ON THE FIRST PASSAGE TIME FOR AUTOREGRESSIVE PROCESSES

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Abstract. The first passage time problem for an autoregressive process AR(p) is examined. When the innovations are gaussian, the determination of the first passage time probability distribution is closely related to computing a multidimensional integral of a suitable gaussian random vector, known in the literature as orthant probability. Recursive equations involving the first passage time probability distribution are given and a numerical scheme is proposed which takes advantage of the recursion. Compared with the existing procedures in the literature, the algorithm we propose is computationally less expensive and reaches a very good accuracy. The accuracy is tested on some closed form expressions we achieve for special choices of the AR(p) parameters.

1 Introduction Let W_t with $t \in \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ be a sequence of independent and identically distributed (i.i.d.) random variables (r.v.'s) on a probability space $(\Omega, \mathbb{F}, P)$. A general linear process $\{X_t\}$ can be defined as

$$X_t = \sum_{t=-\infty}^{\infty} \alpha_t W_t, \tag{1.1}$$

where $\{\alpha_t\}$ is a sequence of real numbers. The autoregressive AR(p), the moving average MA(q) and the autoregressive moving average ARMA(p,q) are all special cases of linear processes. Discrete time series models are commonly used to represent a wide variety of data: from storage models to exchange rates, from growth of populations to human endeavor.

The purpose of the present paper is to determine the probability distribution (p.d.) of the random variable (r.v.) representing the instant when, for the first time, a dynamic system by enters a preassigned critical region of the state space. In making decisions, we may want to know how likely it is that the process will attain a certain high level before it drops back to or even below the present level, or we may want to know the expected time for the process to reach a certain level. So, the attention will be devoted to the instance in which such a critical region collapses into one single point, while the system is described mathematically by a linear process.

As first step of a theory involving more general linear processes, we examine AR(p) models, one of the most important among time series. Studies related to the first passage time (FPT) problem for AR(1) sequences are often employed in application fields such as surveillance analysis [8], signal detection and many other areas. In economy, the role played by the AR(p) models is well known and confirmed by a wide and detailed literature, see for instance [3]. Recently, AR(p) models have been proposed to study neuronal interspike intervals, assuming the greater is the dependency of the neuronal interspike intervals, the stronger is the memory of the process (see [11] and references therein).

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Let us recall that the analytical results on FPT problems are mostly centered on stochastic processes of diffusion type, where the Markov property plays a leading role in handling the related transition probability density function (pdf), see [16] for a review. FPT distributions have explicit analytical expressions also for the class of stochastic processes named the Levy type anomalous diffusion, in which the mean square displacement of the diffusive variable X_t scales with time as t^γ with $0 < \gamma < 2$ (see [15] and references therein). Nevertheless, no closed forms of FPT distributions are available in the literature, apart from few special cases. Thus numerical algorithms or simulation procedures have been resorted to in order to get more information on the FPT problem. In particular, simulation procedures are suitable to being implemented on parallel computers, see [6].

A typical simulation procedure samples n values of the FPT r.v., by a suitable construction of $N \geq n$ time-discrete paths, so that suitable estimators are obtained. However, as shown through examples in the last section, a Monte Carlo simulation via (1.1) is not always a proper method to approximate distribution and expectation of FPT r.v.'s with high accuracy, see also [7]. So different numerical procedures are hoped for.

The martingale technique has already been used in [13] and [14] for deriving analytical approximations to the distribution and expectation of the FPT r.v. for discrete and continuous time AR(1) type processes. An integral-equation approach, together with the state space representation of time series models, are applied in [2] in order to evaluate exit probabilities from bounded regions for AR(p) models. Numerical schemes are used to solve the constructed integral equations.

Here, by using the state space representation of time series models, we derive recursive integral equations for the FPT p.d. Recursive integration methodologies have many statistical applications and significantly reduce the computational time when applied to the evaluation of high dimensional integrals, see [10]. Here, we propose a Gauss-Laguerre quadrature formula, in order to evaluate numerically these recursive equations. The accuracy of the proposed method is tested on some closed form expressions achieved in correspondence of special values of the parameters $\{\alpha_t\}$. Due to the connection of the FPT problem with the evaluation of orthant probabilities [5], the method is analyzed for gaussian AR(p) models.

The paper is structured as follows. In Section 2 we define the FPT r.v. for an autoregressive process AR(p), showing its connection with the computation of the so-called orthant probabilities, when the innovations are gaussian. In Section 3, we recall the state space representation of an AR(p) model, computing the recursive structure of the vector mean and of the covariance matrix. In Section 4, we analyze the asymptotic behavior of the FPT p.d. and evaluate the occupation time. In Section 5, recursive equations are given characterizing the FPT p.d. The fairness of the FPT r.v. is also stated. Orthant probabilities play a special role in expressing the mean of the FPT r.v., when we add more hypotheses on $\{\alpha_t\}$. Section 6 is devoted to the computation of taboo probabilities and Section 7 gives upper and lower bounds for the FPT cumulative distribution function (cdf). In Section 8, we show comparisons of some numerical results with the ones existing in the literature.

2 Autoregressive processes A process $\{X_n\}$ is called an *autoregression* of order $p \in \mathbb{N}$, or AR(p) model, if it satisfies

$$X_n = \alpha_1 X_{n-1} + \cdots + \alpha_p X_{n-p} + W_n \qquad n \geq 1 \tag{2.1}$$

for some $\alpha_1, \ldots, \alpha_p \in \mathbb{R}$, where $(X_0, X_{-1}, \ldots, X_{1-p})$ is the vector of initial values and $\{W_n\}$ is a sequence of i.i.d. r.v.'s, named *innovations*, with finite mean and finite variance. In the following we assume W_n having a gaussian distribution $N(0, \sigma^2)$, which is the most natural choice to describe a noise component.

Define the FPT r.v. T of $\{X_n\}$ through a constant boundary S as

$$T = \min_{n\geq 1}\{X_n \geq S\},$$

where $\mathbb{P}(X_0 < S, X_{-1} < S, \dots , X_{1-p} < S) = 1$. Set $\mathbf{X}_0 = (X_0, X_{-1}, \dots , X_{1-p})^T$. In the following, we assume the process starts from a fixed position under the boundary, i.e.

$$\mathbb{P}(\mathbf{X}_0 = \mathbf{x}_0) = 1, \tag{2.2}$$

with $\mathbf{x}_0 \in (-\infty, S)^p$ and denote by $P_{\mathbf{x}_0}(T = n)$ the probability that the process $\{X_n\}$ reaches the boundary at the time n, being started in $\mathbf{x}_0$. The region $O_p = (-\infty, S)^p$ is known in the literature as *orthant region*, see [17].

In order to evaluate the FPT p.d., observe that, if $p = 1$, $\{X_n\}$ is trivially a Markov chain and it can be viewed in one sense as an extension of a gaussian random walk [12]. If $p > 1$, the Markov property does not hold anymore so that different considerations should be done in order to gain some information on the FPT problem.

First of all, we have

$$P_{\mathbf{x}_0}(T = n) = \mathbb{P}(X_1 < S, X_2 < S, \dots , X_{n-1} < S, X_n \geq S|\mathbf{X}_0 = \mathbf{x}_0) \tag{2.3}$$

and due to the hypotheses on $\{X_n\}$,

$$P_{\mathbf{x}_0}(T = n) = \int_{D_n} \frac{1}{(2\pi)^{n/2}|\Sigma_n|^{1/2}} \exp\left\{-\frac{1}{2}(\mathbf{x} - \mathbf{m}_n)^T \Sigma_n^{-1}(\mathbf{x} - \mathbf{m}_n)\right\} \mathrm{d}\mathbf{x},$$

where $D_n = (-\infty, S)^{n-1} \times [S, \infty)$ and $\mathbf{m}_n$ and Σ_n are respectively mean vector and covariance matrix of the vector $(X_1, X_2, \dots , X_n)^T$. By using (2.1) and (2.2), it can be stated $|\Sigma_n| > 0$ for every $n \geq 1$. In the following, we will provide a proof by using some different technicalities (see Remark 3.1).

