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MICROLOCAL ANALYSIS OF QUASIANALYTIC GELFAND-SHILOV TYPE ULTRADISTRIBUTIONS

MARCO CAPPIELLO AND RENÉ SCHULZ

ABSTRACT. We introduce a global wave front set suitable for the analysis of tempered ultradistributions of quasianalytic Gelfand-Shilov type. We study the transformation properties of the wave front set and use them to give microlocal existence results for pullbacks and products. We further study quasianalytic microlocality for classes of localization and ultradifferential operators, and prove microellipticity for differential operators with polynomial coefficients.

0. INTRODUCTION

In this paper, we establish a global wave front set suitable for the analysis of tempered ultradistributions of Gelfand-Shilov type. The problem of finding such a wave front set has a longstanding history, since several of the methods used to generalize the classical Hörmander wave front set fail in the quasi-analytic setting. In the following, we present a brief overview of the problem.

Starting from [19], the analysis of *singularities* of Schwartz distributions has been based on the study of their wave front set which has been applied successfully to the analysis of propagation of singularities in the theory of partial differential equations. We recall that a distribution $u \in \mathcal{D}'(\mathbb{R}^d)$ is said to be microlocal at a point $(x_0, \xi_0) \in \mathbb{R}^d \times (\mathbb{R}^d \backslash 0)$ if there exists a cut-off function ϕ supported around x_0 such that $\widehat{\phi u}$ is rapidly decaying in a conic neighborhood of ξ_0. The wave front set is then defined as the set of points $(x, \xi) \in \mathbb{R}^d \times (\mathbb{R}^d \setminus 0)$ where u is not microlocal. This notion has been soon extended to the analysis of other types of singularities such as analytic singularities, see e.g. [19, 34], or *local* Gevrey s-singularities, cf. [31] and the references therein. For $s > 1$ these spaces provide a natural scale of spaces between the real analytic and smooth functions. For $s < 1$, however, the local versions of these spaces are problematic to define due to the lack of compactly supported functions therein. One is thus led to consider also *global* phenomena. The Gelfand-Shilov spaces $\mathcal{S}_\theta$, defined in [15] combining Gevrey smoothness with analytic decay, are then the natural functional

setting, being the counterpart to the Schwartz space $\mathcal{S}$ of rapidly decaying functions in the smooth setting.

Many authors started to study global singularities of tempered distributions in $\mathcal{S}'(\mathbb{R}^d)$ and introduced several different notions of wave front sets providing information not only on the local regularity of the distributions but also on their behavior at infinity. Among them we recall the *scattering wave front set* $\mathrm{WF}_{\mathrm{sc}}$, see [24], also known as $\mathcal{S}$-*wave front set* $\mathrm{WF}_{\mathcal{S}}$, see [12, 13], and the *quadratic scattering wave front set* $\mathrm{WF}_{\mathrm{qsc}}$, see [39], which appear as the natural tools to study the propagation of singularities on manifolds with conical ends. Also the analytic counterpart of these wave front sets has been defined in [29, 30].

Another notion of global wave front set for tempered distributions has been introduced in [20]. In this context a distribution $u \in \mathcal{S}'(\mathbb{R}^d)$ is microlocal at a point $(x_0, \xi_0) \in T^*(\mathbb{R}^d) \setminus \{(0,0)\}$ if there exists a Shubin type symbol $a(x, \xi)$ such that $a^w(x, D)u \in \mathcal{S}(\mathbb{R}^d)$ and a is non-characteristic at (x_0, ξ_0), (cf. also [32] for a different characterization of this set in terms of the Gabor transform). More recently, an equivalent notion of wave front set, called *homogeneous wave front set*, has been defined in [26] in the language of semi-classical analysis. The equivalence between the two wave front sets has been indeed proved only very recently in [33]. The homogeneous wave front set, as well as its Gevrey and analytic versions, see [22, 23, 25], has been mainly applied to study the microlocal smoothing effect for Schrödinger equations.

As previously mentionend, in this paper we are interested in the case of tempered ultradistributions of Gelfand-Shilov type. For $\theta > 1$, the microlocal analysis in these spaces is straight-forward to generalize. A notion of scattering wave front set has been introduced in [3] by a translation of the approach used in [13]. Similarly, the extensions of the homogeneous wave front set in [22, 23, 25], encoding Gevrey and analytic singularities of tempered distributions, can be easily extended to the elements of the dual space of $\mathcal{S}_\theta(\mathbb{R}^d)$, $\theta \geq 1$.

For $\theta < 1$, the scattering wave front set cannot be so easily converted, as the space $\mathcal{S}_\theta(\mathbb{R}^d)$ lacks compactly supported functions. A first approach to microlocal analysis in these spaces would be to approximate cut-off functions as done in the local theory in [19]. However, this method is technically very involved and yields several different notions for the case $\theta < 1$, see [28].
In [20] Hörmander proposed a different approach based on the so-called Fourier-Bros-Iagolnitzer (FBI) transform (also known as Bargmann transform) but he restricted his analysis to the limit case $\theta = 1/2$. On the other hand, recently the spaces $\mathcal{S}_\theta(\mathbb{R}^d)$ and their duals, have been widely considered in the literature for $1/2 \leq \theta < 1$, both in the theory of partial differential equations [1, 2, 20] and in time-frequency

analysis [10, 17, 35, 36, 37, 38]. In the latter sources, the compatibility of these spaces with the Bargmann transform has been observed.

For these reasons we consider it promising to state a rigorous microlocal analysis for quasianalytic tempered ultradistributions. As a further novelty, due to the relation between the FBI transform and the short time Fourier transform, we are able to study the properties of our wave front sets under the action of localization operators. This is an entirely new feature for this wave front set, as previously microlocal properties of localization operators were only known through their connection to Weyl pseudodifferential operators.

Our wave front set is modelled similarly to the homogeneous wave front set in [22, 23, 25, 26], in particular it is defined as a conic set in $\mathbb{R}^{2d} \setminus \{(0,0)\}$. Here we focus mainly on the general properties of this wave front set and postpone to a future paper the possible applications to partial differential equations.

The paper is organized as follows. In Section 1 we recall the definition and some basic facts on the Gelfand-Shilov spaces and their duals and we prove some mapping properties of the FBI transform on these spaces. In Section 2 we define the global wave front set for tempered ultradistributions and illustrate its behavior under linear symplectic transformations and standard operations such as pullbacks, tensor products, products and convolution. In Section 3 we study the microlocality properties of the wave front set with respect to localization operators and to differential and ultradifferential operators. Finally we prove a microellipticity result for differential operators with polynomial coefficients in analogy with what has been done in [20] in the case $\theta = 1/2$.

1. PRELIMINARIES

In the sequel we will use the notation $A(x) \lesssim B(x)$ if two maps A and B from some set X to $[0, +\infty)$ fulfill $A(x) \leq CB(x) \ \forall x \in X$ for some positive constant C independent of all possible indices involved.

1.1. **Gelfand-Shilov functions and ultradistributions.** In the following let $\mu, \nu > 0$ with $\mu + \nu \geq 1$.

Definition 1.1. Let $C > 0$. We denote by $\mathcal{S}^{\mu}_{\nu,C}(\mathbb{R}^d)$ the Banach space of all $\mathcal{C}^{\infty}$-functions satisfying

$$(1.1) \qquad \sup_{\alpha,\beta \in \mathbb{N}^d} \sup_{x \in \mathbb{R}^d} C^{-|\alpha|-|\beta|}(\alpha!)^{-\nu}(\beta!)^{-\mu}|x^{\alpha}\partial^{\beta}f(x)| < \infty,$$

equipped with the norm given by the lefthand side.

The space of all Gelfand-Shilov functions of indices μ, ν on $\mathbb{R}^d$ is then defined as

$$\mathcal{S}^{\mu}_{\nu}(\mathbb{R}^d) := \bigcup_{C > 0} \mathcal{S}^{\mu}_{\nu,C}(\mathbb{R}^d)$$

equipped with the inductive limit topology.

There are other equivalent ways of defining the space $\mathcal{S}_\nu^\mu(\mathbb{R}^d)$, cf. [6, 15]. We list some of them in the following lemma.

Lemma 1.2. *Let $\mu > 0, \nu > 0, \mu + \nu \geq 1$. For $f \in \mathcal{S}(\mathbb{R}^d)$ the following conditions are equivalent:*
i) $f \in \mathcal{S}_\nu^\mu(\mathbb{R}^d)$.
ii) There exist positive constants C, c such that

$$|\partial^\alpha f(x)| \lesssim C^{|\alpha|}(\alpha!)^\mu e^{-c|x|^{1/\nu}}, \qquad x \in \mathbb{R}^d.$$

iii) There exists $\varepsilon > 0$ such that

$$|f(x)| \lesssim e^{-\varepsilon|x|^{1/\nu}}, \quad x \in \mathbb{R}^d, \quad and \quad |\hat{f}(\xi)| \lesssim e^{-\varepsilon|\xi|^{1/\mu}}, \quad \xi \in \mathbb{R}^d.$$

In this paper, we are concerned with the quasianalytic case, i.e. $\mu < 1$. In these spaces, we have additional properties concerning holomorphic extensions:

Proposition 1.3. *If $f \in \mathcal{S}_\nu^\mu(\mathbb{R}^d), 0 < \mu < 1, \nu > 0$, then f extends to an entire analytic function $f(x+iy)$ in $\mathbb{C}^d$, which satisfies the following estimate:*

$$|f(x + iy)| \lesssim e^{-\varepsilon|x|^{\frac{1}{\nu}} + \delta|y|^{\frac{1}{1-\mu}}}, \quad x \in \mathbb{R}^d, y \in \mathbb{R}^d,$$

for some $\delta, \epsilon > 0$.

Definition 1.4. We denote by $(\mathcal{S}_\nu^\mu)'(\mathbb{R}^d)$ the topological dual of $\mathcal{S}_\nu^\mu(\mathbb{R}^d)$ (endowed with the weak topology).
More precisely: a linear form $u : \mathcal{S}_\nu^\mu(\mathbb{R}^d) \to \mathbb{C}$ is in $(\mathcal{S}_\nu^\mu)'(\mathbb{R}^d)$ if for every $\varepsilon > 0$ there exists a positive constant C_ε such that

$$(1.2) \qquad |u(f)| \lesssim C_\varepsilon \sup_{\alpha,\beta \in \mathbb{N}^d} \varepsilon^{-|\alpha|-|\beta|}(\alpha!)^{-\nu}(\beta!)^{-\mu} \sup_{x \in \mathbb{R}^d} |x^\alpha \partial^\beta f(x)|$$

for every $f \in S_\nu^\mu(\mathbb{R}^d)$.

