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Published version:

DOI:10.1007/s11854-019-0048-0

ANISOTROPIC SHUBIN OPERATORS AND EIGENFUNCTION EXPANSIONS IN GELFAND-SHILOV SPACES

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Abstract. We derive new results on the characterization of Gelfand–Shilov spaces $\mathcal{S}^\mu_\nu(\mathbb{R}^n)$, $\mu, \nu > 0$, $\mu + \nu \geq 1$ by Gevrey estimates of the L^2 norms of iterates of (m, k) anisotropic globally elliptic Shubin (or Γ) type operators, $(-\Delta)^{m/2} + |x|^k$ with $m, k \in 2\mathbb{N}$ being a model operator, and on the decay of the Fourier coefficients in the related eigenfunction expansions. Similar results are obtained for the spaces $\Sigma^\mu_\nu(\mathbb{R}^n)$, $\mu, \nu > 0$, $\mu+\nu > 1$, cf. (1.2). In contrast to the symmetric case $\mu = \nu$ and $k = m$ (classical Shubin operators) we encounter resonance type phenomena involving the ratio $\kappa := \mu/\nu$; namely we obtain a characterization of $\mathcal{S}^\mu_\nu(\mathbb{R}^n)$ and $\Sigma^\mu_\nu(\mathbb{R}^n)$ in the case $\mu = kt/(k + m), \nu = mt/(k + m), t \geq 1$, that is, when $\kappa = k/m \in \mathbb{Q}$.

1. Introduction and statement of the results

The main goal of the paper is to prove results on the characterization of the non-symmetric ($\mu \neq \nu$) Gelfand–Shilov spaces $\mathcal{S}^\mu_\nu(\mathbb{R}^n)$, $\mu, \nu > 0$, $\mu + \nu \geq 1$ by Gevrey estimates of the L^2 norms of the iterates $P^\ell u$, $\ell = 1, 2, \ldots, u \in \mathscr{S}(\mathbb{R}^n)$, of positive anisotropic globally elliptic Shubin differential operators P of the type (m, k), m, k being even natural numbers, and on the decay of the Fourier coefficients u_j, $j \in \mathbb{N}$, in the eigenfunction expansions $u = \sum_{j=1}^\infty u_j\varphi_j$, where $\{\varphi_j\}_{j=1}^\infty$ stands for an orthonormal basis of eigenfunctions associated to the operator P. The (m, k) Shubin elliptic differential operators are modelled by

$$(1.1) \qquad \mathcal{H}^{m,k}_n := (-\Delta)^{m/2} + |x|^k, \quad |x| = \sqrt{x_1^2 + \ldots + x_n^2},\ k, m \in 2\mathbb{N}.$$

We recall that for $\mu > 0, \nu > 0$, the inductive (respectively, projective) Gelfand-Shilov classes $\mathcal{S}^\mu_\nu(\mathbb{R}^n)$, $\mu + \nu \geq 1$ (respectively, $\Sigma^\mu_\nu(\mathbb{R}^n)$, $\mu + \nu > 1$), are defined as the set of all $u \in \mathscr{S}(\mathbb{R}^n)$ for which there exist $A > 0, C > 0$ (respectively, for every $A > 0$ there exists $C > 0$) such that

$$(1.2) \qquad |x^\beta \partial^\alpha_x u(x)| \leq CA^{|\alpha|+|\beta|}(\alpha!)^\mu(\beta!)^\nu, \quad \alpha, \beta \in \mathbb{N}^n,$$

see [2, 12, 14, 17, 26] and [28, Chapter 6]. These spaces have recently gained a wide importance in view of the fact that they represent a suitable functional setting both for microlocal analysis and PDE and for Fourier and time-frequency analysis [1, 3, 6–10, 13, 21, 36].

Concerning the investigation in the present paper, we can cite different sources of motivations. First, we recall the fundamental work of Seeley [34] on eigenfunction expansions of real analytic functions on compact manifolds (see also the recent paper of Dasgupta and Ruzhansky [15], extending the result of [34] for all Gevrey

2010 *Mathematics Subject Classification.* Primary 46F05; Secondary 34L10, 47F05.
Key words and phrases. anisotropic Shubin-type operators, Gelfand-Shilov spaces, eigenfunction expansions.

spaces G^σ, $\sigma > 1$, on compact Lie groups). Secondly, we mention the work [19] on the characterization of symmetric Gelfand-Shilov spaces $\mathcal{S}^\mu_\mu(\mathbb{R}^n)$ by means of estimates of iterates and the decay of the Fourier coefficients in the eigenfunction expansions associated to globally elliptic (or Γ elliptic) differential operator. We also refer to [38], where general Gevrey sequences M_p are used. Finally, we mention as additional motivation the results on hypoellipticity in $\mathcal{S}^\mu_\nu(\mathbb{R}^n)$ for elliptic operators of the type $\mathcal{H}^{m,k}_n$ for $\mu \geq k/(m+k)$, $\nu \geq m/(m+k)$, k, m being even natural numbers, cf. [7] (see also the older work [6]).

