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Credit Risk Migration Rates Modeling as Open Systems: A Micro-simulation Approach

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Abstract

The last financial crisis of 2008 stimulated the development of new Regulatory Criteria (commonly known as Basel III) that pushed the banking activity to become more prudential, either in the short and the long run. As well known, in 2014 the International Accounting Standards Board (IASB) promulgated the new International Financial Reporting Standard 9 (IFRS 9) for financial instruments that will become effective in January 2018. Since the delayed recognition of credit losses on loans was identified as a weakness in existing accounting standards, the IASB has introduced an Expected Loss model that requires more timely recognition of credit losses. Specifically, new standards require entities to account both for expected losses from when the impairments are recognized for the first time and for full loan lifetime; moreover, a clear preference toward forward looking models is expressed. In this new framework, it is necessary a re-thinking of the widespread standard theoretical approach on which the well known prudential model is founded.

The aim of this paper is then to define an original methodological approach to migration rates modeling for credit risk which is innovative respect to the *standard method* from the point of view of a bank as well as in a regulatory perspective. Accordingly, the proposed not-standard approach considers a portfolio as an open sample allowing for entries, migrations of stayers and exits as well. While being consistent with the empirical observations, this open-sample approach contrasts with the standard closed-sample method. In particular, this paper offers a methodology to integrate the outcomes of the standard closed-sample method within the open-sample perspective while removing some of the assumptions of the standard method.

Three main conclusions can be drawn in terms of economic capital provision: (a) based on the Markovian hypothesis with a-priori absorbing state *at default*, the standard closed-sample method is to be abandoned for

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not to predict lenders' bankruptcy by construction; (b) to meet more reliable estimates along with the new regulatory standards, the sample to estimate migration rates matrices for credit risk should include either entries and exits; (c) the static eigen-decomposition standard procedure to forecast migration rates should be replaced with a stochastic process dynamics methodology while conditioning forecasts to macroeconomic scenarios.
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1. Introduction and motivation

The last financial crisis of 2008 stimulated the development of new Regulatory Criteria (commonly known as Basel III) fostering a more prudential banking activity, either in the short and the long run. Some financial/economic observers and policy makers expect a reprise of the crisis: the sources of uncertainty in the macroeconomic scenario are many and more than ever intimately tied to the financial world. Financial and credit risk are always in the spotlight: in this debate it is worthy of interest to concentrate the attention on migration rates models for credit risk. The reference literature has a long tradition (for discrete as well as continuous time modeling see, e.g. [1], [2] [3], [4], [5], [6], [7], [8], [9], [10], [11],[12], [13]) but the new accounting standards put forth that widespread problems the *standard method* is far from solving. Before answering the questions, a sounder understanding of migration rates phenomena is needed to assess how the banks can properly and sustainably set the Economic Capital Provision consistently with an accounting and prudential compliant System of Rating Classes.

With the press release of 24 July 2014 for financial instruments, the International Accounting Standards Board (IASB), responsible for International Financial Reporting Standards (IFRS), announced the adoption of a new approach for the classification and the measurement of credit losses that will enter in force on January 2018. The current Accounting Standards require that a loss event occurs before that a provision can be made. The advantage of this Incurred Loss provisioning model is its objectivity inasmuch it reduces the necessity to relay on estimations about probability of impairments and recovery rates. The main oddity of this model is its backward-looking nature. During the recent financial crisis, the delayed recognition of credit losses on loans was identified as a weakness in existing accounting standards. In response to this, the IASB has introduced an Expected Loss model that requires more a timely recognition of credit losses. Specifically, new standards require entities to account both for expected losses from when the impairments are recognized for the first time and for full loan lifetime, moreover, a clear preference toward forward looking models is expressed.

The new forward looking model adopted by the IASB is consistent with the expected loss approach advocated by the Basel Committee on Banking Supervision (BCBS) for prudential purposes. A better understanding of the linkages between the accounting and prudential concept of expected loss requires to study the relation between the new Accounting Standards and the Capital Regulation. Although the so called Basel II - Basel III prudential regulation is silent about the level of provisioning, it must be recognized that the introduction of Expected Loss concept in the Accounting Standards provides an important innovation for the alignment of prudential risk measures and accounting.

However, the requirement to evaluate the expected losses with a forward looking and full loan lifetime perspective is something new if compared with the prudential setting where a one year time horizon is adopted and the through the cycle philosophy is preferred. The requirements put forward by the new accounting standards advocate for a re-thinking of the theoretical framework on which the prudential model is founded. Specifically, obtaining the full loan lifetime probability of default through the prudential one year probability of default (for example by rising to powers one year rating migration matrices), the theoretical framework can be questioned given the observed non time homogeneity of the migration probabilities. Moreover, this loss must be conditioned on realistic long-run economic scenarios: this means that not a short-run (i.e. from date $t-1$ to date t) but a full loan lifetime (long-run) forecast is required for each credit-line or financial product. Therefore, in practice, how much do bankers have to set for provision and credit risk according to the IFRS 9?

Referring to the standard model, this relies on few simplifying assumptions: (1) the *cohort method*: samples are periodically updated for migration rates matrices estimation to be projected into the future; (2) the *closed sample approach*: i.e. the Stay-only-model is involved; (3) the *time-homogeneous Markovian hypothesis*: this is rejected by empirical evidence but still widely used.

However, the standard approach is unfortunately based on simplifying assumptions so fundamental for theoretical purposes as controversial for their lack of realism:

- the migration rates matrix for forecasting is defined with reference to a closed sample that does not consider the renewal of the portfolio of a bank excluding entries and exits;
- the default is conventionally regarded as an absorbing state that does not allow the return to performing from the default for the borrower, thus determining a distorted estimation - over-estimated - of the

almost-certain loss for the lender because in the short-medium term all the loans are doomed to get into default;

- as a result of the previous axiomatic and simplifying assumptions, the Markovian hypothesis applied for forecasting necessarily entails that the whole migration rates matrix collapses to default in the short term;
- this then imposes the lender to earmark an excessive capital; thereby, bank failure is expected by construction unless bank decides upon an excessive shrinkage of the credit, for a raising of lending prudential thresholds and for an increasing propensity to profit.

These assumptions are not clearly confirmed in practice and, if introduced as methodological simplifications, they become axioms of a coherent theory but unrealistic and not in accordance with the new regulatory standards, for example in reference to the so-called principle of full loan lifetime.

The aim of this paper is then to define an original methodological approach which is innovative respect to the standard method of migration rates modeling for credit risk from the point of view of a bank and in a regulatory perspective.

Accordingly, since the beginning, the here developing not-standard approach considers a portfolio as an open sample allowing for entries, migrations of stayers and exits as well. While being consistent with the empirical observations, this open-sample approach contrasts with the standard closed-sample approach. In particular, this paper offers a methodology to integrate the outcomes of the closed-sample approach with the open-sample perspective while removing some of the assumptions of the standard method.

Montecarlo simulations are developed to verify the implications of axiomatic assumptions before formalizing an alternative theoretical proposal. Since a numerical algorithm is developed for the simulation of a set of random events, the setting follows a simplified and generic approach but the introduced assumptions are empirically based.

The involved assumptions shape the theoretical framework which is consistent with the normative principles of regulation. The assumptions of this paper grow upon a professional study of real data. Therefore, while embedding future regulatory standards, assumptions are automatically proved simplifying hypotheses. In general, assumptions are introduced without explicit definition of functions involved: simple functional forms are of course introduced for simulation purposes. Therefore, according to the general structure of assumptions, it is possible to specify different sets of functions to test for different effects: this might be useful for regulation, supervision and evaluation purposes.

By means of a controlled simulation environment, this paper is a first methodological alternative for removing those restrictive assumptions which do not match with facts (e.g. the Markov hypothesis, the absorbency of default and a closed sample approach) and it is organized as follows.
In Section 2, the overall assumptions are posed. In Section 3 the E-S-L migration model is outlined where the rating rules of entries, of stayers and of leavers are introduced in Sections 3.1, 3.2 and 3.3, respectively Section 3.5 develops the notion of augmented migration matrix. The simulation algorithm and the results are described in Section 4: Section 4.1 refers to the Part I about macro and micro data simulation; Section 4.2 illustrates the Part II about the E-S-L rating procedure; Section 4.3 explains the Part III about the estimation and the dynamics of migration rates matrices. In Section 5, simulation results and implications are analyzed. Section 6 concludes. Appendix A gathers algorithms' pseudocodes.

2. The overall assumptions

A commercial bank offers several types of contracts to borrowers. Among many, for the ease of simulation purposes, only one technical form is here considered, e.g. mortgages.

Assumption 1. $\mathbb{F}_t = \{i \in \mathbb{N} : 0 < i \leq F_t\}$ *is a technically homogeneous portfolio of* F_t *contracts subscribed by the bank right after date* $t-1$ *and before date t with its clients: i represents the ith client.*[1]

The portfolio is technically homogeneous while each contract i is characterized by specific properties of the client[2] or related to the contract.[3] For such a specific homogeneous credit-line, borrowers are therefore intrinsically heterogeneous, also because they live or operate in a not-homogeneous economic space.[4] Moreover, as it will be lately clarified with Assumption 11, since a contract can be signed during the time interval $(t-1, t)$, it is possible to allow for very short-term contracts to expire right before date t, these kind of contracts do not receive a rating of credit quality.

With respect to the standard approach, it is here explicitly considered that some contracts in the portfolio have been subscribed at date $\tau < t$ while not being yet expired and some others are new entries at date t.

[1] The client, the borrower, the debtor and the contractor are synonymous on the credit demand side, while the bank or the lender are synonymous on the supply side.

[2] E.g.: the creditworthiness history, the assets set as guarantee, the occupational status in case of individuals or, in case of firms, the industrial sector of economic activity, etc.

[3] E.g.: the value, the expiration date, the terms of payment, etc.

[4] Differences might be related to geographic, social and economic local characteristics of the client's location, also with a differentiated variety of credit needs.