1.2. **Integral transforms on Gelfand-Shilov spaces.**

1.2.1. *Fourier transform.* Already in [15] it was shown that the Gelfand-Shilov spaces are invariant under translations and dilations and that they have the following behaviour under Fourier transforms:

Lemma 1.5. *The Fourier transformation maps continuously $\mathcal{S}_\nu^\mu(\mathbb{R}^d)$ into $\mathcal{S}_\mu^\nu(\mathbb{R}^d)$ and $(\mathcal{S}_\nu^\mu)'(\mathbb{R}^d)$ into $(\mathcal{S}_\mu^\nu)'(\mathbb{R}^d)$.*

To have full metaplectic invariance, in the sequel we shall restrict ourselves to the case of $\mu = \nu =: \theta$ with $\frac{1}{2} \leq \theta < 1$ and we shall write $\mathcal{S}_\theta(\mathbb{R}^d), \mathcal{S}'_\theta(\mathbb{R}^d)$ to denote the space $\mathcal{S}_\theta^\theta(\mathbb{R}^d)$ and its dual.

There are several other transforms that have good properties with respect to Gelfand-Shilov spaces. One of them is the following Bargmann- or FBI-transform, stemming from [20]:

1.2.2. *FBI-transform with general phases.*

Definition 1.6. Let φ be a quadratic form in $\mathbb{C}^d \times \mathbb{R}^d$, i.e. $\varphi(z, y) = \frac{1}{2}\langle Az, z \rangle + \langle Bz, y \rangle + \frac{1}{2}\langle Cy, y \rangle$ with A, B and C complex $d \times d$-matrices such that A and C are symmetric. Assume further non-degeneracy conditions, i.e. $\det B = \det \varphi''_{zy} \neq 0$, and $\Im C = \Im \varphi''_{yy} > 0$.

For $u \in \mathcal{S}'_\theta(\mathbb{R}^d)$ we denote by

$$(1.3) \qquad T_\varphi(u)(z) = c_\varphi \langle u_y, e^{i\varphi(z,y)} \rangle,$$

the FBI transform with phase φ of u. Herein,

$$c_\varphi := 2^{-\frac{d}{2}} \pi^{-\frac{3d}{4}} (\det \Im C)^{-\frac{1}{4}} |\det B|$$

and the pairing is with respect to the y-variable.

In the sequel we shall denote $\Phi(z) = \max(-\Im\varphi(z, y))$ and by $y(z)$ the point where such a maximum is attained. We then observe that $y(z)$ is a linear function of z and that

$$-\Im\varphi(z, y) - \Phi(z) = -\frac{1}{2}\langle(\Im C)(y - y(z)), y - y(z)\rangle.$$

Remark 1.7. The most interesting case is that of the phase

$$\phi(z, y) := \langle z, y \rangle + \frac{i}{2}|y|^2,$$

for which we have $c_\phi := 2^{-\frac{d}{2}} \pi^{-\frac{3d}{4}}$. In this case, T_ϕ turns out to coincide with the inverse Fourier-Laplace transform of $v(y) = e^{-\frac{1}{2}|y|^2} u(y)$. Moreover, in this case we have $\Phi(z) = \frac{1}{2}|\Im(z)|^2$.

We now study the properties of the Bargmann transform of tempered ultradistributions. Let us first recall a classic result, which will be useful in the sequel: The Bargmann transform takes L^2-functions to entire functions that are L^2 with respect to a Gaussian measure, the so-called Bargmann-Fock space. A proof in our context can be found in [20].

Lemma 1.8. *Let φ be as in Definition 1.6. Then*

$$(1.4) \qquad \int |T_\varphi u(z)|^2 e^{-2\Phi(z)} d\lambda(z) = \|u\|_{L^2(\mathbb{R}^d)}.$$

where $d\lambda(z)$ denotes the Lebesgue measure on $\mathbb{C}^d$. In other words, the map $u \to T_\varphi u$ is an isometry from $L^2(\mathbb{R}^d)$ into $L^2(\mathbb{C}^d, d\lambda(z)e^{-2\Phi(z)}) \cap \mathcal{H}(\mathbb{C}^d)$, where $\mathcal{H}(\mathbb{C}^d)$ denotes the space of entire functions on $\mathbb{C}^d$.

The following statement is an analogous version with respect to Gelfand-Shilov functions and tempered ultradistributions. Similar statements have been proved in [4, 17, 35, 36] for particular phase functions φ. Here for our purposes we need to consider the general case. In order to go immediately to the core of the paper, we address the reader to the Appendix at the end of the paper for the proof.

Theorem 1.9. *Let $u \in \mathcal{S}'_\theta(\mathbb{R}^d)$ and let φ be a FBI-phase. Then $T_\varphi u$ is an entire analytic function which satisfies for any $\alpha \in \mathbb{N}^d$ the following condition:*

$$(1.5) \qquad \forall \varepsilon > 0 \; \exists C_\varepsilon > 0 \; s.t. |\partial_z^\alpha T_\varphi u(z)| \leq C_\varepsilon^{|\alpha|+1} \alpha!^\theta \; e^{\varepsilon|z|^{1/\theta}+\Phi(z)}.$$

Moreover, if $u \in \mathcal{S}_\theta(\mathbb{R}^d)$, then

$$(1.6) \qquad \exists \varepsilon > 0, C > 0 \; s.t. |\partial_z^\alpha T_\varphi u(z)| \leq C^{|\alpha|+1} \alpha!^\theta \; e^{-\varepsilon|z|^{1/\theta}+\Phi(z)}.$$

Conversely, if there is an analytic function U satisfying the estimate in (1.5) (respectively (1.6)), then there exists a unique $u \in \mathcal{S}'_\theta(\mathbb{R}^d)$ (respectively $u \in \mathcal{S}_\theta(\mathbb{R}^d)$) s.t. $U = T_\varphi u$.

Remark 1.10. In the case $\phi(z, y) = \langle z, y \rangle + \frac{i}{2}|y|^2$, when $u \in \mathcal{S}_\theta(\mathbb{R}^d)$, the estimate (1.6) takes the form

$$|T_\phi u(z)| \leq C e^{-\varepsilon|z|^{1/\theta}+\frac{1}{2}|\Im z|^2}, \qquad z \in \mathbb{C}^d,$$

for some positive constants C, ε, cf. also [4, Proposition 2.5].

We note that in the proof of Theorem 1.9 we use the following identity, which can be seen as an inversion formula:

Lemma 1.11. *Let $u \in \mathcal{S}'_\theta(\mathbb{R}^d)$, $f \in \mathcal{S}_\theta(\mathbb{R}^d)$. Then*

$$(1.7) \quad \langle u, f \rangle = \langle T_\varphi u, e^{-2\Phi(\cdot)} T_{-\overline{\varphi}} f \rangle = \int_{\mathbb{C}^d} T_\varphi u(z) T_{-\overline{\varphi}} f(z) e^{-2\Phi(z)} d\lambda(z).$$

1.3. **Short-time Fourier transform.** The Bargmann or FBI transform is deeply connected to another transform, the *short time Fourier transform* (short: *STFT*). For a broad analysis of this connection, we refer to [16, 35, 36] in the setting of Gelfand-Shilov and modulation spaces. In the STFT, we allow more general window functions than Gaussians. Therefore the holomorphicity properties of the transform are less prominent.

Definition 1.12. Let $g \in \mathcal{S}_\theta(\mathbb{R}^d) \setminus \{0\}$, the so-called *window function*. Then for $u \in \mathcal{S}'_\theta(\mathbb{R}^d)$ the short time Fourier transform $V_g(u)$, is defined as

$$V_g(u)(z) = \underbrace{\frac{1}{(2\pi)^{d/2}\|g\|_2}}_{=:k_g} \langle u, \overline{g(\cdot - x)} e^{-i\langle \xi, \cdot \rangle} \rangle \quad \text{with } z = (x, \xi).$$

In particular, $V_\psi(u)$ denotes the transform with the standard Gaussian window $\psi(y) := \pi^{-d/4} e^{-\frac{y^2}{2}}$.

Lemma 1.13 (Properties of the STFT)**.** *Let $h \in \mathcal{S}_\theta(\mathbb{R}^d)$. Then (see [35]) V_h is a continuous mapping from $\mathcal{S}_\theta(\mathbb{R}^d)$ to $\mathcal{S}_\theta(\mathbb{R}^{2d})$ and $\mathcal{S}'_\theta(\mathbb{R}^d)$ to $\mathcal{S}'_\theta(\mathbb{R}^{2d})$ satisfying $V_h(u) \in \mathcal{S}_\theta(\mathbb{R}^{2d}) \Rightarrow u \in \mathcal{S}_\theta(\mathbb{R}^d)$.*

For the standard Gaussian window ψ and the standard phase ϕ we have the following identity:

(1.8)
$$e^{-\Phi(\chi_\phi^b(x+i\xi))}T_\phi(u)(\chi_\phi^b(x+i\xi)) = e^{-\frac{|x|^2}{2}}T_\phi(u)(-\xi-ix) = V_\psi(u)(x,\xi).$$

Furthermore we have the following estimate for changing to another window $g \in \mathcal{S}_\theta(\mathbb{R}^d)$ (see [16, Lemma 11.3.3]):

(1.9)
$$|V_g u| \lesssim \|h\|_2^{-1}\left(|V_h g| * |V_h u|\right).$$

2. THE GLOBAL WAVE FRONT SET ON GELFAND-SHILOV SPACES

In this section we define the global wave front set for tempered ultradistributions from $\mathcal{S}_\theta'(\mathbb{R}^d)$. We shall define it, for clearness sake, first with respect to a general FBI phase. First of all let us consider the linear canonical transformation associated to a phase function φ and defined as follows:

$$\chi_\varphi : (y, -\varphi_y'(y,z)) \to (z, \varphi_z'(y,z)),$$

and let $\chi_\varphi^b = \chi_\varphi \circ \pi_1$, where $\pi_1 : T^*\mathbb{C}^d \to \mathbb{C}^d$ is the standard projection. It is known from [34] that χ_φ^b is a bijection of $T^*\mathbb{R}^d$ on $\mathbb{C}^d$.

Definition 2.1. Let $(x_0, \xi_0) \in \mathbb{R}^{2d} \setminus \{0\} = T^*\mathbb{R}^d \setminus \{0\}$ and $u \in \mathcal{S}_\theta'(\mathbb{R}^d)$. We say that $(x_0, \xi_0) \notin \mathrm{WF}_{\mathrm{gl}}^\theta(u)$ if there exists an open conic neighbourhood Γ of $\chi_\varphi^b(x_0, \xi_0)$ and $\varepsilon > 0$ such that

(2.1)
$$|T_\varphi u(z)| \lesssim e^{-\varepsilon|z|^{1/\theta}+\Phi(z)}$$

holds for every $z \in \Gamma$.

As a consequence of Theorem 1.9 we get the following result.

