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# Erratum to: A $\mathbb{Q}$ -factorial complete toric variety is a quotient of a poly weighted space

Michele Rossi · Lea Terracini

After the publication of [2], we realized that Proposition 3.1, in that paper, contains an error, whose consequences are rather pervasive along the whole section 3 and for some aspects of examples 5.1 and 5.2. Here we give a complete account of needed corrections.

First of all [2, Prop. 3.1] has to be replaced by the following:

**PROPOSITION 3.1** *Let  $X(\Sigma)$  be a  $\mathbb{Q}$ -factorial complete toric variety and  $Y(\widehat{\Sigma})$  be its universal 1-covering. Let  $\{D_\rho\}_{\rho \in \Sigma(1)}$  and  $\{\widehat{D}_\rho\}_{\rho \in \widehat{\Sigma}(1)}$  be the standard bases of  $\mathcal{W}_T(X)$  and  $\mathcal{W}_T(Y)$ , respectively, given by the torus orbit closures of the rays. Then*

$$D = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho \in \mathcal{C}_T(X) \implies \widehat{D} = \sum_{\rho \in \widehat{\Sigma}(1)} a_\rho \widehat{D}_\rho \in \mathcal{C}_T(Y).$$

Therefore, under the identification  $\mathbb{Z}^{|\Sigma(1)|} \cong \mathcal{W}_T(X) \xrightarrow{\alpha} \mathcal{W}_T(Y) \cong \mathbb{Z}^{|\widehat{\Sigma}(1)|}$  realized by the isomorphism  $D_\rho \xrightarrow{\alpha} \widehat{D}_\rho$ ,

$$\mathcal{C}_T(X) \cong \alpha(\mathcal{C}_T(X)) \leq \mathcal{C}_T(Y) \leq \mathcal{W}_T(Y)$$

is a chain of subgroup inclusions. Moreover the induced morphism  $\bar{\alpha} : \text{Cl}(X) \rightarrow \text{Cl}(Y)$  is injective when restricted to  $\text{Pic}(X)$ , realizing the following further chain of subgroup inclusions

$$\text{Pic}(X) \cong \bar{\alpha}(\text{Pic}(X)) \leq \text{Pic}(Y) \leq \text{Cl}(Y)$$

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*Proof:* Let us fix a basis  $\mathcal{B}$  of the  $\mathbb{Z}$ -module  $M \cong \mathbb{Z}^n$  and let  $V$  and  $\widehat{V}$  be fan matrices representing the standard morphisms

$$\operatorname{div}_X : M \cong \mathbb{Z}^n \xrightarrow{V^T} \mathbb{Z}^{|\Sigma(1)|} \cong \mathcal{W}_T(X) \quad , \quad \operatorname{div}_Y : M \cong \mathbb{Z}^r \xrightarrow{\widehat{V}^T} \mathbb{Z}^{|\widehat{\Sigma}(1)|} \cong \mathcal{W}_T(Y)$$

Let  $\beta \in \operatorname{GL}_n(\mathbb{Q}) \cap \mathbf{M}_n(\mathbb{Z})$  be such that  $V = \beta \widehat{V}$  and so realizing an injective endomorphism of the  $\mathbb{Z}$ -module  $M$ . The result follows by writing down the condition of being locally principal for a Weil divisor and observing that

$$\begin{aligned} \mathcal{I}^\Sigma &= \{I \subseteq \{1, \dots, n+r\} : \langle V^I \rangle \in \Sigma(n)\} \\ &= \{I \subseteq \{1, \dots, n+r\} : \langle \widehat{V}^I \rangle \in \widehat{\Sigma}(n)\} = \mathcal{I}^{\widehat{\Sigma}} \end{aligned} \quad (1)$$

by the construction of  $\widehat{\Sigma} \in \mathcal{SF}(\widehat{V})$ , given the choice of  $\Sigma \in \mathcal{SF}(V)$ . Notice that  $\mathcal{I}^\Sigma$  describes the complements of those sets described by  $\mathcal{I}_\Sigma$ , as defined in [2, Rem. 2.4]. In particular the Weil divisor  $\sum_{j=1}^{n+r} a_j D_j \in \mathcal{W}_T(X)$  is Cartier if and only if

$$\forall I \in \mathcal{I}^\Sigma \quad \exists \mathbf{m}_I \in M : \forall j \notin I \quad \mathbf{v}_j^T \mathbf{m}_I = a_j, \quad (2)$$

where  $\mathbf{v}_j$  is the  $j$ -th column of  $V$ . Then  $\alpha(\sum_{j=1}^{n+r} a_j D_j) = \sum_{j=1}^{n+r} a_j \widehat{D}_j$  is a Cartier divisor since

$$\forall I \in \mathcal{I}^\Sigma \quad \forall j \notin I \quad \widehat{\mathbf{v}}_j^T (\beta^T \mathbf{m}_I) = a_j$$

where  $\widehat{\mathbf{v}}_j$  is the  $j$ -th column of  $\widehat{V}$ .

