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ON THE AUTOMORPHISM GROUP OF A SYMPLECTIC HALF-FLAT 6-MANIFOLD

FABIO PODESTÀ AND ALBERTO RAFFERO

ABSTRACT. We prove that the automorphism group of a compact 6-manifold M endowed with a symplectic half-flat $SU(3)$ -structure has abelian Lie algebra with dimension bounded by $\min\{5, b_1(M)\}$. Moreover, we study the properties of the automorphism group action and we discuss relevant examples. In particular, we provide new complete examples on $T\mathbb{S}^3$ which are invariant under a cohomogeneity one action of $SO(4)$.

1. INTRODUCTION

An $SU(3)$ -structure on a six-dimensional smooth manifold M is the data of an almost Hermitian structure (g, J) with fundamental 2-form $\omega := g(J\cdot, \cdot)$ and a complex volume form $\Psi = \psi + i\hat{\psi} \in \Omega^{3,0}(M)$ such that

$$(1.1) \quad \psi \wedge \hat{\psi} = \frac{2}{3} \omega^3.$$

By [11], the whole data (g, J, Ψ) is completely determined by the real 2-form ω and the real 3-form ψ , provided that they satisfy suitable conditions (see §3 for more details).

An $SU(3)$ -structure (ω, ψ) is said to be *symplectic half-flat* if both ω and ψ are closed. In this case, the intrinsic torsion can be identified with a unique real $(1, 1)$ -form σ which is primitive with respect to ω , i.e., $\sigma \wedge \omega^2 = 0$, and fulfills $d\hat{\psi} = \sigma \wedge \omega$ (see e.g. [4]). This $SU(3)$ -structure is *half-flat* according to [4, Def. 4.1], namely $d(\omega^2) = 0$ and $d\psi = 0$, and the corresponding almost complex structure J is integrable if and only if σ vanishes identically. When this happens, (M, ω, ψ) is a *Calabi-Yau* 3-fold. Otherwise, the symplectic half-flat structure is said to be *strict*.

In recent years, symplectic half-flat structures turned out to be of interest in supersymmetric string theory. For instance, in [10] the authors proved that supersymmetric flux vacua with constant intermediate $SU(2)$ -structure [2] are related to the existence of special classes of half-flat structures on the internal 6-manifold. In particular, they showed that solutions of Type IIA SUSY equations always admit a symplectic half-flat structure. In [12], the definition of symplectic half-flat structures, which are called supersymmetric of Type IIA, is generalized in higher dimensions, and it is proved that semi-flat supersymmetric structures of Type IIA correspond to semi-flat supersymmetric structures of Type IIB via the SYZ and Fourier-Mukai transformations.

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In mathematical literature, symplectic half-flat structures were first introduced and studied in [6] and then in [8], while explicit examples were exhibited in [5, 7, 9, 16, 20]. Most of them consist of simply connected solvable Lie groups endowed with a left-invariant symplectic half-flat structure. Moreover, in [9] it was proved that every six-dimensional compact solvmanifold with an invariant symplectic half-flat structure also admits a solution of Type IIA SUSY equations.

Let M be a 6-manifold endowed with a strict symplectic half-flat structure (ω, ψ) . In the present paper, we are interested in studying the properties of the automorphism group $\text{Aut}(M, \omega, \psi) := \{f \in \text{Diff}(M) \mid f^*\omega = \omega, f^*\psi = \psi\}$, aiming at understanding how to construct non-trivial examples with high degree of symmetry.

In [16], we proved the non-existence of compact homogeneous examples and we classified all non-compact cases which are homogeneous under the action of a semisimple Lie group of automorphisms. Here, in Theorem 2.1 we show that the Lie algebra of $\text{Aut}(M, \omega, \psi)$ is abelian with dimension bounded by $\min\{5, b_1(M)\}$ whenever M is compact. This allows to obtain a direct proof of the aforementioned non-existence result. In the same theorem, we also provide useful information on geometric properties of the $\text{Aut}^0(M, \omega, \psi)$ -action on the manifold, proving in particular that the automorphism group acts by cohomogeneity one only when M is diffeomorphic to a torus. Some relevant examples are then discussed in order to show that the automorphism group can be non-trivial and that the upper bound on its dimension can be actually attained.

As our previous result on non-compact homogeneous spaces suggests, the non-compact ambient might provide a natural setting where looking for new examples. In section 3, we obtain new complete examples of symplectic half-flat structures on the tangent bundle $T\mathbb{S}^3$ which are invariant under the natural cohomogeneity one action of $\text{SO}(4)$. These include also the well-known Calabi-Yau example constructed by Stenzel [19].