Equation (2.3) can be rewritten as

$$P_{\mathbf{x}_0}(T = n) = \begin{cases} 1 - \mathcal{P}_1(S, \Sigma_1), & \text{if } n = 1,\\ \mathcal{P}_{n-1}(S, \Sigma_{n-1}) - \mathcal{P}_n(S, \Sigma_n), & \text{if } n > 1,\end{cases} \tag{2.4}$$

where

$$\begin{aligned} \mathcal{P}_n(S, \Sigma_n) &= \mathbb{P}\left[\cap_{i=1}^n (X_i < S)|\, \mathbf{X}_0 = \mathbf{x}_0\right] \\ &= \int_{O_n} \frac{1}{(2\pi)^{n/2}|\Sigma_n|^{1/2}} \exp\left\{-\frac{1}{2}(\mathbf{x} - \mathbf{m}_n)^T \Sigma_n^{-1}(\mathbf{x} - \mathbf{m}_n)\right\} \mathrm{d}\mathbf{x} \end{aligned} \tag{2.5}$$

are the so-called *orthant* probabilities, due to the integration regions $O_n = (-\infty, S)^n$. A more general orthant probability is $\mathcal{P}_n(\mathbf{S}, \Sigma_n)$ with $O_n = \times_{i=1}^n(-\infty, S_i)$ and $\mathbf{S}$ the vector $(S_1, S_2, \dots , S_n)$.

So, the problem of characterizing $P_{\mathbf{x}_0}(T = n)$ could be bring back to the evaluation of orthant probabilities. Unless $n \leq 3$ or $\Sigma_n = I_n$, where I_n is the identity matrix, no closed form expressions are known for $\mathcal{P}_n(S, \Sigma_n)$ so its evaluation requires numerical methods, see [4] for an updated review. The main problem of numerical evaluations of $\mathcal{P}_n(S, \Sigma_n)$ is the growing complexity of the involved multidimensional integrals depending on n.

The relation between orthant probabilities and the FPT problem discloses a new method to evaluate such multivariate integrals [5]. Here, by means of the state space representation of $\{X_n\}$ as Markov chain on $\mathbb{R}^p$ [3], we give recursive equations for the FPT p.d. which turn out to be useful also in computing orthant probabilities.

3 Representation of an autoregressive process as Markov chain Set

$$F = \begin{pmatrix} \alpha_1 & \ldots & \alpha_{p-1} & \alpha_p \\ 1 & \ldots & 0 & 0 \\ \vdots & \ldots & \vdots & \vdots \\ 0 & \ldots & 1 & 0 \end{pmatrix} \quad \text{and} \quad G = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Then, equation (2.1) can be rewritten as:

$$\mathbf{Y}_n = F\mathbf{Y}_{n-1} + GW_n, \tag{3.1}$$

with $\mathbf{Y}_n = (X_n, X_{n-1}, \ldots, X_{n-p+1})^T$ and $\mathbf{Y}_0 = \mathbf{X}_0$. Being F and G not dependent from n, the Markov chain $\{\mathbf{Y}_n\}$ is temporally homogeneous. The conditional pdf of $\mathbf{Y}_n$ given $\mathbf{Y}_{n-1} = \mathbf{x}$ is

$$f_{\mathbf{Y}_n|\mathbf{Y}_{n-1}}(\mathbf{y}|\mathbf{x}) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2}(\mathbf{y} - F\mathbf{x})^T(\mathbf{y} - F\mathbf{x})\right\}, \tag{3.2}$$

where $\mathbf{x}$ and $\mathbf{y}$ are column vectors of $\mathbb{R}^p$. Due to the temporally homogeneous property, the function $f_{\mathbf{Y}_n|\mathbf{Y}_{n-1}}(\mathbf{y}|\mathbf{x})$ does not depend on n, so it will be denoted by $f(\mathbf{y}|\mathbf{x})$. Moreover, if $\mathbf{x} = (x_1, x_2, \ldots, x_p)^T$ then $\mathbf{y} = (y, x_1, \ldots, x_{p-1})^T$ due to (3.1), so that the conditional pdf (3.2) becomes

$$f(\mathbf{y}|\mathbf{x}) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2\sigma^2}\left(y - \sum_{i=1}^{p} \alpha_i x_i\right)^2\right\} = h(y|\mathbf{x}), \quad y \in \mathbb{R} \tag{3.3}$$

which is crucial in our discussion, as we will see later on.

Proposition 3.1 *If $\mathbb{P}(\mathbf{Y}_0 = \mathbf{x}_0) = 1$, then $\mathbf{Y}_n$ is a multidimensional gaussian vector with mean $\mathbf{m}_n$ and covariance matrix $\Sigma^{\mathbf{Y}_n}$*

$$\mathbf{m}_n = F^n \mathbf{x}_0 \quad \textit{and} \quad \Sigma^{\mathbf{Y}_n} = \sigma^2 C_n C_n^T, \tag{3.4}$$

where C_n is a $p \times n$ matrix built collocating side by side the vectors $F^j G, j = 0, 1, \ldots, n-1$, that is

$$C_n = [F^{n-1}G|F^{n-2}G|\ldots|FG|G]. \tag{3.5}$$

Proof: By induction on n, equation (3.1) gives

$$\mathbf{Y}_n = F^n \mathbf{Y}_0 + \sum_{j=1}^{n} F^{n-j} G W_j = F^n \mathbf{Y}_0 + C_n \mathbf{W}_n, \tag{3.6}$$

where $\mathbf{W}_n = (W_1, W_2, \ldots, W_n)^T$ and C_n is given in (3.5).

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If $p = 1$, then $\mathbf{Y}_n = X_n$ and $\mathbf{y}_0 = x_0$. So we have $E[X_n] = \alpha_1^n x_0$ and $C_n = (\alpha_1^{n-1}, \ldots, \alpha_1, 1)$. The following proposition gives the expression of C_n elements, when $p > 1$.

Proposition 3.2 *Assume C_n given in (3.5), then*

$$(C_n)_{i,j} = \begin{cases} \mathbf{a}_1^{(n-(i+j)+1)}, & \textit{if } j < n - i + 1, \\ 1, & \textit{if } j = n - i + 1, \\ 0, & \textit{if } j > n - i + 1, \end{cases} \tag{3.7}$$

where $\mathbf{a}_1^{(j)}$ *is the first component of a sequence of row vectors* $\{\mathbf{a}^{(j)}\}_{j\geq 1}$, *recursively defined by*

$$\mathbf{a}^{(0)} = G \quad \textit{and} \quad \begin{cases} \mathbf{a}_i^{(j)} = \alpha_i\, \mathbf{a}_1^{(j-1)} + \mathbf{a}_{i+1}^{(j-1)}, & i = 1, 2, \ldots, p-1, \\ \mathbf{a}_p^{(j)} = \alpha_p\, \mathbf{a}_1^{(j-1)}. \end{cases} \tag{3.8}$$

Proof: By induction on n, set $n = 1$. Being $C_1 = G$, then (3.7) follows immediately. Suppose (3.7) holds for $n = k-1$. Being $C_k = [FC_{k-1}|G]$, then (3.7) holds for $i \geq 2$, whereas for $i = 1$ we have

$$(C_k)_{1,k-j} = \sum_{i=1}^{\min\{j,p\}} \alpha_i\, \mathbf{a}_1^{(j-i)} \quad \text{for } j = k-1, k-2, \ldots, 1.$$

From (3.8) for $i = 1$, we have $\mathbf{a}_1^{(j)} = (C_k)_{1,k-j}$ for $j = k-1, k-2, \ldots, 1$ by which the result follows.

