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(Article begins on next page)

ON THE AUTOMORPHISM GROUP OF A CLOSED G_2 -STRUCTURE

FABIO PODESTÀ AND ALBERTO RAFFERO

ABSTRACT. We study the automorphism group of a compact 7-manifold M endowed with a closed non-parallel G_2 -structure, showing that its identity component is abelian with dimension bounded by $\min\{6, b_2(M)\}$. This implies the non-existence of compact homogeneous manifolds endowed with an invariant closed non-parallel G_2 -structure. We also discuss some relevant examples.

1. INTRODUCTION

A seven-dimensional smooth manifold M admits a G_2 -structure if the structure group of its frame bundle can be reduced to the exceptional Lie group $G_2 \subset \mathrm{SO}(7)$. Such a reduction is characterized by the existence of a global 3-form $\varphi \in \Omega^3(M)$ satisfying a suitable non-degeneracy condition and giving rise to a Riemannian metric g_φ and to a volume form dV_φ on M via the identity

$$g_\varphi(X, Y) dV_\varphi = \frac{1}{6} \iota_X \varphi \wedge \iota_Y \varphi \wedge \varphi,$$

for all $X, Y \in \mathfrak{X}(M)$ (see e.g. [2, 13]).

By [10], the intrinsic torsion of a G_2 -structure φ can be identified with the covariant derivative $\nabla^{g_\varphi} \varphi$, and it vanishes identically if and only if both $d\varphi = 0$ and $d *_\varphi \varphi = 0$, $*_\varphi$ being the Hodge operator defined by g_φ and dV_φ . On a compact manifold, this last fact is equivalent to $\Delta_\varphi \varphi = 0$, where $\Delta_\varphi = d^*d + dd^*$ is the Hodge Laplacian of g_φ . A G_2 -structure φ satisfying any of these conditions is said to be *parallel* and its associated Riemannian metric g_φ has holonomy contained in G_2 . Consequently, g_φ is Ricci-flat and the automorphism group $\mathrm{Aut}(M, \varphi) := \{f \in \mathrm{Diff}(M) \mid f^* \varphi = \varphi\}$ of (M, φ) is finite when M is compact and $\mathrm{Hol}(g_\varphi) = G_2$.

Parallel G_2 -structures play a central role in the construction of compact manifolds with holonomy G_2 , and various known methods to achieve this result involve *closed* G_2 -structures, i.e., those whose defining 3-form φ satisfies $d\varphi = 0$ (see e.g. [2, 3, 15, 19]).

A G_2 -structure whose defining 3-form φ satisfies the equation $d *_\varphi \varphi = 0$ is called *co-closed*. On every compact 7-manifold admitting G_2 -structures there exists a co-closed one (cf. [6]), while general results on the existence of closed G_2 -structures are not known.

Due to the recent developments on the G_2 -Laplacian flow and related open problems [11, 14, 16, 17, 19, 20, 21], it is of foremost interest to provide examples of compact manifolds admitting closed G_2 -structures. Most of the known examples consist of simply connected Lie groups endowed with a left-invariant closed G_2 -form φ [5, 8, 9, 12, 17]. Compact

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locally homogeneous examples can be obtained considering the quotient of such groups by a co-compact discrete subgroup, whenever this exists. Further non-homogeneous closed G_2 -structures on the 7-torus can be constructed starting from the symplectic half-flat $SU(3)$ -structure on \mathbb{T}^6 described in [7, Ex. 5.1] (see Example 2.4 for details).

Up to now, the existence of compact homogeneous 7-manifolds admitting an invariant closed non-parallel G_2 -structure was not known (cf. [17, Question 3.1] and [18, 26]). Moreover, among the G_2 -manifolds acted on by a cohomogeneity one simple group of automorphisms studied in [4] no compact examples admitting a closed G_2 -structure occur.

In this short note, we investigate the properties of the automorphism group $\text{Aut}(M, \varphi)$ of a compact 7-manifold M endowed with a closed non-parallel G_2 -structure φ . Our main results are contained in Theorem 2.1, where we show that the identity component $\text{Aut}^0(M, \varphi)$ is necessarily abelian with dimension bounded by $\min\{6, b_2(M)\}$. In particular, this answers negatively [17, Question 3.1] and explains why compact examples cannot occur in [4]. Moreover, we also prove some interesting properties of the automorphism group action, and we describe some relevant examples.

