^+\omega^2 - w_2^+ \wedge \omega + w_5 \wedge \psi_+, \\ d\psi_- &= w_1^-\omega^2 - w_2^- \wedge \omega + Jw_5 \wedge \psi_+, \end{aligned} \tag{2.21}$$

and the component of τ in $\mathcal{W}_k^{(\pm)}$ vanishes identically if and only if $w_k^{(\pm)}$ does.

Definition 2.2.6. The differential forms $w_k^{(\pm)}$, uniquely defined via (2.21), are called *intrinsic torsion forms* of the SU(3)-structure (ω, ψ_+).

The classification of SU(3)-structures can then be stated in terms of the identically vanishing intrinsic torsion forms. This gives rise to $2^7 = 128$ classes overall, which are denoted by the corresponding decomposition of $\mathcal{W}$. When $w_k^{(\pm)} = 0$ for all $k = 1, \ldots, 5$, the intrinsic torsion vanishes identically and the class is $\mathcal{W} = \{0\}$. The next proposition, whose proof follows from the previous observations and Proposition 1.2.22, summarizes the equivalent defining properties of this class.

Proposition 2.2.7. *Let M be a connected six-dimensional manifold endowed with an* SU(3)*-structure (ω, ψ_+) inducing a Riemannian metric g. Then, the following are equivalent:*

i) *the* SU(3)*-structure is torsion-free;*

ii) *the intrinsic torsion forms $w_k^{(\pm)}$ vanish identically;*

iii) *the differential forms ω, ψ_+, ψ_- are closed;*

iv) *the differential forms ω and Ψ are parallel with respect to ∇^g.*

When one of the previous conditions holds, $\mathrm{Hol}(g) \subseteq \mathrm{SU}(3)$.

A further property of a torsion-free SU(3)-structure is that the Riemannian metric g induced by it is Ricci-flat, that is, $\mathrm{Ric}(g) = 0$. This result holds true for all metrics with holonomy in $\mathrm{SU}(n)$, as shown for instance in [110, Prop. 7.1.1]. In the case $n = 3$, a simple proof can be obtained using the description of the Ricci tensor in terms of the intrinsic torsion forms given in [19]. We shall recall it later.

As one would expect from the more general case of almost Hermitian manifolds (cf. Table 2.1), the integrability of the almost complex structure J_{ψ_+} depends on the intrinsic torsion forms $w_1^{\pm}$ and $w_2^{\pm}$. Indeed, $w_1^{\pm} = 0$ and $w_2^{\pm} = 0$ if and only if the $(2, 2)$ part of the exterior derivatives of ψ_+ and ψ_- is zero and this happens if and only if $\overline{A}\Psi = (d\Psi)^{2,2} = 0$. The assertion then follows from

Proposition 2.2.8. *Let (ω, ψ_+) be an* SU(3)*-structure on a 6-manifold M. The almost complex structure $J = J_{\psi_+}$ is integrable if and only if $\overline{A}\Psi = 0$.*

Proof. We know from Proposition 2.1.2 that J is integrable if and only if $\overline{A}\alpha = 0$ for every $\alpha \in \Omega^{1,0}(M)$. Since $\Psi \in \Omega^{3,0}(M)$, $\alpha \wedge \Psi = 0$ and applying the exterior derivative d to both sides of this identity we obtain

$$\overline{A}\alpha \wedge \Psi = \alpha \wedge \overline{A}\Psi.$$

Since $\overline{A}\alpha \wedge \Psi = 0$ implies $\overline{A}\alpha = 0$, we get that $\overline{A}\alpha = 0$ for all $\alpha \in \Omega^{1,0}(M)$ if and only if $\overline{A}\Psi = 0$. □

Furthermore, from the description of Gray and Hervella's class $\mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$ given in [91], it is possible to deduce the following property

Proposition 2.2.9. *Let (ω, ψ_+) be an* SU(3)*-structure on a 6-manifold M and let N_J be the Nijenhuis tensor of the corresponding almost complex structure $J = J_{\psi_+}$. Then, the intrinsic torsion forms $w_2^{\pm}$ vanish identically if and only if N_J is totally skew-symmetric, namely $g(N_J(\cdot,\cdot),\cdot)$ is a 3-form on M.*

Proof. By [91], we have that the component of the intrinsic torsion in $\mathcal{W}_2$ vanishes identically if an only if $g(N_J(X,Y),X) = 0$ for all $X, Y \in \mathfrak{X}(M)$. Thus, $g(N_J(\cdot,\cdot),\cdot)$ is a 3-form, since $N_J(X,Y) = -N_J(Y,X)$ (see (2.1)). □

The next definition was introduced in [40] to denote a distinguished class of SU(3)-structures.

Definition 2.2.10 ([40])**.** A six-dimensional almost Hermitian manifold is *half-flat* if its structure group admits a reduction to SU(3) for which $d\psi_+ = 0$ and $d\omega^2 = 0$. In this case, the SU(3)-structure (ω, ψ_+) is called *half-flat*.

Using the expressions of the exterior derivatives of ω and ψ_+, it is easy to show that the half-flat condition is equivalent to require that the only possibly non-identically vanishing intrinsic torsion forms are w_1^-, w_2^- and w_3. Indeed, from $d\psi_+ = 0$ we get that the forms w_1^+, w_2^+ and w_5 vanish identically, while from

$$0 = d\omega \wedge \omega = w_4 \wedge \omega^2$$

and (2.16) we obtain $w_4 = 0$. Thus, the intrinsic torsion of a half-flat SU(3)-structure is a section of the vector bundle $\mathcal{W}_1^- \oplus \mathcal{W}_2^- \oplus \mathcal{W}_3$, whose rank $1 + 8 + 12 = 21$ is exactly half of the rank of $\mathcal{W}$. In this sense the name *half-flat* refers to the fact that the SU(3)-structure is "half torsion-free" or "half-integrable".

Half-flat SU(3)-structures can be used to construct metrics with holonomy in G_2. This remarkable property was first observed in [102] by Hitchin, who introduced a system of evolution equations for the differential forms ω and ψ_+ which allows to prove it. To explain how this system is obtained, we need to recall some facts about G_2-structures, therefore we will do this in the next chapter (see Section 3.2.1). For the moment, we review the definition and the main properties.

Suppose that the forms ω, ψ_+, ψ_- defining an SU(3)-structure on a 6-manifold M depend on a real parameter t. Then, the system is the following

$$\begin{cases} \frac{\partial}{\partial t}\psi_+(t) = d\omega(t) \\ \frac{\partial}{\partial t}\omega(t) \wedge \omega(t) = -d\psi_-(t) \end{cases} \tag{2.22}$$

and the equations are usually called *Hitchin flow equations* in literature. (2.22) is not a geometric flow in the usual sense (cf. Section 4.2), but it can be obtained as the Hamiltonian flow of a certain functional (see [102]).

A *solution* of (2.22) starting from a given SU(3)-structure (ω, ψ_+) at $t_0 \in \mathbb{R}$ is a one-parameter family of SU(3)-structures $(\omega(t), \psi_+(t))$ with parameter t belonging to an interval $\mathrm{I} \subseteq \mathbb{R}$ containing t_0, which solves the Hitchin flow equations and satisfies $(\omega(t_0), \psi_+(t_0)) = (\omega, \psi_+)$. If the initial condition is half-flat, then the solution is half-flat as long as it exists. Indeed, differentiating $d\left(\psi_+(t)\right)$ with respect to t and using the first evolution equation of (2.22) we get

$$\frac{\partial}{\partial t}(d\psi_+(t)) = d\left(\frac{\partial}{\partial t}\psi_+(t)\right) = d(d\omega(t)) = 0,$$

from which follows $d\left(\psi_+(t)\right) = d\left(\psi_+(t_0)\right) = 0$, while differentiating $d(\omega(t))^2$ with respect to t and using the second evolution equation we obtain

$$\frac{\partial}{\partial t}(d(\omega(t))^2) = 2\,d\left(\frac{\partial}{\partial t}\omega(t) \wedge \omega(t)\right) = -2d(d\psi_-(t)) = 0,$$

and, then, $d(\omega(t))^2 = d(\omega(t_0))^2 = 0$. Moreover, a family of stable forms $(\omega(t), \psi_+(t))$ defined for t in a real interval I and satisfying the Hitchin flow equations is an SU(3)-structure for all $t \in \mathrm{I}$ if the initial condition $(\omega(t_0), \psi_+(t_0)) = (\omega, \psi_+)$ is a half-flat SU(3)-structure. This result was proved in [102] for compact 6-manifolds and was later generalized in [53] in the noncompact case. Finally, it is possible to show that a solution of (2.22) with initial condition a given SU(3)-structure (ω, ψ_+) exists when the latter is half-flat and analytic, but may not exist when the analytic hypothesis is dropped (see [31]).

Clearly, torsion-free SU(3)-structures trivially satisfy the equations defining a half-flat SU(3)-structure. Thus, we may refer to them as *trivial half-flat*. Moreover, we may call *special half-flat* the SU(3)-structures belonging to a subclass of $\mathcal{W}_1^- \oplus \mathcal{W}_2^- \oplus \mathcal{W}_3$, since they satisfy the conditions $d\psi_+ = 0$ and $d\omega^2 = 0$ but have smaller intrinsic torsion. For instance, the *symplectic half-flat* structures introduced in [56] and defined by $d\omega = 0$ and $d\psi_+ = 0$ are special half-flat, as their class is $\mathcal{W}_2^-$. We will consider further special half-flat structures and study their properties in the next sections.

Remark 2.2.11. When SU(3)-structures are described using the spinorial formalism outlined at the end of Section 2.2.1, the classification can be stated in terms of the unit real spinor field and the spinorial field equations it satisfies, as shown in [2]. Some results will be recalled in Section 4.3.2.

2.2.3 The Ricci tensor of an SU(3)-structure

As shown by Bedulli and Vezzoni in [19], the Ricci tensor and the scalar curvature of the Riemannian metric g induced by an SU(3)-structure (ω, ψ_+) can be completely expressed in terms of the intrinsic torsion forms and their derivatives. We recall here their results.

Theorem 2.2.12 ([19])**.** *The scalar curvature of the metric g induced by an* SU(3)*-structure is expressed in terms of the intrinsic torsion forms as*

$$\begin{aligned}\mathrm{Scal}(g) &= \frac{15}{2}\left(w_1^+\right)^2 + \frac{15}{2}\left(w_1^-\right)^2 - \frac{1}{2}\left|w_2^-\right|^2 - \frac{1}{2}\left|w_2^+\right|^2 - \frac{1}{2}|w_3|^2 \\ &\quad +2d^*w_5 + 2d^*w_4 - |w_4|^2 + 4g(w_4, w_5).\end{aligned}$$

It is then possible to obtain informations on the scalar curvature of certain classes of manifolds admitting an SU(3)-structure. For example

Corollary 2.2.13 ([19])**.** *The scalar curvature of a symplectic half-flat manifold is* $\mathrm{Scal}(g) = -\frac{1}{2}|w_2^-|^2$*. Thus, it is everywhere nonpositive and vanishes identically if and only if the* SU(3)*-structure is torsion-free.*

The Ricci tensor belongs to the space $\mathcal{S}^2(M)$ of symmetric 2-covariant tensor fields on M, whose decomposition, induced by the SU(3)-irreducible decomposition of $\mathcal{S}^2((\mathbb{R}^6)^*)$, is

$$\mathcal{S}^2(M) = C^\infty(M)\, g \oplus \mathcal{S}^2_+(M) \oplus \mathcal{S}^2_-(M),$$

where

$$\begin{array}{rcl} \mathcal{S}^2_+(M) & = & \left\{\sigma \in \mathcal{S}^2(M) \mid J\sigma = \sigma \text{ and } \mathrm{tr}_g\sigma = 0\right\}, \\ \mathcal{S}^2_-(M) & = & \left\{\sigma \in \mathcal{S}^2(M) \mid J\sigma = -\sigma\right\}. \end{array}$$

We can write

$$\mathrm{Ric}(g) = \frac{1}{6}\,\mathrm{Scal}(g) g + \mathrm{Ric}^0(g),$$

and the traceless part $\mathrm{Ric}^0(g)$ of the Ricci tensor belongs to $\mathcal{S}^2_+(M) \oplus \mathcal{S}^2_-(M)$. Using the map $\mathrm{i}_+ : \mathcal{S}^2_+(M) \to \left[\Omega^{1,1}_0(M)\right]$ induced by the pointwise SU(3)-module isomorphism

$$\mathrm{i}_+\left(\sigma_{jk} e^j e^k\right) = \sigma_{jr}\omega_{rk} e^{jk},$$

the map $\mathrm{i}_- : \mathcal{S}^2_-(M) \to \left[\!\left[\Omega^{2,1}_0(M)\right]\!\right]$ induced by the pointwise SU(3)-module isomorphism

$$\mathrm{i}_-\left(\sigma_{jk} e^j e^k\right) = \sigma_{jr}\psi_{rkl} e^{jkl},$$

and the projections $\mathrm{E}_1 : \Omega^2(M) \to \left[\Omega^{1,1}_0(M)\right]$ and $\mathrm{E}_2 : \Omega^3(M) \to \left[\!\left[\Omega^{2,1}_0(M)\right]\!\right]$ defined as

$$\begin{array}{rcl} \mathrm{E}_1(\beta) & = & \frac{1}{2}(\beta + J\beta) - \frac{1}{18} * ((*(\beta + J\beta) + (\beta + J\beta)\wedge\omega)\wedge\omega)\,\omega, \\ \mathrm{E}_2(\alpha) & = & \alpha - \frac{1}{2} * (J\alpha\wedge\omega)\wedge\omega - \frac{1}{4} * (\alpha\wedge\psi_-)\,\psi_+ - \frac{1}{4} * (\psi_+\wedge\alpha)\,\psi_-, \end{array}$$

the result for the Ricci tensor can be stated as follows

Theorem 2.2.14 ([19])**.** *The traceless part of the Ricci tensor of the metric g induced by an* SU(3)*-structure is*

$$\mathrm{Ric}^0(g) = \mathrm{i}_+^{-1}(\mathrm{E}_1(\Phi_1)) + \mathrm{i}_-^{-1}(\mathrm{E}_2(\Phi_2)), \tag{2.23}$$

where the 2-form Φ_1 *and the 3-form* Φ_2 *are given by*

$$\begin{aligned}\Phi_1 &= -*(w_4\wedge Jw_3)+\frac{1}{4}*\left(w_2^+\wedge w_2^+\right)+\frac{1}{4}*\left(w_2^-\wedge w_2^-\right)\\ &\quad +d(Jw_5)+\frac{1}{2}d^*w_3+\frac{1}{2}d^*(w_4\wedge\omega)-\frac{1}{4}d^*\left(w_1^+\psi_+\right)+\frac{1}{4}d^*\left(w_1^-\psi_+\right),\\ \Phi_2 &= -2w_1^-w_3-4w_2^-\wedge w_4-2Jdw_2^+-2*Jdw_2^--4d*(w_4\wedge *\psi_+)\\ &\quad -2d*(Jw_5\wedge\psi_+)+2w_1^+Jw_3-2Jd*(w_5\wedge\psi_+)-4w_2^+\wedge Jw_5\\ &\quad +4w_4\wedge *(Jw_5\wedge\psi_+)-2(Jw_4)\wedge *(w_4\wedge\psi_+)-\frac{1}{2}\,\mathrm{Q}(w_3,w_3),\end{aligned}$$

and $\mathrm{Q}:\left[\!\left[\Omega_0^{2,1}(M)\right]\!\right]\times\left[\!\left[\Omega_0^{2,1}(M)\right]\!\right]\to\Omega^3(M)$ *is the bilinear map defined by* $\mathrm{Q}(\alpha,\eta)=\psi_{jkl}(\iota_{e_k}\iota_{e_j}\alpha)\wedge(\iota_{e_l}\eta)$*, being* $(e_1,\ldots,e_6)$ *an adapted basis for* (ω,ψ_+)*.*

From this description of $\mathrm{Ric}(g)$ and Proposition 2.2.7, the following result is immediate

Proposition 2.2.15. *Let M be a 6-manifold endowed with a torsion-free* SU(3)*-structure. Then, the associated Riemannian metric is Ricci-flat.*

Moreover, using (2.23) it is possible to characterize the Einstein condition for g in terms of the intrinsic torsion forms. Indeed, g is Einstein if and only if $\mathrm{Ric}^0(g)$ vanishes identically and this happens if and only if both $\mathrm{E}_1(\Phi_1)$ and $\mathrm{E}_2(\Phi_2)$ are zero. In the general case, these conditions are not very useful to draw conclusions. Nevertheless, some interesting results can be obtained for certain classes of SU(3)-structures. For instance

Proposition 2.2.16 ([19])**.** *A symplectic half-flat manifold is Einstein if and only if its intrinsic torsion vanishes identically.*

2.3 Nearly Kähler SU(3)-structures

We recall that an almost Hermitian manifold (M,g,J) of dimension $2n$ is said to be nearly Kähler if

$$\left(\nabla^g_X J\right)X=0, \tag{2.24}$$

for all $X \in \mathfrak{X}(M)$ (cf. Table 2.1). The Gray and Hervella's class of nearly Kähler manifolds is $\mathcal{W}_1$ and, using identity (2.10), the defining condition is easily seen to be equivalent to

$$\left(\nabla^g_X \omega\right)(X,Y) = 0.$$

Since Kähler manifolds, defined by $\nabla^g J = 0$, satisfy (2.24) trivially, it is usual to call *strict* nearly Kähler those manifolds in $\mathcal{W}_1$ for which $\nabla^g_X J \neq 0$ for all non-vanishing $X \in \mathfrak{X}(M)$.

Remark 2.3.1. In literature, nearly Kähler manifolds are also called *K-spaces* (see for instance [114]) or *almost Tachibana spaces* (see e.g. [144]).

In [90], Gray proved that every complete, simply connected, nearly Kähler manifold is the Riemannian product of a Kähler and a strict nearly Kähler manifold. In particular, in dimension two and four nearly Kähler manifolds are actually Kähler [87], while in dimension six are either Kähler or strict nearly Kähler satisfying

$$\left|\left(\nabla^g_X J\right) Y\right|^2 = r\left(|X|^2|Y|^2 - g(X,Y)^2 - g(JX,Y)^2\right), \quad X,Y \in \mathfrak{X}(M),$$

for some positive constant r (see also [88]). Moreover, the Riemannian metric of a strict nearly Kähler manifold of dimension six is always Einstein [90, 144]. The relevance of the six-dimensional case is clear from the results of [149, 150], where it was proved that any complete, simply connected, nearly Kähler manifold is locally a Riemannian product of Kähler manifolds, twistor spaces over Kähler manifolds and six-dimensional nearly Kähler manifolds.

As noticed by Reyes Carrión in [162], in dimension six the structure group of the frame bundle of a strict nearly Kähler manifold always admits a reduction from U(3) to SU(3) and the intrinsic torsion of the corresponding SU(3)-structure is constrained to lie in $\mathcal{W}_1^-$. We give an idea of the proof here. First, using (2.10) and the identity

$$\left(\nabla^g_X J\right) JY = -J\left(\nabla^g_X J\right) Y, \tag{2.25}$$

from which follows that

$$\left(\nabla^g_X \omega\right)(JY,JZ) = -\left(\nabla^g_X \omega\right)(Y,Z), \tag{2.26}$$

it is possible to prove the following

Proposition 2.3.2. *Let (M, g, J) be a $2n$-dimensional almost Hermitian manifold with fundamental form ω. Then, the following conditions are equivalent and define a nearly Kähler manifold:*

i) $(\nabla^g_X J)X = 0$ *for all* $X \in \mathfrak{X}(M)$*;*

ii) $d\omega = 3\nabla^g\omega$*;*

iii) $d\omega$ *is of type* $(3,0)+(0,3)$ *and* N_J *is totally skew-symmetric.*

Proof. Observe that condition i) is equivalent to

$$(\nabla^g_X J)Y = -(\nabla^g_Y J)X$$

and also to $(\nabla^g_X\omega)(Y,Z) = -(\nabla^g_Y\omega)(X,Z)$ by (2.10). Then, the equivalence between i) and ii) follows from (2.11).

Assume now that i) holds. Then, using (2.26) and the equivalence between i) and ii), we get that

$$d\omega(X, JY, JZ) = 3(\nabla^g_X\omega)(JY, JZ) = -d\omega(X, Y, Z).$$

Thus, $d\omega$ is a real form of type $(3,0)+(0,3)$. Moreover, from the expression of the Nijenhuis tensor in (2.12) and identity (2.25), it follows that

$$N_J(X,Y) = -2(\nabla^g_X J)JY + 2(\nabla^g_Y J)JX = 4J(\nabla^g_X J)Y.$$

Consequently, $g(N_J(\cdot,\cdot),\cdot)$ is a 3-form, since $N_J(X,Y) = -N_J(Y,X)$ and

$$\begin{aligned} g(N_J(X,Y),Z) &= 4g(J(\nabla^g_X J)Y, Z) \\ &= -4(\nabla^g_X\omega)(Y, JZ) \\ &= -g(N_J(X,Z),Y). \end{aligned}$$

Conversely, if iii) holds, then using both the expressions of N_J in (2.1) and (2.12) and identity (2.26), we have

$$\begin{aligned} 0 &= g(N_J(JX,JY),JX) \\ &= -g(N_J(X,Y),JX) \\ &= (\nabla^g_X\omega)(JY,JX) - (\nabla^g_{JY}\omega)(X,JX) + (\nabla^g_{JX}\omega)(Y,JX) - (\nabla^g_Y\omega)(JX,JX) \\ &= (\nabla^g_X\omega)(X,Y) + (\nabla^g_{JX}\omega)(Y,JX), \end{aligned}$$

since $(\nabla^g_{JY}\omega)(X, JX) = 0$ for all $X, Y \in \mathfrak{X}(M)$. Now, using (2.26) again,

$$\begin{aligned} 0 &= d\omega(X, X, Y) \\ &= -d\omega(X, JX, JY) \\ &= -(\nabla^g_X\omega)(JX, JY) - (\nabla^g_{JX}\omega)(JY, X) - (\nabla^g_{JY}\omega)(X, JX) \\ &= (\nabla^g_X\omega)(X, Y) - (\nabla^g_{JX}\omega)(Y, JX). \end{aligned}$$

Comparing these two results, we obtain that $(\nabla^g_X\omega)(X, Y) = 0$ for all $X \in \mathfrak{X}(M)$ and the equivalence between i) and iii) is proved. □

Furthermore, the covariant derivative $\nabla^g\omega$ of the fundamental form of a nearly Kähler manifold is parallel with respect to the minimal connection $\overline{\nabla}$ defined in (2.7) (see for instance [20]).

Lemma 2.3.3. *Let (M, g, J) be a nearly Kähler manifold and consider the minimal connection $\overline{\nabla}$. Then,*

$$\overline{\nabla}(\nabla^g\omega) = 0.$$

As a consequence, by condition ii) of Proposition 2.3.2 and the compatibility of $\overline{\nabla}$ with the Riemannian metric g, the real form $d\omega$ of type $(3,0)+(0,3)$ has constant norm. When the nearly Kähler manifold is strict, there is then a natural way to define ψ_+, namely

$$\psi_+ = \frac{1}{3}d\omega.$$

With this choice, the $(3,0)$-form $\Psi = \psi_+ + iJ\psi_+$ defines a reduction of the structure group of the nearly Kähler manifold to SU(3). The corresponding SU(3)-structure is half-flat, indeed ψ_+ is obviously closed and $d\omega \wedge \omega = 0$, since ω is of type $(1,1)$. Moreover, by Proposition 2.2.9, the Nijenhuis tensor N_J is totally skew-symmetric if and only if the intrinsic torsion forms $w_2^{\pm}$ vanish identically. Consequently,

$$d\psi_- = a\,\omega^2,$$

for some $a \in \mathbb{R}$. Applying now the exterior derivative to both sides of the identity $\omega \wedge \psi_- = 0$ and requiring that the normalization condition is satisfied, we obtain $a = -2$. Thus, (ω, ψ_+) is characterized by the following differential system

$$\begin{aligned} d\omega &= 3\psi_+, \\ d\psi_- &= -2\omega^2, \end{aligned} \tag{2.27}$$

and the only non-identically vanishing intrinsic torsion form is $w_1^- = -2$. Observe now that an SU(3)-structure (ω, ψ_+) with only non-identically vanishing intrinsic torsion form w_1^- can be rescaled to obtain (2.27). Indeed, in this case the exterior derivatives of ω, ψ_+, ψ_- are

$$\begin{aligned} d\omega &= -\frac{3}{2} w_1^- \psi_+, \\ d\psi_+ &= 0, \\ d\psi_- &= w_1^- \omega^2, \end{aligned}$$

and w_1^- is constant on connected manifolds, since

$$0 = d(d\omega) = -\frac{3}{2} dw_1^- \wedge \psi_+$$

and wedging 1-forms by ψ_+ is injective by (2.17). We can then consider the pair $\widehat{\omega} = \frac{(w_1^-)^2}{4}\omega$, $\widehat{\psi}_+ = -\frac{(w_1^-)^3}{8}\psi_+$, which defines an SU(3)-structure satisfying the differential system (2.27) (see the proof of Lemma 2.4.5 for more details). This motivates the

Definition 2.3.4. An SU(3)-structure (ω, ψ_+) is called *nearly Kähler* if the intrinsic torsion forms w_1^+, $w_2^\pm$, w_3, w_4 and w_5 vanish identically.

Nearly Kähler SU(3)-structures belong then to the class $\mathcal{W}_1^-$ and are special half-flat in the sense of the definition introduced earlier. Observe that the intrinsic torsion form w_1^- is a real constant on connected manifolds and that it is equal to zero if and only if the nearly Kähler SU(3)-structure is torsion-free. Moreover, when w_1^- is nonzero the almost Hermitian structure (g, J) underlying a nearly Kähler SU(3)-structure is strict nearly Kähler. Indeed, $d\omega$ is of type $(3,0) + (0,3)$, since it is proportional to ψ_+, and N_J is totally skew-symmetric, as $w_2^\pm = 0$. Then, by Proposition 2.3.2, the almost Hermitian structure (g, J) is nearly Kähler and $\nabla^g\omega$ is proportional to ψ_+. Now, from identity (2.10) and the fact that $\iota_X\psi_+ = 0$ implies $X = 0$ (cf. (2.18)), we get that $\nabla^g_X J \neq 0$ for all non-vanishing $X \in \mathfrak{X}(M)$. This result together with the previous discussion proves the

Proposition 2.3.5. *Let M be a connected six-dimensional manifold endowed with an almost Hermitian structure (g, J) with fundamental form ω. Then, M is strict nearly Kähler if and only if there is a reduction $\Psi = \psi_+ + i\psi_-$ to* SU(3) *such that (ω, ψ_+) is a nearly Kähler* SU(3)*-structure with nonzero w_1^-.*

We already recalled that the Riemannian metric g of a nearly Kähler manifold is Einstein by [90, 144]. A simple proof of this fact for nearly Kähler SU(3)-structures can be obtained using the expression (2.23) of $\mathrm{Ric}^0(g)$. Indeed

$$\mathrm{Ric}^0(g) = \mathrm{i}_+^{-1}(\mathrm{E}_1(\Phi_1)) + \mathrm{i}_-^{-1}(\mathrm{E}_2(\Phi_2)),$$

where

$$\begin{aligned} \Phi_1 &= \frac{1}{4} d^* \left(w_1^- \psi_+ \right) = -\frac{1}{2} (w_1^-)^2 \, \omega, \\ \Phi_2 &= 0. \end{aligned}$$

Thus, $\mathrm{E}_1(\Phi_1) = 0$, $\mathrm{Ric}^0(g) = 0$ and g is an Einstein metric. In particular, the scalar curvature is non-negative,

$$\mathrm{Scal}(g) = \frac{15}{2} \left(w_1^- \right)^2,$$

and vanishes identically if and only if the SU(3)-structure is torsion-free.

As shown by Grunewald in [93], the existence of a strict nearly Kähler structure on a Riemannian 6-manifold (M, g) is related to the existence of a *real Killing spinor*, that is, a non-vanishing spinor field $\phi \in \Gamma(\Sigma M)$ solving the equation

$$\nabla_X \phi = l X \cdot \phi,$$

for every vector field X on M, where $l \in \mathbb{R}$, ∇ is the lifting of the Levi Civita connection ∇^g to the spinor bundle and the dot denotes the *Clifford multiplication* (cf. Section 4.3.1). In detail, every real Killing spinor defines an almost complex structure J on (M, g) such that (M, g, J) is strict nearly Kähler and, conversely, on every connected, simply connected strict nearly Kähler 6-manifold (M, g, J) there exists a real Killing spinor.

Up to now, in literature only few examples of nearly Kähler manifolds are known. In [34], Butruille showed that they are finitely many in the homogeneous case.

Theorem 2.3.6 ([34]). *Six-dimensional, nearly Kähler homogeneous Riemannian manifolds are isomorphic to one of the following spaces:*

i) *the 6-sphere* $S^6 = \mathrm{G}_2/\mathrm{SU}(3)$*;*

ii) *the product of 3-spheres* $S^3 \times S^3 = \mathrm{SU}(2) \times \mathrm{SU}(2)$*;*

iii) *the complex projective space* $\mathbb{CP}^3 = \mathrm{Sp}(2)/\mathrm{SU}(2)\mathrm{U}(1)$*;*

iv) *the flag manifold* $\mathbb{F}(1,2) = \mathrm{U}(3)/\mathrm{U}(1)\times\mathrm{U}(1)\times\mathrm{U}(1)$.

Moreover, each of these spaces admits a unique invariant nearly Kähler structure up to homothety.

Observe that the manifolds appearing in the previous theorem are all compact. There are different ways to define the homogeneous nearly Kähler structure on S^6, some of them are summarized in [34]. We will describe one of the possible constructions in Example 3.2.1 of next chapter. The manifolds $\mathbb{CP}^3$ and $\mathbb{F}(1,2)$ are the twistor spaces of the self-dual Einstein manifolds S^4 and $\mathbb{CP}^2$ endowed with their standard metrics. As we will see in detail in Section 2.5.2, they admit a non-integrable almost complex structure J, a one-parameter family of metrics g_t compatible with J for each positive real number t, and for a suitable choice of t the pair (g_t, J) defines a nearly Kähler structure on them. An alternative description in terms of real Killing spinors on the homogeneous spaces $\mathrm{Sp}(2)/\mathrm{SU}(2)\mathrm{U}(1)$ and $\mathrm{U}(3)/\mathrm{U}(1)\times\mathrm{U}(1)\times\mathrm{U}(1)$ can be found for instance in [18, Sect. 5.4]. Finally, the left-invariant nearly Kähler structure on the Lie group $\mathrm{SU}(2)\times\mathrm{SU}(2)$ will be described in Section 2.5.1.

Remark 2.3.7. Recently, the existence of new non-homogeneous examples on S^6 and $S^3\times S^3$ was proved by Foscolo and Haskins in [73].

Among special half-flat structures there are two classes which generalize the class $\mathcal{W}_1^-$ of nearly Kähler. The corresponding SU(3)-structures can be defined as follows

Definition 2.3.8. Let (ω,ψ_+) be a half-flat SU(3)-structure on a 6-manifold M. It is called *coupled* if $w_3 = 0$, while it is called *double half-flat*, or *co-coupled*, if $w_2^- = 0$.

Thus, coupled structures belong to the class $\mathcal{W}_1^-\oplus\mathcal{W}_2^-$, double half-flat structures belong to $\mathcal{W}_1^-\oplus\mathcal{W}_3$ and the nearly Kähler can be thought as the half-flat structures which are are both coupled and double half-flat.

Remark 2.3.9. To our knowledge, the name *coupled* was introduced by Salamon in [164], while the names *double half-flat* and *co-coupled* were used in [167] and [137], respectively. In physical literature, coupled structures were also called *restricted half-flat* in [123].

As it happens in the nearly Kähler case, the intrinsic torsion form $w_1^- \in C^\infty(M)$ of coupled and double half-flat structures is constant on connected manifolds.

Lemma 2.3.10. *Let M be a six-dimensional, connected manifold endowed with a coupled or a double half-flat* SU(3)*-structure (ω, ψ_+). Then, the intrinsic torsion form w_1^- is constant.*

Proof. If (ω, ψ_+) is coupled, the proof is the same as in the case of nearly Kähler SU(3)-structures. Indeed, taking the exterior derivative of both sides of

$$d\omega = -\frac{3}{2} w_1^- \psi_+$$

and using the fact that ψ_+ is closed, we obtain $0 = dw_1^- \wedge \psi_+$. Therefore, $dw_1^- = 0$, since wedging 1-forms by ψ_+ is injective by (2.17), and the thesis follows from the connectedness of M. In the double half-flat case, we can argue in a similar way: starting from

$$d\psi_- = w_1^- \omega^2,$$

we take the exterior derivative of both sides obtaining

$$0 = dw_1^- \wedge \omega^2 + w_1^- d\omega^2$$

and conclude observing that $d\omega^2 = 0$ and that wedging 1-forms by ω^2 is injective by (2.16). □

The rareness of examples of six-dimensional nearly Kähler manifolds provides a first motivation to study manifolds endowed with a coupled or a double half-flat SU(3)-structure, as both generalize nearly Kähler structures.

Double half-flat SU(3)-structures were considered for instance in [42, 137, 165, 167], where also explicit examples on compact nilmanifolds and on $S^3 \times S^3$ were provided. By [167], they can be characterized as the half-flat structures having totally skew-symmetric Nijenhuis tensor. Thus, as nearly Kähler manifolds, 6-manifolds endowed with a double half-flat structure admit a U(3)-connection whose torsion is totally skew-symmetric by the general result

Theorem 2.3.11 ([78]). *Let (M, g, J) be a $2n$-dimensional almost Hermitian manifold. Then, there exists a* U(n)*-connection with totally skew-symmetric torsion if and only if the Nijenhuis tensor N_J is totally skew-symmetric. In this case, the connection is unique.*

Remark 2.3.12. In the case of $2n$-dimensional nearly Kähler manifolds, the connection is exactly the minimal connection given in (2.7) (see for instance [1, Lemma 2.2] and the references therein).

Natural spaces motivating the study of coupled SU(3)-structures are $S^3 \times S^3$ and the twistor spaces over self-dual Einstein 4-manifolds of positive scalar curvature, since they all admit such structures. Moreover, further motivations come from supersymmetric string theory in physics. In the remaining part of this chapter, we shall discuss the properties of manifolds endowed with a coupled structure, explain in detail the previous motivations and study some additional problems. Most of the content is based on the papers [70, 71, 160].

2.4 Coupled SU(3)-structures

Let us consider a six-dimensional connected manifold M endowed with a coupled SU(3)-structure (ω, ψ_+). The exterior derivatives of ω, ψ_+ and $\psi_- = J\psi_+$ are

$$\begin{aligned} d\omega &= -\tfrac{3}{2} w_1^- \, \psi_+, \\ d\psi_+ &= 0, \\ d\psi_- &= w_1^- \, \omega^2 - w_2^- \wedge \omega, \end{aligned} \tag{2.28}$$

where w_1^- is a real constant and w_2^- a primitive real 2-form of type $(1,1)$. The former is zero if and only if the coupled structure is symplectic half-flat, while the latter vanishes identically if and only if the coupled structure is nearly Kähler. Consequently, using Proposition 2.2.9, we get

Proposition 2.4.1. *Let (ω, ψ_+) be a coupled* SU(3)*-structure. Then, it is nearly Kähler if and only if the Nijenhuis tensor of the corresponding almost complex structure $J = J_{\psi_+}$ is totally skew-symmetric.*

The previous observations clarify which are the possible subclasses of $\mathcal{W}_1^- \oplus \mathcal{W}_2^-$ and how they are characterized in terms of the intrinsic torsion forms w_1^- and w_2^-. That being so, instead of using the adjective *strict* as in the nearly Kähler case, from now on we reserve the name *coupled* for those SU(3)-structures satisfying the differential system (2.28) with nonzero w_1^- and non-identically vanishing w_2^-. They can be described as follows

Proposition 2.4.2. *Let* (ω, ψ_+) *be an* SU(3)*-structure on a connected 6-manifold* M*. Then, it is coupled if and only if there exists a nonzero real constant* c *such that* $d\omega = c\,\psi_+$*. In particular, a coupled* SU(3)*-structure is completely determined by* ω *and the almost complex structure* J *induced by it is never integrable.*

Proof. If (ω, ψ_+) is coupled, then by Lemma 2.3.10 the nonzero real constant is $c = -\frac{3}{2}w_1^-$. Conversely, if there exists a nonzero real constant c such that $d\omega = c\,\psi_+$, then the 3-form ψ_+ is closed and the SU(3)-structure is coupled.

Since ψ_+ is proportional to $d\omega$, the 2-form ω determines the whole SU(3)-structure and, by the discussion preceding Proposition 2.2.8, the almost complex structure induced by it is not integrable. □

An immediate consequence is the next result, which was observed in [165].

Corollary 2.4.3. *Let* (ω, ψ_+) *be a coupled* SU(3)*-structure such that* $d\omega = c\,\psi_+$*, where* $c \neq 0$*. Then, a diffeomorphism* $\nu : M \to M$ *such that* $\nu^*\omega = \omega$ *is an automorphism of the* SU(3)*-structure and, in particular, an isometry.*

For the sake of brevity, we introduce the

Definition 2.4.4. Let (ω, ψ_+) be a coupled SU(3)-structure such that $d\omega = c\,\psi_+$. The nonzero real number c is called *coupled constant.*

Observe that it is alway possible to rescale a coupled structure in order to obtain a different coupled constant.

Lemma 2.4.5. *Let* (ω, ψ_+) *be a coupled* SU(3)*-structure with coupled constant* c*, denote by* g *the associated Riemannian metric and fix a nonzero real number* r*. Then, the pair* $\widehat{\omega} := r^2\,\omega$*,* $\widehat{\psi}_+ := r^3\,\psi_+$ *is a coupled structure with coupled constant* $\widehat{c} = \frac{c}{r}$*. Moreover,* $J_{\widehat{\psi}_+} = J_{\psi_+}$ *and the Riemannian metric* $\widehat{g}$ *induced by* $\widehat{\omega}, \widehat{\psi}_+$ *is* $r^2\,g$*.*

Proof. First, notice that the forms $\widehat{\omega}$ and $\widehat{\psi}_+$ are still stable, since $\widehat{\omega}^3 = r^6\,\omega^3 \neq 0$ and $\lambda(\widehat{\psi}_+) = r^{12}\,\lambda(\psi_+) < 0$. As we observed in Remark 2.2.2, the almost complex structure $J_{\widehat{\psi}_+}$ induced by $\widehat{\psi}_+$ is the same as the one induced by ψ_+, thus

$$\widehat{\psi}_- = J_{\widehat{\psi}_+}\widehat{\psi}_+ = J_{\psi_+}(r^3\,\psi_+) = r^3\,\psi_-.$$

The forms $\widehat{\omega}$ and $\widehat{\psi}_+$ are clearly compatible and the normalization condition is satisfied, indeed

$$\widehat{\psi}_+ \wedge \widehat{\psi}_- = r^6\,\psi_+ \wedge \psi_- = r^6\,\frac{2}{3}\,\omega^3 = \frac{2}{3}\,\widehat{\omega}^3.$$

Moreover,

$$\widehat{g}(\cdot,\cdot) = \widehat{\omega}\left(J_{\widehat{\psi}_+}\cdot,\cdot\right) = r^2\,\omega\left(J_{\psi_+}\cdot,\cdot\right) = r^2\,g(\cdot,\cdot)$$

is a Riemannian metric. Thus, the pair $\widehat{\omega}, \widehat{\psi}_+$ defines an SU(3)-structure. It is moreover coupled with coupled constant $\widehat{c} = \frac{c}{r}$, since

$$d\widehat{\omega} = r^2\,d\omega = cr^2\,\psi_+ = \frac{c}{r}\,\widehat{\psi}_+.$$

□

In Proposition 2.3.5, we reviewed that the almost Hermitian structure (g, J) underlying a nearly Kähler SU(3)-structure with nonzero w_1^- is strict nearly Kähler. When (ω, ψ_+) is coupled, we only have that the exterior derivative of ω is a real form of type $(3,0) + (0,3)$. Therefore:

Proposition 2.4.6. *The almost Hermitian structure (g, J) underlying a coupled* SU(3)*-structure (ω, ψ_+) is quasi-Kähler, i.e., $\overline{\partial}\omega = (d\omega)^{1,2} = 0$.*

Unlike the nearly Kähler and the symplectic half-flat case, the scalar curvature of the Riemannian metric g induced by a coupled structure does not have a definite sign, indeed

$$\mathrm{Scal}(g) = \frac{15}{2}\left(w_1^-\right)^2 - \frac{1}{2}\left|w_2^-\right|^2. \tag{2.29}$$

Moreover, we know that nearly Kähler SU(3)-structures always induce an Einstein metric, while symplectic half-flat structures induce an Einstein metric if and only if they are torsion-free. Thus, since coupled structures generalize both, a natural question is to ask under which conditions on the torsion forms w_1^- and w_2^- the metric induced by a coupled structure is Einstein. In this case, the differential forms appearing in the traceless part of the Ricci tensor are

$$\begin{aligned}\Phi_1 &= \tfrac{1}{4} * (w_2^- \wedge w_2^-) + \tfrac{1}{4} w_1^- d^* \psi_+ \\ &= \tfrac{1}{4} * (w_2^- \wedge w_2^-) - \tfrac{1}{2}(w_1^-)^2\, \omega - \tfrac{1}{4} w_1^-\, w_2^-, \\ \Phi_2 &= -2 * J d w_2^-,\end{aligned} \tag{2.30}$$

where we have used the identity

$$* (w_2^- \wedge \omega) = -w_2^-, \tag{2.31}$$

which holds for every 2-form belonging to $\left[\Omega_0^{1,1}(M)\right]$. A straightforward computations gives then

$$\mathrm{E}_1(\Phi_1) = \frac{1}{4} * \left(w_2^- \wedge w_2^-\right) - \frac{1}{4} w_1^-\, w_2^- + \frac{1}{12}|w_2^-|^2\, \omega,$$

while $\mathrm{E}_2(\Phi_2)$ depends on the component of dw_2^- in $\left[\!\left[\Omega_0^{2,1}(M)\right]\!\right]$. From the decomposition of the space $\Omega^3(M)$ given in (2.20), we know that there exist unique $h^+, h^- \in C^\infty(M)$, $\eta_1 \in \Omega^1(M)$ and $\sigma_3 \in \left[\!\left[\Omega_0^{2,1}(M)\right]\!\right]$ such that

$$dw_2^- = h^+\, \psi_+ + h^-\, \psi_- + \eta_1 \wedge \omega + \sigma_3, \tag{2.32}$$

and we can prove the

Lemma 2.4.7. *Let* (ω, ψ_+) *be a coupled* SU(3)*-structure on a 6-manifold* M*. Then, the intrinsic torsion form* w_2^- *is co-closed. Moreover, the function* h^- *appearing in* (2.32) *vanishes identically, while the function* h^+ *vanishes identically if and only if the* SU(3)*-structure is nearly Kähler.*

Proof. Taking the exterior derivative of both sides of

$$d\psi_- = w_1^-\, \omega^2 - w_2^- \wedge \omega$$

and using (2.31) and $d\omega^2 = 0$, we get $d^* w_2^- = 0$. Moreover, since the decomposition (2.20) of $\Omega^3(M)$ is g-orthogonal, using $*\psi_+ = \psi_-$, $d\psi_+ = 0$, $w_2^- \wedge \psi_\pm = 0$, $w_2^- \wedge \omega^2 = 0$

and $*w_2^- = -w_2^- \wedge \omega$, we have

$$\begin{aligned} 4h^- \, dV_g &= g(h^- \psi_-, \psi_-) dV_g &&= g(dw_2^-, \psi_-) dV_g \\ &= dw_2^- \wedge *\psi_- &&= -dw_2^- \wedge \psi_+ \\ &= -d(w_2^- \wedge \psi_+) + w_2^- \wedge d\psi_+ &&= 0 \end{aligned}$$

and

$$\begin{aligned} 4h^+ \, dV_g &= g(h^+ \psi_+, \psi_+) dV_g &&= g(dw_2^-, \psi_+) dV_g \\ &= dw_2^- \wedge *\psi_+ &&= dw_2^- \wedge \psi_- \\ &= d(w_2^- \wedge \psi_-) - w_2^- \wedge d\psi_- &&= w_2^- \wedge w_2^- \wedge \omega \\ &= -w_2^- \wedge *w_2^- &&= -|w_2^-|^2 dV_g. \end{aligned}$$

Then, $h^- = 0$ and $h^+ = -\frac{|w_2^-|^2}{4}$ vanishes identically if and only if w_2^- does. □

Thus, in the general case coupled structures inducing Einstein metrics can be in principle characterized by two equations involving the intrinsic torsion forms w_1^- and w_2^-.

In Section 2.4.3, we will see that a under the (well-justified) hypothesis $dw_2^- \propto \psi_+$, the characterization is rather simple, while in Section 2.5.2 we will discuss an explicit example of coupled SU(3)-structure inducing an Einstein metric and satisfying that hypothesis.

Remark 2.4.8. In [146], the authors proved that when an SU(3)-structure is nearly Kähler, then every co-closed $\alpha \in \left[\Omega_0^{1,1}(M)\right]$ is such that $d\alpha \in \left[\!\left[\Omega_0^{2,1}(M)\right]\!\right]$. This result does not extend to the coupled case, since from Lemma 2.4.7 we would get that every coupled structure is nearly Kähler.

2.4.1 Examples

We can now look for examples of coupled SU(3)-structures. We begin with the classification of invariant coupled structures on compact nilmanifolds and, then, we describe an example of left-invariant coupled structure on the manifold $S^3 \times S^3$, while we shall give further examples in the next sections.