■

Since C_n is a triangular matrix whose elements on the second diagonal are all different from zero, the covariance matrix $\Sigma^{\mathbf{Y}_n}$ has full rank. Moreover from (3.4), we have

$$(\Sigma^{\mathbf{Y}_n})_{i,j} = \sigma^2 \sum_{t=i}^{n-j+i} \mathbf{a}_1^{(n-t)} \mathbf{a}_1^{(n-t-j+i)} \quad \text{for } j > i,$$

so that

$$\mathrm{Var}(X_k) = \sigma^2 \left(1 + \sum_{i=1}^{k-1} [\mathbf{a}_1^{(i)}]^2 \right) \qquad k = n-p+1, \ldots, n. \tag{3.9}$$

If $p = 1$, then $\mathrm{Var}(X_n) = \sigma^2 \sum_{i=0}^{n-1} \left(\alpha_1^2\right)^i$. Equations (3.8) allow to compute explicitly the mean vector $\mathbf{m}_n$, by calculating the powers of F.

Proposition 3.3 *We have*

$$F^n = \begin{cases} (\mathbf{a}^{(n)}, \mathbf{a}^{(n-1)}, \ldots, \mathbf{a}^{(n-p+1)})^T, & \textit{if} \quad n \geq p, \\ (\mathbf{a}^{(n)}, \mathbf{a}^{(n-1)}, \ldots, \mathbf{a}^{(1)}, \mathbf{e}^{(1)}, \ldots, \mathbf{e}^{(p-n)})^T, & \textit{if} \quad n < p, \end{cases} \tag{3.10}$$

where $\{\mathbf{a}^{(j)}\}_{j\geq 1}$ *is given in (3.8) and*

$$\mathbf{e}_i^{(j)} = \begin{cases} 1, & \textit{if} \quad i = j, \\ 0, & \textit{if} \quad i \neq j. \end{cases}$$

Proof: Observe that $F = (\mathbf{a}^{(1)}, \mathbf{e}^{(1)}, \ldots, \mathbf{e}^{(p-1)})^T$. Equation (3.10) follows by induction on n and by the following identity

$$\mathbf{a}_k^{(j)} = \sum_{i=1}^{\min\{j,p\}} \alpha_i\, \mathbf{a}_1^{(j-i)}, \quad \text{for } j \geq 1 \text{ and } k = 1, 2, \ldots, p.$$

The above identity can be stated by applying recursively (3.8).

■

Remark 3.1 From the first equation in (3.6) and Proposition 3.3, we have

$$X_n = \mathbf{a}^{(n)}\mathbf{x}_0 + \sum_{j=1}^{n} \mathbf{a}_1^{(n-j)} W_j,$$

so that

$$\mathrm{Cov}(X_{k+n}, X_k) = E\left[\left(\sum_{j=1}^{k+n} \mathbf{a}_1^{(k+n-j)} W_j\right)\left(\sum_{j=1}^{k} \mathbf{a}_1^{(k-j)} W_j\right)\right] = \sigma^2 \sum_{j=1}^{k} \mathbf{a}_1^{(k-j)} \mathbf{a}_1^{(k+n-j)},$$

which gives (3.9) when $n = 0$. In particular in (2.5) we have $\Sigma_n = L_n\, L_n^T$ where

$$(L_n)_{i,j} = \begin{cases} 0, & \text{if} \quad i < j, \\ 1, & \text{if} \quad i = j, \\ \mathbf{a}_1^{(i-j)}, & \text{if} \quad i > j. \end{cases}$$

The matrix L_n has full rank, since it is a lower triangular matrix with elements on the main diagonal all equal to 1.

4 Occupation time For every borel set $B \subset \mathbb{R}$, the occupation time η_B is the number of visits of X_n to B after time zero. When $B = (S, \infty)$, we denote by η_S the number of times that the process is over S :

$$\eta_S = \sum_{n=1}^{\infty} \mathbf{1}(X_n > S), \tag{4.1}$$

where $\mathbf{1}(\cdot)$ is the indicator function. From (4.1), we have

$$E[\eta_S | \mathbf{Y}_0 = \mathbf{x}_0] = \sum_{n=1}^{\infty} \mathbb{P}\left(X_n > S | \mathbf{Y}_0 = \mathbf{x}_0\right). \tag{4.2}$$

In order to compute $E[\eta_S|\mathbf{Y}_0 = \mathbf{x}_0]$, we need to evaluating $\mathbb{P}\left(X_n > S|\mathbf{Y}_0 = \mathbf{x}_0\right)$. Due to Remark 3.1, we have

$$\mathbb{P}\left(X_n > S|\mathbf{Y}_0 = \mathbf{x}_0\right) = \frac{1}{2}\left\{1 - \mathrm{Erf}\left[\frac{S - \mathbf{a}^{(n)}\mathbf{x}_0}{\sqrt{2\sigma^2\left(1 + \sum_{i=1}^{n-1}[\mathbf{a}_1^{(i)}]^2\right)}}\right]\right\}, \tag{4.3}$$

where

$$\mathrm{Erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2)\, \mathrm{d}t, \quad x \in \mathbb{R}.$$

Theorem 4.1 *If the eigenvalues of F fall within the open unit disk in $\mathbb{R}$, then*

$$\lim_{n \to \infty} \mathbb{P}\left(X_n > S|\mathbf{Y}_0 = \mathbf{x}_0\right) = \frac{1}{2}\left[1 - \mathrm{Erf}\left(\frac{S}{\sqrt{2\sigma^2 s^2}}\right)\right],$$

where $\lim_{n\to\infty} \mathrm{Var}(X_n) = \sigma^2 s^2 < \infty$.

Proof: If the eigenvalues of F fall within the open unit disk in $\mathbb{R}$, then $\lim_{n\to\infty} F^n = 0$ (cf. [9]) with geometric rate, so that

$$\lim_{n\to\infty} \mathbf{a}^{(n)}\mathbf{x}_0 = 0 \quad \text{and} \quad \lim_{n\to\infty} \mathrm{Var}(X_n) = \sigma^2 s^2 < \infty.$$

The result follows from (4.3).

■

Corollary 4.1 *If the eigenvalues of F fall within the open unit disk in $\mathbb{R}$, then $E[\eta_S|\mathbf{Y}_0 = \mathbf{x}_0]$ is not finite.*

Corollary 4.2 *If the eigenvalues of F fall within the open unit disk in $\mathbb{R}$, then*

$$\lim_{n\to\infty} \frac{\mathcal{P}_{n+1}(S,\Sigma_{n+1})}{\mathcal{P}_n(S,\Sigma_n)} = \frac{1}{2}\left\{1 + \mathrm{Erf}\left[\frac{S}{\sqrt{2\sigma^2 s^2}}\right]\right\}, \tag{4.4}$$

with $\mathcal{P}_n(S,\Sigma_n)$ orthant probability in (2.5).

Note that the hypothesis of Theorem 4.1 is equivalent to require the linear process is causal, a property not of the process $\{X_n\}$ alone, but rather of the relationship between the two processes $\{X_n\}$ and $\{W_n\}$. For a linear process, this property means that $\{X_n\}$ is expressible in terms only of $\{W_k\}$ for $k \leq n$ (cf. [3]). This requirement appears to be natural for a wide range of applications. For Markov chains on countable state spaces, the statement of Corollary 4.1 is sufficient to prove that the chain reaches any region of the state space almost surely. For Markov chains on general state spaces, we need to add some more hypotheses on the model, which will be clarified in the next section. Finally, due to (2.4), Corollary 4.2 states that $P_{\mathbf{x}_0}(T = n)$ goes to zero with geometric rate when n goes to infinity.