The injectivity of  $\bar{\alpha}$  follows from the well-known freeness of  $\operatorname{Pic}(X)$ .  $\square$

As a consequence, parts 1, 4, 5 of [2, Thm. 3.2] still hold, while parts 2, 3, 6, 7 have to be replaced by the following:

**THEOREM 3.2** *Let  $X = X(\Sigma)$  be a  $n$ -dimensional  $\mathbb{Q}$ -factorial complete toric variety of rank  $r$  and  $Y = Y(\widehat{\Sigma})$  be its universal 1-covering. Let  $V$  be a reduced fan matrix of  $X$ ,  $Q = \mathcal{G}(V)$  a weight matrix of  $X$  and  $\widehat{V} = \mathcal{G}(Q)$  be a CF-matrix giving a fan matrix of  $Y$ .*

2. *Define  $\mathcal{I}^\Sigma$  as in (1). For any  $I \in \mathcal{I}^\Sigma$  let  $E_I$  be the  $r \times (n+r)$  matrix admitting as rows the standard basis vectors  $e_i = (0, \dots, 0, \underset{i}{1}, 0, \dots, 0)$ , for  $i \in I$ , representing the  $i$ -th basis divisor  $D_i \in \mathcal{W}_T(X) \cong \mathbb{Z}^{|\Sigma(1)|}$ . Set  $\widetilde{V}_I := (V^I \mid E_I^T) \in \mathbf{M}_{n+r}(\mathbb{Z})$ . Then Cartier divisors give rise to the following maximal rank subgroup of  $\mathcal{W}_T(X)$*

$$\mathcal{C}_T(X) \cong \bigcap_{I \in \mathcal{I}^\Sigma} \mathcal{L}_c(\widetilde{V}_I) \leq \mathbb{Z}^{|\Sigma(1)|} \cong \mathcal{W}_T(X)$$

and a basis of  $\mathcal{C}_T(X) \leq \mathcal{W}_T(X)$  can be explicitly computed by applying the procedure described in [1, § 1.2.3].

3. *Let  $C_X \in \operatorname{GL}_{n+r}(\mathbb{Q}) \cap \mathbf{M}_{n+r}(\mathbb{Z})$  be a matrix whose rows give a basis of  $\mathcal{C}_T(X)$  in  $\mathcal{W}_T(X)$ , as obtained in the previous part 2. Identify  $\operatorname{Cl}(X)$  with  $\mathbb{Z}^r \oplus \bigoplus_{k=1}^s \mathbb{Z}/\tau_k \mathbb{Z}$  by item (c) of part 4 in [2, Thm. 3.2], and represent the morphism  $d_X$  by  $Q \oplus \Gamma$ , according to parts 1 and 5. Let  $A \in \operatorname{GL}_{n+r}(\mathbb{Z})$  be a matrix such that  $A \cdot C_X \cdot Q^T$  is in HNF. Let  $\mathbf{c}_1, \dots, \mathbf{c}_r$  be the first  $r$  rows of the matrix  $A \cdot C_X$  and for  $i = 1, \dots, r$  put  $\mathbf{b}_i = Q \cdot \mathbf{c}_i^T + \Gamma \cdot \mathbf{c}_i^T$ . Then  $\mathbf{b}_1, \dots, \mathbf{b}_r$  is a basis of the free group  $\operatorname{Pic}(X)$  in  $\operatorname{Cl}(X)$ .*

6. *Given the choice of  $\widehat{V}$  and  $V$  as in the previous parts 4 and 5 of [2, Thm. 3.2], consider*

$$U := \begin{pmatrix} rU_Q \\ \widehat{V} \end{pmatrix} \in \operatorname{GL}_{n+r}(\mathbb{Z})$$

$$W \in \operatorname{GL}_{n+r}(\mathbb{Z}) : W \cdot ({}^{n+r-s}U)^T = \operatorname{HNF} \left( ({}^{n+r-s}U)^T \right)$$

$$G := {}_s\widehat{V} \cdot ({}_sW)^T \in \mathbf{M}_s(\mathbb{Z})$$

$$U_G \in \operatorname{GL}_s(\mathbb{Z}) : U_G \cdot G^T = \operatorname{HNF}(G^T).$$