First of all, we recall that an SU(3)-structure (ω, ψ_+) on a Lie group G is *left-invariant* if the differential forms ω, ψ_+ (and, consequently, the tensors J, ψ_-, g) are left-invariant. In this case, the pair (ω, ψ_+) determines an SU(3)-structure on the Lie algebra of G. Conversely, an SU(3)-structure on a six-dimensional Lie algebra $\mathfrak{g}$ is defined by a pair of compatible and normalized stable forms $(\omega, \psi_+) \in \Lambda^2(\mathfrak{g}^*) \times \Lambda^3(\mathfrak{g}^*)$, with $\lambda(\psi_+) < 0$, such that $g(\cdot, \cdot) = \omega(J_{\psi_+} \cdot, \cdot)$ is an inner product, and it induces a left-invariant SU(3)-structure on the corresponding simply connected Lie group. Thus, there is a one-to-one correspondence between left-invariant SU(3)-structures on six-dimensional simply connected Lie groups and SU(3)-structures on the corresponding Lie algebras. Clearly, an SU(3)-structure (ω, ψ_+) on $\mathfrak{g}$ is half-flat if and only if $d\omega^2 = 0$ and $d\psi_+ = 0$, where d denotes the Chevalley-Eilenberg differential on $\mathfrak{g}$, while it is coupled if and only if ω and $d\omega$ are stable and ψ_+ is proportional to $d\omega$.

In Section 1.3, we defined a compact nilmanifold as the quotient of a simply connected nilpotent Lie group N by a lattice $\Gamma \subset$ N. In the six-dimensional case, a left-invariant SU(3)-structure (ω, ψ_+) on N passes to the quotient, defining an *invariant* SU(3)-structure on the compact nilmanifold N/Γ. Moreover, up to isomorphism, there exist 34 six-dimensional nilpotent Lie algebras and each of them gives rise to a compact nilmanifold. Consequently, there is a one-to-one correspondence between invariant SU(3)-structures (ω, ψ_+) on compact nilmanifolds and pairs (ω, ψ_+) defining an SU(3)-structure on their nilpotent Lie algebras. Thus, the classification of six-dimensional compact nilmanifolds admitting an invariant SU(3)-structure of a certain type can be obtained working only with SU(3)-structures on nilpotent Lie algebras.

By Milnor's result stated in Theorem 1.4.3, it follows that there are no strict nearly Kähler SU(3)-structures defined on non-Abelian nilpotent Lie algebras, since the metric induced by a nearly Kähler is always Einstein.

In [49], Conti classified six-dimensional nilpotent Lie algebras admitting a half-flat SU(3)-structure up to isomorphism. In detail, starting from the list of 34 non-isomorphic six-dimensional nilpotent Lie algebras, he gave an explicit example of a half-flat SU(3)-structure on 24 of them and introduced an obstruction to the existence of half-flat structures on Lie algebras, which allowed him to show that the remaining

10 cases do not admit any. The obstruction was later refined by Freibert and Schulte-Hengesbach in [74]. We recall it here.

Proposition 2.4.9 ([74]). *Let $\mathfrak{g}$ be a real six-dimensional Lie algebra with volume form $\Omega \in \Lambda^6(\mathfrak{g}^*)$. If there exists a nonzero $\alpha \in \mathfrak{g}^*$ such that*

$$\alpha \wedge \widetilde{J}_\rho \alpha \wedge \sigma = 0$$

for all closed 3-forms $\rho \in \Lambda^3(\mathfrak{g}^*)$ and closed 4-forms $\sigma \in \Lambda^4(\mathfrak{g}^*)$, where for every $X \in \mathfrak{g}$

$$\widetilde{J}_\rho \alpha(X)\Omega = \alpha \wedge (\iota_X \rho) \wedge \rho,$$

then $\mathfrak{g}$ does not admit any half-flat SU(3)*-structure.*

Referring to the list of six-dimensional nilpotent Lie algebras given in Table 1.1, we can state the result of Conti as follows

Theorem 2.4.10 ([49]). *Let $\mathfrak{n}$ be a six-dimensional nilpotent Lie algebra admitting a half-flat* SU(3)*-structure. Then, $\mathfrak{n}$ is isomorphic to $\mathfrak{n}_k$ for $k = 4, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 21, 22, 24, 25, 27, 28, 29, 30, 31, 32, 33, 34$.*

Further classifications of six-dimensional nilpotent Lie algebras admitting special half-flat SU(3)-structures were studied in literature. For instance, in [42] the authors classified those admitting a double half-flat SU(3)-structure, while the result for symplectic half-flat structures was obtained in [52]. The more general classification of six-dimensional Lie algebras admitting half-flat SU(3)-structures was obtained in [74, 168] in the decomposable case and in [75] for indecomposable Lie algebras with five-dimensional nilradical.

We now focus on the case of coupled SU(3)-structures on nilpotent Lie algebras. Since coupled are in particular half-flat, to classify six-dimensional nilpotent Lie algebras admitting a coupled structure we start from the list given in Theorem 2.4.10 and use the following obstruction, which holds in the more general case of Lie algebras and whose proof is immediate.

Lemma 2.4.11. *Let $\mathfrak{g}$ be a six-dimensional real Lie algebra with volume form $\Omega \in \Lambda^6(\mathfrak{g}^*)$. If $\lambda(d\sigma) \geq 0$ for all $\sigma \in \Lambda^2(\mathfrak{g}^*)$, then $\mathfrak{g}$ does not admit any coupled* SU(3)*-structure.*

Observe that on the six-dimensional Abelian Lie algebra every differential form is closed with respect to the Chevalley-Eilenberg differential. Thus, if we look for coupled SU(3)-structures on nilpotent Lie algebras, we have to exclude the Abelian case. We are now ready to prove the classification result.

Theorem 2.4.12 ([71]). *Let $\mathfrak{n}$ be a six-dimensional, non-Abelian, nilpotent Lie algebra admitting a coupled* SU(3)*-structure. Then, $\mathfrak{n}$ is isomorphic to one of the following*

$$\mathfrak{n}_9 = (0, 0, 0, e^{12}, e^{14} - e^{23}, e^{15} + e^{34}), \quad \mathfrak{n}_{28} = (0, 0, 0, 0, e^{13} - e^{24}, e^{14} + e^{23}).$$

Proof. Let $\mathfrak{n}$ be one of the six-dimensional, non-Abelian, nilpotent Lie algebras admitting a half-flat SU(3)-structure and denote by $(e^1, \ldots, e^6)$ the basis of $\mathfrak{n}^*$ for which the structure equations of $\mathfrak{n}$ are those given in Table 1.1. A 2-form ω on $\mathfrak{n}$ can be written with respect to the corresponding basis of $\Lambda^2(\mathfrak{n}^*)$ as

$$\begin{aligned} \omega \;=\; & b_1e^{12} + b_2e^{13} + b_3e^{14} + b_4e^{15} + b_5e^{16} + b_6e^{23} + b_7e^{24} + b_8e^{25} \\ & + b_9e^{26} + b_{10}e^{34} + b_{11}e^{35} + b_{12}e^{36} + b_{13}e^{45} + b_{14}e^{46} + b_{15}e^{56}, \end{aligned}$$

where $b_i \in \mathbb{R}, i = 1, \ldots, 15$. We fix the volume form $\Omega = e^{123456}$ and compute the quartic invariant $\lambda(d\omega)$ for each nilpotent Lie algebra (observe that the sign of $\lambda(d\omega)$ does not depend on the choice of Ω). The expression of $\lambda(d\omega)$ in each case is given in Table 2.2. Among the 24 nilpotent Lie algebras admitting a half-flat SU(3)-structure we have:

- 1 case ($\mathfrak{n}_{28}$) for which $\lambda(d\omega) < 0$ if $b_{15} \neq 0$,
- 2 cases ($\mathfrak{n}_4$ and $\mathfrak{n}_9$) for which the sign of $\lambda(d\omega)$ depends on ω,
- 21 cases for which $\lambda(d\omega)$ cannot be negative.

Therefore, the 21 algebras having $\lambda(\sigma) \geq 0$ do not admit any coupled SU(3)-structure by Lemma 2.4.11.

Consider $\mathfrak{n}_4$, it has structure equations

$$(0, 0, e^{12}, e^{13}, e^{14} + e^{23}, e^{24} + e^{15}),$$

the sign of $\lambda(d\omega)$ depends on the coefficients b_i and, thus, we cannot apply the obstruction used before. However, we can show that there are no coupled structures on $\mathfrak{n}_4$ in the following way. From the expression of $\lambda(d\omega)$, we get that $d\omega$ is stable if and only if $b_{15} \neq 0$ and $b_{15}(b_{12}+b_{13}) > b_{14}^2$. Imposing the compatibility condition $d\omega \wedge \omega = 0$, we obtain four polynomial equations in the b_i which can be solved using the constraint $b_{15} \neq 0$. We can now compute $J = J_{d\omega}$, the matrix associated with $g(\cdot,\cdot) = \omega(J\cdot,\cdot)$ with respect to the basis $(e_1,\dots,e_6)$ and observe that for the nonzero vector $v = e_4 - \frac{b_{14}}{b_{15}}e_5 + \frac{b_{13}}{b_{15}}e_6$ it holds $g(v,v) = 0$. Therefore, g cannot be an inner product and, as a consequence, it is not possible to find a coupled SU(3)-structure on $\mathfrak{n}_4$.

The Lie algebra $\mathfrak{n}_9$ has structure equations

$$(0,0,0,e^{12},e^{14}-e^{23},e^{15}+e^{34})$$

and the pair of stable forms

$$\begin{aligned}\omega &= -e^{13}-e^{24}+e^{26}+e^{56},\\ \psi_+ &= -e^{125}-e^{146}+e^{236}+e^{234}+e^{345},\end{aligned}$$

defines a coupled SU(3)-structure on it with $\psi_+ = -d\omega$.

On the Lie algebra $\mathfrak{n}_{28}$, whose structure equations are

$$(0,0,0,0,e^{13}-e^{24},e^{14}+e^{23}),$$

the pair of stable forms

$$\begin{aligned}\omega &= e^{12}+e^{34}-e^{56},\\ \psi_+ &= e^{136}-e^{145}-e^{235}-e^{246},\end{aligned}$$

defines a coupled SU(3)-structure such that $\psi_+ = -d\omega$. In particular, it induces the inner product

$$g = (e^1)^2 + \cdots + (e^6)^2,$$

which is a nilsoliton (cf. Example 1.4.17). □

$\mathfrak{n}$.	$\lambda(d\omega)$	Sign of $\lambda(d\omega)$
$\mathfrak{n}_4$	$4b_{15}^2(-b_{15}(b_{12}+b_{13})+b_{14}^2)$	
$\mathfrak{n}_6$	b_{15}^4	≥ 0
$\mathfrak{n}_7$	$(b_{14}^2-b_{15}^2)^2$	≥ 0
$\mathfrak{n}_8$	$(b_{14}^2-b_{15}^2)^2$	≥ 0
$\mathfrak{n}_9$	$4b_{15}^2(-b_{15}(b_9+b_{13})+b_{14}^2)$	
$\mathfrak{n}_{10}$	b_{15}^4	≥ 0
$\mathfrak{n}_{11}$	b_{15}^4	≥ 0
$\mathfrak{n}_{12}$	0	0
$\mathfrak{n}_{13}$	0	0
$\mathfrak{n}_{14}$	b_{14}^4	≥ 0
$\mathfrak{n}_{15}$	$(b_{14}^2-b_{15}^2)^2$	≥ 0
$\mathfrak{n}_{16}$	$(b_{14}^2+b_{15}^2)^2$	≥ 0
$\mathfrak{n}_{21}$	0	0
$\mathfrak{n}_{22}$	b_{15}^4	≥ 0
$\mathfrak{n}_{24}$	0	0
$\mathfrak{n}_{25}$	b_{15}^4	≥ 0
$\mathfrak{n}_{27}$	0	0
$\mathfrak{n}_{28}$	$-4b_{15}^4$	≤ 0
$\mathfrak{n}_{29}$	0	0
$\mathfrak{n}_{30}$	b_{15}^4	≥ 0
$\mathfrak{n}_{31}$	0	0
$\mathfrak{n}_{32}$	0	0
$\mathfrak{n}_{33}$	0	0

Table 2.2: Expression of $\lambda(d\omega)$ for six-dimensional nilpotent Lie algebras admitting a half-flat SU(3)-structure.

Remark 2.4.13. The examples of half-flat SU(3)-structures on $\mathfrak{n}_9$ and $\mathfrak{n}_{28}$ given in [49] are of type $\mathcal{W}_1^- \oplus \mathcal{W}_2^- \oplus \mathcal{W}_3$ and $\mathcal{W}_3$, respectively. Thus, the two examples of coupled structures contained in the proof of the previous theorem did not appear in [49]. Moreover, the fact that $\mathfrak{n}_{28}$ admits a coupled structure was also noticed in the physical paper [136], we will discuss this later in Section 2.4.3.

Observe that the coupled SU(3)-structure on $\mathfrak{n}_{28}$ described before induces an invariant coupled structure on the Iwasawa manifold (Example 1.3.8) whose associated Riemannian metric is a Ricci soliton (cf. Section 1.4.4). It is the unique example of this kind by the next result.

Proposition 2.4.14 ([71]). *Let $\mathfrak{n}$ be a six-dimensional, non-Abelian, nilpotent Lie algebra admitting a coupled* SU(3)*-structure inducing a nilsoliton. Then, $\mathfrak{n}$ is isomorphic to $\mathfrak{n}_{28}$.*

Proof. It is clear from the classification of nilpotent Lie algebras admitting a coupled SU(3)-structure that to prove the assertion it suffices to show that $\mathfrak{n}_9$ does not admit any coupled structure inducing a nilsoliton inner product. If we consider the basis $(e^1, \dots, e^6)$ of $\mathfrak{n}_9^*$ for which the structure equations are

$$\left(0, 0, 0, \frac{\sqrt{5}}{2}e^{12}, e^{14} - e^{23}, \frac{\sqrt{5}}{2}e^{15} + e^{34}\right),$$

then by [179] the inner product $g = \sum_{i=1}^{6}(e^i)^2$ is a nilsoliton on $\mathfrak{n}_9$ and by [124] it is unique up to isometry and scaling. Let ω be a generic 2-form on $\mathfrak{n}_9$, we can write it with respect to the basis $\{e^{ij}\}$ of $\Lambda^2(\mathfrak{n}_9^*)$ as in the proof of the previous theorem. Using this expression, we compute $\lambda(d\omega)$ and impose that it is negative, obtaining the constraints $b_{15} \neq 0$ and $\sqrt{5}b_{14}^2 - 2b_{15}b_9 - 2b_{15}b_{13} < 0$. From the compatibility condition $d\omega \wedge \omega = 0$, we get three polynomial equations in the unknowns b_i which can be solved using $b_{15} \neq 0$. We then compute $J_{d\omega}$ and the matrix G associated with $\omega(J_{d\omega}\cdot, \cdot)$ with respect to the considered basis. By the uniqueness of the nilsoliton up to scaling, we have to impose that G is proportional to the identity matrix. The associated equations do not have solutions under the constraints on the b_i imposed by $\lambda(d\omega)$, as one can check considering for instance the equations $G_{5,6} = 0, G_{4,6} = 0, G_{2,5} = 0$, where $G_{i,j} = \omega(J_{d\omega}e_i, e_j)$. □

Since it is usually more convenient to work with an adapted frame for the SU(3)-structure, we observe that for the nilpotent Lie algebras $\mathfrak{n}_9$ and $\mathfrak{n}_{28}$ there exists a basis $(e^1, \dots, e^6)$ of their dual spaces which is adapted for the coupled SU(3)-structure and such that the structure equations become

$$\mathfrak{n}_9 = \left(0, 0, 0, e^{13}, e^{14} + e^{23}, e^{13} - e^{15} - e^{24}\right), \tag{2.33}$$

$$\mathfrak{n}_{28} = (0, 0, 0, 0, e^{14} + e^{23}, e^{13} - e^{24}). \tag{2.34}$$

In both cases, ω and ψ_+ can be written as in (2.13) and $d\omega = -\psi_+$.

We give now an example of left-invariant coupled SU(3)-structure on the homogeneous manifold $S^3 \times S^3$. As a Lie group, it is $\mathrm{SU}(2) \times \mathrm{SU}(2)$ and any left-invariant SU(3)-structure on it can be identified with a pair of stable forms (ω, ψ_+) defining an SU(3)-structure on its Lie algebra $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$. The *standard basis* of $\mathfrak{su}(2)$ is given by the matrices

$$e_1 = \begin{pmatrix} \frac{i}{2} & 0 \\ 0 & -\frac{i}{2} \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & \frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix},$$

and it is easy to check that the only non-vanishing structure constants are $c_{12}^3 = c_{23}^1 = -1 = -c_{13}^2$.

Consider $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$, denote by (e_1, e_2, e_3) the standard basis of the first copy of $\mathfrak{su}(2)$, by (e_4, e_5, e_6) the standard basis of the second copy and by (e^1, e^2, e^3) and (e^4, e^5, e^6) their dual bases. Then, the Lie algebra $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ has the following structure equations:

$$de^1 = e^{23}, \quad de^2 = e^{31}, \quad de^3 = e^{12},$$
$$de^4 = e^{56}, \quad de^5 = e^{64}, \quad de^6 = e^{45}.$$

Example 2.4.15. The pair

$$\omega = -\sqrt{3}\, e^{16} - e^{24} - e^{25} - e^{35},$$
$$\psi_+ = \sqrt[4]{3}\,(-\sqrt{3}e^{236} + \sqrt{3}e^{145} + e^{134} + e^{256} + e^{135} - e^{246} - e^{125} - e^{346}),$$

defines a coupled SU(3)-structure on $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ such that $\psi_+ = \sqrt[4]{3}\, d\omega$.

2.4.2 Coupled SU(3)-structures and the Hitchin flow

As we observed earlier, a solution of the Hitchin flow equations starting from a half-flat SU(3)-structure is half-flat as long as it exists. We may rephrase this by saying that the torsion class $\mathcal{W}_1^- \oplus \mathcal{W}_2^- \oplus \mathcal{W}_3$ is *preserved* by the Hitchin flow equations. When we restrict our attention to special half-flat structures, in general it is not possible to show that their torsion class is preserved. Anyway, there are some examples proving that in certain situations this happens. For instance, if we consider a nearly Kähler SU(3)-structure (ω, ψ_+) with $w_1^- = -2$ and we look for solutions of the Hitchin flow equations starting from it at $t = 0$ and preserving the nearly Kähler condition, then we always find

$$\begin{aligned}\omega(t) &= (t+1)^2\,\omega,\\ \psi_+(t) &= (t+1)^3\,\psi_+,\end{aligned}$$

which is however quite trivial, since it evolves only by a rescaling of the starting condition. Moreover, in the double half-flat case a solution preserving the torsion class $\mathcal{W}_1^- \oplus \mathcal{W}_3$ was given on the manifold $S^3 \times S^3$ in [137]. It is then a natural question to ask whether there exist examples of solutions of the Hitchin flow equations preserving the torsion class $\mathcal{W}_1^- \oplus \mathcal{W}_2^-$, which we may call *coupled solutions*. More precisely, we introduce the

Definition 2.4.16. Let $(\omega(t), \psi_+(t))$ be a solution of the Hitchin flow equations defined on an interval $\mathrm{I} \subseteq \mathbb{R}$ containing 0 and starting from a coupled structure at $t = 0$. If $(\omega(t), \psi_+(t))$ is a coupled structure for each $t \in \mathrm{I}$, that is,

$$d\omega(t) = c(t)\psi_+(t)$$

for some smooth and nowhere zero function $c : \mathrm{I} \to \mathbb{R}$, we call it a *coupled solution.*

Coupled solutions can be easily characterized and induce an almost complex structure not depending on t. Indeed

Proposition 2.4.17 ([70]). *Let M be a connected 6-manifold and suppose that there exists on it a solution $(\omega(t), \psi_+(t))$ of the Hitchin flow equations starting from a coupled structure $(\omega(0), \psi_+(0))$ and defined on some interval $\mathrm{I} \subseteq \mathbb{R}$ containing 0.*

If $(\omega(t), \psi_+(t))$ is a coupled solution, then there exists a smooth, non-constant and nowhere zero function $f : \mathrm{I} \to \mathbb{R}$ such that

$$\psi_+(t) = f(t)\psi_+(0).$$

Conversely, if the pair $(\omega(t), \psi_+(t))$ is a solution of the Hitchin flow equations with $\psi_+(t) = f(t)\psi_+(0)$, then it is a coupled solution.

Proof. If $(\omega(t), \psi_+(t))$ is a solution of the Hitchin flow equations with $\psi_+(t) = f(t)\psi_+(0)$ and non-constant, nowhere zero $f(t)$, then from $\frac{\partial}{\partial t}\psi_+(t) = d\omega(t)$ we obtain

$$d\omega(t) = \frac{\partial}{\partial t}\left(f(t)\psi_+(0)\right) = \left(\frac{d}{dt}f(t)\right)\psi_+(0).$$

Thus, the solution is a coupled structure with $c(t) = \frac{d}{dt}(\ln f(t))$. Suppose now that the solution is coupled, $d\omega(t) = c(t)\psi_+(t)$. Then, from the flow equation we obtain

$$\frac{\partial}{\partial t}\psi_+(t) = c(t)\psi_+(t).$$

Working in local coordinates on M, it is easy to show that

$$\psi_+(t) = f(t)\psi_+(0),$$

where $f(t) = e^{\int_0^t c(s)ds}$. □

Corollary 2.4.18. *Let $(\omega(t), \psi_+(t))$ be a coupled solution of the Hitchin flow equations on a connected 6-manifold M. Then, the associated almost complex structure is $J(t) = J(0)$. Thus, it does not depend on t.*

Proof. We know that $\psi_+(t) = f(t)\psi_+(0)$, therefore

$$J(t) = J_{\psi_+(t)} = J_{f(t)\psi_+(0)} = J_{\psi_+(0)} = J(0),$$

since the almost complex structure induced by ψ_+ does not change if this 3-form is rescaled by a real constant. □

The case of six-dimensional nilpotent Lie algebras allows us to conclude that coupled solutions may not exist. Indeed, if we consider the nilpotent Lie algebras $\mathfrak{n}_9$

and $\mathfrak{n}_{28}$ with the structure equations given in (2.33) and (2.34), respectively, then in both cases the pair

$$\begin{array}{rcl} \omega & = & e^{12} + e^{34} + e^{56}, \\ \psi_+ & = & e^{135} - e^{146} - e^{236} - e^{245}, \end{array} \tag{2.35}$$

is a coupled SU(3)-structure with $d\omega = -\psi_+$ and we can prove the following

Proposition 2.4.19 ([70]). *Consider the Hitchin flow equations on the nilpotent Lie algebras $\mathfrak{n}_9$ and $\mathfrak{n}_{28}$. Then, on $\mathfrak{n}_{28}$ there exists a coupled solution starting from* (2.35) *at $t = 0$, while on $\mathfrak{n}_9$ there are no coupled solutions starting from* (2.35)*.*

Proof. First of all, observe that in the case of Lie algebras the Hitchin flow equations become a system of ordinary differential equations.

Let us start with $\mathfrak{n}_{28}$, a solution of the Hitchin flow equations which is coupled in the sense of our definition was given in [41], we recover it in our setting starting from a suitable pair $(\omega(t), \psi_+(t))$ and using our previous observations. From Proposition 2.4.17, we know that $(\omega(t), \psi_+(t))$ is a coupled solution if and only if

$$\psi_+(t) = f(t)\psi_+(0) = f(t)(e^{135} - e^{146} - e^{236} - e^{245}),$$

with $f(0) = 1$. It is also clear that $\psi_-(t) = f(t)\left(e^{136} + e^{145} + e^{235} - e^{246}\right)$. Moreover, we consider three smooth functions $a_1(t), a_2(t), a_3(t)$ with $a_i(0) = 1$ and such that

$$\omega(t) = a_1(t)e^{12} + a_2(t)e^{34} + a_3(t)e^{56}.$$

From now on, we omit the t-dependence of the functions for sake of brevity. The forms $\omega(t)$ and $\psi_\pm(t)$ are compatible for each t and from the normalization condition, we get

$$f^2 = a_1a_2a_3. \tag{2.36}$$

From the first Hitchin flow equation in (2.22) we obtain

$$\frac{d}{dt}f = -a_3, \tag{2.37}$$

while from the second one we have

$$\frac{d}{dt}(a_1a_3) = 0, \quad (2.38)$$
$$\frac{d}{dt}(a_2a_3) = 0, \quad (2.39)$$
$$\frac{d}{dt}(a_1a_2) = -4f. \quad (2.40)$$

From (2.38), (2.39) and the initial conditions at $t = 0$ we deduce that

$$a_1 = a_2 = \frac{1}{a_3}.$$

Using this result and (2.36), it holds necessarily

$$f = \frac{1}{\sqrt{a_3}}.$$

Thus, the ODE (2.37) becomes

$$\frac{d}{dt}a_3 = 2a_3^2\sqrt{a_3}$$

and solving it with initial condition $a_3(0) = 1$ we get

$$a_3 = (1 - 3t)^{-\frac{2}{3}}.$$

It is then easy to check that also (2.40) is satisfied. Then, the pair

$$\begin{aligned} \omega(t) &= (1-3t)^{\frac{2}{3}}e^{12} + (1-3t)^{\frac{2}{3}}e^{34} + (1-3t)^{-\frac{2}{3}}e^{56}, \\ \psi_+(t) &= (1-3t)^{\frac{1}{3}}(e^{135} - e^{146} - e^{236} - e^{245}), \end{aligned}$$

is a coupled solution of the Hitchin flow equations.

Consider now $\mathfrak{n}_9$, we shall show that there are no coupled solutions starting from (2.35). Also in this case, we need

$$\psi_+(t) = f(t)\psi_+(0) = f(t)(e^{135} - e^{146} - e^{236} - e^{245}),$$

with $f(0) = 1$, while we introduce 15 smooth real valued functions $b_{ij} = b_{ij}(t)$, $1 \leq i < j \leq 6$, such that

$$\omega(t) = \sum_{1\leq i<j\leq 6} b_{ij}(t)e^{ij},$$

$b_{12}(0) = b_{34}(0) = b_{56}(0) = 1$ and $b_{ij}(0) = 0$ for the remaining functions. We first impose that the equations resulting from the compatibility condition $\omega(t) \wedge \psi_+(t) = 0$ are satisfied. Then, we consider the Hitchin flow equations and we compute the ODEs deriving from them. From $\frac{d}{dt}\psi_+(t) = d\omega(t)$ we obtain

$$b_{24} = b_{26} = b_{45} = b_{46} = 0$$

and

$$\frac{d}{dt} f = -b_{56}.$$

Among the ODEs coming from $\frac{d}{dt}\omega(t) \wedge \omega(t) = -d\psi_-(t)$, we have

$$\frac{d}{dt}(b_{23}\, b_{56}) = 0, \qquad (2.41)$$

$$\frac{d}{dt}(b_{25}\, b_{34}) = 0, \qquad (2.42)$$

$$\frac{d}{dt}(b_{23}\, b_{25}) = -f. \qquad (2.43)$$

(2.41) and (2.42) give $b_{23}\, b_{56} = 0$ and $b_{25}\, b_{34} = 0$ for all t, since $b_{23}(0) = b_{25}(0) = 0$ and $b_{34}(0) = b_{56}(0) = 1$. From these results and the expression

$$(\omega(t))^3 = 6\, b_{12}\, b_{34}\, b_{56}\, e^{123456},$$

we get that b_{23} and b_{25} are identically zero, since b_{34} and b_{56} must be nowhere vanishing. That being so, (2.43) becomes $f(t) = 0$, which is not possible. □

2.4.3 Coupled SU(3)-structures and supersymmetry

Starting from the seminal work [173] of Strominger, six-dimensional manifolds endowed with a non-integrable SU(3)-structure have been frequently considered in supersymmetric string theory, giving rise to a broad literature in this area. We do not claim here to give a rigorous introductory description of this theory, but just some hints which are useful to understand how coupled SU(3)-structures appear in this setting. The reader may refer to the references we cite for more informations.

The space-time of the five consistent superstring theories existing in theoretical physics is modeled on a real ten-dimensional pseudo-Riemannian manifold (M^{10}, g_{10})

endowed with a spin structure and certain spinor fields. This manifold splits into the product of a four-dimensional Lorentzian manifold (M^4, g_4) and a compact six-dimensional manifold M^6 through a process called *compactification*. The requirement that part of the supersymmetry is preserved implies the existence of a globally-defined complex spinor field on the *internal manifold* M^6, which provides a reduction of the structure group of FM^6 to SU(3). The spinor field has to solve certain equations, which constrain the intrinsic torsion of the corresponding SU(3)-structure to lie only in some subclasses of $\mathcal{W}$. The classes of SU(3)-structures which are relevant in the various theories were recently reviewed in [123], among these we find $\mathcal{W}_1^- \oplus \mathcal{W}_2^-$, i.e., coupled SU(3)-structures.

In [136], the authors considered the problem of finding necessary and sufficient conditions for $\mathcal{N} = 1$ compactifications of Type IIA string theory on spaces of the form $\mathrm{AdS}_4 \times M^6$, where AdS_4 is the four-dimensional *anti-de Sitter space*. As a result, they obtained a set of constraints that the intrinsic torsion forms of the SU(3)-structure (ω, ψ_+) on the internal manifold have to satisfy. We recall them here briefly. *Supersymmetry equations* and the so-called *Bianchi identities* constrain the intrinsic torsion to lie in $\mathcal{W}_1^- \oplus \mathcal{W}_2^-$. Furthermore, in absence of *sources*, the Bianchi identities provide a further constraint on the exterior derivative of w_2^-:

$$dw_2^- \propto \psi_+, \tag{2.44}$$

and the norms of w_1^- and w_2^- have to satisfy the following inequality [118]

$$3(w_1^-)^2 \geq |w_2^-|^2, \tag{2.45}$$

where $|\cdot|$ denotes the norm with respect to the metric g induced by the SU(3)-structure. In the *massless limit*, the solutions reduce to $\mathrm{AdS}_4 \times M^6$, being M^6 a compact 6-manifold endowed with a coupled SU(3)-structure for which only (2.44) holds. Moreover, it was observed in [118] that the conditions (2.44) and (2.45) can be relaxed in the presence of sources. A remarkable property of this result is that the constraints are not only necessary, as usually happens, but also sufficient to guarantee the existence of solutions to the problem.

It is then worth studying the properties of 6-manifolds endowed with coupled SU(3)-structures satisfying (2.44) and (2.45) and look for possible examples. The forthcoming discussion is based on the first part of our work [70]. As we did before, we always assume that the considered manifolds are connected.

First, using Lemma 2.4.7, it is possible to show that condition (2.44) forces the proportionality factor between dw_2^- and ψ_+ to be constant and proportional to the squared norm of w_2^-.

Proposition 2.4.20. *Let M be a 6-manifold endowed with a coupled* SU(3)*-structure (ω, ψ_+) such that dw_2^- is proportional to ψ_+. Then, it holds*

$$dw_2^- = -\frac{|w_2^-|^2}{4}\psi_+.$$

Moreover, the norm of w_2^- is constant.

Proof. The identity $dw_2^- = -\frac{|w_2^-|^2}{4}\psi_+$ follows directly from the proof of Lemma 2.4.7 and the hypothesis on dw_2^-. Thus, we only need to prove that $|w_2^-|$ is constant. To do this, observe that if $dw_2^- = f\,\psi_+$ for some function $f \in C^\infty(M)$, then f has to be constant. Indeed, taking the exterior derivatives of both sides and using that ψ_+ is closed, we get

$$df \wedge \psi_+ = 0,$$

which implies $df = 0$ since wedging 1-forms by ψ_+ is injective. □

Remark 2.4.21. The expression of the proportionality factor between dw_2^- and ψ_+ given in the previous result was also obtained in [136] in terms of certain quantities coming from the physical situation.

From Proposition 2.4.20 and the fact that w_1^- is constant, we obtain the following constraint.

Proposition 2.4.22. *Let M be a 6-manifold endowed with a coupled* SU(3)*-structure (ω, ψ_+) such that dw_2^- is proportional to ψ_+. Then, the scalar curvature of the metric g induced by (ω, ψ_+) is constant.*

Proof. Consider the expression (2.29) of the scalar curvature of g and conclude using the fact that both w_1^- and $|w_2^-|$ are constant. □

Consider now condition (2.45), this implies a further constraint on the scalar curvature.

Proposition 2.4.23. *Let M be a 6-manifold endowed with a coupled* $\mathrm{SU}(3)$*-structure* (ω, ψ_+) *whose intrinsic torsion forms satisfy* $3(w_1^-)^2 \geq |w_2^-|^2$. *Then, the scalar curvature of the associated metric g is positive. Moreover, it is also constant if dw_2^- is proportional to ψ_+.*

Proof. Using (2.29) and the inequality $3(w_1^-)^2 \geq |w_2^-|^2$, we get

$$\mathrm{Scal}(g) = \frac{15}{2}(w_1^-)^2 - \frac{1}{2}|w_2^-|^2 \geq 2|w_2^-|^2 > 0.$$

Moreover, if dw_2^- is proportional to ψ_+, then the scalar curvature is constant by Proposition 2.4.22. □

Finally, when dw_2^- is proportional to ψ_+, an easy characterization for coupled structures inducing an Einstein metric can be given.

Proposition 2.4.24. *Let M be a 6-manifold endowed with a coupled* $\mathrm{SU}(3)$*-structure* (ω, ψ_+) *such that dw_2^- is proportional to ψ_+. Then, the induced metric g is Einstein if and only if the following identity holds*

$$*(w_2^- \wedge w_2^-) = w_1^- \, w_2^- - \frac{|w_2^-|^2}{3}\,\omega.$$

Proof. From the expression of Φ_1 in (2.30), we already got that

$$\mathrm{E}_1(\Phi_1) = \frac{1}{4} * (w_2^- \wedge w_2^-) - \frac{1}{4}w_1^- \, w_2^- + \frac{1}{12}|w_2^-|^2\,\omega.$$

When dw_2^- is proportional to ψ_+, from (2.30) we obtain

$$\Phi_2 = -2 * J(dw_2^-) = -\frac{|w_2^-|^2}{2}\,\psi_+$$

and $\mathrm{E}_2(\Phi_2) = 0$, since ψ_+ does not have components in $\left[\!\left[\Omega_0^{2,1}(M)\right]\!\right]$. Therefore, g is Einstein if and only if $\mathrm{Ric}^0(g) = \mathrm{i}_+^{-1}(\mathrm{E}_1(\Phi_1))$ is zero, and from this the assertion follows. □

In physical literature, examples of manifolds endowed with coupled structures satisfying the conditions (2.44) and (2.45) were studied for instance in [38, 118, 136, 175]. The case of compact nilmanifolds is a bit scattered through the literature and only some partial results are stated, sometimes without proofs. Therefore, we consider the problem of finding invariant coupled structures on compact nilmanifolds satisfying (all or in part) the conditions (2.44) and (2.45) and we give a unified description of its solution.

From the classification of nilpotent Lie algebras admitting a coupled structure obtained in Theorem 2.4.12, we know that we have to study only two cases, namely $\mathfrak{n}_9$ and $\mathfrak{n}_{28}$. Since every nilpotent Lie group is solvable, the following result by Milnor holds in the case we are considering.

Theorem 2.4.25 ([145])**.** *Let* S *be a solvable Lie group. Then, every left-invariant metric on* S *is either flat or has strictly negative scalar curvature.*

In particular, if a nilpotent Lie algebra is endowed with an inner product g, then $\mathrm{Scal}(g)$ is nonpositive. Consequently, using Proposition 2.4.23 it is immediate to show the

Proposition 2.4.26. *There are no six-dimensional nilmanifolds admitting an invariant coupled structure whose intrinsic torsion forms satisfy* $3(w_1^-)^2 \geq |w_2^-|^2$.

Thus, we can only look for nilpotent Lie algebras endowed with a coupled structure (ω, ψ_+) having dw_2^- proportional to ψ_+. Let us examine the two examples on $\mathfrak{n}_9$ and $\mathfrak{n}_{28}$ obtained earlier.

Example 2.4.27. Consider the Lie algebra $\mathfrak{n}_{28}$, its structure equations with respect to an adapted basis for the coupled SU(3)-structure are

$$\mathfrak{n}_{28} = (0, 0, 0, 0, e^{14} + e^{23}, e^{13} - e^{24}).$$

The pair

$$\begin{aligned} \omega &= e^{12} + e^{34} + e^{56}, \\ \psi_+ &= e^{135} - e^{146} - e^{236} - e^{245}, \end{aligned}$$

defines then a coupled SU(3)-structure and its coupled constant is $c = -1$. The only non-vanishing intrinsic torsion forms are

$$\begin{aligned} w_1^- &= \frac{2}{3}, \\ w_2^- &= -\frac{4}{3}e^{12} - \frac{4}{3}e^{34} + \frac{8}{3}e^{56}, \end{aligned}$$

and it is easy to check that condition (2.44) is satisfied:

$$dw_2^- = -\frac{8}{3}\,\psi_+.$$

Moreover, $-\frac{1}{4}|w_2^-|^2 = -\frac{8}{3}$, as we expected from Proposition 2.4.20. Finally, the inner product g induced by (ω, ψ_+) is such that the considered basis is orthonormal and its scalar curvature is $\mathrm{Scal}(g) = -2$.

Example 2.4.28. The structure equations of the Lie algebra $\mathfrak{n}_9$ with respect to an adapted basis for the coupled SU(3)-structure are

$$\mathfrak{n}_9 = \left(0, 0, 0, e^{13}, e^{14} + e^{23}, e^{13} - e^{15} - e^{24}\right).$$

Thus, the pair

$$\begin{aligned} \omega &= e^{12} + e^{34} + e^{56}, \\ \psi_+ &= e^{135} - e^{146} - e^{236} - e^{245}, \end{aligned}$$

is a coupled SU(3)-structure on $\mathfrak{n}_9$ and satisfies $d\omega = -\psi_+$. The only non-vanishing intrinsic torsion forms are

$$\begin{aligned} w_1^- &= \frac{2}{3}, \\ w_2^- &= -\frac{4}{3}e^{12} - \frac{4}{3}e^{34} + e^{36} - e^{45} + \frac{8}{3}e^{56}, \end{aligned}$$

and dw_2^- is not proportional to ψ_+. Moreover, the inner product induced by (ω, ψ_+) is $g = \sum_{k=1}^{6}(e^k)^2$ and $\mathrm{Scal}(g) = -3$.

The coupled structure on $\mathfrak{n}_{28}$ gives rise to an invariant coupled structure on the Iwasawa manifold and, as we recalled in Remark 2.4.13, the fact that this manifold admits an invariant coupled structure was also observed in [136], where the authors wrote it was the unique nilmanifold admitting a coupled structure they knew. From Theorem 2.4.12, we know that the non-Abelian nilpotent Lie algebras admitting a coupled structure are, up to isomorphism, $\mathfrak{n}_9$ and $\mathfrak{n}_{28}$. Moreover, as observed in Example 2.4.27, the coupled structure on $\mathfrak{n}_{28}$ satisfies condition (2.44), i.e., dw_2^- is proportional to ψ_+. Thus, we may ask whether $\mathfrak{n}_9$ admits a coupled structure satisfying (2.44) or not. In [38], the authors looked for the possible nilmanifolds admitting an invariant coupled structure satisfying (2.44) and concluded (without giving an explicit proof) that a systematic scan of all of the possible six-dimensional nilmanifolds yields two possibilities: the six-torus and the Iwasawa manifold. The six-torus has Abelian Lie algebra, so it is not considered in Theorem 2.4.12, since every differential form defined on it is closed. Anyway, this result seems to answer negatively our question and we can prove this is actually what happens.

Proposition 2.4.29. *There are no coupled* SU(3)*-structures on* $\mathfrak{n}_9$ *for which the exterior derivative of the intrinsic torsion form* w_2^- *is proportional to* ψ_+*.*

Proof. The idea is to describe all of the possible coupled SU(3)-structures on $\mathfrak{n}_9$ and see whether there exists one whose intrinsic torsion form w_2^- satisfies the required condition. We begin considering the frame $(e^1, \ldots, e^6)$ of $\mathfrak{n}_9^*$ for which the structure equations of $\mathfrak{n}_9$ are those written in Example 2.4.28. Let $\omega \in \Lambda^2(\mathfrak{n}_9^*)$ be a generic 2-form on $\mathfrak{n}_9$, we can write it as

$$\omega = \sum_{1\leq i<j\leq 6} b_{ij} e^{ij},$$

where b_{ij} are real numbers. We may think the 15-tuple $(b_{12}, \ldots, b_{56}) =: (b_{ij})$ as a point in the affine space $\mathbb{A}^{15}_{\mathbb{R}} - \{0\}$. The homogeneous polynomial P_ω of degree 3 in the unknowns b_{ij} appearing as coefficient of e^{123456} in the expression of ω^3 has to be non-vanishing, this gives a first constraint for (b_{ij}). Since we want a coupled structure, we consider a 3-form ψ_+ on $\mathfrak{n}_9$ given by $\psi_+ = cd\omega$, for some nonzero real number c. Assuming

$$\lambda(\psi_+) = -4c^4 b_{56}^2 (b_{36}b_{56} - b_{45}b_{56} - b_{46}^2 + b_{56}^2) < 0,$$

that is, $b_{56} \neq 0$ and $B := b_{36}b_{56} - b_{45}b_{56} - b_{46}^2 + b_{56}^2 > 0$, we can compute the almost complex structure J induced by the stable form ψ_+. Now, we change the basis from

$(e_1, \dots, e_6)$ to a basis $(E_1, \dots, E_6)$ which is adapted for J. To do this, it suffices to define $E_i = e_i$ and $E_{i+1} = Je_i$ for $i = 1, 3, 5$. With respect to $(E_1, \dots, E_6)$, the matrix associated with J is skew-symmetric with non-vanishing entries given by $J^2{}_1 = 1 = J^4{}_3 = J^6{}_5$. We can then compute the new structure equations with respect to the dual basis $(E^1, \dots, E^6)$, obtaining

$$\begin{aligned}
dE^i &= 0, \quad i = 1, 2, 3 \\
dE^4 &= \frac{b_{56}}{\sqrt{B}} E^{13}, \\
dE^5 &= -\frac{b_{46}}{b_{56}} E^{13} + \frac{\sqrt{B}}{b_{56}} \left(E^{14} + E^{23}\right), \\
dE^6 &= -\frac{b_{26}}{b_{56}} E^{12} - \frac{b_{46}}{b_{56}} E^{14} - \frac{b_{56}}{\sqrt{B}} E^{15} - \frac{\sqrt{B}}{b_{56}} E^{24} \\
&\quad -\frac{b_{36}b_{56} + b_{45}b_{56} - b_{46}^2 - b_{56}^2}{b_{56}\sqrt{B}} E^{13}.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
\psi_+ &= -c\,\frac{B}{b_{56}} \left(E^{135} - E^{146} - E^{236} - E^{245}\right), \\
\psi_- &= -c\,\frac{B}{b_{56}} \left(E^{136} + E^{145} + E^{235} - E^{246}\right).
\end{aligned}$$

We can write ω with respect to the new basis and impose it is of type $(1,1)$ with respect to J, obtaining 3 equations in the variables b_{ij} which can be solved under the constraint $\lambda(\psi_+) < 0$. We can then consider the symmetric matrix G associated with $g(\cdot, \cdot) = \omega(\cdot, J\cdot)$ with respect to the basis $(E_1, \dots, E_6)$ and denote by $\mathcal{P} \subset \mathbb{A}^{15}_{\mathbb{R}}$ the set on which it is positive definite. It is immediate to check that $P_\omega \neq 0$ when $(b_{ij}) \in \mathcal{P}$. Now, if we let (b_{ij}) vary in the non-empty set $\mathcal{Q} := \mathcal{P} \cap \{(b_{ij}) \mid \lambda(\psi_+) < 0\}$, we have all of the possible non-normalized coupled SU(3)-structures on $\mathfrak{n}_9$. The intrinsic torsion form w_1^- is always $-\frac{2}{3c}$, while w_2^- can be computed from its defining properties and the expression of $d\psi_-$. We are interested in the coupled structures having dw_2^- proportional to ψ_+. Thus, we can start with a generic 2-form w of type $(1,1)$ with respect to J and write it as

$$\begin{aligned}
w &= w_{12}E^{12} + w_{34}E^{34} + w_{56}E^{56} + w_{13}(E^{13} + E^{24}) + w_{14}(E^{14} - E^{23}) \\
&\quad + w_{15}(E^{15} + E^{26}) + w_{16}(E^{16} - E^{25}) + w_{35}(E^{35} + E^{46}) + w_{36}(E^{36} - E^{45}),
\end{aligned}$$

where w_{ij} are real numbers. Then, we have to impose that w is primitive with respect to ω, i.e., $w \wedge \omega^2 = 0$, and fulfills

$$d\psi_- = -\frac{2}{3c}\omega^2 - w \wedge \omega$$

and that dw is proportional to ψ_+. The last condition gives rise to a set of polynomial equations in the variables w_{ij} with coefficients depending on b_{ij} which can be solved in $\mathcal{Q}$. The condition on $d\psi_-$ gives thirteen equations of the same kind as before. We can solve four of them, namely those obtained comparing the coefficients of $E^{3456}, E^{2356}, E^{1256}, E^{2345}$, but then we get that some of the remaining equations can be solved only if $c = 0$ or $\lambda(\psi_+) = 0$. The assertion is then proved. □

The previous results can be summarized as follows

Proposition 2.4.30. *Let $\mathfrak{n}$ be a six-dimensional, non-Abelian, nilpotent Lie algebra endowed with a coupled* SU(3)*-structure (ω, ψ_+) having dw_2^- proportional to ψ_+. Then, $\mathfrak{n}$ is isomorphic to the Lie algebra $\mathfrak{n}_{28}$.*

2.5 Half-flat SU(3)-structures and Einstein metrics

As we saw in Section 2.2.3, requiring that the Riemannian metric induced by an SU(3)-structure is Einstein gives rise to some constraints on the intrinsic torsion and, in certain cases, allows to obtain non-existence results, like the one recalled in Proposition 2.2.16.