Note that, if $p = 1$, the hypothesis of Theorem 4.1 gives $|\alpha_1| < 1$. From (3.9) and Remark 3.1, we have

$$\mathrm{Var}(X_n) = \sigma^2 \frac{1-\alpha_1^{2n}}{1-\alpha_1^2}, \quad \mathrm{Cov}(X_{k+n}, X_k) = \sigma^2 \frac{\alpha_1^n(1-\alpha_1^{2k})}{1-\alpha_1^2}$$

and $s^2 = (1-\alpha_1^2)^{-1}$. If $|\alpha_1| = 1$, from (3.9), we have $\lim_{n\to\infty} \mathrm{Var}(X_n) = \infty$. Being $\lim_{n\to\infty} E(X_n) = x_0$, from (4.3) we have $\lim_{n\to\infty} \mathbb{P}\left(X_n > S|X_0 = x_0\right) = 1/2$ and the result of Corollary 4.1 still holds.

5 First passage time Denote the one-step transition pdf of the chain $\mathbf{Y}_n$ by

$$P(\mathbf{x}, D) = \mathbb{P}(\mathbf{Y}_n \in D|\mathbf{Y}_{n-1} = \mathbf{x}),$$

with $\mathbf{x} \in \mathbb{R}^p$ and $D = D_1 \times D_2 \times \cdots \times D_p \in \mathcal{B}(\mathbb{R}^p)$, where $\mathcal{B}(\mathbb{R}^p)$ is the σ-field generated by the borel sets of $\mathbb{R}^p$. From (3.3), we have

$$P(\mathbf{x}, D) = \int_D f(\mathbf{y}|\mathbf{x})\,\mathrm{d}\mathbf{y} = \int_{D_1} h(y|\mathbf{x})\,\mathrm{d}y \quad \text{iff } (x_1, \ldots, x_{p-1}) \in D_2 \times \cdots \times D_p, \tag{5.1}$$

otherwise being zero.

Theorem 5.1 *For any $\mathbf{x}_0 \in \mathbb{R}^p$ and $S \in \mathbb{R}$*

$$P_{\mathbf{x}_0}(T = n) = \begin{cases} \int_S^\infty h(y|\mathbf{x}_0)\,\mathrm{d}y, & \text{if} \quad n = 1, \\ \int_{-\infty}^S h(y|\mathbf{x}_0)\, P_{(y;\mathbf{x}_{0,1};\ldots;\mathbf{x}_{0,p-1})^T}(T = n-1)\,\mathrm{d}y, & \text{if} \quad n > 1. \end{cases} \tag{5.2}$$

Proof: From (2.3) and (3.1), we have

$$P_{\mathbf{x}_0}(T=n) = \mathbb{P}(\mathbf{Y}_n \in D, \mathbf{Y}_{n-1} \in O_p, \ldots, \mathbf{Y}_1 \in O_p | \mathbf{Y}_0 = \mathbf{x}_0), \tag{5.3}$$

where $D = [S,\infty) \times (-\infty, S)^{p-1}$ and $O_p = (-\infty, S)^p$. Due to the Markov property of $\{\mathbf{Y}_n\}$, equation (5.3) can be recursively rewritten

$$P_{\mathbf{x}_0}(T=n) = \begin{cases} P(\mathbf{x}_0, D), & \text{if} \quad n=1, \\ \int_{O_p} f(\mathbf{y}|\mathbf{x}_0)\, P_{\mathbf{y}}(T=n-1)\, \mathrm{d}\mathbf{y}, & \text{if} \quad n>1. \end{cases} \tag{5.4}$$

The result follows from (5.1) and (3.3).

■

The following corollary is the first step to implementing recursively equations (5.2).

Corollary 5.1 *If* $\mathbf{x}_0 \in (-\infty, S)^p$, *then*

$$P_{\mathbf{x}_0}(T=n) = \frac{1}{2}\, \phi_n\left(\frac{S - \boldsymbol{\alpha}\mathbf{x}_0}{\sqrt{2\sigma^2}}\right), \quad n \geq 1 \tag{5.5}$$

where $\phi_1(x) = 1 - \mathrm{Erf}(x) = \mathrm{Erfc}(x)$ *and*

$$\phi_n(x) = \frac{1}{\sqrt{\pi}} \int_0^{\infty} \exp[-(x-t)^2]\, \phi_{n-1}\left[\alpha_1 t + \frac{S - \boldsymbol{\alpha}\mathbf{w}}{\sqrt{2\sigma^2}}\right] \mathrm{d}t, \tag{5.6}$$

with $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_p)$ *and* $\mathbf{w} = (S; \mathbf{x}_{0,1}; \ldots; \mathbf{x}_{0,p-1})^T$.

Proof: From (5.2), we have

$$P_{\mathbf{x}_0}(T=1) = \frac{1}{\sqrt{2\,\pi\,\sigma^2}} \int_S^{\infty} \exp\left\{-\frac{(y - \boldsymbol{\alpha}\mathbf{x}_0)^2}{2\,\sigma^2}\right\} \mathrm{d}y = \frac{1}{2}\, \mathrm{Erfc}\left(\frac{S - \boldsymbol{\alpha}\mathbf{x}_0}{\sqrt{2\sigma^2}}\right),$$

by which equation (5.5) follows for $n=1$. By induction on n, suppose equation (5.5) true for $n = k-1$. From (5.2), we have

$$\begin{aligned} P_{\mathbf{x}_0}(T=k) &= \int_{-\infty}^{S} h(y|\mathbf{x}_0)\, P_{(y;\mathbf{x}_{0,1};\ldots;\mathbf{x}_{0,p-1})^T}(T=k-1)\, \mathrm{d}y \\ &= \frac{1}{2\sqrt{2\pi\,\sigma^2}} \int_{-\infty}^{S} \exp\left\{-\frac{(y - \boldsymbol{\alpha}\mathbf{x}_0)^2}{2\sigma^2}\right\} \phi_{k-1}\left[\frac{S - \alpha_1 y - \sum_{i=2}^{p} \alpha_i \mathbf{x}_{0,i-1}}{\sqrt{2\sigma^2}}\right] \mathrm{d}y, \end{aligned} \tag{5.7}$$

where we have replaced $P_{(y;\mathbf{x}_{0,1};\ldots;\mathbf{x}_{0,p-1})^T}(T=k-1)$ by the function obtained from (5.5) for $n=k-1$ and starting point $(y;\mathbf{x}_{0,1};\ldots;\mathbf{x}_{0,p-1})^T \in (-\infty,S)^p$, since $y<S$. The expression (5.5) for $n=k$ follows by replacing $y = S - t\sqrt{2\sigma^2}$ in the integral (5.7).

■

Remark 5.1 Suppose $\alpha_1 = 0$. From equation (5.5) we have:

$$P_{\mathbf{x}_0}(T=n) = \begin{cases} \frac{1}{2}\, \mathrm{Erfc}\left(\frac{S - \boldsymbol{\alpha}\mathbf{x}_0}{\sqrt{2\sigma^2}}\right), & \text{if} \quad n=1 \\ \frac{1}{2^2}\, \mathrm{Erfc}\left(\frac{S - \boldsymbol{\alpha}\mathbf{w}}{\sqrt{2\sigma^2}}\right) \mathrm{Erfc}\left(\frac{\boldsymbol{\alpha}\mathbf{x}_0 - S}{\sqrt{2\sigma^2}}\right), & \text{if} \quad n=2 \\ \frac{1}{2^{n-2}} P_{\mathbf{x}_0}(T=2) \left[\mathrm{Erfc}\left(\frac{\boldsymbol{\alpha}\mathbf{w} - S}{\sqrt{2\sigma^2}}\right)\right]^{n-2}, & \text{if} \quad n \geq 3. \end{cases} \tag{5.8}$$

So $P_{\mathbf{x}_0}(T=n)$ goes to zero with geometric rate when n goes to infinite, relaxing any hypothesis on the eigenvalues of F.