Then a “torsion matrix” representing the “torsion part” of the morphism  $d_X$ , that is,  $\tau_X : \mathcal{W}_T(X) \rightarrow \text{Tors}(\text{Cl}(X))$ , is given by

$$\Gamma = U_G \cdot {}_sW \pmod{\tau} \quad (3)$$

where this notation means that the  $(k, j)$ -entry of  $\Gamma$  is given by the class in  $\mathbb{Z}/\tau_k\mathbb{Z}$  represented by the corresponding  $(k, j)$ -entry of  ${}^sU_G \cdot {}_sW$ , for every  $1 \leq k \leq s$ ,  $1 \leq j \leq n+r$ .

7. Setting  $\delta_\Sigma := \text{lcm}(\det(Q_I) : I \in \mathcal{I}^\Sigma)$  then

$$\delta_\Sigma \mathcal{W}_T(X) \subseteq \mathcal{C}_T(X) \quad \text{and} \quad \delta_\Sigma \mathcal{W}_T(Y) \subseteq \mathcal{C}_T(Y)$$

and there are the following divisibility relations

$$\delta_\Sigma \mid [\text{Cl}(Y) : \text{Pic}(Y)] = [\mathcal{W}_T(Y) : \mathcal{C}_T(Y)] \mid [\text{Cl}(X) : \text{Pic}(X)] = [\mathcal{W}_T(X) : \mathcal{C}_T(X)].$$

*Proof:* (2): Recalling relation (2) in the proof of Proposition 3.1, set

$$\forall I \in \mathcal{I}^\Sigma \quad \mathcal{P}^I = \left\{ L = \sum_{j=1}^{n+r} a_j D_j \in \mathcal{W}_T(X) \mid \exists \mathbf{m} \in M : \forall j \notin I \quad \mathbf{m} \cdot \mathbf{v}_j = a_j \right\}.$$

Then  $\mathcal{P}^I$  contains  $\text{Im}(\text{div}_X : M \rightarrow \mathcal{W}_T(X)) = \mathcal{L}_c(V^T)$  and a  $\mathbb{Z}$ -basis of  $\mathcal{P}^I$  is given by

$$\{D_j, j \in I\} \cup \left\{ \sum_{k=1}^{n+r} v_{ik} D_k, i = 1, \dots, n \right\},$$

where  $\{v_{ik}\}$  is the  $i$ -th entry of  $\mathbf{v}_k$ , so giving the rows of the matrix  $\tilde{V}_I$  defined in the statement.

(3): By definition

$$\text{Pic}(X) = \text{Im}(\mathcal{C}_T(X) \hookrightarrow \mathcal{W}_T(X) \xrightarrow{d_X} \text{Cl}(X))$$

so that  $\text{Pic}(X)$  is generated by the image under  $Q \oplus \Gamma$  of the transposed of the rows of  $C_X$ . Since  $\text{rk}(C_X) = n+r$  and  $\text{rk}(Q) = r$ , the matrix  $C_X \cdot Q^T$  has rank  $r$  and therefore its HNF has the last  $n-r$  rows equal to zero. Therefore the rows of the matrix  $A \cdot C_X$  provide a basis of  $\mathcal{C}_T(X)$  in  $\mathcal{W}_T(X)$  such that its last  $n$  rows are a basis of  $\mathcal{L}_r(\hat{V}) \cap \mathcal{C}_T(X) = \mathcal{L}_r(V)$ . Since  $\text{Pic}(X)$  is free of rank  $r$  it is freely generated by the images under  $d_X$  of the first  $r$  rows.

