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FRACTAL ATTRACTORS IN ECONOMIC GROWTH MODELS WITH RANDOM POLLUTION EXTERNALITIES

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Fractal Attractors in Economic Growth Models with Random Pollution Externalities

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Abstract

We analyze a discrete time two-sector economic growth model where the production technologies in the final and human capital sectors are affected by random shocks both directly (via productivity and factor shares) and indirectly (via a pollution externality). We determine the optimal dynamics in the decentralized economy and show how these dynamics can be described in terms of a two-dimensional affine iterated function system with probability. This allows us to identify a suitable parameter configuration capable of generating exactly the classical Barnsley's fern as the attractor of the log-linearized optimal dynamical system.

Keywords: Two-sector Growth Model, Pollution, Stochastic Factor Shares, Fractal Attractors, Iterated Function Systems, Singular Measures
JEL Classification: C61, O41

1   Introduction

Over the last two decades a large and growing number of studies have tried to characterize the eventual fractal nature of the steady state in economic models. Indeed, since the pioneering work by Montrucchio and Privileggi (1999) it has been well known that traditional macroeconomic models may give rise to random dynamics possibly converging to invariant measures supported on fractal sets. Several works try to identify the conditions under which this might be the case by borrowing from the iterated function systems literature (Hutchinson, 1981; Vrscay, 1991; Barnsley, 1993). Most of these works analyze discrete time stochastic economic growth models with logarithmic utility and Cobb-Douglas production, either in a one-sector or two-sector framework, showing that through appropriate log-transformations their optimal dynamics can be converted into an affine iterated function system converging to a singular measure supported on some fractal set; in the case of unidimensional iterated function systems such an attractor can be the Cantor set (Montrucchio and Privileggi, 1999; Mitra et al., 2003; Mitra and Privileggi, 2004, 2006, 2009; Marsiglio, 2012; Privileggi and Marsiglio, 2013; La Torre et al., 2015), while in the case of two-dimensional iterated function systems it can be either the Sierpinski gasket or distorted-copies of the Barnsley's fern (La Torre et al., 2011;

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La Torre et al., 2015; La Torre et al., 2018). With the exception of La Torre et al. (2018) who consider also shocks affecting factor shares, in all these works randomness affects economic activities through the productivity channel, following the Brock and Mirman (1972) tradition. Some of these works also identify specific parameter configurations under which the invariant probability measure turns out to be either singular or absolutely continuous (Mitra et al., 2003; La Torre et al., 2015; La Torre et al., 2018).

We contribute to this stochastic growth and fractal attractors literature by analyzing an economy in which economic production is affected by random shocks both directly (via productivity and factor shares) and indirectly (via a pollution externality). Specifically, we build on the model by La Torre et al. (2018) in which such direct effects have already been accounted for in order to allow for pollution to be an additional and indirect source of randomness. To the best of our knowledge none of the extant works has ever considered how the presence of externalities complicate aggregate macroeconomic dynamics and what this may imply for the attractor of the associated iterated function system. However, accounting for the existence of such a pollution externality, as widely discussed in the environmental economics literature, is important to better characterize potential economic outcomes. Several papers document and discuss the extent to which the economy and the environment are mutually related (IPCC, 2007; Nordhaus, 2013): on the one hand, economic activities generate pollution which is the primary determinant of environmental problems; on the other hand, environmental degradation precludes pollution absorption which in turn critically determines economic capabilities. Moreover, due to the large degree of uncertainty associated with environmental phenomena, very little is known with precision about such a bilateral economy-environment relation, which is most likely to be random (Soretz, 2003; Marsiglio and La Torre, 2016). In order to take these issues into account in the most intuitive way, we focus on the optimal dynamics in the decentralized economy, where the externality is not internalized yet but it fully affects the evolution of both control and state variables; such a setting is the most appropriate to give rise to a simple but realistic characterization of real world dynamics. We show how the optimal dynamics can be described in terms of a two-dimensional affine iterated function system with probability, whose coefficients can eventually take on negative values, differently from all extant papers. Such a peculiarity of our framework, due to the presence of the pollution externality, allows us to identify a suitable parameter configuration capable of generating exactly the classical Barnsley's fern as the attractor of the log-linearized optimal dynamical system. To the best of our knowledge, no other macroeconomic model has ever been able to give rise to the Barnsley's fern[1] for some specific – but fully admissible – parameter configuration.

This short paper proceeds as follows. In Section 2 we briefly recall some basic results from the mathematical theory on iterated function system, which will help us discuss our following results. In Section 3 we formally introduce our model which consists of a two-sector economic growth model subject to random shocks affecting economic productive activities both directly and indirectly, thanks to the role played by pollution. We also derive the optimal rule for the control variables and the optimal dynamics of the state variables – physical and human capital – in the decentralized economy in which the pollution externality is not internalized. In Section 4 we introduce a log-transformation which allows us to recast the nonlinear optimal dynamical system in terms of a two-dimensional affine iterated function system, allowing us to borrow from the mathematical literature to discuss its eventual convergence to an invariant measure supported on some fractal set. In Section 5 we present two specific model's parameterizations:

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[1]Since the coefficients in their iterated function system can only take on positive values, La Torre et al.'s (2018) model is able to generate only distorted copies of the fern, which ultimately do not even remotely resemble a fern.

the first one allows us to obtain exactly the classical Barnsley's fern as the attractor of the log-linearized optimal dynamics; the second one focuses on a more realistic scenario giving rise to an attractor very different from a fern. In Section 6 as usual we conclude and present directions for future research.

2 Preliminaries on Iterated Function Systems

Let (X, d) denote a compact metric space. An N-map *iterated function system* (IFS) on X, $\mathbf{w} = \{w_1, \ldots, w_N\}$, consists of N contraction mappings on X, i.e., $w_i : X \to X$, $i = 1, \cdots, N$, with contraction factors $c_i \in [0, 1)$ (see Barnsley, 1993; Hutchinson, 1981; Kunze et al., 2012). Associated with an N-map IFS one can define a set-valued mapping $\hat{\mathbf{w}}$ on the space $\mathcal{H}([a, b])$ of nonempty compact subsets of X as follows:

$$\hat{\mathbf{w}}(S) := \bigcup_{i=1}^{N} w_i(S), \qquad S \in \mathcal{H}([a, b]).$$

The following two results state a convergence property of an N-map IFS towards its attractor. More properties and results can be found in Barnsley (1993), Hutchinson (1981), and Kunze et al. (2012).

Theorem 1 (Hutchinson, 1981) *For $A, B \in \mathcal{H}(X)$,*

$$h(\hat{\mathbf{w}}(A), \hat{\mathbf{w}}(B)) \le ch(A, B) \qquad \text{where} \qquad c = \max_{1 \le i \le N} c_i < 1$$

and h denotes the Hausdorff metric on $\mathcal{H}(X)$.

Corollary 1 (Hutchinson, 1981) *There exists a unique set $A \in \mathcal{H}([a, b])$, the* attractor *of the IFS $\mathbf{w}$, such that*

$$A = \hat{\mathbf{w}}(A) = \bigcup_{i=1}^{N} w_i(A).$$

Moreover, for any $B \in \mathcal{H}([a, b])$, $h(A, \hat{\mathbf{w}}^n(B)) \to 0$ as $n \to \infty$.

An N-map *Iterated Function System with (constant) Probabilities* (IFSP) $(\mathbf{w}, \mathbf{p})$ is an N-map IFS $\mathbf{w}$ with associated probabilities $\mathbf{p} = \{p_1, \ldots, p_N\}$, $\sum_{i=1}^{N} p_i = 1$.