Proposition 5.1 *The FPT r.v. T is fair, that is $P_{\mathbf{x}_0}(T<\infty)=1$.*

Proof: Summing $P_{\mathbf{x}_0}(T=n)$ in (5.2) for $n\geq 1$ gives

$$L(\mathbf{x}_0)=P(\mathbf{x}_0,D)+\int_{-\infty}^{S}h(y_1|\mathbf{x}_0)\,L[(y_1;\mathbf{x}_{0,1};\ldots;\mathbf{x}_{0,p-1})^T]\,\mathrm{d}y_1, \tag{5.9}$$

where $D=[S,\infty)\times(-\infty,S)^{p-1}$,

$$L(\mathbf{x}_0)=\sum_{n\geq 1}P_{\mathbf{x}_0}(T=n)\quad\text{and}\quad L[(y_1;\mathbf{x}_{0,1};\ldots;\mathbf{x}_{0,p-1})^T]=\sum_{n\geq 1}P_{(y_1;\mathbf{x}_{0,1};\ldots;\mathbf{x}_{0,p-1})^T}(T=n).$$

Set $z=S-y_1$ in the integral of equation (5.9). We have

$$L(\mathbf{x}_0)=P(\mathbf{x}_0,D)+\int_{0}^{\infty}h(S-z|\mathbf{x}_0)\,L[(S-z;\mathbf{x}_{0,1};\ldots;\mathbf{x}_{0,p-1})^T]\,\mathrm{d}z. \tag{5.10}$$

Define K to be the operator such that

$$K[L(\mathbf{x}_0)]=\int_{0}^{\infty}h(S-z|\mathbf{x}_0)\,L[(S-z;\mathbf{x}_{0,1};\ldots;\mathbf{x}_{0,p-1})^T]\,\mathrm{d}z.$$

Due to (3.3), this operator is compact. Indeed a sufficient condition for K to be compact is to have the kernel bounded and continuous (cf. [1]), which is true for $h(S-z|\mathbf{x}_0)$. So by virtue of the Fredholm alternative, the equation exhibits a unique solution. Since $L(\mathbf{x}_0)=1$ satisfies (5.10), the result follows.

■

5.1 Mean of the FPT r.v. The goal of this section is to prove that if the eigenvalues of F fall within the open unit disk in $\mathbb{R}$, then the FPT mean is finite. To this aim, we first need to prove the following lemma.

Lemma 5.1 *If the eigenvalues of F fall within the open unit disk in $\mathbb{R}$, then*

$$\sum_{n=1}^{\infty}\mathcal{P}_n(S,\Sigma_n)<\infty.$$

Proof: Let $F_{\mathbf{x}_0}(n)$ be the FPT cdf, that is

$$F_{\mathbf{x}_0}(n)=\sum_{i=1}^{n}P_{\mathbf{x}_0}(T=i).$$

From (2.4), we have

$$F_{\mathbf{x}_0}(n)=1-\mathcal{P}_n(S,\Sigma_n). \tag{5.11}$$

As T is a fair r.v., then

$$\lim_{n\to\infty}\mathcal{P}_n(S,\Sigma_n)=0.$$

The result follows from Corollary 4.2, by which the sequence $\{\mathcal{P}_n(S,\Sigma_n)\}$ decreases to zero with a geometric rate, if the eigenvalues of F fall within the open unit disk in $\mathbb{R}$.

■

From (5.11), we have

$$P_{\mathbf{x}_0}(T > n) = \mathcal{P}_n(S, \Sigma_n)$$

and

$$\sum_{n=1}^{\infty} P_{\mathbf{x}_0}(T \geq n) = 1 + \sum_{n=1}^{\infty} \mathcal{P}_n(S, \Sigma_n) < \infty,$$

due to Lemma 5.1 and Proposition 5.1. As T is a r.v. assuming only nonnegative integer values, then

$$E_{\mathbf{x}_0}[T] = \sum_{n=1}^{\infty} P_{\mathbf{x}_0}(T \geq n),$$

by which we state the following theorem.

Theorem 5.2 *If the eigenvalues of F fall within the open unit disk in $\mathbb{R}$, then the FPT mean is finite and*

$$E_{\mathbf{x}_0}[T] = 1 + \sum_{n=1}^{\infty} \mathcal{P}_n(S, \Sigma_n).$$

6 Taboo probability

We define the n-step *taboo probability* as

$${}_A P^n(\mathbf{x}_0, B) = \mathrm{P}(X_n \in B, T_A \geq n | \mathbf{X}_0 = \mathbf{x}_0),$$

where $A, B \in \mathcal{B}(\mathbb{R}), T_A = \min_{n \geq 1}\{X_n \in A\}$ and $\mathbf{x}_0 \in \overline{A}^p$. So the n-step taboo probability ${}_A P^n(\mathbf{x}_0, B)$ denotes the probability of a transition to B in n steps, avoiding the set A.

Theorem 6.1 *Suppose $A = (-\infty, a]$ and $B = [b, \infty)$ with $a < b$. For any $\mathbf{x}_0 \in (a, \infty)^p$, we have*

$$
{}_A P^n(\mathbf{x}_0, B) = \begin{cases} \dfrac{1}{2}\, \varphi_1\left(\dfrac{b - \boldsymbol{\alpha}\mathbf{x}_0}{\sqrt{2\,\sigma^2}}\right), & \text{if} \quad n = 1, \\[2ex] \dfrac{1}{2}\, \varphi_n\left(\dfrac{\boldsymbol{\alpha}\mathbf{x}_0 - a}{\sqrt{2\,\sigma^2}}\right), & \text{if} \quad n > 1, \end{cases} \tag{6.1}
$$

where

$$
\varphi_n(x) = \begin{cases} \mathrm{Erfc}(x), & \text{if} \quad n = 1, \\ \dfrac{1}{\sqrt{\pi}} \displaystyle\int_0^{\infty} \exp[-(x-t)^2]\, \mathrm{Erfc}\left(\frac{b - \boldsymbol{\alpha}\mathbf{w}}{\sqrt{2\sigma^2}} - \alpha_1 t\right) \mathrm{d}t, & \text{if} \quad n = 2, \\ \dfrac{1}{\sqrt{\pi}} \displaystyle\int_0^{\infty} \exp[-(x-t)^2]\, \varphi_{n-1}\left(\frac{\boldsymbol{\alpha}\mathbf{w} - a}{\sqrt{2\sigma^2}} + \alpha_1 t\right) \mathrm{d}t, & \text{if} \quad n > 3, \end{cases}
$$

with $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_p)$ and $\mathbf{w} = (a; \mathbf{x}_{0,1}; \ldots; \mathbf{x}_{0,p-1})^T$.

Proof: As in Theorem 5.1, the taboo probabilities satisfy the iterative relations

$$
{}_A P^n(\mathbf{x}_0, B) = \begin{cases} \int_b^{\infty} h(y|\mathbf{x}_0)\, \mathrm{d}y, & \text{if} \quad n = 1, \\[2ex] \int_a^{\infty} h(y|\mathbf{x}_0)\, {}_A P^{n-1}[(y; \mathbf{x}_{0,1}; \ldots; \mathbf{x}_{0,p-1})^T, B]\, \mathrm{d}y, & \text{if} \quad n > 1. \end{cases} \tag{6.2}
$$

For $n = 1$ the result follows immediately, being ${}_A P^1(\mathbf{x}_0, B) = P(\mathbf{x}_0, B)$. For $n > 1$ the result follows by replacing $y = t\sqrt{2\sigma^2} + a$ in (6.2), as already done in the proof of Corollary 5.1.