Corollary 2.2. *Let $u \in \mathcal{S}_\theta'(\mathbb{R}^d)$. Then $\mathrm{WF}_{\mathrm{gl}}^\theta(u) = \emptyset$ iff $u \in \mathcal{S}_\theta(\mathbb{R}^d)$.*

Proof. If $u \in \mathcal{S}_\theta(\mathbb{R}^d)$, then $\mathrm{WF}_{\mathrm{gl}}^\theta(u) = \emptyset$ since (2.1) holds on all $\mathbb{C}^d$. Viceversa, if $\mathrm{WF}_{\mathrm{gl}}^\theta(u) = \emptyset$, by the invertibility of χ_φ^b, we have that for every $z \in \mathbb{C}^d$ with $|z| = 1$ there exists a conic neighbourhood of z where the estimate in (1.6) holds. Since the set $\{z \in \mathbb{C}^d : |z| = 1\}$ is compact, then we can find $\varepsilon > 0, C > 0$ such that

(2.2)
$$|T_\varphi u(z)| \le C e^{-\varepsilon|z|^{1/\theta}+\Phi(z)}$$

for every $z \in \mathbb{C}^d$ with $|z| \ge 1$. For $|z| < 1$ the estimate is obviously satisfied. Then (2.2) holds on $\mathbb{C}^d$ and this yields $u \in \mathcal{S}_\theta(\mathbb{R}^d)$ by Theorem 1.9. $\square$

Arguing as in [20] it is easy to verify that the set $\mathrm{WF}_{\mathrm{gl}}^\theta(u)$ is independent of the choice of φ. In particular for $\phi(y,z) = \langle z, y \rangle + \frac{i}{2}|y|^2$ we have $\chi_\phi^b(x, \xi) = -\xi - ix$, which gives that a point $(x_0, \xi_0) \notin \mathrm{WF}_{\mathrm{gl}}^\theta(u)$ if

and only if there exist a conic neighbourhood Γ' of (x_0, ξ_0) and $\varepsilon > 0$ such that

$$(2.3) \qquad \left| \langle u_y, e^{-i\langle y, \xi \rangle - \frac{1}{2}|y-x|^2} \rangle \right| \leq C e^{-\varepsilon(|x|^{1/\theta} + |\xi|^{1/\theta})} \qquad (x, \xi) \in \Gamma'.$$

Equivalently, using (1.8), this can be written in terms of the STFT:

$$(2.4) \qquad |V_\psi(u)(x, \xi)| \leq C e^{-\varepsilon(|x|^{1/\theta} + |\xi|^{1/\theta})} \qquad (x, \xi) \in \Gamma'.$$

Lemma 2.3. *Let* $z_0 \in \mathbb{R}^{2d} \setminus \{0\}$, Γ *be an open conic neighbourhood of* z_0. *Let* $U \in \mathcal{C}^\infty(\mathbb{R}^{2d})$ *satisfy the following conditions*

$$\exists \varepsilon, \ C > 0 \ s.t. \ |U(x, \xi)| \leq C e^{-\varepsilon(|x|^{1/\theta} + |\xi|^{1/\theta})} \qquad (x, \xi) \in \Gamma,$$

$$\forall \varepsilon, \exists C_\varepsilon > 0 \ s.t. \ |U(x, \xi)| \leq C_\varepsilon e^{\varepsilon(|x|^{1/\theta} + |\xi|^{1/\theta})} \qquad (x, \xi) \in \Gamma^c,$$

Let $G \in \mathcal{S}_\theta(\mathbb{R}^{2d})$. *Then for every open conic neighbourhood* Γ' *of* z_0 *such that* $\overline{\Gamma'} \subset \Gamma$ *we have*
$$(2.5)$$
$$\exists \varepsilon, \ C > 0 \ s.t. \ |(G * U)(x, \xi)| \leq C e^{-\varepsilon(|x|^{1/\theta} + |\xi|^{1/\theta})} \qquad (x, \xi) \in \Gamma'.$$

Proof. The proof is done by careful splitting of the convolution integral with respect to Γ. As we already carry out a similar argument in detail in the later proof of Proposition 2.10, we omit it here. $\qquad \square$

The following Lemma states what happens if the Gaussian window function used in the transform is replaced by another general element in $\mathcal{S}_{1/2}(\mathbb{R}^d)$.

Lemma 2.4. *Let* $u \in \mathcal{S}'_\theta(\mathbb{R}^d)$ *and let* Γ *be a closed cone in* $T^*\mathbb{R}^d$ *such that* $\Gamma \cap \mathrm{WF}^\theta_{\mathrm{gl}}(u) = \emptyset$. *Then for every* $g \in \mathcal{S}_{1/2}(\mathbb{R}^d)$ *there exist positive constants* M, m *such that*

$$|V_g(u)(x, \xi)| \leq M e^{-m(|x|^{\frac{1}{\theta}} + |\xi|^{\frac{1}{\theta}})}, \quad (x, \xi) \in \Gamma.$$

Conversely, in the previous characterization (2.4) *of the wave front set we can replace* ψ *by any non-zero window function* $g \in \mathcal{S}_{1/2}(\mathbb{R}^d) \setminus \{0\}$.

Proof. By (1.9) we write

$$|V_g u| \lesssim |V_\psi g| * |V_\psi u|.$$

Applying Lemma 2.3 yields the assertion. $\qquad \square$

The following Proposition asserts that for each possible global wave front set, there exists a distribution with such singularities. The construction used is similar to the original one for the classical wave front set in [19] and has first been used in a similar statement for the corner component of the $\mathcal{S}$-wave front set in [14]. For the Gabor wave front set, i.e. in the tempered setting, it has been carried out in [33]. Here we adapt it to the Gelfand-Shilov context.

Proposition 2.5. *Let* $\Gamma \subset \mathbb{R}^{2d} \setminus \{0\}$ *closed, conic. Then there exists* $u \in \mathcal{S}'_\theta(\mathbb{R}^d)$ *such that* $\mathrm{WF}^\theta_{\mathrm{gl}}(u) = \Gamma$.

Proof. Let $(y, \eta) \in \mathbb{R}^{2d} \setminus \{0\}$ and $k \in \mathbb{N}$ be fixed. We define for $x \in \mathbb{R}^d$

$$(2.6) \qquad f_k(x; y, \eta) := \exp\left(-\frac{1}{2}|x + k^2 y|^2 + i k^2 \langle x, \eta \rangle\right) \in \mathcal{S}_{1/2}(\mathbb{R}^d).$$

We may calculate the modulus of its transform:

$$(2.7) \qquad |T_\phi (f_k(\cdot; y, \eta))(x + i\xi)| = \frac{c_\phi}{\pi^{d/2}} e^{-\frac{1}{4}|\xi + k^2 y|^2 - \frac{1}{4}|x + k^2 \eta|^2 + \frac{|\xi|^2}{2}}$$

As the $T_\phi (f_k(\cdot; y, \eta))$ are analytic functions, we can thus conclude that

$$U(x + i\xi; y, \eta) := \sum_{k=1}^\infty T_\phi (f_k(\cdot; y, \eta))$$

is an analytic function satisfying (1.5) everywhere, as the sum is bounded on each compactum.

Now take a sequence (y_j, η_j), dense in $\Gamma \cap S^{d-1}$ and without repetitions[1] and define

$$U(x + i\xi) := \sum_{j=1}^\infty 2^{-j} U(x + i\xi; y_j, \eta_j),$$

which again is an analytic function satisfying the estimate in (1.5). We thus define u as the unique ultradistribution in $\mathcal{S}'_\theta(\mathbb{R}^d)$ such that $T_\phi u \equiv U$.

If $z_0 \notin \Gamma$, we have by a standard scaling inequality for disjoint cones in an open conic neighbourhood L of $\chi_\phi^b(z_0)$:

$$\forall z \in L, \ w \in \Gamma : \quad |\chi_\phi^b(z) - w| \gtrsim |\chi_\phi^b(z)| + |w|$$

Recall that for the standard phase we have $\chi_\phi^b(x + i\xi) = -\xi - ix$. We may thus conclude by (2.7) that on L we have

$$|U(z)| \lesssim \sum_{j=1}^\infty \sum_{k=1}^\infty 2^{-j} \exp\left(-\frac{1}{4}|\chi_\phi^b(z) - k^2(y, \eta)|^2 + \Phi(z)\right)$$

$$\lesssim \sum_{k=1}^\infty \exp\left(-c(k^4 + |z|^2) + \Phi(z)\right),$$

which proves $z_0 \notin \mathrm{WF}_{\mathrm{gl}}(u)$ and thus $\mathrm{WF}_{\mathrm{gl}}(u) \subset \Gamma$.

To prove the opposite inclusion, $\Gamma \subset \mathrm{WF}_{\mathrm{gl}}(u)$, consider a fixed (y_j, η_j). For that we note that for $m \neq j$

$$(2.8) \qquad \left|\left(e^{-\Phi} U\right)(k^2 \chi_\phi^b(y_j, \eta_j); y_j, \eta_j)\right| \overset{k\to\infty}{\to} \frac{c_\phi}{\pi^{d/2}},$$

$$(2.9) \qquad \left|\left(e^{-\Phi} U\right)(k^2 \chi_\phi^b(y_j, \eta_j); y_m, \eta_m)\right| \overset{k\to\infty}{\to} 0.$$

With these identities it is easy to prove that for suitably large k we have $|U(k^2 \chi_\phi^b(y_j, \eta_j))| > 1/2$, cf. [33], meaning $(y_j, \eta_j) \in \mathrm{WF}_{\mathrm{gl}}^\theta(u)$. As the (y_j, η_j) are dense in Γ, and $\mathrm{WF}_{\mathrm{gl}}^\theta(u)$ is a closed set, this concludes the proof. $\square$

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[1]In case of a finite Γ take the finite set of points $\{(y_j, \eta_j)\} = \Gamma \cap S^{d-1}$.

2.1. **Transformation properties.** In this section we list the behavior of $\mathrm{WF}_{\mathrm{gl}}^{\theta}$ under unitary transformations associated with linear symplectomorphisms of $T^*\mathbb{R}^d$, cf. [20, Proposition 6.7].

Proposition 2.6. *Let $u \in \mathcal{S}_{\theta}'(\mathbb{R}^d)$ and let $(x_0, \xi_0) \in \mathbb{R}^{2d} \setminus \{0\}$. Then the following properties hold:*
i) Let $\mathbb{R}^d = \mathbb{R}^{d_v + d_w}$,

$$(v_0, w_0, \zeta_0, \eta_0) \in \mathrm{WF}_{\mathrm{gl}}^{\theta}(u) \Leftrightarrow (\zeta_0, w_0, -v_0, \eta_0) \in \mathrm{WF}_{\mathrm{gl}}^{\theta}(\mathcal{F}_{v \to \zeta} u).$$

ii) Let A be a real, symmetric $d \times d$-matrix. Then

$$(x_0, \xi_0) \in \mathrm{WF}_{\mathrm{gl}}^{\theta}(u) \Leftrightarrow (x_0, \xi_0 + A x_0) \in \mathrm{WF}_{\mathrm{gl}}^{\theta}\left(e^{i x^t A x} u\right).$$

iii) Given a linear invertible map A on $\mathbb{R}^d$ and denoted by ^tA its transpose, we have

$$(x_0, \xi_0) \in \mathrm{WF}_{\mathrm{gl}}^{\theta}(u) \Leftrightarrow (A^{-1}x_0, {}^tA\xi_0) \in \mathrm{WF}_{\mathrm{gl}}^{\theta}(A^* u),$$

where $A^* u(y) = \sqrt{|detA|}u(Ay)$.
iv) $(x_0, \xi_0) \in \mathrm{WF}_{\mathrm{gl}}^{\theta}(u) \Leftrightarrow (x_0, -\xi_0) \in \mathrm{WF}_{\mathrm{gl}}^{\theta}(\bar{u})$.

