Phase Type and Matrix Exponential Distributions in Stochastic Modeling

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Abstract

Since their introduction, properties of Phase Type (PH) distributions have been analyzed and many interesting theoretical results found. Thanks to these results, PH distributions have been profitably used in many modeling contexts where non-exponentially distributed behavior is present. Matrix Exponential (ME) distributions are distributions whose matrix representation is structurally similar to that of PH distributions but represent a larger class. For this reason, ME distributions can be usefully employed in modeling contexts in place of PH distributions using the same computational techniques and similar algorithms, giving rise to new opportunities the fact, they are able to represent different dynamics, e.g., faster dynamics, or the same dynamics but at lower computational cost. In this work, we deal with the characteristics of PH and ME distributions, and their use in stochastic analysis of complex systems. Moreover, the techniques used in the analysis to take advantage of them are revised.

Phase Type and Matrix Exponential Distributions in Stochastic Modeling

Andras Horvath, Marco Scarpa and Miklos Telek

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1 Introduction

Stochastic modeling has been used for performance analysis and optimization of computer systems for more than five decades [19]. The main analysis method behind this effort was the continuous time Markov chains (CTMC) description of the sys-

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tem behavior and the CTMC-based analysis of the performance measures of interest. With the evolution of computing devices, model description languages (e.g., queueing systems, Petri nets, process algebras), and model analysis techniques (a wide range of software tools with efficient analysis algorithm using adequate data representation and memory management) the analysis of more and more complex systems has become possible. One of main modeling limitations of the CTMC-based approach is the limitation on the distribution of the random time durations, which is restricted to be exponentially distributed. Unfortunately, in a wide range of practical applications, the empirical distribution of field data differs significantly from the exponential distribution. The effort to relax this restriction of the CTMC-based modeling on exponentially distributed durations resulted in the development of many alternative stochastic modeling methodologies (semi-Markov and Markov regenerative processes [11], analysis with the use of continuous system parameters [8]), yet all of the alternative modeling methodologies suffer from infeasible computational complexity very quickly when the complexity of the systems considered increases beyond basic examples.

It remains a significant research challenge to relax the modeling restriction of the exponentially distributed duration time and still evaluate complex model behaviors. To this end, one of the most promising approaches is the extension of CTMC-based analysis to non-exponentially distributed durations. Initial steps in this direction date back to the activity of A.K. Erlang in the first decades of the twentieth century as reported in [10]. These initial trials were referred to as the method of phases, which influenced later terminology. M.F. Neuts characterized a set of distributions which can be incorporated into CTMC-based analysis by introducing the set of phase type (PH) distributions [16].

The extension of CTMC-based analysis (where the durations are exponentially distributed) with PH distributed durations requires the generation of a large CTMC, referred to as extended Markov chain (EMC), which combines the system behavior with the description of the PH distributions. In this chapter, we summarize the basics of EMC-based stochastic analysis and provide some application examples. Finally, we note that in this work we restrict our attention to continuous time stochastic models, but that the same approach applies for discrete time stochastic models as well.

1.1 Structure of the Chapter

The next two sections, Sects. 2 and 3, summarize the basic information on PH and ME distributions, respectively. The following two sections, Sects. 4 and 5, discuss the analysis procedure for complex stochastic systems with PH and ME distributed durations, respectively. The tools available to support EMC-based analysis of stochastic systems is presented in Sect. 6. Numerical examples demonstrate the modeling and analysis capabilities of the approach are discussed in Sect. 7 and the main findings and conclusions are given in Sect. 8.

2 PH Distributions and Their Basic Properties

2.1 Assumed Knowledge

Transient behavior of a finite state Markov chain with generator **Q** and initial distribution π, specifically, the transient probability vector $p(t)$, satisfies the ordinary differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}p(t) = p(t)\mathbf{Q}, \text{ with initial condition } p(0) = \pi,$$

whose solution is a matrix exponential function

$$p(t) = \pi \mathrm{e}^{\mathbf{Q}t}, \tag{1}$$

where the matrix exponential term is defined as

$$\mathrm{e}^{\mathbf{Q}t} = \sum_{i=0}^{\infty} \frac{t^i}{i!}\mathbf{Q}^i.$$

The properties of generator **Q** and initial distribution π are as follows. The elements of π are probabilities, i.e., nonnegative numbers not greater than one. The off-diagonal elements of **Q** are transition intensities, i.e., nonnegative numbers. The diagonal elements of **Q** are such that each row sum is zero, i.e., the diagonal elements are non-positive. The elements of π sum to one, that is $\sum_i \pi_i = \pi\mathbf{1} = 1$. Each row of a generator matrix sums to zero, that is $\sum_j Q_{ij} = 0$, or equivalently, in vector form, we can write $\mathbf{Q1} = \mathbf{0}$, where $\mathbf{1}$ is a column vector of ones and $\mathbf{0}$ is a column vector of zeros. Hereafter, the sizes of vector $\mathbf{1}$ and $\mathbf{0}$ are defined by the context such that the dimensions in the vector expressions are compatible.

The stationary distribution of an irreducible finite state Markov chain with generator **Q**, $p \triangleq \lim_{t\to\infty} p(t)$, can be computed as the unique solution of the linear system of equations

$$p\mathbf{Q} = \mathbf{0}, \quad p\mathbf{1} = 1. \tag{2}$$

In this chapter, we focus on the computation of the initial distribution and the generator matrix of the EMC and do not discuss the efficient solution methods for solving (1) and (2).

2.2 Phase Type Distributions

PH distributions are defined by the behavior of a Markov chain, which is often referred to as the background Markov chain behind a PH.

Let $X(t)$ be a Markov chain with n transient and one absorbing states, meaning that the absorbing state is reachable (by a series of state transitions) from all transient states, but when the Markov chain moves to the absorbing state it remains there forever. Let π be the initial distribution of the Markov chain, that is $\pi_i = P(X(0) = i)$. Without loss of generality, we number the states of the Markov chain such that state $1, \ldots, n$ are transient states and state $n + 1$ is the absorbing state. The generator matrix of such a Markov chain has the following structure

$$\mathbf{Q} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{a} \\ \mathbf{0} & 0 \end{bmatrix},$$

where $\boldsymbol{A}$ is a square matrix of size n and $\boldsymbol{a}$ is a column vector of size n. Since the rows of the generator matrix sum to zero, the elements of $\boldsymbol{a}$ can be computed from $\boldsymbol{A}$, that is $\boldsymbol{a} = -\boldsymbol{A}\mathbf{1}$. Similarly, the first n elements of the initial vector π, denoted by $\boldsymbol{\alpha}$, completely defines the initial vector, since the $(n + 1)$st element of π is $1 - \boldsymbol{\alpha}\mathbf{1}$. We note that $\boldsymbol{\alpha}$ defines the initial probabilities of the transient states. With the help of this Markov chain, we are ready to define PH distributions.

Definition 1 The time to reach the absorbing state of a Markov chain with a finite number of transient and an absorbing state

$$T = \min\{t : X(t) = n + 1, t \geq 0\},$$

is phase type distributed.

Throughout this document, we assume that the Markov chain starts from one of the transient states and consequently $\boldsymbol{\alpha}\mathbf{1} = 1$, i.e., there is no probability mass at zero and T has a continuous distribution on $\mathbb{R}^+$. Since the time to reach the absorbing state is a transient measure of the Markov chain, we can evaluate the distribution of random variable T, based on the transient analysis of the Markov chain with initial distribution π and and generator matrix $\mathbf{Q}$

$$F_T(t) = P(T < t) = P(X(t) = n + 1) = \pi \mathrm{e}^{\mathbf{Q}t} e_{n+1},$$

where e_{n+1} is the $(n + 1)$st unit vector (the column vector with zero elements except in position $n + 1$ which is one).