2. THE AUTOMORPHISM GROUP

Let M be a six-dimensional manifold endowed with an $\text{SU}(3)$ -structure (ω, ψ) . The *automorphism group* of (M, ω, ψ) consists of the diffeomorphisms of M preserving the $\text{SU}(3)$ -structure, namely

$$\text{Aut}(M, \omega, \psi) := \{f \in \text{Diff}(M) \mid f^*\omega = \omega, f^*\psi = \psi\}.$$

Clearly, $\text{Aut}(M, \omega, \psi)$ is a closed Lie subgroup of the isometry group $\text{Iso}(M, g)$, as every automorphism preserves the Riemannian metric g induced by the pair (ω, ψ) . The Lie algebra of the identity component $\mathfrak{G} := \text{Aut}^0(M, \omega, \psi)$ is

$$\mathfrak{g} = \{X \in \mathfrak{X}(M) \mid \mathcal{L}_X\omega = 0, \mathcal{L}_X\psi = 0\},$$

and every $X \in \mathfrak{g}$ is a Killing vector field for the metric g . Moreover, the Lie group $\text{Aut}(M, \omega, \psi) \subset \text{Iso}(M, g)$ is compact whenever M is compact.

If (M, ω, ψ) is a Calabi-Yau 3-fold, i.e., if ω , ψ and $\widehat{\psi}$ are all closed, then the Riemannian metric g is Ricci-flat and $\text{Hol}(g) \subseteq \text{SU}(3)$. When M is compact and the holonomy group is precisely $\text{SU}(3)$, it follows from Bochner's Theorem that $\text{Aut}(M, \omega, \psi)$ is finite.

We now focus on *strict symplectic half-flat* structures, namely $SU(3)$ -structures (ω, ψ) such that

$$d\omega = 0, \quad d\psi = 0, \quad d\widehat{\psi} = \sigma \wedge \omega,$$

with $\sigma \in [\Omega_0^{1,1}(M)] := \{\kappa \in \Omega^2(M) \mid J\kappa = \kappa, \kappa \wedge \omega^2 = 0\}$ not identically vanishing. Notice that the condition on σ is equivalent to requiring that the almost complex structure J induced by (ω, ψ) is non-integrable (cf. e.g. [4]). In this case, we can show the following result.

Theorem 2.1. *Let M be a compact six-dimensional manifold endowed with a strict symplectic half-flat structure (ω, ψ) . Then, there exists an injective map*

$$\mathcal{F} : \mathfrak{g} \rightarrow \mathcal{H}^1(M), \quad X \mapsto \iota_X \omega,$$

where $\mathcal{H}^1(M)$ is the space of Δ_g -harmonic 1-forms. Consequently, the following properties hold:

- 1) $\dim(\mathfrak{g}) \leq b_1(M)$;
- 2) \mathfrak{g} is abelian with $\dim(\mathfrak{g}) \leq 5$;
- 3) for every $p \in M$, the isotropy subalgebra \mathfrak{g}_p has dimension $\dim(\mathfrak{g}_p) \leq 2$. If $\dim(\mathfrak{g}_p) = 2$ for some p , then $G_p = G$;
- 4) the G -action is free when $\dim(\mathfrak{g}) \geq 4$. In particular, when $\dim(\mathfrak{g}) = 5$ the manifold M is diffeomorphic to \mathbb{T}^6 .

Before proving the theorem, we show a general lemma.

Lemma 2.2. *Let (ω, ψ) be an $SU(3)$ -structure. Then, for every vector field X the following identity holds*

$$\iota_X \psi \wedge \psi = -2 * (\iota_X \omega),$$

where $*$ denotes the Hodge operator determined by the Riemannian metric g and the orientation $dV_g = \frac{1}{6}\omega^3$.

Proof. From the equation $\iota_X \Psi \wedge \Psi = 0$, which holds for every vector field X , we have

$$\iota_X \psi \wedge \psi = \iota_X \widehat{\psi} \wedge \widehat{\psi}, \quad \iota_X \psi \wedge \widehat{\psi} = -\iota_X \widehat{\psi} \wedge \psi.$$

Using the above identities and the relations $\iota_X \psi = \iota_{JX} \widehat{\psi}$, $\iota_{JX} \psi = -\iota_X \widehat{\psi}$, we get

$$\begin{aligned} \iota_X \psi \wedge \psi &= \iota_{JX} \widehat{\psi} \wedge \psi \\ &= \iota_{JX} (\widehat{\psi} \wedge \psi) + \widehat{\psi} \wedge \iota_{JX} \psi \\ &= \iota_{JX} (\widehat{\psi} \wedge \psi) - \widehat{\psi} \wedge \iota_X \widehat{\psi} \\ &= \iota_{JX} (\widehat{\psi} \wedge \psi) - \psi \wedge \iota_X \psi. \end{aligned}$$

Hence, $2\iota_X \psi \wedge \psi = \iota_{JX} (\widehat{\psi} \wedge \psi)$. Now, from condition (1.1) we know that $\widehat{\psi} \wedge \psi = -\frac{2}{3}\omega^3 = -4dV_g$. Thus,

$$\iota_X \psi \wedge \psi = -2\iota_{JX} dV_g = -2 * (JX)^\flat = -2 * (\iota_X \omega).$$

□

Proof of Theorem 2.1.