Let $\mathcal{M}(X)$ denote the set of probability measures on (Borel subsets of) X and d_{MK} the Monge-Kantorovich distance on this space: For $\mu, \nu \in \mathcal{M}(X)$, with Monge-Kantorovich metric,

$$d_{MK}(\mu, \nu) = \sup_{f \in Lip_1(X)} \left[\int f \, d\mu - \int f \, d\nu \right].$$

where $Lip_1(X) = \{f : X \to \mathbb{R} \mid |f(x) - f(y)| \le d(x, y)\}$. The metric space $(\mathcal{M}(X), d_{MK})$ is complete (Barnsley, 1993; Hutchinson, 1981; Kunze et al., 2012).

The *Markov operator* associated with an N-map IFSP is a mapping $M : \mathcal{M} \to \mathcal{M}$, defined as follows: For any $\mu \in \mathcal{M}(X)$, and for any measurable set $S \subset X$, define a measure $\nu = M\mu$ as:

$$\nu(S) = (M\mu)(S) = \sum_{i=1}^{N} p_i \, \mu(w_i^{-1}(S)).$$

The following results show that the Markov operator has a unique invariant measure $\bar{\nu}$ and it is globally attracting.

Theorem 2 (Hutchinson, 1981) *For $\mu, \nu \in \mathcal{M}(X)$,*

$$d_{MK}(M\mu, M\nu) \leq c\, d_{MK}(\mu, \nu) \,.$$

Corollary 2 (Hutchinson, 1981) *There exists a unique measure $\bar{\nu} \in \mathcal{M}(X)$, the* invariant measure *of the IFSP $(\mathbf{w}, \mathbf{p})$, such that*

$$\bar{\mu}(S) = (M\bar{\mu})(S) = \sum_{i=1}^{N} p_i \bar{\mu}\left(w_i^{-1}(S)\right) \,.$$

Moreover, for any $\nu \in \mathcal{M}(X)$, $d_{MK}(\bar{\mu}, M^n \nu) \to 0$ as $n \to \infty$.

Theorem 3 (Hutchinson, 1981) *The support of the invariant measure $\bar{\mu}$ of an N-map IFSP $(\mathbf{w}, \mathbf{p})$ is the attractor A of the IFS $\mathbf{w}$, i.e.,*

$$supp\, \bar{\mu} = A \,.$$

In order to determine the attractor of an IFSP, the following random dynamical system known as Chaos Game might be implemented: Starting from $x_0 \in X$, let us determine $x_{t+1} = w_\sigma(x_t)$ where σ is chosen in the interval $1 \ldots N$ with probabilities $p_1, \ldots, p_N$. It can be proved (see Kunze et al., 2012, for more details) that the orbit of this random dynamical system is dense in the attractor A.

3 The Model

We analyze a discrete time two-sector economic growth model where the production technologies in the final and human capital sectors are affected by random shocks both directly (via productivity and factor shares) and indirectly (via pollution). While a large literature has considered the implications of random shocks for macroeconomic dynamics by analyzing the direct channel (Brock and Mirman, 1972; Montrucchio and Privileggi, 1999; Mitra et al., 2003; La Torre et al., 2015, 2018), more limited is the number of works analyzing the indirect channel, and in particular the effects of pollution on aggregate macroeconomic dynamics (Privileggi and Marsiglio, 2013; Marsiglio and La Torre, 2016). However, it is now well known that economic activities and environmental outcomes are mutually related, thus taking into account the existence of some economic-environmental feedback is essential to understand macroeconomic dynamics. Several studies discuss that pollution is a by-product of economic activities and how pollution impacts on both aggregate economic activities and health (IPCC, 2007; Nordhaus, 2013). We take these issues into account by (i) allowing output to be the driver of pollution, and (ii) allowing pollution to affect both the productivity of final output and that of human capital (meant as a broad form of capital encompassing not only education but health as well; Barro and Sala–i–Martin, 2004).

We analyze a standard two-sector optimal growth model under uncertainty, as discussed in La Torre et al. (2011), extended to account for such mutual economic-environmental feedback. Specifically, the representative household maximizes its lifetime utility subject to the evolution of physical and human capital. Lifetime utility is the infinite discounted ($\beta > 0$ is the discount factor) sum of instantaneous utilities, and the utility function is assumed to be logarithmic in consumption. At each time t, the household chooses its level of consumption, c_t, and which share of its human capital, u_t, to devote to the production of the final consumption good, which is produced according to a Cobb-Douglas technology combining physical, k_t, and human,

h_t, capital. Also new human capital is produced according to a Cobb-Douglas technology, which however employs only human capital (Lucas, 1988; Rebelo, 1991). As in La Torre et al. (2018), the production technologies of the final good and new human capital are directly affected by exogenous shocks which take both a multiplicative form through coefficients z_t and η_t respectively, and an exponential form affecting the factor shares in both production functions; therefore, output is given by $y_t = z_t A_t k_t^{\alpha_t} (u_t h_t)^{\gamma_t}$, where α_t and γ_t denote the random physical and human capital shares of income respectively, while human capital is given by $h_{t+1} = \eta_t B_t [(1 - u_t) h_t]^{\phi_t}$, with ϕ_t denoting the random human capital share of human capital. In this formulation A_t and B_t denote the pollution-induced productivity levels in the final and human capital sectors, respectively. Specifically, we assume that $A_t = P_t^{\nu_t}$ and $B_t = P_t^{\mu_t}$, where P_t denotes pollution which is a by-product of macroeconomic activities and $\nu_t, \mu_t \in \mathbb{R}$ are random parameters; since these parameters can take on real values, this accounts for the fact that pollution may have positive or negative effects on the production of final output and/or human capital. As in Marsiglio et al. (2016) economic activities generate pollution according to $P_t = k_t^{\chi_t} h_t^{\omega_t}$ in order to represent that the production inputs are characterized by different pollution-intensities; such intensities are set by parameters χ_t and ω_t respectively, which, like ν_t and μ_t, can take on real values and are random. This implies that $A_t = k_t^{\nu_t \chi_t} h_t^{\nu_t \omega_t}$ and $B_t = k_t^{\mu_t \chi_t} h_t^{\mu_t \omega_t}$, suggesting that randomness through the pollution channels affects indirectly the production technologies of the final good and human capital.

The whole $(z_t, \eta_t, \alpha_t, \gamma_t, \phi_t, \nu_t, \mu_t, \chi_t, \omega_t) \in \mathbb{R}^9$ is a random vector which is independent and identically distributed, and can take on m values, *i.e.*, at each time t $(z_t, \eta_t, \alpha_t, \gamma_t, \phi_t, \nu_t, \mu_t, \chi_t, \omega_t) \in \{(z_i, \eta_i, \alpha_i, \gamma_i, \phi_i, \nu_i, \mu_i, \chi_i, \omega_i)\}_{i=1}^m$. While shocks z_t, η_t enter multiplicatively the two Cobb-Douglas production functions and $\alpha_t, \gamma_t, \phi_t$ represent shocks on the factor shares, $\nu_t, \mu_t, \chi_t, \omega_t$ are random externalities affecting final production through two channels: 1) ν_t, μ_t determine how pollution modifies final production, 2) χ_t, ω_t determine how much pollution is generated by the current levels of physical and human capital, k_t and h_t, employed in the production of the composite good. As far as the first five shocks are concerned, we shall assume that $z_t, \eta_t > 0$, $0 < \alpha_i, \gamma_i, \phi_i < 1$ and $\alpha_i + \gamma_i \leq 1$ for all $i = 1, \ldots, m$. We do not impose any restriction on parameters $\nu_i, \mu_i, \chi_i, \omega_i$, but the realization of the shock determines whether physical or human capital is the relatively greener production input (according to how χ_t and ω_t compare). Each vector realization, $(z_i, \eta_i, \alpha_i, \gamma_i, \phi_i, \nu_i, \mu_i, \chi_i, \omega_i)$, occurs with (constant) probability p_i, with $p_i \in (0, 1)$, $i = 1, \ldots, m$, and $\sum_{i=1}^m p_i = 1$.