Moreover, although with different motivations, contracts expired at date $t-1$ exit and do not appear in $\mathbb{F}_t$. Therefore, without loss of generality, one can consider the following

Assumption 2. *The bank offers contracts of value* 1\$*:* $V_{i,t} = 1\$,\ \forall i \in \mathbb{F}_t$*.*

According to assumptions 1 and 2,

$$F_t = \sum_{i \in \mathbb{F}_t} V_{i,t} \equiv |\mathbb{F}_t| \tag{1}$$

represents both the size of the portfolio (i.e the number of contracts size) and its value. Such a value determines the exposition of the bank and it may dynamically change under the effect of the macro-economic scenario or cycle, characterized by some degree of uncertainty, which influences the propensity or need of potential borrowers, so fostering or reducing the entry rate in the portfolio, as well as it may influence the creditworthiness of clients which are already in the portfolio, so affecting the exit rate. Accordingly, as outlined before, the here proposed not-standard approach considers a portfolio as an open sample allowing for entries, migrations of stayers and exits. Therefore, let the following assumption be

Assumption 3. *The bank owns a time series of portfolios* $\mathcal{F}_t = \{\mathbb{F}_\tau : \tau \leq t\}$ *which is continuously updating through time. Each contract* $i \in \mathbb{F}_{t-1}$ *is characterized by a stay-or-leave probability* $Q_{i,t-1}$ *and it is expected to leave the portfolio if* $Q_{i,t-1} \leq q^* \in (0,1)$*, otherwise it remains in the portfolio.*

As a matter of fact, the stay-or-leave probability is dependent on both individual-specific and economic-overall conditions: for the ease of simulation, simply let the stay-or-leave probability be a random variable whose outcomes are identically and independently distributed according to the Uniform distribution 0 and 1

$$Q_{i,t-1} \sim_{iid} U(0,1) \tag{2}$$

Let the bank setting q^* as a reliable threshold to be maintained as homogeneous and constant over the portfolio at each date: this threshold could have been estimated by some accredited or regulatory institution, hence it might be an exogenous information to the bank for that specific credit-line. An indicator then identifies a contract to stay as $X_{i,t-1} = +1$ or to leave as $X_{i,t-1} = -1$: the former contract is maintained in the portfolio $\mathbb{F}_t$, the latter one is deleted[5]. That is,

$$X_{i,t-1} = \begin{cases} -1 \text{ iff } Q_{i,t-1} \leq q^* \\ +1 \text{ iff } Q_{i,t-1} > q^* \end{cases} \Rightarrow X_{i,t-1} \sim_{iid} Bernoulli(q^*)\ ,\ q^* \in (0,1) \tag{3}$$

[5]As it will be show in Section 4.1, it is considered that the bank deletes the borrower by setting all its main quantities at zero from the period of exit onward, while maintaining in a database all the past information.

with $Pr\{X_{i,t-1} = -1\} = q^*$ and $Pr\{X_{i,t-1} = +1\} = 1 - q^*$. Therefore,

$$L_t = \sum_{i \in \mathbb{F}_{t-1}} \#[X_{i,t-1} = -1] \sim_{iid} Binomial(F_{t-1}, q^*) \tag{4}$$

is the number of those who leave the portfolio and

$$S_t = \sum_{i \in \mathbb{F}_{t-1}} \#[X_{i,t-1} = +1] = F_{t-1} - L_t \sim_{iid} Binomial(F_{t-1}, 1 - q^*) \tag{5}$$

is the number of those who stay.

Finally, the number $E - t$ of borrowers within the time interval $(t-1, t)$ is defined as

$$E_t \sim_{iid} Poisson(\theta_t \cdot F_1) \; : \; F_1 \gg 1 \tag{6}$$

Therefore, (2)-(5) introduce a *stay and leave* process, while (6) introduces an *entry* process with entry rate $\theta_t \cdot F_1$, where F_1 is a scaling parameter.

It can then be stated that

Assumption 4. *Given the portfolio $\mathbb{F}_{t-1}$ of F_{t-1} contracts and a set $\mathbb{E}_t$ of E_t new entries, being $\mathbb{S}_t$ the set of S_t not yet expired contracts while $\mathbb{L}_t$ is the set of L_t deleted ones, the current period portfolio*

$$\mathbb{F}_t = \mathbb{E}_t \cup \mathbb{S}_t = \mathbb{E}_t \cup [\mathbb{F}_{t-1} \setminus \mathbb{L}_t] \tag{7}$$

is an open system with renewal.

This representation describes the portfolio ($\mathbb{F}_t$) as a dynamic open to entry ($\mathbb{E}_t$) system which allows for staying ($\mathbb{S}_t$) at least one period or leaving ($\mathbb{L}_t$) even before than a period: this aspect will be clarified while discussing the following Assumption 11. Accordingly, as implied by assumptions 2 and 4, the economic value of the portfolio $\mathbb{F}_t$ is

$$F_t = E_t + S_t = E_t + [F_{t-1} - L_t] \tag{8}$$

As the bank, also the borrowers are not isolated entities, rather they both live embedded in a socioeconomic environment which reshapes time after time. The macro-economic environment influences borrowers and lenders in different ways and with different impacts. Among many possible ways of modeling these effects, for the sake of simplicity let Y_t be the main macro-economic information simulated as an $AR(R)$ process so defined

$$Y_t = \phi_0 + \sum_{r=1}^{R} \phi_r \cdot Y_{t-r} + \epsilon_t \; : \; \epsilon_t \sim_{iid} \mathcal{N}(0, \sigma_\epsilon^2) \; , \; R \geq 1 \tag{9}$$

with growth rate

$$\gamma_t = \frac{Y_t}{Y_{t-1}} - 1 \tag{10}$$

Assumption 5. *$\{Y_t\}$ is the main public information about the economic state of the system.*

Of course, Assumption 5 is a rough simplification, nevertheless it can be accepted while Y_t is summarizing the macroeconomic state of the whole economic system as a state quantity. During *good times*, e.g. when the economy grows, the bank portfolio increases either by the effect of some euphoria of investors/debtors (looking for returns) and because optimistic expectations weaken the prudential criteria of the bank itself (looking for profit), which are usually more tight in *bad times*, [14, 15], [16].

Let now $\theta_t = \theta(\gamma_t)$ be a step function which randomly maps γ_t onto the open set $(0, 1)$

$$\theta_t = \theta(\gamma_t) \sim_{iid} \begin{cases} U(m_0^-, m_1^-) \ iff \ \gamma_t \leq 0 \\ U(m_0^+, m_1^+) \ iff \ \gamma_t > 0 \end{cases} \ : \ 0 < m_0^- < m_1^- \leq m_0^+ < m_1^+ < 1 \tag{11}$$

Therefore, if the growth rate γ_t is positive then Y_t is associated to a random number uniformly drawn between m_0^+ and m_1^+, otherwise it is drawn between m_0^- and m_1^-. Of course, this is a simplification introduced to the end of simulation purposes, nevertheless one can consistently consider the following

Assumption 6. *The entry rate θ_t in $\mathbb{F}_t$ depends on the economic cycle γ_t.*

To consider the consequences of assumptions 5 and 6, let a random variable $Z \sim \mathcal{N}^{tr}_{[a,b]}(1)$ be truncated-normally distributed between a and b with variance 1 such that

$$Z_t^1 \sim_{iid} \begin{cases} \mathcal{N}^{tr}_{(0,2]}(1) \ iff \ \gamma_t > 0 \\ \mathcal{N}^{tr}_{[-2,0)}(1) \ iff \ \gamma_t \leq 0 \end{cases} \tag{12}$$

and

$$Z_{i,t} \sim_{iid} \mathcal{N}(0, 1) \ , \ \forall i \in \mathbb{F}_t \tag{13}$$

And, finally, let the creditworthiness $W_{i,t}$ be defined as a linear combination

$$W_{i,t} = \sqrt{\beta} \cdot Z_t^1 + \sqrt{1 - \beta} \cdot Z_{i,t} \sim_{iid} \mathcal{N}(0, 1) \tag{14}$$

where β is the square-correlation between Z_t^1 and $Z_{i,t}$. According to a standard Contingent Claim Modeling which dates back to [1], the following assumption can be proposed

Assumption 7. *Creditworthiness $W_{i,t}$ of $i \in \mathbb{F}_t$ is influenced by a systemic latent factor Z^1_t, which synthesizes the properties of the economic cycle, and by a contract-specific latent factor $Z_{i,t}$, which synthesizes intrinsic characteristics of the contract in the economy.*

The statements of previous assumptions have been introduced as they plausibly match with real world facts and future regulatory criteria: their formal specification allows to develop a simple simulation algorithm.

3. The Entry-Stay-Leave (E-S-L) model

This section describes the mechanics of the dynamic evolution of a portfolio considered as an open system. In what follows, several tables and matrices will be introduced to account for entries, migrations and exits from the portfolio. Moreover, innovative and realistic rating principles are introduced with respect to the standard closed-sample approach: the developed generic description allows for both a closed and an open-sample to be compared. Finally, it is worth recalling that the aim is to specify a simulation algorithm to meet the needs of the main regulatory criteria which are going to enter in force in the near future.

According to Assumption 7, as creditworthiness of contractors changes with time under the effect of some contract-specific and overall latent factors, where the latter influences both actual and potential clients, and consistently with Assumption 4, the following assumption is introduced

Assumption 8. *By considering two adjacent portfolios, each contract can be classified to belong to one and only one of three main species*

$$\begin{aligned} i \in \mathbb{E}_t \quad &\Leftrightarrow \quad i \in \mathbb{F}_t | i \notin \mathbb{F}_{t-1} \quad (E : entry) \\ i \in \mathbb{S}_t \quad &\Leftrightarrow \quad i \in \mathbb{F}_t | i \in \mathbb{F}_{t-1} \quad (S : stay) \\ i \in \mathbb{L}_t \quad &\Leftrightarrow \quad i \notin \mathbb{F}_t | i \in \mathbb{F}_{t-1} \quad (L : leave) \end{aligned} \tag{15}$$

Therefore, unless it is a short-term contract (see Assumption 11), an *entry* contract is going to be rated for the first time, a *stay* contract is about to be rated anew while a *leave* one cannot be rated anymore, either because it has expired or because the last deadline for payments has been legally overcome or the recovery process has been completed. Clearly, this is consistent with (7) and it allows for specifying the following state-spaces

$$\Sigma_{ES} = \{E, S\}\ ,\ \Sigma_{SL} = \{S, L\}\ ,\ \Sigma_{ESL} = \{E, S, L\} \tag{16}$$

The single contract i is rated by means of an internal rating procedure which differently operates depending on the state $\sigma \in \Sigma_{ES}$, this is because an E-borrower is almost unknown to the bank, differently from a S-debtor: notice that, by definition, an L-debtor may only have an exit class.