(6): A representative matrix of the torsion part  $\tau_X : \mathcal{W}_T(X) \rightarrow \text{Cl}(X)$  of the morphism  $d_X$  is any matrix satisfying the following properties:

- (i)  $\Gamma = (\gamma_{kj})$  with  $\gamma_{kj} \in \mathbb{Z}/\tau_k\mathbb{Z}$ ,
- (ii)  $\Gamma \cdot ({}^rU_Q)^T = \mathbf{0}_{s,r} \pmod{\tau}$ , meaning that  $\Gamma$  kills the generators of the free part  $F \leq \text{Cl}(X)$  defined in display (4) of part 1 of [2, Thm. 3.2],
- (iii)  $\Gamma \cdot V^T = \mathbf{0}_{s,n} \pmod{\tau}$ , where  $V$  is a fan matrix satisfying condition 4.(b) in [2, Thm. 3.2]: this is due to the fact that the rows of  $V$  span  $\ker(d_X)$ ,
- (iv)  $\Gamma \cdot ({}_s\hat{V})^T = \mathbf{I}_s \pmod{\tau}$ , since the rows of  ${}_s\hat{V}$  give the generators of  $\text{Tors}(\text{Cl}(X))$ , as in display (6) of part 5 of [2, Thm. 3.2].

Therefore it suffices to show that the matrix  $U_G \cdot {}_sW$  in (3) satisfies the previous conditions (ii), (iii) and (iv) without any reduction mod  $\tau$ , that is,

$$U_G \cdot {}_sW \cdot ({}^{n+r-s}U)^T = \mathbf{0}_{s,n+r-s} \quad , \quad U_G \cdot {}_sW \cdot ({}_s\hat{V})^T = \mathbf{I}_s .$$

The first equation follows by the definition of  $W$ , in fact

$$W \cdot ({}^{n+r-s}U)^T = \text{HNF} \left( ({}^{n+r-s}U)^T \right) = \begin{pmatrix} \mathbf{I}_{n+r-s} \\ \mathbf{0}_{s,n+r-s} \end{pmatrix} \Rightarrow {}_sW \cdot ({}^{n+r-s}U)^T = \mathbf{0}_{s,n+r-s}$$

The second equation follows by the definition of  $U_G$ , in fact

$$U_G \cdot {}_s W \cdot ({}_s \widehat{V})^T = U_G \cdot G^T = \text{HNF}(G^T) = \mathbf{I}_s.$$

(7): Part (4) of [1, Thm. 2.9] gives that  $\delta_\Sigma \mid [\text{Cl}(Y) : \text{Pic}(Y)] = [\mathcal{W}_T(Y) : \mathcal{C}_T(Y)]$ . On the other hand Proposition 3.1 gives that  $[\mathcal{W}_T(Y) : \mathcal{C}_T(Y)] \mid [\mathcal{W}_T(X) : \mathcal{C}_T(X)] = [\text{Cl}(X) : \text{Pic}(X)]$ .  $\square$

Considerations i, ii, iii, iv, v of [2, Rem. 3.3] still holds, while vi, vii and the remaining part of Remark 3.3 have to be replaced by the following

REMARK 3.3

- vi. apply procedure [1, § 1.2.3], based on the HNF algorithm, to get a  $(n+r) \times (n+r)$  matrix  $C_X$  whose rows give a basis of  $\mathcal{C}_T(X) \leq \mathcal{W}_T(X) \cong \mathbb{Z}^{|\Sigma(1)|}$ ;
- vii. apply procedure described in part 6 of Theorem 3.2 to get a system of generators of  $\text{Pic}(X)$  in  $\text{Cl}(X)$ . Precisely, let  $A \in \text{GL}_{n+r}(\mathbb{Z})$  be a switching matrix such that  $\text{HNF}(C_X \cdot Q^T) = A \cdot C_X \cdot Q^T$ , and put

$$B_X = {}^r(A \cdot C_X \cdot Q^T), \quad \Theta_X = {}^r(A \cdot C_X \cdot \Gamma^T) \quad (4)$$

Then the rows of the matrices  $B_X$  and  $\Theta_X$  represent respectively the free part and the torsion part of a basis of  $\text{Pic}(X)$  in  $\text{Cl}(X)$ , where the latter is identified to  $\mathbb{Z}^r \oplus \bigoplus_{k=1}^s \mathbb{Z}/\tau_k \mathbb{Z}$ .