In literature, conjectures regarding the existence of certain classes of manifolds endowed with special geometric structures inducing an Einstein metric have been formulated. For instance, it was conjectured by Goldberg in [83] that every compact almost Kähler manifold (M, g, J) whose metric is Einstein is actually Kähler. In [169], Sekigawa showed that this is true when the scalar curvature is non-negative. Moreover, there exists a noncompact example of Einstein almost Kähler manifold with negative scalar curvature [7], which is the unique example of six-dimensional Einstein almost Kähler (non-Kähler) solvmanifold by the results contained in [99] and [65].

Motivated by the fact that the metric induced by a strict nearly Kähler structure is always Einstein, we may look for examples of double half-flat and coupled SU(3)-structures inducing Einstein metrics and see whether obstructions to the existence occur. Using some results already appeared in literature together with ours contained in [160], we are able to conclude that in both cases it is possible to find such examples, but there are various situations in which coupled Einstein structures cannot exist.

2.5.1 $S^3 \times S^3$ and its $\mathrm{Ad}(S^1)$-invariant Einstein metrics

In this section, based on [160, Sect. 3], we consider the problem of finding left-invariant special half-flat structures inducing Einstein metrics on $S^3 \times S^3$, identified with the Lie group $\mathrm{SU}(2) \times \mathrm{SU}(2)$.

As we discussed in Section 2.4.1, every left-invariant SU(3)-structure on $S^3 \times S^3$ can be identified with an SU(3)-structure defined on the Lie algebra $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$, whose structure equations with respect to a certain basis $(e^1, \ldots, e^6)$ of its dual space are

$$(e^{23}, e^{31}, e^{12}, e^{56}, e^{64}, e^{45}).$$

By Butruille's result recalled in Theorem 2.3.6, we know that on $S^3 \times S^3$ there exists an example of left-invariant nearly Kähler SU(3)-structure which is unique up to homothety. With respect to the considered basis of $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$, it can be described as follows

Example 2.5.1. The pair of compatible and normalized stable forms on $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$

$$\begin{aligned} \omega &= -\frac{\sqrt{3}}{18}\left(e^{14} + e^{25} + e^{36}\right), \\ \psi_+ &= \frac{\sqrt{3}}{54}\left(-e^{234} + e^{156} + e^{135} - e^{246} - e^{126} + e^{345}\right), \end{aligned}$$

defines a nearly Kähler SU(3)-structure, since $d\omega = 3\psi_+$ and $d\psi_- = -2\omega^2$.

The inner product g induced by the nearly Kähler SU(3)-structure on $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ gives rise to a left-invariant Einstein metric on $S^3 \times S^3$, known in literature as *Jensen metric*. With respect to the basis $(e_1, \ldots, e_6)$ of $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ and up to a scalar constant, the matrix associated with the Jensen metric is

$$\begin{bmatrix} 2 & 0 & 0 & -1 & 0 & 0 \\ 0 & 2 & 0 & 0 & -1 & 0 \\ 0 & 0 & 2 & 0 & 0 & -1 \\ -1 & 0 & 0 & 2 & 0 & 0 \\ 0 & -1 & 0 & 0 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 2 \end{bmatrix}. \qquad (2.46)$$

Together with the *standard metric* $\sum_{i=1}^{6}(e^i)^2$, they constitute the unique known examples of left-invariant Einstein metrics on $S^3 \times S^3$ and the problem of classifying all of the left-invariant Einstein metrics existing on this manifold is still open. Moreover, these two examples are unique in the following sense

Theorem 2.5.2 ([154]). *Let g be a left-invariant Einstein metric on the Lie group* $\mathrm{SU}(2) \times \mathrm{SU}(2)$ *which is* $\mathrm{Ad}(S^1)$*-invariant for some embedding* $S^1 \subset \mathrm{SU}(2) \times \mathrm{SU}(2)$*. Then, g is isometric up to homothety either to the standard metric or to the Jensen metric.*

In [167], the author gave an example of left-invariant double half-flat SU(3)-structure on $S^3 \times S^3$ inducing the standard metric, we recall it here.

Example 2.5.3 ([167]). The pair of compatible, normalized stable forms on $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$

$$\begin{aligned} \omega &= -e^{14} - e^{25} - e^{36}, \\ \psi_+ &= \frac{1}{\sqrt{2}}\left(e^{123} - e^{156} + e^{246} - e^{345} + e^{126} - e^{135} + e^{234} - e^{456}\right), \end{aligned}$$

induces the standard metric. Thus, it defines an SU(3)-structure. Moreover, $d\psi_+ = 0$, $d\omega^2 = 0$, $d\psi_- = \frac{1}{\sqrt{2}}\omega^2$ and $d\omega$ is not proportional to ψ_+, i.e., it is a double half-flat SU(3)-structure.

Moreover, in [160] we gave an example of half-flat SU(3)-structure of class $\mathcal{W}_1^- \oplus \mathcal{W}_2^- \oplus \mathcal{W}_3$ inducing the Jensen metric.

Example 2.5.4 ([160])**.** The pair

$$\begin{aligned}\omega &= \frac{\sqrt[3]{4}\sqrt[6]{3}}{2}\left(-e^{14}+e^{25}+e^{36}\right),\\ \psi_+ &= e^{123}+e^{135}-e^{246}-e^{126}+e^{345}-e^{456},\end{aligned}$$

defines an SU(3)-structure on $\mathfrak{su}(2)\oplus\mathfrak{su}(2)$ and induces a metric which is (proportional to) the Jensen metric. Moreover, this SU(3)-structure is half-flat, since both ψ_+ and ω^2 are closed, and it is neither coupled nor double half-flat, since $d\omega$ is not proportional to ψ_+ and $d\psi_-$ is not proportional to ω^2.

Summarizing, on $S^3\times S^3$ there exist left-invariant half-flat ($\mathcal{W}_1^-\oplus\mathcal{W}_2^-\oplus\mathcal{W}_3$) and nearly Kähler SU(3)-structures ($\mathcal{W}_1^-$) inducing the Jensen metric and left-invariant double half-flat SU(3)-structures ($\mathcal{W}_1^-\oplus\mathcal{W}_3$) inducing the standard metric. It is then natural to ask whether there are left-invariant coupled structures inducing any of the two $\mathrm{Ad}(S^1)$-invariant Einstein metrics existing on this manifold. The answer is negative and it is possible to show this using the theory of algebraic varieties. We shall introduce the objects which are useful for our aim directly in the proof, the reader may refer for instance to [55] for more details.

Theorem 2.5.5 ([160])**.** *$S^3\times S^3$ does not admit left-invariant coupled* SU(3)*-structures (ω,ψ_+) inducing an* $\mathrm{Ad}(S^1)$*-invariant Einstein metric.*

Proof. Let us consider a left-invariant coupled SU(3)-structure (ω,ψ_+) on $S^3\times S^3$, which we identify with a 2-form ω and a 3-form ψ_+ defined on $\mathfrak{su}(2)\oplus\mathfrak{su}(2)$ and such that $\psi_+=c\,d\omega$, $c\in\mathbb{R}-\{0\}$. Since ω^2 is closed and the Lie algebra $\mathfrak{su}(2)$ is simple, it follows that $\omega\in\mathfrak{su}^*(2)\otimes\mathfrak{su}^*(2)$ (cf. [167, Ch. 5, Lemma 1.1]). Thus,

$$\omega=a_{14}e^{14}+a_{15}e^{15}+a_{16}e^{16}+a_{24}e^{24}+a_{25}e^{25}+a_{26}e^{26}+a_{34}e^{34}+a_{35}e^{35}+a_{36}e^{36},$$

where a_{ij} are real coefficients. From this expression, we obtain that of $\psi_+=c\,d\omega$ and from the closedness of ω^2, we know that the compatibility condition $\omega\wedge\psi_+=0$ holds. It is now possible to compute $\lambda=\lambda(\psi_+)$, which turns out to be a homogeneous polynomial of degree 4 in the coefficients a_{ij}, the almost complex structure $J=J_{\psi_+}$ and $g(\cdot,\cdot)=\omega(\cdot,J\cdot)$. With respect to the basis $(e_1,\ldots,e_6)$, the matrix G associated with g is symmetric. Moreover, up to a global sign depending on whether the considered basis is positively oriented or not and not affecting the computations afterwards, the nonzero entries are the following:

$$
\begin{aligned}
G_{i,i} &= \tfrac{-2c^2}{\sqrt{-\lambda}}(a_{14}a_{25}a_{36} - a_{14}a_{26}a_{35} - a_{15}a_{24}a_{36} \\
&\quad + a_{15}a_{26}a_{34} + a_{16}a_{24}a_{35} - a_{16}a_{25}a_{34}), \quad i = 1, \dots, 6 \\
G_{1,4} &= \tfrac{-c^2}{\sqrt{-\lambda}}(a_{14}^3 + a_{14}a_{15}^2 + a_{14}a_{16}^2 + a_{14}a_{24}^2 - a_{14}a_{25}^2 - a_{14}a_{26}^2 + a_{14}a_{34}^2 - a_{14}a_{35}^2 \\
&\quad - a_{14}a_{36}^2 + 2a_{15}a_{24}a_{25} + 2a_{15}a_{34}a_{35} + 2a_{16}a_{24}a_{26} + 2a_{16}a_{34}a_{36}), \\
G_{1,5} &= \tfrac{-c^2}{\sqrt{-\lambda}}(a_{14}^2a_{15} + 2a_{14}a_{24}a_{25} + 2a_{14}a_{34}a_{35} + a_{15}^3 + a_{15}a_{16}^2 - a_{15}a_{24}^2 + a_{15}a_{25}^2 \\
&\quad - a_{15}a_{26}^2 - a_{15}a_{34}^2 + a_{15}a_{35}^2 - a_{15}a_{36}^2 + 2a_{16}a_{25}a_{26} + 2a_{16}a_{35}a_{36}), \\
G_{1,6} &= \tfrac{-c^2}{\sqrt{-\lambda}}(a_{14}^2a_{16} + 2a_{14}a_{24}a_{26} + 2a_{14}a_{34}a_{36} + a_{15}^2a_{16} + 2a_{15}a_{25}a_{26} + a_{16}^3 \\
&\quad + 2a_{15}a_{35}a_{36} - a_{16}a_{24}^2 - a_{16}a_{25}^2 + a_{16}a_{26}^2 - a_{16}a_{34}^2 - a_{16}a_{35}^2 + a_{16}a_{36}^2), \\
G_{2,4} &= \tfrac{-c^2}{\sqrt{-\lambda}}(a_{14}^2a_{24} + 2a_{14}a_{15}a_{25} + 2a_{14}a_{16}a_{26} - a_{15}^2a_{24} - a_{16}^2a_{24} + a_{24}^3 + a_{24}a_{25}^2 \\
&\quad + a_{24}a_{26}^2 + a_{24}a_{34}^2 - a_{24}a_{35}^2 - a_{24}a_{36}^2 + 2a_{25}a_{34}a_{35} + 2a_{26}a_{34}a_{36}), \\
G_{2,5} &= \tfrac{c^2}{\sqrt{-\lambda}}(a_{14}^2a_{25} - 2a_{14}a_{15}a_{24} - a_{15}^2a_{25} - 2a_{15}a_{16}a_{26} + a_{16}^2a_{25} - a_{24}^2a_{25} \\
&\quad - 2a_{24}a_{34}a_{35} - a_{25}^3 - a_{25}a_{26}^2 + a_{25}a_{34}^2 - a_{25}a_{35}^2 + a_{25}a_{36}^2 - 2a_{26}a_{35}a_{36}), \\
G_{2,6} &= \tfrac{c^2}{\sqrt{-\lambda}}(a_{14}^2a_{26} - 2a_{14}a_{16}a_{24} + a_{15}^2a_{26} - 2a_{15}a_{16}a_{25} - a_{16}^2a_{26} - a_{24}^2a_{26} \\
&\quad - 2a_{24}a_{34}a_{36} - a_{25}^2a_{26} - 2a_{25}a_{35}a_{36} - a_{26}^3 + a_{26}a_{34}^2 + a_{26}a_{35}^2 - a_{26}a_{36}^2), \\
G_{3,4} &= \tfrac{-c^2}{\sqrt{-\lambda}}(a_{14}^2a_{34} + 2a_{14}a_{15}a_{35} + 2a_{14}a_{16}a_{36} - a_{15}^2a_{34} - a_{16}^2a_{34} + a_{24}^2a_{34} \\
&\quad + 2a_{24}a_{25}a_{35} + 2a_{24}a_{26}a_{36} - a_{25}^2a_{34} - a_{26}^2a_{34} + a_{34}^3 + a_{34}a_{35}^2 + a_{34}a_{36}^2), \\
G_{3,5} &= \tfrac{c^2}{\sqrt{-\lambda}}(a_{14}^2a_{35} - 2a_{14}a_{15}a_{34} - a_{15}^2a_{35} - 2a_{15}a_{16}a_{36} + a_{16}^2a_{35} + a_{24}^2a_{35} \\
&\quad - 2a_{24}a_{25}a_{34} - a_{25}^2a_{35} - 2a_{25}a_{26}a_{36} + a_{26}^2a_{35} - a_{34}^2a_{35} - a_{35}^3 - a_{35}a_{36}^2), \\
G_{3,6} &= \tfrac{c^2}{\sqrt{-\lambda}}(a_{14}^2a_{36} - 2a_{14}a_{16}a_{34} + a_{15}^2a_{36} - 2a_{15}a_{16}a_{35} - a_{16}^2a_{36} + a_{24}^2a_{36} \\
&\quad - 2a_{24}a_{26}a_{34} + a_{25}^2a_{36} - 2a_{25}a_{26}a_{35} - a_{26}^2a_{36} - a_{34}^2a_{36} - a_{35}^2a_{36} - a_{36}^3),
\end{aligned}
$$

where $G_{i,j} = g(e_i, e_j)$. Observe that up to multiplication by $\sqrt{-\lambda}$, the nonzero terms are all homogeneous polynomials of third degree in the a_{ij}.

We are looking for coupled SU(3)-structures inducing either the standard metric or the Jensen metric, which with respect to the considered basis can be written as the identity matrix and as (2.46), respectively. Thus, since $\omega \wedge \psi_+ = 0, d\psi_+ = 0$ and $d\omega^2 = 0$, we first have to solve the system obtained by imposing that the matrix G is proportional to the identity matrix or to the matrix (2.46) under the assumption $\lambda < 0$ and then, if we find solutions of it, we need to impose that the normalization condition is satisfied in order to obtain what we want.

Case 1: the standard metric.
Since rescaling a metric with a positive constant does not change the Ricci tensor, we are looking for solutions of the equation

$$G = rI,$$

where r is a positive real number.

Since the entries in the diagonal of G are all equal, we only have to solve the system of equations

$$G_{i,j} = 0, \quad i = 1, 2, 3, \; j = 4, 5, 6,$$

under the assumptions $G_{1,1} \neq 0$ and $\lambda < 0$.

For $i, j = 1, \ldots, 6$, we let

$$\widetilde{G}_{i,j} := \sqrt{-\lambda}\, G_{i,j}.$$

Then, as already observed, $\widetilde{G}_{i,j}$ are homogeneous polynomials of degree 3 in a_{ij} and, under our assumptions, $G_{i,j} = 0$ if and only if $\widetilde{G}_{i,j} = 0$ for $i = 1, 2, 3, \; j = 4, 5, 6$.

Since we have a system of equations involving homogeneous polynomials of the same degree and we are looking for solutions defined up to a multiplicative constant, let us consider the projective space $\mathbb{CP}^8$ with coordinate ring

$$\mathbb{C}[a_{14}, a_{15}, a_{16}, a_{24}, a_{25}, a_{26}, a_{34}, a_{35}, a_{36}]$$

and the homogeneous ideals

$$P := \left\langle \widetilde{G}_{1,1} \right\rangle,$$

$$Q := \left\langle \widetilde{G}_{1,4}, \widetilde{G}_{2,4}, \widetilde{G}_{3,4}, \widetilde{G}_{1,5}, \widetilde{G}_{2,5}, \widetilde{G}_{3,5}, \widetilde{G}_{1,6}, \widetilde{G}_{2,6}, \widetilde{G}_{3,6} \right\rangle.$$

What we are looking for is the set of points $[a_{14} : \ldots : a_{36}]$ lying in the projective variety

$$V(Q) = \left\{ [a_{14} : \ldots : a_{36}] \in \mathbb{CP}^8 \mid \widetilde{G}_{i,j}(a_{14}, \ldots, a_{36}) = 0, \; i = 1, 2, 3, \; j = 4, 5, 6 \right\}$$

but not in

$$V(P) = \left\{ [a_{14} : \ldots : a_{36}] \in \mathbb{CP}^8 \mid \widetilde{G}_{1,1}(a_{14}, \ldots, a_{36}) = 0 \right\}$$

and for which $\lambda < 0$. By [55, Ch. 4, Thm. 7], we know that

$$\overline{V(Q) - V(P)} \subseteq V(Q : P),$$

where $Q : P$ is the ideal quotient of Q by P. In our case,

$$Q : P = \langle a_{14}, a_{15}, a_{16}, a_{24}, a_{25}, a_{26}, a_{34}, a_{35}, a_{36} \rangle.$$

Therefore, $V(Q : P) = \emptyset$. This proves that on $S^3 \times S^3$ there are no left-invariant coupled SU(3)-structures inducing the standard metric.

Case 2: the Jensen metric.

Following the same idea of the previous case and looking at the entries of the matrix (2.46), we have now to consider the ideals P and

$$R := \left\langle \widetilde{G}_{1,5}, \widetilde{G}_{1,6}, \widetilde{G}_{2,4}, \widetilde{G}_{2,6}, \widetilde{G}_{3,4}, \widetilde{G}_{3,5}, \widetilde{G}_{2,5} - \widetilde{G}_{3,6}, \widetilde{G}_{3,6} - \widetilde{G}_{1,4}, \widetilde{G}_{1,1} + 2\widetilde{G}_{1,4} \right\rangle$$

and look for those points lying in the projective variety $V(R)$ but not in $V(P)$ and for which $\lambda < 0$. Now,

$$R : P = \langle a_{15}, a_{16}, a_{24}, a_{26}, a_{34}, a_{35}, a_{25} - a_{14}, a_{36} - a_{14} \rangle,$$

then

$$V(R : P) = \{[\gamma : 0 : 0 : 0 : \gamma : 0 : 0 : 0 : \gamma] \mid \gamma \in \mathbb{C} - \{0\}\}$$

is a point in $\mathbb{CP}^8$ and, since $\mathbb{C}$ is algebrically closed and R is a radical ideal,

$$V(R : P) = \overline{V(R) - V(P)}$$

by [55, Ch. 4, Thm. 7]. Moreover, the requested condition on λ is satisfied, indeed

$$\lambda = -3\, c^4 \gamma^4 < 0.$$

The coupled SU(3)-structures we are interested in are defined when γ is a negative real number. In this case, we have

$$\begin{aligned}
\omega &= \gamma(e^{14} + e^{25} + e^{36}),\\
\psi_+ &= c\,\gamma(e^{234} - e^{156} - e^{135} + e^{246} + e^{126} - e^{345}),\\
\psi_- &= \frac{c\,\gamma}{\sqrt{3}}(2e^{123} - e^{126} + e^{135} - e^{156} - e^{234} + e^{246} - e^{345} + 2e^{456}).
\end{aligned}$$

The forms ω and ψ_+ are stable and the normalization condition implies

$$c = \pm\sqrt{\frac{-2\gamma}{\sqrt{3}}}.$$

In both cases, the SU(3)-structure is nearly Kähler. □

2.5.2 Twistor spaces

The theory of twistor spaces is not new and there are several long-standing results on this topic in literature. In this section, we recall those which are useful to show that examples of coupled SU(3)-structures arise from this construction, paying particular attention to their properties.

Let (M^4, g) be an oriented, four-dimensional Riemannian manifold and denote by (Q, π_Q, M^4) the corresponding principal SO(4)-bundle. The set of all almost complex structures on M^4 which are compatible with g and preserve the orientation is parametrized by the bundle $\mathcal{Z} := Q \times_{\mathrm{SO}(4)} \mathrm{SO}(4)/\mathrm{U}(2)$ associated with Q with fiber SO(4)/U(2).

Definition 2.5.6. $\mathcal{Z}$ is called the *twistor space* of (M^4, g).

Remark 2.5.7. The twistor space $\mathcal{Z}$ can be equivalently defined as the 2-sphere bundle over M^4 consisting of the unit (-1)-eigenvectors of the Hodge operator acting on $\Lambda^2(T^*M)$ (see for instance [147] for the details).

Denote by $\pi : \mathcal{Z} \to M^4$ the bundle projection. The vertical subbundle $T^{\mathcal{V}}\mathcal{Z} = \ker(\pi_*)$ of $T\mathcal{Z}$ inherits a complex structure $J^{\mathcal{V}}$ from the canonical complex structure on the fiber $\mathrm{SO}(4)/\mathrm{U}(2) \cong \mathbb{CP}^1$. Moreover, the Levi Civita connection ∇^g on (M^4, g) induces a decomposition $T\mathcal{Z} = T^{\mathcal{H}}\mathcal{Z} \oplus T^{\mathcal{V}}\mathcal{Z}$ of the tangent bundle $T\mathcal{Z}$ into horizontal and vertical subbundles and the former is endowed with a tautological almost complex structure $J^{\mathcal{H}}$ defined at the point (p, J) of $\mathcal{Z}$ by $J^{\mathcal{H}}_{(p,J)} = \pi_*^{-1} \circ J \circ \pi_* : T^{\mathcal{H}}_{(p,J)}\mathcal{Z} \to T^{\mathcal{H}}_{(p,J)}\mathcal{Z}$. It is then possible to define two almost complex structures on $\mathcal{Z}$ preserving the considered decomposition of $T\mathcal{Z}$: the first one is [10]

$$J_1 = J^{\mathcal{H}} + J^{\mathcal{V}},$$

and the second one is [61]

$$J_2 = J^{\mathcal{H}} - J^{\mathcal{V}}.$$

By [10], J_1 is integrable if and only if (M^4, g) is a *self-dual* Riemannian manifold, i.e., the negative part of the Weyl tensor W vanishes identically, while it was shown in [61] that J_2 is never integrable.

The decomposition of $T\mathcal{Z}$ into horizontal and vertical subbundles also allows to define a 1-parameter family of Riemannian metrics on $\mathcal{Z}$ given by

$$g_t = \pi^* g + t g^{\mathcal{V}}, \quad t > 0.$$

where $g^{\mathcal{V}}$ is the standard metric of constant curvature 1 on the fiber $\mathbb{CP}^1$.

It is immediate to check that g_t is compatible with the almost complex structures J_1, J_2 for all $t > 0$, thus both $(\mathcal{Z}, g_t, J_1)$ and $(\mathcal{Z}, g_t, J_2)$ are six-dimensional almost Hermitian manifolds. Their possible types in Gray and Hervella's classification were determined in some particular cases in [10, 61] and were completely described by Muškarov in [147], who characterized them in terms of the properties of the Riemannian manifold (M^4, g). For instance, $(\mathcal{Z}, g_t, J_2)$ is quasi Kähler, i.e., belongs to $\mathcal{W}_1 \oplus \mathcal{W}_2$, if and only if (M^4, g) is a self-dual Einstein manifold.

In [77], the authors studied the conditions for which the Riemannian manifold $(\mathcal{Z}, g_t)$ is Einstein, obtaining the following result.

Theorem 2.5.8 ([77])**.** *Let (M^4, g) be an oriented self-dual Einstein 4-manifold with positive scalar curvature* $\mathrm{Scal}(g) > 0$. *If $t = \frac{48}{\mathrm{Scal}(g)}$ or $t = \frac{24}{\mathrm{Scal}(g)}$, then g_t is an Einstein metric on the twistor space $\mathcal{Z}$. Conversely, if the twistor space $(\mathcal{Z}, g_t)$ of a four-dimensional, oriented Riemannian manifold (M^4, g) is Einstein, then (M^4, g) is a self-dual Einstein manifold with positive scalar curvature and either $t = \frac{48}{\mathrm{Scal}(g)}$ or $t = \frac{24}{\mathrm{Scal}(g)}$.*

Moreover, it was proved independently in [81] and in [100] that the twistor space $(\mathcal{Z}, g_t, J_1)$ is Kähler if and only if (M^4, g) is Einstein self-dual with positive scalar curvature and $t = \frac{48}{\mathrm{Scal}(g)}$, while it was shown in [147] that $(\mathcal{Z}, g_t, J_2)$ is nearly Kähler if and only if (M^4, g) is Einstein self-dual with positive scalar curvature and $t = \frac{24}{\mathrm{Scal}(g)}$. Observe that, using the properties of the scalar curvature, the identity $t = \frac{r}{\mathrm{Scal}(g)}$

can be rewritten as $\mathrm{Scal}\left(\frac{g}{t}\right) = r$ for every real number r and $t > 0$. Thus, since

$$g_t = t\left(\pi^*\left(\frac{g}{t}\right) + g^{\mathcal{V}}\right),$$

instead of seeing g as a fixed metric on M^4, we may think of it as a varying metric on the base 4-manifold and restate the previous results as follows

Proposition 2.5.9. *The twistor space $(\mathcal{Z}, g_t, J_2)$ over a self-dual Einstein 4-manifold (M^4, g) with positive scalar curvature is quasi Kähler. Moreover, it is nearly Kähler if the metric g is rescaled so that* $\mathrm{Scal}\left(\frac{g}{t}\right) = 24$, *while for the rescaling* $\mathrm{Scal}\left(\frac{g}{t}\right) = 48$ *the corresponding Riemannian metric is Einstein.*

By [61, Prop. 8.1], for every oriented Riemannian 4-manifold (M^4, g), the first Chern class of the almost complex manifold $(\mathcal{Z}, J_2)$ is zero. Therefore, there exists on $\mathcal{Z}$ a globally defined $(3,0)$-form and, using this, it is possible to define a reduction of the structure group of $(\mathcal{Z}, g_t, J_2)$ from U(3) to SU(3) (see also [163, Ch. 7]). When (M^4, g) is self-dual Einstein with positive scalar curvature, the corresponding SU(3)-structure is coupled for all values of $t > 0$ but the one giving $\mathrm{Scal}\left(\frac{g}{t}\right) = 24$, which is exactly the nearly Kähler case. There are different ways to show this, a simple one consists in considering the principal SU(3)-bundle over $\mathcal{Z}$ and working with the first structure equations determined by Xu in [181]. This was done for instance in [175] and reviewed in our work [70] with some additional details. Let us now write explicitly the computations.

Denote by $\pi_P : P \to \mathcal{Z}$ the SU(3)-structure on the twistor space. For every $u \in P$ such that $\pi_P(u) = p \in \mathcal{Z}$, we can consider $u^{-1} : T_p\mathcal{Z} \to \mathbb{R}^6 \cong \mathbb{C}^3$ and define

$$\varepsilon_k(u) = \left(\left(u^{-1}\right)^*(dz_k)\right) \circ (\pi_P)_{*u}, \quad k = 1, 2, 3,$$

where (dz_1, dz_2, dz_3) is the standard basis of complex linear 1-forms on $\mathbb{C}^3$ introduced in Remark 2.1.3. This gives three differential 1-forms $\varepsilon_1, \varepsilon_2, \varepsilon_3$ on P called *tautological 1-forms*. In particular, if $\omega(\cdot,\cdot) = g_t(J_2\cdot,\cdot)$ and Ψ are the differential forms associated with the SU(3)-structure on $\mathcal{Z}$, then their pullbacks on P have the following expressions in terms of the frame $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$

$$\begin{aligned}\boldsymbol{\omega} := \pi_P^*(\omega) &= \frac{i}{2}\left(\varepsilon^1 \wedge \overline{\varepsilon^1} + \varepsilon^2 \wedge \overline{\varepsilon^2} + \varepsilon^3 \wedge \overline{\varepsilon^3}\right),\\ \boldsymbol{\Psi} := \pi_P^*(\Psi) &= i\left(\varepsilon^1 \wedge \varepsilon^2 \wedge \varepsilon^3\right).\end{aligned}$$

Remark 2.5.10. The imaginary unit i appearing in the expression of $\boldsymbol{\Psi}$ is due to the fact that we are using a convention which is different from the one used in [181]. In detail, $-\Re(\Psi)$ here is $\Im(\Psi)$ there and $\Im(\Psi)$ here is $\Re(\Psi)$ there.

When the base manifold $\left(M^4, \frac{g}{t}\right)$ is self-dual and Einstein, the first structure equations of P are [181]

$$\mathrm{d}\begin{pmatrix} \varepsilon^1 \\ \varepsilon^2 \\ \varepsilon^3 \end{pmatrix} = -\begin{pmatrix} \alpha & 0 \\ 0 & -\mathrm{tr}(\alpha) \end{pmatrix} \wedge \begin{pmatrix} \varepsilon^1 \\ \varepsilon^2 \\ \varepsilon^3 \end{pmatrix} + \begin{pmatrix} \overline{\varepsilon^2 \wedge \varepsilon^3} \\ \overline{\varepsilon^3 \wedge \varepsilon^1} \\ \sigma\, \overline{\varepsilon^1 \wedge \varepsilon^2} \end{pmatrix},$$

where α is a 2×2 skew-Hermitian matrix of 1-forms and $\sigma := \frac{t\,\mathrm{Scal}(g)}{24}$. The pullbacks $\boldsymbol{w}_k^{(\pm)}$ of the intrinsic torsion forms $w_k^{(\pm)}$ can then be computed from $\mathrm{d}\boldsymbol{\omega}$ and $\mathrm{d}\boldsymbol{\Psi}$. By a straightforward computation, we get that only two of them are non-identically vanishing, namely

$$\begin{aligned} \boldsymbol{w}_1^- &= \frac{2}{3}(\sigma+2), \\ \boldsymbol{w}_2^- &= -\frac{2}{3}\, i\,(\sigma-1)\left(\varepsilon^1 \wedge \overline{\varepsilon^1} + \varepsilon^2 \wedge \overline{\varepsilon^2} - 2\varepsilon^3 \wedge \overline{\varepsilon^3}\right), \end{aligned}$$

and that it holds

$$\mathrm{d}\boldsymbol{w}_2^- = -\frac{8}{3}\,(\sigma-1)^2\,\Re(\boldsymbol{\Psi}).$$

Therefore, the SU(3)-structure (ω, ψ_+) on $\mathcal{Z}$ is nearly Kähler for $\sigma = 1$ and is coupled for the remaining positive values of σ. In the second case, the exterior derivative of the intrinsic torsion form w_2^- is proportional to ψ_+ and this result can be used to provide examples of manifolds satisfying the conditions discussed in Section 2.4.3, as it was observed in the physical paper [175].

Let us now go back to $\mathcal{Z}$. At each point $p \in \mathcal{Z}$, we can consider the linear frame $u \in \pi_P^{-1}(p)$ and the basis $(e^1, \ldots, e^6)$ of $T_p^*\mathcal{Z}$ defined in such a way that $(u^{-1})^*(dz_k) = e^{2k-1} + i\, e^{2k}$, $k = 1, 2, 3$. It is adapted for the SU(3)-structure and at p the differential forms ω, ψ_+, ψ_- can be written as follows

$$\begin{aligned} \omega &= e^{12} + e^{34} + e^{56}, \\ \psi_+ &= e^{246} - e^{136} - e^{145} - e^{235}, \\ \psi_- &= e^{135} - e^{146} - e^{236} - e^{245}. \end{aligned}$$

Moreover, ψ_+ induces the almost complex structure J_2, the complex 1-forms e^1+ie^2, e^3+ie^4, e^5+ie^6 are of type $(1,0)$ with respect to it and the Riemannian metric g_t is such that the considered frame is orthonormal. Using the results of [19] recalled in Section 2.2.3, we can then compute the Ricci and the scalar curvature of g_t in terms of the non-vanishing intrinsic torsion forms. We obtain the following expression for the scalar curvature

$$\mathrm{Scal}(g_t) = -2\sigma^2 + 24\sigma + 8,$$

while the matrix associated with the traceless part of the Ricci tensor of g_t with respect to the basis $(e_1, \dots, e_6)$ is

$$\mathrm{Ric}^0(g_t) = -\frac{2}{3}\,(\sigma-1)(\sigma-2)\,\mathrm{diag}(1,1,1,1,-2,-2).$$

Thus, the metric g_t is Einstein if and only if $\sigma = 1$ or $\sigma = 2$, that is, if and only if the scalar curvature of the Riemannian metric $\frac{g}{t}$ on M^4 is 24 or 48, respectively. These results are consistent with Theorem 2.5.8. As we noticed before, the coupled structure is nearly Kähler for $\sigma = 1$, while for $\sigma = 2$ we get an example of a coupled SU(3)-structure inducing an Einstein metric. More in detail, with respect to the adapted frame, the latter has the following non-identically vanishing intrinsic torsion forms

$$\begin{aligned} w_1^- &= \frac{8}{3}, \\ w_2^- &= -\frac{4}{3}\left(e^{12} + e^{34} - 2e^{56}\right), \end{aligned}$$

the coupled constant is $c = -4$ and the scalar curvature is $\mathrm{Scal}(g_t) = 48$. Working with this frame, it is also easy to check that this example satisfies the characterization given in Proposition 2.4.24.

When (M^4, g) is compact, there exist essentially two examples of manifolds satisfying the properties considered in the previous discussion. Indeed

Theorem 2.5.11 ([81, 100]). *Let (M^4, g) be a four-dimensional, compact, self-dual Einstein manifold with positive scalar curvature. Then, it is isometric either to the sphere S^4 or to the complex projective plane $\mathbb{CP}^2$, both endowed with their standard metric.*

Consequently, the possible manifolds admitting a coupled SU(3)-structure arising from this construction are the complex projective space $\mathcal{Z} = \mathbb{CP}^3$ when $M^4 = S^4$ and the flag manifold $\mathcal{Z} = \mathbb{F}(1,2)$ when $M^4 = \mathbb{CP}^2$. When the coupled SU(3)-structure is nearly Kähler, we obtain the two examples already mentioned in Theorem 2.3.6 (see also the detailed discussion in [34]).

Remark 2.5.12. An alternative way to show that the SU(3)-structure defined via the twistor construction on $\mathbb{CP}^3$ and $\mathbb{F}(1,2)$ is coupled consists in considering the associated spinor field and determine its spinor field equations. The computations were worked out in [2, Ex. 3.15].

2.5.3 Six-dimensional Einstein solvmanifolds

The homogeneous examples of (special) half-flat structures inducing Einstein metrics we considered until this moment are all defined on compact manifolds. Now, we move to the noncompact case, where the only currently known examples of homogeneous Einstein manifolds are Einstein solvmanifolds. We discussed the main results on this topic in Section 1.4.2, here we concentrate on the six-dimensional case.

Six-dimensional Einstein solvmanifolds were classified by Nikitenko and Nikonorov in [153]. The result is recalled in the next theorem. Instead of the Lie algebra structure equations given (in the original formulation of [153]) by the nontrivial Lie brackets of the basis vectors, we write here the structure equations in terms of the Chevalley-Eilenberg differential of the basis 1-forms, since we will use these in our next computations.

Theorem 2.5.13 ([153]). *Let* $(\mathfrak{s}, g)$ *be a six-dimensional nonunimodular metric solvable Lie algebra with Einstein inner product* g *such that* $\mathrm{Ric}(g) = -r^2 g$*, where* $r > 0$*. Then,* $(\mathfrak{s}, g)$ *is isomorphic to one of the metric Lie algebras contained in Table 2.3. For each Lie algebra,* $(e_1, \ldots, e_6)$ *is a* g*-orthonormal basis with dual basis* $(e^1, \ldots, e^6)$*.*

Remark 2.5.14. All of the metric solvable Lie algebras appearing in Table 2.3 are of Iwasawa-type. In particular, the rank of $\mathfrak{s}_k$ is equal to 1 for $k = 1 \ldots 9$, to 2 for $k = 10, 11, 12$ and to 3 for $k = 13$. It is worth emphasizing here that for each $\mathfrak{s}_i$, the inner product g, with respect to which the basis $(e_1, \ldots, e_6)$ is orthonormal, is the only one having the property of being Einstein up to scaling (cf. Section 1.4.2).

$\mathfrak{s}$	Structure equations $(de^1, de^2, de^3, de^4, de^5, de^6)$
$\mathfrak{s}_1$	$\left(\frac{r}{2\sqrt{2}}e^{16},\frac{r}{2\sqrt{2}}e^{26},\frac{r}{2\sqrt{2}}e^{36},\frac{r}{2\sqrt{2}}e^{46},-\frac{r}{\sqrt{2}}e^{12}-\frac{r}{\sqrt{2}}e^{34}+\frac{r}{\sqrt{2}}e^{56},0\right)$
$\mathfrak{s}_2$	$\left(2r\sqrt{\frac{2}{105}}e^{16},r\sqrt{\frac{3}{70}}e^{26},-\frac{2r}{\sqrt{7}}e^{12}+r\sqrt{\frac{7}{30}}e^{36},2r\sqrt{\frac{3}{70}}e^{46},-r\sqrt{\frac{2}{7}}e^{14}-\frac{2r}{\sqrt{7}}e^{23}+r\sqrt{\frac{10}{21}}e^{56},0\right)$
$\mathfrak{s}_3$	$\left(\frac{r}{\sqrt{55}}e^{16},\frac{2r}{\sqrt{55}}e^{26},-r\sqrt{\frac{6}{11}}e^{12}+\frac{3r}{\sqrt{55}}e^{36},-r\sqrt{\frac{6}{11}}e^{13}+\frac{4r}{\sqrt{55}}e^{46},-\frac{2r}{\sqrt{11}}e^{14}-\frac{2r}{\sqrt{11}}e^{23}+\frac{5r}{\sqrt{55}}e^{56},0\right)$
$\mathfrak{s}_4$	$\left(\frac{r\sqrt{6}}{30}e^{16},\frac{3r\sqrt{6}}{20}e^{26},-\frac{r}{\sqrt{2}}e^{12}+\frac{11r\sqrt{6}}{60}e^{36},-r\sqrt{\frac{2}{3}}e^{13}+\frac{13r\sqrt{6}}{60}e^{46},-\frac{r}{\sqrt{2}}e^{14}+\frac{r\sqrt{6}}{4}e^{56},0\right)$
$\mathfrak{s}_5$	$\left(\frac{r}{3\sqrt{2}}e^{16},\frac{r}{2\sqrt{2}}e^{26},\frac{r}{2\sqrt{2}}e^{36},-\frac{r}{\sqrt{2}}e^{12}+\frac{5r}{6\sqrt{2}}e^{46},-\frac{r}{\sqrt{2}}e^{13}+\frac{5r}{6\sqrt{2}}e^{56},0\right)$
$\mathfrak{s}_6$	$\left(\frac{r}{2\sqrt{6}}e^{16},\frac{r}{2\sqrt{6}}e^{26},-r\sqrt{\frac{2}{3}}e^{12}+\frac{r}{\sqrt{6}}e^{36},-\frac{r}{\sqrt{2}}e^{13}+r\frac{\sqrt{6}}{4}e^{46},-\frac{r}{\sqrt{2}}e^{23}+r\frac{\sqrt{6}}{4}e^{56},0\right)$
$\mathfrak{s}_7$	$\left(\frac{r}{\sqrt{39}}e^{16},\frac{2r}{\sqrt{39}}e^{26},-r\sqrt{\frac{2}{3}}e^{12}+\frac{3r}{\sqrt{39}}e^{36},-r\sqrt{\frac{2}{3}}e^{13}+\frac{4r}{\sqrt{39}}e^{46},\frac{3r}{\sqrt{39}}e^{56},0\right)$
$\mathfrak{s}_8$	$\left(r\sqrt{\frac{2}{21}}e^{16},r\sqrt{\frac{2}{21}}e^{26},-r\sqrt{\frac{2}{3}}e^{12}+2r\sqrt{\frac{2}{21}}e^{36},r\sqrt{\frac{3}{14}}e^{46},r\sqrt{\frac{3}{14}}e^{56},0\right)$
$\mathfrak{s}_9$	$\left(\frac{r}{\sqrt{5}}e^{16},\frac{r}{\sqrt{5}}e^{26},\frac{r}{\sqrt{5}}e^{36},\frac{r}{\sqrt{5}}e^{46},\frac{r}{\sqrt{5}}e^{56},0\right)$
$\mathfrak{s}_{10}$	$\left(\frac{2r}{\sqrt{33}}e^{15}+rte^{16}+r\sqrt{\frac{1}{2}-11t^2}e^{26},\frac{2r}{\sqrt{33}}e^{25}+r\sqrt{\frac{1}{2}-11t^2}e^{16}+rte^{26},-r\sqrt{\frac{2}{3}}e^{12}+\frac{4r}{\sqrt{33}}e^{35}+2rte^{36},\frac{3r}{\sqrt{33}}e^{45}-4rte^{46},0,0\right)$
$\mathfrak{s}_{11}$	$\left(\frac{r}{\sqrt{30}}e^{15}+\frac{3r}{\sqrt{30}}e^{16},\frac{2r}{\sqrt{30}}e^{25}-\frac{4r}{\sqrt{30}}e^{26},-r\sqrt{\frac{2}{3}}e^{12}+\frac{3r}{\sqrt{30}}e^{35}-\frac{r}{\sqrt{30}}e^{36},-r\sqrt{\frac{2}{3}}e^{13}+\frac{4r}{\sqrt{30}}e^{45}+\frac{2r}{\sqrt{30}}e^{46},0,0\right)$
$\mathfrak{s}_{12}$	$\left(\frac{r}{2}e^{15}+r\frac{1+s+t}{2\sqrt{1+t^2+s^2}}e^{16},\frac{r}{2}e^{25}+r\frac{1-s-t}{2\sqrt{1+t^2+s^2}}e^{26},\frac{r}{2}e^{35}+r\frac{t-s-1}{2\sqrt{1+t^2+s^2}}e^{36},\frac{r}{2}e^{45}+r\frac{s-t-1}{2\sqrt{1+t^2+s^2}}e^{46},0,0\right)$
$\mathfrak{s}_{13}$	$\left(\frac{r}{\sqrt{3}}e^{14}-\frac{2r}{\sqrt{6}}e^{16},\frac{r}{\sqrt{3}}e^{24}+\frac{r}{\sqrt{2}}e^{25}+\frac{r}{\sqrt{6}}e^{26},\frac{r}{\sqrt{3}}e^{34}-\frac{r}{\sqrt{2}}e^{35}+\frac{r}{\sqrt{6}}e^{36},0,0,0\right)$

Table 2.3: Six-dimensional nonunimodular metric solvable Lie algebras with Einstein inner product $g=\sum_{i=1}^{6}(e^i)^2$. The Lie algebra $\mathfrak{s}_{10}$ depends on a real parameter $0\leq t\leq\frac{1}{\sqrt{22}}$ and $\mathfrak{s}_{12}$ depends on two real parameters $0\leq s\leq t\leq 1$.

Since every connected, simply connected homogeneous Riemannian space of nonpositive sectional curvature is isometric to a standard solvmanifold by [5], the previous classification allowed the authors to prove the

Theorem 2.5.15 ([153]). *Let (M,g) be a six-dimensional connected, simply connected homogeneous Einstein manifold of nonpositive sectional curvature. Then, it is symmetric or isometric to one of the solvmanifolds of negative sectional curvature generated by the metric Lie algebras $\mathfrak{s}_5$, $\mathfrak{s}_8$. Moreover, in the symmetric case, (M,g) is obtained as the solvmanifold corresponding to the metric Lie algebras $\mathfrak{s}_1, \mathfrak{s}_9, \mathfrak{s}_{10}$ for $t = \frac{1}{\sqrt{22}}, \mathfrak{s}_{11}, \mathfrak{s}_{12}$ for $(s,t) = (0,0)$ and $(s,t) = (1,1)$ and $\mathfrak{s}_{13}$.*

Now, we focus on the problem of finding left-invariant half-flat structures on six-dimensional Einstein solvmanifolds inducing the Einstein, non Ricci-flat, metric. They are in one-to-one correspondence with half-flat structures inducing the Einstein inner product on six-dimensional nonunimodular metric solvable Lie algebras.

In [74, 75], the authors completely classified the left-invariant half-flat structures on six-dimensional decomposable Lie groups (using also the classification contained in [168]) and on six-dimensional indecomposable Lie groups with five-dimensional nilradical. These classifications will be useful in the proof of the following

Theorem 2.5.16 ([160]). *There are no half-flat* SU(3)*-structures inducing the Einstein metric on the rank 1 metric solvable Lie algebras $\mathfrak{s}_k, k = 1\dots 9$, and on the rank 2 metric solvable Lie algebra $\mathfrak{s}_{12}$. Moreover, there are no coupled* SU(3)*-structures inducing the Einstein metric on the rank 2 metric solvable Lie algebras $\mathfrak{s}_{10}, \mathfrak{s}_{11}$ and on the rank 3 metric solvable Lie algebra $\mathfrak{s}_{13}$.*

Proof. We prove the theorem as follows: in the list of Einstein metric solvable Lie algebras we first exclude those not admitting a half-flat structure using the results of [74, 75], then we show the result by direct computations in the remaining cases.