Let $X \in \mathfrak{g}$. Then, using the closedness of ω we have $0 = \mathcal{L}_X \omega = d(\iota_X \omega)$. Moreover, since $d\psi = 0$ and $\mathcal{L}_X \psi = 0$, then $d(\iota_X \psi \wedge \psi) = 0$ and Lemma 2.2 implies that $d * (\iota_X \omega) = 0$. Hence, the 1-form $\iota_X \omega$ is Δ_g -harmonic and \mathcal{F} coincides with the injective map $Z \mapsto \iota_Z \omega$ restricted to \mathfrak{g} . From this 1) follows.

In order to prove 2), we begin recalling that every Killing field on a compact manifold preserves every harmonic form. Consequently, for all $X, Y \in \mathfrak{g}$ we have

$$0 = \mathcal{L}_Y(\iota_X \omega) = \iota_{[Y, X]} \omega + \iota_X(\mathcal{L}_Y \omega) = \iota_{[Y, X]} \omega.$$

Since the map $Z \mapsto \iota_Z \omega$ is injective, we obtain that \mathfrak{g} is abelian. Now, G is compact abelian and it acts effectively on the compact manifold M . Therefore, the principal isotropy is trivial and $\dim(\mathfrak{g}) \leq 6$. When $\dim(\mathfrak{g}) = 6$, M can be identified with the 6-torus \mathbb{T}^6 endowed with a left-invariant metric, which is automatically flat. Hence, if (ω, ψ) is strict symplectic half-flat, then $\dim(\mathfrak{g}) \leq 5$.

As for 3), we fix a point p of M and we observe that the image of the isotropy representation $\rho : G_p \rightarrow O(6)$ is conjugate into $SU(3)$. Since $SU(3)$ has rank two and G_p is abelian, the dimension of \mathfrak{g}_p is at most two. If $\dim(\mathfrak{g}_p) = 2$, then the image of ρ is conjugate to a maximal torus of $SU(3)$ and its fixed point set in $T_p M$ is trivial. As $T_p(G \cdot p) \subseteq (T_p M)^{G_p}$, the orbit $G \cdot p$ is zero-dimensional, which implies that $\dim(\mathfrak{g}) = 2$.

Assertion 4) is equivalent to proving that G_p is trivial for every $p \in M$ whenever $\dim(\mathfrak{g}) \geq 4$. In this case, $\dim(\mathfrak{g}_p) \leq 1$ by 3), and therefore $\dim(G \cdot p) \geq 3$. If G_p contains a non-trivial element h , then $\rho(h)$ fixes every vector in $T_p(G \cdot p)$ and, consequently, its fixed point set in $T_p M$ must be non-trivial of dimension at least three. On the other hand, a non-trivial element of $SU(3)$ is easily seen to have a fixed point set of dimension at most two. This shows that $G_p = \{1_G\}$. The last assertion follows immediately from [14]. \square

Point 2) in the above theorem gives a direct proof of a result obtained in [16].

Corollary 2.3. *There are no compact homogeneous 6-manifolds endowed with an invariant strict symplectic half-flat structure.*

It is worth observing here that the non-compact case is less restrictive. For instance, it is possible to exhibit non-compact examples which are homogeneous under the action of a *semisimple* Lie group of automorphisms (see e.g. [16]). Moreover, in the next section we shall construct non-compact examples of cohomogeneity one with respect to a semisimple Lie group of automorphisms.

The next example was given in [8]. It shows that G can be non-trivial, that the upper bound on its dimension given in 2) can be attained, and that 4) is only a sufficient condition.