The optimization problem of the representative household can be summarized as follows:

$$V(k_0, h_0, z_0, \eta_0, \alpha_0, \gamma_0, \phi_0, \nu_0, \mu_0, \chi_0, \omega_0) = \max_{\{c_t, u_t\}} \sum_{t=0}^\infty \beta^t \mathbb{E}_0 \ln c_t \qquad (1)$$

$$\text{s.t.} \begin{cases} k_{t+1} = z_t A_t k_t^{\alpha_t} (u_t h_t)^{\gamma_t} - c_t \\ h_{t+1} = \eta_t B_t [(1 - u_t) h_t]^{\phi_t} \\ k_0 > 0, \ h_0 > 0, \ (z_0, \eta_0, \alpha_0, \gamma_0, \phi_0, \nu_0, \mu_0, \chi_0, \omega_0) \text{ are given}, \end{cases} \qquad (2)$$

As in Benhabib and Perli (1994), in the competitive (decentralized) solution the representative households take A_t and B_t as given, meaning that, because of the pollution externality, this solution will differ from the planning (centralized) solution. Most papers focusing on the effects of random shocks on macroeconomic dynamics analyze the centralized solution where all externalities are internalized by the social planner; we will instead focus on the decentralized solution since it is likely to provide us with a more realistic characterization of real world dynamics. In a competitive solution, since agents do not take into account the existence of the pollution externality, the optimal control problem is concave such that first order conditions

turn out to be also sufficient. However, even if the household does not internalize the externality this will affect macroeconomic dynamics, allowing us to take track of the aggregate effects of pollution. Hence, assuming that parameters A_t and B_t are considered as constants by utility maximizing households when deriving the FOCs, following the same steps as in La Torre et al. (2018), tedious algebra allows us to determine the following optimal policy functions:

$$c_t = [1 - \beta \mathbb{E}(\alpha)] y_t \tag{3}$$

$$u_t = \frac{[1 - \beta \mathbb{E}(\phi)] \gamma_t}{[1 - \beta \mathbb{E}(\phi)] \gamma_t + \beta \mathbb{E}(\gamma) \phi_t}, \tag{4}$$

where $\mathbb{E}(\alpha) = \sum_{i=1}^m p_i \alpha_i$, $\mathbb{E}(\gamma) = \sum_{i=1}^m p_i \gamma_i$, and $\mathbb{E}(\phi) = \sum_{i=1}^m p_i \phi_i$. Substituting (3) and (4) back into the law of motion of physical and human capital and taking into account the pollution-induced productivity levels $A_t = k_t^{\nu_t \chi_t} h_t^{\nu_t \omega_t}$ and $B_t = k_t^{\mu_t \chi_t} h_t^{\mu_t \omega_t}$, yield the optimal dynamics in the competitive economy, which turn out to be characterized by the following equations:

$$\begin{cases} k_{t+1} = \Delta_t z_t k_t^{\alpha_t + \nu_t \chi_t} h_t^{\gamma_t + \nu_t \omega_t} \\ h_{t+1} = \Theta_t \eta_t k_t^{\mu_t \chi_t} h_t^{\phi_t + \mu_t \omega_t} \end{cases} \tag{5}$$

with

$$\Delta_t = \beta \mathbb{E}(\alpha) \left\{ \frac{[1 - \beta \mathbb{E}(\phi)] \gamma_t}{[1 - \beta \mathbb{E}(\phi)] \gamma_t + \beta \mathbb{E}(\gamma) \phi_t} \right\}^{\gamma_t} \tag{6}$$

$$\Theta_t = \left\{ \frac{\beta \mathbb{E}(\gamma) \phi_t}{[1 - \beta \mathbb{E}(\phi)] \gamma_t + \beta \mathbb{E}(\gamma) \phi_t} \right\}^{\phi_t}. \tag{7}$$

4 Log-Transformation

We can apply the same technique developed in La Torre et al. (2018) to build a specific transformation that recasts system (5) into an affine, topologically equivalent system. Specifically, we wish to transform it into a system of the following form:

$$\begin{cases} \varphi_{t+1} = (\alpha_t + \nu_t \chi_t) \varphi_t + (\gamma_t + \nu_t \omega_t) \psi_t + \zeta_t \\ \psi_{t+1} = \mu_t \chi_t \varphi_t + (\phi_t + \mu_t \omega_t) \psi_t + \vartheta_t, \end{cases}, \tag{8}$$

where the coefficients are the exponents of physical and human capital in our original equations (5) plus the parameters characterizing pollution externalities, and the additive random vector $(\zeta_t, \vartheta_t) \in \mathbb{R}^2$ takes on m values corresponding to realizations of the multiplicative shocks (z_t, η_t). We will show that, by imposing some conditions on the parameters, there exists a one-to-one continuous transformation from the dynamics of (k_t, h_t) defined by (5) to those of (φ_t, ψ_t) as in (8). By recalling section 2, the convergence of the random dynamical system in (8) to the steady state can be obtained by noticing that equation (8) is the chaos game associated with an IFS with probabilities whose associated Markov operator will be converging to an invariant measure $\bar{\mu}$.

It may be useful to rewrite (8) in vector terms as follows:

$$\begin{bmatrix} \varphi_{t+1} \\ \psi_{t+1} \end{bmatrix} = \begin{bmatrix} \alpha_t + \nu_t \chi_t & \gamma_t + \nu_t \omega_t \\ \mu_t \chi_t & \phi_t + \mu_t \omega_t \end{bmatrix} \begin{bmatrix} \varphi_t \\ \psi_t \end{bmatrix} + \begin{bmatrix} \zeta_t \\ \vartheta_t \end{bmatrix}, \tag{9}$$

where:

$$Q_t = \begin{bmatrix} \alpha_t + \nu_t \chi_t & \gamma_t + \nu_t \omega_t \\ \mu_t \chi_t & \phi_t + \mu_t \omega_t \end{bmatrix} \tag{10}$$

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is a random 2×2 matrix which, together with the vector $(\zeta_t, \vartheta_t) \in \mathbb{R}^2$, take on m values corresponding to the m shocks realizations.