The rank 1 metric Lie algebra $\mathfrak{s}_9$ is indecomposable and has Abelian nilradical, therefore it does not admit any half-flat structure by [75, Prop. 4]. This happens also for the 2-parameter family of metric Lie algebras $\mathfrak{s}_{12}$, since it satisfies the hypothesis of Proposition 2.4.9 with $\alpha = e^6$. Using the same notations of [74, 75], we have the following isomorphisms of Lie algebras

$$
\begin{aligned}
\mathfrak{s}_1 &\cong A_{6,82}^{1,0,0} = \left(2\mathrm{f}^{16}+\mathrm{f}^{24}+\mathrm{f}^{35},\mathrm{f}^{26},\mathrm{f}^{36},\mathrm{f}^{46},\mathrm{f}^{56},0\right),\\
\mathfrak{s}_2 &\cong A_{6,94}^{4/3} = \left(\tfrac{10}{3}\mathrm{f}^{16}+\mathrm{f}^{25}+\mathrm{f}^{34},\tfrac{7}{3}\mathrm{f}^{26}+\mathrm{f}^{35},\tfrac{4}{3}\mathrm{f}^{36},2\mathrm{f}^{46},\mathrm{f}^{56},0\right),\\
\mathfrak{s}_3 &\cong A_{6,99} = \left(5\mathrm{f}^{16}+\mathrm{f}^{25}+\mathrm{f}^{34},4\mathrm{f}^{26}+\mathrm{f}^{35},3\mathrm{f}^{36}+\mathrm{f}^{45},2\mathrm{f}^{46},\mathrm{f}^{56},0\right),\\
\mathfrak{s}_4 &\cong A_{6,71}^{9/2} = \left(\tfrac{15}{2}\mathrm{f}^{16}+\mathrm{f}^{25},\tfrac{13}{2}\mathrm{f}^{26}+\mathrm{f}^{35},\tfrac{11}{2}\mathrm{f}^{36}+\mathrm{f}^{45},\tfrac{9}{2}\mathrm{f}^{46},\mathrm{f}^{56},0\right),\\
\mathfrak{s}_5 &\cong A_{6,54}^{2/5,1} = \left(\mathrm{f}^{16}+\mathrm{f}^{35},\mathrm{f}^{26}+\mathrm{f}^{45},\tfrac{3}{5}\mathrm{f}^{36},\tfrac{3}{5}\mathrm{f}^{46},\tfrac{2}{5}\mathrm{f}^{56},0\right),\\
\mathfrak{s}_6 &\cong A_{6,76}^{1} = \left(3\mathrm{f}^{16}+\mathrm{f}^{25},2\mathrm{f}^{26}+\mathrm{f}^{45},\mathrm{f}^{24}+3\mathrm{f}^{36},\mathrm{f}^{46},\mathrm{f}^{56},0\right),\\
\mathfrak{s}_7 &\cong A_{6,39}^{3,2} = \left(3\mathrm{f}^{16}+\mathrm{f}^{45},\mathrm{f}^{15}+4\mathrm{f}^{26},3\mathrm{f}^{36},2\mathrm{f}^{46},\mathrm{f}^{56},0\right),\\
\mathfrak{s}_8 &\cong A_{6,13}^{2/3,2/3,1} = \left(\tfrac{4}{3}\mathrm{f}^{16}+\mathrm{f}^{23},\tfrac{2}{3}\mathrm{f}^{26},\tfrac{2}{3}\mathrm{f}^{36},\mathrm{f}^{46},\mathrm{f}^{56},0\right),\\
\mathfrak{s}_{13} &\cong \mathfrak{r}_2\oplus\mathfrak{r}_2\oplus\mathfrak{r}_2 = \left(0,\mathrm{f}^{12},0,\mathrm{f}^{34},0,\mathrm{f}^{56}\right),
\end{aligned}
$$

where in each case the structure equations are written with respect to a basis $(\mathrm{f}^1,\ldots,\mathrm{f}^6)$ of the dual Lie algebra. Consequently, by [75, Thm. 2] the Lie algebras $\mathfrak{s}_k$ with $k=1,2,4,5,7,8$ do not admit any half-flat structure, whereas the Lie algebras $\mathfrak{s}_3,\mathfrak{s}_6$ do. Moreover, by [74, Thm. 1], on $\mathfrak{s}_{13}$ there exist half-flat structures, too.

We can now start with the second part of the proof. For $k=3,6$, let $\omega\in\Lambda^2(\mathfrak{s}_k^*)$ and $\psi_+\in\Lambda^3(\mathfrak{s}_k^*)$ be generic forms. With respect to the basis $(e^1,\ldots,e^6)$ given in Theorem 2.5.13, we can write

$$
\begin{aligned}
\omega = {} & b_1e^{12}+b_2e^{13}+b_3e^{14}+b_4e^{15}+b_5e^{16}+b_6e^{23}+b_7e^{24}+b_8e^{25}\\
&+b_9e^{26}+b_{10}e^{34}+b_{11}e^{35}+b_{12}e^{36}+b_{13}e^{45}+b_{14}e^{46}+b_{15}e^{56}
\end{aligned}
\tag{2.47}
$$

and

$$
\begin{aligned}
\psi_+ = {} & a_1e^{123}+a_2e^{124}+a_3e^{125}+a_4e^{126}+a_5e^{134}+a_6e^{135}+a_7e^{136}\\
&+a_8e^{145}+a_9e^{146}+a_{10}e^{156}+a_{11}e^{234}+a_{12}e^{235}+a_{13}e^{236}+a_{14}e^{245}\\
&+a_{15}e^{246}+a_{16}e^{256}+a_{17}e^{345}+a_{18}e^{346}+a_{19}e^{356}+a_{20}e^{456},
\end{aligned}
\tag{2.48}
$$

where a_i and b_j are real constants.

Let $\beta_{i_1\dots i_5}$ and $\gamma_{i_1\dots i_5}$ denote the components of the 5-forms $\omega \wedge \psi_+$ and $d\omega^2$, respectively, so that

$$\begin{aligned} \omega \wedge \psi_+ &= \sum_{1 \le i_1 < i_2 < \dots < i_5 \le 6} \beta_{i_1 \dots i_5} e^{i_1 \dots i_5}, \\ d\omega^2 &= \sum_{1 \le i_1 < i_2 < \dots < i_5 \le 6} \gamma_{i_1 \dots i_5} e^{i_1 \dots i_5}. \end{aligned}$$

Observe that the non-vanishing β are always homogeneous polynomials of degree 2 in a_i, b_j, while the non-vanishing γ are always homogeneous polynomials of degree 2 in b_j.

For each Lie algebra $\mathfrak{s}_k$, we impose the conditions the forms (2.47) and (2.48) have to satisfy in order to be a half-flat SU(3)-structure inducing the Einstein metric. What we have to do is to solve the equations obtained from

$$\begin{cases} \omega \wedge \psi_+ = 0 \\ d\psi_+ = 0 \\ d\omega^2 = 0 \end{cases} \tag{2.49}$$

under the assumptions $\lambda = \lambda(\psi_+) < 0$, $\omega^3 \neq 0$. Moreover, since we are considering a basis which is orthonormal with respect to the Einstein metric (see Theorem 2.5.13), we have also to impose that the entries $G_{i,j} = g(e_i, e_j)$ of the matrix G associated with $g(\cdot,\cdot) = \omega(\cdot, J_{\psi_+}\cdot)$ with respect to the basis $(e_1, \dots, e_6)$ satisfy

$$\begin{aligned} G_{i,j} &= 0, \text{ for } 1 \le i, j \le 6 \text{ and } i \neq j, \\ G_{i,i} - G_{i+1,i+1} &= 0, \text{ for } 1 \le i \le 5, \end{aligned} \tag{2.50}$$

with $G_{i,i} > 0$. These conditions give us a system of polynomial equations in 35 unknowns to solve under some constraints on them we will specify case by case. Since the expressions of the unknowns we obtain solving the equations are often too long to be written down, in what follows we will point out only from which equation a certain unknown is obtained, specifying its value only if it is zero.

Let us start with the Lie algebra $\mathfrak{s}_6$, whose structure equations are given in Table 2.3. We solve all of the linear equations in the a_i deriving from $d\psi_+ = 0$. Then, looking at the expression of λ, we deduce that $a_6 \neq 0$. We can then solve all of the equations obtained from $\omega \wedge \psi_+ = 0$, $d\omega^2 = 0$ and $G_{i,j} = 0$ for $i \neq j$ using $a_6 \neq 0$ and comparing case by case each equation with $G_{i,i}$ and ω^3. After doing this, G becomes a diagonal matrix and we have to solve the remaining five equations of (2.50), which do not have any solution under the constraints $G_{i,i} \neq 0$ and $\lambda \neq 0$.

For the Lie algebra $\mathfrak{s}_3$, we can argue in a similar way, but instead of working on it, we can show the result on the Lie algebra $A_{6,99} \cong \mathfrak{s}_3$, since the computations are less involved. In this case, we consider the generic forms ω and ψ_+ as in (2.47) and (2.48) with e^i replaced by f^i. Observe that the matrix G associated with the Einstein inner product with respect to the basis $(\mathrm{f}_1, \dots, \mathrm{f}_6)$ is not proportional to the identity anymore, but it is still diagonal. Thus, we still have to solve the equations $G_{i,j} = 0$ for $i \neq j$. First of all, we solve the linear equations in the a_i obtained from $d\psi_+ = 0$. Then, we observe that having $b_1 = 0$ or $a_6 = 0$ leads to a contradiction after solving some equations: if $b_1 = 0$ we can use $G_{1,1}, G_{2,2} \neq 0$ to solve $G_{1,2} = 0$, $G_{1,3} = 0$, $\beta_{12345} = 0$, $\gamma_{12346} = 0$, $\gamma_{12356} = 0$, $\beta_{12346} = 0$, but then γ_{12456} cannot be zero; if $a_6 = 0$ we use $G_{1,1}, G_{2,2} \neq 0$ to solve $G_{1,2} = 0$, $G_{1,3} = 0$, $\beta_{12345} = 0$, $G_{1,5} = 0$, $\beta_{12356} = 0$, $\gamma_{12356} = 0$, $G_{2,3} = 0$, $\beta_{12346} = 0$ and obtain that $\gamma_{12346} = 0$ if and only if $G_{1,1}G_{3,3} = 0$. Thus, we assume $b_1 \neq 0$ and $a_6 \neq 0$. Under these constraints and comparing case by case the polynomial we want to be zero with $G_{i,i}$ and λ, we can get the expression of b_4 from the equation $G_{1,2} = 0$, b_9 from $G_{2,3} = 0$, b_7 from $\beta_{12345} = 0$, b_{10} from $\gamma_{12346} = 0$, b_{11} from $\gamma_{12356} = 0$, a_{18} from $\beta_{12346} = 0$, $b_2 = 0$ from $\beta_{12356} = 0$, $b_6 = 0$ from $G_{1,3} = 0$, $a_8 = 0$ from $G_{3,4} = 0$, $a_{14} = 0$ from $G_{3,5} = 0$, $a_{17} = 0$ from $G_{3,6} = 0$, $a_{10} = 0$ from $G_{1,4} = 0$, $a_{19} = 0$ from $G_{1,6} = 0$, $b_{14} = 0$ from $\beta_{13456} = 0$, a_{20} from $G_{1,5} = 0$, b_3 from $G_{2,4} = 0$, b_8 from $G_{2,6} = 0$. Now, $G_{4,6} = 0$ implies $\omega^3 = 0$.

We can now turn our attention to the Lie algebras $\mathfrak{s}_{10}, \mathfrak{s}_{11}$ and $\mathfrak{s}_{13}$, we shall show that none of these admits a coupled structure inducing the Einstein metric. The way in which we proceed is similar to the one followed for $\mathfrak{s}_6$ and $\mathfrak{s}_3$, but in this case, we consider a generic ω of the form (2.47) and $\psi_+ = c d\omega$ for $c \in \mathbb{R} - \{0\}$. Observe that the second condition of (2.49) is satisfied, since ψ_+ is now an exact 3-form, and that the first and the third condition are actually the same. For each Lie algebra, we consider the structure equations given in Table 2.3.

Consider $\mathfrak{s}_{10}$, this is a 1-parameter family of Lie algebras depending on a real parameter $t \in \left[0, \frac{1}{\sqrt{22}}\right]$. Since $G_{3,3}$ cannot be zero, we have that $b_{10} \neq 0$, $b_2 \neq \pm b_6$ and $t \neq \frac{1}{\sqrt{22}}$. The way in which we solve the equations depends on whether $t = \frac{7}{2\sqrt{330}}$ or not. If $t \neq \frac{7}{2\sqrt{330}}$, we can use $b_{10} \neq 0$ to obtain b_1 from $\gamma_{12345} = 0$, b_{15} from $\gamma_{12456} = 0$, b_5 from $\gamma_{13456} = 0$ and b_9 from $\gamma_{23456} = 0$. Then, b_{12} from $G_{3,4} = 0$, $b_3 = 0$ from $G_{1,3} = 0$, $b_7 = 0$ from $G_{2,3} = 0$, $b_{11} = 0$ from $G_{4,5} = 0$, b_8 from $G_{1,4} = 0$, b_4 from $G_{2,4} = 0$, $b_{13} = 0$ from $G_{3,5} = 0$. Now, $G_{3,6} = 0$ if and only if $\lambda = 0$. If $t = \frac{7}{2\sqrt{330}}$, the computations are the same until we arrive to the equation $G_{2,4} = 0$, which has no solutions since $G_{2,4}$ is proportional to λ.

For $\mathfrak{s}_{11}$ we have that $b_{10} \neq 0$ and $\sqrt{5}b_3b_7 - b_{10}b_{13} - 3b_{10}b_{14} \neq 0$, since otherwise $G_{4,4} = 0$. Using $b_{10} \neq 0$, we can obtain b_1, b_4, b_8, b_{15} from $\gamma_{12345} = 0, \gamma_{13456} = 0, \gamma_{23456} = 0, \gamma_{12456} = 0$, respectively, b_{11} from $G_{3,4} = 0$, b_9 from $G_{2,4} = 0$ and b_5 from $G_{1,4} = 0$. Then, using also the other constraint, we get b_{12} from $G_{4,6} = 0$. Now, $G_{4,5} = 0$ if and only if $b_7 = 0$ or $b_{14} = -2b_{13}$. If $b_7 = 0$, from $G_{2,5} = 0$ and $\lambda \neq 0$ we have $b_3 = 0$ but then $G_{1,2} = 0$ only if either $\lambda = 0$ or $G_{1,1} = 0$. Thus, $b_7 \neq 0$ and $b_{14} = -2b_{13}$. Moreover, $b_6 \neq 0$, otherwise λ would be proportional to $G_{2,3}$. Thus, we can solve $G_{2,3} = 0$ to get b_3 and use $\lambda \neq 0$ to solve $G_{1,2} = 0$ and obtain b_6. Now, $G_{3,5}$ is proportional to λ, therefore it cannot be zero.

In the last case $\mathfrak{s}_{13}$, we can see that $b_1, b_2, b_6 \neq 0$ and $b_3 \neq \sqrt{2}\, b_5$ from the fact that the entries in the diagonal of G cannot be zero. Solving the equations $\gamma_{12456} = 0, \gamma_{12345} = 0, \gamma_{12346} = 0, \gamma_{13456} = 0, \gamma_{23456} = 0$ under the previous constraints, we obtain the expressions of $b_{14}, b_8, b_9, b_{13}, b_{15}$, respectively. Then, we get b_{12} from $G_{1,2} = 0$, b_{10} from $G_{1,3} = 0$, b_{11} from $G_{1,5} = 0$, b_5 from $G_{2,5} = 0$ and $b_4 = 0$ from $G_{2,6} = 0$. Now, $G_{2,4} = 0$ if and only if $\lambda = 0$. □

Remark 2.5.17. In the previous proof, it is in principle possible to use the properties of algebraic varieties to find solutions as we did in the proof of Theorem 2.5.5. However, the computations here are more involved, since we have more unknowns (35 or 15 instead of 9) and more equations arising from the fact that some defining conditions for an SU(3)-structure that were easily verified in the case of $S^3 \times S^3$ have to be imposed in this case.

From the fact that the class of coupled SU(3)-structures is a subclass of the half-flat one, we can use the result of the previous theorem together with Theorem 2.5.13

to obtain the

Corollary 2.5.18. *Let* $(\mathfrak{s}, g)$ *be a six-dimensional nonunimodular metric solvable Lie algebra with* g *Einstein. Then, on* $\mathfrak{s}$ *there are no coupled* $\mathrm{SU}(3)$*-structures inducing the Einstein inner product.*

Moreover, from the previous theorem and the Theorem 2.5.15, we obtain a constraint for the existence of coupled structures inducing Einstein metrics on homogeneous spaces.

Corollary 2.5.19. *Let* (M, g) *be a six-dimensional connected, simply connected homogeneous Einstein manifold of nonpositive sectional curvature. Then, there are no left-invariant coupled* $\mathrm{SU}(3)$*-structures on* M *inducing the Einstein metric.*

Chapter 3

Locally conformal calibrated G_2-manifolds

In this chapter, we focus on seven-dimensional manifolds endowed with a G_2-structure. The main properties are reviewed in the first part, while the second part is devoted to the study of locally conformal calibrated G_2-manifolds, for which we show the results obtained in the papers [66, 70, 71].

3.1 G_2-structures

3.1.1 The group G_2 as stabilizer of tensors on $\mathbb{R}^7$

Let us begin considering the real vector space $\mathbb{R}^7$ endowed with an inner product g inducing the norm $|v| = g(v,v)^{\frac{1}{2}}$, $v \in \mathbb{R}^7$. On $(\mathbb{R}^7, g)$ there exists an analogue of the usual vector cross product defined on three-dimensional vector spaces:

Definition 3.1.1. A *two-fold vector cross product* on $(\mathbb{R}^7, g)$ is a bilinear map $\mathrm{P} : \mathbb{R}^7 \times \mathbb{R}^7 \to \mathbb{R}^7$ satisfying the following properties for every $v, w \in \mathbb{R}^7$

i) $g(\mathrm{P}(v,w), v) = 0 = g(\mathrm{P}(v,w), w)$;

ii) $|\mathrm{P}(v,w)|^2 = |v|^2|w|^2 - g(v,w)^2$.

It follows from the definition that P is skew-symmetric in its two entries, i.e., $P(v, w) = -P(w, v)$, and that the 3-covariant tensor φ defined as

$$\varphi(v, w, z) = g(P(v, w), z), \quad v, w, z \in \mathbb{R}^7,$$

is a 3-form on $\mathbb{R}^7$, called the *fundamental 3-form* of P. Moreover, the inner product on $\mathbb{R}^7$ is completely determined by the two-fold vector cross product. Indeed, from the identity [67, Cor. 2.2]

$$P(v, P(v, P(v, w))) = -|v|^2 P(v, w), \quad v, w \in \mathbb{R}^7,$$

it is possible to obtain $|v|^2$ by choosing v and w linearly independent and, then, to get the inner product g by means of

$$g(v, w) = \frac{1}{4}\left(|v + w|^2 - |v - w|^2\right).$$

Unlike the three-dimensional case, a two-fold vector cross product on $\mathbb{R}^7$ is not unique up to sign. For instance, $\mathbb{R}^7$ can be endowed with the inner product g_0 for which the canonical basis $(e_1, \ldots, e_7)$ is orthonormal and with the two-fold vector cross product P_0 described in Table 3.1.

$P_0(\downarrow, \rightarrow)$	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	0	e_7	e_5	$-e_6$	$-e_3$	e_4	$-e_2$
e_2	$-e_7$	0	$-e_6$	$-e_5$	e_4	e_3	e_1
e_3	$-e_5$	e_6	0	e_7	e_1	$-e_2$	$-e_4$
e_4	e_6	e_5	$-e_7$	0	$-e_2$	$-e_1$	e_3
e_5	e_3	$-e_4$	$-e_1$	e_2	0	e_7	$-e_6$
e_6	$-e_4$	$-e_3$	e_2	e_1	$-e_7$	0	e_5
e_7	e_2	$-e_1$	e_4	$-e_3$	e_6	$-e_5$	0

Table 3.1: Example of two-fold vector cross product on $\mathbb{R}^7$.

With this choice of P_0, the fundamental 3-form φ_0 associated with it is

$$\varphi_0 = e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245}, \tag{3.1}$$

where $(e^1, \ldots, e^7)$ is the dual basis of $(e_1, \ldots, e_7)$, and if we introduce the symbol φ_{ijk} which is skew-symmetric in its three indices and whose values are uniquely determined via the identity

$$\varphi_0 = \frac{1}{6}\varphi_{ijk}e^{ijk},$$

then with respect to the considered basis we have

$$\mathrm{P}_0(e_i, e_j) = \sum_{k=1}^{7} \varphi_{ijk} e_k.$$

The group G_2 can be defined equivalently in terms of P_0 or in terms of φ_0 as follows.

Definition 3.1.2 ([30, 67])**.**

$$\begin{aligned}\mathrm{G}_2 &= \left\{a \in \mathrm{O}(7) \mid a\,\mathrm{P}_0\left(a^{-1}\cdot, a^{-1}\cdot\right) = \mathrm{P}_0(\cdot,\cdot)\right\} \\ &= \left\{a \in \mathrm{GL}(7,\mathbb{R}) \mid a^*\varphi_0 = \varphi_0\right\}.\end{aligned}$$

By [29], G_2 is a connected, simply connected, compact, simple Lie group of dimension 14 and it is a subgroup of $\mathrm{SO}(7)$. In particular, G_2 preserves the inner product g_0 and the volume form dV_0 on $\mathbb{R}^7$ for which $(e_1, \ldots, e_7)$ is an oriented orthonormal basis, and the 4-form

$$*_{\varphi_0}\varphi_0 = e^{3456} + e^{1256} + e^{1234} - e^{2467} + e^{2357} + e^{1457} + e^{1367} = \frac{1}{24}\varphi_{ijkl}e^{ijkl},$$

where $*_{\varphi_0}$ is the Hodge operator determined by g_0 and dV_0 and φ_{ijkl} is the skew-symmetric symbol uniquely defined via the previous identity.

The $\mathrm{GL}(7,\mathbb{R})$-orbit $\Lambda^3_+((\mathbb{R}^7)^*)$ of φ_0 in $\Lambda^3((\mathbb{R}^7)^*)$ is open, since it is isomorphic to $\mathrm{GL}(7,\mathbb{R})/\mathrm{G}_2$. Thus, φ_0 is a stable form in the sense of Definition 2.2.1. The volume form determined by it is dV_0, which can be obtained together with g_0 from

$$g_0(v,w)\,dV_0 = \frac{1}{6}\,(\iota_v\varphi_0) \wedge (\iota_w\varphi_0) \wedge \varphi_0, \quad v, w \in \mathbb{R}^7.$$

Moreover, every stable 3-form belonging to $\Lambda^3_+((\mathbb{R}^7)^*)$ defines an inner product and a volume form as above and has stabilizer isomorphic to G_2.

Remark 3.1.3. We recalled in the previous chapter that in dimension seven stability occurs for 3-forms and 4-forms. In the setting we are considering, $*_{\varphi_0}\varphi_0$ is a stable 4-form with open $GL(7,\mathbb{R})$-orbit $\Lambda^4_+((\mathbb{R}^7)^*)$. The stabilizer of any element in this orbit is isomorphic to $G_2 \cup \{a \circ (-\mathrm{Id}_{\mathbb{R}^7}) \mid a \in G_2\}$ and, consequently, every stable 4-form belonging to $\Lambda^4_+((\mathbb{R}^7)^*)$ defines an inner product but not an orientation. This is due to the fact that the map $\Lambda^3_+((\mathbb{R}^7)^*) \to \Lambda^4_+((\mathbb{R}^7)^*)$ given by $\sigma \mapsto *_\sigma\sigma$ is a double covering.

G_2 acts irreducibly on $\mathbb{R}^7$ and, then, also on $\Lambda^1((\mathbb{R}^7)^*)$ and $\Lambda^6((\mathbb{R}^7)^*)$, while the action is not irreducible on $\Lambda^k((\mathbb{R}^7)^*)$, $k = 2,3,4,5$. The G_2-irreducible decompositions of these spaces are completely described by the irreducible decompositions of $\Lambda^2((\mathbb{R}^7)^*)$ and $\Lambda^3((\mathbb{R}^7)^*)$, since the Hodge operator $*_{\varphi_0}$ is an isomorphism between the G_2-modules $\Lambda^k((\mathbb{R}^7)^*)$ and $\Lambda^{7-k}((\mathbb{R}^7)^*)$. By [29, 67], it holds

$$\begin{aligned}\Lambda^2((\mathbb{R}^7)^*) &= \Lambda^2_7((\mathbb{R}^7)^*) \oplus \Lambda^2_{14}((\mathbb{R}^7)^*),\\ \Lambda^3((\mathbb{R}^7)^*) &= \Lambda^3_1((\mathbb{R}^7)^*) \oplus \Lambda^3_7((\mathbb{R}^7)^*) \oplus \Lambda^3_{27}((\mathbb{R}^7)^*),\end{aligned}$$

where $\Lambda^k_r((\mathbb{R}^7)^*)$ denotes an irreducible G_2-module of dimension r and the irreducible summands in the cases $k = 4,5$ are obtained from these as $\Lambda^k_r((\mathbb{R}^7)^*) = *_{\varphi_0}(\Lambda^{7-k}_r((\mathbb{R}^7)^*))$. The G_2-modules appearing in the decompositions above can be described as follows.

$$\begin{aligned}\Lambda^2_7((\mathbb{R}^7)^*) &= \left\{\kappa \in \Lambda^2((\mathbb{R}^7)^*) \mid *_{\varphi_0}(\kappa \wedge \varphi_0) = 2\,\kappa\right\}\\ &= \left\{\kappa \in \Lambda^2((\mathbb{R}^7)^*) \mid *_{\varphi_0}(*_{\varphi_0}\varphi_0 \wedge (*_{\varphi_0}(*_{\varphi_0}\varphi_0 \wedge \kappa))) = 3\,\kappa\right\},\\ \Lambda^2_{14}((\mathbb{R}^7)^*) &= \left\{\kappa \in \Lambda^2((\mathbb{R}^7)^*) \mid \kappa \wedge *_{\varphi_0}\varphi_0 = 0\right\}\\ &= \left\{\kappa \in \Lambda^2((\mathbb{R}^7)^*) \mid *_{\varphi_0}(\kappa \wedge \varphi_0) = -\kappa\right\},\end{aligned}$$

and

$$\begin{aligned}\Lambda^3_1((\mathbb{R}^7)^*) &= \langle\varphi_0\rangle = \mathbb{R}\,\varphi_0,\\ \Lambda^3_7((\mathbb{R}^7)^*) &= \left\{*_{\varphi_0}(\alpha \wedge \varphi_0) \mid \alpha \in \Lambda^1((\mathbb{R}^7)^*)\right\}\\ &= \left\{\beta \in \Lambda^3((\mathbb{R}^7)^*) \mid *_{\varphi_0}(\varphi_0 \wedge *_{\varphi_0}(\varphi_0 \wedge \beta)) = -4\,\beta\right\},\\ \Lambda^3_{27}((\mathbb{R}^7)^*) &= \left\{\beta \in \Lambda^3((\mathbb{R}^7)^*) \mid \beta \wedge \varphi_0 = 0,\ \beta \wedge *_{\varphi_0}\varphi_0 = 0\right\}.\end{aligned}$$

Moreover, $\Lambda^2_{14}((\mathbb{R}^7)^*)$ is isomorphic to the Lie algebra $\mathfrak{g}_2$ of G_2 and $\Lambda^3_{27}((\mathbb{R}^7)^*)$ is isomorphic to the space of traceless symmetric $(0,2)$-tensors $\mathcal{S}^2_0((\mathbb{R}^7)^*)$ (see [30]).

3.1.2 G_2-structures and their classification

Let M be a seven-dimensional manifold. A G_2-structure on M is by definition a reduction of the structure group of the frame bundle FM from $GL(7,\mathbb{R})$ to G_2. By the results of the previous section and Proposition 1.2.2, the existence of a G_2-structure on M is equivalent to the existence of a 3-form φ which is a global section of the open subbundle $\Lambda^3_+(T^*M) \subset \Lambda^3(T^*M)$ defined as the union of the spaces $\Lambda^3_+(T^*_pM)$. This motivates the following

Definition 3.1.4. A G_2-*structure* on a seven-dimensional manifold M is a stable 3-form $\varphi \in \Omega^3_+(M) := \Gamma(\Lambda^3_+(T^*M))$. A 7-manifold M endowed with a G_2-structure φ is denoted by (M, φ).

Since G_2 is a connected, simply connected subgroup of $SO(7)$, every 7-manifold M endowed with a G_2-structure is orientable and has a spin structure (see for instance [110, Prop. 3.6.2]). Moreover, using an observation due to Gray [87], it is possible to prove that these two necessary conditions are also sufficient, as shown in [30]. In particular, there is a one-to-one correspondence between G_2-structures and real spinor fields of length one on M. For more details on the description of G_2-structures from the spinorial point of view we refer the reader to [2, 80].

Starting from a G_2-structure φ, it is possible to define a Riemannian metric g_φ and a volume form dV_φ on M via

$$g_\varphi(X, Y)\, dV_\varphi = \frac{1}{6}(\iota_X\varphi) \wedge (\iota_Y\varphi) \wedge \varphi, \tag{3.2}$$

for every pair of vector fields $X, Y \in \mathfrak{X}(M)$. The Hodge operator determined by g_φ and dV_φ is denoted by $*_\varphi$. Moreover, on M there exists a two-fold vector cross product $P \in \mathcal{T}^1_2(M)$ defined from φ and g_φ in the following way

$$\varphi(X, Y, Z) = g_\varphi(P(X, Y), Z), \quad X, Y, Z \in \mathfrak{X}(M).$$

Remark 3.1.5. Clearly, the existence of a two-fold vector cross product P on a seven-dimensional Riemannian manifold (M, g) provides a reduction of the structure group of FM from $O(7)$ to G_2. This yields an equivalent definition of G_2-structures.

The 3-form φ_0 on $\mathbb{R}^7$ described in (3.1) can be chosen as model tensor of φ. Thus, at each point p of M there exists a basis $(e_1, \dots, e_7)$ of T_pM with dual basis $(e^1, \dots, e^7)$ which is adapted for φ, i.e., at p we can write

$$\varphi = e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245}$$

and $g_\varphi = \sum_{i=1}^{7}(e^i)^2$. We call equally $(e_1, \dots, e_7)$ and $(e^1, \dots, e^7)$ a G_2-*basis* for the G_2-structure φ at the point p. A simple computation with respect to such a basis proves that for every $\alpha \in \Omega^1(M)$ the following useful identities hold

$$*_\varphi(*_\varphi(\alpha \wedge \varphi) \wedge \varphi) = -4\,\alpha, \tag{3.3}$$

$$*_\varphi(*_\varphi\varphi \wedge *_\varphi(*_\varphi\varphi \wedge \alpha)) = 3\,\alpha. \tag{3.4}$$

Furthermore, for every $\alpha \in \Omega^1(M)$ and $\kappa \in \Omega^2(M)$

$$\alpha \wedge \varphi = 0 \iff \alpha = 0, \tag{3.5}$$

$$\kappa \wedge \varphi = 0 \iff \kappa = 0, \tag{3.6}$$

$$\alpha \wedge *_\varphi\varphi = 0 \iff \alpha = 0. \tag{3.7}$$

The decompositions of the spaces $\Lambda^k((\mathbb{R}^7)^*)$ into irreducible G_2-modules induce the following decompositions of the spaces of differential forms on the manifold

$$\begin{aligned}\Omega^2(M) &= \Omega^2_7(M) \oplus \Omega^2_{14}(M),\\ \Omega^3(M) &= \Omega^3_1(M) \oplus \Omega^3_7(M) \oplus \Omega^3_{27}(M),\end{aligned}$$

where $\Omega^k_r(M)$ is the space of sections of the bundle $\Lambda^k_r(T^*M)$ defined as the union of the spaces $\Lambda^k_r(T^*_pM)$. For instance,

$$\begin{aligned}\Omega^2_{14}(M) &= \left\{\kappa \in \Omega^2(M) \mid \kappa \wedge *_\varphi\varphi = 0\right\}\\ &= \left\{\kappa \in \Omega^2(M) \mid \ *_\varphi(\kappa \wedge \varphi) = -\kappa\right\},\end{aligned}$$

and

$$\Omega^3_{27}(M) = \left\{\beta \in \Omega^3(M) \mid \beta \wedge \varphi = 0,\ \beta \wedge *_\varphi\varphi = 0\right\}.$$

The decompositions of the spaces $\Omega^4(M)$ and $\Omega^5(M)$ are obtained applying the Hodge operator $*_\varphi$ to $\Omega^3(M)$ and $\Omega^2(M)$, respectively.

The classification of manifolds endowed with a G_2-structures was first described by Fernández and Gray in [67], where the authors considered the space $\mathcal{X}$ of tensors satisfying the same symmetries as $\nabla^{g_\varphi}\varphi$, which is pointwise the subspace of $T_p^*M \otimes \Lambda^3(T_p^*M)$

$$\mathcal{X}_p = \left\{\alpha \in T_p^*M \otimes \Lambda^3(T_p^*M) \mid \alpha(X, Y, Z, P(Y, Z)) = 0, \forall\, X, Y, Z \in T_pM\right\},$$

and showed that, according to the decomposition of $\mathcal{X}_p$ into the sum of irreducible G_2-modules, it splits as

$$\mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \mathcal{X}_3 \oplus \mathcal{X}_4,$$

where $\mathcal{X}_{1p}, \mathcal{X}_{2p}, \mathcal{X}_{3p}, \mathcal{X}_{4p}$ have dimension 1, 14, 27 and 7, respectively. The covariant derivative of φ can then be decomposed accordingly and 7-manifolds endowed with a G_2-structure are divided into sixteen classes depending on which summands of $\nabla^{g_\varphi}\varphi$ vanish identically. For instance, (M, φ) belongs to the class $\mathcal{X}_1$ if and only if $(\nabla^{g_\varphi}\varphi)_p \in \mathcal{X}_{1p}$ for every $p \in M$. In this case, φ is said to be of G_2-*type* $\mathcal{X}_1$.

In a similar way to what happens for SU(3)-structures, the components of $\nabla^{g_\varphi}\varphi$ are completely determined by those of $d\varphi$ and $d *_\varphi \varphi$ arising from the irreducible decompositions of $\Omega^4(M)$ and $\Omega^5(M)$, let us see how. Following Bryant's notations of [30], we have

Proposition 3.1.6 ([30])**.** *Let φ be a G_2-structure on a 7-manifold M. Then, there exist unique differential forms $\tau_0 \in C^\infty(M)$, $\tau_1 \in \Omega^1(M)$, $\tau_2 \in \Omega^2_{14}(M)$ and $\tau_3 \in \Omega^3_{27}(M)$ such that*

$$\begin{aligned} d\varphi &= \tau_0 *_\varphi \varphi + 3\,\tau_1 \wedge \varphi + *_\varphi \tau_3, \\ d *_\varphi \varphi &= 4\,\tau_1 \wedge *_\varphi \varphi + \tau_2 \wedge \varphi, \end{aligned} \tag{3.8}$$

Proof. The only non-trivial part consists in proving that the 1-form τ_1 appears in both $d\varphi$ and $d*_\varphi\varphi$. A priori, $d\varphi = \tau_0 *_\varphi \varphi + \alpha\wedge\varphi + *_\varphi\tau_3$ and $d*_\varphi\varphi = \beta\wedge *_\varphi\varphi + \tau_2\wedge\varphi$, for unique $\tau_0 \in C^\infty(M)$, $\alpha, \beta \in \Omega^1(M)$, $\tau_2 \in \Omega^2_{14}(M)$ and $\tau_3 \in \Omega^3_{27}(M)$. Using the identity [29]

$$*_\varphi\varphi \wedge *_\varphi(d *_\varphi \varphi) + *_\varphi(d\varphi) \wedge \varphi = 0,$$

together with (3.3) and (3.4), it follows that $3\,\beta = 4\,\alpha$. Thus, $\tau_1 := \frac{1}{4}\beta = \frac{1}{3}\alpha$. □

Moreover, it is possible to show (see e.g. [141]) that $d\varphi = s_4(\nabla^{g_\varphi}\varphi)$ and $d *_\varphi \varphi = s_5(\nabla^{g_\varphi}\varphi)$, where s_4 and s_5 are induced by the G_2-equivariant maps

$$s_4 : (\mathbb{R}^7)^* \otimes \Lambda^3((\mathbb{R}^7)^*) \longrightarrow \Lambda^4((\mathbb{R}^7)^*), \quad e^i \otimes e^{jkl} \longmapsto e^{ijkl}$$

and

$$s_5 : (\mathbb{R}^7)^* \otimes \Lambda^3((\mathbb{R}^7)^*) \longrightarrow \Lambda^5((\mathbb{R}^7)^*), \quad e^i \otimes e^{jkl} \longmapsto *(\delta_{ij}\, e^{kl} - \delta_{ik}\, e^{jl} + \delta_{il} e^{jk}),$$

being $(e^1, \ldots, e^7)$ a G_2-basis of $\mathbb{R}^7$ for φ_0. Applying now Schur's Lemma to the restrictions of s_4 and s_5 to $\mathcal{X}$, it follows that the component of $\nabla^{g_\varphi}\varphi$ in $\mathcal{X}_1$ corresponds to τ_0, the component in $\mathcal{X}_2$ to τ_2, the component in $\mathcal{X}_3$ to τ_3 and the component in $\mathcal{X}_4$ to τ_1. This motivates the

Definition 3.1.7. The differential forms $\tau_0, \tau_1, \tau_2, \tau_3$ uniquely defined by (3.8) are called *intrinsic torsion forms* of the G_2-structure φ.

By the correspondence above, the sixteen classes of manifolds endowed with a G_2-structure φ can be completely characterized in terms of $d\varphi$ and $d *_\varphi \varphi$, that is, in terms of the identically vanishing intrinsic torsion forms. The full list can be found in [67, 141], while in Table 3.2 are summarized the classes of G_2-structures appearing in this thesis. We conclude this section with some remarks on them.

From general results on G-structures (see Section 1.2.3), the correspondence between the components of $\nabla^{g_\varphi}\varphi$ and the intrinsic torsion forms and [87, Thm. 4.1], we obtain the following equivalent defining properties for the class $\mathcal{X} = \{0\}$.

Proposition 3.1.8. *Let M be a connected 7-manifold endowed with a G_2-structure φ with Riemannian metric g_φ and two-fold vector cross product P. Then, denoted by ∇^{g_φ} the Levi Civita connection of g_φ, the following are equivalent:*

i) *the G_2-structure is torsion-free;*

ii) *the intrinsic torsion forms vanish identically;*

iii) *the differential forms φ and $*_\varphi\varphi$ are closed;*

iv) *the differential form φ is parallel with respect to ∇^{g_φ};*

v) *the tensor* P *is parallel with respect to* ∇^{g_φ}*;*

vi) $\mathrm{Hol}(g_\varphi)$ *is a subgroup of* G_2*.*

Manifolds endowed with a G_2-structure satisfying one of the previous properties are usually called G_2-*manifolds* and the corresponding G_2-structure is called *parallel*. By a result of Bonan [26], every G_2-manifold (M, φ) is Ricci-flat, i.e., $\mathrm{Ric}(g_\varphi) = 0$.

In literature, the first examples of complete metrics with holonomy G_2 were constructed by Bryant and Salamon in [32], while compact examples of Riemannian manifolds with holonomy G_2 were obtained first by Joyce [109] and then by Kovalev [119] and by Corti, Haskins, Nordström, Pacini [54]. As we mentioned in the previous chapter, noncompact examples of G_2-manifolds can also be constructed starting from 6-manifolds endowed with a half-flat SU(3)-structure. We will examine this construction in detail in Section 3.2.1.

Class	Name	Defining conditions
$\{0\}$	Parallel	$d\varphi = 0$, $d *_\varphi \varphi = 0$
$\mathcal{X}_1$	Nearly parallel	$d\varphi = \tau_0 *_\varphi \varphi$, $d *_\varphi \varphi = 0$
$\mathcal{X}_2$	Closed, calibrated	$d\varphi = 0$
$\mathcal{X}_4$	Locally conformal parallel	$d\varphi = 3\,\tau_1 \wedge \varphi$, $d *_\varphi \varphi = 4\,\tau_1 \wedge *_\varphi\varphi$
$\mathcal{X}_1 \oplus \mathcal{X}_4$	Locally conformal nearly parallel	$d\varphi = \tau_0 *_\varphi \varphi + 3\,\tau_1 \wedge \varphi$, $d *_\varphi \varphi = 4\,\tau_1 \wedge *_\varphi\varphi$
$\mathcal{X}_2 \oplus \mathcal{X}_4$	Locally conformal calibrated	$d\varphi = 3\,\tau_1 \wedge \varphi$
$\mathcal{X}_1 \oplus \mathcal{X}_3$	Co-closed, co-calibrated	$d *_\varphi \varphi = 0$
$\mathcal{X}_1 \oplus \mathcal{X}_3 \oplus \mathcal{X}_4$	G_2 with (skew) torsion	$d *_\varphi \varphi = 4\,\tau_1 \wedge *_\varphi\varphi$

Table 3.2: Some classes of manifolds endowed with a G_2-structure.

The function τ_0 is the only possibly non-identically vanishing intrinsic torsion form of a nearly parallel G_2-structure φ. Using (3.5), it is immediate to show that τ_0 is constant on connected manifolds, since

$$0 = d(d\varphi) = d\tau_0 \wedge *_\varphi \varphi.$$

As we will see later, the Riemannian metric underlying a nearly parallel G_2-structure is always Einstein. Moreover, it is Ricci-flat if and only if τ_0 vanishes identically.

The name *calibrated* for the G_2-structures of type $\mathcal{X}_2$ is due to the fact that a closed 3-form $\varphi \in \Omega^3_+(M)$ defines a calibration on M by [95]. This means that for every $p \in M$ and for all oriented three-dimensional subspaces W_p of T_pM, it holds

$$\varphi\,\big|_{W_p} \leq dV,$$

where dV is the volume form of W_p. Properties of manifolds endowed with a calibrated G_2-structure were studied for instance in [30, 47] and examples were provided in [50, 64, 140]. We will recall some results in Section 3.1.3.

By [78], a G_2-structure φ on a 7-manifold is of G_2-type $\mathcal{X}_1 \oplus \mathcal{X}_3 \oplus \mathcal{X}_4$ if and only if there exists a linear connection with totally skew-symmetric torsion preserving φ. Moreover, such a connection is unique. This class of manifolds can then be seen as the G_2-analogue of the class $\mathcal{G}_1$ of almost Hermitian manifolds (see Table 2.1 and Theorem 2.3.11).

Finally, it is possible to prove that a manifold (M, φ) belonging to one of the classes $\mathcal{X}_4$, $\mathcal{X}_1 \oplus \mathcal{X}_4$ and $\mathcal{X}_2 \oplus \mathcal{X}_4$ has closed intrinsic torsion form τ_1. Consequently, by Poincaré Lemma, on a neighborhood $\mathcal{U}$ of each point p of M there exists a local function $f \in C^\infty(\mathcal{U})$ such that $\tau_1 = df$ and the local conformal change $\widehat{\varphi} = e^{-3f}\varphi$ gives rise to a G_2-structure on $\mathcal{U}$ which is parallel if $(M, \varphi) \in \mathcal{X}_4$, nearly parallel if $(M, \varphi) \in \mathcal{X}_1 \oplus \mathcal{X}_4$ and calibrated if $(M, \varphi) \in \mathcal{X}_2 \oplus \mathcal{X}_4$. This motivates the names of the structures having these G_2-types. In the second part of this chapter we study more in depth the class $\mathcal{X}_2 \oplus \mathcal{X}_4$.

3.1.3 The Ricci tensor of a G_2-structure

In [30], Bryant proved that the Ricci tensor and the scalar curvature of the Riemannian metric g_φ induced by a G_2-structure φ can be expressed in terms of the intrinsic torsion forms and their derivatives. More precisely

Theorem 3.1.9 ([30]). *Let M be a 7-manifold endowed with a* G_2*-structure φ. Then, the scalar curvature and the Ricci tensor of the Riemannian metric g_φ are expressed in terms of the intrinsic torsion forms as*

$$\begin{aligned}
\mathrm{Scal}(g_\varphi) &= 12d^*\tau_1 + \frac{21}{8}\tau_0^2 + 30|\tau_1|^2 - \frac{1}{2}|\tau_2|^2 - \frac{1}{2}|\tau_3|^2, \\
\mathrm{Ric}(g_\varphi) &= -\left(\frac{3}{2}d^*\tau_1 - \frac{3}{8}\tau_0^2 + 15|\tau_1|^2 - \frac{1}{4}|\tau_2|^2 + \frac{1}{2}|\tau_3|^2\right) g_\varphi \\
&\quad + \mathrm{j}\left(-\frac{5}{4}d(*_\varphi(\tau_1 \wedge *_\varphi\varphi)) - \frac{1}{4}d\tau_2 + \frac{1}{4} *_\varphi d\tau_3\right. \\
&\quad + \frac{5}{2}\tau_1 \wedge *_\varphi(\tau_1 \wedge *_\varphi\varphi) - \frac{1}{8}\tau_0\,\tau_3 + \frac{1}{4}\tau_1 \wedge \tau_2 \\
&\quad \left. + \frac{3}{4} *_\varphi (\tau_1 \wedge \tau_3) + \frac{1}{8} *_\varphi (\tau_2 \wedge \tau_2) + \frac{1}{64}\,\mathrm{Q}(\tau_3, \tau_3)\right),
\end{aligned}$$

where $d^* = - *_\varphi d *_\varphi$ on 1-forms, the map $\mathrm{j} : \Omega^3(M) \to \mathcal{S}^2(M)$ is defined for every $\beta \in \Omega^3(M)$ and $X, Y \in \mathfrak{X}(M)$ as

$$\mathrm{j}(\beta)(X, Y) = *_\varphi ((\iota_X\varphi) \wedge (\iota_Y\varphi) \wedge \beta),$$

and $\mathrm{Q} : \Omega^3(M) \times \Omega^3(M) \to \Omega^3(M)$ is given for $\alpha, \beta \in \Omega^3(M)$ by

$$\mathrm{Q}(\alpha, \beta) := *_\varphi \left(\varphi_{ijkl}\left(\iota_{e_j}(\iota_{e_i} *_\varphi \alpha)\right) \wedge \left(\iota_{e_l}(\iota_{e_k} *_\varphi \beta)\right)\right),$$

being $(e_1, \ldots, e_7)$ a G_2*-basis for φ.*

Remark 3.1.10. The description of the Ricci tensor of an SU(3)-structure in terms of the intrinsic torsion forms obtained in [19] and recalled in Section 2.2.3 is the SU(3)-analogue of the previous theorem and was obtained following Bryant's approach in the G$_2$-case.