Moreover:

- recall that, for the universal 1-covering  $Y$  of  $X$ , once fixed the basis  $\{\widehat{D}_j\}_{j=1}^{n+r}$  of  $\mathcal{W}_T(Y) \cong \mathbb{Z}^{n+r}$  and the basis  $\{d_Y(\widehat{L}_i)\}_{i=1}^r$  of  $\text{Cl}(Y) \cong \mathbb{Z}^r$ , (see (11) in [1, Thm. 2.9]), one gets the following commutative diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & M & \xrightarrow{\begin{pmatrix} \mathbf{0}_{n,r} & \mathbf{I}_n \end{pmatrix}} & \mathcal{C}_T(Y) \cong \text{Pic}(Y) \oplus M & \xrightarrow{\begin{pmatrix} \mathbf{I}_r & \mathbf{0}_{r,n} \end{pmatrix}} & \text{Pic}(Y) \longrightarrow 0 \\
 & & \parallel & & \downarrow C_Y^T & & \downarrow B_Y^T \\
 0 & \longrightarrow & M & \xrightarrow[\widehat{V}^T]{\text{div}_Y} & \mathcal{W}_T(Y) = \bigoplus_{j=1}^{n+r} \mathbb{Z} \cdot D_j & \xrightarrow[Q]{d_Y} & \text{Cl}(Y) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & \longrightarrow & \mathcal{T}_Y & \xrightarrow{\cong} & \mathcal{T}_Y \longrightarrow 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0
 \end{array}$$

where  $B_Y$  is the  $r \times r$  matrix constructed in [1, Thm. 2.9(3)] and

$$C_Y = \begin{pmatrix} B_Y & \mathbf{0}_{r,n} \\ \mathbf{0}_{n,r} & \mathbf{I}_n \end{pmatrix} \cdot U_Q = \begin{pmatrix} B_Y \cdot {}^r U_Q \\ \widehat{V} \end{pmatrix},$$

- once fixed the basis  $\{D_j\}_{j=1}^{n+r}$  for  $\mathcal{W}_T(X) \cong \mathbb{Z}^{n+r}$  and the basis  $\{d_X(L_i)\}_{i=1}^r$  of the free part  $F \cong \mathbb{Z}^r$  of  $\text{Cl}(X)$ , constructed in part 1 of [2, Thm. 3.2], one gets the following commutative diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & M & \xrightarrow{\begin{pmatrix} \mathbf{0}_{n,r} & \mathbf{I}_n \end{pmatrix}} & \mathcal{C}_T(X) \cong \text{Pic}(X) \oplus M & \xrightarrow{\begin{pmatrix} \mathbf{I}_r & \mathbf{0}_{r,n} \end{pmatrix}} & \text{Pic}(X) \longrightarrow 0 \\
 & & \parallel & & \downarrow C_X^T & & \downarrow B_X^T \oplus \Theta_X^T \\
 0 & \longrightarrow & M & \xrightarrow[\mathbf{V}^T]{\text{div}_X} & \mathcal{W}_T(X) = \bigoplus_{j=1}^{n+r} \mathbb{Z} \cdot D_j & \xrightarrow[\mathbf{Q} \oplus \Gamma]{d_X = f_X \oplus \tau_X} & \text{Cl}(X) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & \longrightarrow & \mathcal{T}_X & \xrightarrow{\cong} & \mathcal{T}_X \longrightarrow 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0
 \end{array}$$

Moreover:

- recall the following commutative diagram of short exact sequences

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & & & & & \ker(\bar{\alpha}) = \text{Tors}(\text{Cl}(X)) \\
 0 & \longrightarrow & M & \xrightarrow[\mathbf{V}^T]{\text{div}_X} & \mathcal{W}_T(X) = \mathbb{Z}^{|\Sigma(1)|} & \xrightarrow{d_X} & \text{Cl}(X) \longrightarrow 0 \\
 & & \downarrow \beta^T & & \downarrow \mathbf{I}_{n+r} \alpha & & \downarrow \bar{\alpha} \\
 0 & \longrightarrow & M & \xrightarrow[\widehat{\mathbf{V}}^T]{\text{div}_Y} & \mathcal{W}_T(Y) = \mathbb{Z}^{|\widehat{\Sigma}(1)|} & \xrightarrow{d_Y} & \text{Cl}(Y) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \text{coker}(\beta^T) \cong \text{Tors}(\text{Cl}(X)) & & 0 & & 0 \\
 & & \downarrow & & & & \\
 & & 0 & & & & 
 \end{array}$$