Proposition 1 *There exists a one-to-one logarithmic transformation $(k_t, h_t) \to (\varphi_t, \psi_t)$ defined by*

$$\begin{cases} \varphi_t = \rho_1 \ln k_t + \rho_2 \ln h_t + \rho_3 \\ \psi_t = \rho_4 \ln k_t + \rho_5 \ln h_t + \rho_6 \end{cases} \tag{11}$$

that is topologically conjugate to the nonlinear system (5) provided that the model's parameters $z_i, \eta_i, \alpha_i, \gamma_i, \phi_i, \nu_i, \mu_i, \chi_i, \omega_i$, the constants ζ_i, ϑ_i in the IFS (8), together with the coefficients $\rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \rho_6$, satisfy the following conditions:

$$\begin{cases} (\mu_i \chi_i)\, \rho_2 = (\gamma_i + \nu_i \omega_i)\, \rho_4 \\ (\gamma_i + \nu_i \omega_i)\, \rho_1 = (\alpha_i + \nu_i \chi_i - \phi_i - \mu_i \omega_i)\, \rho_2 + (\gamma_i + \nu_i \omega_i)\, \rho_5 \\ (\alpha_i + \nu_i \chi_i - \phi_i - \mu_i \omega_i)\, \rho_4 + \mu_i \chi_i \rho_5 = \mu_i \chi_i \rho_1 \\ (\ln \Delta_i + \ln z_i)\, \rho_1 + (\ln \Theta_i + \ln \eta_i)\, \rho_2 + (1 - \alpha_i - \nu_i \chi_i)\, \rho_3 - (\gamma_i + \nu_i \omega_i)\, \rho_6 = \zeta_i \\ (\ln \Delta_i + \ln z_i)\, \rho_4 + (\ln \Theta_i + \ln \eta_i)\, \rho_5 + (1 - \phi_i - \mu_i \omega_i)\, \rho_6 - \mu_i \chi_i \rho_3 = \vartheta_i \end{cases} \tag{12}$$

$$\text{for all } i = 1, \dots, m.$$

Proof. We use (11) to rewrite both sides of (8):

$$\begin{aligned} \rho_1 \ln k_{t+1} + \rho_2 \ln h_{t+1} + \rho_3 = {} & (\alpha_t + \nu_t \chi_t)\,(\rho_1 \ln k_t + \rho_2 \ln h_t + \rho_3) \\ & + (\gamma_t + \nu_t \omega_t)\,(\rho_4 \ln k_t + \rho_5 \ln h_t + \rho_6) + \zeta_t \\ \rho_4 \ln k_{t+1} + \rho_5 \ln h_{t+1} + \rho_6 = {} & \mu_t \chi_t\,(\rho_1 \ln k_t + \rho_2 \ln h_t + \rho_3) \\ & + (\phi_t + \mu_t \omega_t)\,(\rho_4 \ln k_t + \rho_5 \ln h_t + \rho_6) + \vartheta_t. \end{aligned}$$

Then, use (5) to rewrite the LHS in each equation above in order to obtain the following two equations:

$$\begin{aligned} & \rho_1 \ln \Delta_t + \rho_1 \ln z_t + \rho_1 (\alpha_t + \nu_t \chi_t) \ln k_t + \rho_1 (\gamma_t + \nu_t \omega_t) \ln h_t \\ & + \rho_2 \ln \Theta_t + \rho_2 \ln \eta_t + \rho_2 \mu_t \chi_t \ln k_t + \rho_2 (\phi_t + \mu_t \omega_t) \ln h_t + \rho_3 \\ & = (\alpha_t + \nu_t \chi_t)\, \rho_1 \ln k_t + (\alpha_t + \nu_t \chi_t)\, \rho_2 \ln h_t + (\alpha_t + \nu_t \chi_t)\, \rho_3 \\ & + (\gamma_t + \nu_t \omega_t)\, \rho_4 \ln k_t + (\gamma_t + \nu_t \omega_t)\, \rho_5 \ln h_t + (\gamma_t + \nu_t \omega_t)\, \rho_6 + \zeta_t, \end{aligned} \tag{13}$$

$$\begin{aligned} & \rho_4 \ln \Delta_t + \rho_4 \ln z_t + \rho_4 (\alpha_t + \nu_t \chi_t) \ln k_t + \rho_4 (\gamma_t + \nu_t \omega_t) \ln h_t \\ & + \rho_5 \ln \Theta_t + \rho_5 \ln \eta_t + \rho_5 \mu_t \chi_t \ln k_t + \rho_5 (\phi_t + \mu_t \omega_t) \ln h_t + \rho_6 \\ & = \mu_t \chi_t \rho_1 \ln k_t + \mu_t \chi_t \rho_2 \ln h_t + \mu_t \chi_t \rho_3 \\ & + (\phi_t + \mu_t \omega_t)\, \rho_4 \ln k_t + (\phi_t + \mu_t \omega_t)\, \rho_5 \ln h_t + (\phi_t + \mu_t \omega_t)\, \rho_6 + \vartheta_t. \end{aligned} \tag{14}$$

As these equations must hold for all $t \geq 0$, under the i.i.d. assumption it is sufficient that they hold for all parameters' values, that is, for all $i = 1, \dots, m$; hence, in the sequel we replace the time index t of each term involving only the model's parameters with the index $i = 1, \dots, m$ (clearly, the state variables k_t and h_t remain indexed by t). By equating the corresponding coefficients in the LHS and the RHS, equations (13) and (14) become independent of values taken by the variables $\ln k_t$ and $\ln h_t$; this is equivalent to the following conditions (corresponding to system (25) in La Torre et al., 2018):

$$\begin{cases} (\mu_i \chi_i)\, \rho_2 = (\gamma_i + \nu_i \omega_i)\, \rho_4 \\ (\gamma_i + \nu_i \omega_i)\, \rho_1 = (\alpha_i + \nu_i \chi_i - \phi_i - \mu_i \omega_i)\, \rho_2 + (\gamma_i + \nu_i \omega_i)\, \rho_5 \qquad \text{for all } i = 1, \dots, m. \\ (\alpha_i + \nu_i \chi_i - \phi_i - \mu_i \omega_i)\, \rho_4 + \mu_i \chi_i \rho_5 = \mu_i \chi_i \rho_1 \end{cases}$$

After joining them with the terms that do not depend on $\ln k_t$ and $\ln h_t$ left in equations (13) and (14) we obtain the system of $5m$ equations in (12) and the proof is complete. ∎

System (12) has $5m$ equations and 6 unknowns, $\rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \rho_6$; clearly such system has no solution whenever there is more than one state of nature, *i.e.*, when $m \geq 2$, as already with $m = 2$ it has 10 equations in 6 unknowns. Hence, following an approach similar to that pursued in La Torre et al. (2015, 2018), in the next section we will add more constraints on the parameters' values so to increase the number of unknowns; specifically we will treat some values for the coefficients z_i, η_i as unknowns in order to have as many unknowns as the number of equations in (12). A complication ensuing in this case is that system (12) ceases to be linear, as the new unknowns (even in their log-expression) $\ln z_i, \ln \eta_i$ enter multiplicatively the other unknowns of the type ρ_i. Therefore, to solve (12) we will rely on numerical methods.

5   Some Specific Parameterizations

We now present two different model's parameterizations. The first allow us to generate exactly the classical Barnsley's fern, but this requires to impose somehow questionable parameters values. The second tries to fix this issue by focusing on more realistic parameter values; this however gives rise to an attractor very different from a fern.