■

Close to taboo probabilities, there are the probabilities of first entrance in B avoiding the set A. If $A = (-\infty, a]$ and $B = [b, \infty)$ with $a < b$, these probabilities are related to the upper exit time from $[a, b]$, i.e. the probabilities of crossing the level b before the level a :

$$P_n^{a,b}(\mathbf{x}_0) = \mathbb{P}(a < X_1 < b, \ldots, a < X_{n-1} < b, X_n \geq b | \mathbf{X}_0 = \mathbf{x}_0) = P_{\mathbf{x}_0}(T_A \geq n, T_B = n).$$

For $A = (-\infty, a]$ and $B = [b, \infty)$, we have

$$_A P^n(\mathbf{x}_0, B) \geq P_n^{a,b}(\mathbf{x}_0).$$

The proof of the next theorem is similar to the one of Theorem 6.1.

Theorem 6.2 *Suppose $a < b$. Then for any $\mathbf{x}_0 \in (a, \infty)^p$ the upper exit time p.d. is*

$$P_n^{a,b}(\mathbf{x}_0) = \frac{1}{2}\, \psi_n \left(\frac{b - \boldsymbol{\alpha} \mathbf{x}_0}{\sqrt{2\,\sigma^2}} \right), \tag{6.3}$$

where $\psi_1(x) = \mathrm{Erfc}(x)$ and for $n \geq 2$

$$\psi_n(x) = \frac{1}{\sqrt{\pi}} \int_0^{\frac{b-a}{\sqrt{2\sigma^2}}} \exp[-(x-t)^2]\, \psi_{n-1} \left(\frac{b - \boldsymbol{\alpha} \mathbf{w}}{\sqrt{2\sigma^2}} + \alpha_1 t \right) \mathrm{d}t, \tag{6.4}$$

with $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_p)$ and $\mathbf{w} = (b; \mathbf{x}_{0,1}; \ldots; \mathbf{x}_{0,p-1})^T$.

Remark 6.1 If $\alpha_1 = 0$, equation (6.4) may be rewritten

$$\psi_n(x) = \frac{1}{2} \left[\mathrm{Erf}(x) - \mathrm{Erf}\left(x - \frac{b-a}{\sqrt{2\sigma^2}} \right) \right] \psi_{n-1} \left(\frac{b - \boldsymbol{\alpha} \mathbf{w}}{\sqrt{2\sigma^2}} \right), \tag{6.5}$$

and in particular for $n \geq 2$

$$\psi_n \left(\frac{b - \boldsymbol{\alpha} \mathbf{w}}{\sqrt{2\sigma^2}} \right) = \frac{1}{2^{n-1}} \left[\mathrm{Erf}\left(\frac{b - \boldsymbol{\alpha} \mathbf{w}}{\sqrt{2\sigma^2}} \right) - \mathrm{Erf}\left(\frac{a - \boldsymbol{\alpha} \mathbf{w}}{\sqrt{2\sigma^2}} \right) \right]^{n-1} \mathrm{Erfc}\left(\frac{b - \boldsymbol{\alpha} \mathbf{w}}{\sqrt{2\sigma^2}} \right), \tag{6.6}$$

applying (6.5) recursively. Therefore, replacing (6.6) in (6.3), we have

$$P_n^{a,b}(\mathbf{x}_0) = \frac{1}{2^n} \left[\mathrm{Erf}\left(\frac{b - \boldsymbol{\alpha} \mathbf{w}}{\sqrt{2\sigma^2}} \right) - \mathrm{Erf}\left(\frac{a - \boldsymbol{\alpha} \mathbf{w}}{\sqrt{2\sigma^2}} \right) \right]^{n-1} \mathrm{Erfc}\left(\frac{b - \boldsymbol{\alpha} \mathbf{w}}{\sqrt{2\sigma^2}} \right), \tag{6.7}$$

for $n \geq 2$.

7 Bounds for the first passage time cumulative distribution function In the following, we give some bounds for the FPT cdf, depending on some properties of orthant probabilities. First, we recall a result due to Plackett and later stated by Slepian in a more general form (see [17]).

Theorem 7.1 *Let $\mathbf{X} \equiv (X_1, X_2, \ldots, X_n)^T$ and $\mathbf{Y} \equiv (Y_1, Y_2, \ldots, Y_n)^T$ be standard gaussian vectors, with $(\Sigma_n^X)_{ij} = \rho_{ij}^X$ and $(\Sigma_n^Y)_{ij} = \rho_{ij}^Y$ the correlation matrix respectively of $\mathbf{X}$ and $\mathbf{Y}$. If $\rho_{ij}^X \geq \rho_{ij}^Y$ for all $i, j = 1, 2, \ldots, n$ then*

$$\mathcal{P}_k(\mathbf{S}, \Sigma_k^X) \geq \mathcal{P}_k(\mathbf{S}, \Sigma_k^Y), \quad k = 1, 2, \ldots, n. \tag{7.1}$$

The previous theorem states that $\mathcal{P}_k(\mathbf{S}, \Sigma_k)$ is a non-decreasing function of $\rho_{ij} \in I_{(i,j)} = \{\rho_{ij} : \Sigma_k \text{ is positive definite}\}$.

Proposition 7.1 [Lower Bound] *Set*

$$\rho_{\max}^{(n)} = \max_{i,j=1,2,\ldots,n; i\neq j} \left\{ \frac{\sum_{t=1}^{\min\{i,j\}} \mathbf{a}_1^{(i-t)} \mathbf{a}_1^{(j-t)}}{\sqrt{\left(1+\sum_{t=1}^{i-1}[\mathbf{a}_1^{(t)}]^2\right)\left(1+\sum_{t=1}^{j-1}[\mathbf{a}_1^{(t)}]^2\right)}} \right\}.$$

If $\mathbf{x}_0 \in (-\infty, S)^p$, *then for* $n \geq 2$

$$F_{\mathbf{x}_0}(n) \geq \begin{cases} 1 - U_n[S_{\max}^{(n)}, \rho_{\max}^{(n)}], & \text{if} \quad \rho_{\max}^{(n)} \in [0,1), \\ 1 - \Phi^n[S_{\max}^{(n)}], & \text{if} \quad \rho_{\max}^{(n)} \leq 0, \end{cases}$$

where $S_{\max}^{(n)}$ *is the maximum among the components of the vector* $\mathbf{S}^{(n)}$ *such that*

$$\mathbf{S}^{(n)} = \left(\frac{S - \mathbf{a}^{(1)}\mathbf{x}_0}{\sigma}, \frac{S - \mathbf{a}^{(2)}\mathbf{x}_0}{\sigma\sqrt{\left(1+[\mathbf{a}_1^{(1)}]^2\right)}}, \ldots, \frac{S - \mathbf{a}^{(n)}\mathbf{x}_0}{\sigma\sqrt{\left(1+\sum_{j=1}^{n-1}[\mathbf{a}_1^{(j)}]^2.\right)}} \right), \tag{7.2}$$

$\Phi(x)$ *is the standard gaussian cdf with pdf* $\phi(x)$, *and*

$$U_n(s,t) = \int_{\mathbb{R}} \Phi^n\left(\frac{s + z\sqrt{t}}{\sqrt{1-t}}\right)\phi(z)\,\mathrm{d}z, \quad s \in \mathbb{R},\, t \in (-1,1). \tag{7.3}$$