Example 2.4. On \mathbb{R}^6 with standard coordinates (x^1, \dots, x^6) consider three smooth functions $a(x^1)$, $b(x^2)$, $c(x^3)$ in such a way that

$$\lambda_1 := b(x^2) - c(x^3), \quad \lambda_2 := c(x^3) - a(x^1), \quad \lambda_3 := a(x^1) - b(x^2),$$

are \mathbb{Z}^6 -periodic. Then, the following pair of \mathbb{Z}^6 -invariant differential forms on \mathbb{R}^6 induces an $SU(3)$ -structure on $\mathbb{T}^6 = \mathbb{R}^6 / \mathbb{Z}^6$:

$$\omega = dx^{14} + dx^{25} + dx^{36}, \quad \psi = -e^{\lambda_3} dx^{126} + e^{\lambda_2} dx^{135} - e^{\lambda_1} dx^{234} + dx^{456},$$

where $dx^{ijk\dots}$ is a shorthand for the wedge product $dx^i \wedge dx^j \wedge dx^k \wedge \dots$. It is immediate to check that (ω, ψ) is strict symplectic half-flat whenever the functions λ_i are not all constant. The automorphism group of $(\mathbb{T}^6, \omega, \psi)$ is \mathbb{T}^3 when $a(x^1) b(x^2) c(x^3) \neq 0$, while it becomes \mathbb{T}^4 (\mathbb{T}^5) when one (two) of them vanishes identically.

Finally, we observe that there exist examples where the upper bound on the dimension of \mathfrak{g} given in 1) is more restrictive than the upper bound given in 2).

Example 2.5. In [5], the authors obtained the classification of six-dimensional nilpotent Lie algebras admitting symplectic half-flat structures. The only two non-abelian cases are described up to isomorphism by the following structure equations

$$(0, 0, 0, 0, e^{12}, e^{13}), \quad (0, 0, 0, e^{12}, e^{13}, e^{23}).$$

Denote by N the simply connected nilpotent Lie group corresponding to one of the above Lie algebras, and endow it with a left-invariant strict symplectic half-flat structure (ω, ψ) . By [13], there exists a co-compact discrete subgroup $\Gamma \subset N$ giving rise to a compact nilmanifold $\Gamma \backslash N$. Moreover, the left-invariant pair (ω, ψ) on N passes to the quotient defining an $SU(3)$ -structure of the same type on $\Gamma \backslash N$. By [15], we have that $b_1(\Gamma \backslash N)$ is either four or three.

3. NON-COMPACT COHOMOGENEITY ONE EXAMPLES

In this section, we construct complete examples of strict symplectic half-flat structures on a non-compact 6-manifold admitting a cohomogeneity one action of a semisimple Lie group of automorphisms. This points out the difference between the compact and the non-compact case, and together with the results in [16, §4.3] it suggests that the non-compact ambient provides a natural setting to obtain new examples.

From now on, we consider the natural cohomogeneity one action on $M = TS^3 \cong \mathbb{S}^3 \times \mathbb{R}^3$ induced by the transitive $SO(4)$ -action on \mathbb{S}^3 . Then, we have

$$TS^3 \cong SO(4) \times_{SO(3)} \mathbb{R}^3.$$

We refer the reader to [1, 14, 17, 18] for basic notions on cohomogeneity one isometric actions. Following the notation of [18], we consider the Lie algebra $\mathfrak{so}(4) \cong \mathfrak{su}(2) + \mathfrak{su}(2)$ and we fix the following basis of $\mathfrak{su}(2)$

$$H := \frac{1}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad E := \frac{1}{2\sqrt{2}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad V := \frac{1}{2\sqrt{2}} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Let $\gamma : \mathbb{R} \rightarrow M$ be a normal geodesic such that $p := \gamma(0) \in \mathbb{S}^3$ and $\gamma_t := \gamma(t)$ is a regular point for all $t \neq 0$. The singular isotropy subalgebra is $\mathfrak{so}(4)_p = \mathfrak{su}(2)_{\text{diag}}$, while the principal isotropy subalgebra $\mathfrak{k} := \mathfrak{so}(4)_{\gamma_t}$, $t \neq 0$, is one-dimensional and spanned by (H, H) . We consider the following basis of $\mathfrak{so}(4) \cong \mathfrak{su}(2) + \mathfrak{su}(2)$

$$\begin{aligned} E_1 &:= (E, 0), & V_1 &:= (V, 0), & E_2 &:= (0, E), & V_2 &:= (0, V), \\ U &:= (H, H), & A &:= (H, -H). \end{aligned}$$

We let $\xi := \frac{\partial}{\partial t}$, and for any $Z \in \mathfrak{so}(4)$ we denote by \widehat{Z} the corresponding fundamental vector field on M . Then, a basis of $T_{\gamma_t}M$ for $t \neq 0$ is given by

$$(\xi, \widehat{A}, \widehat{E}_1, \widehat{V}_1, \widehat{E}_2, \widehat{V}_2)_{\gamma_t}.$$

We shall denote the dual coframe along γ_t by $(\xi^*, A^*, E_1^*, V_1^*, E_2^*, V_2^*)_{\gamma_t}$, where $\xi^* := dt$.