These results shed some light on the structure of compact 7-manifolds admitting closed G_2 -structures and can be of some help in the construction of new examples.

2. THE AUTOMORPHISM GROUP

Let M be a seven-dimensional manifold endowed with a closed G_2 -structure φ , and consider its automorphism group

$$\text{Aut}(M, \varphi) := \{f \in \text{Diff}(M) \mid f^*\varphi = \varphi\}.$$

Notice that $\text{Aut}(M, \varphi)$ is a closed Lie subgroup of the isometry group $\text{Iso}(M, g_\varphi)$ of g_φ , and that the Lie algebra of its identity component $\mathfrak{g} := \text{Aut}^0(M, \varphi)$ is

$$\mathfrak{g} = \{X \in \mathfrak{X}(M) \mid \mathcal{L}_X\varphi = 0\}.$$

In particular, every $X \in \mathfrak{g}$ is a Killing vector field for the metric g_φ (cf. [19, Lemma 9.3]).

When M is compact, the Lie group $\text{Aut}(M, \varphi) \subset \text{Iso}(M, g_\varphi)$ is also compact, and we can show the following.

Theorem 2.1. *Let M be a compact seven-dimensional manifold endowed with a closed non-parallel G_2 -structure φ . Then, there exists an injective map*

$$F : \mathfrak{g} \rightarrow \mathcal{H}^2(M), \quad X \mapsto \iota_X\varphi,$$

where $\mathcal{H}^2(M)$ is the space of Δ_φ -harmonic 2-forms. As a consequence, the following properties hold:

- 1) $\dim(\mathfrak{g}) \leq b_2(M)$;
- 2) \mathfrak{g} is abelian with $\dim(\mathfrak{g}) \leq 6$;
- 3) for every $p \in M$, the isotropy subalgebra \mathfrak{g}_p has dimension $\dim(\mathfrak{g}_p) \leq 2$, with equality only when $\dim(\mathfrak{g}) = 2, 3$;
- 4) the G -action is free when $\dim(\mathfrak{g}) \geq 5$. Moreover, when $\dim(\mathfrak{g}) = 6$ the manifold M is diffeomorphic to \mathbb{T}^7 .

Proof. Let $X \in \mathfrak{g}$. Then, $0 = \mathcal{L}_X \varphi = d(\iota_X \varphi)$, as φ is closed. We claim that $\iota_X \varphi$ is co-closed (see also [19, Lemma 9.3]). Indeed, for every $p \in M$ the 2-form $\iota_X \varphi|_p$ belongs to the unique seven-dimensional G_2 -irreducible submodule $\Lambda_7^2(T_p^*M) \subset \Lambda^2(T_p^*M)$, and therefore (see e.g. [1, p. 541]) we have

$$\iota_X \varphi \wedge \varphi = 2 *_{\varphi} (\iota_X \varphi),$$

from which it follows that

$$0 = d(\iota_X \varphi \wedge \varphi) = 2 d *_{\varphi} (\iota_X \varphi).$$

Consequently, the 2-form $\iota_X \varphi$ is Δ_{φ} -harmonic and F is the restriction of the injective map $Z \mapsto \iota_Z \varphi$ to \mathfrak{g} . From this 1) follows.

As for 2), we begin observing that $\mathcal{L}_Y(\iota_X \varphi) = 0$ for all $X, Y \in \mathfrak{g}$, since every Killing field on a compact manifold preserves every harmonic form. Hence, we have

$$0 = \mathcal{L}_Y(\iota_X \varphi) = \iota_{[Y, X]} \varphi + \iota_X(\mathcal{L}_Y \varphi) = \iota_{[Y, X]} \varphi.$$

This proves that \mathfrak{g} is abelian, the map $Z \mapsto \iota_Z \varphi$ being injective. Now, G is compact abelian and it acts effectively on the compact manifold M . Therefore, the principal isotropy is trivial and $\dim(\mathfrak{g}) \leq 7$. When $\dim(\mathfrak{g}) = 7$, M can be identified with the 7-torus \mathbb{T}^7 endowed with a left-invariant metric, which is automatically flat. Hence, if φ is closed non-parallel, then $\dim(\mathfrak{g}) \leq 6$.