To each linear symplectomorphism $\chi : T^*\mathbb{R}^d \to T^*\mathbb{R}^d$ there exists an associated unitary transform $U : L^2(\mathbb{R}^{2d}) \to L^2(\mathbb{R}^{2d})$, see [20]. As $i) - iii)$ yield the generators of the symplectic group, we get from Proposition 2.6 the following Corollary:

Corollary 2.7. *Let $\chi : T^*\mathbb{R}^d \to T^*\mathbb{R}^d$ linear, symplectic, then*

$$(x_0, \xi_0) \in \mathrm{WF}_{\mathrm{gl}}^{\theta}(u) \Leftrightarrow \chi(x_0, \xi_0) \in \mathrm{WF}_{\mathrm{gl}}^{\theta}(Uu),$$

where U is the unitary transform associated to χ.

Remark 2.8. The preceding corollary underlines the usefulness of the notion of $\mathrm{WF}_{\mathrm{gl}}^{\theta}(u)$ to describe global propagation of singularities under Schrödinger equations.
For ultradistributions of the form $u(t, \cdot) = \mathcal{F}^{-1}\left(e^{i\xi^2 t}\mathcal{F}(u_0)\right) \in \mathcal{S}_{\theta}'(\mathbb{R}^d)$, $u_0 \in \mathcal{S}_{\theta}'(\mathbb{R}^d)$, we have

$$(x, \xi) \in \mathrm{WF}_{\mathrm{gl}}^{\theta}(u_0) \Leftrightarrow (x + t\xi, \xi) \in \mathrm{WF}_{\mathrm{gl}}^{\theta}(u(t, \cdot)).$$

These distributions solve the homogeneous initial value problem for the Schrödinger equation

$$\begin{cases} -i\partial_t u(t, \cdot) + \Delta u(t, \cdot) & = 0 \\ \left. u \right|_{t=0} & = u_0. \end{cases}$$

The metaplectic invariance of the (Gabor) wave front set was used in [8] to study more general Schrödinger operators. We note however that the counter-example of [9, Proposition 4.1] limits the class of interesting operators in the super-exponential setting. For more on propagation

of these singularities under Schrödinger operators in various functional settings consider [22, 23, 25, 26].

2.2. **Behaviour under operations.** In the following we will study the behaviour of $\mathrm{WF}^{\theta}_{\mathrm{gl}}(u)$ under operations such as pull-backs, tensor products, etc. For that, we first introduce a notion of continuity on the space of distributions with wave front set in a given cone. When speaking about continuity in the sequel, we will mean sequential continuity with respect to the following notion of convergence.

Definition 2.9. Let Γ be a closed sub-cone of $T^*(\mathbb{R}^d) \setminus 0$. We denote by $\mathcal{S}'_{\theta,\Gamma}(\mathbb{R}^d)$ the space

$$\mathcal{S}'_{\theta,\Gamma}(\mathbb{R}^d) := \{u \in \mathcal{S}'_{\theta}(\mathbb{R}^d) | \mathrm{WF}^{\theta}_{\mathrm{gl}}(u) \subset \Gamma\}$$

endowed with the following notion of convergence:
We say that $u_n \overset{\mathcal{S}'_{\theta,\Gamma}}{\to} 0$ if

 (1) $u_n \overset{\mathcal{S}'_{\theta}}{\to} 0$,
 (2) For all $z \in \Gamma^c$ there exists a conic neighbourhood L of $\chi^b_{\varphi}(z)$, $C > 0$ and $\varepsilon > 0$ such that for all $n \in \mathbb{N}$, $z' \in L$ we have $|T_{\varphi}(u_n)(z')| \leq C_{\varepsilon} e^{-\varepsilon|z'|^{1/\theta}+\Phi(z')}$.

Proposition 2.10. $\mathcal{S}_{\theta}(\mathbb{R}^d)$ *is dense in* $\mathcal{S}'_{\theta,\Gamma}(\mathbb{R}^d)$.

First let us note that an ultradistribution in $\mathcal{S}'_{\theta}(\mathbb{R}^d)$ satisfies the estimate in (1.6) for the standard phase ϕ in a cone $\Gamma \ni \chi^b_{\phi}(z)$, if and only if the following short time Fourier transform of u, which differs from the standard one by only a phase,

$$(2.10) \qquad Vu(x,\xi) = \underbrace{2^{-\frac{d}{2}} \pi^{-\frac{3d}{4}}}_{=:c} \langle u, e^{i(x-y)\cdot\xi - \frac{1}{2}|x-y|^2} \rangle$$

satisfies on Γ

$$(2.11) \qquad |Vu(x,\xi)| \leq C e^{-\epsilon(|x|^{1/\theta}+|\xi|^{1/\theta})}$$

For the proof of Proposition 2.10 we need to understand the transform of a regularizing pseudodifferential operator acting on a distribution u via the action of another pseudodifferential operator on Vu. The above transform enjoys the following identity:

Lemma 2.11. *Let* $a \in \mathcal{S}_{\theta}(\mathbb{R}^{2d})$, $u \in \mathcal{S}'_{\theta}(\mathbb{R}^d)$. *Then we have the following identity*

$$V(a(y,D_y)u) = \widetilde{a}(x,\xi,D_x,D_{\xi})V(u),$$

where $\widetilde{a}(x,\xi,x^*,\xi^*) = a(x - \xi^*, x^*)$ *and* (x^*, ξ^*) *denotes the covariable to* (x,ξ).

Proof. The statement can be verified for $u \in \mathcal{S}_{\theta}(\mathbb{R}^d)$ by repeating the computation in [21, Proposition 3.3.1]. The assertion for general $u \in \mathcal{S}'_{\theta}(\mathbb{R}^d)$ then follows by a density argument. $\square$

Proof of Proposition 2.10. Let $u \in \mathcal{S}'_{\theta,\Gamma}(\mathbb{R}^d)$, i.e. $\mathrm{WF}^\theta_{\mathrm{gl}}(u) \subset \Gamma$ and choose $a_\varepsilon(x,\xi) = e^{-\frac{\varepsilon}{2}|x|^2}e^{-\frac{\varepsilon}{2}|\xi|^2}$. Then $a(x,\xi,\varepsilon) \in \mathcal{S}_\theta(\mathbb{R}^{2d})$ for every $\varepsilon > 0$ and thus also $u_\varepsilon := a_\varepsilon(x, D_x)u \in \mathcal{S}_\theta(\mathbb{R}^d)$.

It is easy to verify that $a_\varepsilon(x, D_x)u \xrightarrow{\mathcal{S}'_\theta} u$ as $\varepsilon \to 0$.

It remains to show, that for all $(x_0, \xi_0) \in \Gamma^c$ we can find a δ, C_δ such that in a conic neighbourhood Γ_0 of (x_0, ξ_0) we have for every $\epsilon > 0$

$$(2.12) \qquad Vu_\varepsilon(x,\xi) \le C_\delta e^{-\delta(|x|^{1/\theta} + |\xi|^{1/\theta})}.$$

To do that, we use

$$|Vu_\varepsilon(x,\xi)| = |V\left(a_\varepsilon(y, D_y)u\right)(x,\xi)|$$
$$= |(\widetilde{a}_\varepsilon(x,\xi, D_x, D_\xi)Vu)(x,\xi)|$$
$$\lesssim \left| \int a_\varepsilon(x - \xi^*, x^*)e^{i(x-x')x^* + i(\xi-\xi')\xi^*}Vu(x',\xi')\ dx' dx^* d\xi' d\xi^* \right|$$
$$\lesssim \varepsilon^{-d/2} \int a_{\varepsilon^{-1}}(x - x', \xi - \xi')\,|Vu(x',\xi')|\ dx' d\xi'$$
$$= \varepsilon^{-d/2}\left(a_{\varepsilon^{-1}} * |Vu|\right)$$

In order to estimate the convolution, we split the integral in two parts: As $\mathrm{WF}^\theta_{\mathrm{gl}}(u) \subset \Gamma$, we pick an open cone $\Gamma_1 \ni (x_0, \xi_0)$, such that $\overline{\Gamma_1} \cap \Gamma = \emptyset$. We can then pick an intermediate closed cone Γ_2 such that $\Gamma_1 \subset \Gamma_2$ and $\Gamma_2 \cap \Gamma = \emptyset$. We then write, with $z = (x,\xi)$:

$$C\varepsilon^{-d/2}\left(a_{\varepsilon^{-1}} * |Vu|\right)(z) = C\varepsilon^{-d/2}\int_{\Gamma_2} e^{\frac{-1}{2\varepsilon}|z-w|^2}|Vu(w)|\ dw$$
$$+ C\varepsilon^{-d/2}\int_{\Gamma_2^c} e^{\frac{-1}{2\varepsilon}|z-w|^2}|Vu(w)|\ dw$$
$$= I_{\Gamma_2}(z) + I_{\Gamma_2^c}(z).$$

Let us first study $I_{\Gamma_2^c}$. There we have, for $z \in \Gamma_1$, due to a standard scaling estimate for disjoint cones $|z - w| \gtrsim |z| + |w|$ and therefore for some constants $\delta_i > 0$

$$e^{\frac{-1}{2\varepsilon}|z-w|^2} \le e^{\frac{-\delta_1}{2\varepsilon}|z|^2}e^{\frac{-\delta_1}{2\varepsilon}|w|^2},$$

and consequently

$$|I_{\Gamma_2^c}(z)| \lesssim e^{-\delta_2|z|^2}.$$

On Γ_2 we can assume, due to compactness of $\Gamma_2 \cap S^{d-1}$, that there exist a single constant $\delta_3 > 0$ such that

$$Vu(w) \le e^{-\delta_3|w|^{1/\theta}}.$$

Using this, we conclude that

$$|I_{\Gamma_2}(z)| \lesssim e^{-\delta_4|z|^{1/\theta}}$$

and thus

$$|Vu_\varepsilon(x,\xi)| \lesssim \varepsilon^{-d/2} \left(a_{\varepsilon^{-1}} * |Vu|\right)(z) \lesssim e^{-\delta_2|z|^2} + e^{-\delta_4|z|^{1/\theta}} \lesssim e^{-\delta|z|^{1/\theta}},$$

which proves the assertion. □

We now study tensor products and pullbacks of ultradistributions and their resulting wave front set, following [20].