5.1   A Model Generating the Classical Barnsley's Fern

It is well known that the classical Barnsley's fern (Barnsley, 1993) is produced by the following affine IFS:

$$
\begin{cases}
w_1(\varphi, \psi) = \begin{bmatrix} 0 & 0 \\ 0 & 0.16 \end{bmatrix} \begin{bmatrix} \varphi \\ \psi \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} & \text{with prob. } p_1 = 0.01 \\[2ex]
w_2(\varphi, \psi) = \begin{bmatrix} -0.15 & 0.28 \\ 0.26 & 0.24 \end{bmatrix} \begin{bmatrix} \varphi \\ \psi \end{bmatrix} + \begin{bmatrix} 0 \\ 0.44 \end{bmatrix} & \text{with prob. } p_2 = 0.07 \\[2ex]
w_3(\varphi, \psi) = \begin{bmatrix} 0.20 & -0.26 \\ 0.23 & 0.22 \end{bmatrix} \begin{bmatrix} \varphi \\ \psi \end{bmatrix} + \begin{bmatrix} 0 \\ 1.60 \end{bmatrix} & \text{with prob. } p_3 = 0.07 \\[2ex]
w_4(\varphi, \psi) = \begin{bmatrix} 0.85 & 0.04 \\ -0.04 & 0.85 \end{bmatrix} \begin{bmatrix} \varphi \\ \psi \end{bmatrix} + \begin{bmatrix} 0 \\ 1.60 \end{bmatrix} & \text{with prob. } p_4 = 0.85.
\end{cases}
\tag{15}
$$

We now look for a configuration of parameters' values for the economic growth model discussed in Section 3 such that the dynamics described by the IFS (15), generating the Barnsley's fern through the chaos game, are obtained as the result of some logarithmic transformation of the form in (11) applied to the optimal nonlinear dynamics defined by (5). In other words, following the arguments developed in Section 4 we look for suitable values of the parameters $z_i, \eta_i, \alpha_i, \gamma_i, \phi_i, \nu_i, \mu_i, \chi_i, \omega_i$ that allow the existence of coefficients $\rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \rho_6$ solving system (12). Note that the existence of a pollution externality allows some of the parameters in the random matrix (10) to take on negative values, which is specifically required in order to generate the IFS (15).

The realization of the random matrix Q_1 representing the first map in the IFS (15), w_1, imposes a constraint that yields immediately the values of ρ_2, ρ_4 and ρ_5 for all $i = 1, \ldots, 4$. In fact, according to (10), $\mu_1 \chi_1 = 0$ must hold for $i = 1$; as $(\gamma_i + \nu_i \omega_i) \neq 0$ in the w_i maps for $i = 2, \ldots, 4$, replacing $\mu_1 \chi_1 = 0$ in the first equation of system (12) for $i = 1$ yields $\rho_4 = 0$, which, when substituted in the third equation of system (12) leads to $\mu_i \chi_i \rho_5 = \mu_i \chi_i \rho_1$, which,

in turn implies that $\rho_5 = \rho_1$ must hold, as $\mu_i \chi_i \neq 0$ in the w_i maps for $i = 2, \ldots, 4$. Finally, using $\rho_5 = \rho_1$ in the second set of equations of system (12) and noting that, besides having $(\gamma_i + \nu_i \omega_i) \neq 0$ in the w_i maps for $i = 2, \ldots, 4$, $(\alpha_i + \nu_i \chi_i - \phi_i - \mu_i \omega_i) \neq 0$ at least in the w_1 map (for $i = 1$), we observe that also $\rho_2 = 0$ must hold. Therefore, with $\rho_2 = \rho_4 = 0$ and $\rho_5 = \rho_1$ satisfying the first three equations in (12) for all $i = 1, \ldots, 4$, after substituting such values in the remaining equations the whole system boils down to the following 8 equations:

$$\begin{cases} (\ln \Delta_i + \ln z_i)\, \rho_1 + (1 - \alpha_i - \nu_i \chi_i)\, \rho_3 - (\gamma_i + \nu_i \omega_i)\, \rho_6 = \zeta_i \\ (\ln \Theta_i + \ln \eta_i)\, \rho_1 + (1 - \phi_i - \mu_i \omega_i)\, \rho_6 - \mu_i \chi_i \rho_3 = \vartheta_i \end{cases} \quad \text{for } i = 1, \ldots, 4, \quad (16)$$

having only the three coefficients ρ_1, ρ_3 and ρ_6 as unknowns.

As anticipated at the end of Section 4, we will choose arbitrarily 3 out of the 8 values for the multiplicative shocks z_i, η_i (or, equivalently, $\ln z_i, \ln \eta_i$) and leave the remaining 5 values for z_i, η_i ($\ln z_i, \ln \eta_i$) as unknowns so that, together with the 3 coefficients ρ_1, ρ_3, ρ_6, (16) becomes a system of 8 equations in 8 unknowns. Specifically, we will set

$$\ln z_1 = -0.4, \qquad \ln \eta_1 = -0.2 \qquad \text{and} \qquad \ln \eta_2 = 0 \qquad (17)$$

and leave $\ln z_2$, $\ln z_3$, $\ln z_4$, $\ln \eta_3$ and $\ln \eta_4$ as unknowns, to be found as part of the solution for system (16).

As far as all other parameters are concerned, we set $\beta = 0.96$, while the constant terms ζ_i and ϑ_i in system (16) are clearly given by the coordinates of the constant vectors in the maps w_i of the IFS (15). To choose all the exponents involved in the optimal dynamics (5) we must equate the random matrix Q_i defined in (10) to the 4 values considered in the IFS (15), that is, we must choose values for the exponents $\alpha_i, \gamma_i, \phi_i, \nu_i, \mu_i, \chi_i, \omega_i$ that satisfy the following 4 sets of conditions:

$$1. \begin{cases} \alpha_1 + \nu_1 \chi_1 = 0 \\ \gamma_1 + \nu_1 \omega_1 = 0 \\ \mu_1 \chi_1 = 0 \\ \phi_1 + \mu_1 \omega_1 = 0.16 \end{cases} \qquad 2. \begin{cases} \alpha_2 + \nu_2 \chi_2 = -0.15 \\ \gamma_2 + \nu_2 \omega_2 = 0.28 \\ \mu_2 \chi_2 = 0.26 \\ \phi_2 + \mu_2 \omega_2 = 0.24 \end{cases}$$

$$3. \begin{cases} \alpha_3 + \nu_3 \chi_3 = 0.20 \\ \gamma_3 + \nu_3 \omega_3 = -0.26 \\ \mu_3 \chi_3 = 0.23 \\ \phi_3 + \mu_3 \omega_3 = 0.22 \end{cases} \qquad 4. \begin{cases} \alpha_4 + \nu_4 \chi_4 = 0.85 \\ \gamma_4 + \nu_4 \omega_4 = 0.04 \\ \mu_4 \chi_4 = -0.04 \\ \phi_4 + \mu_4 \omega_4 = 0.85. \end{cases}$$

1. The third equation for the first shock realization requires that either μ_1 or χ_1 be zero; we opt for the former solution, $\mu_1 = 0$, as the latter would imply that α_1 is zero as well from the first equation, which is ruled out by the basic assumptions on the model's fundamentals. Such choice implies that, in the first shock realization, the production of human capital is not being affected by pollution. Setting $\alpha_1 = 0.3$ and $\nu_1 = -0.5$ (output production decreases in the stock of pollution) the first equation yields $\chi_1 = -0.3/(-0.5) = 0.6$ (pollution increases in the stock of physical capital); setting $\gamma_1 = 0.6$, the second equation yields $\omega_1 = -0.6/(-0.5) = 1.2$ (pollution increases in the stock of human capital as well); finally, with $\mu_1 = 0$ the last equation implies that $\phi_1 = 0.16$. This configuration envisages $\chi_1 < \omega_1$, so that human capital turns out to be more pollution-intense than physical capital.

2. In the second set of conditions we set $\mu_2 = -0.5$ (human capital production decreases in the stock of pollution), so that the third equation yields $\chi_2 = 0.26/(-0.5) = -0.52$ (pollution decreases in the stock of physical capital), which, by setting $\alpha_2 = 0.15$ in

the first equation, leads to $\nu_2 = (-0.15 - 0.15)/(-0.52) = 0.3/(0.52) = 0.577$ (output production increases in the stock of pollution). By setting $\phi_2 = 0.1$ the last equation yields $\omega_2 = (0.24 - 0.1)/(-0.5) = 0.14/(-0.5) = -0.28$ (pollution decreases in the stock of human capital), which, after substituting into the second equation, in turn yields $\gamma_2 = 0.28 - 0.577(-0.28) = 0.28 + 0.162 = 0.442$. In this case again $\chi_2 < \omega_2$, but now both have negative values, which implies that the stock of pollution is more sensitive to changes in the stock of physical capital than in the stock of human capital.