(5)

then, putting all together, one gets the following 3-dimensional commutative diagram

$$\begin{array}{ccccc}
 M^{\mathbb{C}} & \xrightarrow{\text{div}_X} & \mathcal{C}_T(X) & \xrightarrow{d_{X|}} & \text{Pic}(X) \\
 \downarrow \beta^T & & \downarrow \alpha_1 & & \downarrow \bar{\alpha}_1 \\
 M^{\mathbb{C}} & \xrightarrow{\text{div}_Y} & \mathcal{C}_T(Y) & \xrightarrow{d_{Y|}} & \text{Pic}(Y) \\
 \downarrow \beta^T & & \downarrow \alpha_1 & & \downarrow \bar{\alpha}_1 \\
 M^{\mathbb{C}} & \xrightarrow{\text{div}_X} & \mathcal{W}_T(X) & \xrightarrow{d_X = f_X \oplus \tau_X} & \text{Cl}(X) \\
 \downarrow \beta^T & & \downarrow \alpha & & \downarrow \bar{\alpha} \\
 M^{\mathbb{C}} & \xrightarrow{\text{div}_Y} & \mathcal{W}_T(Y) & \xrightarrow{d_Y} & \text{Cl}(Y) \\
 \downarrow \beta^T & & \downarrow \alpha & & \downarrow \bar{\alpha} \\
 \mathcal{K} & \xrightarrow{\cong} & \mathcal{K} & \xrightarrow{\cong} & \mathcal{K} \\
 \downarrow & & \downarrow & & \downarrow \\
 \mathcal{T}_X & \xrightarrow{\cong} & \mathcal{T}_X & \xrightarrow{\cong} & \mathcal{T}_X \\
 \downarrow & & \downarrow & & \downarrow \\
 \mathcal{T}_Y & \xrightarrow{\cong} & \mathcal{T}_Y & \xrightarrow{\cong} & \mathcal{T}_Y
 \end{array}$$

(6)

The Snake Lemma implies

$$\begin{aligned}
 \text{coker}(\beta^T) &\cong \ker(\bar{\alpha}) \cong \text{Tors}(\text{Cl}(X)) \\
 \mathcal{K} &\cong \text{coker}(\alpha_1) \cong \mathcal{C}_T(Y)/\mathcal{C}_T(X)
 \end{aligned}$$

so giving the following short exact sequences on torsion subgroups

$$\begin{array}{ccccccc}
 & & 0 & & & & (7) \\
 & & \downarrow & & & & \\
 0 & \longrightarrow & \text{Tors}(\text{Cl}(X)) & \longrightarrow & \mathcal{C}_T(Y)/\mathcal{C}_T(X) & \longrightarrow & \text{Pic}(Y)/\text{Pic}(X) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & & & \text{Cl}(X)/\text{Pic}(X) & & \\
 & & & & \downarrow & & \\
 & & & & \text{Cl}(Y)/\text{Pic}(Y) & & \\
 & & & & \downarrow & & \\
 & & & & 0 & & 
 \end{array}$$

For what concerns the examples given in section 5, considerations related with parts v, vi and vii of Remark 3.3 have to be replaced as follows

EXAMPLE 5.1

v. A matrix  $W \in \mathrm{GL}_4(\mathbb{Z})$  such that  $\mathrm{HNF}(({}^3U)^T) = W \cdot ({}^3U)^T$  is given by

$$W = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & -2 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

giving

$$G := {}_1\widehat{V} \cdot ({}_1W)^T = \begin{pmatrix} 1 \end{pmatrix}$$

Therefore

$$\Gamma = {}_1W \pmod{5} = \begin{pmatrix} [0]_5 & [4]_5 & [2]_5 & [1]_5 \end{pmatrix}.$$

Consequently display (16) in [2], giving the action of  $\mathrm{Hom}(\mathrm{Tors}(\mathrm{Cl}(X)), \mathbb{C}^*) \cong \mu_5$  on  $Y = \mathbb{P}^3$ , should be replaced by the following (equivalent) one:

$$\begin{aligned} \mu_5 \times \mathbb{P}^3 &\longrightarrow \mathbb{P}^3 \\ (\varepsilon, [x_1 : \dots : x_4]) &\mapsto [x_1 : \varepsilon^4 x_2 : \varepsilon^2 x_3 : \varepsilon x_4] . \end{aligned} \quad (8)$$

vi. Applying procedure [1, § 1.2.3] as described in part 2 of Theorem 3.2, one gets a  $4 \times 4$  matrix  $C_X$  whose rows give a basis of  $\mathcal{C}_T(X)$  inside  $\mathcal{W}_T(X) \cong \mathbb{Z}^{|\Sigma(1)|}$ . Namely

$$C_X = \begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ -3 & -3 & 1 & 0 \\ -2 & -4 & 0 & 1 \end{pmatrix}$$

meaning that

$$\mathcal{C}_T(X) = \mathcal{L}(5D_1, 5D_2, -3D_1 - 3D_2 + D_3, -2D_1 - 4D_2 + D_4) .$$