As a consequence, using Proposition 3.1.8, it is immediate to get an alternative proof of Bonan's result mentioned earlier.

Proposition 3.1.11 ([26]). *Let (M, φ) be a* G_2*-manifold. Then, g_φ is Ricci-flat.*

Further properties of the metric underlying non-integrable G_2-structures and non-existence results can be obtained using these expressions of $\mathrm{Ric}(g_\varphi)$ and $\mathrm{Scal}(g_\varphi)$. For instance

Proposition 3.1.12. *Let φ be a nearly parallel G_2-structure. Then, g_φ is an Einstein metric, as*

$$\mathrm{Ric}(g_\varphi) = \frac{3}{8}\,\tau_0^2\,g_\varphi.$$

Moreover, it is Ricci-flat if and only if the G_2-structure is parallel.

Proposition 3.1.13 ([30])**.** *Let (M,φ) be a manifold endowed with a calibrated G_2-structure φ. Then, the scalar curvature of the associated metric g_φ is nonpositive,*

$$\mathrm{Scal}(g_\varphi) = -\frac{1}{2}|\tau_2|^2,$$

and vanishes identically if and only if the G_2-structure is parallel.

Moreover, as an analogous of Goldberg conjecture in almost Kähler geometry (cf. Section 2.5), one may ask whether there exist compact seven-dimensional manifolds endowed with a calibrated G_2-structure whose underlying Riemannian metric is Einstein. In [47], Cleyton and Ivanov showed that the answer is negative.

Proposition 3.1.14 ([30],[47])**.** *Let M be a compact 7-manifold endowed with a calibrated G_2-structure φ. If g_φ is Einstein, then $d*_\varphi\varphi = 0$, that is, the G_2-structure is parallel.*

An alternative proof was given by Bryant in [30] using the above description of the Ricci tensor. In detail, the metric g_φ induced by a calibrated G_2-structure φ is Einstein, i.e., $\mathrm{Ric}^0(g_\varphi) = 0$, if and only if the intrinsic torsion form τ_2 satisfies

$$d\tau_2 = \frac{3}{14}\,|\tau_2|^2\,\varphi + \frac{1}{2}\,*_\varphi(\tau_2\wedge\tau_2),$$

from which follows that

$$d\left(\frac{1}{3}\tau_2^3\right) = \frac{2}{7}\,|\tau_2|^4\,dV_\varphi,$$

since $|\tau_2\wedge\tau_2|^2 = |\tau_2|^4$. When M is compact, Stokes' Theorem gives

$$\int_M |\tau_2|^4\,dV_\varphi = \int_M d\left(\frac{7}{6}\tau_2^3\right) = 0,$$

as $\partial M = \emptyset$. Consequently, τ_2 must vanish identically and the G_2-structure is parallel.

By [47], the previous result can be extended to the noncompact case under the additional assumption that the Einstein metric induced by the calibrated G_2-structure is also $*$-*Einstein*, that is, the traceless part of the $*$-*Ricci tensor* ρ^* vanishes identically, where

$$\rho^*_{rs} := R_{ijkl}\,\varphi_{ijr}\,\varphi_{kls}. \tag{3.9}$$

Up to now, there are no known examples of (even incomplete) Einstein non-Ricci-flat metrics underlying calibrated G_2-structures. Recently, some negative results were proved in the case of noncompact homogeneous spaces in [65], where the authors showed that a seven-dimensional solvmanifold cannot admit any left-invariant calibrated G_2-structure φ inducing an Einstein metric g_φ unless g_φ is flat.

3.2 The relation between G_2- and SU(3)-structures

In this section, we review the relation between 6-manifolds endowed with an SU(3)-structure and 7-manifolds endowed with a G_2-structure. In order to make no confusion with the symbols, from now on we use the following

Notation. A six-dimensional manifold endowed with an SU(3)-structure (ω, ψ_+) is denoted by $\overline{M}$. The Riemannian metric induced by (ω, ψ_+) is denoted by g, the associated almost Hermitian structure by J and $\psi_- = J\psi_+$ is the imaginary part of the complex volume form $\Psi = \psi_+ + i\psi_-$. The Riemannian volume form of g is $dV_g = \frac{\omega^3}{6}$ and the corresponding Hodge operator is denoted by $*$.

A seven-dimensional manifold endowed with a G_2-structure φ is denoted by M, the underlying Riemannian metric and volume form are g_φ and dV_φ, respectively, and the associated Hodge operator is $*_\varphi$.

3.2.1 Hypersurfaces of 7-manifolds with a G_2-structure

It is well-known that the group G_2 acts transitively on the 6-sphere $S^6 \subset \mathbb{R}^7$ with isotropy subgroup SU(3), reflecting the fact that $S^6 = G_2/\mathrm{SU}(3)$ as G_2-homogeneous space. Here, considering $\mathbb{R}^7 = \mathbb{R}^6 \oplus \mathbb{R}$ with basis $(e_1, \dots, e_6)$ for $\mathbb{R}^6$ and e_7 for $\mathbb{R}$, SU(3) is embedded into G_2 as the subgroup whose elements fix e^7, since

$$\begin{aligned}\varphi_0 &= \left(e^{12}+e^{34}+e^{56}\right)\wedge e^7 + e^{135} - e^{146} - e^{236} - e^{245}\\ &= \omega_0 \wedge e^7 + \Re(\Psi_0)\end{aligned}$$

and SU(3) can be defined as the stabilizer in $GL(6,\mathbb{R})$ of the pair $(\omega_0, \Re(\Psi_0))$ by the results of Section 2.2.1. As a consequence of this fact, a G_2-structure on a seven-dimensional manifold induces an SU(3)-structure on every oriented hypersurface. This is a long-standing result, due to Calabi [35] and Gray [86, 87]. In their papers, the authors studied the properties of the almost Hermitian structure underlying the SU(3)-structure, while a detailed study of the latter was carried out by Martín Cabrera in [143], where some of the possible classes of G_2-structures on the ambient manifold were described in relation with the classes of SU(3)-structures on the hypersurfaces and the second fundamental form.

To see how the SU(3)-structure is defined, let us consider a seven-dimensional manifold M endowed with a G_2-structure φ with associated Riemannian metric g_φ and two-fold vector cross product P. Let $\overline{M}$ be an oriented hypersurface of M, $\dim(\overline{M}) = 6$, denote by $\iota : \overline{M} \to M$ the inclusion map and by N a unit normal vector field to $\overline{M}$ with respect to g_φ. Then, $\overline{M}$ is a Riemannian manifold with Riemannian metric $g = \iota^*(g_\varphi)$, the almost complex structure on it is defined by

$$JX = \mathrm{P}(N, \iota_* X),$$

for all $X \in \mathfrak{X}(\overline{M})$, the fundamental form associated with (g, J) is

$$\omega = \iota^*(\iota_N \varphi) \tag{3.10}$$

and the real and imaginary part of the complex volume form are

$$\begin{aligned}\psi_+ &= \iota^*\varphi,\\ \psi_- &= -\iota^*(\iota_N *_\varphi \varphi).\end{aligned} \tag{3.11}$$

Furthermore, the following identity holds

$$\omega^2 = 2\,\iota^*(*_\varphi \varphi). \tag{3.12}$$

The proof that the previous data define an SU(3)-structure on $\overline{M}$ consists in considering at each point p of $\overline{M} \subset M$ a G_2-basis $(e_1, \ldots, e_7)$ of T_pM such that $N_p = e_7$ and observing that, with respect to the previous definitions, $(e_1, \ldots, e_6)$ is an SU(3)-basis of $T_p\overline{M}$.

As an example, we describe the invariant nearly Kähler SU(3)-structure on the 6-sphere $S^6 = G_2/SU(3)$ appearing in Butruille's Theorem 2.3.6.

Example 3.2.1. Consider the vector space $\mathbb{R}^7$ and let $(x^k) = (x^1, \ldots, x^7)$ denote its standard coordinates. The G_2-structure on $\mathbb{R}^7$ is then given by the 3-form

$$\varphi = dx^{127} + dx^{347} + dx^{567} + dx^{135} - dx^{146} - dx^{236} - dx^{245},$$

where dx^{jkl} is a shorthand for the wedge product $dx^j \wedge dx^k \wedge dx^l$, and induces the Riemannian metric $g_\varphi = \sum_{k=1}^{7}(dx^k)^2$. The 6-sphere is embedded in $\mathbb{R}^7$ as the set

$$S^6 = \left\{x \in \mathbb{R}^7 \mid \left(x^1\right)^2 + \cdots + \left(x^7\right)^2 = 1\right\},$$

and the unit normal N at each point x of S^6 is the restriction of the radial vector field of $\mathbb{R}^7$ to the 6-sphere, i.e.,

$$N_x = x^i \frac{\partial}{\partial x^i}.$$

If $\iota : S^6 \to \mathbb{R}^7$ is the standard embedding, then the differential forms

$$\omega = \iota^*(\iota_N \varphi), \quad \psi_+ = \iota^*\varphi, \quad \psi_- = -\iota^*(\iota_N *_\varphi \varphi),$$

obtained as pullback of G_2-invariant forms on $\mathbb{R}^7$, define the invariant SU(3)-structure on S^6. A straightforward computation in coordinates on $\mathbb{R}^7$ gives

$$d(\iota_N \varphi) = 3\,\varphi, \quad d(\iota_N *_\varphi \varphi) = 4 *_\varphi \varphi.$$

Consequently,

$$d\omega = d\,\iota^*(\iota_N \varphi) = \iota^* d(\iota_N \varphi) = 3\,\iota^*\varphi = 3\,\psi_+$$

and

$$d\psi_- = -d\,\iota^*(\iota_N *_\varphi \varphi) = -\iota^* d(\iota_N *_\varphi \varphi) = -4\,\iota^* *_\varphi \varphi = -2\,\omega^2.$$

Thus, the SU(3)-structure is nearly Kähler.

Among the results of [143] mentioned before, we recall that when (M, φ) is a G_2-manifold, its oriented hypersurfaces are endowed with a half-flat SU(3)-structure, as one can check immediately using the definitions (3.11), (3.12) and the fact that both φ and $*_\varphi\varphi$ are closed. Moreover, special half-flat SU(3)-structures arise from this construction and are characterized by the *scalar second fundamental form* $s(X,Y) = -g_\varphi(\nabla^{g_\varphi}_X N, Y)$ of the hypersurface. For instance, the SU(3)-structure on a hypersurface $\overline{M}$ of a G_2-manifold is coupled if and only if the second fundamental form is J-invariant, it is double half-flat if and only if $s + Js = 2\,Hg$, where H is the mean curvature, while it is nearly Kähler if and only if $\overline{M}$ is totally umbilic, that is, if and only if the shape operator $s^\sharp$ is pointwise a multiple of the identity on the tangent space to $\overline{M}$.

At this point, the obvious question is whether every SU(3)-structure on a six-dimensional manifold $\overline{M}$ is induced by an embedding into a G_2-manifold M. In the general case the answer is negative and some additional hypothesis on the SU(3)-structure are needed.

As observed by Bryant in [31], when $\overline{M}$ is an embedded and normally oriented hypersurface of M, there is an open neighborhood $\mathcal{U} \subset M$ of $\overline{M}$ which can be identified with $\overline{M} \times \mathrm{I}$, being I an open interval of $\mathbb{R}$ containing 0. Consequently, at least locally the G_2-structure φ on M can be thought as a 1-parameter family of SU(3)-structures on $\overline{M}$. Indeed, if t denotes the coordinate on I and $h : \overline{M} \times \mathrm{I} \to \mathcal{U}$ is the map defined by $h(p,t) = \exp_p(tN_p)$, then from (3.10) and (3.11) we get

$$h^*(\varphi) = \omega \wedge dt + \psi_+, \quad h^*(*_\varphi\varphi) = \frac{1}{2}\,\omega^2 + \psi_- \wedge dt,$$

where ω, ψ_+ and ψ_- are differential forms on $\overline{M}$ depending on t. Now, requiring that the G_2-structure is parallel for each t fixed, we have

$$0 = dh^*(\varphi) = d\psi_+, \quad 0 = dh^*(*_\varphi\varphi) = \frac{1}{2}\,d\omega^2$$

and the SU(3)-structure is half-flat. If we let t vary, then the same request gives

$$0 = dh^*(\varphi) = d\omega \wedge dt - \frac{\partial}{\partial t}\psi_+ \wedge dt$$

and

$$0 = dh^*(*_\varphi\varphi) = \frac{\partial}{\partial t}\omega \wedge \omega \wedge dt + d\psi_- \wedge dt,$$

from which we obtain the Hitchin flow equations introduced in Section 2.2.2

$$\begin{cases} \frac{\partial}{\partial t}\psi_+(t) = d\omega(t) \\ \frac{\partial}{\partial t}\omega(t)\wedge\omega(t) = -d\psi_-(t) \end{cases}.$$

A solution of the previous system starting from a given SU(3)-structure gives then rise to a parallel G_2-structure on $\overline{M}\times \mathrm{I}$ and can be interpreted as an SU(3)-structure induced on an oriented hypersurface of a G_2-manifold (M,φ) by the parallel G_2-structure φ. Such a solution exists when the initial condition $(\omega(0),\psi_+(0))$ is a real-analytic half-flat SU(3)-structure by [31, Thm. 4], while it need not to exist when $(\omega(0),\psi_+(0))$ is non-analytic and half-flat by [31, Thm. 5].

3.2.2 Construction of G_2-structures from SU(3)-structures

Until now, we have considered SU(3)-structures induced on hypersurfaces of 7-manifolds by a G_2-structure. It is possible to reverse the point of view, starting from a 6-manifold endowed with an SU(3)-structure and constructing examples of 7-manifolds with a G_2-structure, possibly with non-identically vanishing torsion.

Noncompact examples can be achieved in the following way (see for instance [113], paying attention to the different convention used for (3.2)).

Proposition 3.2.2. *Let $\overline{M}$ be a six-dimensional manifold endowed with an* SU(3)*-structure (ω,ψ_+) with associated Riemannian metric g and complex volume form $\Psi=\psi_+ + i\psi_-$. Consider an open interval $\mathrm{I}\subseteq\mathbb{R}$ and two smooth functions $F:\mathrm{I}\to\mathbb{C}-\{0\}$ and $G:\mathrm{I}\to(0,+\infty)$. Then, the following 3-form defines a G_2-structure on $\overline{M}\times\mathrm{I}$*

$$\varphi = G|F|^2\,\omega\wedge dt + \Re\left(F^3\,\Psi\right), \tag{3.13}$$

where t is the coordinate on I *and $|F|^2 = F\,\overline{F}$. Moreover,*

$$\begin{aligned} g_\varphi &= |F|^2\,g + G^2\,dt^2, \\ dV_\varphi &= G\,|F|^6\,dV_g\wedge dt, \\ *_\varphi\varphi &= G\,\Im\left(F^3\Psi\right)\wedge dt + \frac{1}{2}\,|F|^4\,\omega^2, \end{aligned}$$

where $dV_g = \frac{\omega^3}{6}$.

Remark 3.2.3. Observe that the situation in the previous definition is different from the one described at the end of Section 3.2.1. Indeed, here the differential forms defining the SU(3)-structure do not depend on t.

For suitable choices of the interval I and the functions F and G, the following noncompact manifolds endowed with the G_2-structure (3.13) are obtained:

- the *cylinder* $\mathcal{C}yl(\overline{M})$ over $\overline{M}$ with metric $g_\varphi = g + dt^2$, if $\mathrm{I} = \mathbb{R}$ and $G(t) = 1$, $F(t) = 1$, where
$$\varphi = \omega \wedge dt + \psi_+;$$
- the *cone* $\mathcal{C}(\overline{M})$ over $\overline{M}$ with metric $g_\varphi = t^2\, g + dt^2$, if $\mathrm{I} = (0, +\infty)$ and $G(t) = 1$, $F(t) = t$, where
$$\varphi = t^2\, \omega \wedge dt + t^3\, \psi_+;$$
- the *sine-cone* $\mathcal{SC}(\overline{M})$ over $\overline{M}$ with metric $g_\varphi = \sin^2(t)\, g + dt^2$, if $\mathrm{I} = (0, \pi)$ and $G(t) = 1$, $F(t) = \sin(t)e^{i\frac{t}{3}}$, where
$$\varphi = \sin^2(t)\, \omega \wedge dt + \sin^3(t)\cos(t)\, \psi_+ - \sin^4(t)\, \psi_-.$$

Clearly, the intrinsic torsion of the G_2-structure defined via this construction depends on the intrinsic torsion of the SU(3)-structure. Thus, it is in principle possible to get plenty of examples of non-integrable G_2-structures.

Remark 3.2.4. The correspondence between the irreducible components of the intrinsic torsion of an SU(3)-structure (ω, ψ_+) on $\overline{M}$ and the irreducible components of the intrinsic torsion of the G_2-structure induced by (ω, ψ_+) on $\mathcal{C}yl(\overline{M})$ was described in [40, Thm. 3.1].

Observe that with the choice $G(t) = 1$, the manifold $\overline{M} \times \mathrm{I}$ with metric $|F|^2\, g + dt^2$ is the *warped product* of $\overline{M}$ and I with *warping function* $|F|$. Using the expression of the Ricci tensor of a warped product metric [156], it is possible to show the following general properties (see also [27]).

Proposition 3.2.5. *Let (M^m, g) be a Riemannian manifold of dimension m. Then, the cone metric $t^2\, g + dt^2$ is Ricci-flat if and only if the metric g is Einstein with* $\mathrm{Ric}(g) = (m-1)\, g$.

Proposition 3.2.6. *Let (M^m, g) be a Riemannian manifold of dimension m with Einstein metric g such that* $\mathrm{Ric}(g) = (m-1)\, g$. *Then, the sine-cone metric* $\sin^2(t)g + dt^2$ *is Einstein with Einstein constant m.*

As we will see, these results are useful to provide noncompact examples of non-integrable G_2-structures inducing Einstein metrics.

Compact examples can be obtained, for instance, starting from a compact 6-manifold $\overline{M}$ endowed with an SU(3)-structure (ω, ψ_+) and considering the product manifold $\overline{M} \times S^1$ with the G_2-structure

$$\varphi = \omega \wedge dt + \psi_+,$$

where t denotes the angle coordinate on the circle defined by identifying each point of $S^1 \subset \mathbb{C}$ with $e^{2\pi i t}$. This example can be seen as a particular case of a more general construction, let us introduce it.

Definition 3.2.7. Let M be a connected manifold of dimension m, let $\nu : M \to M$ be a diffeomorphism and let $\Gamma_{\widetilde{\nu}}$ denote the infinite cyclic group of diffeomorphisms of $M \times \mathbb{R}$ generated by

$$\begin{array}{rccc} \widetilde{\nu}: & M \times \mathbb{R} & \longrightarrow & M \times \mathbb{R} \\ & (p,t) & \longmapsto & (\nu(p), t+1) \end{array}.$$

$\Gamma_{\widetilde{\nu}}$ acts freely and proper discontinuously on the product manifold $M \times \mathbb{R}$, thus the quotient

$$M_\nu := (M \times \mathbb{R}) / \Gamma_{\widetilde{\nu}}$$

is a smooth manifold of dimension $m+1$, called the *mapping torus* of ν.

Observe that when $\nu = \mathrm{Id}_M$, then $M_\nu = M \times S^1$. The properties of the mapping torus are summarized in the next

Proposition 3.2.8. *Let M be a connected manifold and let M_ν be the mapping torus of a diffeomorphism $\nu : M \to M$. Then,*

i) *If M is compact, then also M_ν is compact.*

ii) *M_ν is the total space of a locally trivial fiber bundle over S^1 with typical fiber M, monodromy ν and projection $\pi : M_\nu \to S^1$ defined by $\pi(p,t) = e^{2\pi i t}$. In particular, we have the following diagram*

$$\begin{array}{ccccc} M & \xleftarrow{\mathrm{p}_1} & M \times \mathbb{R} & \xrightarrow{\mathrm{q}} & M_\nu \\ & & \downarrow \mathrm{p}_2 & & \downarrow \pi \\ & & \mathbb{R} & \xrightarrow{\Pi} & S^1 \end{array}$$

where p_1 *and* p_2 *are the projections from $M \times \mathbb{R}$ onto the first and the second factor, respectively,* q *is the quotient map, and $\Pi(t) = e^{2\pi i t}$ is the universal covering map.*

iii) *Every differential form $\alpha \in \Omega^k(M)$ which is ν-invariant, i.e., $\nu^*\alpha = \alpha$, defines a differential form $\widetilde{\alpha} \in \Omega^k(M_\nu)$, since the pullback $\mathrm{p}_1^*\alpha \in \Omega^k(M \times \mathbb{R})$ is invariant by the diffeomorphism $\widetilde{\nu}$.*

iv) *The 1-form $\mathrm{p}_2^*(dt)$ on $M \times \mathbb{R}$ is invariant by $\widetilde{\nu}$, thus it induces a closed 1-form $\eta \in \Omega^1(M_\nu)$, called characteristic 1-form of M_ν. Moreover, there exists a distinguished vector field $\xi \in \mathfrak{X}(M_\nu)$ induced by the vector field $\frac{d}{dt}$ on $\mathbb{R}$ and such that $\eta(\xi) = 1$.*

Using these properties, we can prove the following result (see also [140]).

Proposition 3.2.9. *Let $\overline{M}$ be a six-dimensional connected manifold endowed with an* SU(3)*-structure (ω, ψ_+) and let $\nu : \overline{M} \to \overline{M}$ be a diffeomorphism such that $\nu^*\omega = \omega$ and $\nu^*\psi_+ = \psi_+$. Then, the 3-form*

$$\widetilde{\varphi} = \widetilde{\omega} \wedge \eta + \widetilde{\psi}_+$$

defines a G_2*-structure on the mapping torus $\overline{M}_\nu$ with associated metric $g_{\widetilde{\varphi}} = \widetilde{g} + \eta^2$.*

Proof. We recalled in Section 2.2.1 that a diffeomorphism ν preserving the stable forms ω and ψ_+ is an automorphism of the SU(3)-structure and preserves, in particular, the associated Riemannian metric g. We also know that on the product $\overline{M} \times \mathbb{R}$ there is a G_2-structure defined by the 3-form

$$\varphi = \omega \wedge dt + \psi_+$$

and inducing the Riemannian metric $g_\varphi = g + dt^2$. Now, both φ and g_φ are invariant by $\widetilde{\nu}$, thus they define a 3-form $\widetilde{\varphi} = \widetilde{\omega} \wedge \eta + \widetilde{\psi}_+$ and a Riemannian metric $\widetilde{g_\varphi} = \widetilde{g} + \eta^2$ on $\overline{M}_\nu$, respectively. Moreover, at each point of $\overline{M} \times \mathbb{R}$ there exists a G_2-basis for φ, which gives rise to a G_2-basis for $\widetilde{\varphi}$ on the corresponding point of $\overline{M}_\nu$. Consequently, $\widetilde{\varphi}$ is a G_2-structure on $\overline{M}_\nu$ inducing the Riemannian metric $g_{\widetilde{\varphi}} = \widetilde{g_\varphi}$. □

3.3 Locally conformal calibrated G_2-structures

We now focus on the class of locally conformal calibrated G_2-structures and study some related problems. Let us start recalling the

Definition 3.3.1. A G_2-structure φ is called *locally conformal calibrated* if the intrinsic torsion forms τ_0 and τ_3 vanish identically.

A 7-manifold endowed with a locally conformal calibrated G_2-structure is said to be a *locally conformal calibrated* G_2*-manifold.*

Every locally conformal calibrated G_2-manifold (M, φ) belongs then to Fernández and Gray's class $\mathcal{X}_2 \oplus \mathcal{X}_4$ and its only possibly non-identically vanishing intrinsic torsion forms are $\tau_1 \in \Omega^1(M)$ and $\tau_2 \in \Omega^2_{14}(M)$. When $\tau_1 = 0$ the G_2-structure is calibrated (G_2-type $\mathcal{X}_2$), while when $\tau_2 = 0$ the G_2-structure is locally conformal parallel (G_2-type $\mathcal{X}_4$). Thus, we may sometimes emphasize when the intrinsic torsion forms τ_1 and τ_2 are both non-identically vanishing to distinguish this case from $\mathcal{X}_2$ and $\mathcal{X}_4$.

The condition given in Definition 3.3.1 can be completely characterized in terms of the exterior derivative of φ. Before writing the precise statement of the result, it is convenient to introduce the

Definition 3.3.2. The *Lee form* of a G_2-structure φ is the 1-form

$$\theta := \frac{1}{4} *_\varphi (*_\varphi d\varphi \wedge \varphi).$$

Using identity (3.3) and the definition of $\Omega^3_{27}(M)$, a simple computation shows that $\theta = -3\,\tau_1$ for every G_2-structure φ. It is then immediate to prove the

Proposition 3.3.3. *Let M be a 7-manifold endowed with a G_2-structure φ. Then, φ is locally conformal calibrated if and only if*

$$d\varphi = -\theta \wedge \varphi, \tag{3.14}$$

where θ is the Lee form of φ. In this case, θ is closed.

Proof. The characterization follows comparing (3.8) with (3.14). Taking the exterior derivative of both sides of (3.14), we get

$$0 = d(d\varphi) = -d\theta \wedge \varphi + \theta \wedge d\varphi = -d\theta \wedge \varphi$$

and by (3.6) the previous identity holds if and only if $d\theta = 0$. □

Corollary 3.3.4.

i) *The Lee form of a locally conformal parallel G_2-structure is closed.*

ii) *A locally conformal calibrated G_2-structure is calibrated if and only if the Lee form vanishes identically.*

Remark 3.3.5. In view of Proposition 3.3.3, locally conformal calibrated G_2-structures represent the G_2-analogue of *locally conformal symplectic structures* on even-dimensional manifolds, i.e., $\mathrm{Sp}(n, \mathbb{R})$-structures whose defining non-degenerate 2-form ω satisfies the identity $d\omega = -\theta \wedge \omega$ for some closed 1-form θ (see for instance [176]).

As we mentioned in Section 3.1.2, the name *locally conformal calibrated* refers to the fact that, at least locally, the G_2-structure is conformally equivalent to a calibrated one. Let us see the computation in detail. Since the Lee form of a locally conformal calibrated G_2-structure φ on a 7-manifold M is closed, by Poincaré Lemma we have that for each point p of M there exist an open neighborhood $\mathcal{U} \subseteq M$ of p and a smooth function $f \in C^\infty(\mathcal{U})$ such that $\theta = df$ on $\mathcal{U}$. Now, considering $\widehat{\varphi} := e^f \varphi$, which is a stable form defined on $\mathcal{U}$, we have

$$d\widehat{\varphi} = e^f df \wedge \varphi + e^f d\varphi = e^f \theta \wedge \varphi - e^f (\theta \wedge \varphi) = 0$$

and the claim is proved.

When we consider a *conformal change* $\widehat{\varphi} = e^f \varphi$ of the stable 3-form φ defining a G_2-structure, the Riemannian metric and orientation of the G_2-structure $\widehat{\varphi}$ can be obtained from those of φ by an appropriate conformal change, as the next result shows.

Lemma 3.3.6. *Let M be a 7-manifold endowed with a G_2-structure φ inducing a Riemannian metric g_φ and volume form dV_φ and consider a smooth function $f \in C^\infty(M)$. Then, the 3-form*

$$\widehat{\varphi} := e^f \varphi$$

is still stable and the associated Riemannian metric $g_{\widehat{\varphi}}$ and volume form $dV_{\widehat{\varphi}}$ are related to those of φ via

$$\begin{aligned} g_{\widehat{\varphi}} &= e^{\frac{2}{3}f} g_\varphi, \\ dV_{\widehat{\varphi}} &= e^{\frac{7}{3}f} dV_\varphi. \end{aligned}$$

Proof. First, observe that for every $X, Y \in \mathfrak{X}(M)$ we have

$$\begin{aligned} g_{\widehat{\varphi}}(X,Y)\, dV_{\widehat{\varphi}} &= \tfrac{1}{6}(\iota_X \widehat{\varphi}) \wedge (\iota_Y \widehat{\varphi}) \wedge \widehat{\varphi} \\ &= e^{3f}\, g_\varphi(X,Y)\, dV_\varphi. \end{aligned} \tag{3.15}$$

From this relation, working in local coordinates we get

$$\left(\det(g_{\widehat{\varphi}})\right)^{\frac{1}{2}}\ g_{\widehat{\varphi}} = e^{3f}\, (\det(g_\varphi))^{\frac{1}{2}}\ g_\varphi.$$

Taking the determinant of both sides, it follows that

$$\left(\det(g_{\widehat{\varphi}})\right)^{\frac{1}{2}} = e^{\frac{7}{3}f}\, (\det(g_\varphi))^{\frac{1}{2}}\ .$$

Thus, $dV_{\widehat{\varphi}} = e^{\frac{7}{3}f} dV_\varphi$ and from (3.15) we obtain $g_{\widehat{\varphi}} = e^{\frac{2}{3}f} g_\varphi$. □

Remark 3.3.7. By [79, Thm. 3.1], given a compact 7-manifold M admitting a G_2-structure φ, there exists a unique (up to homothety) conformal G_2-structure $e^{3f}\varphi$ such that the corresponding Lee form is co-closed. A G_2-structure with co-closed Lee form is also called a *Gauduchon* G_2*-structure*.

Properties of locally conformal calibrated G_2-manifolds were studied for instance by Fernández and Ugarte in [69], where such manifolds were characterized as those endowed with a G_2-structure φ for which the following sequence is a complex, called G_2-*coeffective complex*,

$$0 \longrightarrow \mathcal{B}^3(M) \xrightarrow{\tilde{d}} \mathcal{B}^4(M) \xrightarrow{\tilde{d}} \Omega^5(M) \xrightarrow{d} \Omega^6(M) \xrightarrow{d} \Omega^7(M) \longrightarrow 0,$$

where $\mathcal{B}^k(M) = \{\alpha \in \Omega^k(M) \mid \alpha \wedge \varphi = 0\}$ and $\tilde{d}$ denotes the restriction to $\mathcal{B}^k(M)$ of the exterior derivative d for $k = 3, 4$. Moreover, in the same paper the ellipticity of the G_2-coeffective complex was studied and the relations between its cohomology groups and the de Rham cohomology groups were established.

3.3.1 Examples

We now use the constructions described in Section 3.2.2 to provide examples of locally conformal calibrated G_2-manifolds. As we will see, coupled SU(3)-structures play a central rôle.

Proposition 3.3.8. *Let* $\overline{M}$ *be a connected 6-manifold endowed with a coupled* SU(3)-*structure* (ω, ψ_+) *with coupled constant* $c \in \mathbb{R} - \{0\}$ *and intrinsic torsion form* w_2^- *non-identically vanishing. Then*

i) *The* G_2-*structure* $\varphi = \omega \wedge dt + \psi_+$ *defined on the cylinder* $\mathcal{C}yl(\overline{M})$ *is locally conformal calibrated of type* $\mathcal{X}_2 \oplus \mathcal{X}_4$. *Its Lee form is* $\theta = -3\,\tau_1 = c\,dt$ *and the intrinsic torsion form* τ_2 *is* $-w_2^-$.

ii) *The* G_2-*structure* $\varphi = t^2\,\omega \wedge dt + t^3\,\psi_+$ *defined on the cone* $\mathcal{C}(\overline{M})$ *has intrinsic torsion forms* $\tau_1 = \frac{3-c}{3\,t}\,dt$, $\tau_2 = -t\,w_2^-$, $\tau_0 = 0$, $\tau_3 = 0$. *Thus, it is locally conformal calibrated when* $c \neq 3$, *while it is calibrated when* $c = 3$.

Proof. In order to distinguish the exterior derivative on $\overline{M}$ and on the 7-manifolds $\mathcal{C}yl(\overline{M})$ and $\mathcal{C}(\overline{M})$, we use a subscript containing the dimension. For instance, the coupled condition reads $d_6\,\omega = c\,\psi_+$.

Let us begin our computations with the cylinder $\mathcal{C}yl(\overline{M})$.

$$\begin{aligned} d_7\,\varphi &= d_7(\omega \wedge dt + \psi_+) = d_6\,\omega \wedge dt + d_6\,\psi_+ \\ &= c\,\psi_+ \wedge dt = -c\,dt \wedge \varphi. \end{aligned}$$

Thus, by Proposition 3.3.3, φ is locally conformal calibrated with Lee form $\theta = c\,dt$. Moreover,

$$\begin{aligned} d_7 *_\varphi \varphi &= d_7\left(\psi_- \wedge dt + \tfrac{1}{2}\,\omega^2\right) &&= d_6\,\psi_- \wedge dt + \tfrac{1}{2}\,d_6\,\omega^2 \\ &= \left(-\tfrac{2}{3}c\,\omega^2 - w_2^- \wedge \omega\right) \wedge dt &&= -\tfrac{4}{3}\,\theta \wedge *_\varphi\varphi - w_2^- \wedge \varphi, \end{aligned}$$

from which follows that $\tau_2 = -w_2^-$.

Consider now the cone $\mathcal{C}(\overline{M})$.

$$\begin{aligned} d_7\,\varphi &= d_7(t^2\,\omega \wedge dt + t^3\,\psi_+) &&= t^2\,d_6\,\omega \wedge dt + 3t^2\,dt \wedge \psi_+ \\ &= c\,t^2\,\psi_+ \wedge dt + 3t^2\,dt \wedge \psi_+ &&= -\tfrac{c-3}{t}\,dt \wedge \varphi \end{aligned}$$

and

$$\begin{aligned} d_7 *_\varphi \varphi &= d_7\left(t^3\,\psi_- \wedge dt + \tfrac{1}{2}\,t^4\,\omega^2\right) &&= t^3\,d_6\,\psi_- \wedge dt + 2\,t^3\,dt \wedge \omega^2 \\ &= t^3\,\left(\tfrac{6-2c}{3}\,\omega^2 - w_2^- \wedge \omega\right) \wedge dt &&= -\tfrac{4}{3}\,\tfrac{c-3}{t}\,dt \wedge *_\varphi\varphi - t\,w_2^- \wedge \varphi. \end{aligned}$$

Consequently, φ is locally conformal calibrated with Lee form $\theta = \frac{c-3}{t}\,dt$ and intrinsic torsion form $\tau_2 = -t\,w_2^-$. Moreover, the G_2-structure is calibrated if $c = 3$. □

Since a coupled SU(3)-structure with nonzero coupled constant c and identically vanishing w_2^- is nearly Kähler with $w_1^- = -\frac{2}{3}\,c$, the following result can be seen as a consequence of the previous proposition.

Corollary 3.3.9. *Let $\overline{M}$ be a connected six-dimensional manifold endowed with a nearly Kähler* SU(3)*-structure (ω, ψ_+). Then*

i) *The G_2-structure $\varphi = \omega \wedge dt + \psi_+$ defined on the cylinder $\mathcal{C}yl(\overline{M})$ is locally conformal parallel.*

ii) *The G_2-structure $\varphi = t^2\,\omega \wedge dt + t^3\,\psi_+$ defined on the cone $\mathcal{C}(\overline{M})$ is locally conformal parallel if $w_1^- \neq -2$, while it is parallel otherwise.*

Remark 3.3.10. In fact, it is possible to prove (cf. [14, Lemma 7]) that an SU(3)-structure (ω, ψ_+) on a 6-manifold $\overline{M}$ is nearly Kähler with $w_1^- = -2$, i.e., satisfies

$$d\omega = 3\,\psi_+, \quad \psi_- = -2\,\omega^2,$$

if and only if the G_2-structure induced by it on the cone $\mathcal{C}(\overline{M})$ is parallel.

If we consider the mapping torus construction, we can provide compact examples of locally conformal calibrated G_2-manifolds.

Proposition 3.3.11 ([66]). *Let $\overline{M}$ be a six-dimensional compact, connected manifold endowed with a coupled* $SU(3)$*-structure* (ω, ψ_+) *with nonzero coupled constant c and let $\nu : \overline{M} \to \overline{M}$ be a diffeomorphism such that $\nu^*\omega = \omega$. Then, the mapping torus $\overline{M}_\nu$ admits a locally conformal calibrated* G_2*-structure $\widetilde{\varphi}$ with Lee form $\theta = c\eta$. Moreover, $\mathcal{L}_\xi \widetilde{\varphi} = 0$ and the vector field ξ is the $g_{\widetilde{\varphi}}$-dual of the closed 1-form η.*

Proof. By Corollary 2.4.3, we know that a diffeomorphism preserving the 2-form ω is an automorphism of the coupled SU(3)-structure $\left(\omega, \frac{1}{c}\, d\omega\right)$. Thus, by Proposition 3.2.9, the 3-form

$$\widetilde{\varphi} = \widetilde{\omega} \wedge \eta + \widetilde{\psi}_+$$

defines a G_2-structure on the mapping torus $\overline{M}_\nu$ with associated metric $g_{\widetilde{\varphi}} = \widetilde{g} + \eta^2$.

Observe that $\widetilde{\omega}$ and $\widetilde{\psi}_+$ are obtained gluing up the pullbacks $p_1^*(\omega) \in \Omega^2(\overline{M} \times \mathbb{R})$ and $p_1^*(\psi_+) \in \Omega^3(\overline{M} \times \mathbb{R})$, as ν preserves both ω and ψ_+. Consequently, since $d(p_1^*\omega) = c\, p_1^*\psi_+$, we have

$$d\widetilde{\omega} = c\, \widetilde{\psi}_+,$$

and using this identity we get

$$d\widetilde{\varphi} = d\widetilde{\omega} \wedge \eta + \widetilde{\omega} \wedge d\eta + d\widetilde{\psi}_+ = -c\eta \wedge \widetilde{\psi}_+ = -c\eta \wedge \widetilde{\varphi}.$$

Therefore, $\widetilde{\varphi}$ is locally conformal calibrated with Lee form $\theta = c\eta$. Moreover, since both $\widetilde{\omega}$ and $\widetilde{\psi}_+$ derive from differential forms defined on $\overline{M}$, we have $\iota_\xi \widetilde{\omega} = 0$ and $\iota_\xi \widetilde{\psi}_+ = 0$. From these conditions, it follows that

$$\iota_\xi \widetilde{\varphi} = \iota_\xi \widetilde{\omega} \wedge \eta + \widetilde{\omega}\, \eta(\xi) + \iota_\xi \widetilde{\psi}_+ = \widetilde{\omega}.$$

Then,

$$\begin{aligned} \mathcal{L}_\xi \widetilde{\varphi} &= \iota_\xi(d\widetilde{\varphi}) + d(\iota_\xi \widetilde{\varphi}) = \iota_\xi(-c\,\eta \wedge \widetilde{\varphi}) + d\,\widetilde{\omega} \\ &= -c\,\widetilde{\varphi} + c\,\eta \wedge (\iota_\xi \widetilde{\varphi}) + c\,\widetilde{\psi}_+ = -c\,\widetilde{\varphi} + c\,(\eta \wedge \widetilde{\omega} + \widetilde{\psi}_+) = 0. \end{aligned}$$

□

Remark 3.3.12. In [140], Manero showed that the mapping torus of a diffeomorphism preserving a symplectic half-flat SU(3)-structure is endowed with a closed G_2-structure. This situation corresponds to the case $c = 0$ in the previous proposition.

The previous result can be applied, for instance, to compact nilmanifolds admitting an invariant coupled SU(3)-structure.

Example 3.3.13 ([66])**.** Let us consider the Iwasawa manifold H/Γ introduced in Example 1.3.8, where

$$\mathrm{H} = \left\{ \begin{pmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix},\ z_k \in \mathbb{C},\ k = 1, 2, 3 \right\}$$

is the complex Heisenberg group and Γ is the lattice defined as the subgroup of H for which z_i are Gaussian integers. H can be seen as a real Lie group of dimension six with basis of left-invariant 1-forms $(e^1, \ldots, e^6)$ obtained from

$$e^1 + ie^2 = dz_1, \quad e^3 + ie^4 = dz_2, \quad e^5 + ie^6 = -dz_3 + z_1 \wedge dz_2,$$

and the pair

$$\omega = e^{12} + e^{34} - e^{56}, \qquad \psi_+ = e^{136} - e^{145} - e^{235} - e^{246},$$

defines an invariant coupled SU(3)-structure on the compact nilmanifold H/Γ with coupled constant $c = -1$, as we saw in the proof of Theorem 2.4.12. It is easy to check that the automorphism

$$\nu : \mathrm{H} \to \mathrm{H}, \qquad \begin{pmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix} \overset{\nu}{\mapsto} \begin{pmatrix} 1 & z_1 & -iz_3 \\ 0 & 1 & -iz_2 \\ 0 & 0 & 1 \end{pmatrix},$$

is such that

$$\nu^* e^1 = e^1, \quad \nu^* e^2 = e^2, \quad \nu^* e^3 = e^4, \quad \nu^* e^4 = -e^3, \quad \nu^* e^5 = e^6, \quad \nu^* e^6 = -e^5.$$

Consequently, $\nu^*\omega = \omega$ and it is possible to apply Proposition 3.3.11 obtaining that the mapping torus $(\mathrm{H}/\Gamma)_\nu$ is a compact manifold admitting a locally conformal calibrated G_2-structure with Lee form $\theta = -\eta$.

If we start with a 6-manifold endowed with a nearly Kähler SU(3)-structure and we consider the mapping torus of a certain diffeomorphism preserving the defining differential forms, we obtain the following

Proposition 3.3.14 ([66])**.** *Let $\overline{M}$ be a six-dimensional compact, connected manifold endowed with a nearly Kähler* SU(3)*-structure* (ω, ψ_+) *with* $w_1^- = -2$ *and let* $\nu : \overline{M} \to \overline{M}$ *be a diffeomorphism such that* $\nu^*\omega = \omega$. *Then, the mapping torus* $\overline{M}_\nu$ *admits a locally conformal parallel* G_2*-structure.*

Proof. As in the proof of Proposition 3.3.11, we can define the differential forms $\widetilde{\omega} \in \Omega^2(\overline{M}_\nu)$ and $\widetilde{\psi}_\pm \in \Omega^3(\overline{M}_\nu)$, which in this case satisfy the relations

$$\begin{aligned} d\widetilde{\omega} &= 3\,\widetilde{\psi}_+, \\ d\widetilde{\psi}_- &= -2\,\widetilde{\omega}^2. \end{aligned}$$

The stable 3-form

$$\widetilde{\varphi} = \widetilde{\omega} \wedge \eta + \widetilde{\psi}_+$$

defines a G_2-structure on $\overline{M}_\nu$ with Hodge dual

$$*_{\widetilde{\varphi}}\widetilde{\varphi} = \widetilde{\psi}_- \wedge \eta + \frac{1}{2}\,\widetilde{\omega}^2.$$

It follows from computations that

$$\begin{aligned} d\widetilde{\varphi} &= 3(-\eta) \wedge \widetilde{\varphi}, \\ d *_{\widetilde{\varphi}} \widetilde{\varphi} &= 4(-\eta) \wedge *_{\widetilde{\varphi}}\widetilde{\varphi}. \end{aligned}$$

Therefore, $\widetilde{\varphi}$ is a locally conformal parallel G_2-structure defined on $\overline{M}_\nu$. □

Example 3.3.15 ([66])**.** In Example 2.5.1, we described the left-invariant nearly Kähler SU(3)-structure on SU(2)×SU(2). It is induced by the nearly Kähler structure

$$\begin{aligned} \omega &= -\tfrac{\sqrt{3}}{18}\left(e^{14} + e^{25} + e^{36}\right), \\ \psi_+ &= \tfrac{\sqrt{3}}{54}\left(-e^{234} + e^{156} + e^{135} - e^{246} - e^{126} + e^{345}\right), \end{aligned}$$

defined on the Lie algebra $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ and having $w_1^- = -2$.

Let $\nu : \mathrm{SU}(2) \times \mathrm{SU}(2) \to \mathrm{SU}(2) \times \mathrm{SU}(2)$ be the diffeomorphism such that

$$\nu^*e^1 = e^1, \quad \nu^*e^2 = e^3, \quad \nu^*e^3 = -e^2, \quad \nu^*e^4 = e^4, \quad \nu^*e^5 = e^6, \quad \nu^*e^6 = -e^5,$$

it preserves the 2-form ω. Therefore, the mapping torus $(\mathrm{SU}(2)\times\mathrm{SU}(2))_\nu$ is endowed with a locally conformal parallel G_2-structure by the previous proposition.

We consider now locally conformal calibrated G_2-structures defined on seven-dimensional Lie algebras. We will show that they are closely related to coupled SU(3)-structures on six-dimensional Lie algebras, generalizing the result proved in [140] for calibrated G_2-structures obtained from symplectic half-flat structures on Lie algebras. As we did before, we fix the notations to distinguish easily the six-dimensional case from the seven-dimensional one in what follows:

Notation. $\hat{\mathfrak{g}}$ denotes a six-dimensional real Lie algebra and $\hat{d}$ its Chevalley-Eilenberg differential, while $\mathfrak{g}$ denotes a seven-dimensional real Lie algebra with Chevalley-Eilenberg differential d.

From the discussion at the beginning of Section 2.4.1, we know that an SU(3)-structure on a six-dimensional Lie algebra $\hat{\mathfrak{g}}$ is a pair $(\omega,\psi_+)\in\Lambda^2(\hat{\mathfrak{g}}^*)\times\Lambda^3(\hat{\mathfrak{g}}^*)$ of differential forms which can be expressed as

$$\omega = e^{12}+e^{34}+e^{56}, \qquad \psi_+ = e^{135}-e^{146}-e^{236}-e^{245},$$

with respect to some basis $\left(e^1,\dots,e^6\right)$ of the dual space $\hat{\mathfrak{g}}^*$, called SU(3)-basis for (ω,ψ_+). An SU(3)-structure (ω,ψ_+) on $\hat{\mathfrak{g}}$ is coupled if

$$\hat{d}\omega = c\,\psi_+,$$

for some nonzero real constant c.