3. In the third set of conditions again we set $\mu_3 = -0.5$ (human capital production decreases in the stock of pollution), so that the third equation yields $\chi_3 = 0.23/(-0.5) = -0.46$ (pollution decreases in the stock of physical capital), which, by setting $\alpha_3 = 0.05$ in the first equation, leads to $\nu_3 = (0.20 - 0.05)/(-0.46) = 0.15/(-0.46) = -0.326$ (output production decreases in the stock of pollution). By setting $\gamma_3 = 0.05$ the second equation yields $\omega_3 = (-0.26 - 0.05)/(-0.326) = 0.31/(0.326) = 0.951$ (pollution increases in the stock of human capital), which, after substituting into the last equation, in turn yields $\phi_3 = 0.22 - (-0.5)(0.951) = 0.22 + 0.475 = 0.695$. Also in this scenario $\chi_3 < \omega_3$, but now χ_3 and ω_3 have opposite signs, meaning that they affect pollution in opposite directions.

4. In the fourth shock realization we set $\mu_4 = -0.1$ (human capital production decreases in the stock of pollution), so that the third equation yields $\chi_4 = -0.04/(-0.1) = 0.4$ (pollution increases in the stock of physical capital), which, by setting $\alpha_4 = 0.89$ in the first equation, leads to $\nu_4 = (0.85 - 0.89)/(0.4) = -0.04/(0.4) = -0.1$ (output production decreases in the stock of pollution with the same intensity as human capital production). By setting $\gamma_4 = 0.06$ the second equation yields $\omega_4 = (0.04 - 0.06)/(-0.1) = -0.02/(-0.1) = 0.2$ (pollution increases in the stock of human capital), which, after substituting into the last equation, in turn yields $\phi_4 = 0.85 - (-0.1)(0.2) = 0.85 + 0.02 = 0.87$. The fourth configuration thus has $\chi_4 > \omega_4$, so that physical capital turns out to be more pollution-intense than human capital.

Table 1 summarizes all parameters' values discussed so far.

i	α_i	γ_i	ϕ_i	ν_i	μ_i	χ_i	ω_i	ζ_i	ϑ_i	p_i
1	0.3	0.6	0.16	-0.5	0	0.6	1.2	0	0	0.01
2	0.15	0.442	0.1	0.557	-0.5	-0.52	-0.28	0	0.44	0.07
3	0.05	0.05	0.695	-0.326	-0.5	-0.46	0.951	0	1.60	0.07
4	0.89	0.06	0.87	-0.1	-0.1	0.4	0.2	0	1.60	0.85

TABLE 1: parameters characterizing our model, additive constants and probability values corresponding to the IFS (15) generating the classical Barnsley's (1993) fern.

Under these parameters' choice,

$$\mathbb{E}(\alpha) = \sum_{i=1}^{4} \alpha_i p_i = 0.7735, \quad \mathbb{E}(\gamma) = \sum_{i=1}^{4} \gamma_i p_i = 0.0914 \quad \text{and} \quad \mathbb{E}(\phi) = \sum_{i=1}^{4} \phi_i p_i = 0.7968,$$

such that the coefficients (6) and (7) defining the optimal policy of the model at the end of Section 3 become:

i	Δ_i	Θ_i
1	0.7015	0.6809
2	0.7164	0.7747
3	0.6779	0.8847
4	0.6642	0.8629

Recall that, following the discussion at the beginning of this section on the parameters' constraints due to the first shock realization, $\rho_2 = \rho_4 = 0$ and $\rho_1 = \rho_5$. Hence, using the choice in (17) for the values of $\ln z_1, \ln \eta_1, \ln \eta_2$, we find the following unique solution for system (16) by means of the standard (symbolic, not numerical) 'solve' routine in Maple:

$$\rho_1 = \rho_5 = 5.6984$$
$$\rho_2 = \rho_4 = 0$$
$$\rho_3 = 4.3$$
$$\rho_6 = 3.9638,$$

with all multiplicative shocks configurations:

i	z_i	η_i
1	0.6703	0.8187
2	0.7123	1
3	0.6731	1.0345
4	1.3824	1.3414

(18)

where the values of z_1, η_1 and η_2 correspond to the choice in (17) and all other values are found as a solution of (16).

Figure 1(a) reports the standard Barnsley's fern by tracing 50,000 random iterations[2] of the affine IFS defined in (15) expressed in terms of the log-transformed variables φ, ψ. It is well known that the unique invariant measure supported on the fern is singular (Theorem 1 in La Torre et al., 2018). Figure 1(b) portraits the attractor of the corresponding nonlinear IFS defined in (5) for the parameters' values characterizing our growth model reported in Table 1, expressed in terms of physical and human capital, k, h; such attractor turns out to be a downsized version of the fern in Figure 1(a) (rescaled by a factor 10 on the horizontal axis and by a factor 4 on the vertical axis) which is being translated into the positive orthant (k and h cannot be negative) and somewhat straighten up along its main branch.

Some comments on the parametrization employed to generate the classical fern are needed. Leaving aside the productivity parameter values in (18), the four-shocks parameters' configuration reported in Table 1 does not only clearly satisfy all the assumptions on the model's fundamentals, but may even be considered not totally unrealistic, as the following tentative arguments try to clarify.

The first scenario foresees reasonable values for all three factor shares (perhaps the human capital share, ϕ_1, in human capital production is too low), an output production function which decreases in the stock of pollution ($\nu_1 < 0$), while the latter increases in both the physical and human capital stocks ($\chi_1, \omega_1 > 0$). The only anomalies may appear to be an output production process that is not being affected by pollution ($\mu_1 = 0$) and a human capital production that generates pollution with and intensity which is the double of that of physical capital ($\omega_1 = 2\chi_1$)—i.e., human capital is a dirtier input than physical capital. The latter feature may be characteristic in an economy that is more efficient in producing physical goods than in recycling waste generated by human activity.

In the second shock configuration, besides perhaps too low values for the physical and human capital shares in output production and a human capital share in human capital production even lower than in the first case, the most apparent anomalies are given by a positive relationship between the pollution stock and output production ($\nu_2 > 0$)—i.e., pollution becomes itself some exogenous production factor—together with a negative relationship between both physical and