Let $K \subset \mathrm{SO}(4)$ be the principal isotropy subgroup corresponding to the Lie algebra \mathfrak{k} . The space of K -invariant 2-forms on $T_{\gamma_t}M$, $t \neq 0$, is spanned by

$$\begin{aligned} \omega_1 &:= \xi^* \wedge A^*, & \omega_2 &:= E_1^* \wedge V_1^*, & \omega_3 &:= E_2^* \wedge V_2^*, \\ \omega_4 &:= E_1^* \wedge E_2^* + V_1^* \wedge V_2^*, & \omega_5 &:= E_1^* \wedge V_2^* - V_1^* \wedge E_2^*. \end{aligned}$$

These forms extend as $\mathrm{SO}(4)$ -invariant 2-forms on the regular part $M_0 := \mathbb{S}^3 \times \mathbb{R}^+$. By [18], their differentials along γ_t are

$$(3.1) \quad \begin{aligned} d\omega_1|_{\gamma_t} &= \frac{1}{4} \xi^* \wedge (\omega_2 - \omega_3), & d\omega_2|_{\gamma_t} &= d\omega_3|_{\gamma_t} = 0, \\ d\omega_4|_{\gamma_t} &= -2 A^* \wedge \omega_5, & d\omega_5|_{\gamma_t} &= 2 A^* \wedge \omega_4. \end{aligned}$$

We now describe the general $\mathrm{SO}(4)$ -invariant symplectic 2-form ω on M . Along γ_t , $t \neq 0$, we have

$$\omega|_{\gamma_t} = \sum_{i=1}^5 f_i(t) \omega_i,$$

for suitable smooth functions $f_i \in \mathcal{C}^\infty(\mathbb{R}^+)$. By [18, Prop. 6.1], the $\mathrm{SO}(4)$ -invariant 2-form ω on M_0 corresponding to $\omega|_{\gamma_t}$ admits a smooth extension to the whole M if and only if the functions f_i extend smoothly on \mathbb{R} as follows:

- i) f_1 and f_4 are even and f_2, f_3, f_5 are odd;
- ii) $f_3'(0) = \frac{1}{2} f_1(0) + f_2'(0)$, $f_5'(0) = -\frac{1}{4} f_1(0) - f_2'(0)$, and $f_4(0) = 0$.

Moreover, $\omega|_p$ is non-degenerate if and only if $f_1(0) \neq 0$.

Using (3.1), we compute $d\omega$ and we see that ω is closed if and only if

$$f_4, f_5 \equiv 0, \quad \begin{cases} f_2' = -\frac{1}{4} f_1 \\ f_3' = \frac{1}{4} f_1 \end{cases}.$$

Combining this with the extendability conditions, we obtain that every $\mathrm{SO}(4)$ -invariant symplectic 2-form ω on M can be written as

$$(3.2) \quad \omega|_{\gamma_t} = f_1(t) \omega_1 + f_2(t) \omega_2 + f_3(t) \omega_3, \quad t \neq 0,$$

with $f_1 \in \mathcal{C}^\infty(\mathbb{R})$ even and nowhere vanishing, and

$$f_2(t) = -\frac{1}{4} \int_0^t f_1(s) ds = -f_3(t).$$

Notice that $\omega^3|_{\gamma_t} = -6 f_1 f_2^2 \omega_1 \wedge \omega_2 \wedge \omega_3$ at every regular point of the geodesic γ_t . As f_1 is nowhere zero, we may assume that $f_1 < 0$, so that the volume form $\xi^* \wedge A^* \wedge E_1^* \wedge V_1^* \wedge E_2^* \wedge V_2^*$ defines the same orientation on $T_{\gamma_t}M$ as $\frac{1}{6} \omega^3|_{\gamma_t}$ for all $t \in \mathbb{R}^+$.

We now search for an $\text{SO}(4)$ -invariant closed 3-form $\psi \in \Omega^3(M)^{\text{SO}(4)}$ so that the pair (ω, ψ) defines an $\text{SO}(4)$ -invariant symplectic half-flat structure on M . For the sake of simplicity, we make the following Ansatz

$$\psi = du, \quad u \in \Omega^2(M)^{\text{SO}(4)}.$$

As before, along γ_t , $t \neq 0$, we can write

$$(3.3) \quad u|_{\gamma_t} = \sum_{i=1}^5 \phi_i(t) \omega_i,$$

for some smooth functions $\phi_i \in \mathcal{C}^\infty(\mathbb{R}^+)$ satisfying the same extendability conditions as the f_i 's. Then, omitting the dependence on t for brevity, we have

$$(3.4) \quad \psi|_{\gamma_t} = \psi_2 \xi^* \wedge \omega_2 + \psi_3 \xi^* \wedge \omega_3 + \phi'_4 \xi^* \wedge \omega_4 + \phi'_5 \xi^* \wedge \omega_5 + 2A^* \wedge (\phi_5 \omega_4 - \phi_4 \omega_5),$$

where $\psi_2 := \frac{1}{4} \phi_1 + \phi'_2$ and $\psi_3 := \phi'_3 - \frac{1}{4} \phi_1$.