Assumption 9. *A system of rating classes $\Lambda = \{\lambda_j : j \leq J\}$ is determined such that $\lambda_1 = A$ collects borrowers with the best performing rate while, through intermediate classes of decreasing quality, the not-performing ones belong to the last $\lambda_J = D$ class: rating classes are decreasingly ordered $\lambda_1 \succ \ldots \succ \lambda_J$ as inversely related to their quality.*[6] *By conditioning on the origin state $\sigma \in \Sigma_{ES}$, a rating class is assigned to $i \in \mathbb{F}_t \cup \mathbb{F}_{t-1}$ depending on $W_{i,t}$ according to a rule $\rho_\sigma(W_{i,t}) \to \lambda_j$:*

- $\rho_\sigma(W_{i,t}) \to \lambda_1 \Rightarrow i \in A$*: in bonis with a good performing rate,*
- $\rho_\sigma(W_{i,t}) \to \lambda_2 \Rightarrow i \in B$*: in bonis with a medium/low performing rate,*
- $\rho_\sigma(W_{i,t}) \to \lambda_3 \Rightarrow i \in D$*: at default due to a not-performing rate*

where $\rho_E(W_{i,t})$ is the rating rule for $i \in \mathbb{E}_t$ and $\rho_S(W_{i,t})$ is that for $i \in \mathbb{S}_t$.

At a very simplified level of description three main rating classes can be considered in both cases: $\lambda_1 = A$ for best performing, $\lambda_2 = B$ for medium/low performing and $\lambda_3 = D$ for not-performing positions.[7]
The following sections 3.1 and 3.2 are inspired to [17] and they explain the rationale of the two rules of Assumption 9.

3.1. The rating rule for entries $\mathbb{E}_t$

Although $i \in \mathbb{E}_t$ is almost an unknown borrower for the lender, the lender has just screened it to assign i a creditworthiness $W_{i,t}$: thus, i must receive a rating class. Intuitively enough, each lender is interested in selecting the best borrowers, i.e. those to be rated in class A: this is because they have a very high probability to pay back both the principal and the interest on debt while having a very low probability to default. Unfortunately not all the new debtors have such a good performing rate. Indeed, it may happen that

[6] From here on, $A \succ B$ means A is preferred or better than B.

[7] Of course, there may be many rating classes, as described in the Basel II accord (see Regulation (EU) No 575/2013 of the European Parliament and of the Council of 26 June 2013 on prudential requirements for credit institutions and investment firms and amending Regulation (EU) No 648/2012) or as developed by accreditated institutions like Standard and Poor's, JP Morgan, Morgan Stanley, Moody's and, in Italy, CERVED, just to mention a few among the most important rating agencies. Reducing such a variety to three classes it is a deep simplifcation but it does not affect the experimental principle developed in this paper.

$p(y\|E)$	A	B	D	L
E	0.80	0.15	0.04	0.01
$Pr\{\bullet \leq y\|E\}$	1.00	0.20	0.05	0.01

Table 1: A representative example for shares of new debtors over $\Lambda = \{A, B, D\}$ and L in the open-sample case.

some contractors are to be placed in class B while some others may even be placed into class D. Although it is likely to be understood that the majority of new clients should be *in bonis*, it is not always possible to exclude that some others are *at default*: as a matter of fact this happens due to mergers or securtizations. Therefore, if the new potential borrower is truly unknown to the lender it then clearly screens and selects it only to be placed into A or B, but if the lender inherits the portfolio of another it may happen that some new debtors have to be placed *at default* in class D because they were so.

To the end of simulation purposes the entry rating Table 1 is considered, which might be provided by the R&D office or by some accredited institution.

The meaning of the table is the following: $p(y|E)$ is the observed or desired share of new borrowers in rating class y, the only exception is $p(L|E)$ which is introduced to take care of short-term contracts (see Assumption 11), which are those who may enter and leave during the interval of time between date $t-1$ and date t without receiving a rating. Table 1 is therefore almost not-standard but it is consistent to what can be observed.

Let then $\rho_E(W_{i,t}) \to \lambda$ be the rating class to be assigned to client $i \in \mathbb{E}_t$ at date t depending on $W_{i,t}$ by means of a give rule ρ_E. The problem is to specify this rule, which means to find a set of bins $\ell = \{\ell_E^y\}$ to classify $W_{i,t}$ for $i \in \mathbb{E}_t$. To this end, the event $\{\rho_E(W_{i,t}) \to \lambda\} \leq y|E$ is considered, therefore

$$Pr\{\rho_E(W_{i,t}) \leq y|E\} = \sum_{z \leq y} p(z|E) \tag{17}$$

For instance, $p(\rho_E(W_{i,t}) \leq A|E) = 1.00$, because each new debtor has a rating class, which at best is A and D at worst, the short-term contracts without rating are the only exception because they enter and leave within $(t-1, t)$. Not only, $p(\rho_E(W_{i,t}) \leq B|E) = 0.20$ because the cumulative share of new clients which are at best of class B is 0.20. Of course $p(\rho_E(W_{i,t}) \leq D|E) = 0.05$ because there cannot be borrowers with a rating class lower than D: since L has no current period rating, it is not considered for classification but its has to be involved.

Since (14) implies that

$$W_{i,t} \sim \mathcal{N}(0,1) \Rightarrow Pr\{a \leq W_{i,t} < b\} = \Phi(b) - \Phi(a) \tag{18}$$

then

$$
\begin{aligned}
\ell_E^A &= \Phi^{-1}\left(Pr\{\rho_E(W_{i,t}) \leq A|E\}\right) = \Phi^{-1}(1.00) = +\infty \\
\ell_E^B &= \Phi^{-1}\left(Pr\{\rho_E(W_{i,t}) \leq B|E\}\right) = \Phi^{-1}(0.20) = -0.8416 \\
\ell_E^D &= \Phi^{-1}\left(Pr\{\rho_E(W_{i,t}) \leq D|E\}\right) = \Phi^{-1}(0.05) = -1.6449 \\
\ell_E^{\emptyset} &= -\infty
\end{aligned} \tag{19}
$$

where $\ell_E^{\emptyset}$ means that no rating is possible below D, which is technically an empty class $\emptyset$, also because short-term contracts are not rated (see Assumption 11). Therefore, given the defined bins, the rule ρ_E is

$$
\begin{aligned}
&\rho_E(W_{i,t}) \to \lambda_1 \; iff \; \ell_E^B \leq W_{i,t} < \ell_E^A \quad \Rightarrow \quad C_{i,t} = A \; : \; good\ performing \\
&\rho_E(W_{i,t}) \to \lambda_2 \; iff \; \ell_E^D \leq W_{i,t} < \ell_E^B \quad \Rightarrow \quad C_{i,t} = B \; : \; medium/low\ performing \\
&\rho_E(W_{i,t}) \to \lambda_3 \; iff \; \ell_E^{\emptyset} < W_{i,t} < \ell_E^D \quad \Rightarrow \quad C_{i,t} = D \; : \; at\ detault\ not-performing
\end{aligned} \tag{20}
$$

Notice that (20) fulfills Assumption 9. Moreover, as long as creditworthiness $W_{i,t}$ may change through time after the entrance in the portfolio, the classification $C_{i,t}$ of i may change as well, but the classification of those who stay obeys a different rule explained in the following Section 3.2.

3.2. The rating rule for stayers $\mathbb{S}_t$

For those debtors who have already been in the portfolio, that is $i \in \mathbb{S}_t$, a rating rule ρ_S assigns a current period class depending on the current period creditworthiness while considering the previous period rating. Therefore, it might be the case that some debtors who had been rated as λ_h at date $t-1$ are rated as λ_k at date t. If $h = k$ it means the creditworthiness changed not so sensibly to induce the lender to change its opinion on the borrower. If $h \neq k$ two possibilities are at hands: being $\lambda_h \succ \lambda_k$ the migration from $(i \in \lambda_h, t-1)$ to $(i \in \lambda_k, t)$ is a *deterioration*, while the migration from $(i \in \lambda_k, t-1)$ to $(i \in \lambda_h, t)$ is an *improvement*.
This transitory mechanics is often considered as the main justification of the Markovian hypothesis: unfortunately there is no empiric evidence for this and, rather than being an hypothesis or a simplifying assumption, it should be better to say it is a (strong) axiom.[8] Since this lack of empirical justification, the Markovian hypothesis is not considered here.

[8] On the Markovianity hypothesis see [4], [18], [19].

The standard approach, which is based on a closed-sample hypothesis,[9] considers all the possible deterioration migrations (e.g. $A \rightarrow B$, $B \rightarrow D$ and $A \rightarrow D$) while *cures* are considered to be possible only in the case of performing rates (e.g. $A \leftarrow B$) while no cure is allowed from the not-performing one (e.g. $B \leftarrow D$ and $A \leftarrow D$ are considered as impossible) by assumption, i.e. axiomatically. As such, the standard approach considers the default class $\lambda_3 = D$ to be an absorbing state: once a borrower enters such a state it cannot leave state D for entering in the portfolio again, it can only exit to leave it due to the end of the recovery process, which implies a loss for the bank. More precisely:

$$p(A|D) = p(B|D) = 0 \Rightarrow p(D|D) = 1 \tag{21}$$

Although this seams to be a plausible simplification, it is not what can be empirically observed. Indeed, reclassifications or cures from default can be empirically observed. Moreover, the main consequence of considering D to be absorbing under the Markovian hypothesis is that, in the medium-long run, in the standard closed-sample approach the migration matrix will collapse to default. As a consequence, this implies that the bank should prepare an economic capital provision not less than the value of the portfolio: under such standard hypotheses, the bank is therefore doomed to fail by construction. To overcome such an over-prudential inconsistency, it can be generically stated that

Assumption 10. *Each debtor $i \in \mathbb{S}_t$ has a positive probability of migration over the whole system Λ and there is no a-priori absorbing state.*

It is noteworthy that this assumption does not exclude the possibility for a *temporally* absorbing state happens,[10] e.g. at a time it may happen that $p(D|D) = 1$, but nothing guarantees this will be true repeatedly in the mid-long run. Therefore, this assumption rejects an *a-priori* identification of an absorbing state at each point in time. Therefore, the problem is how to fulfill Assumption 10 when the previous period rating class is $\lambda_3 = D$. The solution comes by considering that $\lambda_3 \prec \lambda_2 \prec \lambda_1 \in \Lambda$, therefore

$$A \succ B \succ D \Rightarrow 0 < p(A|D) < p(B|D) < p(D|D) < 1 \tag{22}$$

[9] Differently from the here proposed open-sample one, in a standard closed-sample approach the entry and leave possibilities are not considered in estimating migration probabilities. This aspect clearly has a great impact on the capital requirement estimation problem. That is, a standard closed-sample approach under the Markovian hypothesis overestimates the capital requirement as a consequence of the presence of an absorbing state D.