Proof: First observe that

$$\mathcal{P}_n(S, \Sigma_n) = \mathcal{P}_n[\mathbf{S}^{(n)}, R_n] = \int_{O_n^*} \frac{1}{(2\pi)^{n/2}|R_n|^{1/2}} \exp\left\{-\frac{1}{2}\mathbf{y}^T R_n^{-1}\mathbf{y}\right\} \mathrm{d}\mathbf{y},$$

where R_n is the correlation matrix of $(X_1, X_2, \ldots, X_n)$ and $O_n^* = \{\mathbf{y} \in \mathbb{R}^n : \mathbf{y} < \mathbf{S}^{(n)}\}$. So, due to the Theorem 7.1, $\mathcal{P}_n[\mathbf{S}^{(n)}, R_n] \leq \mathcal{P}_n[\mathbf{S}^{(n)}, R_n^*]$, where R_n^* is a correlation matrix with entries are all equal to $\rho_{\max}^{(n)}$, referring to a gaussian random vector with zero mean. Moreover, we have $\mathcal{P}_n[\mathbf{S}^{(n)}, R_n^*] \leq \mathcal{P}_n[S_{\max}^{(n)}, R_n^*]$. If $\rho_{\max}^{(n)} \geq 0$, then $\mathcal{P}_n[S_{\max}^{(n)}, R_n^*]$ are orthant probabilities associated to exchangeable r.v.'s (see [17]) and in particular $\mathcal{P}_n[S_{\max}^{(n)}, R_n^*] = U_n[S_{\max}^{(n)}, \rho_{\max}^{(n)}]$. If $\rho_{\max}^{(n)} \leq 0$, then $\mathcal{P}_n[\mathbf{S}^{(n)}, R_n] \leq \mathcal{P}_n[\mathbf{S}^{(n)}, I_n] \leq \mathcal{P}_n[S_{\max}^{(n)}, I_n] = \Phi^n[S_{\max}^{(n)}]$, since the gaussian r.v.'s involved in $\mathcal{P}_n[S_{\max}^{(n)}, I_n]$ are independents.

∎

By similar arguments one can prove the following result.

Proposition 7.2 [Upper Bound] *Set*

$$\rho_{\min}^{(n)} = \min_{i,j=1,2,\ldots,n; i\neq j} \left\{ \frac{\sum_{t=1}^{\min\{i,j\}} \mathbf{a}_1^{(i-t)} \mathbf{a}_1^{(j-t)}}{\sqrt{\left(1+\sum_{t=1}^{i-1}[\mathbf{a}_1^{(t)}]^2\right)\left(1+\sum_{t=1}^{j-1}[\mathbf{a}_1^{(t)}]^2\right)}} \right\}.$$

For $\rho_{\min}^{(n)} \in [0,1)$ *and* $\mathbf{x}_0 \in (-\infty, S)^p$, *we have*

$$F_{\mathbf{x}_0}(n) \leq 1 - U_n[S_{\min}^{(n)}, \rho_{\min}^{(n)}] \quad \text{for } n \geq 2,$$

where $S_{\min}^{(n)}$ *is the minimum among the components of the vector* $\mathbf{S}^{(n)}$ *given in (7.2) and* $U_n(s,t)$ *is the function given in (7.3).*

8 Numerical examples As shown in the previous sections, the evaluation of $P_{\mathbf{x}_0}(T=n)$ can be treated only numerically, as well as its mean and taboo probabilities. In this section, we suggest a numerical scheme to implement equations (5.5), (6.1) and (6.3), which is substantially different from the numerical schemes proposed in [2]. In the following, we focus the attention on the AR(2) process:

$$X_n = 0.2X_{n-1} + 0.3X_{n-2} + W_n \tag{8.1}$$

with $W_n \approx N(0,1)$. We assume $\mathbf{x}_0 = (0.5, 0.5)$.

In order to compute $P_{\mathbf{x}_0}(T=n)$ numerically, first notice that the functions $\phi_n(x)$ in (5.6) could be rewritten as

$$\phi_n(x) = \frac{\exp(-x^2)}{2\sqrt{\pi\sigma^2}} \int_0^\infty \frac{\exp(-t)}{\sqrt{t}} \exp(2x\sqrt{t})\, \phi_{n-1}\left[\alpha_1\sqrt{t} + \frac{S-\boldsymbol{\alpha}\mathbf{w}}{\sqrt{2\sigma^2}}\right] \mathrm{d}t, \tag{8.2}$$

by which we have $|\phi_n(x)| \leq \phi_n(0) < \infty$, for all $n \in \mathbb{N}$ and $\lim_{x\to\pm\infty} \phi_n(x) = 0$. So a generalized Gauss-Laguerre quadrature formula may be implemented in order to estimate (8.2), that is

$$\phi_n(x) \approx \frac{\exp(-x^2)}{2\sqrt{\pi\sigma^2}} \sum_{k=1}^{m} w_k \exp(2x\sqrt{n_k})\, \phi_{n-1}\left[\alpha_1\sqrt{n_k} + \frac{S-\boldsymbol{\alpha}\mathbf{w}}{\sqrt{2\sigma^2}}\right], \tag{8.3}$$

with $\{n_k, w_k\}_{k=1}^m$ respectively nodes and weights of the quadrature formula. For all figures, we have used $m=8$ nodes.

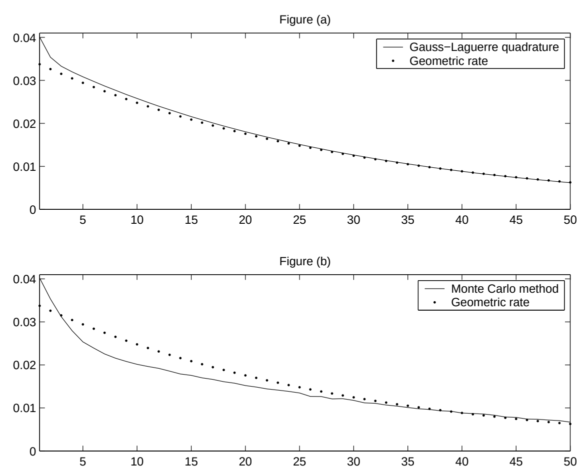


Figure 1: Plots refer to $\{P_{\mathbf{x}_0}(T=n)\}$ through the boundary $S=2$ for the AR(2) process (8.1). In Figure 1(a), the sequence $\{P_{\mathbf{x}_0}(T=n)\}$, computed via (5.5) with ϕ_n approximated through (8.3), has been plotted together with the p.d.$\{p_n\}$ of a geometric r.v. with failure probability q equal to the resulting limit in (4.4). In Figure 1(b), the same p.d. $\{p_n\}$ has been plotted together with the sequence $\{P_{\mathbf{x}_0}(T=n)\}$ estimated via a Monte Carlo method with 10^6 simulated paths.

In Figure 1(a), we compare the p.d. $\{P_{\mathbf{x}_0}(T=n)\}$, computed via the generalized Gauss-Laguerre quadrature formula (8.3) when $S=2$, with the p.d. $\{p_n\}$ of a geometric r.v. having failure probability q equal to the resulting limit in (4.4). Here, the value 1.196

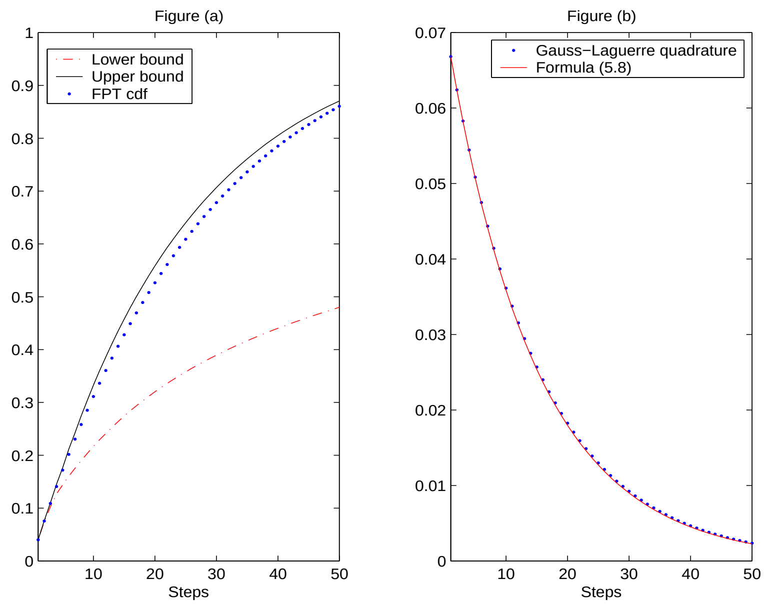


Figure 2: In Figure 2(a), for the AR(2) process (8.1) and $S = 2$, plots of $F_{\mathbf{x}_0}(n)$ and its upper and lower bounds are shown (see Propositions 7.1 and 7.2). In Figure 2(b), plot of $\{P_{\mathbf{x}_0}(T = n)\}$, given in (5.5) with ϕ_n computed via (8.3), is shown together with the same p.d. computed via (5.8), when $\alpha_1 = 0, \alpha_2 = 1$ and $S = 2, \mathbf{x}_0 = (0.5, 0.5)$.