Before stating our main results we need some preliminaries.

As counterpart of an elliptic operator in a compact manifold, we consider in $\mathbb{R}^n$ the decay of the Fourier coefficients in the eigenfunction expansions associated to $\mathcal{H}^{m,k}_n$. In contrast to the symmetric case $\mu = \nu$ and $k = m$ (classical Shubin operators) we encounter new resonance type phenomena involving $\kappa := \mu/\nu$, namely we can characterize the spaces $\mathcal{S}^\mu_\nu(\mathbb{R}^n)$, $\mu+\nu \geq 1$ (respectively $\Sigma^\mu_\nu(\mathbb{R}^n)$, $\mu+\nu > 1$) by iterates and eigenfunction expansions defined by $\mathcal{H}^{m,k}_n$ iff κ is rational number, $\kappa = k/m$.

Our basic example of operator will be the anisotropic quantum harmonic oscillator appearing in Quantum Mechanics

$$\mathcal{H}^{2,k}_n = -\triangle + |x|^k, \qquad k \in 2\mathbb{N}, \tag{1.3}$$

with recovering for $k = 2$ the standard harmonic oscillator whose eigenfunctions are the Hermite functions

$$h_\alpha(x) = H_\alpha(x) e^{-|x|^2/2}, \quad \alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}^n, \tag{1.4}$$

where $H_\alpha(x)$ is the α-th Hermite polynomial. See for example [25,30,32] for related Hermite expansions as well as [18, 39] for connections with a degenerate harmonic oscillator.

Here we shall consider a more general class of operators with polynomial coefficients in $\mathbb{R}^n$, namely (m, k) anisotropic operators:

$$P = \sum_{\frac{|\alpha|}{m} + \frac{|\beta|}{k} \leq 1} c_{\alpha\beta} x^\beta D^\alpha_x, \quad D^\alpha = (-i)^{|\alpha|} \partial^\alpha_x. \tag{1.5}$$

Set

$$\Lambda_{m,k}(x,\xi) \quad = \quad (1 + |x|^{2k} + |\xi|^{2m})^{1/2}, \quad (x,\xi) \in \mathbb{R}^{2n}, \ m, k \in 2\mathbb{N}. \tag{1.6}$$

The global ellipticity for P in (1.5) is defined by imposing

$$p_{m,k}(x,\xi) := \sum_{\frac{|\alpha|}{m} + \frac{|\beta|}{k} = 1} c_{\alpha\beta} x^\beta \xi^\alpha \neq 0 \quad \text{for} \ \ (x,\xi) \neq (0,0). \tag{1.7}$$

or equivalently, there exist $C_1 > 0, C_2 > 0, R > 0$ such that the full symbol

$$p(x,\xi) = \sum_{\frac{|\alpha|}{m} + \frac{|\beta|}{k} \leq 1} c_{\alpha\beta} x^\beta \xi^\alpha \tag{1.8}$$

satisfies the condition

$$C_2 \leq \frac{|p(x,\xi)|}{\Lambda_{m,k}(x,\xi)} \leq C_1, \quad |(x,\xi)| \geq R. \tag{1.9}$$

Under the assumption (1.7) (or (1.9)), the following estimate holds for every $u \in \mathscr{S}(\mathbb{R}^n)$:

$$\sum_{\frac{|\alpha|}{m}+\frac{|\beta|}{k}\leq 1} \|x^\beta D_x^\alpha u\|_{L^2} \leq C(\|Pu\|_{L^2} + \|u\|_{L^2}), \tag{1.10}$$

cf. [4].

For these operators, the counterpart of the standard Sobolev spaces are the spaces $Q^s_{m,k}(\mathbb{R}^n)$, $s \in \mathbb{R}$, defined, for example, by requiring that

$$\|\Lambda(x,D)^s u\|_{L^2} < \infty, \tag{1.11}$$

where

$$\Lambda(x,\xi) = (1+|x|^{2k}+|\xi|^{2m})^{1/2\max\{k,m\}}, \quad k,m \in 2\mathbb{N}. \tag{1.12}$$

Under the global ellipticity assumption (1.7),

$$P : Q^s_{m,k}(\mathbb{R}^n) \to L^2(\mathbb{R}^n), \ s = \max\{k,m\},$$

is a Fredholm operator. The finite-dimensional null-space $\operatorname{Ker} P$ is given by functions in the Schwartz space $\mathscr{S}(\mathbb{R}^n)$.