This straight forward description of the distribution of T is not widely used due to the redundancy of matrix $\mathbf{Q}$ and vector π. Indeed, matrix $\boldsymbol{A}$ and the initial vector associated with the transient states, $\boldsymbol{\alpha}$, define all information about the distribution of T and the analytical description based on $\boldsymbol{\alpha}$ and $\boldsymbol{A}$ is much simpler to use in more complex stochastic models. To obtain the distribution based on $\boldsymbol{\alpha}$ and $\boldsymbol{A}$, we carry on the block structure of matrix $\mathbf{Q}$ in the computation.

$$
\begin{aligned}
F_T(t) &= P\left(T < t\right) = P\left(X(t) = n+1\right) = 1 - \sum_{i=1}^{n} P\left(X(t) = n+1\right) \\
&= 1 - [\boldsymbol{\alpha}, 0]\mathrm{e}^{\mathbf{Q}t}\begin{bmatrix}\mathbf{1}\\ 0\end{bmatrix} = 1 - [\boldsymbol{\alpha}, 0]\sum_{i=0}^{\infty}\frac{t^i}{i!}\begin{bmatrix}\boldsymbol{A} & \boldsymbol{a}\\ \mathbf{0} & 0\end{bmatrix}^i\begin{bmatrix}\mathbf{1}\\ 0\end{bmatrix} \\
&= 1 - [\boldsymbol{\alpha}, 0]\sum_{i=0}^{\infty}\frac{t^i}{i!}\begin{bmatrix}\boldsymbol{A}^i & \bullet\\ \mathbf{0} & 0\end{bmatrix}\begin{bmatrix}\mathbf{1}\\ 0\end{bmatrix} = 1 - \boldsymbol{\alpha}\sum_{i=0}^{\infty}\frac{t^i}{i!}\boldsymbol{A}^i\mathbf{1} = 1 - \boldsymbol{\alpha}\mathrm{e}^{\boldsymbol{A}t}\mathbf{1},
\end{aligned}
$$

where $\bullet$ indicates irrelevant matrix block whose elements are multiplied by zero. The PDF of T can be obtained from the derivative of its CDF.

$$
\begin{aligned}
f_T(t) &= \frac{\mathrm{d}}{\mathrm{d}t}F_T(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(1 - \boldsymbol{\alpha}\sum_{i=0}^{\infty}\frac{t^i}{i!}\boldsymbol{A}^i\mathbf{1}\right) = -\boldsymbol{\alpha}\sum_{i=0}^{\infty}\frac{\mathrm{d}}{\mathrm{d}t}\frac{t^i}{i!}\boldsymbol{A}^i\mathbf{1} \\
&= -\boldsymbol{\alpha}\sum_{i=1}^{\infty}\frac{t^{i-1}}{(i-1)!}\boldsymbol{A}^{i-1}\boldsymbol{A}\mathbf{1} = -\boldsymbol{\alpha}\mathrm{e}^{\boldsymbol{A}t}\boldsymbol{A}\mathbf{1} = \boldsymbol{\alpha}\mathrm{e}^{\boldsymbol{A}t}\boldsymbol{a},
\end{aligned}
$$

where we used $\boldsymbol{a} = -\boldsymbol{A}\mathbf{1}$ in the last step.

Before computing the remaining properties of PH distributions we need to classify the eigenvalues of $\boldsymbol{A}$. The i, j element of matrix $\mathrm{e}^{\boldsymbol{A}t}$ contains the probability that starting from transient state i the Markov chain is in transient state j at time t. If states $1, \ldots, n$ are transient states then as t tends to infinity $\mathrm{e}^{\boldsymbol{A}t}$ tends to zero, which means that the eigenvalues of $\boldsymbol{A}$ have negative real part and, as a consequence, $\boldsymbol{A}$ is non-singular.

The Laplace transform of T, $E\left(\mathrm{e}^{-sT}\right)$, can be computed as

$$
\begin{aligned}
f_T^*(s) &= E\left(\mathrm{e}^{-sT}\right) = \int_{t=0}^{\infty}\mathrm{e}^{-st}f_T(t)\,\mathrm{d}t = \int_{t=0}^{\infty}\mathrm{e}^{-st}\boldsymbol{\alpha}\mathrm{e}^{\boldsymbol{A}t}\boldsymbol{a}\,\mathrm{d}t \\
&= \boldsymbol{\alpha}\int_{t=0}^{\infty}\mathrm{e}^{(-s\mathbf{I}+\boldsymbol{A})t}\,\mathrm{d}t\boldsymbol{a} = \boldsymbol{\alpha}(s\mathbf{I} - \boldsymbol{A})^{-1}\boldsymbol{a},
\end{aligned}
$$

where we note that the integral surely converges for $\mathcal{R}(s) \geq 0$ because in this case the eigenvalues of $-s\mathbf{I} + \boldsymbol{A}$ also possess a negative real part.

To compute the kth moment of T, $E\left(T^k\right)$, we need the following integral relation

$$
\left[t^k\mathrm{e}^{\boldsymbol{A}t}\right]_0^{\infty} = \int_{t=0}^{\infty}kt^{k-1}\mathrm{e}^{\boldsymbol{A}t}\mathrm{d}t + \int_{t=0}^{\infty}t^k\mathrm{e}^{\boldsymbol{A}t}\boldsymbol{A}\mathrm{d}t,
$$

whose left-hand side is zero because the eigenvalues of $\boldsymbol{A}$ possess a negative real part. Multiplying both side with $(-\boldsymbol{A})^{-1}$ we get

$$\int_{t=0}^{\infty} t^k \mathrm{e}^{\boldsymbol{A}t}\mathrm{d}t = k \int_{t=0}^{\infty} t^{k-1} \mathrm{e}^{\boldsymbol{A}t}\mathrm{d}t(-\boldsymbol{A})^{-1}.$$

Using this relation, the kth moment of T is

$$E\left(T^k\right) = \int_{t=0}^{\infty} t^k f_T(t)\mathrm{d}t = \boldsymbol{\alpha}\int_{t=0}^{\infty} t^k \mathrm{e}^{\boldsymbol{A}t}\mathrm{d}t(-\boldsymbol{A})\mathbf{1} = k\boldsymbol{\alpha}\int_{t=0}^{\infty} t^{k-1}\mathrm{e}^{\boldsymbol{A}t}\mathrm{d}t\mathbf{1}$$
$$= k(k-1)\boldsymbol{\alpha}\int_{t=0}^{\infty} t^{k-2}\mathrm{e}^{\boldsymbol{A}t}\mathrm{d}t(-\boldsymbol{A})^{-1}\mathbf{1} = \cdots = k!\boldsymbol{\alpha}(-\boldsymbol{A})^{-k}\mathbf{1}.$$

These four properties of PH distributions (CDF, PDF, Laplace transform, and moments) have several interesting consequences and some of which we summarize below.

- Matrix $(-\boldsymbol{A})^{-1}$ has an important stochastic meaning. Let T_{ij} be the time spent in transient state j before moving to the absorbing state when the Markov chain starts from state i. For $E\left(T_{ij}\right)$, we have

$$E\left(T_{ij}\right) = \frac{\delta_{ij}}{-\boldsymbol{A}_{ii}} + \sum_{k,k\neq i} \frac{\boldsymbol{A}_{ik}}{-\boldsymbol{A}_{ii}} E\left(T_{kj}\right),$$

  where δ_{ij} is the Kronecker delta symbol. The first term of the left-hand side is the time spent in state j while the Markov chain is in the initial state, and the second term is the time spent in state j during later visits to j. Multiplying both sides by $-\boldsymbol{A}_{ii}$ and adding $E\left(T_{ij}\right)\boldsymbol{A}_{ii}$ gives

$$0 = \delta_{ij} + \sum_k \boldsymbol{A}_{ik} E\left(T_{kj}\right),$$

  whose matrix form is

$$\mathbf{0} = \mathbf{I} + \boldsymbol{A}\overline{\mathbf{T}} \quad \longrightarrow \quad \overline{\mathbf{T}} = (-\boldsymbol{A})^{-1},$$

  where $\overline{\mathbf{T}}$ is the matrix composed of the elements $E\left(T_{ij}\right)$. Consequently, the (ij) element of $(-\boldsymbol{A})^{-1}$ is $E\left(T_{ij}\right)$, which is a nonnegative number.
- $f_T^*(s)$ is a rational function of s whose numerator is at most order $n-1$ and denominator is at most order n. This is because

$$f_T^*(s) = \boldsymbol{\alpha}(s\mathbf{I}-\boldsymbol{A})^{-1}\boldsymbol{a} = \sum_i\sum_j \boldsymbol{\alpha}_i(s\mathbf{I}-\boldsymbol{A})^{-1}_{ij}\boldsymbol{a}_j$$
$$= \sum_i\sum_j \boldsymbol{\alpha}_i\left[\frac{\det_{ji}(s\mathbf{I}-\boldsymbol{A})}{\det(s\mathbf{I}-\boldsymbol{A})}\right]\boldsymbol{a}_j = \frac{\sum_i\sum_j \boldsymbol{\alpha}_i\boldsymbol{a}_j\det_{ji}(s\mathbf{I}-\boldsymbol{A})}{\det(s\mathbf{I}-\boldsymbol{A})}.$$

$\det_{ji}(\mathbf{M})$ denotes the determinant of the matrix obtained by removing row j and column i of matrix $\mathbf{M}$. The denominator of the last expression is an order n polynomial of s, while the numerator is the sum of order $n-1$ polynomials, which is at most an order $n-1$ polynomial of s.