In order to prove 3), we fix a point p of M and we observe that the image of the isotropy representation $\rho : G_p \rightarrow O(7)$ is conjugate into G_2 . Since G_2 has rank two and G_p is abelian, the dimension of \mathfrak{g}_p is at most two. If $\dim(\mathfrak{g}_p) = 2$, then the image of ρ is conjugate to a maximal torus of G_2 and its fixed point set in $T_p M$ is one-dimensional. As $T_p(G \cdot p) \subseteq (T_p M)^{G_p}$, the dimension of the orbit $G \cdot p$ is at most one, which implies that $\dim(\mathfrak{g})$ is either two or three.

The first assertion in 4) is equivalent to proving that G_p is trivial for every $p \in M$ whenever $\dim(\mathfrak{g}) \geq 5$. In this case, $\dim(\mathfrak{g}_p) \leq 1$ by 3) and therefore the dimension of the orbit $G \cdot p$ is at least four. Then, for every $h \in G_p$, the element $\rho(h) \in G_2$ has a fixed point set containing $T_p(G \cdot p)$, hence with dimension at least four. On the other hand, a non-trivial element in G_2 is easily seen to have a fixed point set in \mathbb{R}^7 of dimension at most three. Indeed, every $u \in G_2$ is conjugate to an element of a maximal torus of G_2 contained in the maximal rank subgroup $SU(3) \subset G_2$, i.e., it can be supposed to be of the form

$$\text{diag}(z, w, \bar{z} \cdot \bar{w}) \in SU(3),$$

for some $z, w \in \mathbb{C}$ of unit norm. Thus, u fixes at least the real line $V \subset \mathbb{R}^7$ that is fixed by $SU(3)$. Moreover, if u is non-trivial, its fixed point set in the $SU(3)$ -module V^{\perp} has complex dimension at most one. This shows that $G_p = \{1_G\}$. The last assertion follows from [23]. \square

The following corollary answers negatively a question posed by Lauret in [17].

Corollary 2.2. *There are no compact homogeneous 7-manifolds endowed with an invariant closed non-parallel G_2 -structure.*

Proof. The assertion follows immediately from point 2) of Theorem 2.1. \square

In contrast to the last result, it is possible to exhibit non-compact homogeneous examples. Consider for instance a six-dimensional non-compact homogeneous space H/K endowed with an invariant symplectic half-flat $SU(3)$ -structure, namely an $SU(3)$ -structure (ω, ψ) such that $d\omega = 0$ and $d\psi = 0$ (see [25] for the classification of such spaces when H is semisimple and for more information on symplectic half-flat structures). If (ω, ψ) is not torsion-free, i.e., if $d(J\psi) \neq 0$, then the non-compact homogeneous space $(H \times \mathbb{S}^1)/K$ admits an invariant closed non-parallel G_2 -structure defined by the 3-form

$$\varphi := \omega \wedge ds + \psi,$$

where ds denotes the global 1-form on \mathbb{S}^1 .

Remark 2.3. In [4], the authors investigated G_2 -manifolds acted on by a cohomogeneity one simple group of automorphisms. Theorem 2.1 explains why compact examples in the case of closed non-parallel G_2 -structures do not occur.

The next example shows that G can be non-trivial, that the upper bound on its dimension given in 2) can be attained, and that 4) is only a sufficient condition.

Example 2.4. In [7], the authors constructed a symplectic half-flat $SU(3)$ -structure (ω, ψ) on the 6-torus \mathbb{T}^6 as follows. Let (x^1, \dots, x^6) be the standard coordinates on \mathbb{R}^6 , and let $a(x^1)$, $b(x^2)$ and $c(x^3)$ be three smooth functions on \mathbb{R}^6 such that

$$\lambda_1 := b(x^2) - c(x^3), \quad \lambda_2 := c(x^3) - a(x^1), \quad \lambda_3 := a(x^1) - b(x^2),$$

are \mathbb{Z}^6 -periodic and non-constant. Then, the following pair of \mathbb{Z}^6 -invariant differential forms on \mathbb{R}^6 induces an $SU(3)$ -structure on $\mathbb{T}^6 = \mathbb{R}^6/\mathbb{Z}^6$:

$$\begin{aligned} \omega &= dx^{14} + dx^{25} + dx^{36}, \\ \psi &= -e^{\lambda_3} dx^{126} + e^{\lambda_2} dx^{135} - e^{\lambda_1} dx^{234} + dx^{456}, \end{aligned}$$

where $dx^{ijk\dots}$ is a shorthand for the wedge product $dx^i \wedge dx^j \wedge dx^k \wedge \dots$. It is immediate to check that both ω and ψ are closed and that $d(J\psi) \neq 0$ whenever at least one of the functions $a(x^1)$, $b(x^2)$, $c(x^3)$ is not identically zero. Thus, the pair (ω, ψ) defines a symplectic half-flat $SU(3)$ -structure on the 6-torus. The automorphism group of $(\mathbb{T}^6, \omega, \psi)$ is \mathbb{T}^3 when $a(x^1)b(x^2)c(x^3) \neq 0$, while it becomes \mathbb{T}^4 (\mathbb{T}^5) when one (two) of them vanishes identically.

Now, we can consider the closed G_2 -structure on $\mathbb{T}^7 = \mathbb{T}^6 \times \mathbb{S}^1$ defined by the 3-form $\varphi = \omega \wedge ds + \psi$. Depending on the vanishing of none, one or two of the functions $a(x^1)$, $b(x^2)$, $c(x^3)$, φ is a closed non-parallel G_2 -structure and the automorphism group of (\mathbb{T}^7, φ) is \mathbb{T}^4 , \mathbb{T}^5 or \mathbb{T}^6 , respectively.

Finally, we observe that there exist examples where the upper bound on the dimension of \mathfrak{g} given in 1) is more restrictive than the upper bound given in 2).

Example 2.5. In [5], the authors obtained the classification of seven-dimensional nilpotent Lie algebras admitting closed G_2 -structures. An inspection of all possible cases shows that the Lie algebras whose second Betti number is lower than seven are those appearing in Table 1.

Let \mathfrak{n} be one of the Lie algebras in Table 1, and consider a closed non-parallel G_2 -structure φ on it. Then, left multiplication extends φ to a left-invariant G_2 -structure of the same type on the simply connected nilpotent Lie group N corresponding to \mathfrak{n} . Moreover, as the

nilpotent Lie algebra \mathfrak{n}	$b_2(\mathfrak{n})$
$(0, 0, e^{12}, e^{13}, e^{23}, e^{15} + e^{24}, e^{16} + e^{34})$	3
$(0, 0, e^{12}, e^{13}, e^{23}, e^{15} + e^{24}, e^{16} + e^{34} + e^{25})$	3
$(0, 0, e^{12}, 0, e^{13} + e^{24}, e^{14}, e^{46} + e^{34} + e^{15} + e^{23})$	5
$(0, 0, e^{12}, 0, e^{13}, e^{24} + e^{23}, e^{25} + e^{34} + e^{15} + e^{16} - 3e^{26})$	6

TABLE 1.

structure constants of \mathfrak{n} are integers, there exists a co-compact discrete subgroup $\Gamma \subset \mathbb{N}$ giving rise to a compact nilmanifold $\Gamma \backslash \mathbb{N}$ [22]. The left-invariant 3-form φ on \mathbb{N} passes to the quotient defining an invariant closed non-parallel G_2 -structure on $\Gamma \backslash \mathbb{N}$. By Nomizu Theorem [24], the de Rham cohomology group $H_{\text{dR}}^k(\Gamma \backslash \mathbb{N})$ is isomorphic to the cohomology group $H^k(\mathfrak{n}^*)$ of the Chevalley-Eilenberg complex of \mathfrak{n} . Hence, $b_2(\Gamma \backslash \mathbb{N}) = b_2(\mathfrak{n})$.