We assume, as in [19], that P is a positive anisotropic elliptic operator, which implies that k and m are even numbers. This guarantees the existence of an orthonormal basis of eigenfunctions φ_j, $j \in \mathbb{N}$, with eigenvalues λ_j, $\lim\limits_{j\to\infty} \lambda_j = +\infty$ (see [35]). Moreover we have that

$$\lambda_j \sim C j^{\frac{mk}{n(m+k)}} \qquad as \quad j \to +\infty. \tag{1.13}$$

for some $C > 0$, cf. [4,35]. Hence, given $u \in L^2(\mathbb{R}^n)$, or $u \in \mathscr{S}'(\mathbb{R}^n)$, we can expand

$$u = \sum_{j=1}^{\infty} u_j \varphi_j \tag{1.14}$$

where the Fourier coefficients $u_j \in \mathbb{C}$ are defined by

$$u_j = (u, u_j)_{L^2}, \ \ j = 1, 2, \dots \tag{1.15}$$

with convergence in $L^2(\mathbb{R}^n)$ or $\mathscr{S}'(\mathbb{R}^n)$ for (1.14). Note that the positivity of P is for example granted if we assume $p(x,\xi)$ of the form (1.8) such that (1.7) holds and $p(x,\xi) \geq 0$ on $\mathbb{R}^{2n}$, and we regard P as the Toeplitz operator (localization operator) with symbol $p(x,\xi)$, see e.g. [20] and its analysis. Observe that passing to the classical left quantization, the symbol remains of the form (1.8) with the same principal part $p_{m,k}(x,\xi)$, then the condition (1.7) still holds, whereas the lower order terms may change, cf. [28].

By the hypoellipticity results of [7] the eigenfunctions φ_j belong to $\mathcal{S}^{k/(m+k)}_{m/(m+k)}(\mathbb{R}^n)$.

We first state an assertion on the characterization of the anisotropic Sobolev spaces $Q^s_{m,k}(\mathbb{R}^n)$ and the Schwartz class $\mathscr{S}(\mathbb{R}^n)$.

Theorem 1.1. *Suppose that P is (m,k)-globally elliptic cf.* (1.5), (1.7), *and positive. Then:*

(i) $u \in Q^s_{m,k}(\mathbb{R}^n) \iff \sum\limits_{j=1}^{\infty} |u_j|^2 \lambda_j^{s/\max\{m,k\}} < \infty$, $s \in \mathbb{N}$.

(ii) $u \in \mathscr{S}(\mathbb{R}^n) \iff |u_j| = O(\lambda_j^{-s})$, $j \to \infty \iff |u_j| = O(j^{-s})$, $j \to \infty$ *for all* $s \in \mathbb{N}$.

Let us now come to the characterization of the spaces $\mathcal{S}^\mu_\nu(\mathbb{R}^n)$ and $\Sigma^\mu_\nu(\mathbb{R}^n)$ in the case $\kappa := \mu/\nu \in \mathbb{Q}$. We may link μ, ν with an operator of the form (1.5) for a suitable choice of k and m. In fact, observe first that we may write $\mu = t\mu_o, \nu = t\nu_o$ for some $t > 0$ with $\mu_o = \kappa/(1+\kappa), \nu_0 = 1/(1+\kappa)$ so that $\mu_o + \nu_o = 1$. If $\mu + \nu \geq 1$ we have $t \geq 1$, if $\mu + \nu > 1$ then $t > 1$. On the other hand, for any given $\mu_o \in \mathbb{Q}$ we may write $\mu_o = k/(k+m)$ for two positive integers k and m, and consequently $\nu_o = 1 - \mu_o = m/(k+m)$. Multiples of k and m work as well, in particular we may assume k and m to be even natural numbers so that the symbol of $\Lambda_{m,k}$ in (1.6) is a smooth function which is necessary for the proof of the hypoellipticity result of [7]. So we have

$$\mu = \frac{kt}{k+m}, \quad \nu = \frac{mt}{k+m}.$$

For given even integers k and m, an example of globally elliptic positive operator is given by (1.1).

The first main result of the paper characterizes the Gelfand-Shilov spaces in terms of estimates of the iterates of P and reads as follows.

Theorem 1.2. *Let P be an operator of the form* (1.5) *for some integers $k \geq 1, m \geq 1$, be globally elliptic, namely satisfy* (1.7) *and let $u \in \mathscr{S}(\mathbb{R}^n)$. Then $u \in \mathcal{S}^{\frac{kt}{k+m}}_{\frac{mt}{k+m}}(\mathbb{R}^n), t \geq 1$ (respectively $u \in \Sigma^{\frac{kt}{k+m}}_{\frac{mt}{k+m}}(\mathbb{R}^n), t > 1$) if and only if there exist $C > 0, R > 0$ (respectively for every $C > 0$ there exists $R > 0$) such that:*

$$\|P^M u\|_{L^2} \leq RC^M (M!)^{\frac{kmt}{k+m}} \tag{1.16}$$

for every integer $M \geq 1$.

Remark 1.3. Theorem 1.2 suggests the possibility of considering new function spaces defined by the estimates (1.16) also for $0 < t < 1$ (respectively $0 < t \leq 1$). Corresponding Gelfand-Shilov classes are empty in that case as well known from [17] and the equivalence in Theorem 1.2 fails. Nevertheless such definition in terms of (1.16) deserves interest, cf. also [11, 37].