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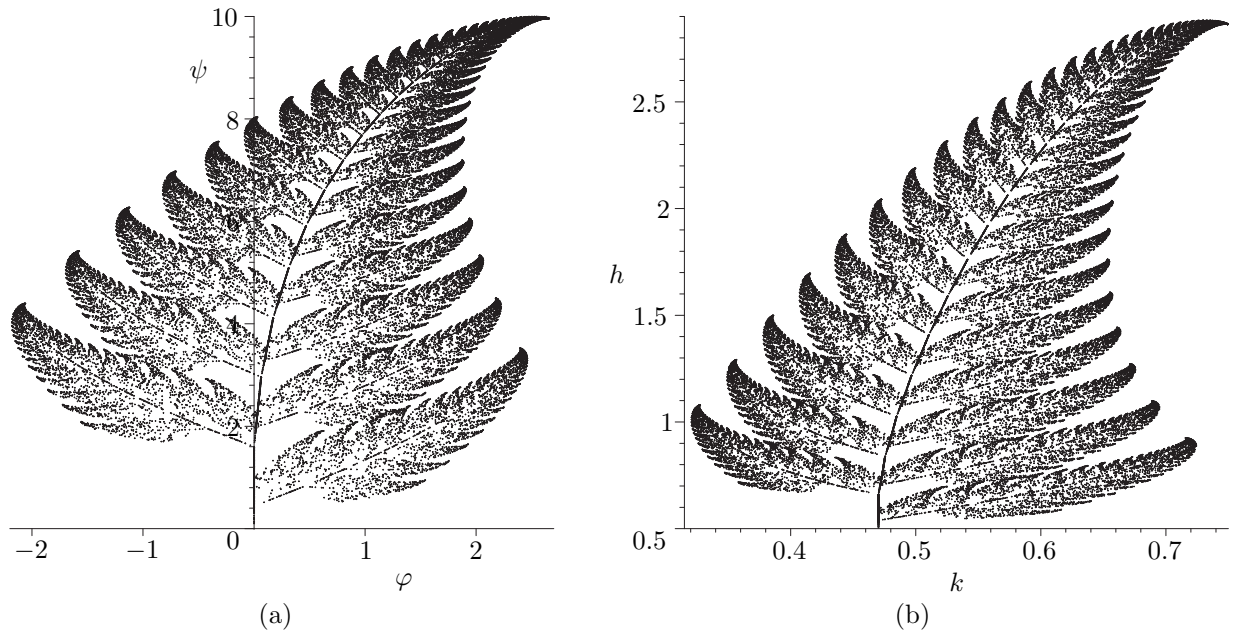[2]The Maple code is available from the authors upon request.

FIGURE 1: approximation through 50000 random iterations of (a) the IFS in (15) generating the standard Barnsley's fern as attractor and (b) the attractor of the corresponding nonlinear IFS in (5) for the parameters' values listed in Table 1.

human capital and pollution stock ($\chi_2, \omega_2 < 0$)—*i.e.*, physical and human capital accumulation reduces pollution. On the other hand, the negative relationship between pollution and human capital ($\mu_2 < 0$) has an intuitive and appealing interpretation if one attributes to the latter a broader meaning that includes health. Such scenario may describe a virtuous economy that is capable of recycling pollution as a production enhancer ($\nu_2 > 0$), while physical together with human capital accumulation may be thought as general progress capable of bringing about a better and more efficient treatment and recycling of polluting by-products.

The third shock environment is characterized by even smaller values for the physical and human capital shares in output production while the human capital share in human capital production more realistically becomes larger. The pollution parameters envisage a reasonable negative relationship between the pollution stock and both physical and human capital production ($\nu_3, \mu_3 < 0$) together with a positive relationship between human capital accumulation and pollution ($\omega_3 > 0$), while, again, the relationship between physical capital accumulation and the pollution stock is negative ($\chi_3 < 0$)—*i.e.*, physical capital accumulation reduces pollution. In this case human capital turns out to be a dirty input while physical capital accumulation has a beneficial effect on the environment. This may occur in an economy that invests a lot in renewable technologies.

The last exogenous shock includes an output production function with a very large physical capital share compared to the human capital share as well as a quite large human capital share in human capital production, while the pollution stock decreases both physical and human capital production ($\nu_4, \mu_4 < 0$) and both physical and human capital accumulation increase the pollution stock ($\chi_4, \omega_4 > 0$). Besides the excessive displacement in the final production factor shares,[3] this scenario seems to describe a somewhat more realistic economy, as human capital

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[3]There is, however, a growing discussion on the reduction of the labor income share due to automation, machines and AI replacing labor in tasks that it used to perform, implying that scenarios envisaging very low intensities of human capital in final production may well deemed plausible. See, *e.g.*, Korinek and Stiglitz (2017)

happens to be a greener input than physical capital ($\omega_4 < \chi_4$).

5.2 A Tentative More Realistic Example

Considering the possible drawbacks of our previous parametrization, we now rely on a set of parameter values allowing for a more intuitive interpretation of typical real world situations. Such a configuration is summarized in Table 2.

i	α_i	γ_i	ϕ_i	ν_i	μ_i	χ_i	ω_i	ζ_i	ϑ_i	p_i
1	0.05	0.94	0.05	-0.1	-0.7	0.2	0.7	0	0	0.15
2	0.15	0.84	0.15	-0.3	-0.5	0.3	0.5	0	0.44	0.30
3	0.45	0.54	0.45	-0.5	-0.3	0.5	0.3	0	1.60	0.50
4	0.70	0.29	0.70	-0.7	0	0.7	0.2	0	1.60	0.05

TABLE 2: factor shares, additive constants and probability values corresponding to the IFS (19).

To such configuration corresponds the following affine IFS:

$$
\begin{cases}
w_1(\varphi, \psi) = \begin{bmatrix} 0.03 & 0.87 \\ -0.14 & -0.44 \end{bmatrix} \begin{bmatrix} \varphi \\ \psi \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} & \text{with prob. } p_1 = 0.15 \\[2mm]
w_2(\varphi, \psi) = \begin{bmatrix} 0.06 & 0.69 \\ -0.15 & -0.10 \end{bmatrix} \begin{bmatrix} \varphi \\ \psi \end{bmatrix} + \begin{bmatrix} 0 \\ 0.44 \end{bmatrix} & \text{with prob. } p_2 = 0.30 \\[2mm]
w_3(\varphi, \psi) = \begin{bmatrix} 0.20 & 0.39 \\ -0.15 & 0.36 \end{bmatrix} \begin{bmatrix} \varphi \\ \psi \end{bmatrix} + \begin{bmatrix} 0 \\ 1.60 \end{bmatrix} & \text{with prob. } p_3 = 0.50 \\[2mm]
w_4(\varphi, \psi) = \begin{bmatrix} 0.21 & 0.15 \\ 0 & 0.70 \end{bmatrix} \begin{bmatrix} \varphi \\ \psi \end{bmatrix} + \begin{bmatrix} 0 \\ 1.60 \end{bmatrix} & \text{with prob. } p_4 = 0.05.
\end{cases}
\tag{19}
$$

In order to solve the first $3m$ equations in system (12) in one shot we exploit the same property that we used at the beginning of Section 5.1 of having zero as the left bottom element in one of the four stochastic matrices Q_i. Specifically we set $\mu_4 = 0$ for $i = 4$, so that in the fourth shock realization the production of human capital is not being affected by pollution. The same argument followed in Section 5.1 then immediately yields $\rho_2 = \rho_4 = 0$ and $\rho_5 = \rho_1$ satisfying the first three equations in (12) for all $i = 1, \ldots, 4$; after substituting such values in the remaining equations the whole system again boils down to the same 8 equations as in (16) having only the three coefficients ρ_1, ρ_3 and ρ_6 as unknowns. To have 8 unknowns in total as well, again we choose arbitrarily 3 out of the 8 values for the multiplicative shocks z_i, η_i (or, equivalently, $\ln z_i, \ln \eta_i$) and leave the remaining 5 values for z_i, η_i ($\ln z_i, \ln \eta_i$) as unknowns; in this example we will set

$$\ln z_1 = \ln \eta_1 = \ln \eta_2 = 0 \tag{20}$$

and leave $\ln z_2$, $\ln z_3$, $\ln z_4$, $\ln \eta_3$ and $\ln \eta_4$ as unknowns, to be found as part of the solution for system (16).