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REFERENCES

- [1] R. L. Bryant. Metrics with exceptional holonomy. *Ann. of Math.*, **126**(3), 525–576, 1987.
- [2] R. L. Bryant. Some remarks on G_2 -structures. In *Proceedings of Gökova Geometry-Topology Conference 2005*, pages 75–109. Gökova Geometry/Topology Conference (GGT), Gökova, 2006.
- [3] R. L. Bryant and F. Xu. Laplacian flow for closed G_2 -structures: Short time behavior. arXiv:1101.2004.
- [4] R. Cleyton and A. Swann. Cohomogeneity-one G_2 -structures. *J. Geom. Phys.*, **44**(2-3), 202–220, 2002.
- [5] D. Conti and M. Fernández. Nilmanifolds with a calibrated G_2 -structure. *Differential Geom. Appl.*, **29**(4), 493–506, 2011.
- [6] D. Crowley and J. Nordström. New invariants of G_2 -structures. *Geom. Topol.*, **19**(5), 2949–2992, 2015.
- [7] P. de Bartolomeis and A. Tomassini. On the Maslov index of Lagrangian submanifolds of generalized Calabi-Yau manifolds. *Internat. J. Math.*, **17**(8), 921–947, 2006.
- [8] M. Fernández. An example of a compact calibrated manifold associated with the exceptional Lie group G_2 . *J. Differential Geom.*, **26**(2), 367–370, 1987.
- [9] M. Fernández. A family of compact solvable G_2 -calibrated manifolds. *Tohoku Math. J. (2)*, **39**(2), 287–289, 1987.
- [10] M. Fernández and A. Gray. Riemannian manifolds with structure group G_2 . *Ann. Mat. Pura Appl. (4)*, **132**, 19–45, 1982.
- [11] J. Fine and C. Yao. Hypersymplectic 4-manifolds, the G_2 -Laplacian flow and extension assuming bounded scalar curvature. arXiv:1704.07620.
- [12] A. Fino and A. Raffero. Closed G_2 -structures on non-solvable Lie groups. arXiv:1712.09664.
- [13] N. Hitchin. Stable forms and special metrics. In *Global differential geometry: the mathematical legacy of Alfred Gray (Bilbao, 2000)*, v. 288 of *Contemp. Math.*, pp. 70–89. Amer. Math. Soc., 2001.
- [14] H. Huang, Y. Wang and C. Yao. Cohomogeneity-one G_2 -Laplacian flow on the 7-torus. *J. London Math. Soc.*, 2018. doi:10.1112/jlms.12137
- [15] D. D. Joyce. Compact Riemannian 7-manifolds with holonomy G_2 . I, II. *J. Differential Geom.*, **43**(2), 291–328, 329–375, 1996.
- [16] J. Lauret. Laplacian flow of homogeneous G_2 -structures and its solitons. *Proc. Lond. Math. Soc. (3)*, **114**(3), 527–560, 2017.

- [17] J. Lauret. Laplacian solitons: questions and homogeneous examples. *Differential Geom. Appl.*, **54**(B), 345–360, 2017.
- [18] H. V. Lê and M. Munir. Classification of compact homogeneous spaces with invariant G_2 -structures. *Adv. Geom.*, **12**(2), 302–328, 2012.
- [19] J. D. Lotay and Y. Wei. Laplacian flow for closed G_2 structures: Shi-type estimates, uniqueness and compactness. *Geom. Funct. Anal.*, **27**(1), 165–233, 2017.
- [20] J. D. Lotay and Y. Wei. Stability of torsion-free G_2 structures along the Laplacian flow. arXiv:1504.07771. To appear in *J. Differential Geom.*
- [21] J. D. Lotay and Y. Wei. Laplacian flow for closed G_2 structures: real analyticity. arXiv:1601.04258. To appear in *Comm. Anal. Geom.*
- [22] A. I. Malčev. On a class of homogeneous spaces. *Amer. Math. Soc. Translation*, **1951**(39), 33 pp., 1951.
- [23] P. S. Mostert. On a compact Lie group action on a manifold. *Ann. of Math.*, **65**(2), 447–455, 1957; Errata, *ibid.* **66**(2), 589, 1957.
- [24] K. Nomizu. On the cohomology of compact homogeneous spaces of nilpotent Lie groups. *Ann. of Math. (2)*, **59**, 531–538, 1954.
- [25] F. Podestà, A. Raffero. Homogeneous symplectic half-flat 6-manifolds. To appear in *Ann. Global Anal. Geom.* doi: 10.1007/s10455-018-9615-3.
- [26] F. Reidegeld. Spaces admitting homogeneous G_2 -structures. *Differential Geom. Appl.*, **28**(3), 301–312, 2010.

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