Using Theorem 1.2 we can prove the following result.

Theorem 1.4. *Let P be a positive operator of the form* (1.5) *for some integers $k \geq 1, m \geq 1$, satisfying* (1.7) *and let $u \in \mathscr{S}(\mathbb{R}^n)$. Let the eigenvalues λ_j and the Fourier coefficients u_j be defined as before. The following conditions are equivalent:*
i) $u \in \mathcal{S}^{\frac{kt}{k+m}}_{\frac{mt}{k+m}}(\mathbb{R}^n), t \geq 1$ (respectively $u \in \Sigma^{\frac{kt}{k+m}}_{\frac{mt}{k+m}}(\mathbb{R}^n), t > 1$);
ii) there exists $\varepsilon > 0$ such that (respectively for every $\varepsilon > 0$) we have

$$\sum_{j=1}^{\infty} |u_j|^2 e^{\epsilon \lambda_j^{\frac{k+m}{kmt}}} < \infty; \tag{1.17}$$

iii) there exists $\varepsilon > 0$ such that (respectively for every $\varepsilon > 0$) we have

$$\sup_{j \in \mathbb{N}} |u_j|^2 e^{\epsilon \lambda_j^{\frac{k+m}{kmt}}} < \infty. \tag{1.18}$$

iv) there exists $\varepsilon > 0$ such that (respectively for every $\varepsilon > 0$) we have for some $C > 0$:

$$|u_j| \leq Ce^{-\varepsilon j^{\frac{1}{tn}}}, \qquad j \in \mathbb{N}.$$

The somewhat surprising fact that in *iv*) the estimates do not depend on the couple (m, k), that is on (μ, ν), may find intuitive explanation in the $\mathcal{S}^{\mu}_{\nu}$ regularity of the eigenfunctions φ_j, cf. [7].

2. Proof of the main results

Proof of Theorem 1.1. The proof of Theorem 1.1 is easy, by using the r-th power of $P, r \in \mathbb{R}$, that we may define as

$$P^r u = \sum_{j=1}^{\infty} \lambda_j^r u_j \varphi_j,$$

and by observing that the norms $\|P^r u\|_{L^2}, r = s/\max\{k, m\}$ and $\|\Lambda(x, D)^s u\|_{L^2}$ are equivalent, see [4, 28, 35]. On the other hand, by Parseval identity

$$\|P^r u\|^2_{L^2} = \|\sum_{j=1}^{\infty} \lambda_j^r u_j \varphi_j\|^2_{L^2} = \sum_{j=1}^{\infty} \lambda_j^{2r} |u_j|^2$$

and *i*) follows. Since $\mathscr{S}(\mathbb{R}^n) = \bigcap\limits_{s \in \mathbb{N}} Q^s_{m,k}(\mathbb{R}^n)$ we also obtain *ii*). □

The proof of Theorem 1.2 needs some preparation. We first define, for fixed $r \geq 0$ and $u \in L^2(\mathbb{R}^n)$:

$$|u|_r = \sum_{\frac{|\alpha|}{m} + \frac{|\beta|}{k} = r} \|x^\beta D^\alpha u\|_{L^2} \tag{2.1}$$

First it is useful to characterize Gelfand-Shilov spaces in terms of the norms $|u|_r$ as follows.

Proposition 2.1. *Let $u \in L^2(\mathbb{R}^n)$. Then $u \in \mathcal{S}^{\frac{kt}{k+m}}_{\frac{mt}{k+m}}(\mathbb{R}^n), t \geq 1$ (respectively $u \in \Sigma^{\frac{kt}{k+m}}_{\frac{mt}{k+m}}(\mathbb{R}^n), t > 1$) if and only if there exist $C > 0, R > 0$ (respectively for every $C > 0$ there exists $R > 0$) such that*

$$|u|_r \leq R C^r r^{\frac{kmrt}{k+m}} \tag{2.2}$$

for every $r > 0$.

We have the following preliminary result.

Lemma 2.2. *There exists a constant $C > 0$ such that, for any given $p \in \mathbb{N}, (\alpha, \beta) \in \mathbb{N}^{2n}$, with $|\alpha|/m + |\beta|/k = r, p < r < p + 1$, and for every $\varepsilon > 0$, the following estimate holds true:*

$$|u|_r \leq \varepsilon |u|_{p+1} + C\varepsilon^{-\frac{r-p}{p+1-r}} |u|_p + C^p (p+1)!^{\frac{km}{k+m}} |u|_0 \tag{2.3}$$

for all $u \in \mathscr{S}(\mathbb{R}^n)$.

The proof follows the same lines as the proof of Proposition 2.1 in [5], cf. also [24], and it is omitted.