[10] Clearly, a rigorous definition of absorbing state implies that the state is definitive, here it is considered that sometime it may happen that the share the debtors *at default* in the portfolio is constant at 1 for at least two periods, but not forever, at least *a-priori*.

$p(y|x)$	A	B	D
A	0.75	0.10	0.04
B	0.25	0.60	0.09
D	0.02	0.03	0.90

Table 2: An example of not-standard migration rates matrix over Λ in the open-sample case.

should replace (21). Hence, as an example, let a R&D office of a bank or an accredited institution provide the *not-standard migration matrix* in Table 2. In this matrix where $p(A|D) = 0.02$ and $p(B|D) = 0.03$ are small enough but positive while $p(D|D) = 0.90$ is very close but different from 1, where the rows are not normalized to 1 because, in the here proposed open-sample approach, the possibility to leave has to be considered as described in the following Section 3.3.

Given the migration matrix, the problem is then to find a set of bins like (20) to classify stayers from the past rating classes in the new, and maybe different, rating classes according to the current period creditworthiness.

The procedure to follow is the same developed in Section 3.1 to classify the entries, the difference is that it is applied to each row of the not-standard migration matrix in Table 2. That is, the row referred to the origin state x at date $t-1$ is fixed and then (19) is applied as follows

$$\begin{aligned}
\ell_x^A &= \Phi^{-1}\left(Pr\{\rho_S(W_{i,t}) \leq A|x\}\right) = \Phi^{-1}(1) = +\infty \\
\ell_x^B &= \Phi^{-1}\left(Pr\{\rho_S(W_{i,t}) \leq B|x\}\right) = \Phi^{-1}(p(B|x)) \\
\ell_x^D &= \Phi^{-1}\left(Pr\{\rho_S(W_{i,t}) \leq D|x\}\right) = \Phi^{-1}(p(D|x)) \\
\ell_x^{\emptyset} &= -\infty
\end{aligned} \tag{23}$$

This has to be repeated for all the rows in Table 2 to find the bins for the x-th row which allows for a correct classification according to Assumption 9:

$$\begin{aligned}
&\rho_S(W_{i,t}) \to \lambda_1 \; iff \; \ell_x^B \leq W_{i,t} < \ell_x^A \quad \Rightarrow \quad C_{i,t} = A \; : \; good\ performing \\
&\rho_S(W_{i,t}) \to \lambda_2 \; iff \; \ell_x^D \leq W_{i,t} < \ell_x^B \quad \Rightarrow \quad C_{i,t} = B \; : \; medium/low\ performing \\
&\rho_S(W_{i,t}) \to \lambda_3 \; iff \; \ell_x^{\emptyset} < W_{i,t} < \ell_x^D \quad \Rightarrow \quad C_{i,t} = D \; : \; at\ detault\ not-performing
\end{aligned} \tag{24}$$

Numerically, equation (24) applied to Table 2 gives Table 3.

x	ℓ_x^A	ℓ_x^B	ℓ_x^D	$\ell_x^{\emptyset}$
A	$+\infty$	-1.0803	-1.7507	$-\infty$
B	$+\infty$	0.4959	-1.3408	$-\infty$
D	$+\infty$	1.4758	1.2816	$-\infty$

Table 3: The bins for rating debtors in $\mathbb{S}_t$ according to rule ρ_S and Table 2.

It is now worth pointing out that in the case of a standard closed-sample approach, based on the Markovian hypothesis with D as an absorbing state, the last row of Table 2 would simply disappear: this is because cures from default are considered as impossible events. On the contrary, and consistently with empirical observation, the E-S-L modeling allows for such cures.

3.3. The rating rule for leavers $\mathbb{L}_t$

The last set to consider is that of those debtors who leave the portfolio. In this case, there is no precise rule to follow other than counting how many individuals exit the portfolio at date t while being in a given class at date $t-1$. Nevertheless it is important to distinguish those with a regular exit, which are those *in-bonis* that have reached the maturity of the contract at date t, from those who leave the portfolio from *default*, which are those for which the recovery process had been closed at date t. The relevance of such a distinction is almost intuitive: those who exit from default determine the losses of the bank.
Moreover, as discussed so far, empirical observation suggests that there are debtors who enter the portfolio with a very short-term contract, provided that the technical form allows for this. Differently said, there can be contracts which have been signed at date $t-1$, or just after, with a maturity right before date t, or precisely at date t: such short-term contracts live for less than a period and are not rated.

Assumption 11. *The technical form of the portfolio $\mathbb{F}_t$ allows for short-term contracts with maturity at less than a period after the subscription: such contracts are not rated.*

Consistently with Assumption 11, the classification then involves four states: an additional entry-leave state is set as λ_0 to augment Λ defined in Assumption 9. Therefore, as a consequence of Assumption 11, the system Λ of Assumption 9 is augmented as

$$\Lambda_0 = \{\lambda_0 = L, \lambda_1 = A, \lambda_2 = B, \lambda_3 = D\} \tag{25}$$

x	$p(L\|x)$
E	0.01
A	0.11
B	0.06
D	0.05

Table 4: An example of the share of debtors leaving the protfolio by state.

Notice that $\lambda_0 > \lambda_1$ because the bank is almost sure to recover what lent to the borrower because the contract is ruled within a very short interval of time and, usually, the value of such contracts is very small: however, as a simplification, Assumption 2 is here applied also to such contracts.

Therefore, in the end, as a numerical example consistent with the previous ones, the leave-shares of Table 4 may be observed, where $p(L|E) = 0.01$ is the observed share of new contracts which leave the portfolio right before the next period, $p(L|A) = 0.11$ is the exit rate from class A, $p(L|B) = 0.06$ is the exit rate from B and, finally, $p(L|D) = 0.05$ means that the %5 of contracts exits from default so accounting for $0.05 \cdot F_t$ losses for the bank. From a different point of view, shares in Table 4 might be considered as desired or expected exit rates.

3.4. The open sample migration matrix

At this point, the first row of Table 1, and tables 2 and 4 can be put together to form a single not-standard migration matrix as in Table 5, which considers the open-sample structure by including E and L.

Table 5 is clearly a stochastic matrix and it can always be calculated by the bank: as a matter of fact, according to Assumption 3, the bank owns a time series of such not-standard migration matrices because it owns a time series of portfolios. Hence, according to the following Section 3.5, the bank has all the information needed to estimate it: this kind of augmented migration table will be found as the main outcome of the E-S-L model simulation algorithm.

3.5. The augmented migration matrix

The E-S-L modeling suggests the estimation of an augmented or not-standard migration matrix, which takes care both of the rating for *enteries* and *stayers* and for *leavers* as well. To this end, a migration table

$x \rightarrow y$	A	B	D	L	$\rightarrow$	*sum.*
E	0.80	0.15	0.04	0.01	$\rightarrow$	1
A	0.75	0.10	0.04	0.11	$\rightarrow$	1
B	0.25	0.60	0.09	0.06	$\rightarrow$	1
D	0.02	0.03	0.90	0.05	$\rightarrow$	1

Table 5: The not-standard migration matrix: share of debtors allocations, migrations and exits from the protfolio at date t by state of origin at date $t-1$.

is needed, hence it is assumed the bank already knows a not-standard migration matrix as Table 5 to be used in the classification of borrowers.

The elements of a *migration table* are absolute joint frequencies. Basically, by means of two adjacent portfolios, the classification rules ρ_E and ρ_S allow for a cross-tabulation of rating classes:

$$N_x^y = \#\{(i \in x, t-1) \wedge (i \in y, t)\} \geq 0 \ : \ x, y \in \Lambda_0 \tag{26}$$

On the other hand, the elements of a *migration matrix* are row-normalized frequencies, which can be obtained from the migration table as follows

$$0 < p(y|x) = \frac{N_x^y}{\sum_{z \in \Lambda_0} N_z^y} < 1 \ : \ \sum_{y \in \Lambda_0} p(y|x) = 1 \quad \forall x \in \Lambda_0 \tag{27}$$

Clearly, these are conditional o migration probabilities. Notice that a migration matrix is always a stochastic matrix, both in the standard closed and in the not-standard open-sample approaches but, consistently with Assumption 10, in the latter no *a-priori* absorbing state is considered.
The problem is that the migration matrix depends on the migration table which, as previously discussed, needs of a migration matrix to be estimated according to rules ρ_E and ρ_S. Therefore, for the sake of simulation purposes, Table 5 is assumed to be known to the bank, maybe as the time-average of the time series of not-standard migration matrices. From a different point of view, it can also be the product of the R&D office which has adjusted a standard migration matrix[11] to account for E, D and L, which is a private information of the bank. As said, in any case, the bank has all the information needed for this calculation.

[11] A standard migration matrix follows the closed-sample approach: it considers rows A and B and columns A, B and D only to be normalized by row.

Assumption 12. *By considering two adjacent portfolios $\mathbb{F}_{t-1}$ and $\mathbb{F}_t$ and an available not-standard migration matrix, the bank computes a migration table to account for how entries allocate over Λ_0 at date t according to rule ρ_E, how stayers migrate over Λ between date $t-1$ and date t according to rule ρ_S, and how individuals leave from Λ_0 at t just by counting the exits.*

$t-1 \to t$		E	S			L
	$x \to y$		A	B	D	
E		0	N_E^A	N_E^B	N_E^D	N_E^L
	A	0	N_A^A	N_A^B	N_A^D	N_A^L
S	B	0	N_B^A	N_B^B	N_B^D	N_B^L
	D	0	N_D^A	N_D^B	N_D^D	N_D^L
L		0	0	0	0	N_L^L

Table 6: The augmented migration table of the E-S-L model.

Notice that Assumption 12 is consistent with assumptions 4 and 9 and it states that, in the time spanned between date $t-1$ and date t, the bank computes the augmented migration Table 6 according to a given known migration matrix as Table 5. Not only, the migration table is also consistent with Assumption 10. As such, the not-standard migration matrix is the key notion of the E-S-L modeling. Moreover, it is also worth stressing that distinction is made between the notion of migration table and that of migration matrix.