On the other hand, by part (3) of [1, Thm. 2.9], a basis of  $\mathcal{C}_T(Y) \subseteq \mathcal{W}_T(Y)$  is given by the rows of

$$C_Y = \mathbf{I}_4 \cdot U_Q = U_Q \in \mathrm{GL}_n(\mathbb{Z})$$

giving  $\mathcal{C}_T(Y) = \mathcal{W}_T(Y)$ , as expected for  $Y = \mathbb{P}^3$ .

vii. A basis of  $\mathrm{Pic}(X)$  inside  $\mathrm{Cl}(X)$  is then obtained by applying part 6 of Theorem 3.2. With the notation of Remark 3.3 vii, a switching matrix  $A$  such that  $A \cdot C_X \cdot Q^T$  is in HNF is

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

so that

$$\begin{aligned} B_X &= {}^1(A \cdot C_X \cdot Q^T) = \begin{pmatrix} 5 \end{pmatrix} \\ \Theta_X &= {}^1(A \cdot C_X \cdot \Gamma^T) = \begin{pmatrix} 0 \end{pmatrix} \end{aligned}$$

Then

$$\mathrm{Pic}(X) \cong \mathbb{Z}[5d_X(D_1)] \leq \mathbb{Z}[d_X(D_1)] \oplus \mathbb{Z}/5\mathbb{Z}[d_X(D_3 - D_4)] \cong \mathrm{Cl}(X) \Rightarrow \mathrm{Cl}(X)/\mathrm{Pic}(X) \cong \mathbb{Z}/5\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z} .$$



## EXAMPLE 5.2

v. A matrix  $U$  as defined in part 6 of Theorem 3.2 is given by

$$U = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -6 & 3 & 1 & 0 & 0 & 0 \\ 521 & -251 & -168 & -2 & 14 & 28 \\ 388 & -222 & -112 & 7 & 45 & 3 \\ -184 & 105 & 53 & -2 & -23 & -1 \\ 191 & -109 & -55 & 2 & 24 & 1 \end{pmatrix}$$

A matrix  $W \in \text{GL}_6(\mathbb{Z})$  such that  $\text{HNF}(({}^4U)^T) = W \cdot ({}^4U)^T$  is given by

$$W = \begin{pmatrix} -57 & -115 & 3 & -549 & 17 & 0 \\ 4 & 8 & 1 & 3 & 7 & 0 \\ -125 & -250 & 0 & -1090 & 14 & 0 \\ -170 & -340 & 0 & -1482 & 19 & 0 \\ -188 & -376 & 0 & -1639 & 21 & 0 \\ -126 & -252 & 0 & -1092 & 13 & 1 \end{pmatrix}$$

then

$$G = {}_2\widehat{V}' \cdot ({}_2W)^T = \begin{pmatrix} -2093 & -1392 \\ 2302 & 1531 \end{pmatrix}$$

A matrix  $U_G \in \text{GL}_2(\mathbb{Z})$  such that  $\text{HNF}(G^T) = U_G \cdot G^T$  is given by

$$U_G = \begin{pmatrix} 1531 & -2302 \\ 1392 & -2093 \end{pmatrix}$$

hence giving

$$\begin{aligned} \Gamma &= U_G \cdot {}_2W \pmod{\tau} \\ &= \begin{pmatrix} 2224 & 4448 & 0 & 4475 & 2225 & -2302 \\ 2022 & 4044 & 0 & 4068 & 2023 & -2093 \end{pmatrix} \pmod{\begin{pmatrix} 3 \\ 15 \end{pmatrix}} \\ &= \begin{pmatrix} [1]_3 & [2]_3 & [0]_3 & [2]_3 & [2]_3 & [2]_3 \\ [12]_{15} & [9]_{15} & [0]_{15} & [3]_{15} & [13]_{15} & [7]_{15} \end{pmatrix} \end{aligned}$$

Consequently display (20) in [2] should be replaced by the following (equivalent) one

$$g(((t_1, t_2), \varepsilon, \eta), (x_1, \dots, x_6)) := \begin{pmatrix} t_1^2 t_2 \varepsilon \eta^{12} x_1, t_1^4 t_2 \varepsilon^2 \eta^9 x_2, t_1 t_2^3 x_3, t_1^5 t_2^2 \varepsilon^2 \eta^3 x_4, t_1^4 t_2^3 \varepsilon^2 \eta^{13} x_5, t_1^3 t_2^7 \varepsilon^2 \eta^7 x_6 \end{pmatrix} \quad (9)$$

vi. Depending on the choice of the fan  $\Sigma_i \in \mathcal{SF}(V)$ , by applying procedure [1, § 1.2.3] as described in part 2 of Theorem 3.2, one gets a  $6 \times 6$  matrix  $C_{X,i}$  whose rows give a basis of  $\mathcal{C}_T(X_i)$  inside  $\mathcal{W}_T(X_i) \cong \mathbb{Z}^{|\Sigma_i(1)|}$ .