Next, fixed $\lambda > 0, p \in \mathbb{N}$ and $u \in L^2(\mathbb{R}^n)$, we set:

$$\sigma_p(u, \lambda) = \lambda^{-p} (p!)^{-\frac{kmt}{k+m}} |u|_p. \tag{2.4}$$

Lemma 2.3. *For every $p \in \mathbb{N}$ and for $\lambda > 0$ sufficiently large, we have:*

$$\sigma_{p+1}(u,\lambda) \leq (p+1)^{-\frac{kmt}{k+m}} \sigma_p(Pu,\lambda) + \sum_{h=0}^{p} \sigma_h(u,\lambda) \tag{2.5}$$

for every $u \in \mathscr{S}(\mathbb{R}^n)$.

Proof. For $p = 0$ the assertion is a direct consequence of (1.10) if λ is large enough. Fix now $p \in \mathbb{N}, p \geq 1$ and let $\alpha, \beta \in \mathbb{N}^n$ such that $|\alpha|/m + |\beta|/k = p+1$. It is easy to verify that we can find $\gamma, \delta \in \mathbb{N}^n$, with $\gamma \leq \alpha, \delta \leq \beta$ such that $|\gamma|/m + |\delta|/k = p$ and $|\alpha - \gamma|/m + |\beta - \delta|/k = 1$. Then by (1.10) we can write

$$\begin{aligned} \|x^\beta D^\alpha u\|_{L^2} &\leq \|x^{\beta-\delta} D^{\alpha-\gamma}(x^\delta D^\gamma u)\|_{L^2} + \|x^{\beta-\delta}[x^\delta, D^{\alpha-\gamma}] D^\gamma u\|_{L^2} \\ &\leq C\|P(x^\delta D^\gamma u)\|_{L^2} + \|x^{\beta-\delta}[x^\delta, D^{\alpha-\gamma}] D^\gamma u\|_{L^2} \\ &\leq I_1 + I_2 + I_3, \end{aligned}$$

where

$$I_1 = C\|x^\delta D^\gamma (Pu)\|_{L^2}, \qquad I_2 = C\|[P, x^\delta D^\gamma]u\|_{L^2}, \qquad I_3 = \|x^{\beta-\delta}[x^\delta, D^{\alpha-\gamma}]D^\gamma u\|_{L^2}.$$

Let now

$$J_h = \sum_{\frac{|\alpha|}{m} + \frac{|\beta|}{k} = p+1} I_h, \qquad Y_h = \lambda^{-p-1}(p+1)!^{-\frac{kmt}{k+m}} J_h, \quad h = 1,2,3.$$

Then, obviously we have

$$|u|_{p+1} \leq J_1 + J_2 + J_3, \qquad \sigma_{p+1}(\lambda, u) \leq Y_1 + Y_2 + Y_3.$$

Now, since $J_1 \leq C_1 |Pu|_p$ for some $C_1 > 0$, then we have $Y_1 \leq (p+1)^{-\frac{kmt}{k+m}} \sigma_p(\lambda, Pu)$, if $\lambda \geq C_1^{-1}$. To estimate J_2 and Y_2 we observe that

$$[P, x^\delta D^\gamma]u = \sum_{\frac{|\tilde\alpha|}{m} + \frac{|\tilde\beta|}{k} \leq 1} c_{\tilde\alpha\tilde\beta}[x^{\tilde\beta} D^{\tilde\alpha}, x^\delta D^\gamma]u,$$

and that

$$[x^{\tilde\beta} D^{\tilde\alpha}, x^\delta D^\gamma]u = \sum_{0 \neq \tau \leq \tilde\alpha, \tau \leq \delta} C_{\tilde\alpha\delta\tau} x^{\delta+\tilde\beta-\tau} D^{\gamma+\tilde\alpha-\tau}u - \sum_{0 \neq \tau \leq \tilde\beta, \tau \leq \gamma} C_{\tilde\beta\gamma\tau} x^{\delta+\tilde\beta-\tau} D^{\gamma+\tilde\alpha-\tau}u.$$

where the constants $|C_{\tilde\alpha\delta\tau}|$ and $|C_{\tilde\beta\gamma\tau}|$ can be estimated by $C_2\, p^{|\tau|}$ for some positive constant C_2 independent of p. We observe now that in both the sums above we have

$$r = \frac{|\gamma + \tilde\alpha - \tau|}{m} + \frac{|\delta + \tilde\beta - \tau|}{k} = p + \frac{|\tilde\alpha|}{m} + \frac{|\tilde\beta|}{k} - \frac{m+k}{km}|\tau| \leq p + 1 - \frac{m+k}{km}|\tau|,$$

hence in particular we have $0 \leq r < p+1$ since $|\tau| > 0$. Moreover, we have

$$|\tau| \leq \frac{km}{m+k}(p+1-r).$$

In view of these considerations, we easily obtain

$$J_2 \leq C_3(J_2' + p^{\frac{km}{k+m}}|u|_p + J_2''),$$

where

$$J_2' = \sum_{p<r<p+1} p^{\frac{km}{k+m}(p+1-r)}|u|_r,$$

$$J_2'' = \sum_{0\le r<p} p^{\frac{km}{k+m}(p+1-r)}|u|_r.$$

Now, applying Lemma 2.2 to J_2' with

$$\varepsilon = (4C_3)^{-1}p^{-\frac{km}{k+m}(p+1-r)},$$

and using standard factorial inequalities we obtain

$$J_2' \le (4C_3)^{-1}|u|_{p+1} + C_4 p^{\frac{km}{k+m}}|u|_p + C_5^{p+1}(p+1)!^{\frac{km}{k+m}}|u|_0.$$