of s^2 has been numerically estimated, by using (3.9) and by observing that, for the AR(2) process in (8.1), we have

$$\mathbf{a}_1^{(0)} = 1, \quad \mathbf{a}_1^{(1)} = \alpha_1, \quad \mathbf{a}_1^{(j)} = \alpha_1\, \mathbf{a}_1^{(j-1)} + \alpha_2\, \mathbf{a}_1^{(j-2)} \quad \text{for} \quad j \geq 2.$$

The asymptotic behavior is matched even at lower values of T. As shown in Figure 1(b), if we estimate the p.d. $\{P_{\mathbf{x}_0}(T = n)\}$ via a Monte Carlo method, the fitting with the geometric asymptotic behavior is not so early. Here we have generated 10^6 paths.

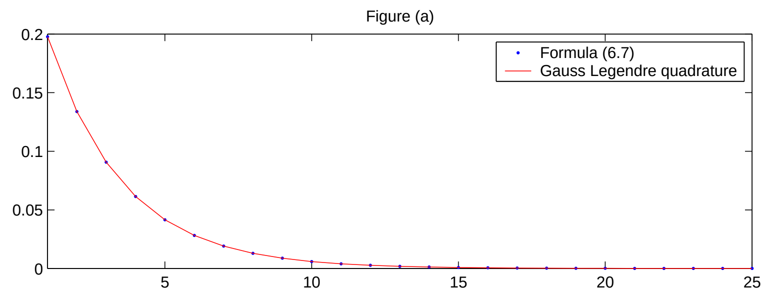
We remark that the computational cost of the generalized Gauss-Laguerre quadrature is very low. Indeed, in order to compute $P_{\mathbf{x}_0}(T = n)$ for all n, we need to compute the required m nodes and weights just one time, at the beginning of the procedure. Due to the recursive structure of (8.3), at each step we need to compute

$$\phi_n\left(\frac{S - \boldsymbol{\alpha}\mathbf{x}_0}{\sqrt{2\sigma^2}}\right) \quad \text{and} \quad \phi_n\left[\alpha_1\sqrt{n_k} + \frac{S - \boldsymbol{\alpha}\mathbf{w}}{\sqrt{2\sigma^2}}\right] \quad k = 1, \ldots, m.$$

No comparisons have been made with the numerical procedure proposed in [2], where it is not suggested an algorithm to compute $P_{\mathbf{x}_0}(T = n)$, being unbounded the integration interval.

In Figure 2(a), we have plotted the cdf of the AR(2) process in (8.1) together with its bounds. In Figure 2(b), we have plotted $\{P_{\mathbf{x}_0}(T = n)\}$, given in (5.5) with ϕ_n computed via (8.3), together with $\{P_{\mathbf{x}_0}(T = n)\}$ given in (5.8). In this case we have chosen $\alpha_1 = 0$ and $\alpha_2 = 1$, so that the F eigenvalues fall within the closed unit disk in $\mathbb{R}$. As it is evident, the matching is very good.

In Figures 3, we have plotted $\{P_n^{a,b}(\mathbf{x}_0)\}$ given in (6.7) through the levels $a = -1$ and $b = 1$ for an AR(2) process with $\alpha_1 = 0.0, \alpha_2 = 0.3$ assuming $\mathbf{x}_0 = (0.5, 0.5)$. In Figure 3(a),



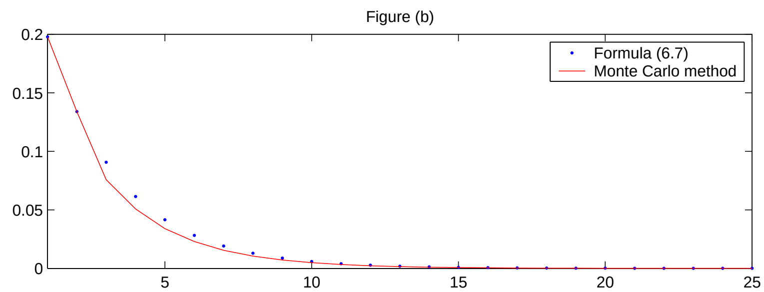


Figure 3: Plots refer to $\{P_n^{a,b}(\mathbf{x}_0)\}$ through the boundaries $a = -1$ and $b = 1$ for an AR(2) process with $\alpha_1 = 0.0, \alpha_2 = 0.3$ assuming $\mathbf{x}_0 = (0.5, 0.5)$. In Figure 3(a) we show $\{P_n^{a,b}(\mathbf{x}_0)\}$ given in (6.3), with ψ_n computed via a Gauss-Legendre quadrature formula applied to (6.4), and $\{P_n^{a,b}(\mathbf{x}_0)\}$ given in (6.7). In Figure 3(b), we show $\{P_n^{a,b}(\mathbf{x}_0)\}$ given in (6.7) and the same p.d. estimated via a Monte Carlo method with 10^6 simulated paths.

we have also plotted $\{P_n^{a,b}(\mathbf{x}_0)\}$ given in (6.3). In order to use the functions in (6.4), we have implemented a Gauss-Legendre quadrature formula, by a suitable transformation of the integration interval $\left(0, \frac{b-a}{\sqrt{2\sigma^2}}\right)$ in $(-1, 1)$. In Figure 3(b), we have also plotted $\{P_n^{a,b}(\mathbf{x}_0)\}$ built by using a Monte Carlo method with 10^6 simulated paths. It is evident that the Monte Carlo method has not a great accuracy.

Finally, we compare the computations of the overall upper exit time probability, that is $\sum_n P_n^{a,b}(\mathbf{x}_0)$, via a Gauss-Legendre quadrature formula with the numerical scheme proposed in [2]. This numerical scheme consists in a solution of a system of linear equations plus an interpolation algorithm. So the computational cost is greater than the one of the Gauss-Legendre quadrature formula we propose. The results are given in Table 1. The results given in [2] are in agreement with those obtained via a Monte Carlo method. But, we have already stressed that the Monte Carlo method does not allow to compute such probabilities with high accuracy (see Figures 3). Notice that, for the choice of the parameters in Figures 3, the overall upper exit time probability is 0.6132, if we use the Gauss-Legendre quadrature formula, while it results 0.5628 if we use the Monte Carlo method.

References

[1] Anselone, P.M. and Sloan, I.H. (1987) Numerical solutions of integral equations on the half line. *Numer. Math.* **51**, 599–614.

[2] Basak, G.K. and Ho, K. R. (2004) Level-crossing probabilities and first- passage times for linear processes. *Adv. Appl. Probab.* **36**, 643–666.

b	Quadrature Formula	Monte Carlo Method	Basak et al. procedure
1	0.6402	0.5920	0.5914
2	0.2589	0.2416	0.2426
3	0.0283	0.0344	0.0343
4	0.0010	0.0016	0.0016

Table 1: Comparisons among the Gauss-Legendre quadrature formula, the Monte Carlo method and the procedure suggested in [2] in the evaluation of the overall upper exit probability of the AR(2) process in (8.1) when $a = -1$ and for various values of b.

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