Namely

$$C_{X,1} = \begin{pmatrix} 265926375 & 0 & 0 & 0 & 0 & 0 \\ -148978500 & 825 & 0 & 0 & 0 & 0 \\ -58474020 & -375 & 15 & 0 & 0 & 0 \\ 37 & -18 & -7 & 1 & 0 & 0 \\ -58473933 & -417 & -3 & 0 & 3 & 0 \\ 19 & -8 & -5 & 0 & -1 & 1 \end{pmatrix}$$

$$C_{X,2} = \begin{pmatrix} 43543500 & 0 & 0 & 0 & 0 & 0 \\ -34716000 & 15 & 0 & 0 & 0 & 0 \\ -594165 & 0 & 30 & 0 & 0 & 0 \\ -34715963 & -3 & -7 & 1 & 0 & 0 \\ 17655087 & -12 & -18 & 0 & 3 & 0 \\ 19 & -8 & -5 & 0 & -1 & 1 \end{pmatrix}$$

$$C_{X,3} = \begin{pmatrix} 43543500 & 0 & 0 & 0 & 0 & 0 \\ -37009500 & 825 & 0 & 0 & 0 & 0 \\ -6534165 & -750 & 30 & 0 & 0 & 0 \\ 37 & -18 & -7 & 1 & 0 & 0 \\ 87 & -42 & -18 & 0 & 3 & 0 \\ 19 & -8 & -5 & 0 & -1 & 1 \end{pmatrix}$$

vii. A basis of  $\text{Pic}(X_i)$  inside  $\text{Cl}(X_i)$  is then obtained by applying part 6 of Theorem 3.2. For  $i = 1, 2, 3$ , matrices  $A_i$  switching  $C_{X_i} \cdot Q^T$  in Hermite normal form are respectively

$$A_1 = \begin{pmatrix} -351039 & -449987 & -449987 & 0 & 0 & 0 \\ -502913 & -644670 & -644670 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} -93838 & -117699 & 0 & 0 & 0 & 0 \\ -1157199 & -1451450 & 0 & 0 & 0 & 0 \\ 4 & 5 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ -2 & -2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$A_3 = \begin{pmatrix} -10317 & -12139 & 0 & 0 & 0 & 0 \\ -22429 & -26390 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

giving

$$B_{X_1} = {}^2(A_1 \cdot C_{X_1} \cdot Q^T) = \begin{pmatrix} 825 & 185620050 \\ 0 & 265926375 \end{pmatrix}$$

$$B_{X_2} = {}^2(A_2 \cdot C_{X_2} \cdot Q^T) = \begin{pmatrix} 60 & 1765515 \\ 0 & 21771750 \end{pmatrix}$$

$$B_{X_3} = {}^2(A_3 \cdot C_{X_3} \cdot Q^T) = \begin{pmatrix} 3300 & 10016325 \\ 0 & 21771750 \end{pmatrix}$$

$$\Theta_{X_i} = {}^2(A_i \cdot C_{X_i} \cdot \Gamma^T) = \begin{pmatrix} [0]_3 & [0]_{15} \\ [0]_3 & [0]_{15} \end{pmatrix}, \quad \text{for } i = 1, 2, 3.$$

**References**

1. Rossi M. and Terracini L.  *$\mathbb{Z}$ -linear Gale duality and poly weighted spaces (PWS)* Linear Algebra Appl. **495** (2016), 256-288; DOI:10.1016/j.laa.2016.01.039; [arXiv:1501.05244](#)
2. Rossi, M., and Terracini, L. *A  $\mathbb{Q}$ -factorial complete toric variety is a quotient of a poly weighted space* Ann. Mat. Pur. Appl. **196** (2017), 325–347; DOI:10.1007/s10231-016-0574-7; [arXiv:1502.00879](#).