Similarly, writing

$$J_2'' = p^{\frac{km}{k+m}(p+1)}|u|_0 + \sum_{q=0}^{p-1}\sum_{q<r<q+1} p^{\frac{km}{k+m}(p+1-r)}|u|_r$$

and applying Lemma 2.2 to each term of the sum above with

$$\varepsilon = p^{-\frac{km}{k+m}(q+1-r)},$$

we get

$$\begin{aligned} J_2'' &\le C_6^{p+1}(p+1)!^{\frac{km}{k+m}}|u|_0 + C_7\sum_{q=0}^{p-1}\left[p^{\frac{km}{k+m}(p-q)}|u|_{q+1} + p^{\frac{km}{k+m}(p-q+1)}|u|_q\right] \\ &\le C_8^{p+1}(p+1)!^{\frac{km}{k+m}}|u|_0 + C_9\sum_{q=1}^{p} p^{\frac{km}{k+m}(p-q+1)}|u|_q, \end{aligned}$$

from which we get

$$J_2 \le \frac{1}{4}|u|_{p+1} + \tilde{C}^{p+1}(p+1)!^{\frac{km}{k+m}}|u|_0 + C'\sum_{q=1}^{p} p^{\frac{km}{k+m}(p-q+1)}|u|_q$$

for some positive constants $C', \tilde{C}$ independent of p. From the estimates above, taking λ sufficiently large and using the fact that $t \ge 1$, we obtain

$$Y_2 = \lambda^{-p-1}(p+1)!^{-\frac{kmt}{k+m}}J_2 \le \frac{1}{4}\sum_{h=0}^{p+1}\sigma_h(\lambda, u).$$

Analogous estimates can be derived for Y_3 and yield (2.5). We leave the details for the reader. □

Starting from (2.5) and arguing by induction on p it is easy to prove the following result. We omit the proof for the sake of brevity.

Lemma 2.4. *For every $p \in \mathbb{N}, t \ge 1$ and $\lambda > 0$ sufficiently large we have*

$$\sigma_p(u,\lambda) \le 2^p\sigma_0(u,\lambda) + \sum_{\ell=1}^{p} 2^{p-\ell}\binom{p}{\ell}(\ell!)^{-\frac{kmt}{k+m}}\sigma_0(P^\ell u, \lambda).$$

Proof of Theorem 1.2. The fact that the Gelfand-Shilov regularity of u implies (1.16) is easy to prove and we omit the details. In the opposite direction, by Proposition 2.1 it is sufficient to prove that u satisfies (2.2) for every $r > 0$. From the previous estimate, we have, for every $p \in \mathbb{N}$:

$$\sigma_p(u,\lambda) \le C + \sum_{\ell=1}^{p} 2^{p-\ell}\binom{p}{\ell}C^{\ell+1} \le C(2+C)^{p+1}.$$

Therefore

$$|u|_p \le C^{p+1} p!^{\frac{kmt}{k+m}}$$

for a new constant $C > 0$, which gives (2.2) in the case $r \in \mathbb{N}$. If $r > 0$ is not integer, then $p < r < p+1$ for some $p \in \mathbb{N}$ and we can apply Lemma 2.2 which yields

$$\begin{aligned} |u|_r &\le \varepsilon |u|_{p+1} + C\varepsilon^{-\frac{r-p}{p+1-r}} |u|_p + C^p (p!)^{\frac{km}{k+m}} |u|_0 \\ &\le \varepsilon C_1^{p+1} (p+1)!^{\frac{kmt}{k+m}} + C_1^p \varepsilon^{-\frac{r-p}{p+1-r}} (p+1)!^{\frac{kmt}{k+m}} + C_1^p (p+1)!^{\frac{kmt}{k+m}} \le C_2^{r+1} r^{\frac{kmrt}{k+m}}. \end{aligned}$$

Then, by Proposition 2.1 we conclude that $u \in \mathcal{S}^{\frac{kt}{k+m}}_{\frac{mt}{k+m}}(\mathbb{R}^n)$. Similarly we argue for $u \in \Sigma^{\frac{kt}{k+m}}_{\frac{mt}{k+m}}(\mathbb{R}^n)$. □