- This rational Laplace transform representation indicates that a PH distribution with n transient state can be represented by $2n-1$ independent parameters. A polynomial of order n is defined by $n+1$ coefficients, and a rational function of order $n-1$ numerator, and order n denominator is defined by $2n+1$ parameters. Normalizing the denominator such that the coefficient of s^n is 1 and considering that $\int_t f_T(t)\mathrm{d}t = \lim_{s\to 0} f_T^*(s) = 1$ adds two constraints for the coefficients, from which the number of independent parameters is $2n-1$.
- The PDF of a PH distribution is the sum of exponential functions. Let $\boldsymbol{A} = \mathbf{B}^{-1}\Lambda\mathbf{B}$ be the Jordan decomposition[1] of $\boldsymbol{A}$ and let $u = \boldsymbol{\alpha}\mathbf{B}^{-1}$ and $v = \mathbf{B}\boldsymbol{a}$. Then,

$$f_T(t) = \boldsymbol{\alpha} e^{\boldsymbol{A}t}\boldsymbol{a} = \boldsymbol{\alpha}\mathbf{B}^{-1}e^{\Lambda t}\mathbf{B}\boldsymbol{a} = ue^{\Lambda t}v\ .$$

  At this point, we distinguish two cases.

  - The eigenvalues of $\boldsymbol{A}$ are different and Λ is a diagonal matrix. In this case, $f_T(t)$ is a sum of exponential functions because

$$f_T(t) = u\mathrm{e}^{\Lambda t}v = \sum_i u_i v_i \mathrm{e}^{\lambda_i t} = \sum_i c_i \mathrm{e}^{\lambda_i t},$$

    where $c_i = u_i v_i$ is a constant coefficient of the exponential function.
    Here the eigenvalues (λ_i) as well as the associated coefficients (c_i) can be real or complex conjugate pairs. For a complex conjugate pair of eigenvalues, we have

$$c_i\mathrm{e}^{\lambda_i t} + \bar{c}_i\mathrm{e}^{\bar{\lambda}_i t} = 2|c_i|\mathrm{e}^{\mathcal{R}(\lambda_i)t}\cos(\mathcal{I}(\lambda_i)t - \varphi_i),$$

    where $c_i = |c_i|\mathrm{e}^{\imath\varphi_i}$, $\mathcal{R}(\lambda_i)$ and $\mathcal{I}(\lambda_i)$ are the real and the imaginary part of λ_i and $\imath$ is the imaginary unit.
  - There are eigenvalues of $\boldsymbol{A}$ with higher multiplicity and Λ contains real Jordan blocks. The matrix exponent of a Jordan block is

$$\exp\left[\begin{pmatrix} \lambda & 1 & & \\ & \lambda & 1 & \\ & & \ddots & \ddots \\ & & & \lambda \end{pmatrix} t\right] = \begin{pmatrix} \mathrm{e}^{\lambda t} & t\mathrm{e}^{\lambda t} & \frac{1}{2!}t^2\mathrm{e}^{\lambda t} & \frac{1}{3!}t^3\mathrm{e}^{\lambda t} \\ & \mathrm{e}^{\lambda t} & t\mathrm{e}^{\lambda t} & \frac{1}{2!}t^2\mathrm{e}^{\lambda t} \\ & & \ddots & \ddots \\ & & & \mathrm{e}^{\lambda t} \end{pmatrix}.$$

---

[1] The case of different Jordan blocks with identical eigenvalue is not considered here, because it cannot occur in non-redundant PH representations.

Consequently, the density function takes the form

$$f_T(t) = \sum_{i=1}^{\#\lambda} \sum_{j=1}^{\#\lambda_i} c_{ij} t^{j-1} \mathrm{e}^{\lambda_i t},$$

where $\#\lambda$ is the number of different eigenvalues and $\#\lambda_i$ is the multiplicity of λ_i. Similar to the previous case, the eigenvalues (λ_i) as well as the associated coefficients ($c_{i,j}$) can be real or complex conjugate pairs. For a complex conjugate pair of eigenvalues, we have

$$c_{i,j} t^{j-1} \mathrm{e}^{\lambda_i t} + \bar{c}_{i,j} t^{j-1} \mathrm{e}^{\bar{\lambda}_i t} = 2|c_{i,j}| t^{j-1} \mathrm{e}^{\mathcal{R}(\lambda_i)t} \cos(\mathcal{I}(\lambda_i)t - \varphi_{i,j}),$$

where $c_{i,j} = |c_{i,j}| \mathrm{e}^{\imath \varphi_{i,j}}$.

As a result of all of these cases, the density function of a PH distribution possesses the form

$$f_T(t) = \sum_{i=1}^{\#\lambda_R} \sum_{j=1}^{\#\lambda_i^R} c_{ij} t^{j-1} \mathrm{e}^{\lambda_i^R t} + \sum_{i=1}^{\#\lambda_C} \sum_{j=1}^{\#\lambda_i^C} 2|c_{i,j}| t^{j-1} \mathrm{e}^{\mathcal{R}(\lambda_i^C)t} \cos(\mathcal{I}(\lambda_i^C)t - \varphi_{i,j}) \tag{3}$$

where $\#\lambda_R$ is the number of different real eigenvalues and $\#\lambda_C$ is the number of different complex conjugate eigenvalue pairs.

- In general, infinitely many Markov chains can represent the same PH distribution.
  - The following *similarity transformation* generates representations with identical size.
    Let $\mathbf{T}$ be a non-singular matrix with unit row sums ($\mathbf{T}\mathbf{1} = \mathbf{1}$). The vector–matrix pairs $(\boldsymbol{\alpha}, \boldsymbol{A})$ and $(\boldsymbol{\alpha}\mathbf{T}, \mathbf{T}^{-1}\boldsymbol{A}\mathbf{T})$ are two different vector–matrix representations of the same PH distribution, since

    $$F_T(t) = 1 - \boldsymbol{\alpha}\mathbf{T}\mathrm{e}^{\mathbf{T}^{-1}\mathbf{A}\mathbf{T}t}\mathbf{1} = 1 - \boldsymbol{\alpha}\mathbf{T}\mathbf{T}^{-1}\mathrm{e}^{\boldsymbol{A}t}\mathbf{T}\mathbf{1} = 1 - \boldsymbol{\alpha}\mathrm{e}^{\boldsymbol{A}t}\mathbf{1}.$$

  - Representations with different sizes can be obtained as follows.
    Let matrix $\mathbf{V}$ of size $m \times n$ be such that $\mathbf{V}\mathbf{1} = \mathbf{1}$.
    The vector–matrix pairs $(\boldsymbol{\alpha}, \boldsymbol{A})$ of size n and $(\boldsymbol{\gamma}, \mathbf{G})$ of size m are two different vector–matrix representations of the same PH distribution if $\boldsymbol{A}\mathbf{V} = \mathbf{V}\mathbf{G}$ and $\boldsymbol{\alpha}\mathbf{V} = \boldsymbol{\gamma}$ because

    $$F_T(t) = 1 - \boldsymbol{\gamma}\mathrm{e}^{\mathbf{G}t}\mathbf{1} = 1 - \boldsymbol{\alpha}\mathbf{V}\mathrm{e}^{\mathbf{G}t}\mathbf{1} = 1 - \boldsymbol{\alpha}\mathrm{e}^{\boldsymbol{A}t}\mathbf{V}\mathbf{1} = 1 - \boldsymbol{\alpha}\mathrm{e}^{\boldsymbol{A}t}\mathbf{1}$$

    in this case.