Proposition 2.12 (Behaviour under tensor products). *Given* $u \in \mathcal{S}'_{\theta,\Gamma_1}(\mathbb{R}^{d_1})$, $v \in \mathcal{S}'_{\theta,\Gamma_2}(\mathbb{R}^{d_2})$ *and let*

$$\Gamma = ((\Gamma_1 \cup \{0\}) \times (\Gamma_2 \cup \{0\})) \setminus \{(0,0)\} \subset \mathbb{R}^{d_1+d_2}.$$

Then $u \otimes v \in \mathcal{S}'_{\theta,\Gamma}(\mathbb{R}^{d_1+d_2})$.

Proof. This is a consequence of $T_\phi(u \otimes v) = T_\phi(u) \otimes T_\phi(v)$. □

Theorem 2.13 (Behaviour under the pullback by a linear map). *Let* A *be a linear map* $\mathbb{R}^m \to \mathbb{R}^d$. *Let* Γ *be a closed cone such that*

$$(2.13) \qquad\qquad \Gamma \cap \{(0,\xi)|{}^tA\xi = 0\} = \emptyset.$$

Then the pull-back $A^* : \mathcal{S}_\theta(\mathbb{R}^d) \to \mathcal{S}_\theta(\mathbb{R}^m)$ *can be uniquely extended to a continuous map* $\mathcal{S}'_{\theta,\Gamma}(\mathbb{R}^d) \to \mathcal{S}'_{\theta,A^*\Gamma}(\mathbb{R}^m)$, *where*

$$A^*\Gamma = \{(x,{}^tA\xi)|(Ax,\xi) \in \Gamma\}.$$

Proof. For the purpose of self-containedness, we give a shortened proof with respect to [20, Proposition 6.15].

Due to Proposition 2.6 it suffices to show this for the maps $\iota : \mathbb{R}^{d-1} \to \mathbb{R}^d$ $y \mapsto (y,0)$ and $\pi : \mathbb{R}^{d+1} \to \mathbb{R}^d$ $(x,x') \mapsto x$. For the second case, we can define $\pi^*u = u \otimes \mathbb{1}$.

We are therefore reduced to the case of ι. Formally, we want to define $\langle u, f \otimes \delta(x_d)\rangle$. In view of Lemma 1.11, we therefore make the following formal calculation, with the notation $\phi = \phi_d = \phi_{d-1} + \phi_1$ and $(z_1,\ldots,z_{d-1},z_d) = (z',z_d)$,

$$(2.14) \quad \langle \pi^*u, f\rangle = \int_{\mathbb{C}^d} T_{\phi_d}u(z)T_{-\bar\phi_d}(f \otimes \delta(x_d))(z)e^{-2\Phi(z)}d\lambda(z)$$

$$= \int_{\mathbb{C}^d} T_{\phi_d}u(z)T_{-\bar\phi_{d-1}}f(z')T_{-\bar\phi_1}(\delta(x_d))(z_d)e^{-2\Phi_{d-1}(z')-2\Phi_1(z_d)}d\lambda(z)$$

$$= c_{\phi_1}\int_{\mathbb{C}^{d-1}\times\mathbb{C}^1} T_{\phi_d}u(z',z_d)T_{-\bar\phi_{d-1}}f(z')e^{-2\Phi_{d-1}(z')-2\Phi_1(z_d)}d\lambda(z')d\lambda(z_d).$$

Again in light of Lemma 1.11 we consider the expression

$$(2.15) \qquad U(z') = c_{\phi_1}\int_{\mathbb{C}} T_{\phi_d}u(z',z_d)e^{-2\Phi_1(z_d)}d\lambda(z_d).$$

In the situation $A = \iota$ the condition (2.13) takes the form

$$\mathrm{WF}^\theta_{\mathrm{gl}}(u)\cap\{(0,\xi)|\xi_1 = \cdots = \xi_{d-1} = 0\} = \emptyset \Rightarrow (0,\ldots,0,\pm1) \notin \mathrm{WF}^\theta_{\mathrm{gl}}(u).$$

We calculate $\chi_\phi^b(0 + i(0, \ldots, 0, \pm 1)) = (0, \ldots, 0, \mp 1)$ and conclude that there exist open cones $\Gamma_\pm \ni (0, \ldots, 0, \pm 1)$ such that $T_{\phi_d} u$ satisfies the estimate in (1.6) on $\Gamma_+ \cup \Gamma_-$. Using (1.5) and $\Phi_1(z_d) = \frac{1}{2}|\Im(z_d)|^2$ we conclude that the integrand of (2.15) is bounded by

$$e^{-\varepsilon|\Re(z_d)|^{1/\theta} - \frac{1}{2}|\Im(z_d)|^2} e^{\delta|z'|^{1/\theta} + \Phi_{d-1}(z')}$$

for some $\varepsilon > 0$ and any $\delta > 0$. Therefore the integral (2.15) converges for any z', and yields an entire function satisfying

$$\forall \varepsilon > 0 \; \exists C_\varepsilon > 0 \text{ s.t. } |U(z')| \leq C_\varepsilon e^{\varepsilon|z'|^{1/\theta} + \Phi(z')}.$$

We can therefore define $A^* u$ as the ultradistribution $v \in \mathcal{S}'_\theta(\mathbb{R}^{d-1})$, granted by Theorem 1.9, such that $T_{\phi_{d-1}} v = U$.

The estimate for the resulting wave front set follows by careful splitting of the integral into regions where the integrand satisfies the stronger estimates (1.6). The continuity is immediate from dominated convergence of the integrals. $\square$

With this notion of pullback and the tensor product it is now possible to introduce products, convolutions, restrictions and pairings of tempered ultradistributions.

Corollary 2.14 (Products and convolutions). *Let* Γ_1, $\Gamma_2 \subset T^*\mathbb{R}^d \backslash \{0\}$,

$$\Gamma_3 = \{(x, \xi + \eta) | (x, \xi) \in \Gamma_1 \text{ and } (x, \eta) \in \Gamma_2\} \cup \Gamma_1 \cup \Gamma_2.$$

Then the product of two ultradistributions $u \in \mathcal{S}'_{\theta, \Gamma_1}(\mathbb{R}^d)$, $v \in \mathcal{S}'_{\theta, \Gamma_2}(\mathbb{R}^d)$ *is well-defined under the assumption that* $0 \notin \Gamma_3$, *i.e.*

$$(0, \xi) \in \Gamma_1 \Rightarrow (0, -\xi) \notin \Gamma_2.$$

Under these assumptions, we have the inclusion $\mathrm{WF}^\theta_{\mathrm{gl}}(u \cdot v) \subset \Gamma_3$ *and the product is a continuous mapping* $\mathcal{S}'_{\theta, \Gamma_1}(\mathbb{R}^d) \times \mathcal{S}'_{\theta, \Gamma_2}(\mathbb{R}^d) \to \mathcal{S}'_{\theta, A^*\Gamma}(\mathbb{R}^d)$,

Similarly, the convolution of two ultradistributions $u, v \in \mathcal{S}'_\theta(\mathbb{R}^d)$ *is well-defined under the assumption that*

$$(\xi, 0) \in \mathrm{WF}^\theta_{\mathrm{gl}}(u) \Rightarrow (-\xi, 0) \notin \mathrm{WF}^\theta_{\mathrm{gl}}(v).$$

We then have the inclusion

$$\mathrm{WF}^\theta_{\mathrm{gl}}(u*v) \subset \{(x+y, \xi) | (x, \xi) \in \mathrm{WF}^\theta_{\mathrm{gl}}(u) \cap \{0\} : (y, \xi) \in \mathrm{WF}^\theta_{\mathrm{gl}}(u) \cap \{0\}\}.$$

Proof. Use Theorem 2.13 and Proposition 2.12 to define the product of two distributions u and v by $(u \cdot v)(x) = \delta^*\big(u(x) \otimes v(y)\big)$ where δ is the diagonal map $x \mapsto (x, x)$.

The statement about convolution follows directly by Fourier transformation and $i)$ of Proposition 2.6. $\square$

Corollary 2.15 (Pairings of ultradistributions). *Under the assumption that* $(x, \xi) \in \mathrm{WF}^\theta_{\mathrm{gl}}(u) \Rightarrow (x, -\xi) \notin \mathrm{WF}_{\mathrm{gl}}(v)$ *we can define the pairing of u and v as the unique continuous extension of the pairing of two test functions.*

Proof. We define the pairing as the image of $\mathcal{F}(u \cdot v)$ under the pull-back via $0 \hookrightarrow \mathbb{R}^d$. $\qquad\square$

3. Microlocality and microellipticity properties

In this section we prove microlocality and microellipticity properties for some classes of operators with respect to $\mathrm{WF}_{\mathrm{gl}}^{\theta}(u)$.

3.1. **Localization operators.** Localization operators, or Anti-Wick quantized operators, have appeared in many contexts, ranging from Quantum field theory to signal analysis. Quite recently, localization operators in the setting of various function spaces have been an active field of research. For a history and survey on the subject, consider [7] and the references therein. The function spaces considered include Bargmann-Fock spaces, modulation spaces with exponential weights [11, 37] and Gelfand-Shilov spaces [36], in particular quasi-analytic ones [10], using in particular the good transformation behaviour of these spaces with respect to the short time Fourier transform. It is therefore only natural to consider their microlocal properties with respect to our global wave front set. We will do so in this section, proving a microlocality result.

Definition 3.1. Let $a \in \mathcal{S}_{\theta}'(\mathbb{R}^{2d})$. The localization operator A_a^{ψ} with respect to the standard window ψ with symbol a is weakly defined by (for $u, \ v \in \mathcal{S}_{\theta}(\mathbb{R}^d)$):

$$\langle A_a^{\psi} u, v \rangle = \langle a, V_{\psi} u \cdot V_{\psi} v \rangle.$$

We have already noted that $(x_0, \xi_0) \notin \mathrm{WF}_{\mathrm{gl}}^{\theta}(u)$ if the short-time Fourier transform $V_{\psi} u(x, \xi)$ satisfies (2.5) on an open cone $\Gamma \ni (x_0, \xi_0)$. We now recall the result of [36, Proposition 5.11], stating that if $\theta \neq 1/2$ and $a \in L_{\mathrm{loc}}^{\infty}(\mathbb{R}^{2d})$ satisfies that $\forall \varepsilon > 0$ there exists $C_{\varepsilon} > 0$ such that

$$(3.1) \qquad\qquad |a(x, \xi)| \leq C_{\varepsilon} e^{\varepsilon(|x|^{1/\theta} + |\xi|^{1/\theta})},$$

then A_a^{ψ} is continuous on both $\mathcal{S}_{\theta}(\mathbb{R}^d)$ and $\mathcal{S}_{\theta}'(\mathbb{R}^d)$. We can prove the following microlocal improvement of this statement.