Proof of Theorem 1.4. The equivalence between *ii*) and *iii*) is obvious. Moreover *iii*) is equivalent to *iv*) in view of (1.13). The arguments are similar for $\mathcal{S}^{\frac{kt}{k+m}}_{\frac{mt}{k+m}}(\mathbb{R}^n)$ and $\Sigma^{\frac{kt}{k+m}}_{\frac{mt}{k+m}}(\mathbb{R}^n)$ classes. To conclude the proof we will show the equivalence between *i*) and *iv*). We first observe that

$$\|P^M u\|^2_{L^2} = \|\sum_{j=1}^{\infty} u_j P^M \varphi_j\|^2_{L^2} = \sum_{j=1}^{\infty} \lambda_j^{2M} |u_j|^2,$$

in view of Parseval identity. By (1.13) it follows that

$$C_1 \|P^M u\|^2_{L^2} \le \sum_{j=1}^{\infty} j^{2Mkm/(n(k+m))} |u_j|^2 \le C_2 \|P^M u\|^2_{L^2} \tag{2.6}$$

for suitable positive constants C_1, C_2. Now if *iv*) holds, then we have

$$|u_j|^2 \le e^{-\epsilon j^{1/(nt)}}$$

for some new constant $\epsilon > 0$. Then from the first estimate in (2.6) we have for some $C > 0$

$$\|P^M u\|^2_{L^2} \le C \sum_{j=1}^{\infty} j^{2Mkm/(n(m+k))} e^{-\epsilon j^{1/(nt)}} \tag{2.7}$$

$$\le \tilde{C} \sup_{j \in \mathbb{N}} j^{2Mmk/(n(m+k))} e^{-\epsilon j^{1/(nt)}} \tag{2.8}$$

with

$$\tilde{C} = C \sum_{j=1}^{\infty} e^{-\epsilon j^{1/(nt)}}.$$

Moreover, for any fixed $\omega > 0$ we have

$$e^{\omega j^{1/(nt)}} = \sum_{M=0}^{\infty} \frac{\omega^M j^{M/(nt)}}{M!}.$$

This implies that for every $M \in \mathbb{N}$:

$$j^{M/(nt)} e^{-\omega j^{1/(nt)}} \le \omega^{-M} M! \tag{2.9}$$

Taking the $2kmt/(k+m)$-th power of both sides of (2.9) and applying in the last estimate in (2.8) with

$$\omega = 2\epsilon kmt/(k+m),$$

we obtain

$$\|P^M u\|_{L^2}^2 \leq \tilde{C}\omega^{-\frac{2Mkmt}{k+m}}(M!)^{\frac{2mkt}{m+k}},$$

which gives $i)$ in view of Theorem 1.2.

$i) \Rightarrow ii)$ Viceversa assume that $u \in \mathcal{S}_{\frac{mt}{k+m}}^{\frac{kt}{k+m}}(\mathbb{R}^n)$. In view of $iv)$ it is sufficient to show that

$$\sup_{j\in\mathbb{N}} |u_j|^2 e^{\epsilon j^{\frac{1}{nt}}} < +\infty. \tag{2.10}$$

Theorem 1.2 and the second inequality in (2.6) imply that

$$\frac{j^{\frac{2Mkm}{n(k+m)}}}{C^M (M!)^{\frac{2kmt}{k+m}}} |u_j|^2 \leq C$$

for every $j, M \in \mathbb{N}$ and for some C independent of j and M. Taking the supremum of the left-hand side over M we get (2.10) with $\epsilon = \frac{2kmt}{k+m} C^{-\frac{k+m}{2kmt}}$. This concludes the proof. □

3. Generalizations

We list some possible generalizations of the preceding results. First, one can replace the hypothesis of positivity for the operator P by assuming that P is normal, i.e. $P^*P = PP^*$. This guarantees the existence of an orthonormal basis of eigenfunctions $\varphi_j, j \in \mathbb{N}$, with eigenvalues λ_j, $\lim_{j\to\infty} |\lambda_j| = +\infty$, see [35], and we may then proceed as before, cf. [34].

Another possible generalization consists in replacing L^2 norms with L^p norms, $1 < p < \infty$. Let us observe that the basic estimate (1.10) is valid also for L^p norms, see [16, 27], and it seems easy to extend Theorem 1.2 in this direction.

A much more challenging problem is an analogous characterization of the classes $\mathcal{S}_\nu^\mu(\mathbb{R}^n)$ when $\kappa = \mu/\nu = k/m$ is irrational. First difficulty, in this case, is given by an appropriate choice of the operator P. In fact, the natural candidates

$$P = (-\Delta)^{m/2} + (1 + |x|^2)^{k/2}, \qquad m \in 2\mathbb{N}, k > 0, k \notin 2\mathbb{N}$$

can be easily treated in the setting of temperate distributions but results of Gelfand-Shilov regularity, extending those in [7], are missing for them.

Note. With great sorrow, Marco Cappiello, Stevan Pilipovic and Luigi Rodino inform that their friend Todor Gramchev passed away on October 18, 2015. He inspired and collaborated to the initial version of the present paper and appears here as co-author.