3 Matrix Exponential Distributions and Their Basic Properties

In the definition of PH distributions, vector $\boldsymbol{\alpha}$ is a probability vector with nonnegative elements and matrix $\boldsymbol{A}$ is a generator matrix with negative diagonal and nonnegative off-diagonal elements. Relaxing these sign constraints for the vector and matrix elements and maintaining the matrix exponential distribution (and density) function results in the set of matrix exponential (ME) distributions.

Definition 2 Random variable T with distribution function

$$F_T(t) = 1 - \boldsymbol{\alpha} \mathrm{e}^{\boldsymbol{A}t} \mathbf{1},$$

where $\boldsymbol{\alpha}$ is a finite real vector and $\boldsymbol{A}$ is a finite real matrix, is matrix exponentially distributed.

The size of $\boldsymbol{\alpha}$ and $\boldsymbol{A}$ plays the same role as the number of transient states in case of PH distributions. By definition, the set of PH distributions with a given size is a subset of the set of PH distributions with the same size.

ME distributions share the following basic properties with PH distributions: matrix exponential distribution function, matrix exponential density function, moments, rational Laplace transform, the same set of functions as in (3), and non-unique representation. The main difference between the matrix exponential and the PH classes comes from the fact that the sign constraints on the elements of generator matrixes restrict the eigenvalue structure of such matrixes, while such restrictions do not apply in case of ME distributions. For example, the eigenvalues of an order three PH distribution with dominant eigenvalue θ satisfy $\mathcal{R}(\lambda_i) \leq \theta$ and $|\mathcal{I}(\lambda_i)| \leq (\theta - \mathcal{R}(\lambda_i)/\sqrt{3}$, while the eigenvalues of an order three ME distribution with dominant eigenvalue θ satisfy $\mathcal{R}(\lambda_i) \leq \theta$ only. This flexibility of the eigenvalues has significant consequence on the flexibility of the set of order three PH and ME distributions. For example, the minimal squared coefficient of variation among the order three PH and ME distributions are $1/3$ and 0.200902, respectively.

The main difficulty encountered when working with ME distributions is that a general vector–matrix pair does not always define a nonnegative density function, while a vector–matrix pair with the sign constraints of PH distributions does. Efficient numerical methods have been proposed recently to check the nonnegativity of a matrix exponential function defined by a general vector–matrix pair, but general symbolic conditions are still missing.

4 Analysis of Models with PH Distributed Durations

If all durations (service times, interarrival times, repair times, etc.) in a system are distributed according to PH distributions, then its overall behavior can be captured by a continuous time Markov chain, referred to as extended Markov chain (EMC).

In this section, we show how to derive the infinitesimal generator of this EMC using Kronecker operations. The methodology here described has been originally presented in the case of Discrete PHs in [17] and more recently in the case of Continuous PHs in [13]

To this end we first introduce the notation used to describe the model. By $\mathcal{S}$, we denote the set of states and by $N = |\mathcal{S}|$ the number of states. The states themselves are denoted by $s_1, s_2, ..., s_N$. The set of activities is denoted by $\mathcal{A}$ and the set of those that are active in state s_i is denoted by $\mathcal{A}_i$. The activities are denoted by $a_1, a_2, ..., a_M$ with $M = |\mathcal{A}|$. When activity a_i is completed in state s_j then the system moves from state s_j to state $n(j, i)$, i.e., n is the function that provides the next state. We assume that the next state is a deterministic function of the current state and the activity that completes. We further assume that there does not exist a triple, k, i, j, for which $s_k \in \mathcal{S}, a_i \in \mathcal{A}, a_j \in \mathcal{A}$ and $n(k, i) = n(k, j)$. These two assumptions, which make the formulas simpler, are easy to relax in practice. There can be activities that end when the system moves from state s_i to state s_j even if they do not complete and are active both in s_i and in s_j. These activities are collected in the set $e(i, j)$. The PH distribution that is associated with activity a_i is characterized by the initial vector $\boldsymbol{\alpha}_i$ and matrix $\boldsymbol{A}_i$. As before, we use the notation $\boldsymbol{a}_i = -\boldsymbol{A}_i\mathbf{1}$ to refer to the vector containing the intensities that lead to completion of activity a_i. The number of phases of the PH distribution associated with activity a_i is denoted by n_i.

Example 1 PH/PH/1/K queue with server break-downs. As an example, we consider, using the above-described notation, a queue in which the server is subject to failure only if the queue is not empty. The set of states is $\mathcal{S} = \{s_1, s_2, ..., s_{2K+1}\}$ where s_1 represents the empty queue, s_{2i} with $1 \leq i \leq K$ represents the state with i clients in the queue and the server up, and s_{2i+1} with $1 \leq i \leq K$ represents the state with i clients and the server down. There are four activities in the system: a_1 represents the arrival activity, a_2 the service activity, a_3 the failure activity,[2] and a_4 the repair activity. The vectors and matrices that describe the associated PH distributions are $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3, \boldsymbol{\alpha}_4$ and $\boldsymbol{A}_1, \boldsymbol{A}_2, \boldsymbol{A}_3, \boldsymbol{A}_4$. In this example, we assume that the arrival activity is active if the system is not full and it is inactive if the system is full. The service and the failure activities are active if the queue is not empty and the server is up. The repair activity is active if the queue is not empty and the server is down. Accordingly, we have $\mathcal{A}_1 = \{a_1\}$, $\mathcal{A}_{2i} = \{a_1, a_2, a_3\}$ for $1 \leq i \leq K - 1$, $\mathcal{A}_{2i+1} = \{a_1, a_4\}$ for $1 \leq i \leq K - 1$, $\mathcal{A}_{2K} = \{a_2, a_3\}$, and $\mathcal{A}_{2K+1} = \{a_4\}$. The next state function is as follows: for arrivals we have $n(1, 1) = s_2$ and $n(i, 1) = s_{i+2}$ with $2 \leq i \leq 2K - 1$; for services $n(2, 2) = s_1$ and $n(2i, 2) = s_{2i-2}$ with $2 \leq i \leq K$; for failures $n(2i, 3) = s_{2i+1}$ with $1 \leq i \leq K$; for repairs $n(2i + 1, 4) = s_{2i}$ with $1 \leq i \leq K$. We assume that the failure activity ends every time when a service activity completes, i.e., failure is connected to single jobs and not to the aging of the server. Other activities end only when they complete or when such a state is reached in which they are not active. Accordingly, $\mathrm{e}(2i, 2i - 2) = \{a_3\}$ for $2 \leq i \leq K$.

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[2] Failure is more like an event than an activity but, in order to keep the discussion clearer, we refer to it as failure activity.

Based on the description of the ingredients of the model, it is possible to derive blocks of the initial probability vector and the blocks of the infinitesimal generator of the corresponding CTMC. Let us start with the infinitesimal generator, which we denote by $\boldsymbol{Q}$, composed of $N \times N$ blocks. The block of $\boldsymbol{Q}$ that is situated in the ith row of blocks and in the jth column of blocks is denoted by $\boldsymbol{Q}_{ij}$. A block in the diagonal, $\boldsymbol{Q}_{ii}$ describes the parallel execution of the activities that are active in s_i. The parallel execution of CTMCs can be captured by the Kronecker-sum operator ($\oplus$), and thus we have

$$\boldsymbol{Q}_{ii} = \bigoplus_{j: s_j \in \mathcal{A}_i} \boldsymbol{A}_j \ .$$

An off-diagonal block, $\boldsymbol{Q}_{ij}$, is not a zero matrix only if there exists an activity whose completion moves the system from state s_i to state s_j. Let us assume that the completion of activity a_k moves the system from state s_i to state s_j, i.e., $n(i,k) = s_j$. The corresponding block, $\boldsymbol{Q}_{ij}$, must

- reflect the fact that activity a_k completes and restarts if a_k is active in s_j,
- reflect the fact that activity a_k completes and does not restart if a_k is not active in s_j,
- end activities that are active in s_i but not in s_j,
- start those activities that are not active in s_i but are active in s_j,
- end and restart those activities that are active both in s_i and in s_j but are in $\mathrm{e}(i,j)$,
- and maintain the phase of those that are active both in s_i and in s_j and are not in $\mathrm{e}(i,j)$.