We keep the individual discount factor value $\beta = 0.96$ and all the values of the constant terms ζ_i and ϑ_i in system (12)—i.e., in the IFS (19)—are the same as the coordinates of the constant vectors in the maps w_i of the IFS (15). We choose four values for each factor share (the

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and Acemoglu and Restrepo (2918), who also propose policies aimed at countervailing such tendency and the inequality it involves.

first three columns in Table 2) that, together with the probabilities listed in the last column of Table 2, on average resemble empirical evidence:

$$\mathbb{E}\left(\alpha\right) = \sum_{i=1}^{4} \alpha_i p_i = \sum_{i=1}^{4} \phi_i p_i = 0.3125, \quad \text{and} \quad \mathbb{E}\left(\gamma\right) = \sum_{i=1}^{4} \gamma_i p_i = 0.6775.$$

Under these assumptions the coefficients (6) and (7) defining the optimal policy of the model at the end of Section 3 become:

i	Δ_i	Θ_i
1	0.2867	0.8583
2	0.2637	0.7464
3	0.2201	0.6886
4	0.2133	0.7725

As far as the four pollution parameters, $\nu_i, \mu_i, \chi_i, \omega_i$, are concerned, we propose an increasing pattern for both the (absolute value of the) intensity of pollution affecting output production, ν_i, and the intensity of physical capital accumulation adding to the pollution stock, χ_i, as the physical capital share α_i in output production increases; similarly, also the (absolute value of the) intensity of pollution affecting human capital production, μ_i, and the intensity of human capital accumulation adding to the pollution stock, ω_i, increase as the human capital share γ_i in human capital production increases. The latter property is consistent with the assumption that in the fourth shock configuration pollution does not affect the production of human capital, $\mu_4 = 0$, for the technical reasons explained before.

Having set $\rho_2 = \rho_4 = 0$ and $\rho_1 = \rho_5$ and using the choice in (20) for the values of $\ln z_1, \ln \eta_1, \ln \eta_2$, we find the following unique solution for system (16) by means of the standard (symbolic, not numerical) 'solve' routine in Maple:

$$\begin{aligned} \rho_1 &= \rho_5 = -3.6368 \\ \rho_2 &= \rho_4 = 0 \\ \rho_3 &= -4.6267 \\ \rho_6 &= 0.0640, \end{aligned} \tag{21}$$

with all multiplicative shocks configurations:

i	z_i	η_i
1	1	1
2	1.1331	1
3	1.6307	0.7816
4	1.7117	0.8381

where the values of z_1, η_1 and η_2 correspond to the choice in (20) and all other values are found as a solution of (16).

Figure 2(a) traces 50,000 random iterations to approximate the attractor of the affine IFS (19) expressed in terms of the log-transformed variables φ, ψ. By applying condition (5) of Theorem 1 in La Torre et al. (2018) to the random matrix Q_i defined in (10) we find that also for the parameters' values reported in Table 2 the invariant measure supported on the attractor in Figure 2(a) is singular, as

$$|\det\left(Q_1\right)|^{p_1} |\det\left(Q_2\right)|^{p_2} |\det\left(Q_3\right)|^{p_3} |\det\left(Q_4\right)|^{p_4} < p_1^{p_1} p_2^{p_2} p_3^{p_3} p_4^{p_4} \quad \Longleftrightarrow \quad 0.1170 < 0.3191.$$

Figure 2(b) portrays the attractor of the corresponding nonlinear IFS defined in (5) for the parameters' values characterizing our growth model reported in Table 2, expressed in terms of

physical and human capital, k, h; such attractor turns out to be a downsized version of the fern in Figure 2(a) which is being translated into the positive orthant (k and h cannot be negative) and rotated by 180°, consistently with both the magnitude and the (negative) signs of the nonzero coefficients ρ_1, ρ_3 and ρ_5 in (21).
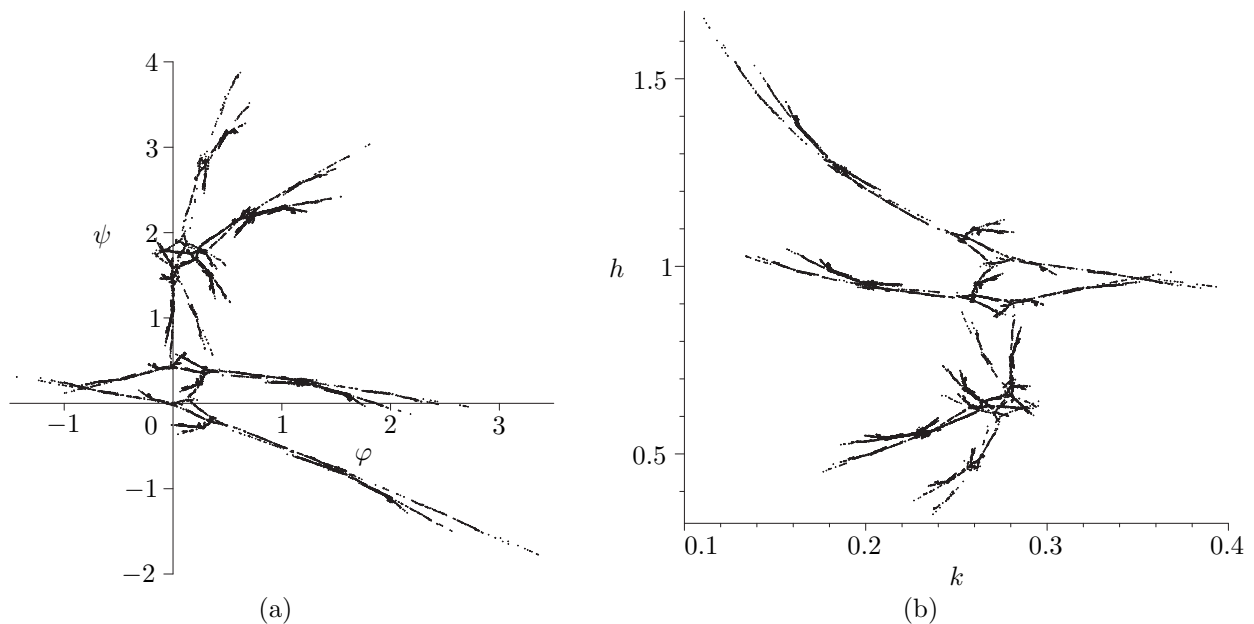

(a)   (b)

FIGURE 2: approximation through 50000 random iterations of (a) the attractor of the affine IFS in (19) and (b) the attractor of the corresponding nonlinear IFS in (5) for the parameters' values listed in Table 2.

6   Conclusions

We extend the analysis of the fractal nature of steady states in macroeconomic models, by considering a stochastic two-sector discrete-time economic growth model in which shocks affect the production function not only directly (via productivity and factor shares) but also indirectly (via a pollution externality). This extension is meaningful from both economic and mathematical points of view, since it allows to capture important economy-environment feedback and it gives rise to an IFS with potential negative coefficients. This latter characteristic, completely lacking in extant literature and due to the presence of the pollution externality, is essential in order to prove our main result: a fully-fledged macroeconomic model can generate optimal dynamics which, through an appropriate log-transformation, converges to a singular measure supported on the classical Barnsley's fern. To the best of our knowledge, no other paper is able to generate the Barnsley's fern as an attractor of the optimal dynamics emerging from a meaningful macroeconomic model.

This paper closes an open gap in the stochastic growth and fractal attractors literature. Two important questions still remain open: (i) how to characterize also absolute continuity of the invariant measure in a two-dimensional affine IFS; and (ii) how to extend the analysis in a framework in which probability are not constant but place-dependent. These further issues are on top of our future research agenda.

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