Similarly, a G_2-structure on a seven-dimensional Lie algebra $\mathfrak{g}$ is a 3-form $\varphi\in\Lambda^3(\mathfrak{g}^*)$ which can be written as

$$\varphi = e^{127}+e^{347}+e^{567}+e^{135}-e^{146}-e^{236}-e^{245},$$

with respect to some basis $\left(e^1,\dots,e^7\right)$ of $\mathfrak{g}^*$, called G_2-basis for φ. φ is a *locally conformal calibrated* G_2-structure on $\mathfrak{g}$ if

$$d\varphi = -\theta\wedge\varphi,$$

for some d-closed 1-form θ on $\mathfrak{g}$.

If $\hat{\mathfrak{g}}$ is a six-dimensional Lie algebra with Lie bracket $[\cdot,\cdot]_{\hat{\mathfrak{g}}}$ and $D \in \mathrm{Der}(\hat{\mathfrak{g}})$ is a derivation of $\hat{\mathfrak{g}}$, then the vector space

$$\mathfrak{g} = \hat{\mathfrak{g}} \oplus_D \mathbb{R}\xi$$

is a Lie algebra with the Lie bracket $[\cdot,\cdot]_{\mathfrak{g}}$ given by

$$[X,Y]_{\mathfrak{g}} = [X,Y]_{\hat{\mathfrak{g}}}, \qquad [\xi, X]_{\mathfrak{g}} = DX, \tag{3.16}$$

for every $X, Y \in \hat{\mathfrak{g}}$. It is useful to observe how the Chevalley-Eilenberg differential of $\mathfrak{g}$ is related to that of $\hat{\mathfrak{g}}$.

Lemma 3.3.16. *Let $\hat{\mathfrak{g}}$ be a six-dimensional real Lie algebra with Chevalley-Eilenberg differential $\hat{d}$, consider $D \in \mathrm{Der}(\hat{\mathfrak{g}})$ and let d denote the Chevalley-Eilenberg differential of $\mathfrak{g} = \hat{\mathfrak{g}} \oplus_D \mathbb{R}\xi$. Then, for every $\alpha \in \Lambda^k(\mathfrak{g}^*)$*

$$d\alpha = \hat{d}\alpha + \beta \wedge \eta,$$

for a certain $\beta \in \Lambda^k(\hat{\mathfrak{g}}^*)$, where η is the 1-form on $\mathfrak{g}$ such that $\eta(X) = 0$ for all $X \in \hat{\mathfrak{g}}$ and $\eta(\xi) = 1$.

Proof. Let $(e_1, \ldots, e_6)$ denote a basis of $\hat{\mathfrak{g}}$ with dual basis $(e^1, \ldots, e^6)$. Then, the 7-tuple $(e_1, \ldots, e_6, \xi)$ is a basis of $\mathfrak{g}$ with dual basis $(e^1, \ldots, e^6, \eta)$. To simplify the computations, let $e_7 := \xi$ and $e^7 := \eta$. If we denote by c^j_{kl} the structure constants of $\mathfrak{g}$ with respect to the considered basis and by $\hat{c}^j_{kl}$ those of $\hat{\mathfrak{g}}$, it follows from (3.16) that $c^j_{7k} = D^j{}_k$ for every $j, k = 1, \ldots, 6$, since

$$[e_7, e_k] = De_k = D^j{}_k e_j,$$

and that for all $j, k, l = 1, \ldots, 6$

$$c^j_{kl} = \hat{c}^j_{kl}, \quad c^7_{kl} = 0.$$

Consequently,

$$de^7 = 0$$

and for $j = 1, \ldots, 6$,

$$\begin{aligned} de^j &= \textstyle\sum_{1\le k<l\le 7}(-c^j_{kl})e^{kl} = \sum_{1\le k<l\le 6}(-c^j_{kl})e^{kl} + \sum_{k=1}^{6}(-c^j_{k7})e^{k7} \\ &= \hat{d}e^j + \left(\textstyle\sum_{k=1}^{6} D^j{}_k e^k\right) \wedge e^7. \end{aligned}$$

The assertion follows then writing every $\alpha \in \Lambda^k(\mathfrak{g}^*)$ with respect to the considered basis of $\mathfrak{g}^*$ and applying the properties of the differential d together with the previous expressions. □

It is clear from the discussion in Remark 1.2.10 that there exists a real representation of 3×3 complex matrices

$$\rho : \mathfrak{gl}(3, \mathbb{C}) \longrightarrow \mathfrak{gl}(6, \mathbb{R})$$

which sends a matrix $a \in \mathfrak{gl}(3, \mathbb{C})$ to the matrix $\rho(a)$ obtained by replacing each complex entry ${a^j}_k$ of a with the 2×2 real matrix

$$\begin{pmatrix} \Re({a^j}_k) & -\Im({a^j}_k) \\ \Im({a^j}_k) & \Re({a^j}_k) \end{pmatrix}.$$

Now, suppose that (ω, ψ_+) is a coupled SU(3)-structure on a six-dimensional Lie algebra $\hat{\mathfrak{g}}$ and let D be a derivation of $\hat{\mathfrak{g}}$ such that $D = \rho(a)$, where $a \in \mathfrak{sl}(3, \mathbb{C})$. Then, the matrix associated with D with respect to an SU(3)-basis $(e_1, \ldots, e_6)$ of $\hat{\mathfrak{g}}$ for (ω, ψ_+) is of the form

$$D = \left(\begin{array}{cc|cc|cc} b_{11} & -b_{12} & b_{13} & -b_{14} & b_{15} & -b_{16} \\ b_{12} & b_{11} & b_{14} & b_{13} & b_{16} & b_{15} \\ \hline b_{21} & -b_{22} & b_{23} & -b_{24} & b_{25} & -b_{26} \\ b_{22} & b_{21} & b_{24} & b_{23} & b_{26} & b_{25} \\ \hline b_{31} & -b_{32} & b_{33} & -b_{34} & -b_{11} - b_{23} & b_{12} + b_{24} \\ b_{32} & b_{31} & b_{34} & b_{33} & -b_{12} - b_{24} & -b_{11} - b_{23} \end{array}\right), \tag{3.17}$$

where $b_{jk} \in \mathbb{R}$. This gives a sufficient condition for the existence of a locally conformal calibrated G_2-structure on $\hat{\mathfrak{g}} \oplus_D \mathbb{R}\xi$:

Proposition 3.3.17 ([66]). *Let (ω, ψ_+) be a coupled* SU(3)*-structure on a Lie algebra $\hat{\mathfrak{g}}$ of dimension six and let $D = \rho(a)$, $a \in \mathfrak{sl}(3, \mathbb{C})$, be a derivation of $\hat{\mathfrak{g}}$ whose matrix representation with respect to an* SU(3)*-basis $(e_1, \ldots, e_6)$ of $\hat{\mathfrak{g}}^*$ for (ω, ψ_+) is as in* (3.17)*. Then, the Lie algebra*

$$\mathfrak{g} = \hat{\mathfrak{g}} \oplus_D \mathbb{R}\xi,$$

with the Lie bracket given by (3.16)*, has a locally conformal calibrated G_2-structure.*

Proof. On $\mathfrak{g} = \hat{\mathfrak{g}} \oplus_D \mathbb{R}\xi$ consider the 3-form

$$\varphi = \omega \wedge \eta + \psi_+, \tag{3.18}$$

where η is the 1-form on $\mathfrak{g}$ introduced in Lemma 3.3.16 and defined in such a way that $\eta(X) = 0$ for all $X \in \hat{\mathfrak{g}}$ and $\eta(\xi) = 1$. φ defines a G_2-structure on $\mathfrak{g}$, since $(e^1, \dots, e^6, \eta)$ is a G_2-basis of $\mathfrak{g}^*$. We shall see that

$$d\varphi = d\omega \wedge \eta + d\psi_+ = -c\eta \wedge \varphi,$$

where c is the coupled constant of the coupled SU(3)-structure on $\hat{\mathfrak{g}}$, i.e., $\hat{d}\omega = c\,\psi_+$.

Suppose that $X, Y, Z, W \in \hat{\mathfrak{g}}$. Then, it is clear that

$$(d\omega \wedge \eta)(X, Y, Z, W) = 0.$$

Consequently, by Lemma 3.3.16, we have

$$d\varphi(X, Y, Z, W) = d\psi_+(X, Y, Z, W) = \hat{d}\psi_+(X, Y, Z, W) = 0,$$

as ψ_+ is $\hat{d}$-closed. Let us consider $X, Y, Z \in \hat{\mathfrak{g}}$. Using (3.18), we obtain

$$\begin{aligned} d\varphi(X, Y, Z, \xi) &= -\varphi([X, Y], Z, \xi) + \varphi([X, Z], Y, \xi) - \varphi([X, \xi], Y, Z) \\ &\quad -\varphi([Y, Z], X, \xi) + \varphi([Y, \xi], X, Z) - \varphi([Z, \xi], X, Y) \\ &= -\omega([X, Y], Z) + \omega([X, Z], Y) - \omega([Y, Z], X) \\ &\quad -\psi_+([X, \xi], Y, Z) + \psi_+([Y, \xi], X, Z) - \psi_+([Z, \xi], X, Y) \\ &= \psi_+(D(X), Y, Z) + \psi_+(X, D(Y), Z) + \psi_+(X, Y, D(Z)) \\ &\quad +d\omega(X, Y, Z). \end{aligned}$$

Taking into account the expressions of D and ψ_+ in terms of the SU(3)-basis, it is easy to check that

$$\psi_+(D(e_j), e_k, e_l) + \psi_+(e_j, D(e_k), e_l) + \psi_+(e_j, e_k, D(e_l)) = 0,$$

for every triple $\{e_j, e_k, e_l\}$ of elements of the SU(3)-basis. Therefore,

$$d\varphi(X, Y, Z, \xi) = d\omega(X, Y, Z) = \hat{d}\omega(X, Y, Z) = c\,\psi_+(X, Y, Z).$$

Using (3.18) again, we get

$$d\varphi(X, Y, Z, \xi) = -(c\eta \wedge \varphi)(X, Y, Z, \xi),$$

which completes the proof that the 3-form φ given by (3.18) defines a locally conformal calibrated G_2-structure on $\mathfrak{g}$. □

As an application of the previous proposition, we describe two examples of non-isomorphic solvable Lie algebras endowed with a locally conformal calibrated G_2-structure. They are obtained considering two different derivations on the nilpotent Lie algebra $\mathfrak{n}_{28}$ (cf. Table 1.1).

Example 3.3.18 ([66])**.** Consider the six-dimensional nilpotent Lie algebra $\mathfrak{n} := \mathfrak{n}_{28}$. We know that it admits a coupled SU(3)-structure and that its structure equations with respect to the corresponding SU(3)-basis $(e^1, \dots, e^6)$ of $\mathfrak{n}^*$ are (see (2.34))

$$(0, 0, 0, 0, e^{14} + e^{23}, e^{13} - e^{24}).$$

The coupled SU(3)-structure is then defined by the pair

$$\omega = e^{12} + e^{34} + e^{56}, \qquad \psi_+ = e^{135} - e^{146} - e^{236} - e^{245},$$

and has coupled constant $c = -1$.

Let D be the derivation of $\mathfrak{n}$ defined as follows

$$De_1 = -e_3, \quad De_2 = -e_4, \quad De_3 = e_1, \quad De_4 = e_2, \quad De_5 = 0, \quad De_6 = 0.$$

The Lie algebra $\mathfrak{s} = \mathfrak{n} \oplus_D \mathbb{R}e_7$ has the following structure equations with respect to the basis $\left(e^1, \dots, e^6, e^7\right)$ of $\mathfrak{s}^*$

$$\left(e^{37}, e^{47}, -e^{17}, -e^{27}, e^{14} + e^{23}, e^{13} - e^{24}, 0\right).$$

By Proposition 3.3.17, the 3-form

$$\varphi = \omega \wedge e^7 + \psi_+$$

defines a locally conformal calibrated G_2-structure on $\mathfrak{s}$ with Lee form $\theta = -e^7$.

Let S denote the simply connected solvable Lie group with Lie algebra $\mathfrak{s}$, let N denote the simply connected nilpotent Lie group such that $\mathrm{Lie}(\mathrm{N}) = \mathfrak{n}$ and let $e \in \mathrm{N}$ denote the identity element. Observe that $\mathrm{S} = \mathbb{R} \ltimes_{\mu} \mathrm{N}$, where μ is the unique smooth action of $\mathbb{R}$ on N such that $\mu(t)_{*e} = \exp(tD)$, for every $t \in \mathbb{R}$, and where exp denotes the map $\exp : \mathrm{Der}(\mathfrak{n}) \to \mathrm{Aut}(\mathfrak{n})$. Hence, being S the semi-direct product of $\mathbb{R}$ and its nilradical N, it is *almost nilpotent* in the sense of [85].

Now, in order to show a lattice of S we proceed as follows. The considered SU(3)-basis $(e_1, \ldots, e_6)$ is a rational basis of $\mathfrak{n}$ and with respect to it we have

$$\exp(tD) = \begin{bmatrix} \cos(t) & 0 & \sin(t) & 0 & 0 & 0 \\ 0 & \cos(t) & 0 & \sin(t) & 0 & 0 \\ -\sin(t) & 0 & \cos(t) & 0 & 0 & 0 \\ 0 & -\sin(t) & 0 & \cos(t) & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

In particular, $\exp(\pi D)$ is an integer matrix. Therefore, denoted by $\exp^{\mathrm{N}} : \mathfrak{n} \to \mathrm{N}$ the exponential map, $\exp^{\mathrm{N}}(\mathbb{Z}\langle e_1, \ldots, e_6\rangle)$ is a lattice of N preserved by $\mu(\pi)$ and, consequently,

$$\Gamma = \pi\mathbb{Z} \ltimes_{\mu} \exp^{\mathrm{N}}(\mathbb{Z}\langle e_1, \ldots, e_6\rangle) \tag{3.19}$$

is a lattice in S (see [25]). Thus, the compact quotient S/Γ is a compact solvmanifold endowed with an invariant locally conformal calibrated G_2-structure φ.

Example 3.3.19 ([66])**.** Let us consider the coupled SU(3)-structure (ω, ψ_+) on $\mathfrak{n} := \mathfrak{n}_{28}$ described in the previous example and the derivation $D \in \mathrm{Der}(\mathfrak{n})$ given by

$$De_1 = 2e_3, \quad De_2 = 2e_4, \quad De_3 = e_1, \quad De_4 = e_2, \quad De_5 = 0, \quad De_6 = 0,$$

with respect to the SU(3)-basis $(e_1, \ldots, e_6)$ of $\mathfrak{n}$. Then, the Lie algebra $\mathfrak{q} = \mathfrak{n} \oplus_D \mathbb{R}e_7$ has the following structure equations with respect to the basis $\left(e^1, \ldots, e^6, e^7\right)$ of $\mathfrak{q}^*$

$$\left(e^{37}, e^{47}, 2e^{17}, 2e^{27}, e^{14} + e^{23}, e^{13} - e^{24}, 0\right).$$

The 3-form

$$\varphi = \omega \wedge e^7 + \psi_+$$

defines a locally conformal calibrated G_2-structure on $\mathfrak{q}$ with Lee form $\theta = -e^7$ by Proposition 3.3.17.

As in the previous example, we have an almost nilpotent Lie group $\mathrm{Q} = \mathbb{R} \ltimes_\mu \mathrm{N}$, where Q is the simply connected Lie group with solvable Lie algebra $\mathfrak{q}$ and μ is the unique smooth action of $\mathbb{R}$ on N such that $\mu(t)_{*e} = \exp(tD)$, for every $t \in \mathbb{R}$. With respect to the rational basis $(X_1, \ldots, X_6)$ of $\mathfrak{n}$ given by $X_1 = -\frac{1}{\sqrt{2}}e_2 + e_4, X_2 = -\frac{1}{\sqrt{2}}e_1 + e_3, X_3 = \frac{1}{\sqrt{2}}e_1 + e_3, X_4 = \frac{1}{\sqrt{2}}e_2 + e_4, X_5 = \sqrt{2}e_5, X_6 = \sqrt{2}e_6$, the matrix associated with $\exp\left(\sqrt{2}D\right)$ is integer. More in detail, we have

$$\exp\left(\sqrt{2}D\right) = \mathrm{diag}(-2, -2, 2, 2, 0, 0).$$

Thus, $\exp^{\mathrm{N}}(\mathbb{Z}\langle X_1, \ldots, X_6\rangle)$ is a lattice of N preserved by $\mu\left(\sqrt{2}\right)$ and, as a consequence,

$$\Gamma = \sqrt{2}\,\mathbb{Z} \ltimes_\mu \exp^{\mathrm{N}}(\mathbb{Z}\langle X_1, \ldots, X_6\rangle)$$

is a lattice in Q. The compact quotient Q/Γ is then a compact solvmanifold endowed with an invariant locally conformal calibrated G_2-structure φ.

Further results on the existence of locally conformal calibrated G_2-structures on Lie algebras can be obtained using Lemma 3.3.16 also when the derivation D is not of the form $\rho(a)$ for some $a \in \mathfrak{sl}(3, \mathbb{C})$. In detail

Proposition 3.3.20 ([71]). *Let $\hat{\mathfrak{g}}$ be a six-dimensional Lie algebra admitting a coupled* $\mathrm{SU}(3)$*-structure (ω, ψ_+) with coupled constant $c \in \mathbb{R} - \{0\}$ and let D be a derivation of $\hat{\mathfrak{g}}$. Consider the seven-dimensional Lie algebra $\mathfrak{g} = \hat{\mathfrak{g}} \oplus_D \mathbb{R}\xi$ with Lie bracket given by* (3.16) *and Chevalley-Eilenberg differential d. Then, the 3-form*

$$\varphi = \omega \wedge \eta + \psi_+$$

defines a G_2-structure on $\mathfrak{g}$. Moreover, there exists $\beta \in \Lambda^3(\hat{\mathfrak{g}}^*)$ such that $d\psi_+ = \beta \wedge \eta$ and

i) *$\beta = 0$ if and only if φ is locally conformal calibrated with Lee form $\theta = c\eta$.*

ii) $\beta = -2\,c\,\psi_+$ *if and only if* φ *is locally conformal calibrated with Lee form* $\theta = -c\eta$.

Proof. We already know from the proof of Proposition 3.3.17 that φ is a G_2-structure on $\mathfrak{g} = \hat{\mathfrak{g}} \oplus_D \mathbb{R}\xi$. By Lemma 3.3.16, there exists $\beta \in \Lambda^3(\hat{\mathfrak{g}}^*)$ such that

$$d\psi_+ = \hat{d}\psi_+ + \beta \wedge \eta = \beta \wedge \eta,$$

since $\hat{d}\psi_+ = 0$. Moreover,

$$\begin{aligned} d\varphi &= d\omega \wedge \eta + d\psi_+ &&= \hat{d}\omega \wedge \eta + \beta \wedge \eta \\ &= c\,\psi_+ \wedge \eta + \beta \wedge \eta &&= -c\eta \wedge \varphi + \beta \wedge \eta \end{aligned} \tag{3.20}$$

and from this the first point follows immediately.

Let us now prove the second point. First, suppose that $\beta = -2\,c\,\psi_+$. Then, from (3.20) we obtain

$$d\varphi = -c\eta \wedge \varphi + \beta \wedge \eta = -(-c\eta) \wedge \varphi.$$

Conversely, if $d\varphi = -\theta \wedge \varphi$ with $\theta = -c\eta$, then from (3.20) we get

$$c\eta \wedge \varphi = d\varphi = -c\eta \wedge \varphi + \beta \wedge \eta,$$

which implies $\beta = -2\,c\,\psi_+$. □

3.4 Einstein locally conformal calibrated G_2-structures

From the results recalled in Section 3.1.3, we know that calibrated G_2-structures inducing an Einstein metric cannot exist on compact 7-manifolds [30, 47] and that the same holds true in the noncompact case for left-invariant calibrated G_2-structures inducing an Einstein non-flat metric on solvmanifolds [65]. It is then natural to ask whether these results extend to manifolds endowed with an *Einstein locally conformal calibrated* G_2-structure, that is, a G_2-structure of type $\mathcal{X}_2 \oplus \mathcal{X}_4$ whose underlying metric is Einstein.

A useful tool to study the problem is the *conformal Yamabe constant*, let us recall its definition.

Definition 3.4.1. Let (M, g) be a Riemannian manifold of dimension $m \geq 3$, let $a_m := \frac{4(m-1)}{m-2}$, $p_m := \frac{2m}{m-2}$ and let $C_c^\infty(M)$ denote the set of compactly supported smooth real valued functions on M. Then, the *conformal Yamabe constant* of (M, g) is

$$Q(M,g) := \inf_{u \in C_c^\infty(M), u \not\equiv 0} \left\{ \frac{\int_M \left(a_m |du|^2 + u^2 \operatorname{Scal}(g)\right) dV_g}{\left(\int_M |u|^{p_m} dV_g\right)^{\frac{2}{p_m}}} \right\}.$$

The sign of $Q(M, g)$ is a conformal invariant. In particular, the following characterization holds.

Proposition 3.4.2 ([166])**.** *Let (M, g) be a compact Riemannian manifold of dimension $m \geq 3$. Then, $Q(M, g)$ is negative/zero/positive if and only if g is conformal to a Riemannian metric of negative/zero/positive scalar curvature.*

Moreover, it is possible to show the

Proposition 3.4.3 ([132])**.** *Let (M, g) be a complete Riemannian manifold of nonpositive scalar curvature. If the volume of M is finite, then $Q(M, g) \leq 0$.*

We are now ready to prove the first result.

Theorem 3.4.4 ([71])**.** *Let M be a seven-dimensional compact manifold endowed with an Einstein locally conformal calibrated* G_2*-structure φ. Then,* $\operatorname{Scal}(g_\varphi) \leq 0$*. Moreover, if M is connected,* $\operatorname{Scal}(g_\varphi)$ *is either zero or negative.*

Proof. Suppose that $\operatorname{Scal}(g_\varphi) > 0$, then the Lee form $\theta \in \Omega^1(M)$ is exact. Indeed, since $d\theta = 0$, we can consider the de Rham class $[\theta] \in H^1_{\mathrm{dR}}(M)$ and take the harmonic 1-form α representing $[\theta]$, that is, $\theta = \alpha + df$, where $\Delta\alpha = (dd^* + d^*d)\alpha = 0$ and $f \in C^\infty(M)$. α has to vanish everywhere on M, since it is compact, oriented and has positive Ricci curvature (cf. [24]). Then, $\theta = df$. Let us consider $\widehat{\varphi} := e^f \varphi$, by Lemma 3.3.6 and the discussion preceding it, we know that $\widehat{\varphi}$ is a closed G_2-structure on M with associated Riemannian metric $g_{\widehat{\varphi}} = e^{\frac{2}{3}f} g_\varphi$ conformal to the metric g_φ of positive scalar curvature. Consequently, the conformal Yamabe constant $Q(M, g_{\widehat{\varphi}})$ is positive by Proposition 3.4.2. Since $(M, g_{\widehat{\varphi}})$ is compact, it has finite volume and is complete. Moreover, it has nonpositive scalar curvature by Proposition 3.1.13. Therefore, by Proposition 3.4.3, we have $Q(M, g_{\tilde{\varphi}}) \leq 0$, which is in contrast with the previous result. □

As a consequence, we obtain the

Corollary 3.4.5 ([71]). *A seven-dimensional compact, homogeneous manifold M cannot admit any invariant Einstein locally conformal calibrated* G_2*-structure φ, unless the underlying metric g_φ is flat.*

Proof. A homogeneous Einstein manifold with negative scalar curvature is noncompact by [22, Thm. 7.56]. Thus, every seven-dimensional compact, homogeneous manifold M with an invariant G_2-structure φ whose associated metric is Einstein has $\mathrm{Scal}(g_\varphi) \geq 0$. Combining this with the previous theorem, we have $\mathrm{Scal}(g_\varphi) = 0$ and, in particular, g_φ is Ricci-flat. The statement then follows recalling that in the homogeneous case Ricci-flatness implies flatness [6]. □

That being so, in the compact homogeneous case there are no invariant locally conformal calibrated G_2-structures whose underlying metric is Einstein non-flat. We show now that this is not true in the noncompact case, providing an example of a left-invariant Einstein (non-flat) locally conformal calibrated G_2-structure on a seven-dimensional solvmanifold. This tells us, in particular, that the aforementioned result of [65] does not extend to the case of locally conformal calibrated G_2-structures.

Example 3.4.6 ([71]). Consider the six-dimensional nilpotent Lie algebra

$$\mathfrak{n}_{28} = (0, 0, 0, 0, e^{13} - e^{24}, e^{14} + e^{23})$$

endowed with the coupled SU(3)-structure

$$\omega = e^{12} + e^{34} - e^{56}, \quad \psi_+ = e^{136} - e^{145} - e^{235} - e^{246},$$

whose coupled constant is $c = -1$. As we observed in the proof of Theorem 2.4.12, the inner product $g = \sum_{k=1}^{6} (e^k)^2$ induced by (ω, ψ_+) is a nilsoliton (cf. Section 1.4.4) with Ricci operator

$$\mathrm{Rc}(g) = -3I + 4\,\mathrm{diag}\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1, 1\right),$$

where $D = \mathrm{diag}\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1, 1\right)$ is a symmetric derivation of $\mathfrak{n}_{28}$. By Lauret's result recalled in Proposition 1.4.15, the metric rank-one solvable extension $\mathfrak{s} = \mathfrak{n}_{28} \oplus \mathbb{R}e_7$

of $\mathfrak{n}_{28}$ with structure equations

$$\left(\frac{1}{2}e^{17}, \frac{1}{2}e^{27}, \frac{1}{2}e^{37}, \frac{1}{2}e^{47}, e^{13} - e^{24} + e^{57}, e^{14} + e^{23} + e^{67}, 0\right)$$

is endowed with the Einstein inner product $g+(e^7)^2$. This is exactly the inner product g_φ induced by the 3-form

$$\varphi = \omega \wedge e^7 + \psi_+,$$

which defines a locally conformal calibrated G_2-structure on $\mathfrak{s}$ with Lee form $\theta = e^7$ by the second point of Proposition 3.3.20, as $d\psi_+ = 2\,\psi_+ \wedge e^7$. A simple computation shows that the non-vanishing intrinsic torsion forms of φ are

$$\tau_1 = -\frac{1}{3}e^7, \quad \tau_2 = -\left(\frac{5}{3}e^{12} + \frac{5}{3}e^{34} + \frac{10}{3}e^{56}\right).$$

Moreover, φ is not $*$-Einstein, since by direct computation with respect to the orthonormal basis $(e_1, \ldots, e_7)$, we get the following expression of the $*$-Ricci tensor (cf. (3.9))

$$\rho^* = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 22 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 22 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -6 \end{pmatrix}.$$

Summarizing, φ gives rise to a left-invariant Einstein (non-flat) locally conformal calibrated G_2-structure on the simply connected solvable Lie group S with Lie algebra $\mathfrak{s}$, which is not unimodular ($\mathrm{tr}(\mathrm{ad}_{e_7}) \neq 0$) and, so, does not admit any compact quotient by [145, Lemma 6.2].

We can also give an example of noncompact homogeneous manifold admitting an Einstein (non-flat) locally conformal parallel G_2-structure.

Example 3.4.7 ([71])**.** The Einstein rank-one solvable extension of the Abelian Lie algebra $\mathfrak{a} = (0, 0, 0, 0, 0, 0)$ of dimension six is the solvable Lie algebra $\mathfrak{e}$ with structure equations

$$\left(ae^{17}, ae^{27}, ae^{37}, ae^{47}, ae^{57}, ae^{67}, 0\right),$$

where a is a nonzero real number. The inner product on $\mathfrak{e}$

$$g = \sum_{k=1}^{7} (e^k)^2$$

is Einstein with Ricci tensor given by $\mathrm{Ric}(g) = -6a^2\, g$. The 3-form

$$\varphi = -e^{125} - e^{136} - e^{147} + e^{237} - e^{246} + e^{345} - e^{567}$$

defines a G_2-structure on $\mathfrak{e}$ with G_2-basis $\left(-e^1, e^4, e^2, e^3, e^5, -e^6, e^7\right)$ and such that $g_\varphi = g$. From the expressions

$$\begin{aligned} d\varphi &= 3a\left(-e^{2467} + e^{3457} - e^{1257} - e^{1367}\right), \\ d *_\varphi \varphi &= 4a\left(e^{23567} + e^{12347} - e^{14567}\right), \end{aligned}$$

it is immediate to see that $\tau_1 = -ae^7$ and $\tau_0 = 0, \tau_2 = 0, \tau_3 = 0$. Then, the G_2-structure φ is locally conformal parallel.

Starting from a 6-manifold endowed with a suitable coupled SU(3)-structure inducing an Einstein metric, it is possible to construct a noncompact manifold endowed with a locally conformal calibrated G_2-structure inducing a Ricci-flat metric. To our knowledge, the next example is the first of this kind.

Example 3.4.8 ([70])**.** Let us consider the coupled Einstein SU(3)-structure (ω, ψ_+) obtained on the twistor space $\mathcal{Z}$ in Section 2.5.2. First of all, we rescale it in the following way

$$\widehat{\omega} = \frac{8}{5}\,\omega, \quad \widehat{\psi}_+ = \left(\frac{8}{5}\right)^{\frac{3}{2}} \psi_+.$$

Then, by Lemma 2.4.5, the pair $\widehat{\omega}, \widehat{\psi}_+$ is a coupled SU(3)-structure with coupled constant $\widehat{c} = -\sqrt{10}$ and inducing the metric $\widehat{g} = \frac{8}{5}\,g$. Consequently, since g is Einstein with Einstein constant 48, we have $\mathrm{Scal}(\widehat{g}) = 30$ and $\mathrm{Ric}(\widehat{g}) = 5\,\widehat{g}$.

If we consider the G_2-structure φ induced on the cone $\mathcal{C}(\mathcal{Z})$ by $\widehat{\omega}, \widehat{\psi}_+$, then the metric $g_\varphi = t^2\,\widehat{g} + dt^2$ is Ricci-flat by Proposition 3.2.5. Moreover, by Proposition 3.3.8, the only non-identically vanishing intrinsic torsion forms of the G_2-structure are

$$\tau_1 = \frac{3+\sqrt{10}}{3\,t}\,dt, \quad \tau_2 = -t\,w_2^-.$$

Therefore, the coupled Einstein SU(3)-structure $\widehat{\omega}, \widehat{\psi}_+$ induces a locally conformal calibrated G$_2$-structure on the cone $\mathcal{C}(\mathcal{Z})$ whose underlying metric is Ricci-flat.

Remark 3.4.9. It is worth observing here that calibrated G$_2$-structures inducing a Ricci-flat metric are actually parallel as a consequence of Proposition 3.1.13. The previous example shows that a result of this kind is not true anymore for locally conformal calibrated G$_2$-structures.

Since the sine-cone over an m-dimensional Einstein manifold with Einstein constant $(m-1)$ is still Einstein by Proposition 3.2.6, on the sine-cone over the coupled Einstein manifold of the previous example there exists a G$_2$-structure inducing an Einstein metric. Its G$_2$-type is described in the next example.

Example 3.4.10. Let $\widehat{\omega}, \widehat{\psi}_+$ be the coupled Einstein SU(3)-structure on $\mathcal{Z}$ considered in Example 3.4.8. Then, the Riemannian metric g_φ underlying the G$_2$-structure

$$\varphi = \sin^2(t)\,\widehat{\omega} \wedge dt + \sin^3(t)\cos(t)\,\widehat{\psi}_+ - \sin^4(t)\,\widehat{\psi}_-$$

on the sine-cone $\mathcal{SC}(\mathcal{Z})$ is Einstein by Proposition 3.2.6.

A long but straightforward computation gives the following expressions for the intrinsic torsion forms of the G$_2$-structure φ induced on the sine-cone by a coupled SU(3)-structure (ω, ψ_+) with coupled constant c and possibly non-identically vanishing w_2^-:

$$\begin{aligned}
\tau_0 &= \frac{8c+4}{7},\\
\tau_1 &= \left(1 - \frac{c}{3}\right)\cot(t)dt,\\
\tau_2 &= -\frac{\sin(2t)}{2} w_2^-,\\
\tau_3 &= \frac{c-3}{7}\left(\sin^4(t)\psi_- - \sin^3(t)\cos(t)\psi_+ + \frac{4}{3}\sin^2(t)dt\wedge\omega\right) - \sin^2(t)dt \wedge w_2^-.
\end{aligned}$$

It is possible to crosscheck this result computing $d\varphi$ and $d *_\varphi \varphi$ and comparing them with the expression of the right-hand side of (3.8) when the differential forms appearing there are those written above.

Thus, since in our case $c = -\sqrt{10}$ and $w_2^- \neq 0$, the sine-cone $\mathcal{SC}(\mathcal{Z})$ is endowed with a G$_2$-structure of type $\mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \mathcal{X}_3 \oplus \mathcal{X}_4$ and inducing an Einstein metric.

Moreover, observe that from the above expressions of the intrinsic torsion forms we get that a nearly Kähler SU(3)-structure with $w_1^- = -2$ induces a nearly parallel G_2-structure with $\tau_0 = 4$ on the sine-cone, recovering a result of [23].

3.5 A structure result

In Proposition 3.3.11, we proved that, given a compact, connected 6-manifold $\overline{M}$ endowed with a coupled SU(3)-structure (ω, ψ_+) preserved by a diffeomorphism ν of $\overline{M}$, there exists a locally conformal calibrated G_2-structure $\widetilde{\varphi}$ on the compact mapping torus $\overline{M}_\nu$ of ν. Moreover, we observed that the Lee form of $\widetilde{\varphi}$ is $c\eta$, where c is the coupled constant of (ω, ψ_+) and η is the characteristic 1-form of $\overline{M}_\nu$, $\mathcal{L}_\xi \widetilde{\varphi} = 0$ and the vector field ξ is the $g_{\widetilde{\varphi}}$-dual of η. In addition to this, we emphasize here also that the fibers of the fibration $\pi : \overline{M}_\nu \to S^1$ are compact 6-manifolds endowed with a coupled SU(3)-structure.

A natural question arising from this result is whether it is possible to find a converse and, more precisely, under which conditions a compact, connected 7-manifold M endowed with a locally conformal calibrated G_2-structure is fibered over S^1 with fibers endowed with a coupled SU(3)-structure. Our purpose in this section is to find a solution to this problem.

Similar problems have been studied in literature before. For instance, in [13] Banyaga showed that special types of exact locally conformal symplectic manifolds are fibered over S^1 with each fiber carrying a contact form. In this context, *exact* means *d_θ-exact* in the sense of the following

Definition 3.5.1. Let M be a manifold and consider a closed 1-form θ on it. A differential form $\alpha \in \Omega^k(M)$ is said to be *d_θ-exact* if there exists some $\beta \in \Omega^{k-1}(M)$ such that

$$\alpha = d\beta + \theta \wedge \beta =: d_\theta \beta.$$

Examples of exact locally conformal symplectic structures are given by those called *of the first kind* in Vaisman's paper [176], where the author proved that a manifold M^{2n} endowed with such a structure is a 2-contact manifold and has a vertical two-dimensional foliation. Moreover, when this foliation is regular, he showed that M^{2n} is a T^2-principal bundle over a symplectic manifold.

More in general, by [12, Prop. 3.3], every compact manifold of dimension $2k+2$ admitting a generalized contact pair of type $(k,0)$, that is, a pair of 1-forms (α,β) such that $\alpha\wedge(d\alpha)^k\wedge\beta$ is a volume form, $d\beta=0$ and $(d\alpha)^{k+1}=0$, fibers over the circle with fiber a contact manifold and the monodromy acting by a contactomorphism. Conversely, every mapping torus of a contactomorphism admits a generalized contact pair of type $(k,0)$ and an induced locally conformal symplectic form. Note also that a contact pair (α,β) of type $(k,0)$ gives rise to a locally conformal symplectic form defined by $d\alpha+\alpha\wedge\beta$.

In [133], Li proved that odd-dimensional co-symplectic and co-Kähler manifolds can be characterized as mapping tori over symplectic and Kähler manifolds, respectively.

Finally, a characterization of compact locally conformal parallel G_2-manifolds as fiber bundles over S^1 with compact nearly Kähler fiber was obtained in [104] (see also [177]). It was also shown there that for compact seven-dimensional locally conformal parallel G_2-manifolds (M,φ) with co-closed Lee form θ, the Lee flow preserves the Gauduchon G_2-structure, i.e., $\mathcal{L}_{\theta^\sharp}\varphi=0$, where $\theta^\sharp$ is the dual of θ with respect to g_φ.

3.5.1 d_θ-exact G_2-structures

Some of the results just recalled suggest that having a G_2-structure whose defining 3-form is exact in the sense of Definition 3.5.1 might be a good hypothesis for our aim. This observation is strenghtened by the fact that every d_θ-exact G_2-structure is locally conformal calibrated.

Proposition 3.5.2. *Let M be a 7-manifold endowed with a G_2-structure φ which is d_θ-exact for a certain closed 1-form θ on M. Then, φ is locally conformal calibrated with Lee form θ.*

Proof. We know that $\varphi=d_\theta\beta=d\beta+\theta\wedge\beta$ for some $\beta\in\Omega^2(M)$. Then,

$$d\varphi=d(d\beta+\theta\wedge\beta)=d\theta\wedge\beta-\theta\wedge d\beta=-\theta\wedge\varphi$$

and the assertion is proved. □

We already encountered an example of d_θ-exact G_2-structure, namely that constructed on the mapping torus in the proof of Proposition 3.3.11 and recalled at the beginning of this section. Indeed, it satisfies

$$\widetilde{\varphi} = \widetilde{\omega} \wedge \eta + \widetilde{\psi}_+ = \widetilde{\omega} \wedge \eta + \frac{1}{c} d\widetilde{\omega} = d_{c\eta}\left(\frac{\widetilde{\omega}}{c}\right).$$

Remark 3.5.3. Given a locally conformal calibrated G_2-structure φ, we can consider the class

$$\left\{ e^f \varphi \mid f \in C^\infty(M) \right\}$$

of locally conformal calibrated G_2-structures which are conformally equivalent to φ. As $d\varphi = -\theta \wedge \varphi$, we have

$$d\left(e^f \varphi\right) = (df - \theta) \wedge e^f \varphi$$

and φ is d_θ-exact if and only if $e^f\varphi$ is $d_{(\theta - df)}$-exact. Thus, being d_θ-exact is a conformal property for locally conformal calibrated G_2-structures.

It is a general fact that the $\mathbb{R}$-linear map

$$d_\theta : \Omega^k(M) \to \Omega^{k+1}(M), \quad d_\theta \beta = d\beta + \theta \wedge \beta,$$

satisfies the property $d_\theta \circ d_\theta = 0$ when θ is a closed 1-form. Thus, $\{\Omega^\cdot(M), d_\theta\}$ is a differential complex and gives rise to the cohomology groups

$$H^k_\theta(M) = \ker\left[d_\theta : \Omega^k(M) \to \Omega^{k+1}(M)\right] \Big/ \operatorname{Im}\left[d_\theta : \Omega^{k-1}(M) \to \Omega^k(M)\right],$$

which are conformal invariants of a locally conformal calibrated G_2-manifold (M, φ) with Lee form θ as a consequence of Remark 3.5.3. Moreover, when φ is locally conformal calibrated with $d\varphi = -\theta \wedge \varphi$, it is immediate to check that $d_\theta \varphi = 0$. Then, the obstruction for φ to be d_θ-exact is represented by the group $H^3_\theta(M)$, meaning that φ is d_θ-exact when $H^3_\theta(M) = \{0\}$, while it might not be d_θ-exact otherwise.

Now, we look for conditions guaranteeing the d_θ-exactness of a locally conformal calibrated G_2-structure with Lee form θ.

Recall that a vector field X on M is a *conformal infinitesimal automorphism* of φ if an only if there exists a smooth function $h_X \in C^\infty(M)$ such that $\mathcal{L}_X \varphi = h_X \varphi$. If h_X is identically zero, then X is an *infinitesimal automorphism* of φ. We start proving the following

Lemma 3.5.4 ([66]). *Let (M, φ) be a locally conformal calibrated* G_2*-manifold with Lee form θ. A vector field X on M is a conformal infinitesimal automorphism of φ if and only if there exists a smooth function $f_X \in C^\infty(M)$ such that $d_\theta \sigma = f_X \varphi$, where $\sigma := \iota_X \varphi$. Moreover, if M is connected, f_X is constant.*

Proof. Let us compute the expression of the Lie derivative of φ with respect to X

$$\begin{aligned}
\mathcal{L}_X \varphi &= d(\iota_X \varphi) + \iota_X(d\varphi) \\
&= d\sigma + \iota_X(-\theta \wedge \varphi) \\
&= d\sigma - \theta(X)\varphi + \theta \wedge (\iota_X \varphi) \\
&= d\sigma + \theta \wedge \sigma - \theta(X)\varphi \\
&= d_\theta \sigma - \theta(X)\varphi,
\end{aligned}$$

where $\sigma := \iota_X \varphi$. Therefore, X is a conformal infinitesimal automorphism of φ with $\mathcal{L}_X \varphi = h_X \varphi$ if and only if $d_\theta \sigma = f_X \varphi$, where f_X is a smooth real valued function on M such that $f_X = h_X + \theta(X)$.

Suppose now that M is connected and let X be a conformal infinitesimal automorphism of φ. We have just shown that $d_\theta \sigma = f_X \varphi$ for some $f_X \in C^\infty(M)$. Using the general property $d_\theta \circ d_\theta = 0$, we have

$$\begin{aligned}
0 &= d_\theta(d_\theta \sigma) \\
&= d_\theta(f_X \varphi) \\
&= d(f_X \varphi) + \theta \wedge (f_X \varphi) \\
&= df_X \wedge \varphi + f_X d\varphi + f_X(\theta \wedge \varphi) \\
&= df_X \wedge \varphi + f_X d\varphi - f_X d\varphi \\
&= df_X \wedge \varphi.
\end{aligned}$$

By (3.5), we obtain $df_X = 0$ and from this the assertion follows. □

Corollary 3.5.5. *If $X \in \mathfrak{X}(M)$ is a conformal infinitesimal automorphism of a locally conformal calibrated* G_2*-structure φ with f_X a nonzero constant, then φ is d_θ-exact. Indeed,*

$$\varphi = \frac{1}{f_X} d_\theta \sigma = d_\theta \left(\frac{\sigma}{f_X} \right).$$

Recall the integral identity shown in [135].

Lemma 3.5.6 ([135])**.** *Let M be a seven-dimensional compact manifold. Then, for every G_2-structure φ on M, every vector field X on M and $f \in C^\infty(M)$, it holds*

$$\int_M \mathcal{L}_X\varphi \wedge *_\varphi f\varphi = -3\int_M df \wedge *_\varphi X^\flat. \tag{3.21}$$

From (3.21) with f identically equal to 1 and X conformal infinitesimal automorphism of φ with $\mathcal{L}_X\varphi = h_X\varphi$, we have

$$\int_M h_X dV_\varphi = 0.$$

Thus, thinking at the proof of Lemma 3.5.4, we get

$$\int_M \theta(X) dV_\varphi = \int_M f_X dV_\varphi = f_X \mathrm{Vol(M)},$$

which proves the following

Lemma 3.5.7. *Let (M,φ) be a compact, connected locally conformal calibrated G_2-manifold with Lee form θ and let $X \in \mathfrak{X}(M)$ be a conformal infinitesimal automorphism of φ. Then, the Riemannian integral of the function $\theta(X)$ over M is constant.*

In conclusion, we can show a characterization for d_θ-exact locally conformal calibrated G_2-structures.

Proposition 3.5.8 ([66])**.** *Let (M,φ) be a connected locally conformal calibrated G_2-manifold with non-vanishing Lee form θ. Let $X = \theta^\sharp$ be the g_φ-dual vector field of θ, i.e., $\theta(\cdot) = g_\varphi(X,\cdot)$, and define the 2-form $\sigma := \iota_X\varphi$. Then, $\mathcal{L}_X\varphi = 0$ if and only if $\theta(X)\varphi = d_\theta\sigma$. Moreover, if $\mathcal{L}_X\varphi = 0$, then $\theta(X) = |X|^2$ is a nonzero constant.*

Proof. We have

$$\begin{aligned} \mathcal{L}_X\varphi &= d(\iota_X\varphi) + \iota_X d\varphi \\ &= d\sigma + \iota_X(-\theta\wedge\varphi) \\ &= d\sigma - \theta(X)\varphi + \theta\wedge\sigma. \end{aligned}$$

Therefore, $\mathcal{L}_X\varphi = 0$ if and only if $\theta(X)\varphi = d_\theta\sigma$.

If $\mathcal{L}_X\varphi = 0$, from Lemma 3.5.4 we have that $\theta(X) = |X|^2$ is a nonzero constant, since $\theta(X)\varphi = d_\theta\sigma$ and $X = \theta^\sharp$, where the map $\cdot^\sharp : \Omega^1(M) \to \mathfrak{X}(M)$ is an isomorphism. □

Remark 3.5.9. Notice that the locally conformal calibrated G_2-structure on the mapping torus studied in Proposition 3.3.11 satisfies the previous characterization. It is then quite natural to presume that $\mathcal{L}_{\theta^\sharp}\varphi = 0$ with θ non-vanishing might be the right hypothesis to find a solution to the problem we are studying.

The examples given at the end of Section 3.3.1 are useful to understand better how restrictive is the situation described in the previous result. Indeed, they allow us to conclude that locally conformal calibrated G_2-structures satisfying the characterization of Proposition 3.5.8 constitute a subset of the set of d_θ-exact G_2-structures. In detail:

Example 3.5.10. Consider the seven-dimensional Lie algebra introduced in Example 3.3.18

$$\mathfrak{s} = \left(e^{37}, e^{47}, -e^{17}, -e^{27}, e^{14} + e^{23}, e^{13} - e^{24}, 0\right).$$

It admits a locally conformal calibrated G_2-structure defined by

$$\varphi = e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245},$$

and whose Lee form is $\theta = -e^7$. φ gives rise to an invariant locally conformal calibrated G_2-structure on the compact solvmanifold S/Γ, where S is the simply connected solvable Lie group with Lie algebra $\mathfrak{s}$ and Γ is the lattice (3.19). If $X = -e_7$ denotes the g_φ-dual vector field of the Lee form $\theta = -e^7$, then it is easy to check that $\mathcal{L}_X\varphi = 0$ and $\varphi = d_\theta\sigma$, where $\sigma = i_X\varphi$. Thus, S/Γ is an example of manifold satisfying the results of Proposition 3.5.8.