The joint treatment of the above cases can be carried out by the Kronecker-product operator and thus we have

$$\boldsymbol{Q}_{ij} = \bigotimes_{l: 1 \le l \le M} \boldsymbol{R}_l$$

with

$$\boldsymbol{R}_l = \begin{cases} \boldsymbol{a}_k & \text{if } l = k \text{ and } k \notin \mathcal{A}_j \\ \boldsymbol{a}_k \boldsymbol{\alpha}_k & \text{if } l = k \text{ and } k \in \mathcal{A}_j \\ \mathbf{1}_{n_l} & \text{if } l \neq k \text{ and } k \in \mathcal{A}_i \text{ and } k \notin \mathcal{A}_j \\ \boldsymbol{\alpha}_l & \text{if } l \neq k \text{ and } k \notin \mathcal{A}_i \text{ and } k \in \mathcal{A}_j \\ \mathbf{1}_{n_l} \boldsymbol{\alpha}_l & \text{if } l \neq k \text{ and } k \in \mathcal{A}_i \text{ and } k \in \mathcal{A}_j \text{ and } k \in e(i,j) \\ \mathbf{I}_{n_l} & \text{if } l \neq k \text{ and } k \in \mathcal{A}_i \text{ and } k \in \mathcal{A}_j \text{ and } k \notin e(i,j) \\ 1 & \text{otherwise} \end{cases}$$

where the subscripts to $\mathbf{1}$ and $\mathbf{I}$ indicate their size.

The initial probability vector of the CTMC, $\boldsymbol{\pi}$, is a row vector composed of N blocks which must reflect the initial probabilities of the states of the system and the initial probabilities of the PH distributions of the active activities. Denoting by π_i the initial probability of state s_i, the ith block of the initial probability vector, $\boldsymbol{\pi}_i$, is given as

$$\boldsymbol{\pi}_i = \bigotimes_{j:s_j \in \mathcal{A}_i} \boldsymbol{\alpha}_j \, .$$

Example 2 For the previous example, the diagonal blocks, which must reflect the ongoing activities, are the following:

$$\begin{aligned}
&\boldsymbol{Q}_{1,1} = \boldsymbol{A}_1, \;\; \boldsymbol{Q}_{2i,2i} = \boldsymbol{A}_1 \bigoplus \boldsymbol{A}_2 \bigoplus \boldsymbol{A}_3, \;\; \boldsymbol{Q}_{2i+1,2i+1} = \boldsymbol{A}_1 \bigoplus \boldsymbol{A}_4, \\
&\boldsymbol{Q}_{2K,2K} = \boldsymbol{A}_2 \bigoplus \boldsymbol{A}_3, \;\; \boldsymbol{Q}_{2K+1,2K+1} = \boldsymbol{A}_4 \quad \text{with } 1 \le i \le K-1
\end{aligned}$$

Arrival in state s_1 takes the system to state s_2. The corresponding block must complete and restart the arrival activity and must restart both the service and the failure activity:

$$\boldsymbol{Q}_{12} = \boldsymbol{a}_1 \boldsymbol{\alpha}_1 \bigotimes \boldsymbol{\alpha}_2 \bigotimes \boldsymbol{\alpha}_3 \tag{4}$$

Arrival in state s_{2i} (server up) takes the system to state s_{2i+2}. If the system does not become full then the corresponding block must complete and restart the arrival activity and must maintain the phase of both the service and the failure activity. If the system becomes full, the arrival activity is not restarted. Accordingly, we have

$$\begin{aligned}
\boldsymbol{Q}_{2i,2i+2} &= \boldsymbol{a}_1 \boldsymbol{\alpha}_1 \bigotimes \mathbf{I}_{n_2} \bigotimes \mathbf{I}_{n_3} \quad \text{with } 1 \le i \le K-2 \\
\boldsymbol{Q}_{2K-2,2K} &= \boldsymbol{a}_1 \bigotimes \mathbf{I}_{n_2} \bigotimes \mathbf{I}_{n_3}
\end{aligned}$$

An arrival in state s_{2i+1} (server down) takes the system to state s_{2i+3}. If the system does not become full then the corresponding block must complete and restart the arrival activity and must maintain the phase of the repair activity. If the system becomes full, the arrival activity is not restarted. Accordingly, we have

$$\begin{aligned}
\boldsymbol{Q}_{2i+1,2i+3} &= \boldsymbol{a}_1 \boldsymbol{\alpha}_1 \bigotimes \mathbf{I}_{n_4} \quad \text{with } 1 \le i \le K-2 \\
\boldsymbol{Q}_{2K-1,2K+1} &= \boldsymbol{a}_1 \bigotimes \mathbf{I}_{n_4}
\end{aligned}$$

Service completion can take place in three different situations. If the system becomes empty then the phase of the arrival activity is maintained, the service activity is completed and the failure activity is put to an end. If the system neither becomes empty nor was full then the phase of the arrival activity is maintained, the service activity is completed and restarted, and the failure activity ends and restarts. Finally, if the queue was full then the arrival activity is restarted, the service activity is completed and restarted, and the failure activity is put to an end and restarted. Accordingly, we have

$$\boldsymbol{Q}_{2,1} = \mathbf{I}_{n_1} \bigotimes \boldsymbol{a}_2 \bigotimes \mathbf{1}_{n_3}$$

$$Q_{2i,2i-1} = \mathbf{I}_{n_1} \bigotimes \boldsymbol{a}_2 \boldsymbol{\alpha}_2 \bigotimes \mathbf{1}_{n_3} \boldsymbol{\alpha}_3 \quad \text{with } 1 < i < K$$

$$Q_{2K,2K-2} = \boldsymbol{\alpha}_1 \bigotimes \boldsymbol{a}_2 \boldsymbol{\alpha}_2 \bigotimes \mathbf{1}_{n_3} \boldsymbol{\alpha}_3 \tag{5}$$

The failure activity can be completed in two different situations. If the system is not full, then the phase of the arrival activity is maintained. If the system is full then the arrival activity is not active. In both cases, the service activity ends, the failure activity is completed and the repair activity is initialized.

$$Q_{2i,2i+1} = \mathbf{I}_{n_1} \bigotimes \mathbf{1}_{n_2} \bigotimes \boldsymbol{a}_3 \bigotimes \boldsymbol{\alpha}_4 \quad \text{with } 1 \le i < K$$

$$Q_{2K,2K+1} = \mathbf{1}_{n_2} \bigotimes \boldsymbol{a}_3 \bigotimes \boldsymbol{\alpha}_4$$

Similarly to the failure activity, also the repair activity can be completed in two different situations because the arrival activity can be active or inactive. In both cases, the service activity and the failure activity must be initialized and the repair activity completes.

$$Q_{2i+1,2i} = \mathbf{I}_{n_1} \bigotimes \boldsymbol{\alpha}_2 \bigotimes \boldsymbol{\alpha}_3 \bigotimes \boldsymbol{a}_4 \quad \text{with } 1 \le i < K$$

$$Q_{2K+1,2K} = \boldsymbol{\alpha}_2 \bigotimes \boldsymbol{\alpha}_3 \bigotimes \boldsymbol{a}_4$$

5 Analysis of Stochastic Systems with ME Distributed Durations

The most important observation to take from this section is that all steps of the method of EMCs (as explained in the previous section) remain directly applicable in case of ME distributed durations (where the $(\boldsymbol{\alpha}_i, \boldsymbol{A}_i)$ vector–matrix pairs describe ME distributions). In that case, the only difference is that the signs of the vector and matrix elements are not restricted to be nonnegative in case of the vector elements and off-diagonal matrix elements and to be negative in case of the diagonal matrix elements. Consequently, the model description does not allow a probabilistic interpretation via Markov chains.