Theorem 3.2. *Let $\theta > 1/2$ and A_a^{ψ} be a localization operator with symbol $a \in L_{\mathrm{loc}}^{\infty}(\mathbb{R}^{2d})$ satisfying (3.1). Then we have $\mathrm{WF}_{\mathrm{gl}}^{\theta}(A_a^{\psi} u) \subset \mathrm{WF}_{\mathrm{gl}}^{\theta}(u)$.*

Proof. We estimate $V_\psi(A_a^\psi u)$ in terms of $V_\psi u$ and a as follows:

$$V_\psi A_a^\psi u(v,\eta) = V_\psi V_\psi^*(a \cdot (V_\psi u))(v,\eta)$$

$$= \int e^{-\frac{1}{2}|v-y|^2 + i\langle y,\eta\rangle} e^{-\frac{1}{2}|x-y|^2 - i\langle \xi,y\rangle} a(x,\xi) V_\psi u(x,\xi) \; dxd\xi dy$$

$$= \int e^{-|y-\frac{x+v}{2}|^2} e^{-\frac{1}{4}|x-v|^2} e^{i\langle y,\eta-\xi\rangle} a(x,\xi) V_\psi u(x,\xi) \; dxd\xi dy$$

$$= 2^{-d/2} \int e^{-\frac{1}{4}|\eta-\xi|^2} e^{-\frac{1}{4}|x-v|^2} e^{\frac{i}{2}\langle x+v,\eta-\xi\rangle} a(x,\xi) V_\psi u(x,\xi) \; dxd\xi$$

We can conclude that

$$|V_\psi A_a^\psi u(x,\xi)| \lesssim \left(e^{-\frac{|\cdot|^2}{4}} * |(a \cdot V_\psi u)| \right)(x,\xi).$$

Then the assertion follows by applying Lemma 2.3.   □

We note that the method of the proof is not limited to the quasianalytic case but it can applied also for $\theta \geq 1$.

Remark 3.3. We can conclude that if A_a^ψ is an invertible operator then we have the equality

$$\mathrm{WF}_{\mathrm{gl}}^\theta(A_a^\psi u) = \mathrm{WF}_{\mathrm{gl}}^\theta(u).$$

3.2. Ultradifferential operators.

In general, localization operators can have symbols that are not analytic. Motivated by Theorem 1.9 we can instead consider the following class of operators, where the coefficients are multipliers and partial derivatives appear.
Denote the space of analytic functions $h : \mathbb{C}^d \to \mathbb{C}$ satisfying

$$(3.2) \qquad \forall \varepsilon > 0 \; \exists D_\varepsilon > 0 \text{ s.t.} |h(z)| \leq D_\varepsilon \; e^{\varepsilon|z|^{1/\theta}}$$

by $MS^\theta(\mathbb{C}^d)$.

Theorem 3.4. *Let $\{h_\alpha\}$ be a family of elements of $MS^\theta(\mathbb{C}^d)$ such that the constants $D_{\epsilon,\alpha}$ in (3.2) satisfy*

$$(3.3) \qquad\qquad D_{\epsilon,\alpha} \leq \frac{D_\varepsilon' \varepsilon^{|\alpha|}}{\alpha!^\theta}$$

for any ε and for some constant $D_\varepsilon' > 0$. Then we can define the following operator

$$Au(z) = \sum_{\alpha \in \mathbb{N}^d} T_\varphi^* h_\alpha(z)(\partial_z^\alpha T_\varphi u)(z)$$

as a continuous map $\mathcal{S}'_{\theta,\Gamma}(\mathbb{R}^d) \to \mathcal{S}'_{\theta,\Gamma}(\mathbb{R}^d)$.
In particular it fulfills $\mathrm{WF}_{\mathrm{gl}}^\theta(Au) \subset \mathrm{WF}_{\mathrm{gl}}^\theta(u)$ and maps $\mathcal{S}_\theta$ into itself.

We can compute the following relations for the transform $T_\varphi u$ of $u \in \mathcal{S}_\theta(\mathbb{R}^d)$ with the standard phase:

$$z T_\varphi u(z) = T_\varphi\big((-iy + i\partial_y)u\big)(z)$$
$$\partial_z T_\varphi u(z) = T_\varphi\big((i\partial_y)u\big)(z)$$

Therefore ultradifferential operators with polynomial coefficients can be understood as a subclass of the operators just considered, if their coefficients satisfy (3.3). In particular, differential operators with polynomial coefficients are microlocal. For these, we can also get the reverse wave front set inclusion in terms of the principal symbol of the operator, known as microellipticity, following [20].

3.2.1. *Microellipticity of differential operators.* Let $P = p(x, D)$ be a differential operator with polynomial coefficients. We can write it as follows

$$p(x, D) = \sum_{|\alpha| + |\beta| \le m} c_{\alpha\beta} x^\beta D^\alpha.$$

In the sequel we shall denote by $p_m(x, \xi)$ the following principal symbol

$$p_m(x, \xi) = \sum_{|\alpha| + |\beta| = m} c_{\alpha\beta} x^\beta \xi^\alpha$$

which is homogeneous of order m in (x, ξ) and define the characteristic set of P as follows:

$$\mathrm{Char}(P) = \{(x, \xi) \in T^*\mathbb{R}^d : p_m(x, \xi) = 0\}.$$

We have the following result.

Theorem 3.5. *Let $u \in \mathcal{S}'_\theta(\mathbb{R}^d)$ and let $p(x, D)u = f$. Then the following inclusions hold:*

$$(3.4) \qquad \mathrm{WF}^\theta_{\mathrm{gl}}(u) \subset \mathrm{WF}^\theta_{\mathrm{gl}}(Pu) \cup \mathrm{Char}(P).$$

Remark 3.6. We observe that Theorem 3.5 represents a generalization of Theorem 1.1 in [1] for the case $\mu = \nu = \theta$. As a matter of fact, if the operator P is globally elliptic, i.e. $\mathrm{Char}(P) = \emptyset$, we obtain that $\mathrm{WF}^\theta_{\mathrm{gl}}(u) = \mathrm{WF}^\theta_{\mathrm{gl}}(Pu)$. In particular, Corollary 2.2 implies that if $Pu \in \mathcal{S}_\theta(\mathbb{R}^d)$, then $u \in \mathcal{S}_\theta(\mathbb{R}^d)$.

Proof of Theorem 3.5. We follow the outline of a proof given in [20] for the case $\theta = 1/2$, which uses the idea from [19] of estimating $|\langle u, e^{-i\langle y, \xi\rangle - \frac{1}{2}|x-y|^2}\rangle|$ in terms of $|\langle Pu, w\rangle|$ by constructing an approximate solution for the equation $P^*w = e^{-i\langle y, \xi\rangle - \frac{1}{2}|x-y|^2}$.
Let $(x_0, \xi_0) \notin \mathrm{WF}^\theta_{\mathrm{gl}}(Pu) \cup \mathrm{Char}(P)$. By this assumption there exists a closed conic set $\Gamma \subset T^*(\mathbb{R}^d)$ such that $p_m(x, \xi) \ne 0$ on Γ and there exist positive constants C, δ such that

$$(3.5) \qquad |\langle f, e^{-i\langle y, \xi\rangle - \frac{1}{2}|x-y|^2}\rangle| \le C e^{-\delta(|x|^{\frac{1}{\theta}} + |\xi|^{\frac{1}{\theta}})}$$

for any $(x, \xi) \in \Gamma$. We consider, for fixed $(x, \xi) \in \Gamma$ the equation

$$(3.6) \qquad p^*(y, D_y)w_{x,\xi}(y) = e^{-i\langle y,\xi\rangle - \frac{1}{2}|x-y|^2},$$

where $p^*(y, D_y)$ denotes the adjoint of P and look for suitable approximate solutions. For that we turn it into a more standard form which can then be approximately solved by a Neumann series. Setting $z = y - x$, we can re-write the equation (3.6) in the following form

$$(3.7) \qquad p^*(x + z, D_z - \xi + iz)W_{x,\xi}(z) = 1,$$

with

$$W_{x,\xi}(z) = e^{i\langle x+z,\xi\rangle + \frac{1}{2}|z|^2} w_{x,\xi}(x + z).$$

Since $p_m(y, -\eta)$ is the principal symbol of P^*, then $p^*(x+z, -\xi+iz) - p_m(x+z, \xi-iz)$ is a polynomial of degree strictly less than m in (x, ξ, z). Moreover, denoting $\rho = \sqrt{|x|^2 + |\xi|^2}$, we have that, since $p_m(x, \xi) \neq 0$ on Γ, then there exist positive constants $M, c > 0$ such that

$$|p^*(x + z, -\xi + iz)| \geq c\rho^m$$

on the set

$$\Gamma_{c,M} = \{(x, \xi, z) \in \mathbb{R}^{3d} : (x, \xi) \in \Gamma, \rho > M, |z| < c\rho\}.$$

In other words, $(1/p^*)(x + z, -\xi + iz)$ behaves like a symbol of order $-m$ for $(x, \xi, z) \in \Gamma_{c,M}$. Setting now

$$(3.8) \qquad G_{x,\xi}(z) = p^*(x + z, -\xi + iz)W_{x,\xi}(z),$$

the equation (3.7) takes the form

$$(3.9) \qquad G_{x,\xi}(z) - RG_{x,\xi}(z) = 1$$

for some operator $R = \sum_{|\alpha| \geq 0} R_\alpha(x, \xi, z)D_z^\alpha$, where R_α is an operator of order $-|\alpha|$ with analytic coefficients on $\Gamma_{c,M}$. Moreover, for $z = \rho\zeta$, we have that

$$R = \sum_{|\alpha| \neq 0} R_\alpha(x, \xi, \rho\zeta)\rho^{-|\alpha|}D_\zeta^\alpha$$

with $|R_\alpha(x, \xi, \rho\zeta)\rho^{-|\alpha|}| \leq C\rho^{-2|\alpha|}$.