By applying rules ρ_E and ρ_S to two adjacent portfolios, a migration table can be obtained by counting how many contracts N_x^y fulfill the classification conditions starting from the class of origin. The migration table is therefore made of several components, which can be described by splitting the *instrumental* and the *effective* parts. Each element of the table is identified by a past-state x and a current-state y, such that the occupation number $N_x^y \geq 0$ is the number of past-state x contracts which migrated into the current-state y according to a given classification rule operating on the current period creditworthiness. Not depending on how many rating classes are introduced in Λ, the description of the migration table always includes 9 blocks. The *instrumental* blocks are:

- $E - E$: not meaningful and set at 0;
- $S - E$: before a borrower enters a portfolio it has no rating, hence it is set at 0;

- $L - E$: not meaningful and set at 0;
- $L - S$: those who have left the portfolio do not receive a rating, hence it is set at 0;
- $L - L$: the number N_L^L accumulates the number of exits from period to period, through time this occupation number follows a pure-birth jump process, i.e. a Poisson process.

The *effective* blocks are:

- the block $S - S$ is a not-standard migration matrix accounting for the number of migrations of stayers, it fulfills Assumption 10;
- the block $E - S$ describes how entries allocate over Λ according to the rule ρ_E;
- the block $S - L$ describe how individual exit from the portfolio after having received a rating in two adjacent periods;
- the block $E - L$ accounts for the number of short-term contracts consistently with Assumption 11.

Therefore, in the end, what really matters to the bank are the following blocks: $E - S$, $E - L$, $S - S$ and $S - L$. By using (27) the not-standard migration matrix consistent with the migration table of the E-S-L-model is like Table 7, which has the same structure of Table 5.

$t-1 \to t$		S			L
	$x \to y$	A	B	D	
E		$p(A\|E)$	$p(B\|E)$	$p(D\|E)$	$p(L\|E)$
	A	$p(A\|A)$	$p(B\|A)$	$p(D\|A)$	$p(L\|A)$
S	B	$p(A\|B)$	$p(B\|B)$	$p(D\|B)$	$p(L\|B)$
	D	$p(A\|D)$	$p(B\|D)$	$p(D\|D)$	$p(L\|D)$

Table 7: The augmented migration matrix of the E-S-L model.

Few remarks are now worth stressing:

- $p(A|D)$ and $p(B|D)$ are *cure* rates, i.e. migration probabilities from *at default* into *in-bonis*;
- $p(D|D)$ is the persistence probability *at default*;

- $p(D|E)$, $p(D|A)$ and $p(D|B)$ are default probabilities from *in-bonis*;
- $p(L|E)$, $p(L|A)$ and $p(L|B)$ are exit rates from *in-bonis*;
- $p(L|D)$) is the exit probability from *at default*;
- all the other probabilities are migration or allocation probabilities.

Finally, notice that $p(L|D)$ is a key indicator. Indeed, due to Assumption 2, N_D^L is the value of losses for the bank, that is what it cannot recover.

4. The simulation algorithm and results

Consistently with the previous sections 2 and 3, this section describes the *pseudocode* reported in Appendix A for the simulation algorithm[12] of the E-S-L model. The whole algorithm is structured in three parts, each of which is made of sub-algorithms with specific aims:

- Part I: Micro and macro data simulation, (algorithms 1, 2 and 3);
- Part II: The E-S-L- rating procedure (algorithms 4 and 5);
- Part III: Estimation and dynamics of migration rates matrices (algorithm 6).

The outcomes of sub-algorithms are mainly matrices and vectors: both are indicated with bold capital letters. Vectors represent macro-data time series or micro-data time-varying samples of different size at each time, while matrices represent panel of such micro-data vectors, the only exceptions are the migration matrices obtained from micro-data but they refer to portfolios.

4.1. Part I: macro and micro data simulation

Based on the assumptions of Section 2, Part I of the algorithm is concerned with the simulation of macro and micro data to be used in the following parts.

Algorithm 1 simulates the time series $\mathbf{Y}$ of the main indicator about the economic cycle and the series $\boldsymbol{\gamma}$ of its growth rate, which is required to simulate both the entry rate series $\boldsymbol{\theta}$, needed to simulate the series $\mathbf{E}$ of new borrowers, and the time series $\mathbf{Z}^1$ of the systemic latent factor.

[12]The programs had been developed with MATLAB [20].

To proceed further, Algorithm 2 allocates matrices to host micro-data through the iterations of the simulation.

Algorithm 3 simulates micro-data. At each iteration t, a sample $\mathbf{z}_t \in \mathcal{M}_{N_t,1}(\mathbb{R})$ of N_t observations is simulated according to (13). According to (14), a sample $\mathbf{w}_t \in \mathcal{M}_{N_t,1}(\mathbb{R})$ is obtained. Equation (2) is then applied to simulate stay-or-leave probabilities $\mathbf{q}_t \in \mathcal{M}_{N_t,1}(\mathbb{R})$ to be classified as $\mathbf{x}_t \in \mathcal{M}_{N_t,1}(\mathbb{R})$ by means of (3). Finally, the number L_t of leavers is computed with (4) as the number S_t of stayers according to (5).

Therefore, $\mathbf{z}_t, \mathbf{w}_t, \mathbf{q}_t, \mathbf{x}_t \in \mathcal{M}_{N_t,1}(\mathbb{R})$ are samples of the contracts-specific latent factor, their creditworthiness, the stay-or-leave probabilities and its stay-or-leave classification respectively at iteration t. Each of the so simulated samples overwrites the null t-column in the corresponding matrices allocated with Algorithm 2. Once all the T iterations have been completed, the matrices $\mathbf{Z} = (\mathbf{z}_1, \dots \mathbf{z}_T), \mathbf{W} = (\mathbf{w}_1, \dots \mathbf{w}_T), \mathbf{Q} = (\mathbf{q}_1, \dots \mathbf{q}_T), \mathbf{X} = (\mathbf{x}_1, \dots \mathbf{x}_T)$, and the vectors $\mathbf{E} = (E_1, \dots, E_T)'$, $\mathbf{S} = (S_1, \dots, S_T)'$ and $\mathbf{L} = (L_1, \dots, L_T)'$ are simulated.

Therefore, after Part I has been run, time series of macro-data and a panel data structure for micro-data are at hands to be classified according to the E-S-L rating procedure developed in Part II.

4.2. Part II: the E-S-L rating procedure

The underlying principle of the E-S-L procedure is to adapt the standard closed-sample approach to a not-standard open-sample one.
Basically, this implies to assume that the R&D office of the lender provides a not-standard closed-sample migration matrix like (45) as defined in Algorithm 4. According to the standard simplifying assumption (21) the standard migration matrix (46) is obtained. Moreover, as if it were an optimal or a long-run empirical estimation, an entry rates vector like (47) is assumed to be known.
By applying rule ρ_E as in equation (19) to the entry rates vector $\mathbf{e}$ the first row of the matrix $\mathbf{\Gamma}$ in (48) is obtained. By applying rule ρ_S as in equation (23) to $\mathbf{\Pi}$ the rows A, B and D of $\mathbf{\Gamma}$ follow.

Therefore, up to this stage, macro and micro data have been simulated, as if the bank has organized the series $\mathcal{F}_T$ of portfolios from $t = 1$ to $t = T$ consistently with Assumption 3. All such micro-data are ready to be classified by using the E-S-L rating procedure.

Algorithm 5 describes the E-S-L rating procedure. At the first iteration no preceding classification is available, as if $t = 1$ were the day the credit-line of the bank is opened for that specific type of contracts. As such, being $\mathbb{S}_1 = \emptyset$, (7) of Assumption 4 gives $\mathbb{F}_1 \equiv \mathbb{E}_1$ and the rating rule ρ_E defined in (20) operates on $\mathbf{w}_1$ according to $\mathbf{x}_1$ to determine $\mathbf{c}_1$. This is represented by lines $1 - 10$.

If $t \geq 2$ a preceding portfolio $\mathbb{F}_{t-1}$ exists and a new sample of entries $\mathbb{E}_t$ appears, hence Assumption 4 is fulfilled with (7). Therefore, the new entries allowed to stay are rated according to ρ_E while those destined to leave are classified as L, (see lines 12 – 18). The occupation numbers N_E^y are computed, (see line 19).

Since $t \geq 2$ then there are borrowers of the current portfolio coming from the previous one. Therefore $\mathbb{S}_t$ is to be rated according to ρ_S by means of (24). Some of the stayers may now leave, while others may proceed. Therefore, past period borrowers, which do not leave in the current period, are rated depending on their past period rating class as in lines 22 – 34. Past period debtors who leave now are rated as L at line 36: line 39 accumulates leavers from period to period.

Notice that line 24 involves row A of $\mathbf{\Pi}$ and $\mathbf{\Gamma}$, while line 29 refers to row B and line 34 rates those with a past period rating D. These last cases are very relevant because $D - A$ and $D - B$ account for *cures from default* while $D - D$ are *persistence at default*. Moreover, line 36 defines exits from *default* $D - L$, which account for losses of the bank. On the contrary, line 26 for $A - L$ and line 31 for $B - L$ account for *exits from in-bonis*, said regular exits.

Therefore, at the end of the t-th iteration the sample $\mathbf{c}_t$ is defined: each element $C_{i,t}$ is the current period rating class conditioned on the previous period state. Once all the T iterations have been processed, the matrix $\mathbf{C}$ is evaluated and a sequence of not-standard migration rates tables and matrices can be defined.

4.3. Part III: estimation and dynamics of migration rates matrices

Part III is concerned with the estimation of open-sample not-standard migration rates matrices. The first step to this end is using the classification vectors $\mathbf{c}_t$ in the panel $\mathbf{C}$.