This general conclusion was obtained through serious research efforts. Following the results in [12], it was suspected that in a stochastic model ME distributions could be used in place of PH distributions and several results would carry over, but it was not easy to prove these conjectured results in the general setting because probabilistic arguments associated with PH distributions no longer hold. In [1], it was shown that matrix geometric methods can be applied for quasi-birth–death processes (QBDs) with rational arrival processes (RAPs) [3], which can be viewed as an extension of ME distributions to arrival processes. To prove that the matrix geometric relations hold, the authors of [1] use an interpretation of RAPs proposed in [3]. However, the models considered are limited to QBDs. For the model class of SPNs with ME distributed firing times, the applicability of the EMC-like analysis was proved in [2] and refined for the special case when the ME distribution has no PH representation in [4].

6 Analysis tools

Based on the common representation of the EMC through the Kronecker algebra, smart algorithms have been developed recently to optimize memory usage. These algorithms build the EMC in a completely symbolic way, both at the process state space level and at the expanded state space level, as deeply explained in [13] that we use as reference.

The algorithm presented in [13] is based on two high level steps:

1. to generate the reachability graph of the model (which collects the system states in a graph according to their reachability from an initial set of states) using a symbolic technique;
2. to enrich the symbolically stored reachability graph with all the necessary information to evaluate Kronecker expressions representing the expanded state space.

Step 1 is performed using symbolic technique based on complex data structures like Multi-Valued Decision Diagram (MDD) [18] to encode the model state space; step 2 adds information related to each event memory policy to the encoded state space. In manner it is possible to use on the fly expressions introduced in Sects. 4 and 5 to compute various probability measures of the model.

6.1 Symbolic Generation of Reachability Graph

Both traditional performance or dependability evaluation techniques and more recent model checking-based approaches are grounded in the knowledge of the set of states that the system considered can reach starting from a particular initial state (or in general from a set of initial states). Symbolic techniques [5] focus on generating a compact representation of huge state spaces by exploiting a model's structure and regularity. A model has a structure when it is composed of K sub-models, for some $K \in \mathbb{N}$. In this case, a global system state can be represented as a K-tuple $(q^1, \ldots, q^K)$, where q^k is the local state of sub-model k (having some finite size n^k).

The use of (MDDs) for the encoding of model state spaces was introduced by Miner and Ciardo in [14]. MDDs are rooted, directed, acyclic graphs associated with a finite ordered set of integer variables. When used to encode a state space, an MDD has the following structure:

- nodes are organized into $K+1$ levels, where K is the number of sub-models;
- level K contains only a single non-terminal node, the root, whereas levels $K-1$ through 1 contain one or more non-terminal nodes;
- a non-terminal node at level k has n^k arcs pointing to nodes at level $k-1$;

A state $s=(q^1,\ldots,q^K)$ belongs to S if and only if a path exists from the root node to the terminal node 1 such that, at each node, the arc corresponding to the local state q^k is followed. In [6], and then in [7], Ciardo et al. proposed the *Saturation* algorithm for the generation of reachability graphs using MDDs. Such an iteration strategy improves both memory and execution time efficiency.

An efficient encoding of the reachability graph is built in the form of a set of Kronecker matrices $\mathbf{W}_{e,k}$ with $e \in \mathcal{A}$ and $k=1,\ldots,K$, where $\mathcal{A}$ is the set collecting all the system events or activities. $\mathbf{W}_{e,k}[i_k, j_k] = 1$ if state j_k of sub-model k is reachable from state i_k due to event e. According to such a definition, the next state function of the model can be encoded as the incidence matrix given by the boolean sum of Kronecker products $\sum_{e\in\mathcal{A}} \bigotimes_{K\geq k\geq 1} \mathbf{W}_{e,k}$. As a consequence, the matrix representation $\mathbf{R}$ of the reachability graph of the model can be obtained by filtering the rows and columns of such a matrix corresponding to the reachable global states encoded in the MDD and replacing each non-null element with the labels of the events that cause the corresponding state transition.

Saturation Unbound is a very effective way to represent the model state space and the related reachability graph of a model. In any case, the methodology we are dealing with is not strictly dependent on any particular algorithm to efficiently store the reachability graph. We refer to the *Saturation Unbound* algorithm simply because its efficiency is well known [7].

6.2 Annotating the Reachability Graph

The use of Saturation together with the Kronecker representation presented in previous sections enable solution of the derived stochastic process. However, knowledge of the reachability graph of the untimed system as produced by Saturation is not sufficient to manage the infinitesimal generator matrix $\mathbf{Q}$ on the fly according to the symbolic representation. Considering that the information about the enabled events for all the system states is contained in the high level description of the model and it can be evaluated on the fly when needed with a negligible overhead, the only additional information needed is knowledge about the sets of active but not enabled events in each state s ($T_a^{(s)}$). Using Saturation for the evaluation of the reachability graph requires an additional analysis step for the computation of such an information and use of a different data structure for storage. Multi Terminal Multi-Valued Decision Diagram (MTMDD) [15] is used for this purpose.

The main differences with respect to MDDs are that: (1) more than two terminal nodes are present in an MTMDD and (2) such nodes can be labeled with arbitrary integer values, rather than just 0 and 1. An MTMDD can efficiently store both the system state space S and the sets $T_a^{(s)}$ of active but not enabled events for all $s \in \mathcal{S}$; this is necessary, in our approach, to correctly evaluate non-null blocks of $\mathbf{Q}$ matrix. In fact, while an MDD is only able to encode a state space, an MTMDD is also able to associate an integer to each state. Thus, the encoding of sets $T_a^{(s)}$ can be done associating to each possible set of events an integer code that unambiguously represents it. Let us associate to each event an unique index n such that $1 \leq n \leq \|\mathcal{A}\|$. Then the integer value associated to one of the possible sets $T_a^{(s)}$ is computed starting from the indices associated with the system events that belong to it in the following way:

$$b_M \cdot 2^A + \cdots + b_n \cdot 2^n + \cdots b_1 \cdot 2^1 + 1 = \textstyle\sum_{i=1}^{M} b_i 2^i + 1$$

where

$$b_i = \begin{cases} 1, & \text{if event } e_i \in T_a^{(s)} \\ 0, & \text{otherwise} \end{cases}$$

In this manner all the necessary information to apply the Kronecker-based expressions on the fly are provided; the only remaining need is a method to evaluate the set $T_a^{(s)}$ given a referring state s.

In [13], the following theorem has been proved.

Theorem 1 *Given a model $\mathcal{M}$, a state $s_0 \in S$ and an event $\overline{e} \in \mathcal{A}$ with an age memory policy associated, then $\overline{e} \in T_a^{(s_0)}$ iff $\overline{e} \notin T_e^{(s_0)}$ and one of the following statements holds:*

1. $\exists\, s_1 \in \mathcal{S},\ \exists\, e_1 \in \mathcal{A}, s_1 \neq s_0,\ e_1 \neq \overline{e} \mid s_0 \in \mathcal{N}_{e_1}(s_1)\ \wedge\ \overline{e} \in T_e^{(s_1)}$
2. $\exists\, s_1 \in \mathcal{S}, s_1 \neq s_0 \mid s_0 \in \mathcal{N}(s_1)\ \wedge\ \overline{e} \in T_a^{(s_1)}$

where $\mathcal{N}_{e_1}$ is the next state function associated to event e_1.

Note that function $\mathcal{N}$ is the equivalent to the $n(\cdot,\cdot)$ defined in Sect. 4; function $\mathcal{N}_e$ instead differs for the restriction to the firing of a specific event e. We use this notation because it is less cumbersome in this specific context.

Theorem 1 gives a way to evaluate if an event e belongs to the set $T_a^{(s_0)}$ or not. In fact, according to the statements proved, it is possible to characterize a state s_0 with respect to the system event memory policies by exploring its reachability graph. Exploration can be performed using classical bread-first search and depth-first search algorithms, easily applicable to an explicitly stored reachability graph; it is more complicated to apply classical search algorithms when the graph is stored in implicit manner as is the case when MTMDD data structures are used.