By [11], the pair (ω, ψ) defines an $\text{SU}(3)$ -structure if and only if the following conditions hold:

- a) the compatibility condition $\omega \wedge \psi = 0$;
- b) the stability condition $P(\psi) < 0$, P being the characteristic quartic polynomial defined on 3-forms (see below for the definition);
- c) denoted by J the almost complex structure induced by (ω, ψ) , then the complex volume form $\Psi := \psi + i \widehat{\psi}$ with $\widehat{\psi} := J\psi$ fulfills the normalization condition (1.1);
- d) the symmetric bilinear form $g := \omega(\cdot, J\cdot)$ is positive definite.

The compatibility condition a) along γ_t reads $f_2 \psi_3 + f_3 \psi_2 = 0$. Since $f_2 = -f_3 \neq 0$, this implies

$$(3.5) \quad \psi_2 = \psi_3.$$

Recall that at each point $q \in M$ the 3-form ψ gives rise to an endomorphism $S \in \text{End}(T_q M)$ defined as follows for every $\theta \in T_q^* M$ and every $v \in T_q M$

$$\iota_v \psi \wedge \psi \wedge \theta = \theta(S(v)) \frac{\omega^3}{6}.$$

The endomorphism S satisfies $S^2 = P(\psi)\text{Id}$, and it gives rise to the almost complex structure $J := \frac{1}{\sqrt{|P(\psi)|}} S$ when $P(\psi) < 0$.

From the expressions

$$\begin{aligned} \iota_\xi \psi \wedge \psi|_{\gamma_t} &= 2(\psi_2^2 - (\phi'_4)^2 - (\phi'_5)^2) \xi^* \wedge \omega_2 \wedge \omega_3 - 4(\phi'_4 \phi_5 - \phi_4 \phi'_5) A^* \wedge \omega_2 \wedge \omega_3, \\ \iota_{\widehat{A}} \psi \wedge \psi|_{\gamma_t} &= 4(\phi_4 \phi'_5 - \phi'_4 \phi_5) \xi^* \wedge \omega_2 \wedge \omega_3 - 8(\phi_4^2 + \phi_5^2) A^* \wedge \omega_2 \wedge \omega_3, \end{aligned}$$

we see that the endomorphism $S \in \text{End}(T_{\gamma_t} M)$ maps the subspace of $T_{\gamma_t} M$ spanned by ξ and $\widehat{A}|_{\gamma_t}$ into itself with associated matrix given by

$$(3.6) \quad -\frac{1}{f_1 f_2^2} \begin{pmatrix} 4(\phi'_4 \phi_5 - \phi_4 \phi'_5) & 8(\phi_4^2 + \phi_5^2) \\ 2(\psi_2^2 - (\phi'_4)^2 - (\phi'_5)^2) & -4(\phi'_4 \phi_5 - \phi_4 \phi'_5) \end{pmatrix}.$$

Since the curve γ_t must be a normal geodesic for the metric g induced by (ω, ψ) , it follows that the tangent vector ξ is orthogonal to the orbit $\text{SO}(4) \cdot \gamma_t$ at every regular point of γ_t . In particular, we have

$$0 = g(\xi, \widehat{A}) = \omega(\xi, J(\widehat{A})) = \frac{1}{\sqrt{|P(\psi)|}} \omega(\xi, S(\widehat{A})) = \frac{4}{f_2^2 \sqrt{|P(\psi)|}} (\phi_4' \phi_5 - \phi_4 \phi_5'),$$

from which we get

$$(3.7) \quad \phi_4' \phi_5 = \phi_4 \phi_5'.$$

Using (3.5), (3.7) and the definition of $P(\psi)$, we obtain

$$(3.8) \quad P(\psi) = \frac{16}{f_1^2 f_2^4} (\phi_4^2 + \phi_5^2) (\psi_2^2 - (\phi_4')^2 - (\phi_5')^2).$$

Consequently, the stability condition b) reads

$$(3.9) \quad \psi_2^2 - (\phi_4')^2 - (\phi_5')^2 < 0, \quad \phi_4^2 + \phi_5^2 \neq 0,$$

for all $t \in \mathbb{R}^+$.