Example 3.5.11. The seven-dimensional Lie algebra introduced in Example 3.3.19

$$\mathfrak{q} = \left(e^{37}, e^{47}, 2e^{17}, 2e^{27}, e^{14} + e^{23}, e^{13} - e^{24}, 0\right)$$

is endowed with the locally conformal calibrated G_2-structure

$$\varphi = e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245}$$

with Lee form $\theta = -e^7$. If $X = -e_7$ denotes the g_φ-dual vector field of θ, a simple computation shows that $\mathcal{L}_X\varphi \neq 0$ and, according to Proposition 3.5.8, $\varphi \neq d_\theta(\iota_X\varphi)$. However, φ is d_θ-exact. Indeed, $\varphi = d_\theta\gamma$, where

$$\gamma = \frac{5}{7}e^{12} - \frac{3}{7}e^{14} + \frac{3}{7}e^{23} - \frac{1}{7}e^{34} - e^{56}.$$

3.5.2 The main theorem

Let us now consider a seven-dimensional compact, connected manifold M endowed with a locally conformal calibrated G_2-structure φ. We shall show that when the Lee form θ is non-vanishing and $\mathcal{L}_{\theta^\sharp}\varphi = 0$, the manifold M is fibered over S^1 and each fiber is endowed with a coupled SU(3)-structure.

We begin recalling some known results which we will use in the proof. The first one follows immediately from the discussion of Section 3.2.1.

Proposition 3.5.12 ([53]). *Let V be a seven-dimensional real vector space endowed with a* G_2*-structure φ inducing the inner product g_φ. Moreover, let $\mathbf{n} \in V$ be a unit vector with $g_\varphi(\mathbf{n},\mathbf{n}) = 1$ and let $W := \langle \mathbf{n}\rangle^\perp$ denote the g_φ-orthogonal complement of the subspace $\langle \mathbf{n}\rangle \subset V$. Then, the pair (ω, ψ_+) defined by*

$$\omega = (\iota_{\mathbf{n}}\varphi)\,|_W \ , \quad \psi_+ = \varphi\,|_W$$

is an SU(3)*-structure on W inducing the inner product $g = g_\varphi\,|_W$.*

A result due to Tischler [174] characterizes compact manifolds which are fibered over the circle.

Theorem 3.5.13 ([174]). *Let M be a compact manifold of dimension m. Then, M is the total space of a fiber bundle over the circle if and only if there exists a non-vanishing closed 1-form on it.*

Proof. We give an idea of the proof focusing only on the results which are of interest for us. The reader can refer to [174] for the details.

If $\pi : M \to S^1$ is a fiber bundle and t denotes the angle coordinate on S^1, then the pullback $\pi^*(dt)$ defines a non-vanishing closed 1-form on M.

Conversely, let $\theta \in \Omega^1(M)$ be a non-vanishing closed 1-form on M. Since M is compact, θ is not exact and its de Rham cohomology class $[\theta] \in H^1_{\mathrm{dR}}(M)$ is nonzero. Let $\alpha_1, \ldots, \alpha_k$ be closed 1-forms on M defining a basis $\{[\alpha_1], \ldots, [\alpha_k]\}$ of $H^1_{\mathrm{dR}}(M)$. We can write

$$\theta = \sum_{i=1}^{k} x_i \alpha_i + dh$$

for certain $x_i \in \mathbb{R}$ and $h \in C^\infty(M)$. S^1 is an Eilenberg-Maclane space, as it has only one non-trivial homotopy group, namely $\pi_1(S^1) \cong \mathbb{Z}$. In this case, there is a bijection between the set of homotopy classes of maps from M into the circle and $H^1(M, \mathbb{Z})$, i.e., the first singular cohomology group of M with coefficients in $\mathbb{Z}$. Using this fact and the de Rham isomorphisms of M and S^1, it is possible to obtain k smooth functions $f_i : M \to S^1$, $1 \leq i \leq k$, such that $f_i^*(dt) = \alpha_i + dh_i$, where $h_i \in C^\infty(M)$. Then,

$$\theta = \sum_{i=1}^{k} x_i f_i^*(dt) + \sum_{i=1}^{k} x_i dh_i + dh$$

and the last two summands can be absorbed in the first one, since for every smooth function $f : M \to S^1$ and $h \in C^\infty(M)$ it holds

$$f^*(dt) + dh = (f + \Pi \circ h)^* (dt),$$

where $\Pi : \mathbb{R} \to S^1$ is the universal covering map and the addition in the right-hand side of the identity is induced by the group structure on S^1. Now, for an appropriate choice of rational numbers $\frac{n_i}{q}$, $1 \leq i \leq k$, the quantity

$$\left| \theta - \frac{1}{q} \sum_{i=1}^{k} n_i f_i^*(dt) \right|$$

can be made arbitrarily small, where the norm $|\cdot|$ is induced by some Riemannian metric on M (cf. Remark 1.2.6). Consequently, the closed 1-form with integral periods

$$\widehat{\theta} := \sum_{i=1}^{k} n_i f_i^*(dt)$$

is non-vanishing. Since $n_i \in \mathbb{Z}$, the smooth map $f : M \to S^1$ given by $f = \sum_{i=1}^{k} n_i f_i$ is well-defined and satisfies $f^*(dt) = \widehat{\theta}$. f is then a smooth submersion between the compact manifold M and the connected manifold S^1. Thus, by Ehresmann's result [62, Prop. p. 154], f is a fiber map whose fibers are compact and connected. □

Now, we state some further lemmas which will be also useful in the proof of the theorem.

Lemma 3.5.14. *Let (M, g) be a Riemannian manifold and consider two differential forms $\alpha \in \Omega^1(M), \kappa \in \Omega^2(M)$. Then,*

$$|\alpha \wedge \kappa|^2 = 3\,|\alpha|^2|\kappa|^2 - 6\,|\gamma|^2,$$

where $|\cdot|$ is the pointwise norm induced by g and $\gamma \in \Omega^1(M)$ is defined locally as $\gamma = \gamma_i dx^i$, $\gamma_i = g^{rj}\alpha_r\kappa_{ji}$. From this follows

$$|\alpha \wedge \kappa|^2 \leq 3|\alpha|^2|\kappa|^2.$$

When M is compact, with respect to the L^2-norm $\|\cdot\|$ induced by the L^2-inner product of differential forms $\langle \alpha, \beta \rangle = \int_M \alpha \wedge *\beta = \int_M g(\alpha, \beta) * 1$, we then have

$$\|\alpha \wedge \kappa\|^2 \leq 3 \int_M |\alpha|^2|\kappa|^2 * 1.$$

Proof. Using the conventions introduced in Section 1.1.1, in local coordinates we have

$$(\alpha \wedge \kappa)_{ijr} = \alpha_i\kappa_{jr} - \alpha_j\kappa_{ir} + \alpha_r\kappa_{ij}.$$

We can now start with the computations:

$$\begin{aligned}
|\alpha \wedge \kappa|^2 &= (\alpha \wedge \kappa)_{ijr} g^{ia} g^{jb} g^{rc} (\alpha \wedge \kappa)_{abc} \\
&= 3\,\alpha_i\kappa_{jr} g^{ia} g^{jb} g^{rc} \alpha_a\kappa_{bc} - 6\,\alpha_i\kappa_{jr} g^{ia} g^{jb} g^{rc} \alpha_b\kappa_{ac} \\
&= 3(\alpha_i g^{ia}\alpha_a)(\kappa_{jr} g^{jb} g^{rc}\kappa_{bc}) - 6\,(g^{ia}\alpha_i\kappa_{ac}) g^{cr} (g^{bj}\alpha_b\kappa_{jr}) \\
&= 3\,|\alpha|^2|\kappa|^2 - 6\,\gamma_c g^{cr}\gamma_r \\
&= 3\,|\alpha|^2|\kappa|^2 - 6\,|\gamma|^2.
\end{aligned}$$

□

For manifolds endowed with a G_2-structure, we can prove the following

Lemma 3.5.15. *Let M be a 7-manifold endowed with a G_2-structure φ. Consider a vector field $X \in \mathfrak{X}(M)$ and define the 2-form $\sigma := \iota_X\varphi$. Then,*

$$|\sigma|^2 = 3|X|^2,$$

where $|\cdot|$ is the norm induced by g_φ.

Proof. Using the identity $\varphi \wedge (\iota_X\varphi) = 2 *_\varphi (\iota_X\varphi)$ (see [111] for a proof), we have

$$|\sigma|^2 *_\varphi 1 = \sigma \wedge *_\varphi\sigma = \frac{1}{2}\sigma \wedge \varphi \wedge \sigma = \frac{1}{2}(\iota_X\varphi) \wedge (\iota_X\varphi) \wedge \varphi = 3|X|^2 *_\varphi 1.$$

□

Finally, we show the following result on vector spaces

Lemma 3.5.16. *Let V be a real vector space of dimension m endowed with an inner product g inducing the norm $|v| = g(v,v)^{\frac{1}{2}}$, $v \in V$. Consider two vector subspaces $W_1, W_2 \subset V$ of dimension $m-1$ defined as the g-orthogonal complement of two unit vectors $\mathbf{n}_1$ and $\mathbf{n}_2$, respectively. If the angle between $\mathbf{n}_1$ and $\mathbf{n}_2$ is close to zero, then the subspaces W_1 and W_2 thought as points in the Grassmannian $\mathrm{Gr}(m-1,V)$ are close with respect to the distance induced by the operator norm*

$$\begin{aligned} d_{\mathrm{Gr}(m-1,V)}(W_1,W_2) &= \left\|\mathrm{pr}_{W_1} - \mathrm{pr}_{W_2}\right\|_{\mathrm{op}} \\ &= \sup_{v\in V,\ |v|=1} \left\{|\mathrm{pr}_{W_1}v - \mathrm{pr}_{W_2}v|\right\}, \end{aligned}$$

where $\mathrm{pr}_{W_i} : V \to W_i$ is the projection $\mathrm{pr}_{W_i}v = v - g(v,\mathbf{n}_i)\mathbf{n}_i$, $i = 1,2$. Moreover, there exists an invertible linear map $a : V \to V$ which is close to the identity with respect to the operator norm and satisfies $a\,|_{W_1} : W_1 \to W_2$.

Proof. By hypothesis, the angle between $\mathbf{n}_1$ and $\mathbf{n}_2$ is a certain $\varepsilon > 0$ which is close to zero. Then,

$$g(\mathbf{n}_1,\mathbf{n}_2) = \cos(\varepsilon),$$

since both $\mathbf{n}_1$ and $\mathbf{n}_2$ have unit norm. In particular, $\cos(\varepsilon)$ is close to one and $\sin(\varepsilon)$ is close to zero.

Let $(e_1,\ldots,e_m)$ be a g-orthonormal basis of V. Without loss of generality, we can assume that

$$\mathbf{n}_1 = e_m, \quad \mathbf{n}_2 = \sin(\varepsilon)e_{m-1} + \cos(\varepsilon)e_m.$$

Let us now compute the distance $d_{\mathrm{Gr}(m-1,V)}(W_1,W_2)$. Consider a generic unit vector v of V, we can write $v = v^ie_i$, where v^i are m real numbers such that

$$|v|^2 = \left(v^1\right)^2 + \cdots + (v^m)^2 = 1.$$

Now,

$$\mathrm{pr}_{W_1} v = v - g(v, \mathbf{n}_1)\mathbf{n}_1 \;=\; v - g(v, e_m)e_m = \sum_{i=1}^{m-1} v^i e_i,$$

and

$$\begin{aligned}\mathrm{pr}_{W_2} v &= v - g(v, \mathbf{n}_2)\mathbf{n}_2 \;=\; v - (v^{m-1}\sin(\varepsilon) + v^m \cos(\varepsilon))\mathbf{n}_2 \\ &= \sum_{i=1}^{m-2} v^i e_i + (v^{m-1} - B\sin(\varepsilon))e_{m-1} + (v^m - B\cos(\varepsilon))e_m,\end{aligned}$$

where $B := v^{m-1}\sin(\varepsilon) + v^m \cos(\varepsilon)$. Thus,

$$\begin{aligned}\left|\mathrm{pr}_{W_1} v - \mathrm{pr}_{W_2} v\right|^2 &= |B\sin(\varepsilon)e_{m-1} - (v^m - B\cos(\varepsilon))e_m|^2 \\ &= B^2\sin^2(\varepsilon) + (v^m - B\cos(\varepsilon))^2 \\ &= B^2 + (v^m)^2 - 2\,v^m\,B\cos(\varepsilon) \\ &= (v^{m-1})^2\sin^2(\varepsilon) - (v^m)^2\cos^2(\varepsilon) + (v^m)^2 \\ &= \left((v^{m-1})^2 + (v^m)^2\right)\sin^2(\varepsilon) \\ &\le |v|^2\sin^2(\varepsilon) = \sin^2(\varepsilon).\end{aligned}$$

Consequently, for every unit vector v of V, we get

$$\left|\mathrm{pr}_{W_1} v - \mathrm{pr}_{W_2} v\right| \le \sin(\varepsilon),$$

which clearly implies

$$d_{\mathrm{Gr}(m-1,V)}(W_1, W_2) = \left\|\mathrm{pr}_{W_1} - \mathrm{pr}_{W_2}\right\|_{\mathrm{op}} \le \sin(\varepsilon).$$

Since $\sin(\varepsilon)$ is close to zero, the first assertion is proved.

We have now to prove that there exists an invertible linear map $a : V \to V$ which is close to the identity with respect to the operator norm and satisfies $a\,|_{W_1} : W_1 \to W_2$. Let us consider the vector space W_1, if $(w_1, \ldots, w_{m-1})$ is g-orthonormal basis of it, then it is clear that $(w_1, \ldots, w_{m-1}, \mathbf{n}_1)$ is a g-orthonormal basis of V. Notice that if we choose $\mathbf{n}_1 = e_m$ as we did before, then, up to an orthogonal transformation, $(w_1, \ldots, w_{m-1})$ is $(e_1, \ldots, e_{m-1})$. For $i = 1, \ldots, m-1$, consider the vectors of W_2

$$z_i := \mathrm{pr}_{W_2} w_i,$$

they define a basis of W_2. Moreover, since the vectors w_i have unit norm, by the computation above we have

$$|w_i - z_i| = |\mathrm{pr}_{W_1} w_i - \mathrm{pr}_{W_2} w_i| \leq \sin(\varepsilon),$$

that is, the vectors z_i and w_i are close for every $i = 1, \dots, m-1$. The invertible linear map $a : V \to V$ is then defined by sending the basis $(w_1, \dots, w_{m-1}, \mathbf{n}_1)$ to the basis $(z_1, \dots, z_{m-1}, \mathbf{n}_2)$ in the following way

$$a : w_i \mapsto z_i, \quad a : \mathbf{n}_1 \mapsto \mathbf{n}_2.$$

The proof that $\|a - I\|_{\mathrm{op}}$ is close to zero is obtained by computations similar to those worked out previously. □

We can now prove the main result of this section.

Theorem 3.5.17. *Let M be a compact, connected seven-dimensional manifold endowed with a locally conformal calibrated* G_2*-structure φ with non-vanishing Lee form θ and such that $\mathcal{L}_X\varphi = 0$, where X is the g_φ-dual vector field of θ. Then*

i) *M is the total space of a fiber bundle over S^1 and each fiber is endowed with a coupled* $\mathrm{SU}(3)$*-structure.*

ii) *M has a locally conformal calibrated* G_2*-structure $\widehat{\varphi}$ such that its Lee form is a 1-form with integral periods.*

Proof.
i) First of all, observe that the distribution $\ker(\theta)$ is integrable, since the closed 1-form θ is nowhere vanishing. Thus, it gives rise to a foliation $\mathcal{F}_\theta$. We shall prove that the pair

$$\omega := \frac{1}{|X|}\,\iota_X\varphi, \quad \psi_+ := \frac{1}{|X|}\,d\omega$$

defines a coupled SU(3)-structure when restricted to each leaf of this foliation. To do this, at each point of the leaves of $\mathcal{F}_\theta$ we consider the tangent space and apply Proposition 3.5.12.

Under our hypothesis, we have a stable 3-form φ such that $d\varphi = -\theta\wedge\varphi$, $X = \theta^\sharp$ and $\theta(X)\varphi = d\sigma + \theta\wedge\sigma$, where $\sigma := i_X\varphi$ and $\theta(X) = |X|^2$ is a nonzero constant (see Proposition 3.5.8). Let L be a leaf of the foliation $\mathcal{F}_\theta$, then for every point p of L

$$T_pL = \ker(\theta_p) = \{Y_p \in T_pM \mid \theta_p(Y_p) = 0\} \subset T_pM$$

and, as $\theta(\cdot) = g_\varphi(X,\cdot)$, it is clear that

$$\ker(\theta) = \langle X\rangle^\perp.$$

Therefore, $T_pL = \langle X_p\rangle^\perp$ is a six-dimensional subspace of T_pM with unit normal $N_p := \frac{X_p}{|X|}$. Since φ_p defines a G_2-structure on the vector space T_pM, by Proposition 3.5.12 we have that the pair

$$\omega := \left(i_{N_p}\varphi\right)\big|_{T_pL}\,, \quad \psi_+ := \varphi\,\big|_{T_pL}$$

defines an SU(3)-structure on T_pL. Now,

$$\left(i_{N_p}\varphi\right)\big|_{T_pL} = \frac{1}{|X|}\,\sigma_p\,\big|_{T_pL}$$

and for every choice of tangent vectors $U_p, Y_p, Z_p \in T_pL$ we have

$$\begin{aligned}\varphi_p(U_p,Y_p,Z_p) &= \frac{1}{|X|^2}\,(d\sigma_p + \theta_p\wedge\sigma_p)(U_p,Y_p,Z_p)\\ &= \frac{1}{|X|^2}\,(d\sigma_p)(U_p,Y_p,Z_p),\end{aligned}$$

since θ_p evaluated on any vector of T_pL is zero. Consequently,

$$\varphi_p\,\big|_{T_pL} = \frac{1}{|X|}d\omega.$$

Summarizing, the pair (ω,ψ_+) defines a coupled SU(3)-structure with coupled constant $|X|$ when restricted to each leaf L of the foliation.

Let us now observe that M is the total space of a fiber bundle $f: M\to S^1$ by Tischler's result recalled in Theorem 3.5.13. In particular, there exist a closed 1-form $\widehat{\theta}\in\Omega^1(M)$ with integral periods and an integer q such that, by construction, $q\widehat{\theta}$ can be made arbitrarily close to θ with respect to the norm induced by g_φ.

For every point p of M, we can then consider the fiber F of f containing p and the leaf L of the foliation $\mathcal{F}_\theta$ such that $p \in L$. The tangent space to the former at p is defined by $\ker(q\widehat{\theta}_p)$, while the tangent space to the latter by $\ker(\theta_p)$. Moreover, as we did before, it is easy to check that $T_pF = \langle \hat{\theta}_p^\sharp \rangle^\perp$, $T_pL = \langle \theta_p^\sharp \rangle^\perp$ and that the angle between the vectors $\theta_p^\sharp$ and $\hat{\theta}_p^\sharp$ is close to zero, since θ and $q\hat{\theta}$ can be made arbitrarily close. Consequently, up to normalizing these two vectors, we can apply Lemma 3.5.16 and get that, for every point p of M, the tangent space to the fiber containing p is close to the tangent space to the leaf through p, when they are thought as points in the Grassmannian $\mathrm{Gr}(6, T_pM)$ with the distance induced by the operator norm.

We can now show that the restriction of ω and ψ_+ to the fibers of f defines a coupled SU(3)-structure. Let F and L be defined as above, consider the exponential map $\exp_p : T_pM \to M$ and the invertible linear map $a : T_pM \to T_pM$ which is arbitrarily close to the identity map of T_pM and satisfies $a\left|_{T_pF} : T_pF \to T_pL\right.$. Since $(\exp_p)_{*p} = \mathrm{Id}_{T_pM}$, there exist an open neighborhood $\mathcal{U}$ of the origin in T_pM and an open neighborhood $\mathcal{V}$ of p in M such that $\exp_p : \mathcal{U} \to \mathcal{V}$ is a local diffeomorphism. The composition

$$\exp_p \circ a \circ \exp_p^{-1} : \mathcal{V} \to \mathcal{V},$$

restricted to the open set $\mathcal{V} \cap F$ of F, defines a smooth map from an open neighborhood of p in F to an open neighborhood of p in L which fixes p and whose differential at p is close to the identity. Then, we can apply the inverse function theorem to obtain a local diffeomorphism Υ from a neighborhood D_F of p in the fiber F to a neighborhood D_L of p in the leaf L such that $\Upsilon(p) = p$ and Υ_{*p} is close to the identity. Since (ω, ψ_+) defines a coupled SU(3)-structure when restricted to the leaf L, there exists an SU(3)-basis $\left(e^1, \ldots, e^6\right)$ of T_p^*L such that

$$\omega = e^{12} + e^{34} + e^{56}, \quad \psi_+ = e^{135} - e^{146} - e^{236} - e^{245}.$$

Considering the basis $\left(\Upsilon^* e^1, \ldots, \Upsilon^* e^6\right)$ of T_p^*F, we then have that $(\Upsilon^*(\omega), \Upsilon^*(\psi_+))$ defines a coupled SU(3)-structure on T_pF.

ii) From Lemma 3.5.15, we know that

$$|\sigma|^2 = 3\,|X|^2.$$

Define the 3-form $\widehat{\varphi} := d\sigma + q\widehat{\theta} \wedge \sigma$, it is a positive 3-form. Indeed, using Lemma 3.5.14 and the previous observation we have

$$\begin{aligned} |\widehat{\varphi} - \varphi|^2 &= \left|(q\hat{\theta} - \theta) \wedge \sigma\right|^2 \\ &\leq 3 \left|q\hat{\theta} - \theta\right| |\sigma|^2 \\ &= 9\,|X|^2 \left|q\hat{\theta} - \theta\right|^2 . \end{aligned}$$

Then

$$\|\widehat{\varphi} - \varphi\|^2 = \int_M |\hat{\varphi} - \varphi|^2 *_\varphi 1 \leq 9|X|^2 \int_M |q\hat{\theta} - \theta|^2 *_\varphi 1 = 9|X|^2 \left\|q\hat{\theta} - \theta\right\|^2$$

and $\|\widehat{\varphi} - \varphi\|$ is close to zero, since by construction $\left|q\widehat{\theta} - \theta\right|$ can be made arbitrarily small. Therefore, $\widehat{\varphi}$ is stable, as it lies in an arbitrarily small neighborhood of the stable form φ and being a stable form is an open condition. Since $d\widehat{\varphi} = -q\widehat{\theta} \wedge \widehat{\varphi}$, the 3-form $\widehat{\varphi}$ defines a locally conformal calibrated G_2-structure with Lee form $q\widehat{\theta}$, which is a 1-form with integral periods. □

Remark 3.5.18. By Tischler's theorem, we have that M is the mapping torus of a diffeomorphism ν of a certain 6-manifold (see also [133]), but ν in general does not preserve the coupled SU(3)-structure on the fiber.

The previous theorem applies for instance to the compact locally conformal calibrated G_2-manifold $(S/\Gamma, \varphi)$ obtained in Example 3.3.18, Indeed, the Lee form θ is non-vanishing and $\mathcal{L}_{\theta^\sharp}\varphi = 0$, as we also observed in Example 3.5.10. Therefore, S/Γ is fibered over S^1 and each fiber is endowed with a coupled SU(3)-structure.

Chapter 4

Perspectives on flows

In this final chapter, we consider evolution equations (flows) of special geometric structures. We begin with the study of a generalization of the Hitchin flow, then we review the definition of geometric flows and related properties and, finally, we explain some ideas which could be useful to study a currently open problem regarding the existence of geometric flows evolving SU(3)-structures.

4.1 Generalized Hitchin flow

In Section 3.2.1, we reviewed how the Hitchin flow equations of a t-depending SU(3)-structure $(\omega(t), \psi_+(t))$, $t \in \mathrm{I} \subseteq \mathbb{R}$, are obtained. Leaving aside the problem of existence of solutions and using the notations fixed in that section, we may rewrite the result in the following way

Proposition 4.1.1. *An* $\mathrm{SU}(3)$*-structure* $(\omega(t), \psi_+(t))$ *defined on a 6-manifold* $\overline{M}$ *and depending on a real parameter* $t \in \mathrm{I} \subseteq \mathbb{R}$ *can be evolved to a parallel* G_2*-structure on* $\overline{M} \times \mathrm{I}$ *defined by* $\varphi = \omega \wedge dt + \psi_+$ *if and only if it is half-flat for each* t *and the following evolution equations hold*

$$\begin{cases} \frac{\partial}{\partial t}\psi_+ = d\omega \\ \frac{\partial}{\partial t}\omega \wedge \omega = -d\psi_- \end{cases} . \tag{4.1}$$

Furthermore, since the half-flat condition

$$d\psi_+ = 0, \quad d\omega \wedge \omega = 0,$$

is preserved by the Hitchin flow equations, in Section 2.4.2 we restricted our attention to special half-flat SU(3)-structures, observing that there are examples of solutions of (4.1) belonging to the same subclass of $\mathcal{W}_1^- \oplus \mathcal{W}_2^- \oplus \mathcal{W}_3$ as long as they exist, but that in general this need not to be true. Indeed, it is not possible to show that the conditions defining a special half-flat SU(3)-structure (e.g. $d\omega = c\,\psi_+$ in the coupled case) are preserved by the evolution equations (4.1), as long as we do not know how ω and ψ_- evolve. We may then try to consider a suitable generalization of these evolution equations and study the behaviour of a certain class of SU(3)-structures with respect to it. First of all, we need to specify what does generalization mean in this context. Let us consider a result of [68] which is explanatory for this aim.

Recall that an SU(3)-structure (ω, ψ_+) is said to be *nearly half-flat* if its intrinsic torsion belongs to $\mathcal{W}_1^- \oplus \mathcal{W}_1^+ \oplus \mathcal{W}_2^- \oplus \mathcal{W}_3$. This is equivalent to the requirement

$$d\psi_+ = k\,\omega^2,$$

where $k = w_1^+$ is a real constant which is zero if and only if the SU(3)-structure is half-flat. In a similar way as in Proposition 4.1.1, it is possible to show that nearly half-flat SU(3)-structures can be evolved to nearly parallel G_2-structures. In detail

Proposition 4.1.2 ([68])**.** *An* SU(3)*-structure* $(\omega(t), \psi_+(t))$ *defined on a 6-manifold* $\overline{M}$ *and depending on a real parameter* $t \in \mathrm{I} \subseteq \mathbb{R}$ *can be evolved to a nearly parallel* G_2*-structure on* $\overline{M} \times \mathrm{I}$ *defined by* $\varphi = \omega \wedge dt + \psi_+$ *if and only if it is nearly half-flat with* $d\psi_+ = k\,\omega^2$ *for each* t *and the following evolution equations hold*

$$\begin{cases} \frac{\partial}{\partial t}\psi_+ = d\omega - 2k\,\psi_- \\ \frac{\partial}{\partial t}\omega \wedge \omega = -d\psi_- \end{cases} . \tag{4.2}$$

Proof. Denoted by d_7 the exterior derivative on $\overline{M} \times \mathrm{I}$ and by d the exterior derivative on $\overline{M}$, we have

$$\begin{aligned} d_7\varphi &= d_7(\omega \wedge dt + \psi_+) = d\omega \wedge dt + d\psi_+ + dt \wedge \frac{\partial}{\partial t}\psi_+ \\ &= d\psi_+ + \left(d\omega - \frac{\partial}{\partial t}\psi_+\right) \wedge dt \end{aligned}$$

and

$$d_7 *_\varphi \varphi = d_7 \left(\frac{1}{2}\omega^2 + \psi_- \wedge dt \right) = \frac{1}{2} d\omega^2 + \left(\frac{\partial}{\partial t}\omega \wedge \omega + d\psi_- \right) \wedge dt.$$

Now, if φ is nearly parallel with $d\varphi = \tau_0 *_\varphi \varphi$ and $d *_\varphi \varphi = 0$, then (ω, ψ_+) is nearly half-flat for each t with $d\psi_+ = \frac{1}{2}\tau_0 \,\omega^2$ and it satisfies the evolution equations (4.2) with $k = \frac{1}{2}\tau_0$. Conversely, if the pair (ω, ψ_+) satisfies the evolution equations (4.2), then it is nearly half-flat for each t, as

$$\frac{\partial}{\partial t}\left(d\psi_+ - k\,\omega^2 \right) = d\left(\frac{\partial}{\partial t}\psi_+ \right) - 2k\frac{\partial}{\partial t}\omega \wedge \omega = 0,$$

and φ is nearly parallel with $\tau_0 = 2k$. □

It is clear that the evolution equations (4.2) obtained in the last result are a generalization of the Hitchin flow equations which arise when the G_2-structure $\varphi = \omega \wedge dt + \psi_+$ is non-integrable. In this sense, we may call *generalized Hitchin flow equations* those obtained requiring that a t-dependent SU(3)-structure defined on a 6-manifold $\overline{M}$ can be evolved to a non-integrable G_2-structure on $\overline{M} \times \mathrm{I}$.

Remark 4.1.3. As shown in [165], a half-flat SU(3)-structure (ω, ψ_+) has totally skew-symmetric Nijenhuis tensor if and only if the pair (ω, ψ_-) is nearly half-flat. The SU(3)-structures which are contemporarily half-flat, nearly half-flat and have totally skew-symmetric Nijenhuis tensor are precisely the double half-flat SU(3)-structures, which can then be evolved both to parallel and nearly parallel G_2-structures by the previous results (see also [167]).

4.1.1 An example from physics

In this section, based on [70, Sect. 5.2], we consider a generalization of the Hitchin flow equations introduced in [57] and we show that it can be used to define a system of evolution equations for an SU(3)-structure which preserves the SU(3)-type $\mathcal{W}_1^- \oplus \mathcal{W}_2^-$ of coupled SU(3)-structures.

In the paper [57], a generalized Hitchin flow was used to study the moduli space of manifolds endowed with an SU(3)-structure. The starting point to define the generalization is to consider the embedding of an SU(3)-structure into a noncompact 7-manifold with a G_2-structure with torsion, i.e., having G_2-type $\mathcal{X}_1 \oplus \mathcal{X}_3 \oplus \mathcal{X}_4$. This is motivated by the subject the authors are interested in, namely four-dimensional *domain wall* solutions of *heterotic string theory* that preserve $\mathcal{N} = \frac{1}{2}$ supersymmetry (see also [92] and refer to it and to [57] for more details on the objects coming from supersymmetric string theory mentioned in this section). In this case, the internal six-dimensional manifold is endowed with an SU(3)-structure and one can combine it with the direction perpendicular to the domain wall in the four-dimensional noncompact space-time to get a seven-dimensional noncompact manifold endowed with a G_2-structure. The physical setting provides further constraints on the intrinsic torsion forms of the G_2-structure, which we will recall in due course. One can then study under which conditions a certain class of SU(3)-structures is preserved by this generalized flow. In [57], this was done for instance for torsion-free, nearly Kähler, half-flat and symplectic half-flat SU(3)-structures. We investigate here the case of coupled SU(3)-structures.

Let $\overline{M}$ be a connected 6-manifold endowed with an SU(3)-structure $(\omega(t), \psi_+(t))$ depending on a real parameter $t \in \mathrm{I} \subseteq \mathbb{R}$. Following [57], on the 7-manifold $M := \overline{M} \times \mathrm{I}$ we consider the G_2-structure defined by

$$\varphi = n_t dt \wedge \omega + \Re(F\Psi),$$

where $n_t \in C^\infty(M)$ is nowhere zero and F is a nonzero complex valued smooth function defined on I and having constant module 1 (see also Proposition 3.2.2). Observe that the Riemannian metric defined by φ is

$$g_\varphi = g + n_t^2 dt^2,$$

where $g = g(t)$ is the metric induced by $(\omega(t), \psi_+(t))$.

In the case of $\mathcal{N} = \frac{1}{2}$ domain wall solutions, the possibly non-identically vanishing intrinsic torsion forms of the considered G_2-structure are τ_0, τ_1, τ_3. On the product manifold $\overline{M} \times \mathrm{I}$, τ_1 and τ_3 can be decomposed as

$$\begin{aligned} \tau_1 &= \overline{\tau}_1 + u_t dt, \\ \tau_3 &= \overline{\tau}_3 + dt \wedge \kappa_t, \end{aligned}$$

where u_t is a smooth function on M, $\overline{\tau}_1$ is a 1-form on $\overline{M}$, κ_t is a 2-form depending on t and defined on $\overline{M}$ and $\overline{\tau}_3$ is a 3-form on $\overline{M}$. Moreover, the following constraints hold

$$u_t = \frac{1}{2}\frac{\partial}{\partial t}\phi, \quad \overline{\tau}_1 = \frac{1}{2}d\phi, \quad d_7\tau_0 = 0,$$

where ϕ denotes the *ten-dimensional dilaton*, d_7 denotes the exterior derivative on M and d denotes it on $\overline{M}$.

A general argument similar to that used in the proof of Proposition 4.1.2 allows one to write down the flow equations for the SU(3)-structure, namely the generalized Hitchin flow equations associated with the considered embedding, and some relations between the intrinsic torsion forms of the SU(3)-structure and those of the G_2-structure. In particular, it is possible to show that

$$w_4 = 2\,\overline{\tau}_1.$$

Therefore, if we have an SU(3)-structure with identically vanishing intrinsic torsion form w_4, we get $d\phi = 2\,\overline{\tau}_1 = 0$.

As in [57], we work in the gauge $F = 1$. In this case, the embedding of the SU(3)-structure on $\overline{M}$ into the integrable G_2-structure on $\overline{M} \times \mathrm{I}$ is given by

$$\varphi = n_t dt \wedge \omega + \psi_+.$$

If we suppose that the SU(3)-structure is coupled for each t, i.e.,

$$\begin{array}{rcl} d\omega(t) & = & c(t)\psi_+(t), \\ d\psi_+(t) & = & 0, \\ d\psi_-(t) & = & -\frac{2}{3}c(t)(\omega(t))^2 - w_2^-(t) \wedge \omega(t), \end{array} \tag{4.3}$$

where $c : \mathrm{I} \to \mathbb{R}$ is a nonzero smooth function such that $w_1^-(t) = -\frac{2}{3}c(t)$, then the intrinsic torsion forms τ_1 and τ_3 take the following expressions

$$\tau_1 = u_t dt, \quad \tau_3 = \overline{\tau}_3 - \frac{\tau_0}{n_t}\, dt \wedge \omega.$$

Moreover, the 2-form $\omega(t)$ evolves as

$$\frac{\partial}{\partial t}\omega(t) = f_t\omega(t) + h_t, \tag{4.4}$$

where

$$f_t \;=\; 2u_t - n_t w_1^-(t), \tag{4.5}$$

$$h_t \;=\; n_t w_2^-(t) - *(dn_t \wedge *\psi_+(t)). \tag{4.6}$$

It follows from a general argument involving the flow equations that

$$df_t = 0,$$

and using one of the constraints recalled earlier, we get

$$du_t = \frac{1}{2} d\left(\frac{\partial}{\partial t}\phi\right) = \frac{1}{2}\frac{\partial}{\partial t}(d\phi) = 0.$$

Taking the exterior derivative of both sides of (4.5), we then have

$$dn_t = 0.$$

Thus, n_t is actually a function of t and (4.6) becomes $h_t = n_t w_2^-$.

Remark 4.1.4. With our convention, w_2^- here is $-w_2^-$ in the paper [57].

The flow equations for $\psi_+(t)$ and $\psi_-(t)$ determined in [57] from the embedding and the results of [117] reduce to the following in the coupled case

$$\frac{\partial}{\partial t}\psi_+(t) = \frac{3}{2} f_t \psi_+(t) - \frac{7}{4}\tau_0 n_t \psi_-(t) - n_t \gamma, \tag{4.7}$$

$$\frac{\partial}{\partial t}\psi_-(t) = \frac{7}{4}\tau_0 n_t \psi_+(t) + \frac{3}{2} f_t \psi_-(t) + n_t J\gamma, \tag{4.8}$$

where γ is the component of $*\overline{\tau}_3 \in \Omega^3(\overline{M})$ in $\left[\!\left[\Omega_0^{2,1}(\overline{M})\right]\!\right]$.

We derive now all of the conditions that arise requiring these flow equations to preserve the coupled condition. We may sometimes omit the t-dependence of the forms for brevity.

First of all, suppose that for each t the coupled condition $d\omega(t) = c(t)\psi_+(t)$ holds. Differentiating both sides with respect to t, we have

$$d\left(\frac{\partial}{\partial t}\omega\right) = \dot{c}\psi_+ + c\left(\frac{3}{2} f_t \psi_+ - \frac{7}{4}\tau_0 n_t \psi_- - n_t\gamma\right).$$

Moreover, taking the exterior derivative of both sides of (4.4), using $dn_t = 0$ and the hypothesis on the coupled condition, we obtain

$$d\left(\frac{\partial}{\partial t}\omega\right) = f_t c\psi_+ + n_t dw_2^-.$$

Comparing the two equations, it follows

$$n_t dw_2^- = \dot{c}\psi_+ + \frac{1}{2}cf_t\psi_+ - \frac{7}{4}c\tau_0 n_t\psi_- - cn_t\gamma.$$

Wedging both sides by ψ_- and using the fact that $\gamma \wedge \psi_- = 0$, since $\gamma \in \left[\!\left[\Omega_0^{2,1}(\overline{M})\right]\!\right]$, we get

$$n_t dw_2^- \wedge \psi_- = \frac{2}{3}\dot{c}\,\omega^3 + \frac{1}{3}cf_t\omega^3. \tag{4.9}$$

Since for each t it holds $dw_2^- \wedge \psi_- = -|w_2^-|^2\frac{\omega^3}{6}$ (cf. Lemma 2.4.7), where the norm is induced by $g(t)$, equation (4.9) becomes

$$-n_t|w_2^-|^2\frac{\omega^3}{6} = \frac{2}{3}\dot{c}\,\omega^3 + \frac{1}{3}cf_t\omega^3$$

and the following result is proved.

Proposition 4.1.5. *Suppose that the generalized Hitchin flow preserves the coupled condition $d\omega(t) = c(t)\psi_+(t)$. Then, the function $c(t)$ must evolve in the following way*

$$\frac{\partial}{\partial t}c(t) = -\frac{1}{2}c(t)f_t - \frac{1}{4}n_t|w_2^-(t)|^2_{g(t)}.$$

Moreover, for each t, it must hold

$$dw_2^- = -\frac{1}{4}|w_2^-|^2\psi_+ - \frac{7}{4}c\tau_0\psi_- - c\gamma.$$

In order to preserve the closedness of $\psi_+(t)$, we need

$$d\left(\frac{\partial}{\partial t}\psi_+\right) = 0.$$

Moreover, taking the exterior derivative of both sides of the flow equation (4.7) of ψ_+, we have

$$d\left(\frac{\partial}{\partial t}\psi_+\right) = -\frac{7}{4}\tau_0 n_t d\psi_- - n_t d\gamma.$$

Comparing the two equations, it follows

$$d\gamma = -\frac{7}{4}\tau_0 n_t d\psi_-. \tag{4.10}$$

Observe now that $d\gamma \wedge \omega = 0$, since γ is a primitive real form of type $(2,1)+(1,2)$. Therefore, wedging both sides of (4.10) by ω and recalling that $d\psi_- \wedge \omega = -\frac{2}{3}c\,\omega^3$, we get

$$\tau_0 n_t c = 0,$$

and then $\tau_0 = 0$, since both c and n_t cannot be zero. In particular, γ is closed and, by [57, Prop. 2],

$$*\overline{\tau}_3 = \gamma.$$

We can summarize the results in the following

Proposition 4.1.6. *If the closedness of ψ_+ is preserved by the generalized Hitchin flow, then the intrinsic torsion form τ_0 vanishes identically and the 3-form γ is closed and satisfies $*\gamma = -\overline{\tau}_3$.*

Let us now consider the expression of $d\psi_-$ in (4.3) and differentiate it with respect to t, having in mind the results already obtained:

$$d\left(\frac{\partial}{\partial t}\psi_-\right) = \left(-\frac{2}{3}\dot{c} - \frac{4}{3}cf_t\right)\omega^2 + \left(-\frac{4}{3}cn_t - f_t\right)w_2^- \wedge \omega - \frac{\partial}{\partial t}w_2^- \wedge \omega - n_t w_2^- \wedge w_2^-.$$

Taking the exterior derivative of both sides of the flow equation (4.8), we get

$$d\left(\frac{\partial}{\partial t}\psi_-\right) = -f_t c\,\omega^2 - \frac{3}{2}f_t w_2^- \wedge \omega + n_t d(J\gamma).$$

Comparing the two identities, we obtain that the evolution of w_2^- must satisfy the following equation

$$\frac{\partial}{\partial t}w_2^- \wedge \omega = \frac{1}{6}n_t|w_2^-|^2\omega^2 + \left(-\frac{4}{3}cn_t + \frac{1}{2}f_t\right)w_2^- \wedge \omega - n_t w_2^- \wedge w_2^- - n_t d(J\gamma).$$

We also know that the following conditions deriving from the *Bianchi identity* $d_7\hat{H} = 0$ must hold

$$d\overline{S} = 0, \tag{4.11}$$

$$dS_t = \frac{\partial}{\partial t}\overline{S}, \tag{4.12}$$

where $\hat{H} = dt \wedge S_t + \overline{S}$ is the component of the *ten-dimensional flux* along M. Using the previous results, it follows from [57] that for a coupled SU(3)-structure

$$\overline{S} = n_t^{-1} u_t \psi_- + J\gamma, \qquad S_t = 0.$$

From the identity (4.11), we then get

$$d(J\gamma) = -n_t^{-1} u_t d\psi_-. \tag{4.13}$$

Observe that $d(J\gamma) \wedge \omega = 0$. Thus, if we wedge both sides of (4.13) by ω, we obtain

$$n_t^{-1} u_t c = 0,$$

from which follows $u_t = 0$ and, as a consequence, $d(J\gamma) = 0$. The identity (4.12) now reads

$$\frac{\partial}{\partial t}(J\gamma) = 0.$$

Summarizing, after imposing all of the conditions, we get that the only possibly non-identically vanishing intrinsic torsion form of the G_2-structure is $\tau_3 = - * \gamma$, the 3-form γ is closed and satisfies $d(J\gamma) = 0$, $f_t = \frac{2}{3} n_t c(t)$ and the evolution equations of the differential forms defining the coupled SU(3)-structure become

$$\begin{aligned} \frac{\partial}{\partial t}\omega(t) &= \frac{2}{3} n_t c(t)\omega(t) + n_t w_2^-(t), \\ \frac{\partial}{\partial t}\psi_+(t) &= n_t c(t)\psi_+(t) - n_t \gamma, \\ \frac{\partial}{\partial t}\psi_-(t) &= n_t c(t)\psi_-(t) + n_t J\gamma. \end{aligned}$$

Moreover, the intrinsic torsion forms of the coupled SU(3)-structure must evolve as

$$\begin{aligned} \frac{\partial}{\partial t}c(t) &= -\frac{1}{3} n_t (c(t))^2 - \frac{1}{4} n_t |w_2^-(t)|^2_{g(t)}, \\ \frac{\partial}{\partial t}w_2^-(t) \wedge \omega(t) &= \frac{1}{6} n_t |w_2^-(t)|^2_{g(t)} (\omega(t))^2 - n_t c(t) w_2^-(t) \wedge \omega(t) - n_t (w_2^-(t))^2, \end{aligned}$$

and for each t it must hold

$$dw_2^- = -\frac{1}{4}|w_2^-|^2 \psi_+ - c\gamma.$$

As a particular case, if we suppose that $n_t = 1$ and $\gamma = 0$, then

$$\varphi = dt \wedge \omega + \psi_+$$

is a parallel G_2-structure on $\overline{M} \times \mathrm{I}$. Under these hypothesis, the evolution equations of the differential forms $\omega(t), \psi_+(t), \psi_-(t)$ read

$$\begin{aligned} \tfrac{\partial}{\partial t}\omega(t) &= \tfrac{2}{3}c(t)\omega(t) + w_2^-(t), \\ \tfrac{\partial}{\partial t}\psi_+(t) &= c(t)\psi_+(t), \\ \tfrac{\partial}{\partial t}\psi_-(t) &= c(t)\psi_-(t), \end{aligned} \tag{4.14}$$

the evolution equations of the intrinsic torsion forms of the coupled structure must be

$$\begin{aligned} \tfrac{\partial}{\partial t}c(t) &= -\tfrac{1}{3}(c(t))^2 - \tfrac{1}{4}|w_2^-(t)|^2_{g(t)}, \\ \tfrac{\partial}{\partial t}w_2^-(t) \wedge \omega(t) &= \tfrac{1}{6}|w_2^-(t)|^2_{g(t)}(\omega(t))^2 - c(t)w_2^-(t) \wedge \omega(t) - (w_2^-(t))^2, \end{aligned} \tag{4.15}$$

and for each t the 2-form w_2^- has to satisfy the following identity

$$dw_2^- = -\frac{1}{4}|w_2^-|^2\psi_+, \tag{4.16}$$

which is one of the conditions widely discussed in Section 2.4.3.

It is easy to check that a solution of these equations which is coupled for each t is also a coupled solution of the Hitchin flow equations in the sense of Definition 2.4.16 and vice-versa. For instance, the coupled solution of the Hitchin flow on the Lie algebra $\mathfrak{n}_{28}$ obtained in the proof of Proposition 2.4.19 satisfies (4.14) and the conditions (4.15), (4.16). In the general case, the presence of $w_2^-(t)$ in the flow equations makes rather complicated any attempt to solve them. However, we can show that a solution of them starting from a coupled SU(3)-structure stays coupled as long as it exists.