Let us now consider the equation (3.9). To go further with the proof we need to study the equation (3.9) for $z \in \mathbb{C}^d$, that is for $y \in \mathbb{C}^d$. A formal solution for the equation would be given by the Neumann series $\sum_{j \geq 0} R^j 1$. For $N \in \mathbb{N}$, let $G_{x,\xi}^N(z)$ be the sum of all terms in the series involving derivatives with respect to z of order less or equal than N. Then we have

$$G_{x,\xi}^N(z) - RG_{x,\xi}^N(z) = 1 - e_{x,\xi}^N(z)$$

for some functions $e_{x,\xi}^N(z)$. Let now, as an approximate solution candidate for (3.6),

$$w_{x,\xi}^N(y) = e^{-i\langle y,\xi\rangle - \frac{1}{2}|x-y|^2} \frac{G_{x,\xi}^N(y - x)}{p^*(y, -\xi + i(y - x))}.$$

Then we obtain

$$(3.10) \qquad p^*(y, D_y)w_{x,\xi}^N(y) = e^{-i\langle y,\xi\rangle - \frac{1}{2}|x-y|^2}\left(1 - e_{x,\xi}^N(y-x)\right).$$

Arguing as in [19], we obtain the following estimates:

$$(3.11) \qquad |D_z^\beta G_N^{x,\xi}(z)| \le C^N N^{|\beta|}, \qquad |\beta| \le N < \rho^2$$

$$(3.12) \qquad |D_z^\beta e_N^{x,\xi}(z)| \le C^N N^{N+|\beta|}\rho^{-2N}, \qquad |\beta| \le N.$$

For every $(x,\xi) \in \Gamma, |z| < c\rho$ and with $c\rho > 1$. Let now $\chi \in C_0^\infty(\mathbb{C}^d)$ such that $\chi(z) = 1$ for $|z| < c/2$ and $\chi(z) = 0$ for $|z| > c$ and consider the function

$$h_{x,\xi}^N(z) = \chi\left(\frac{z}{\rho}\right)e^{-\frac{1}{2}\langle z,z\rangle}\frac{G_{x,\xi}^N(z)}{p^*(x+z, -\xi+iz)}.$$

We observe that

$$(3.13) \qquad |h_{x,\xi}^N(z)| \le C^{N+1}\rho^{-m}\exp\left(-\frac{1}{2}|\Re z|^2 + \frac{1}{2}|\Im z|^2\right), \quad z \in \mathbb{C}^d.$$

Due to the cut-off function involved, $h_{x,\xi}^N$ is not holomorphic. We now construct a function $\widetilde{h}_{x,\xi}^N$ which shares Gaussian decay such that $h_{x,\xi}^N - \widetilde{h}_{x,\xi}^N$ is holomorphic.
By (3.11) for $|\beta| = 1$ the coefficient functions of $\bar{\partial}_z h_{x,\xi}^N$ satisfy, for $c\rho/2 < |z| < c\rho$:

$$(3.14) \qquad \left|\frac{\partial}{\partial\bar{z}_j}h_{x,\xi}^N(z)\right| \le C^{N+1}\rho^{-m}\exp\left(-\frac{1}{2}|\Re z|^2 + \frac{1}{2}|\Im z|^2\right)$$

$$\le C^{N+1}\rho^{-m}\exp\left(-\frac{1}{4}|\Re z|^2 + \frac{3}{4}|\Im z|^2 - \frac{1}{4}|z|^2\right)$$

$$\le C^{N+1}\rho^{-m}\exp\left(-\frac{1}{4}|\Re z|^2 + \frac{3}{4}|\Im z|^2 - \frac{c^2\rho^2}{16}\right).$$

From the last estimate we obtain that

$$(3.15) \qquad \sum_{j=1}^d \int_{\mathbb{C}^d}\left|\frac{\partial}{\partial\bar{z}_j}h_{x,\xi}^N(z)\right|^2 e^{-\kappa(z)}\,d\lambda(z) \le 2C''C^{2N}\rho^{-2m+2d}e^{-\frac{c^2\rho^2}{8}}$$

with $\kappa(z) = -\frac{1}{2}|\Re z|^2 + \frac{3}{4}|\Im z|^2$. Therefore the coefficient functions of $\bar{\partial}h_{x,\xi}^N$ are elements of $L^2(\mathbb{C}^d, e^{-\kappa}d\lambda)$. We observe that $\kappa(z)$ is plurisubharmonic, see [18, Definition 2.6.1]. As $\bar{\partial}^2 = 0$, we can apply Theorem 4.4.2 in [18] to the equation $\bar{\partial}v = \bar{\partial}h_{x,\xi}^N$. It therefore admits a solution $\tilde{h}_{x,\xi}^N(z)$ such that

$$\int_{\mathbb{C}^d}|\tilde{h}_{x,\xi}^N(z)|^2 e^{-\kappa(z)}(1+|z|^2)^{-2}\,d\lambda(z) \le \frac{C''}{2}C^{2N}\rho^{2d-2m}e^{-\frac{c^2\rho^2}{8}}.$$

By the Cauchy's inequalities we then get
(3.16)
$$|\tilde{h}_{x,\xi}^N(z)| \le C'''C^N \rho^{d-m} e^{-\frac{c^2\rho^2}{16}}(1+|z|)^d \exp\left(-\frac{1}{4}|\Re z|^2 + \frac{3}{4}|\Im z|^2\right), \ z \in \mathbb{C}^d.$$

By construction, the function $H_{x,\xi}^N(z) = h_{x,\xi}^N(z) - \tilde{h}_{x,\xi}^N(z)$ is holomorphic on $\mathbb{C}^d$ because $\bar{\partial} H_{x,\xi}^N(z) = 0$. Moreover, the estimates (3.13) and (3.16) imply that its restriction on the real domain is in $\mathcal{S}_{1/2}(\mathbb{R}^d)$. Then taking now $z = y - x \in \mathbb{R}^d$, we can write

$$(3.17) \quad \langle u, e^{-i\langle \cdot, \xi\rangle - \frac{1}{2}|\cdot - x|^2}\rangle = \langle f, e^{-i\langle \cdot, \xi\rangle} H_{x,\xi}^N(\cdot - x)\rangle$$
$$+ \langle u, e^{-i\langle \cdot, \xi\rangle - \frac{1}{2}|\cdot - x|^2}\rangle - \langle f, e^{-i\langle \cdot, \xi\rangle} H_{x,\xi}^N(\cdot - x)\rangle =$$
$$\langle f, e^{-i\langle \cdot, \xi\rangle} H_{x,\xi}^N(\cdot - x)\rangle + \langle u, e^{-i\langle \cdot, \xi\rangle - \frac{1}{2}|\cdot - x|^2} - p^*(y, D_y)(e^{-i\langle \cdot, \xi\rangle} H_{x,\xi}^N(\cdot - x))\rangle.$$

To conclude the proof we need to estimate properly the two terms in the right-hand side of (3.17). Concerning the first one we observe that since $H_{x,\xi}^N \in S_{1/2}(\mathbb{R}^d)$, then its short time Fourier transform is in $S_{1/2}(\mathbb{R}^{2d})$. In particular we have

$$|V_\psi H_{x,\xi}^N(x, \xi)| \le C_1 C^N e^{-\delta(|x|^2 + |\xi|^2)}, \qquad (x, \xi) \in \mathbb{R}^{2d},$$

for some constants $C_1 > 0, 0 < \delta < 1$ independent of N and where C is the same constant appearing in the estimates (3.12), (3.11), (3.13), (3.16). Choosing now N such that

$$(3.18) \qquad \frac{\delta\rho^2}{Ce} - 1 \le N \le \frac{\delta\rho^2}{Ce},$$

with $\rho = \sqrt{|x|^2 + |\xi|^2}$, we obtain

$$|V_\psi H_{x,\xi}^N(x, \xi)| \le C_1' e^{\frac{\delta\rho^2}{e} - \delta\rho^2} = C_1' e^{-\delta(1-e^{-1})(|x|^2 + |\xi|^2)}.$$

Then by the condition (3.5) and by Lemma 2.4 we get

$$|\langle f, e^{-i\langle \cdot, \xi\rangle} H_{x,\xi}^N(\cdot - x)\rangle| \le |V_\psi H_{x,\xi}^N(x, \xi)| * |\langle f, e^{-i\langle \cdot, \xi\rangle - \frac{1}{2}|\cdot - x|^2}\rangle|$$
$$\le C_2 e^{-\delta_2(|x|^{1/\theta} + |\xi|^{1/\theta})}$$

for some positive constants C_2, δ_2 and for $(x, \xi) \in \Gamma$. Now we want to prove that the second term in the right-hand side of (3.17) is $\mathcal{O}(e^{-\delta_3\rho^2})$ for some $\delta_3 > 0$ uniformly with respect to N. To do this we need to estimate the function

$$(3.19) \qquad e^{-i\langle y, \xi\rangle - \frac{1}{2}|y-x|^2} - p^*(y, D_y)(e^{-i\langle y, \xi\rangle} H_{x,\xi}^N(y - x))$$

and its derivatives. For $|y - x| < c\rho/2$ we have $e^{-i\langle y, \xi\rangle} h_{x,\xi}^N(y - x) = w_{x,\xi}^N(y - x)$. Then

$$e^{-i\langle y, \xi\rangle - \frac{1}{2}|y-x|^2} - p^*(y, D_y)(e^{-i\langle y, \xi\rangle} H_{x,\xi}^N(y - x))$$
$$= e^{-i\langle y, \xi\rangle - \frac{1}{2}|y-x|^2} e_{x,\xi}^N(y - x) + p^*(y, D_y)e^{-i\langle y, \xi\rangle}\tilde{h}_{x,\xi}^N(y - x).$$

Now, by (3.12) and (3.18) we have

$$|e^{-i\langle y,\xi\rangle - \frac{1}{2}|y-x|^2} e^N_{x,\xi}(y-x)| \leq |e^N_{x,\xi}(y-x)| \leq C^N N^N \rho^{-2N} \leq e^{-N} \leq e^{1-\frac{\delta\rho^2}{Ce}},$$

since $0 < \delta < 1$, whereas the term $p^*(y, D_y)e^{-i\langle y,\xi\rangle}\tilde{h}^N_{x,\xi}(y-x)$ satisfies a similar bound by (3.16). For $|y-x| \geq c\rho/2$, we obviously have $e^{-i\langle y,\xi\rangle - \frac{1}{2}|y-x|^2} = \mathcal{O}(e^{-c^2\rho^2/8})$. Moreover, arguing as for (3.14), we obtain for some $\gamma > 0$ that

$$|H^N_{x,\xi}(y-x)| \leq C_3 e^{-\delta_3\rho^2} \exp\left(-\gamma|y-x|^2\right),$$

and a similar estimate holds for $p^*(y, D_y)(e^{-i\langle y,\xi\rangle}H^N_{x,\xi}(y-x))$. In conclusion, for $|y-x| \geq c\rho/2$ we obtain

$$|e^{-i\langle y,\xi\rangle - \frac{1}{2}|y-x|^2} - p^*(y, D_y)(e^{-i\langle y,\xi\rangle}H^N_{x,\xi}(y-x)|$$
$$\leq C_4 e^{-\delta_3\rho^2} \exp\left(-\gamma|y-x|^2\right).$$

The estimate of the derivatives of the function (3.19) can be obtained by estimating the derivatives of its entire extension by Cauchy's inequalities arguing as in the proof of Lemma 4.1. The details are left to the reader, cf. [20].

$\square$

4. APPENDIX. THE PROOF OF THEOREM 1.9

We can reduce the proof to the case when $\alpha = 0$ thanks to the following Lemma, a variant of Cauchy's inequality.