We now note that the vector field $J(\xi)$ is tangent to the $\text{SO}(4)$ -orbits and it belongs to the space of K -fixed vectors in $T_{\gamma_t}(\text{SO}(4) \cdot \gamma_t)^{\text{K}}$, which is spanned by $\widehat{A}|_{\gamma_t}$. Since the geodesic γ_t has unit speed, we see that

$$(3.10) \quad 1 = g(\xi, \xi) = \omega(\xi, J(\xi)) = -\frac{2}{f_2^2 \sqrt{|P(\psi)|}} (\psi_2^2 - (\phi_4')^2 - (\phi_5')^2).$$

Using (3.8), the relation (3.10) implies that

$$(3.11) \quad 4(\phi_4^2 + \phi_5^2) = f_1^2 ((\phi_4')^2 + (\phi_5')^2 - \psi_2^2).$$

Let us now focus on c). From (3.6) and (3.11), we obtain $J(\xi) = \frac{1}{f_1} \widehat{A}|_{\gamma_t}$. Using this and the identity $\widehat{\psi} = J\psi = -\psi(J \cdot, \cdot)$, we have

$$(3.12) \quad \widehat{\psi}|_{\gamma_t} = \xi^* \wedge \left(2 \frac{\phi_4}{f_1} \omega_5 - 2 \frac{\phi_5}{f_1} \omega_4 \right) + f_1 A^* \wedge (\psi_2 (\omega_2 + \omega_3) + \phi_4' \omega_4 + \phi_5' \omega_5).$$

Now, the normalization condition $\psi \wedge \widehat{\psi} = \frac{2}{3} \omega^3$ gives

$$4(\phi_4^2 + \phi_5^2) - f_1^2 (\psi_2^2 - (\phi_4')^2 - (\phi_5')^2) = 2 f_1^2 f_2^2.$$

Combining this with (3.11), we obtain

$$(3.13) \quad \phi_4^2 + \phi_5^2 = \frac{1}{4} (f_1 f_2)^2.$$

Note that (3.8), (3.11) and (3.13) imply

$$P(\psi) \equiv -4$$

along the geodesic γ_t . Thus, the stability of ψ holds also at $t = 0$.

Going back to (3.7), we see that either $\phi_4 = \lambda \phi_5$ or $\phi_5 = \lambda \phi_4$ for some $\lambda \in \mathbb{R} \setminus \{0\}$. Since ϕ_4 and ϕ_5 extend as an even and an odd function on \mathbb{R} , respectively, we see that either $\phi_4 \equiv 0$ or $\phi_5 \equiv 0$. As $f_1 f_2$ is an odd function on \mathbb{R} , (3.13) implies that

$$(3.14) \quad \phi_4 \equiv 0, \quad \phi_5 = \pm \frac{1}{2} f_1 f_2.$$

The matrix associated with the symmetric bilinear form $\omega(\cdot, J\cdot)$ along γ_t , $t \in \mathbb{R}^+$, is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & f_1^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 \frac{\phi'_5 \phi_5}{f_1 f_2} & 0 & -2 \frac{\psi_2 \phi_5}{f_1 f_2} & 0 \\ 0 & 0 & 0 & -2 \frac{\phi'_5 \phi_5}{f_1 f_2} & 0 & -2 \frac{\psi_2 \phi_5}{f_1 f_2} \\ 0 & 0 & -2 \frac{\psi_2 \phi_5}{f_1 f_2} & 0 & -2 \frac{\phi'_5 \phi_5}{f_1 f_2} & 0 \\ 0 & 0 & 0 & -2 \frac{\psi_2 \phi_5}{f_1 f_2} & 0 & -2 \frac{\phi'_5 \phi_5}{f_1 f_2} \end{pmatrix},$$

and condition d) can be written as

$$-2 \frac{\phi'_5 \phi_5}{f_1 f_2} > 0, \quad \psi_2^2 < (\phi'_5)^2.$$

The former condition is equivalent to $(f_2^2)'' > 0$, while the latter is satisfied whenever ψ is stable (cf. (3.9)).

Note that the metric g extends smoothly over the singular orbit \mathbb{S}^3 to a Hermitian symmetric bilinear form. The restriction of g on $T_p \mathbb{S}^3$ is positive definite as $g_p(\widehat{A}, \widehat{A}) = f_1^2(0) > 0$ and the orbit $\text{SO}(4) \cdot p$ is isotropy irreducible. Moreover, $T_p M = T_p \mathbb{S}^3 \oplus J(T_p \mathbb{S}^3)$, and from this we see that g_p is positive definite.

Summing up, the existence of a complete $\text{SO}(4)$ -invariant symplectic half-flat structure (ω, ψ) on M is equivalent to the existence of a smooth function $f_1 \in C^\infty(\mathbb{R})$ satisfying the following conditions:

- 1) f_1 is even and negative;
- 2) the function $f_2(t) := -\frac{1}{4} \int_0^t f_1(s) ds$ satisfies $(f_2^2)'' > 0$;
- 3) there exists an even smooth function $\psi_2 \in C^\infty(\mathbb{R})$ satisfying $\psi_2^2 = [(f_2^2)']^2 - f_2^2$.