As Algorithm 6 shows this is an easy task, it only takes considering two adjacent classification vectors and compute the table of joint-frequencies: a table $\mathbf{N}_t$ has therefore the same structure of Table 6, that is

$$\mathbf{N}_t = \{N_t(y|x) \equiv N_{x,t}^y : x, y \in \{E, A, B, D, L\}\} \in \mathcal{M}_{5,5}(\mathbb{N}) \tag{28}$$

from which it follows

$$entries\ to\ stay : E_t^1 = \sum_{y \in \{A,B,D\}} N_t(y|E) \tag{29}$$

$$entries\ to\ leave : E_t^0 = N_t(L|E) \tag{30}$$

$$leavers\ in-bonis : L_t^1 = \sum_{x \in \{E,A,B\}} N_t(L|x) \tag{31}$$

$$leavers\ at\ default : L_t^0 = N_t(L|D) \tag{32}$$

$$entries : E_t = E_t^1 + E_t^0 \tag{33}$$

$$
\begin{aligned}
stayers: S_t &= \sum_{x,y\in\{E,A,B,D\}} N_t(y|x) &(34)\\
leavers: L_t &= L_t^0 + L_1 &(35)\\
size: F_t &= E_t + S_t &(36)\\
cumulative\ exits: K_t &= \sum_{t<\tau} N_\tau(L|L) &(37)
\end{aligned}
$$

Moreover, from $\{\mathbf{N}_t\}_2^T$ the *unconditional* time average is obtained

$$
\mathbf{N} = \frac{1}{T-1}\sum_{t=2}^{T}\mathbf{N}_t \tag{38}
$$

which, in the end, is the most important outcome of the E-S-L simulation. Four distinct tables can be generated from $\mathbf{N}$, together with their migration matrices:[13]

- the not-standard closed-sample migration table $\mathbf{M}_S$ is obtained by considering the block $S-S$ as it is; the associated migration matrix is $\mathbf{P}_S$;
- the standard closed-sample migration table $\mathbf{M}_C \in \mathcal{M}_{3,3}(\mathbb{N})$ is obtained from $\mathbf{M}_S$ while substituting row D for $(0,0,\sum_y N_t(y|D))$, as if it were an *a-priori* absorbing state; the associated migration matrix is $\mathbf{P}_C$;
- the not-standard open-to-exit migration table $\mathbf{M}_{SL}$ is obtained by augmenting $\mathbf{M}_S$ with column and row L of $\mathbf{N}$; the migration matrix is $\mathbf{P}_{SL}$;
- the not-standard open-sample migration table $\mathbf{M}_{ESL}$ is obtained by augmenting $\mathbf{M}_{SL}$ with row and column E of $\mathbf{N}$, therefore $\mathbf{M}_{ESL} \equiv \mathbf{N}$; the associated migration matrix is $\mathbf{P}_{ESL}$.

Notice that all such migration tables are square and their associated migration matrices are also stochastic. Therefore, for any the migration matrix $\mathbf{P}_m$ the eigen-decomposition $\mathbf{P}_m = \mathbf{V}_m\mathbf{A}_m\mathbf{V}_m^{-1}$ holds. Accordingly, now the standard practice to forecast future dynamics can be applied

$$
\mathbf{P}_m = \mathbf{V}_m\mathbf{A}_m\mathbf{V}_m^{-1} \Rightarrow \widehat{\mathbf{P}}_m(t+k) = \mathbf{V}_m\mathbf{A}_m^{t+k}\mathbf{V}_m^{-1}\ :\ k>1 \tag{39}
$$

where $m = S, C, SL, ESL$.

[13] It is worth recalling that a migration matrix is obtained by row-normalization of a migration table, hence it is square and stochastic.

5. Simulation results and implications

The whole simulation algorithm has been run with the following parameters: (a) initial portfolio size: $F_1 = 100,000$; (b) $T_0 = 10$ and $T = 20$; (c) $\beta = 0.5$; (d) $\phi_0 = 1.50$, $\phi_1 = 0.85$, $\phi_2 = 0.05$; (e) $m_0^- = 0.10$, $m_1^- = 0.20$, $m_0^+ = 0.20$, $m_1^+ = 0.50$. Together with matrices of Algorithm 4, such parameters lead to figures 1 and 2 to show the evolution of the simulated portfolio.

As it can be seen in the second panel of Figure 1, the size F_t of the portfolio increases with some fluctuations due to the effects of entries E_t and exits L_t reported in the first panel: such a renewal is the consequence of considering the portfolio as an open system and it spreads effects on the number of stayers S_t. As the size of the portfolio, also the cumulative number of exits K_t increases by a factor of 10, although on a different scale: this is shown in the second panel of Figure 1.

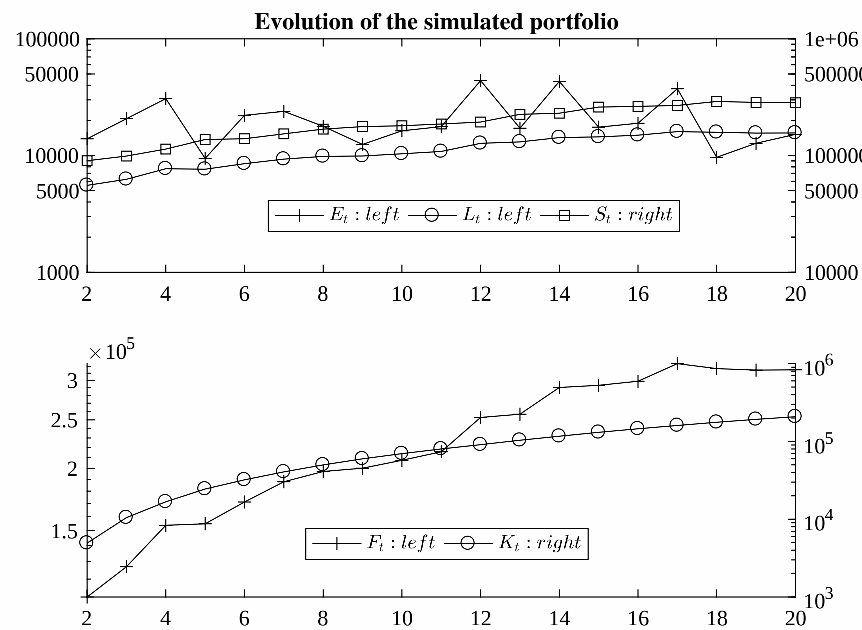

Figure 1: Evolution of the simulated portfolio. First panel: entries, stayers and leavers. Second panel: size and cumulative exits.

This description is macroscopic and it tells almost nothing about how the portfolio reshapes through time over the system of rating classes: indeed, this is due to changes in creditworthiness of stayers and the allocation of new borrower, either in the algorithm as well as in reality. Nevertheless, at each iteration t a migration table has been simulated, and it accounts for such micro-level dynamic stochastic phenomena.

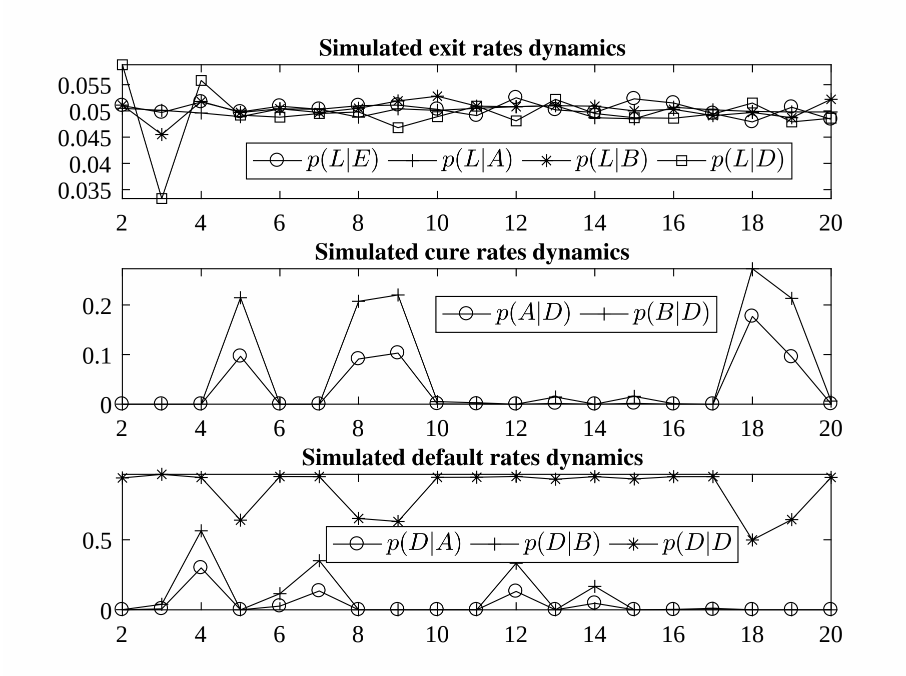

Figure 2: Simulates rates.

The outcome of the simulation algorithm is therefore the matrix **N** defined in (38) and reported with numerical results in (40).

N = (40)

		E	S			L
			A	B	D	
E		0	16,196	5,883	1,810	1,274
	A	0	66,543	24,348	5,294	5,062
S	B	0	16,120	32,305	6,938	2,928
	D	0	370	844	46,554	2,512
L		0	0	0	0	90,375

Consistently with the simulation, migration rates can be *observed* through iterations: see Figure 2. The classification of the micro-simulated data suggests that exits rates are almost stationary and not so evidently differentiated by the rating class of origin: this is clearly due to that no economic-environmental force and micro-economic strategy attributed to individuals.[14]

Wider differences emerge concerning cure rates: interestingly enough they are often very close *zero*, as the standard method *a-priori* assumes, but there are periods for which the *a-priori* absorbing state at D has to be rejected, manly because it seems not to depend on the cycle itself but it can be argued that the major source of such an unexpected dynamics is the superimposition of latent factors in defining the creditworthiness. As a matter of fact, due to this simplified modeling, the economic cycle only influences the number of entries without affecting the stayers' creditworthiness: clearly, this is not true in the real economies.[15]

Finally, default rates show that the persistence at default is pretty close to 1, so suggesting that the standard assumption of D as an absorbing state seems not to be so implausible. Nevertheless, at a careful investigation, it cannot be *a-priori* assumed: first of all it is never precisely fulfilled and, moreover, there are periods for which it is patently false. According to the assumptions made, motivated by the direct observation of real data, this rebuttal is therefore plausible: there can be periods of high persistence at default but nothing justifies that D has to be *a-priori* considered as absorbing.

All such simulation results describe the *empirical* framework, what follows is therefore an analysis of the implications of assuming portfolios as open systems compared to the standard method based on the closed-sample hypothesis with an *a-priori* absorbing state D.

Form (40) the migration matrix (41) can be obtained to represent what the standard method deals with.

$$\mathbf{P}_C = \left(\begin{array}{cc|ccc} & & & \text{S} & \\ & & \text{A} & \text{B} & \text{D} \\ \hline\hline & \text{A} & 0.6918 & 0.2531 & 0.0550 \\ \text{S} & \text{B} & 0.2912 & 0.5835 & 0.1253 \\ & \text{D} & 0 & 0 & 1 \end{array} \right) \tag{41}$$

As clear, being concerned with the $S - S$ block only, the so called standard closed-sample migration matrix $\mathbf{P}_C$, with D as an absorbing state, is the worst solution to the economic capital provision problem. This is because in the medium-long run matrix $\mathbf{P}_C$ in (41) deterministically foresees that $\widehat{\mathbf{P}}_C(t+k)$ will collapse. That is, all the migration rates are doomed to vanish while $p(D|x) \approx 1$ whatever x is: the first panel of Figure 3

[14]A better description would follow by developing an ABM (Agent Based Model), as an example see [21]; although relevant in terms of evaluation of the effects of the normative and standard policy constraints in rating creditworthiness, this topic has not been developed here.