In this case, a different approach can be used by resorting to Computational Tree Logic (CTL) formulas that have been shown to be very efficient for data structures like MDD and MTMDD. The use of CTL formulas to evaluate sets $T_a^{(s)}$ is justified by a theorem introduced in [13]. Before recalling this theorem, we need to introduce a CTL operator.

Definition 3 Let $s_0 \in \mathcal{S}$ be a state of a discrete state process with state space $\mathcal{S}$, and let p and q be two logical conditions on the states. Let also $\mathcal{F}(s) \subseteq \mathcal{N}(s) \cup \mathcal{N}^{-1}(s)$ be a reachability relationship between two states in $\mathcal{S}$ that defines a desired condition over the paths. Then s_0 satisfies the formula $E_{\mathcal{F}}[pUq]$, and we will write $s_0 \vDash E_{\mathcal{F}}[pUq]$, iff $\exists\ n \geq 0, \exists\ s_1 \in \mathcal{F}(s_0), \ldots, \exists\ s_n \in \mathcal{F}(s_{n-1}) \mid (s_n \vDash q) \wedge (\forall m < n, s_m \vDash p)$.

In definition above, we used the path quantifier E with the meaning *there exists a path* and the tense operator U with the meaning *until*, as usually adopted in CTL formulas.

Given Definition 3, the following theorem holds:

Theorem 2 *An event $\overline{e} \in \mathcal{E}$, with an age memory policy associated, belongs to $T_a^{(s_0)}$, with $s_0 \in \mathcal{S}$, iff $s_0 \vDash E_{\mathcal{F}}[pUq]$ over a path at least one long, where p and q are the statements "$\overline{e}$ is not enabled" and "$\overline{e}$ is enabled," respectively, and $\mathcal{F}(s) = \mathcal{N}^{-1}(s) \setminus \mathcal{N}_{\overline{e}}^{-1}(s)$.*

Thanks to Theorem 2, evaluation of the CTL formula $E_{\mathcal{F}}[pUq]$ makes possible to evaluate whether an event $\overline{e}$ is active but not enabled in state s_0 or not by setting condition p as $\overline{e}$ *is not enable* and q as $\overline{e}$ *is enabled*. This is the last brick to build an algorithm able to compute state probabilities of a model, where the event are PH or ME distributed; in fact, it is possible to characterize all the active and/or enabled events in all the different states and to apply the Kronecker expressions with this information to solve the derived EMC.

7 Examples

In this section, we present two examples where non-exponentially distributed durations are present. In the first example, these durations are approximated by PH distributions, while in the second example they are described by ME distributions.

7.1 Reliability Model of Computer System

We introduce a reliability model where we use PH distributions as failure times. The model is specified according to the Petri net depicted in Fig. 1, where the usual graphical notation for the places, transitions, and arcs has been adopted.

The system under study is a distributed computing system composed of a cluster of two computers. Each of them has three main weak points: the motherboard, CPU, and disk. Interconnections inside the cluster are provided by a manager in such a way that the overall system is seen as a single unit. In the distributed system, the two computers work independently, driven by the manager that acts as a load balancer to split the work between them. Since the manager represents a single point of failure, a second instance is deployed for redundancy in the system; this latter instance operates in cold standby when the main computer manager works and it is powered on when it fails.

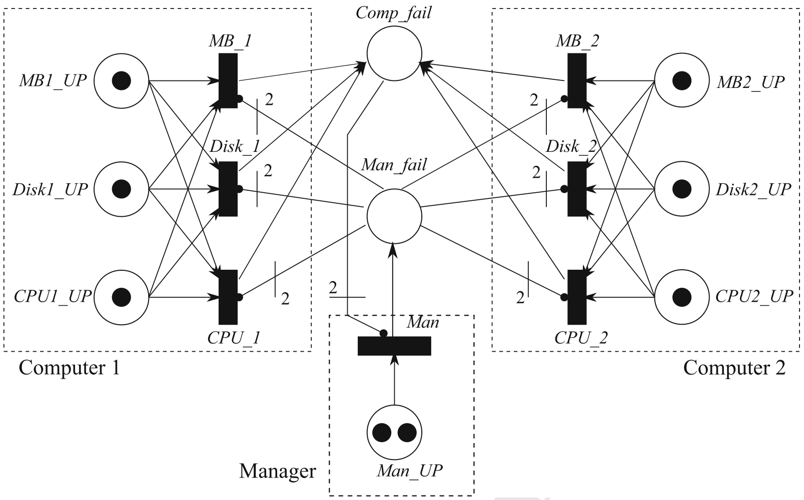


Fig. 1 Computer system reliability model

Due to this configuration, the distributed system works when at least one of the two computers works and the computer manager properly operates. The main components of each computational unit (CPU, motherboard, and disk) may fail rendering the unit inoperable. In the Petri net model, faults in the CPU, motherboard, and disk are modeled by the timed transitions MB_i, $Disk\_i$, and CPU_i whose firing represents the respective faulty event in the i-th Computer; the operating conditions of components are represented by a token in the places $CPUi\_UP$, MBi_UP, and $Diski\_UP$. When one of the transitions above fires a token is flushed out of the place and a token is put in the place $Comp\_fail$. At the same time, all the other transitions related to the faulty events in the same unit become disabled because the unit is considered down and thus no more faults can occur. Two tokens in the place $Comp\_fail$ means that the two computational units are both broken and the overall distributed system is not operational. Similarly, transition Man models the fault of a manager unit. Its firing flushes a token out of the place Man_UP and puts a token in the place Man_fail. Thanks to the redundancy, the first manager unit fault is tolerated whereas the system goes down when a second fault occurs. This state is represented in the Petri net by two tokens in the place Man_fail. In both faulty states, all the transitions are disabled and an absorbing state is reached. In terms of Petri net objects, the not operational condition is expressed by the following statement:

$$(\#Comp\_fail = 2) \vee (\#Man\_fail = 2) \ , \tag{6}$$

where the symbol $\#P$ states the number of token in place P.

Table 1 Failure time distribution parameters

	Weibull			
Transition	β_f	η_f	E	λ
MB_1, MB_2	0.5965	1.20	1.82	0.55
$Disk\_1$, $Disk\_2$	0.5415	1.00	1.71	0.59
CPU_1, CPU_2	0.399	1.46	3.42	0.29
Man	0.5965	1.20	1.82	0.55

As usual in reliability modeling, the time to failure of the components has been modeled using Weibull distributions whose cumulative distribution function is

$$F(t) = 1 - e^{(1/\eta_f)^{\beta_f}} .$$

This choice has been also supported by measures done on real systems such as those analyzed in [9]. The parameters of the Weibull distributions used for the Petri net transitions of Fig. 1 are reported in Table 1.

Weibull distributions have been introduced in the model through the use of 10-phase PH distributions, approximating them by evaluating the formula (6). The results obtained are depicted in Fig. 2. To better highlight the usefulness of the modeling approach presented here, the Petri net model was solved by imposing exponential distributions as transition firing times. In fact, the use of exponential distributions is quite common to obtain a more tractable model. The value of the parameters λ used in this second run was computed as the reciprocal of the expected value, E, of the corresponding Weibull distributions (listed in Table 1). The result obtained are also depicted in Fig. 2. As can be easily noted, the use of exponential distributions produces optimistic results compared to the use of Weibull distributions, making the system appear more reliable than it is in reality.