Indeed, given f_1 we define the symplectic form ω on M as in (3.2), with $f_3 = -f_2$. As for ψ , we let $\psi_3 := \psi_2$, $\phi_4 := 0$, and $\phi_5 := \pm \frac{1}{2} f_1 f_2$ in (3.4). Then, (3.11) and (3.13) imply $\psi_2^2 = (\phi'_5)^2 - f_2^2$, and we can choose the sign in the definition of ϕ_5 so that the extendability condition $\phi'_5(0) = -\psi_2(0)$ given in ii) is satisfied. It is also easy to see that we may choose ϕ_1, ϕ_2, ϕ_3 so that $\psi_2 = \frac{1}{4} \phi_1 + \phi'_2$ and $\psi_3 = \phi'_3 - \frac{1}{4} \phi_1$, and the corresponding u as in (3.3) extends to a global 2-form on M . The resulting 3-form ψ is then stable by condition 3) and (3.8). The stability condition together with the inequality in 2) implies that the induced bilinear form g is everywhere positive definite. Hence, we have proved the following result.

Proposition 3.1. *The existence of a complete $\text{SO}(4)$ -invariant symplectic half-flat structure (ω, ψ) on $T\mathbb{S}^3 = \text{SO}(4) \times_{\text{SO}(3)} \mathbb{R}^3$ with $\psi \in d\Omega^2(M)$ is equivalent to the existence of a smooth function $f_1 \in C^\infty(\mathbb{R})$ satisfying conditions 1), 2), 3).*

Recall that the symplectic half-flat structure (ω, ψ) is strict if and only if the unique 2-form $\sigma \in [\Omega_0^{1,1}(M)]$ fulfilling $d\widehat{\psi} = \sigma \wedge \omega$ is not identically zero. Starting from (3.12),

using (3.1) and the identity $dA^*|_{\gamma_t} = \frac{1}{4}(\omega_3 - \omega_2)$ (cf. [18, (3.27)]), we obtain

$$d\widehat{\psi}|_{\gamma_t} = \left((f_1\phi'_5)' - 4\frac{\phi_5}{f_1} \right) \omega_1 \wedge \omega_5 + (f_1\psi_2)' \omega_1 \wedge (\omega_2 + \omega_3),$$

whence

$$\sigma|_{\gamma_t} = \frac{1}{f_1} (f_1\psi_2)' (\omega_2 + \omega_3) + \frac{1}{f_1} \left((f_1\phi'_5)' - 4\frac{\phi_5}{f_1} \right) \omega_5.$$

By [3], we know that the scalar curvature of the metric g induced by a symplectic half-flat structure is given by $\text{Scal}(g) = -\frac{1}{2}|\sigma|^2$. Hence, in our case we have

$$\text{Scal}(g)|_{\gamma_t} = -\frac{1}{f_1^2 f_2^2} \left[((f_1\psi_2)')^2 - \left((f_1\phi'_5)' - 4\frac{\phi_5}{f_1} \right)^2 \right] = -\left(\frac{(f_1\psi_2)'}{f_1\phi'_5} \right)^2,$$

where the second equality follows from the relations obtained so far.

We may construct plenty of complete $\text{SO}(4)$ -invariant strict symplectic half-flat structures on M by choosing a suitable f_1 as above. For instance, the function

$$f_1(t) := -\cosh(t), \quad t \in \mathbb{R},$$

fits in with conditions 1), 2), 3). With this choice, the scalar curvature is

$$\text{Scal}(g)|_{\gamma_t} = -\tanh^2(t) \frac{(6\cosh^2(t) - 5)^2}{4\cosh^4(t) - 8\cosh^2(t) + 5}.$$

This shows that the resulting symplectic half-flat structure is strict and non-homogeneous.

Note that the vanishing of σ is equivalent to the vanishing of $\text{Scal}(g)$. Hence, setting $(f_1\psi_2)' = 0$, the resulting $\text{SU}(3)$ -structure (ω, ψ) is Calabi-Yau and the associated metric is the well-known Stenzel's Ricci-flat metric on $T\mathbb{S}^3$ (cf. [19]).

Finally, we remark that the scalar curvature always vanishes at $t = 0$. Indeed, $(f_1\psi_2)'(0) = 0$, as $f_1\psi_2$ is even, while $f_1(0)\phi'_5(0) \neq 0$. This implies that an $\text{SO}(4)$ -invariant symplectic half-flat structure (ω, ψ) with ψ exact has constant scalar curvature if and only if it is Calabi-Yau.

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