[15]This topic is left by the authors as a further developments.

shows the cure rates from D to *in bonis* are zero while the first panel of Figure 4 shows that default rates from *in bonis* increase. This is what the currently followed standard approach predicts and the consequence is that, to cover credit risk implied by the default-probability in the medium-long run, a bank should precautionary set aside an economic capital provision which is about the value of the whole portfolio, not depending on the distinction from *in bonis* or *at-defauld* of its clients. Therefore, since this standard model predicts all the debtors are doomed to fail, as a consequence of the standard closed-sample hypothesis with absorbing state *at default*, the bank itself is unrealistically and inconsistently doomed to fail. Clearly, this cannot meet the requirements of the regulatory authority, either because it is inconsistent, unrealistic and unattainable with respect to the commercial activity of the bank and to the (present and future) regulatory standards.

A second matrix (42) can be obtained from (40) to remove the *a-priori* absorbing state *at default* assumption.

$$\mathbf{P}_S = \left(\begin{array}{cc|ccc} & & & \text{S} & \\ & & \text{A} & \text{B} & \text{D} \\ \hline\hline & \text{A} & 0.6918 & 0.2531 & 0.0550 \\ \text{S} & \text{B} & 0.2912 & 0.5835 & 0.1253 \\ & \text{D} & 0.0077 & 0.0177 & 0.9746 \end{array}\right) \tag{42}$$

This is a less-worst solution to the economic capital provision problem. Indeed, the usual forecasting methodology predicts that $\widehat{\mathbf{P}}_S(t+k)$ will approach $\widehat{\mathbf{P}}_C(t+k)$: it is just a matter of time, but the lender is anyway doomed to fail. Indeed, even though the second panel of Figure 3 predicts the possibility of significant cure rates from default, the second panel of Figure 4 predicts that the default rates will increase. As a consequence, this solution is not so satisfactory to meet the so called *full loan lifetime* requirement.

A better solution may come by considering that the sample is not closed: the simplest case is to consider exit-rates both from *in-bonis* and *at default*.

$$\mathbf{P}_{SL} = \left(\begin{array}{cc|ccc|c} & & & \text{S} & & \\ & & \text{A} & \text{B} & \text{D} & \text{L} \\ \hline\hline & \text{A} & 0.6572 & 0.2405 & 0.0523 & 0.0500 \\ \text{S} & \text{B} & 0.2765 & 0.5542 & 0.1190 & 0.0502 \\ & \text{D} & 0.0074 & 0.0168 & 0.9252 & 0.0500 \\ \hline & \text{L} & 0 & 0 & 0 & 1 \end{array}\right) \tag{43}$$

As it comes clear from (43), no absorbing state is allowed for migrations, as it was in the previous case, except of the $L-L$ case, which is not a rating migration and it obeys an accumulation law by definition. As (43) shows, cure rates from default are smaller than those predicted by (42) but, more sensibly, the persistence at default is lower because it is predicted that $p(L|x)$ is about 5%, remarkably not depending on x. As a consequence of the just found results, by removing the standard closed-sample hypothesis, together with the axiomatic perception that D is an absorbing state, it can be observed that opening the rate systems to exits is more realistic than the closed-sample standard; unfortunately, without an immigration process that replaces the exits, sooner or later the whole portfolio would be destined to extinction, at least according to the usual forecasting method.[16]

Therefore, a better description may come by opening the system also to entries into the portfolio. That is, while allowing for migrations of stayers, by considering a portfolio as a system either open to exits as well as entries, according to the usual forecasting method the more realistic solution comes by applying the E-S-L approach, which ends up with the estimator $\widehat{\mathbf{P}}_{ESL}(t+k)$ by applying (39) to (44).

$\mathbf{P}_{ESL} =$ (44)

		E	S			L
			A	B	D	
E		0	0.6437	0.2338	0.0719	0.0506
	A	0	0.6572	0.2405	0.0523	0.0500
S	B	0	0.2765	0.5542	0.1190	0.0502
	D	0	0.0074	0.0168	0.9259	0.0500
L		0	0	0	0	1

Notice that $\mathbf{P}_{SL}$ is embedded into $\mathbf{P}_{ESL}$, hence the third panels of figures 3 and 4 are appropriate also for $\mathbf{P}_{SL}$. Therefore, to the end of providing a solution to the economic capital provision, $\mathbf{P}_{SL}$ and $\mathbf{P}_{ESL}$ are equivalent. Indeed, the only difference is in considering entries but the usual forecasting procedure would not find a big difference. This is because, open to exits or to entries too, the $\mathbf{P}_{SL}$ and $\mathbf{P}_{ESL}$ are anyway static estimates and, in the usual forecasting procedure (39), they meet the same result in forecasting exits from *in bonis* and *default*.[17]

However, according to the third panel of Figure 3, it is interesting to see that cure rates from default start by increasing up to a given point in time, after which they proceed by decreasing. Moreover, migration from D

[16]Further comments on this topic follow in concluding Section 6.

[17]In Section 6 this point will be stressed and futher developments under study by the authors are discussed.

to B is almost flat and dominated by the migration from D to A. Furthermore, the same shape can be also found in the third panel of Figure 4 while leading to a different conclusion: in the long run the default rates are expected to converge with each other to a share about 9%. Hence, the E-S-L description allows for a more comfortable solution to the bank, and this is the consequence of considering the renewal process in shaping migration rates as open systems under the usual forecasting procedure (39).

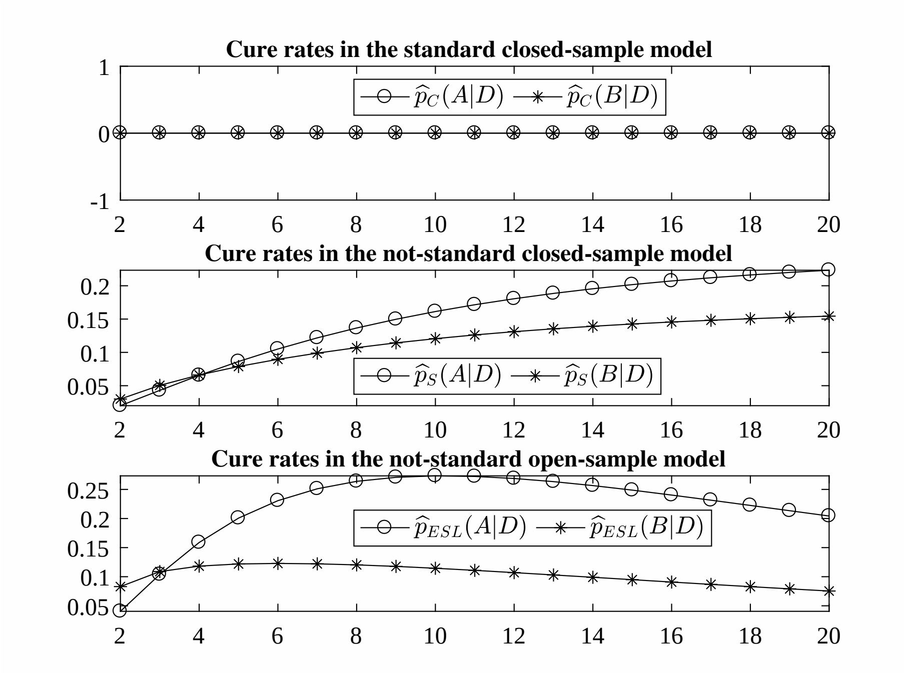

Figure 3: Cure rates from default forecast: comparison of $\widehat{p}_C(y|D)$, $\widehat{p}_S(y|D)$ and $\widehat{p}_{ESL}(y|D)$.

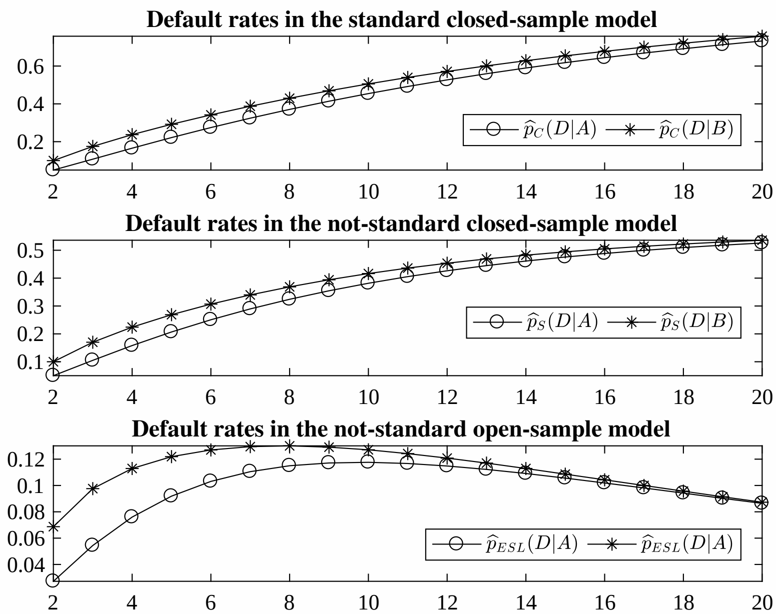

Figure 4: Default rates forecast: comparison of $\widehat{p}_C(D|x)$, $\widehat{p}_S(D|x)$ and $\widehat{p}_{ESL}(D|x)$.

6. Concluding remarks

The paper presentes both a micro-simulation algorithm, which is based on assumptions motivated by empirical observation, and theoretical outcomes to improve the so called standard approach. Usually, lenders classify their borrowers according to a closed-sample approach while *a-priori* assuming that the default state D is absorbing. Moreover, a widely shared forecasting method among lenders has been considered: that is, the deterministic forecasting eigen-decomposition approach has been involved.

As widely known, if this is the starting point, the standard method could not predict other than the whole portfolio will collapse to default in the short-medium run, so predicting an overestimation of the economic capital provision which, in the end, would induce the collapse of the considered credit-line of the bank too. As it can be easily understood, the greater the portfolio the more the bankruptcy of the lender is expected in the short run. Clearly, this cannot meet the normative requirements on the economic capital provision of the bank and, as a mater of fact, it is a puzzling point for commercial banks and for supervisors.