Proposition 4.1.7. *Let $(\omega(t), \psi_+(t), c(t), w_2^-(t))$ be a solution of the equations* (4.14), (4.15), (4.16), *with initial condition a coupled* SU(3)*-structure* $(\omega(0), \psi_+(0))$ *with coupled constant* $c(0)$*. Then,* $(\omega(t), \psi_+(t))$ *is a coupled* SU(3)*-structure as long as it exists.*

Proof. Consider $d\omega(t) - c(t)\psi_+(t)$, differentiating it with respect to t and using the hypothesis, we get (omitting the t-dependence for brevity)

$$\begin{aligned}
\frac{\partial}{\partial t}(d\omega - c\psi_+) &= d\left(\frac{\partial}{\partial t}\omega\right) - \dot{c}\psi_+ - c\frac{\partial}{\partial t}\psi_+ \\
&= \frac{2}{3}cd\omega + dw_2^- + \frac{1}{3}c^2\psi_+ + \frac{1}{4}|w_2^-|^2\psi_+ - c^2\psi_+ \\
&= \frac{2}{3}c(d\omega - c\psi_+).
\end{aligned}$$

Thus, if we let $\beta(t) := d\omega(t) - c(t)\psi_+(t)$, we have that $\frac{\partial}{\partial t}\beta(t) = \frac{2}{3}c(t)\beta(t)$. Therefore, $\beta(t) = q(t)\beta(0)$, where $q(t) = e^{\int_0^t \frac{2}{3}c(s)ds}$. Moreover, $\beta(0) = d\omega(0) - c(0)\psi_+(0) = 0$, since $(\omega(0), \psi_+(0))$ is coupled. Then, $0 = \beta(t) = d\omega(t) - c(t)\psi_+(t)$ and, as a consequence, $d\psi_+(t) = 0$. □

4.2 Geometric flows

Geometric flows are partial differential equations describing the evolution of geometric structures on manifolds. We already encountered an example in the first chapter, namely the Ricci flow

$$\frac{\partial}{\partial t}g(t) = -2\,\mathrm{Ric}(g(t)),$$

where $g(t)$ is a family of Riemannian metrics depending smoothly on a real parameter t (cf. Section 1.4.3). Since the geometric structures we have considered so far can all be defined by global sections of vector bundles over a manifold (possibly satisfying certain compatibility conditions), we recall in this section how it is possible to describe the *evolution* of these objects and which hypothesis guarantee the (short-time) existence and uniqueness of solutions of an *initial value problem*. For more details on this topic, the reader may refer for instance to Aubin's book [11].

Let $\pi : E \to M$ be a rank k vector bundle over a Riemannian manifold (M, g) and let ∇ be a linear connection on E which is compatible with g.

Definition 4.2.1. A *differential operator* of *order* r acting on sections of E is an operator $L : \Gamma(E) \to \Gamma(E)$ such that for every $u \in \Gamma(E)$ and $p \in M$

$$L(u)(p) = F\left(p, u(p), \nabla u(p), \dots, \nabla^r u(p)\right) \in E_p.$$

L is called *smooth* if the function F is smooth in its arguments. L is called *linear* if it is linear in u and *nonlinear* otherwise. A *smooth family* of differential operators $L_t : \mathrm{Dom}(L_t) \to \Gamma(E)$ depending on $t \in [0,T)$, where $\mathrm{Dom}(L_t) \subseteq \Gamma(E)$, is defined in a similar way as before with the function F depending smoothly also on t.

If L is a linear differential operator of order r, in any local coordinate chart on M inducing local coordinates $(x^i) = (x^1, \dots, x^m)$ and trivializing E, we can write

$$L = \sum_{|\alpha|=r} L^{\alpha_1 \dots \alpha_r}(p) \frac{\partial^r}{\partial x^{\alpha_1} \cdots \partial x^{\alpha_r}} + \text{lower order terms},$$

where the sum is over all of the possible multi-indices $\alpha = (\alpha_1, \dots, \alpha_r)$ of length $|\alpha| = r$, each $L^{\alpha_1 \dots \alpha_r}(p)$ belongs to $\mathrm{End}(E_p)$ and *lower order terms* gathers all of the summands appearing in the local expression of L and involving derivatives of order less or equal than $r-1$.

Definition 4.2.2. The *principal symbol* of a linear differential operator L of order r is a bundle map $\sigma(L) : T^*M \times E \to E$ defined for each $p \in M$ by

$$\sigma(L)_p(\xi_p) = \sum_{|\alpha|=r} L^{\alpha_1 \dots \alpha_r}(p) \xi_{\alpha_1} \cdots \xi_{\alpha_r} \in \mathrm{End}(E_p),$$

where $\xi_p \in T^*_pM$ is a nonzero covector having the expression $\xi_j dx^j$ in local coordinates on M.

It is possible to show that the previous definition does not depend on the coordinates, thus it is well-posed. Moreover, it is also possible to give a coordinate-free definition as follows

Proposition 4.2.3. *Let f be a smooth, real valued function defined around a point p of M and such that $df_p = \xi_p \in T^*_pM$. Then, the principal symbol of a linear differential operator $L : \Gamma(E) \to \Gamma(E)$ of order r is given for every $u \in \Gamma(E)$ by*

$$\sigma(L)_p(\xi_p)u(p) = \lim_{s\to\infty} \frac{1}{s^r} e^{-sf(p)} L(e^{sf}u)(p). \tag{4.17}$$

Using (4.17), it is easy to show that given two differential operators L_1, L_2 such that the composition $L_1 \circ L_2$ is defined, then for every nonzero $\xi \in T^*M$

$$\sigma(L_1 \circ L_2)(\xi) = \sigma(L_1)(\xi) \circ \sigma(L_2)(\xi). \tag{4.18}$$

Definition 4.2.4. Let E be a vector bundle over a Riemannian manifold (M, g) and let L be a linear differential operator of order r acting on sections of E. L is said to be *elliptic* if for each point p of M and each nonzero $\xi_p \in T_p^*M$ the linear map $\sigma(L)_p(\xi_p)$ is invertible.

When the order r is even, L is called *strongly elliptic* if there exists a real constant $\mathrm{C} > 0$ such that

$$g(\sigma(L)(\xi)u, u) \geq \mathrm{C}\,|\xi|^r |u|^2$$

for all nonzero $\xi \in T^*M$ and $u \in \Gamma(E)$.

When a differential operator L is nonlinear, it is possible to define the *linearization* of it at $u \in \Gamma(E)$ in the direction of $v \in \Gamma(E)$ as

$$L_{*u}(v) := \left.\frac{d}{ds}\right|_{s=0} L(u + sv) = \lim_{s \to 0} \frac{L(u + sv) - L(u)}{s}$$

and the principal symbol of L as the principal symbol of its linearization. A nonlinear differential operator L is *strongly elliptic at* $u \in \Gamma(E)$ if its linearization L_{*u} is strongly elliptic in the sense of Definition 4.2.4.

Example 4.2.5. The Ricci tensor $\mathrm{Ric}(g)$ of a Riemannian manifold (M, g) can be regarded as a second order nonlinear differential operator

$$\mathrm{Ric} : \mathcal{M}et(M) \to \mathcal{S}^2(M), \quad \mathrm{Ric} : g \mapsto \mathrm{Ric}(g),$$

where $\mathcal{M}et(M) \subset \mathcal{S}^2(M)$ denotes the space of Riemannian metrics on M. It is possible to show that Ric is not elliptic, as its principal symbol has non-trivial kernel (see [94]).

Let u be a smooth section of E depending smoothly on a real parameter $t \in [0, T)$, we can see it as a smooth map $u : M \times [0, T) \to E$ such that $u(p, t) = u_t(p) \in E_p$ for every point p of M. The *evolution equation* (*flow equation*) of $u(\cdot, t)$ in terms of a smooth family of differential operators L_t is defined by

$$\frac{\partial}{\partial t} u(\cdot, t) = L_t(u(\cdot, t))$$

and is called *strictly parabolic* if L_t is a smooth family of strongly elliptic operators. An *initial value problem* consists in considering a given section $u_0 \in \Gamma(E)$ and looking for solutions of the system

$$\begin{cases} \frac{\partial}{\partial t} u(\cdot, t) = L_t(u(\cdot, t)) \\ u(\cdot, 0) = u_0 \end{cases} . \tag{4.19}$$

A sufficient condition guaranteeing short-time existence and uniqueness of solutions of (4.19) is given in the next result (see for instance [11, Thm. 4.51] and the references following it).

Theorem 4.2.6. *Let $\pi : E \to M$ be a vector bundle over a compact Riemannian manifold (M, g), let $L_t : \mathrm{Dom}(L_t) \to \Gamma(E)$ be a smooth family of differential operators and let $u_0 \in \Gamma(E)$ be a smooth section of E. If L_0 is strongly elliptic at u_0, then there exists a unique smooth solution of the system* (4.19) *defined on $M \times [0, \varepsilon)$ for some $\varepsilon > 0$.*

Remark 4.2.7. The previous result cannot be applied in the case of the Ricci flow, as the operator Ric is not elliptic. This explains why it is necessary to use Nash-Moser inverse function Theorem or to modify the flow equation in order to prove short-time existence and uniqueness of solutions (see the discussion in Section 1.4.3).

When a special geometric structure is defined by one or more tensor fields, a geometric flow of it consists in a set of evolution equations for (at least one of) the defining tensors. Clearly, in this case the solution of an initial value problem has to define the same kind of special geometric structure as long as it exists.

In the last decades, after the introduction of the Ricci flow in [94] and the development of the Kähler-Ricci flow on complex manifolds (see [43, Ch. 2] and the references therein for more informations), geometric flows have widely been considered in literature and it is arduous to provide an exhaustive list of references. For instance, in [171] Streets and Tian introduced a geometric flow for the Riemannian metric of a Hermitian manifold (M, g_0, J_0) such that the solution $g(t)$ is compatible with the complex structure J_0 for each t and is moreover Kähler if (g_0, J_0) is Kähler. The generalization of this flow in the almost Hermitian case was obtained by Vezzoni in [178]. In [172], the same authors of [171] studied a family of flows evolving the fundamental form ω and the almost complex structure J of an almost Hermitian manifold, generalizing the flow contained in their previous work. Moreover, they defined a flow for almost Hermitian structures which preserves the almost Kähler condition $d\omega = 0$. In the G_2-case, examples of flows have been studied by Bryant and Xu [30, 33], by Karigiannis [112] and by Kozhasov [120].

A currently open question is whether it is possible to define a geometric flow for SU(n)-structures. The main problem in this context is to find a suitable system of evolution equations for the tensors defining such a structure which has local existence and uniqueness of solutions for a given initial data and preserves all of the compatibility conditions between the tensors. In what follows, we explain some ideas aiming to provide a way to study this problem for SU(3)-structures. This is based on a joint work-in-progress with A. Fino and L. Vezzoni [72].

4.3 A spinor flow

In this section, after reviewing the definition of spin structures on Riemannian manifolds and the correspondence between spinor fields and SU(3)-structures in the six-dimensional case, we study a geometric flow for spinors on 6-manifolds and discuss related properties and consequences.

4.3.1 Spin structures on Riemannian manifolds

We summarize here the main definitions and properties concerning spin structures on Riemannian manifolds. A detailed description and the proofs of the results can be found for instance in [16, 76, 129].

Consider the Euclidean space $(\mathbb{R}^m, g)$ and let $(e_1, \ldots, e_m)$ be a g-orthonormal basis of it. The *real Clifford algebra* $\mathcal{C}_m$ of $\mathbb{R}^m$ with quadratic form $-g(v, v)$, $v \in \mathbb{R}^m$, is an algebra over $\mathbb{R}$ multiplicatively generated by the basis vectors with the relations

$$\begin{aligned} e_k \cdot e_l &= -e_l \cdot e_k, \quad k \neq l, \\ e_k \cdot e_k &= -1, \end{aligned}$$

where $\cdot$ denotes the product on $\mathcal{C}_m$.

The complexification $\mathcal{C}_m^{\mathrm{c}} = \mathcal{C}_m \otimes_{\mathbb{R}} \mathbb{C}$ of the Clifford algebra $\mathcal{C}_m$ is isomorphic to $\mathrm{End}(\mathbb{C}^{2^n})$ when $m = 2n$ is even and to $\mathrm{End}(\mathbb{C}^{2^n}) \oplus \mathrm{End}(\mathbb{C}^{2^n})$ when $m = 2n+1$ is odd.

Definition 4.3.1. The vector space $\Delta_m := \mathbb{C}^{2^n}$, defined for $m = 2n, 2n+1$, is the vector space of *complex m-spinors.*

The group $\mathrm{Spin}(m)$ can be defined as a subgroup of $\mathcal{C}_m$ in the following way

$$\mathrm{Spin}(m) := \{v_1 \cdot \ldots \cdot v_{2k} \mid v_i \in \mathbb{R}^m,\ |v_i| = 1\} \subset \mathcal{C}_m \subset \mathcal{C}_m^{\mathrm{c}}.$$

For $m \geq 3$, $\mathrm{Spin}(m)$ is the universal (double) covering of the group $\mathrm{SO}(m)$ and the covering map is defined by

$$\mathrm{Ad} : \mathrm{Spin}(m) \to \mathrm{SO}(m), \quad \mathrm{Ad}(\varsigma)v = \varsigma\, v\, \varsigma^{-1},$$

for every $\varsigma \in \mathrm{Spin}(m)$ and $v \in \mathbb{R}^m$. In particular, Ad is surjective and $\ker(\mathrm{Ad}) = \{\pm 1\}$.

The *spinor representation* of $\mathrm{Spin}(m)$ on Δ_m

$$\rho : \mathrm{Spin}(m) \to \mathrm{GL}(\Delta_m)$$

is a faithful representation defined as the restriction to $\mathrm{Spin}(m)$ of the isomorphism $\rho_m : \mathcal{C}_m^{\mathrm{c}} \to \mathrm{End}(\Delta_m)$ when $m = 2n$ and of the composition of the isomorphism $\rho_m : \mathcal{C}_m^{\mathrm{c}} \to \mathrm{End}(\Delta_m) \oplus \mathrm{End}(\Delta_m)$ with the projection p_1 onto the first factor when $m = 2n+1$.

In the even-dimensional case $m = 2n$, the endomorphism

$$i^n \rho(e_1 \cdot \ldots \cdot e_{2n}) : \Delta_{2n} \to \Delta_{2n} \tag{4.20}$$

is an involution. Thus, it induces a decomposition $\Delta_{2n} = \Delta_{2n}^+ \oplus \Delta_{2n}^-$, where $\Delta_{2n}^\pm$ are the eigenspaces of complex dimension 2^{n-1} corresponding to the eigenvalues ± 1. Moreover, $\Delta_{2n}^\pm$ are irreducible representations of the group $\mathrm{Spin}(2n)$.

Since $\mathbb{R}^m \subset \mathcal{C}_m$, it is possible to introduce a multiplication of vectors and spinors using the isomorphism ρ_m.

Definition 4.3.2. The *Clifford multiplication* of vectors and spinors is the linear map $\mu : \mathbb{R}^m \times \Delta_m \to \Delta_m$ defined as follows

$$v \cdot \phi := \mu(v, \phi) = \begin{cases} \rho_m(v)\phi, & m = 2n \\ \mathrm{p}_1(\rho_m(v))\phi, & m = 2n+1 \end{cases} .$$

The Clifford multiplication μ is equivariant with respect to the action of $\mathrm{Spin}(m)$ and for $m = 2n$ it satisfies $v \cdot \phi^\pm \in \Delta_{2n}^\mp$ for every $v \in \mathbb{R}^{2n}$ and $\phi^\pm \in \Delta_{2n}^\pm$.

Finally, we recall the

Proposition 4.3.3. *On the vector space Δ_m there exists a positive definite Hermitian product $\langle \cdot, \cdot \rangle$ with the property*

$$\langle v \cdot \phi_1, \phi_2 \rangle = -\langle \phi_1, v \cdot \phi_2 \rangle,$$

for every $v \in \mathbb{R}^m$, $\phi_1, \phi_2 \in \Delta_m$. With respect to this Hermitian product, the spinor representation ρ becomes a unitary representation satisfying $\rho : \mathrm{Spin}(m) \to \mathrm{SU}(\Delta_m)$.

We can now introduce spin structures on Riemannian manifolds.

Definition 4.3.4. Let (M, g) be an oriented Riemannian manifold of dimension m and let $\mathrm{SO}(M)$ denote the principal $\mathrm{SO}(m)$-bundle over M. A *spin structure* on (M, g) is a pair (Q, Θ), where Q is a $\mathrm{Spin}(m)$-principal bundle over M and $\Theta : Q \to \mathrm{SO}(M)$ is a double covering of $\mathrm{SO}(M)$ for which the following diagram commutes

$$\begin{array}{ccccc} Q \times \mathrm{Spin}(m) & \stackrel{\cdot}{\longrightarrow} & Q & \stackrel{\pi_Q}{\longrightarrow} & M \\ \Big\downarrow \Theta \times \mathrm{Ad} & & \Big\downarrow \Theta & & \Big\downarrow \mathrm{Id} \\ \mathrm{SO}(M) \times \mathrm{SO}(m) & \stackrel{\cdot}{\longrightarrow} & \mathrm{SO}(M) & \stackrel{\pi_{\mathrm{SO}(M)}}{\longrightarrow} & M \end{array}$$

where the dots denote the right actions of $\mathrm{Spin}(m)$ and $\mathrm{SO}(m)$ on the corresponding principal bundles. A Riemannian manifold with a spin structure is called *Riemannian spin manifold.*

Remark 4.3.5. It is worth recalling here that a Riemannian manifold (M, g) is orientable if and only if its *first Stiefel-Whitney class* $w_1(M)$ vanishes, while it admits a spin structure if and only if its *second Stiefel-Whitney class* $w_2(M)$ vanishes. Both $w_1(M)$ and $w_2(M)$ are homotopy invariant, thus the existence of a spin structure on an oriented Riemannian manifold depends only on its topology.

Using the spinor representation ρ, it is possible to define the complex vector bundle

$$\Sigma M := Q \times_{\mathrm{Spin}(m)} \Delta_m$$

over M with fiber Δ_m, which is called the *spinor bundle* of (M, g). ΣM is endowed with a complex scalar product $\langle \cdot, \cdot \rangle$ defined from the Hermitian product on Δ_m and with a real scalar product $(\cdot, \cdot) := \Re\langle \cdot, \cdot \rangle$. Moreover, when $m = 2n$ it splits into the direct sum of two subbundles $\Sigma M = \Sigma^+ M \oplus \Sigma^- M$, where

$$\Sigma^{\pm} M := Q \times_{\mathrm{Spin}(2n)} \Delta^{\pm}_{2n}.$$

Definition 4.3.6. A smooth section $\phi \in \Gamma(\Sigma M)$ of ΣM is called (*complex*) *spinor field* on M.

The Clifford multiplication on the fibers of the vector bundle $\pi : \Sigma M \to M$ gives rise to a bundle map

$$\mu : TM \times \Sigma M \to \Sigma M, \quad \mu(X, \phi) = X \cdot \phi,$$

which satisfies the following properties

Proposition 4.3.7. *For every $X, Y \in \mathfrak{X}(M)$ and $\phi, \phi_1, \phi_2 \in \Gamma(\Sigma M)$ the following results hold:*

i) *If ϕ is a spinor field without zeroes, then $X \cdot \phi = 0$ implies $X = 0$;*

ii) *$X \cdot Y \cdot \phi + Y \cdot X \cdot \phi = -2\, g(X, Y)\phi$;*

iii) *$\langle X \cdot \phi_1, \phi_2 \rangle = -\langle \phi_1, X \cdot \phi_2 \rangle$;*

iv) *$(X \cdot \phi, Y \cdot \phi) = g(X, Y)|\phi|^2$;*

v) *if m is even, $\mu : TM \times \Sigma^{\pm} M \to \Sigma^{\mp} M$.*

The Levi Civita connection ∇^g on (M, g) induces a connection on the spinor bundle, which we denote by $\nabla : \Gamma(\Sigma M) \to \Gamma(T^*M \otimes \Sigma M)$. The covariant derivative associated with ∇ is called *spinor derivative* and has the following local expression with respect to a local orthonormal frame $(e_1, \dots, e_m)$ for TM

$$\nabla_X \phi = X(\phi) + \frac{1}{2} \sum_{1 \le k < l \le m} g(\nabla^g_X e_k, e_l)\, e_k \cdot e_l \cdot \phi.$$

Proposition 4.3.8. *The spinor derivative satisfies the following properties for every* $X, Y \in \mathfrak{X}(M)$ *and* $\phi, \phi_1, \phi_2 \in \Gamma(\Sigma M)$

i) $X\langle \phi_1, \phi_2 \rangle = \langle \nabla_X \phi_1, \phi_2 \rangle + \langle \phi_1, \nabla_X \phi_2 \rangle$, *i.e.,* ∇ *is metric;*

ii) $\nabla_X (Y \cdot \phi) = \nabla^g_X Y \cdot \phi + Y \cdot \nabla_X \phi$.

Using the Riemannian metric g to identify the tangent bundle TM with the cotangent bundle T^*M, we can see $\nabla : \Gamma(\Sigma M) \to \Gamma(TM \otimes \Sigma M)$. It is then possible to introduce the following differential operator

Definition 4.3.9. The *Dirac operator* of (M, g) is the first order linear differential operator

$$D : \Gamma(\Sigma M) \to \Gamma(\Sigma M), \quad D := \mu \circ \nabla.$$

Its expression in terms of a local orthonormal frame $(e_1, \dots, e_m)$ for TM is

$$D\phi = \sum_{k=1}^{m} e_k \cdot \nabla_{e_k} \phi.$$

Proposition 4.3.10. *Let* D *be the Dirac operator of a Riemannian manifold* (M, g) *of dimension* m*. Then, for every* $f \in C^\infty(M)$ *and* $\phi \in \Gamma(\Sigma M)$ *the following results hold:*

i) $D(f\phi) = f D\phi + \mathrm{grad}(f) \cdot \phi$;

ii) *if* m *is even,* D *exchanges the positive and the negative part* $\Sigma^+ M$ *and* $\Sigma^- M$ *of* ΣM*;*

iii) D *is elliptic with principal symbol* $\sigma(D)(\xi)\phi = \xi^\sharp \cdot \phi$.

Proof. Points i) and ii) follow from the local expression of D and the properties of the Clifford multiplication and of the spinor derivative. We prove here the assertion iii) using the definition (4.17) for the principal symbol and the identity i). Let p be a given point of M, consider a smooth function f defined around p and let $df_p = \xi_p \in T_p^*M$. Then, for every $\phi \in \Gamma(\Sigma M)$ we have

$$\begin{aligned}
\sigma(D)_p(\xi_p)\phi(p) &= \lim_{s\to\infty} \frac{1}{s}\, e^{-sf(p)} D(e^{sf}\phi)(p) \\
&= \lim_{s\to\infty} \frac{1}{s} e^{-sf(p)} \left(e^{sf} D\phi + \mathrm{grad}(e^{sf})\cdot\phi\right)(p) \\
&= \lim_{s\to\infty} \frac{1}{s}(D\phi)(p) + \lim_{s\to\infty} \frac{1}{s} e^{-sf(p)} \left((de^{sf})^\sharp \cdot \phi\right)(p) \\
&= \left((df)^\sharp \cdot \phi\right)(p) \\
&= \xi_p^\sharp \cdot \phi(p),
\end{aligned}$$

since $\mathrm{grad}(f)(p) = (df)^\sharp(p) = \xi_p^\sharp$. □

The square of the Dirac operator $D^2 : \Gamma(\Sigma M) \to \Gamma(\Sigma M)$ is a second order linear differential operator. By [134], it satisfies the identity

$$D^2 = \Delta^{\Sigma M} + \frac{1}{4}\mathrm{Scal}(g),$$

where $\Delta^{\Sigma M} : \Gamma(\Sigma M) \to \Gamma(\Sigma M)$ is the *Bochner-Laplace operator* on spinors defined in terms of any local orthonormal frame $(e_1, \dots, e_m)$ for TM by

$$\Delta^{\Sigma M} = -\sum_{k=1}^{m} \left(\nabla_{e_k}\nabla_{e_k} + \mathrm{div}(e_k)\nabla_{e_k}\right).$$

The operator $-D^2$ is strongly elliptic. Indeed, using the property (4.18), point iii) of Proposition 4.3.10 and point ii) of Proposition 4.3.7, we get that its principal symbol is

$$\sigma(-D^2)(\xi)\phi = -\sigma(D)(\xi)(\sigma(D)(\xi)\phi) = -\xi^\sharp \cdot \xi^\sharp \cdot \phi = |\xi|^2\,\phi.$$

Consequently, we have

$$\langle \sigma(-D^2)(\xi)\phi, \phi\rangle = |\xi|^2|\phi|^2.$$

4.3.2 The spinors - SU(3)-structures correspondence

We already recalled at the end of Section 2.2.1 that on 6-manifolds there is a correspondence between SU(3)-structures and real spinor fields of length one, which is one-to-one up to a sign in the definition of the spinor field. Here, following in part the notations of [2], we review how the differential forms defining an SU(3)-structure can be obtained starting form a unit real spinor. The reader may refer also to [48, Sect. 2.7].

In dimension six, the real Clifford algebra $\mathcal{C}_6$ is isomorphic to $\mathrm{End}(\mathbb{R}^8)$ and the spinor representation is real and eight-dimensional. Denoted by $\Delta := \mathbb{R}^8$ the corresponding vector space, we have

$$\Delta \otimes_{\mathbb{R}} \mathbb{C} = \Delta_6 = \Delta_6^+ \oplus \Delta_6^-,$$

and

$$\Delta = \left\{\phi \in \Delta_6 \mid \phi = \overline{\phi}\right\}.$$

Consider the vector space $\mathbb{R}^6$ endowed with an inner product g and let $(e_1, \ldots, e_6)$ be an orthonormal basis of it. One possible realization of the real representation of $\mathcal{C}_6$ on Δ is the following (cf. [18])

$$\begin{aligned} e_1 &= +E_{18} + E_{27} - E_{36} - E_{45}, & e_2 &= -E_{17} + E_{28} + E_{35} - E_{46},\\ e_3 &= -E_{16} + E_{25} - E_{38} + E_{47}, & e_4 &= -E_{15} - E_{26} - E_{37} - E_{48},\\ e_5 &= -E_{13} - E_{24} + E_{57} + E_{68}, & e_6 &= +E_{14} - E_{23} - E_{58} + E_{67}, \end{aligned} \tag{4.21}$$

where $E_{kl} \in \mathfrak{so}(8)$ is the standard basis element mapping e_k to e_l, e_l to $-e_k$ and the remaining basis vectors to zero.

The space Δ is endowed with the inner product $(\cdot, \cdot)$ and with a Spin(6)-invariant endomorphism $j : \Delta \to \Delta$ defined by the element $j := e_1 \cdot \ldots \cdot e_6 \in \mathcal{C}_6$. The latter satisfies $j^2 = -\mathrm{Id}_\Delta$, and anti-commutes with the Clifford multiplication by vectors of $\mathbb{R}^6$, i.e., $j(v \cdot \phi) = -v \cdot j(\phi)$. In particular, j is the Spin(6)-invariant complex structure on Δ realizing the well-known isomorphism $\mathrm{Spin}(6) \cong \mathrm{SU}(4)$.

Remark 4.3.11. Comparing j with (4.20), it follows that the spaces $\Delta_6^\pm$ correspond to the $\pm i$-eigenspaces of the $\mathbb{C}$-linear extension of j to $\Delta_6 = \Delta \otimes_{\mathbb{R}} \mathbb{C}$.

Fix a nonzero real spinor $\phi \in \Delta$ of length one, i.e., $(\phi, \phi) = 1$. With respect to the scalar product $(\cdot, \cdot)$, there is an orthogonal decomposition of Δ given by

$$\Delta = \mathbb{R}\phi \oplus \mathbb{R}j(\phi) \oplus \left\{ v \cdot \phi \mid v \in \mathbb{R}^6 \right\}. \tag{4.22}$$

The endomorphism j preserves the subspace $\{v \cdot \phi \mid v \in \mathbb{R}^6\} \cong \mathbb{R}^6$ and it is possible to define an $\mathbb{R}$-linear map $J : \mathbb{R}^6 \to \mathbb{R}^6$ depending on ϕ in the following way

$$J(v) \cdot \phi = -j(v \cdot \phi).$$

J is well-defined by point i) of Proposition 4.3.7, it is a complex structure on $\mathbb{R}^6$, as $j^2 = -\mathrm{Id}_\Delta$, and it is g-orthogonal by point ii) of Proposition 4.3.7. Moreover, ϕ induces a stable 3-form on $\mathbb{R}^6$ given for every $v, w, z \in \mathbb{R}^6$ by

$$\psi_+(v, w, z) = -(v \cdot w \cdot z \cdot \phi, j(\phi)),$$

and (g, J, ψ_+) is an SU(3)-structure on $\mathbb{R}^6$. Its fundamental form is $\omega(v, w) = g(J(v), w)$ and the corresponding complex $(3,0)$-form Ψ has imaginary part

$$\psi_-(v, w, z) = J\psi_+(v, w, z) = -\psi_+(Jv, w, z) = -(v \cdot w \cdot z \cdot \phi, \phi).$$

Conversely, starting from an SU(3)-structure (ω, ψ_+) it is possible to construct a unit real spinor, which turns out to be unique up to a sign.

Remark 4.3.12. The previous definitions are consistent with our conventions fixed in Chapter 2, but differ slightly from those given in [2]. In particular, J here is $-J_\phi$ there, and the complex volume form there is $\psi_\phi + i\psi_\phi^J$ with $\psi_\phi = \psi_-$ and $\psi_\phi^J = \psi_+$.

Consider now a Riemannian spin manifold (M, g) of dimension six with a real spinor field ϕ of length one. The differential forms defining the SU(3)-structure associated with ϕ are obtained from it as described before. Moreover, it follows from decomposition (4.22) that there exist a unique 1-form $\eta \in \Omega^1(M)$ and a unique $A \in \Gamma(T^*M \otimes TM)$ such that for every $X \in \mathfrak{X}(M)$

$$\nabla_X \phi = \eta(X) j(\phi) + A(X) \cdot \phi. \tag{4.23}$$

The intrinsic torsion of the SU(3)-structure corresponding to ϕ is determined by η and A and all of the classes of SU(3)-structures can be characterized using them.

This was done for several classes in the recent paper [2], while some partial results have been shown before. Let us recall some of them pointing out their relation with (4.23).

The SU(3)-structure induced by ϕ is nearly Kähler (SU(3)-type $\mathcal{W}_1^-$) if and only if $A = l\,\mathrm{Id}$, $l \in \mathbb{R} - \{0\}$, and $\eta = 0$, that is, if and only if

$$\nabla_X \phi = lX \cdot \phi.$$

Thus, ϕ is a real Killing spinor and the result of [93] is recovered. In this case, ϕ is an eigenspinor of the Dirac operator with constant eigenfunction, indeed

$$D\phi = \sum_{k=1}^{6} e_k \cdot \nabla_{e_k} \phi = l \sum_{k=1}^{6} e_k \cdot e_k \cdot \phi = -6\,l\,\phi.$$

When A is symmetric and $\eta = 0$, ϕ is called *generalized Killing spinor* (cf. [15]). The corresponding SU(3)-structure is half-flat, as shown in [51], while it is coupled if and only if A also commutes with J, as one can deduce from [2, Lemma 3.5]. Furthermore, the spinor defining a half-flat SU(3)-structure is an eigenspinor of D,

$$D\phi = f\,\phi,$$

and from the general expression of $D\phi$ given in [2] and [2, Lemma 3.11], we deduce that $f = \frac{3}{2}\,w_1^-$. From Lemma 2.3.10, we then have the following

Proposition 4.3.13. *Let ϕ be a unit real spinor field defining a coupled* SU(3)*-structure with coupled constant c on a connected Riemannian 6-manifold (M, g). Then, $D\phi = -c\,\phi$.*

Remark 4.3.14. Given a real spinor ϕ, the general expression of $D\phi$ has the form

$$D\phi = \frac{3}{2}\,w_1^-\phi + \beta\,j(\phi),$$

where β depends on the intrinsic torsion forms w_1^+, w_4 and w_5 (cf. [2]). In particular, $D\phi$ is still a real spinor.

4.3.3 The $(-D^2)$-flow

Due to the correspondence between real spinor fields and SU(3)-structures on a Riemannian 6-manifold (M, g), instead of studying evolution equations for the differential forms ω and ψ_+, we may look for flows evolving a spinor field. The advantage of this approach is that we have to control only one object instead of two objects and the compatibility conditions they have to satisfy. We describe here some preliminary results of [72].

Let (M, g) be a compact Riemannian spin manifold and let $\phi(t) \in \Gamma(\Sigma M)$ be a family of real spinor fields depending smoothly on a real parameter t. It is quite natural to consider the evolution equation for $\phi(t)$ (see also [76, Ch. 4] for the four-dimensional case)

$$\frac{\partial}{\partial t}\phi(t) = -D^2\phi(t),$$

which we may call the $(-D^2)$-*flow*.

Definition 4.3.15. Let ϕ_0 be a real spinor field of length one on M. A one-parameter family of real spinor fields $\phi(t) \in \Gamma(\Sigma M)$ is a *solution* of the $(-D^2)$-flow with initial condition ϕ_0 if

$$\begin{cases} \frac{\partial}{\partial t}\phi(t) = -D^2\phi(t) \\ \phi(0) = \phi_0 \end{cases} . \tag{4.24}$$

Since $-D^2$ is a strongly elliptic second order linear differential operator, the proof of the following result is immediate

Theorem 4.3.16. *Given a compact Riemannian spin manifold (M, g) and a real spinor field of unit length $\phi_0 \in \Gamma(\Sigma M)$, there exists a unique solution of the $(-D^2)$-flow defined on $[0, \varepsilon)$ for a certain $\varepsilon > 0$.*

Observe that, under the hypothesis of the previous theorem, the solution $\phi(t)$ of (4.24) is non-vanishing for each $t \in [0, \varepsilon)$, as $\phi(t)$ depends smoothly on t and being non-vanishing is an open condition. When M is six-dimensional, we can then normalize $\phi(t)$ using the metric $(\cdot, \cdot)$ and get an SU(3)-structure on M depending on t. Therefore, the flow at the spinor level translates into a flow of SU(3)-structures on M leaving the metric g fixed. Using the general identities relating the spinor field to

J and ψ_+, we should then be able to obtain the evolution equations of the tensors defining the SU(3)-structure.

The main problem with (4.24) and this kind of approach is represented by the following type of solutions

Definition 4.3.17. We say that a solution $\phi(t)$ of (4.24) is *self-similar* if there exists a smooth non-vanishing function $h : [0, \varepsilon) \to \mathbb{R}$ such that

$$\phi(t) = h(t)\phi_0.$$

In this case, the normalization of $\phi(t)$ is exactly ϕ_0 and the corresponding SU(3)-structure does not evolve.

Proposition 4.3.18. *Suppose that ϕ_0 is an eigenspinor of the Dirac operator D with constant eigenfunction α, i.e., $D\phi_0 = \alpha\phi_0$ and $\alpha \in \mathbb{R}$. Then, the solution of* (4.24) *starting from ϕ_0 is still an eigenspinor of D with eigenfunction α.*

Proof. Let $\phi(t)$ be the unique solution of the flow (4.24) starting from ϕ_0 and consider the spinor $\hat{\phi}(t) = D\phi(t) - \alpha\phi(t) + \phi(t)$. Observe that $\hat{\phi}(0) = \phi_0$ and

$$\frac{\partial}{\partial t}\hat{\phi}(t) = D(-D^2\phi(t)) + \alpha D^2\phi(t) - D^2\phi(t) = -D^2(\hat{\phi}(t)),$$

as D does not depend on t. By the uniqueness of solutions of (4.24), we then get $D\phi(t) = \alpha\phi(t)$. □

Corollary 4.3.19. *If ϕ_0 is an eigenspinor of the Dirac operator D with constant eigenfunction $\alpha \in \mathbb{R}$, then the solution of* (4.24) *starting from ϕ_0 is self-similar with $h(t) = \mathrm{e}^{-\alpha^2 t}$.*

Proof. We know that $D\phi(t) = \alpha\phi(t)$, thus from the flow equation we obtain

$$\frac{\partial}{\partial t}\phi(t) = -\alpha^2\phi(t)$$

and from this follows that

$$\phi(t) = \mathrm{e}^{-\alpha^2 t}\phi_0.$$

□

As a consequence of this result, on 6-manifolds solutions of (4.24) starting from an eigenspinor ϕ_0 of D with constant eigenfunction cannot be used to construct a family of SU(3)-structures depending on t in the way previously described. In particular, if ϕ_0 induces a coupled SU(3)-structure, then we do not get a family of SU(3)-structures depending on t and starting from the coupled SU(3)-structure by Proposition 4.3.13.

We examine now some examples on six-dimensional real Lie algebras. First, observe that if $\mathfrak{g}$ is a six-dimensional metric Lie algebra with inner product g and $(e_1, \ldots, e_6)$ is a g-orthonormal basis of $\mathfrak{g}$, then for a fixed spinor $\phi \in \Delta = \mathbb{R}^8$, we have

$$\nabla_{e_r}\phi = \frac{1}{2}\sum_{1\leq k<l\leq 6} \Gamma^l_{rk}\, e_k \cdot e_l \cdot \phi.$$

Moreover, the expression of the Christoffel symbols Γ^l_{rk} on $\mathfrak{g}$ with respect to the basis $(e_1, \ldots, e_6)$ can be obtained from the identity

$$2g(\nabla_X Y, Z) = g([X, Y], Z) + g([Z, X], Y) + g([Z, Y], X).$$

It is

$$\Gamma^l_{rk} = \frac{1}{2}g^{nl}(c^s_{rk}g_{sn} + c^s_{nr}g_{sk} + c^s_{nk}g_{sr}),$$

where c^s_{rk} are the structure constants of $\mathfrak{g}$ with respect to the considered basis.

Example 4.3.20. Consider the Lie algebra $\mathfrak{n}_{28}$ with structure equations

$$(0, 0, 0, 0, e^{14} + e^{23}, e^{13} - e^{24})$$

with respect to a basis $(e^1, \ldots, e^6)$ of $\mathfrak{n}^*_{28}$. Endow $\mathfrak{n}_{28}$ with the inner product g for which the dual basis $(e_1, \ldots, e_6)$ of $(e^1, \ldots, e^6)$ is orthonormal. The corresponding Clifford algebra is multiplicatively generated by $e_1, \ldots, e_6$ and we can choose the real representation of it on $\Delta = \mathbb{R}^8$ described in (4.21). The spinor

$$\phi_0 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0, 0, 0, 0, 0\right)^{\mathrm{T}} \in \Delta,$$

where T denotes matrix transposition, is an eigenspinor of D with eigenfunction $\alpha = 1$. A simple computation shows that the SU(3)-structure on $\mathfrak{n}_{28}$ associated with

ϕ_0 is the coupled SU(3)-structure we widely considered in the previous chapters, namely

$$\omega = e^{12} + e^{34} + e^{56}, \quad \psi_+ = e^{135} - e^{146} - e^{236} - e^{245},$$

with coupled constant $c = -1 = -\alpha$.

In this situation, we know that the solution of the $(-D^2)$-flow starting from ϕ_0 is self-similar. More precisely, it is

$$\phi(t) = \mathrm{e}^{-t}\phi_0.$$

Remark 4.3.21. Observe that in the case of Lie algebras the flow equation in (4.24) is a system of linear ODEs. Therefore, the solution starting from a given spinor $\phi_0 \in \Delta$ is

$$\phi(t) = \exp(-tD^2)\phi_0,$$

where we have identified D^2 with the matrix associated with it with respect to the canonical basis of $\Delta = \mathbb{R}^8$. In particular, if ϕ_0 is an eigenspinor of D with eigenfunction $\alpha \in \mathbb{R}$, we obtain again that

$$\phi(t) = \exp(-tD^2)\phi_0 = \sum_{k=0}^{+\infty} \frac{1}{k!}(-\alpha^2)^k t^k \phi_0 = \mathrm{e}^{-\alpha^2 t}\phi_0.$$

We conclude examining a non-trivial example and the behaviour of the solution.

Example 4.3.22. Consider the nilpotent Lie algebra $\mathfrak{n}_9$ with structure equations

$$\left(0, 0, 0, e^{13}, e^{14} + e^{23}, e^{13} - e^{15} - e^{24}\right)$$

with respect to a basis $(e^1, \ldots, e^6)$ of $\mathfrak{n}_9^*$, and endow it with the inner product $g = \sum_{k=1}^{6}(e^k)^2$. The spinor

$$\phi_0 = \left(\frac{1}{\sqrt{2}}, 0, 0, 0, \frac{1}{\sqrt{2}}, 0, 0, 0\right)^{\mathrm{T}} \in \Delta$$

induces the following SU(3)-structure on $\mathfrak{n}_9$

$$\begin{aligned} \omega &= -e^{15} + e^{34} + e^{26}, \\ \psi_+ &= e^{124} - e^{136} - e^{235} + e^{456}, \\ \psi_- &= -e^{123} - e^{146} - e^{245} - e^{356}, \end{aligned}$$

and the corresponding almost complex structure is

$$J = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

It is easy to check that the only vanishing intrinsic torsion form of this SU(3)-structure is w_1^- and that ϕ_0 is not an eigenspinor of the Dirac operator.

Let $\phi = (a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8)^{\mathrm{T}} \in \Delta$ be a generic real spinor, where $a_i = a_i(t)$ are real valued functions depending on t. Then,

$$D^2\phi = \left(a_1, a_2, \frac{1}{4}a_3, \frac{1}{4}a_4, \frac{1}{4}a_5, \frac{1}{4}a_6, 0, 0\right)$$

and from the flow equation in (4.24) we obtain the following system of ODEs

$$\begin{cases} \frac{d}{dt}a_k = a_k, & k = 1, 2, \\ \frac{d}{dt}a_k = \frac{1}{4}a_k, & k = 3, 4, 5, 6, \\ \frac{d}{dt}a_k = 0, & k = 7, 8. \end{cases}$$

The solution of the $(-D^2)$-flow starting from ϕ_0 is then

$$\phi(t) = \left(\frac{1}{\sqrt{2}}\mathrm{e}^t, 0, 0, 0, \frac{1}{\sqrt{2}}\mathrm{e}^{\frac{t}{4}}, 0, 0, 0\right),$$

it is defined for every $t \in \mathbb{R}$ and normalizing it, we get

$$\Phi(t) = \left(\frac{\sqrt{2}\mathrm{e}^t}{\sqrt{2\,\mathrm{e}^{2\,t} + 2\,\mathrm{e}^{\frac{t}{2}}}}, 0, 0, 0, \frac{\sqrt{2}\mathrm{e}^{\frac{t}{4}}}{\sqrt{2\,\mathrm{e}^{2\,t} + 2\,\mathrm{e}^{\frac{t}{2}}}}, 0, 0, 0\right).$$

$\Phi(t)$ gives rise to a one-parameter family of SU(3)-structures on $\mathfrak{n}_9$ inducing the inner product g for each t. We summarize the corresponding tensors here:

$$\begin{aligned}
\omega(t) &= \frac{e^{\frac{3}{2}t}-1}{e^{\frac{3}{2}t}+1}(e^{12}+e^{56}) - 2\,\frac{e^{\frac{3}{4}t}}{e^{\frac{3}{2}t}+1}(e^{15}-e^{26}) + e^{34},\\
\psi_+(t) &= 2\,\frac{e^{\frac{3}{4}t}}{e^{\frac{3}{2}t}+1}(e^{124}+e^{456}) - e^{136} - e^{235} + \frac{e^{\frac{3}{2}t}-1}{e^{\frac{3}{2}t}+1}(-e^{145}+e^{246}),\\
\psi_-(t) &= -2\,\frac{e^{\frac{3}{4}t}}{e^{\frac{3}{2}t}+1}(e^{123}+e^{356}) - e^{146} - e^{245} + \frac{e^{\frac{3}{2}t}-1}{e^{\frac{3}{2}t}+1}(e^{135}-e^{236}),
\end{aligned}$$

$$J(t) = \begin{bmatrix}
0 & -\frac{e^{\frac{3}{2}t}-1}{e^{\frac{3}{2}t}+1} & 0 & 0 & 2\,\frac{e^{\frac{3}{4}t}}{e^{\frac{3}{2}t}+1} & 0\\
\frac{e^{\frac{3}{2}t}-1}{e^{\frac{3}{2}t}+1} & 0 & 0 & 0 & 0 & -2\,\frac{e^{\frac{3}{4}t}}{e^{\frac{3}{2}t}+1}\\
0 & 0 & 0 & -1 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0\\
-2\,\frac{e^{\frac{3}{4}t}}{e^{\frac{3}{2}t}+1} & 0 & 0 & 0 & 0 & -\frac{e^{\frac{3}{2}t}-1}{e^{\frac{3}{2}t}+1}\\
0 & 2\,\frac{e^{\frac{3}{4}t}}{e^{\frac{3}{2}t}+1} & 0 & 0 & \frac{e^{\frac{3}{2}t}-1}{e^{\frac{3}{2}t}+1} & 0
\end{bmatrix}.$$

Observe that for every $t \in \mathbb{R}$ we have

$$\frac{1}{6}(\omega(t))^3 = \frac{1}{4}\psi_+(t) \wedge \psi_-(t) = e^{123456}$$

and

$$\lambda(\psi_+(t)) = -4\left(e^{123456}\right)^{\otimes 2}.$$

Moreover,

$$w_1^-(t) = 0.$$

If we let $t \to +\infty$, the *limit solution* is

$$\begin{aligned}
\omega^\infty &= -e^{12} + e^{34} - e^{56},\\
\psi_+^\infty &= e^{145} - e^{136} - e^{235} - e^{246},\\
\psi_-^\infty &= -e^{135} - e^{146} - e^{245} + e^{236},
\end{aligned}$$

and its vanishing intrinsic torsion forms are w_1^+, w_1^-, w_2^+, w_4.

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