Lemma 4.1. *Let $g(z)$ be an entire function and let $z_0 \in \mathbb{C}^d \setminus \{0\}$. Let open conic subsets of $\mathbb{C}^d \setminus \{0\}$ containing z_0 be denoted by Γ. Then the following conditions are equivalent:*
i) $\exists \Gamma \ \forall\varepsilon > 0 \ \exists C_\varepsilon > 0$ s.t. $|g(z)| \leq C_\varepsilon e^{\varepsilon|z|^{1/\theta}+\Phi(z)} \quad \forall\, z \in \Gamma$
ii) $\exists\Gamma \ \forall\varepsilon > 0 \ \exists C_\varepsilon > 0$ s.t. $|\partial^\alpha_z g(z)| \leq C_\varepsilon^{|\alpha|+1}\alpha!^\theta \ e^{\varepsilon|z|^{1/\theta}+\Phi(z)} \quad \forall\, z \in \Gamma, \alpha \in \mathbb{N}^d$.
Similarly, the following two conditions are equivalent:
a) $\exists\Gamma \ \exists\varepsilon > 0 \ \exists C > 0$ s.t. $|g(z)| \leq C e^{-\varepsilon|z|^{1/\theta}+\Phi(z)} \quad \forall\, z \in \Gamma$
b) $\exists\Gamma \ \exists\varepsilon > 0 \ \exists C > 0$ s.t. $|\partial^\alpha_z g(z)| \leq C^{|\alpha|+1}\alpha!^\theta \ e^{-\varepsilon|z|^{1/\theta}+\Phi(z)} \quad \forall\, z \in \Gamma, \alpha \in \mathbb{N}^d$.

Proof. The proof is a variant of a calculation done in [15, Section 5.2]. Let $g(z)$ satisfy i) on an open cone Γ and let $z \in \Gamma$. Then, by Cauchy's formula we can write

$$\partial^\alpha_z g(z) = \frac{\alpha!}{(2\pi i)^d} \int_{\partial\mathbb{B}_\delta(z)} g(a) \prod_{i=1}^d (a_i - z_i)^{-1-\alpha_i} \, da,$$

where we denote by $\mathbb{B}_\delta(z)$ the polydisc of radius δ centered in z. For fixed δ we can assume $|z|$ large enough so that $\mathbb{B}_\delta(z) \subset \Gamma$. As $\Phi(z)$ is a

polynomial in z, $\overline{z}$ of degree 2 we can estimate it on $\partial\mathbb{B}_\delta(z)$ by Taylor's formula and Young's inequality for products:

$$\Phi(a) \le \Phi(z) + C_1|z|\delta + C_2\delta^2 \le \Phi(z) + \varepsilon|z|^{1/\theta} + c_\epsilon\delta^{1/(1-\theta)},$$

for any $\epsilon > 0$. Then, using i) we can estimate as follows

$$|\partial_z^\alpha g(z)| \le \alpha! \sup_{a\in\partial\mathbb{B}_\delta(z)} |g(a)|$$

$$\le \sup_{a\in\partial\mathbb{B}_\delta(z)} \frac{\alpha!\, C_\varepsilon}{|\delta|^{|\alpha|}} e^{\varepsilon|a|^{1/\theta}+\Phi(a)}$$

$$\le \frac{\alpha!\, C_{\varepsilon'}}{|\delta|^{|\alpha|}} e^{\varepsilon'|z|^{1/\theta}+\Phi(z)} e^{c'_\varepsilon\delta^{1/(1-\theta)}}$$

We now pick $\delta = \delta_0$ to minimize $\frac{\alpha!\, C_{\varepsilon'}}{|\delta|^{|\alpha|}} e^{c'_\varepsilon\delta^{1/(1-\theta)}} := C(\varepsilon,\alpha,\delta)$. We have $\delta_0 = \left(\frac{|\alpha|(1-\theta)}{c'_\varepsilon}\right)^{1-\theta}$ and using Stirling's formula we obtain

$$C(\varepsilon,\alpha,\delta_0) \le (C''_\epsilon)^{|\alpha|+1} \alpha!^\theta,$$

which proves estimate ii).
The second part of the lemma follows in complete analogy. $\square$

Proof of Theorem 1.9. Following [20], we can assume $A = 0$ and $\Re C = 0$ in the definition of φ. By a change of notation this leads to consider a phase function of the form

$$\varphi(y,z) = \langle Bz, y\rangle + \frac{i}{2}\langle Cy, y\rangle,$$

where B is a non degenerate matrix and C is positive definite. Let now $u \in \mathcal{S}'_\theta(\mathbb{R}^d)$. We have that for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$|T_\varphi u(z)| \le C_\varepsilon \sup_{\alpha,\beta\in\mathbb{N}^d} \varepsilon^{-|\alpha|-|\beta|}(\alpha!\beta!)^{-\theta} \sup_{y\in\mathbb{R}^d} |y^\beta D_y^\alpha e^{i\varphi(y,z)}|.$$

Now observe that $e^{-\frac{1}{2}\langle Cy, y\rangle} \in \mathcal{S}_{1/2}(\mathbb{R}^d)$ and that

$$|D_y^\alpha e^{-\frac{1}{2}\langle Cy,y\rangle}| \le M^{|\alpha|+1}\alpha!^{1/2} e^{-\frac{1}{2}\langle Cy,y\rangle}$$

for some positive constant M. Then we have

$$|y^\beta D_y^\alpha e^{i\varphi(y,z)}| \le |y|^{|\beta|} \sum_{\gamma\le\alpha} \binom{\alpha}{\gamma} |(Bz)^\gamma e^{i\langle Bz,y\rangle} D_y^{\alpha-\gamma} e^{-\frac{1}{2}\langle Cy,y\rangle}|$$

$$\le C_1^{|\alpha|+1} \sum_{\gamma\le\alpha} \binom{\alpha}{\gamma} |y|^{|\beta|}|z|^{|\gamma|}(\alpha-\gamma)!^{1/2} e^{\Phi(z)-\frac{1}{2}\langle C(y-y(z)),y-y(z)\rangle}.$$

Since $\theta \geq 1/2$ and $y(z)$ is a linear function of z we have that for every $\varepsilon' > 0$ there exists $C_{\varepsilon'} > 0$ such that

$$|z|^{|\gamma|} \sup_{y \in \mathbb{R}^d} |y|^{|\beta|} e^{-\frac{1}{2}\langle C(y-y(z)), y-y(z) \rangle} = |z|^{|\gamma|} \sup_{y \in \mathbb{R}^d} (|y + y(z)|)^{|\beta|} e^{-\frac{1}{2}\langle Cy, y \rangle}$$

$$\leq C \sup_{y \in \mathbb{R}^d} (|y| + |z|)^{|\beta| + |\gamma|} e^{-\frac{1}{2}\langle Cy, y \rangle} \leq C_{\varepsilon'}^{|\beta| + |\gamma| + 1} (\beta! \gamma!)^{\theta} e^{\varepsilon'|z|^{1/\theta}}.$$

Then, taking $\varepsilon > C_{\varepsilon'} C$, we obtain (1.5).

Let now $u \in \mathcal{S}_\theta(\mathbb{R}^d)$. We observe that for every $\alpha \in \mathbb{N}^d$:

$$(Bz)^\alpha T_\varphi u(z) = (-1)^{|\alpha|} \int_{\mathbb{R}^d} e^{i\langle Bz, y \rangle} D_y^\alpha \left(e^{-\frac{1}{2}\langle Cy, y \rangle} u(y) \right) dy.$$

Then, since $\det B \neq 0$ and $\theta \geq 1/2$, we have by Leibniz and Faà di Bruno formulas

$$|z^\alpha T_\varphi u(z)| \leq C_1^{|\alpha|+1} (\alpha!)^\theta e^{\Phi(z)} \int_{\mathbb{R}^d} e^{-\frac{1}{2}\langle C(y-y(z)), y-y(z) \rangle} dy$$

for every $\alpha \in \mathbb{N}^d$. This gives (1.6).

To prove the second part of the Proposition, we need Lemma 1.8. Let $U(z)$ be an entire function satisfying (1.6) and let $u = T_\varphi^* U$, where T_φ^* is the L^2-adjoint of T_φ. Then $T_\varphi u = U$. Moreover we have

$$T_\varphi^* U(y) = c_\varphi \int_{\mathbb{C}^d} e^{-i\overline{\varphi(y,z)} - 2\Phi(z)} U(z) d\lambda(z) \in \mathcal{S}_\theta(\mathbb{R}^d).$$

As a matter of fact, given $\alpha, \beta \in \mathbb{N}^d$ and arguing as in the proof of (1.5), we have:

$$|y^\beta D_y^\alpha T_\varphi^* U(y)| = c_\varphi \left| y^\beta \int_{\mathbb{C}^d} D_y^\alpha e^{-i\overline{\varphi(y,z)}} e^{-2\Phi(z)} U(z) d\lambda(z) \right|$$

$$\leq C^{|\alpha|} \sum_{\gamma \leq \alpha} \binom{\alpha}{\gamma} (\alpha - \gamma)!^{1/2} \int_{\mathbb{C}^d} |y|^{|\beta|} |z|^{|\gamma|} \left| e^{-i\overline{\varphi(y,z)}} e^{-2\Phi(z)} U(z) \right| d\lambda(z)$$

$$\leq C_\varepsilon C^{|\alpha|} \sum_{\gamma \leq \alpha} \frac{\alpha!}{\gamma!(\alpha-\gamma)!^{1/2}} \cdot$$

$$\cdot \int_{\mathbb{C}^d} |y|^{|\beta|} |z|^{|\gamma|} e^{-\frac{1}{2}\langle C(y-y(z)), y-y(z) \rangle} e^{-\varepsilon|z|^{1/\theta}} d\lambda(z).$$

Arguing as before we have that

$$\sup_{y \in \mathbb{R}^d} (|y| + |z|)^{|\beta| + |\gamma|} e^{-\frac{1}{2}\langle C(y-y(z)), y-y(z) \rangle} e^{-\varepsilon|z|^{1/\theta}} \leq C_\varepsilon C^{|\alpha| + |\gamma|} (\beta! \gamma!)^\theta e^{-\frac{\varepsilon}{2}|z|^{1/\theta}}$$

Then, we obtain that $u \in \mathcal{S}_\theta(\mathbb{R}^d)$. Finally, if $U \in \mathcal{H}(\mathbb{C}^d)$ satisfies (1.5), then

$$(u, v) = \int_{\mathbb{C}^d} U(z) \overline{T_\varphi v(z)} e^{-2\Phi(z)} d\lambda(z), \qquad v \in \mathcal{S}_\theta(\mathbb{R}^d),$$

defines an ultradistribution in $\mathcal{S}'_\theta(\mathbb{R}^d)$. Moreover we have $T_\varphi u = U$, cf. [20]. □

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TORINO, VIA CARLO ALBERTO 10, 10123 TORINO, ITALY.
  E-mail address: marco.cappiello@unito.it

LEIBNIZ UNIVERSITÄT HANNOVER, INSTITUT FÜR ANALYSIS, WELFENPLATZ 1, D–30167 HANNOVER, GERMANY
  E-mail address: rschulz@math.uni-hannover.de