To sheed more light on the evident and unrealistic limitations of the standard approach, a micro-simulation algorithm has been developed to account either for entries and exits while both removing any sort of *a-priori*

absorbing state and considering migrations of stayers. As discussed, the big difference has been found by allowing for exits with *no absorbing* state.
Indeed, it has been shown that even by considering a fully open rating system, the problem is not completely solved. This is because, if the rating procedure is based on static migration table (or matrix) estimate, as the usual eigen-decomposition forecasting procedure suggests, sooner or later all the borrowers will approach a critical state, so predicting a critical state for the lender too. Differently said, let the base migration matrix obtained from a closed or open sample approach, even without an absorbing state, the eigen-decomposition forecasting will anyway predict the collapse of the portfolio and the bankruptcy of the lender: it would be just a matter of time.

The paper then concludes with few main remarks. First of all, it is not true that once one enters the *D* state she cannot leave it: sooner or later she will it even though, and more often than not, she will leave it by exiting the portfolio while being in *at-default*, so generating a real loss to the bank. Therefore, to obtain sounder and more consistent predictions, the closed-sample hypothesis has to be removed together with the simplifying axiom of any *a-priori* absorbing state. Indeed, to approach the regulatory requirements which are about to enter into force in the near future, both hypotheses have to be definitely rejected not to predict the bank is doomed to fail by construction and to allow for a more reliable estimate of losses under the so called *full loan lifetime* requirement for each credit-line of the lender.

But this is not enough. The economic capital provision problem will meet a sounder solution only if the eigen-decomposition forecasting is abandoned as well. This is because any sort of deterministic forecasting based on a static migration matrix estimates, either found with a closed or an open sample approach, will predict a collapse of the portfolio and the bankruptcy of the lender. Therefore, the question is: *what is to be done?*

At the best of the authors knowledge, at least at the time of writing this note, there is no sound alternative to debate on in literature. On one hand, some approaches suggest following a stochastic dynamic approach but they do not remove the assumption of absorbing state at default. On the other hand it seems widely shared an opinion that the rating systems should be considered as opens systems, but it is unclear how to link the micro-level of creditworthiness of borrowers to the macro-level of the rating matrix available to lenders. This paper provides an attempt in this direction. Along with this line, and in the perspective of developing instruments to protect economies from the systemic credit risk, to meet the new regulatory standards, the link between the micro and the macro level should be set by conditioning the dynamic forecast of creditworthiness to macroeconomic-cycle scenarios. .

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A. The algorithm *pseudocodes*

Algorithm 1 Macro data Y_t, γ_t, Z_t^1, θ_t and E_t simulation.

Require: $1 \leq r \leq R, R \geq 1, \{\phi_r\}_1^R, \sigma_\epsilon^2 > 0, T \gg T_0 > 1, 0 < m_0^- < m_1^+ \leq m_0^+ < m_1^+ < 1, F_1 \gg 1.$

Apply (9): $\{\widetilde{Y}_\tau\}_1^{T_0+T+1}$

Apply (10): $\{\widetilde{\gamma}_\tau\}_1^{T_0+T+1}$: $\widetilde{\gamma}_1 = 1$

Delete $1, \ldots, T_0 + 1$ in $\{\widetilde{Y}_\tau\}_1^{T_0+T+1}$, $\{\widetilde{\gamma}_\tau\}_1^{T_0+T+1}$: $\mathbf{Y} = \{Y_t\}_1^T, \boldsymbol{\gamma} = \{\gamma_t\}_1^T$

Apply (12): $\mathbf{Z^1} = \{Z_t^1\}_1^T$

Apply (11): $\boldsymbol{\theta} = \{\theta_t\}_1^T$

Apply (6): $\mathbf{E} = \{E_t\}_1^T$

return $\mathbf{Y}, \boldsymbol{\gamma}, \mathbf{Z^1}, \boldsymbol{\theta}, \mathbf{E} \in \mathcal{M}_{1,T}(\mathbb{R})$.

Algorithm 2 Allocating matrices and vectors $\mathbf{Z}, \mathbf{W}, \mathbf{Q}, \mathbf{X}, \mathbf{C}, \mathbf{S}, \mathbf{L}$ and $\mathbf{E}$.

Require: $T, \{E_t\}_1^T$

Calculate $N_T = \sum_t^T E_t$

Define $\mathbf{Z} = \mathbf{W} = \mathbf{Q} = \mathbf{X} = \mathbf{C} \in \mathcal{M}_{N_T,T}(0)$

Define $\mathbf{S} = \mathbf{L} \in \mathcal{M}_{1,T}(0)$

Algorithm 3 Simulating $Z_{i,t}$, $W_{i,t}$, $Q_{i,t}$, $X_{i,t}$, S_t and L_t.

Require: $T, \beta \in (0,1), q^* \in (0,1), \{Z_t^1\}_1^T$.

for $t = 1$ to T **do** ▹ Micro-simulation begins
  if t=1 **then** ▹ First iteration: initialization
    for $i = 1$ to $N_1 = F_1 = E_1$ **do**
      Apply (13):$Z_{i,1}$, (14):$W_{i,1}$, (2):$Q_{i,1}$, (3):$X_{i,1}$, (4):L_1, (5):S_1
    end for
  else ▹ From second iteration onward
    for $i = 1$ to N_t **do**
      if $X_{i,t-1} = -1$ **then** ▹ Those who leave are deleted
        $X_i, t = -1 \Rightarrow Z_{i,t} = W_{i,t} = Q_{i,t} = 0$
      else ▹ Those who stay proceed
        Apply (13):$Z_{i,t}$, (14):$W_{i,t}$, (2):$Q_{i,t}$, (3):$X_{i,t}$, (4):L_t, (5):S_t
      end if
    end for
  end if
end for ▹ Micro-simulation ends
return $\mathbf{Z}, \mathbf{W}, \mathbf{Q}, \mathbf{X}, \mathbf{E}, \mathbf{S}, \mathbf{L}$

Algorithm 4 Specifying *closed-sample* matrices $\mathbf{\Pi}$, $\mathbf{\Sigma}$, $\mathbf{e}$ and $\mathbf{\Gamma}$.

1: Define the *not-standard* migration matrix

$$\mathbf{\Pi} = \left(\begin{array}{c|ccc} & \text{A} & \text{B} & \text{D} \\ \hline \text{A} & 0.75 & 0.20 & 0.05 \\ \text{B} & 0.30 & 0.60 & 0.10 \\ \text{D} & 0.02 & 0.03 & 0.95 \end{array}\right) \tag{45}$$

2: Define the *standard* migration matrix

$$\mathbf{\Sigma} = \left(\begin{array}{c|ccc} & \text{A} & \text{B} & \text{D} \\ \hline \text{A} & 0.75 & 0.20 & 0.05 \\ \text{B} & 0.30 & 0.60 & 0.10 \\ \text{D} & 0.00 & 0.00 & 1.00 \end{array}\right) \tag{46}$$

3: Define the *entry rates* vector

$$\mathbf{e} = \left(\begin{array}{c|ccc} & \text{A} & \text{B} & \text{D} \\ \hline \text{E} & 0.80 & 0.15 & 0.05 \end{array}\right) \tag{47}$$

4: Define the *bins* upon $\mathbf{\Pi}$ and $\mathbf{e}$

$$\mathbf{\Gamma} = \left(\begin{array}{cc|cccc} & x & \ell_x^A & \ell_x^B & \ell_x^D & \ell_x^\emptyset \\ \hline \mathbf{e} & \text{E} & +\infty & \text{-0.8416} & \text{-1.6449} & -\infty \\ \hline & \text{A} & +\infty & \text{-0.6754} & \text{-1.6449} & -\infty \\ \mathbf{\Pi} & \text{B} & +\infty & 0.5244 & 1.2816 & -\infty \\ & \text{D} & +\infty & 2.0537 & 1.6449 & -\infty \end{array}\right) \tag{48}$$

Algorithm 5 The E-S-L rating procedure.

Require: $T, \mathbf{X}, \mathbf{W}, \mathbf{\Gamma}$

for $t = 1$ to T **do**
  ———————————————————— ▷ **Rating entries**
  if $t = 1$ **then** ▷ *Initial entries*
    for $i \in \mathbb{E}_1$ **do**
      if $X_{i,t} = +1$ **then** ▷ The $E - S$ cases
        $\rho_E(W_{i,t}) \Rightarrow C_{i,t} = A \vee B \vee D$
      else
        $C_{i,t} = L$ ▷ The $E - L$ cases
      end if
    end for
  else ▷ $2 \leq t \leq T$
    for $i \in \mathbb{E}_t$ **do** ▷ *Current entries*
      if $X_{i,t} = +1$ **then** ▷ The $E - S$ cases
        $\rho_E(W_{i,t}) \Rightarrow C_{i,t} = A \vee B \vee D$
      else
        $C_{i,t} = L$ ▷ The $E - L$ cases
      end if
    end for
    Compute: $\mathbf{N}'_{E,t} = (0\ N^A_{E,t}\ N^B_{E,t}\ N^D_{E,t}\ N^L_{E,t})$
    ———————————————————— ▷ **Rating stayers**
    for $i \in \mathbb{S}_t$ **do**
      if $X_{i,t} = +1$ **then** ▷ The $S - S$ cases
        if $C_{i,t-1} = A$ **then** ▷ *Migrations from A*
          $\rho_S(W_{i,t}) \Rightarrow C_{i,t} = A \vee B \vee D$
        else ▷ The $A - L$ exits
          $C_{i,t} = L$
        end if
        if $C_{i,t-1} = B$ **then** ▷ *Migrations form B*
          $\rho_S(W_{i,t}) \Rightarrow C_{i,t} = A \vee B \vee D$
        else ▷ The $B - L$ exits
          $C_{i,t} = L$
        end if
        if $C_{i,t-1} = D$ **then** ▷ *Migrations form D*
          $\rho_S(W_{i,t}) \Rightarrow C_{i,t} = A \vee B \vee D$
        else ▷ The $D - L$ exits
          $C_{i,t} = L$
        end if
      else ▷ The $L - L$ cases
        $C_{i,t} = L$
      end if
    end for
  end if
end for ▷ $\forall t, \mathbf{w}_t \to \mathbf{c}_t \Rightarrow \mathbf{W} \to \mathbf{C}$
return $\mathbf{C} \in \mathcal{M}_{N_T,T}(\mathbb{R})$

Algorithm 6 The sequence of migration tables $\mathbf{N}_t$.

Require: $T, \mathbf{C}, \mathbf{1}_5 = \mathcal{M}_{5,1}(1)$

for $t = 2$ to T **do**

  $\mathbf{N}_t = CrossTabulate(\mathbf{c}_{t-1}, \mathbf{c}_t)$ ▷ The migration table

end for

return $\{\mathbf{N}_t\}_2^T$