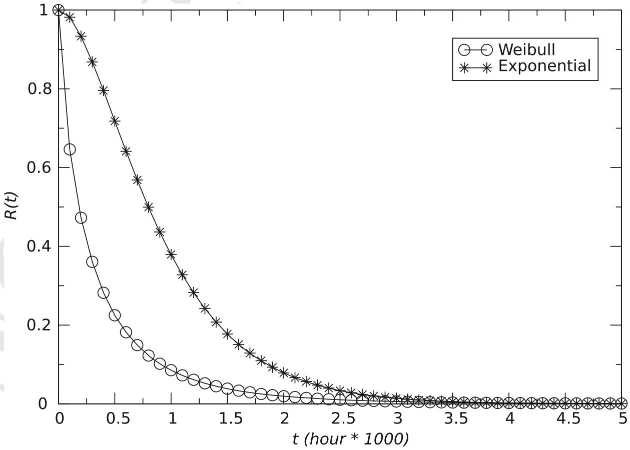


Fig. 2 Computer system reliability $R(t)$

7.2 Numerical Example with "Oscillating" ME Distribution

For our second example, we consider the Activity Network depicted in Fig. 3, which represents a "mission" composed of five activities and the constraints on the order in which the five activities can be carried out. Initially, activities 1 and 2 are active. If activity 1 finishes then activities 3 and 4 start and thus there are three activities under execution, namely, activities 2, 3, and 4. If activity 3 is the first first among these three activities to finish then no new activity starts because in order to start activity 5, both activity 2 and 3 must finish. The graph of all the possible states of the Activity Network is shown in Fig. 4, where in every node we report the activities that are under execution in the node. The label on the edges indicates the activity whose completion triggers the edge. The duration of the activities are modeled with ME distributions and we denote the vector and matrix that represent the duration of activity i by $\boldsymbol{\alpha}_i$ and $\boldsymbol{A}_i$, respectively. Further, we use the notation $\boldsymbol{a}_i = (-\boldsymbol{A}_i)\mathbf{1}$ and denote by I_i the identity matrix whose dimension is equal to that of $\boldsymbol{A}_i$.

Following the approach described in Sect. 4, one can determine the infinitesimal generator of the model. Its first seven block-columns are given as (the left side of the matrix)

$$\left|\begin{array}{ccccccc}
\boldsymbol{A}_1\bigoplus\boldsymbol{A}_2 & \boldsymbol{a}_1\bigotimes I_2\bigotimes\boldsymbol{A}_3\bigotimes\boldsymbol{A}_4 & 0 & 0 & I_1\bigotimes\boldsymbol{a}_2 & 0 & 0 \\
0 & \boldsymbol{A}_2\bigoplus\boldsymbol{A}_3\bigoplus\boldsymbol{A}_4 & I_2\bigotimes I_3\bigotimes\boldsymbol{a}_4 & 0 & 0 & I_2\bigotimes\boldsymbol{a}_3\bigotimes I_4 & \boldsymbol{a}_2\bigotimes I_3\bigotimes I_4 \\
0 & 0 & \boldsymbol{A}_2\bigoplus\boldsymbol{A}_3 & I_2\bigotimes\boldsymbol{a}_3 & 0 & 0 & 0 \\
0 & 0 & 0 & \boldsymbol{A}_2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \boldsymbol{A}_1 & 0 & \boldsymbol{a}_1\bigotimes\boldsymbol{A}_3\bigotimes\boldsymbol{A}_4 \\
0 & 0 & 0 & I_2\bigotimes\boldsymbol{a}_4 & 0 & \boldsymbol{A}_2\bigoplus\boldsymbol{A}_4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \boldsymbol{A}_3\bigoplus\boldsymbol{A}_4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right.$$

and the remaining five block-columns are (the right side of the matrix)

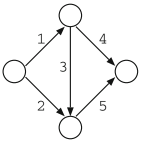


Fig. 3 An activity network

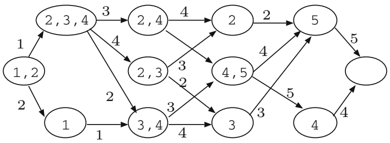


Fig. 4 CTMC of the activity network in Fig. 3

$$
\begin{array}{ccccc|}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\boldsymbol{a}_2 \bigotimes I_3 & 0 & 0 & 0 & 0 \\
0 & \boldsymbol{a}_2 \bigotimes \boldsymbol{A}_5 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & \boldsymbol{a}_2 \bigotimes I_4 \bigotimes \boldsymbol{A}_5 & 0 & 0 \\
I_3 \bigotimes \boldsymbol{a}_4 & 0 & \boldsymbol{a}_3 \bigotimes I_4 \bigotimes \boldsymbol{A}_5 & 0 & 0 \\
\boldsymbol{A}_3 & \boldsymbol{a}_3 \bigotimes \boldsymbol{A}_5 & 0 & 0 & 0 \\
0 & \boldsymbol{A}_5 & 0 & 0 & \boldsymbol{a}_5 \\
0 & \boldsymbol{a}_4 \bigotimes I_5 & \boldsymbol{A}_4 \bigoplus \boldsymbol{A}_5 & I_4 \bigotimes \boldsymbol{a}_5 & 0 \\
0 & 0 & 0 & \boldsymbol{A}_4 & \boldsymbol{a}_4 \\
0 & 0 & 0 & 0 & 0
\end{array}
$$

The vector that provides the initial configuration is $|\boldsymbol{A}_1 \otimes \boldsymbol{A}_2, 0, ..., 0|$.

In order to illustrate a feature of ME distributions that cannot be exhibited by PH distributions, we applied an ME distribution with "oscillating" PDF to describe the duration of activities 1, 2, 4, and 5. The vector–matrix pair of this ME distribution is

$$\boldsymbol{A}_1 = \boldsymbol{A}_2 = \boldsymbol{A}_4 = \boldsymbol{A}_5 = |1.04865, -0.0340166, -0.0146293| \,,$$

$$\boldsymbol{A}_1 = \boldsymbol{A}_2 = \boldsymbol{A}_4 = \boldsymbol{A}_5 = \begin{vmatrix} -1 & 0 & 0 \\ 0 & -1 & -20 \\ 0 & 20 & -1 \end{vmatrix} \,,$$

and its PDF is depicted in Fig. 5. The duration of the remaining activity, namely activity 3, is distributed according to an Erlang distribution with four phases and average execution time equal to 1, i.e.,

$$\boldsymbol{A}_3 = |1, 0, 0, 0| \,, \quad \boldsymbol{A}_3 = \frac{1}{4} \begin{vmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{vmatrix} \,.$$

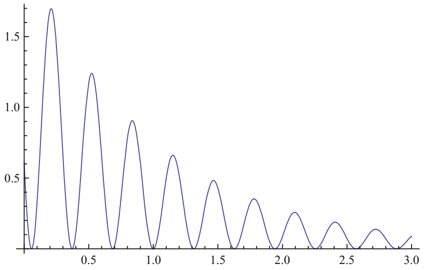
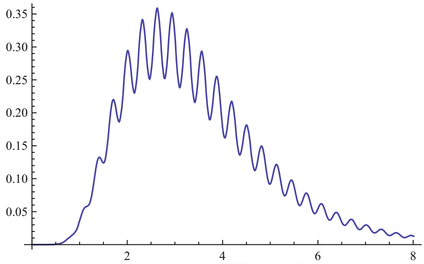


Fig. 5 Oscillating activity duration pdf

Fig. 6 Overall accomplishment time pdf



The model was then used to characterize the PDF of the time that is needed to accomplish the whole mission. The resulting PDF is shown in Fig. 6 and one can observe that the oscillating nature of the distribution of the activity durations carries over into the overall completion time distribution.

8 Conclusions

While the evolution of computing devices and analysis methods resulted in a sharp increase in the complexity of computable CTMC models, CTMC-based analysis had been restricted to the analysis of stochastic models with exponentially distributed duration times. A potential extension of CTMC-based analysis is the inclusion of PH distributed duration times, which enlarges the state space, but still has a feasible computational complexity. We surveyed the basics of PH distributions and the analysis approach to generate the EMC.

A more recent development in this field is the extension of the EMC-based analysis with ME distributed duration times. With respect to the steps of the analysis method, the EMC-based analysis and its extension with ME distributions are identical. However, because ME distributions are more flexible than the PH distributions (more precisely, the set of PH distributions of a given size is a subset of the set of ME distributions of the same size) this extension increases the modeling flexibility of the set of models which can be analyzed with a given computational complexity.

Apart of the steps of the EMC-based analysis method, we discussed the tool support available for the automatic execution of the analysis method. Finally, application examples demonstrate the abilities of the modeling and analysis methods.

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Chapter 1

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