Acknowledgements. L. Rodino was supported by the Ministry of Education and Science of the Russian Federation (Agreement number N. 02.a03.21.0008).

References

1. A.Ascanelli, M.Cappiello, *Hölder continuity in time for SG hyperbolic systems*, J. Differential Equations **244** (2008), 2091–2121.
2. A. Avantaggiati, *S-spaces by means of the behaviour of Hermite-Fourier coefficients*, Boll. Un. Mat. Ital. **6** (1985), 487–495.
3. H.A. Biagioni, T. Gramchev, *Fractional derivative estimates in Gevrey spaces, global regularity and decay for solutions to semilinear equations in* $\mathbb{R}^n$, J. Differential Equations **194** (2003), 140–165.
4. P. Boggiatto, E. Buzano, L. Rodino, *Global hypoellipticity and spectral theory.* Math. Res. **92**, Akademie Verlag, Berlin, 1996.
5. D. Calvo, L. Rodino, *Iterates of operators and Gelfand-Shilov functions*, Int. Transf. Spec. Funct. **22** (2011), 269–276.
6. M. Cappiello, T. Gramchev, L. Rodino, *Super-exponential decay and holomorphic extensions for semilinear equations with polynomial coefficients.* J. Funct. Anal. **237** (2006), 634–654.
7. M. Cappiello, T. Gramchev, L. Rodino, *Entire extensions and exponential decay for semilinear elliptic equations*, J. Anal. Math. **111** (2010), 339–367.
8. M. Cappiello, T. Gramchev, L. Rodino, *Sub-exponential decay and uniform holomorphic extensions for semilinear pseudodifferential equations*, Comm. Partial Differential Equations **35** (2010), n. 5, 846-877.
9. M. Cappiello, L. Rodino, *SG-pseudo-differential operators and Gelfand-Shilov spaces*, Rocky Mountain J. Math.**36** (2006) n. 4, 1117–1148.
10. M. Cappiello, J. Toft, *Pseudo-differential operators in a Gelfand-Shilov setting*, Math. Nachr. (2016). Available online DOI: 10.1002/mana.201500465.
11. Y. Chen, M. Signahl, J. Toft, *Factorizations and singular value estimates of operators with Gelfand–Shilov and Pilipović kernels*, arXiv:1511.06257 (2016).
12. J. Chung, S. Y. Chung, D. Kim, *Characterization of the Gelfand-Shilov spaces via Fourier transforms*, Proc. Am. Math. Soc. **124** (1996), 2101–2108.
13. E. Cordero, F. Nicola, L. Rodino, *Wave packet analysis of Schrödinger equations in analytic function spaces*, Adv. Math. *278* (2015), 182–209.
14. E. Cordero, S. Pilipović, L. Rodino, N. Teofanov, *Localization operators and exponential weights for modulation spaces*, Mediterranean J. Math. **2** (2005), 381–394.
15. A. Dasgupta, M. Ruzhansky, *Eigenfunction expansions of ultradifferentiable functions and ultradistributions*, Trans. Amer. Math. Soc. **368** (2016) n. 12, 8481-8498.
16. G. Garello, A. Morando, *L^p-bounded pseudo-differential operators and regularity for multi-quasi-elliptic equations*, Integral Equations Operator Theory **51** (2005), 501–517.
17. I.M. Gelfand, G.E. Shilov, *Generalized functions II.* Academic Press, New York, 1968.
18. T. Gramchev, S. Pilipović, L. Rodino, *Global Regularity and Stability in S-Spaces for Classes of Degenerate Shubin Operators.* Pseudo-Differential Operators: Complex Analysis and Partial Differential Equations Operator Theory: Advances and Applications **205** (2010), 81-90.
19. T. Gramchev, S. Pilipović, L. Rodino, *Eigenfunction expansions in* $\mathbb{R}^n$, Proc. Amer. Math. Soc. **139** (2011), 4361–4368.
20. K. Gröchenig, J. Toft, *Isomorphism properties of Toeplitz operators and pseudo-differential operators between modulation spaces*, J. Anal. Math. **114** (2011), 255–283.
21. K. Gröchenig, G. Zimmermann, *Spaces of test functions via the STFT*, J. Funct. Spaces Appl. **2** (2005), 1671–1716.
22. B. Helffer, *Théorie spectrale pour des opérateurs globalement elliptiques.* Astérisque **112**, Société Mathématique de France, Paris, 1984.
23. H. Komatsu, *A proof of Kotake and Narashiman's Theorem.* Proc. Japan Acad. **38** (1962), 615-618.
24. T. Kotake, M.S. Narasimhan, *Regularity theorems for fractional powers of a linear elliptic operator.* Bull. Soc. Math. France, **90** (1962), 449-471.
25. M. Langenbruch, *Hermite functions and weighted spaces of generalized functions.* Manuscripta Math. **119** (2006), 269–285.
26. B.S. Mitjagin, *Nuclearity and other properties of spaces of type S*, Amer. Math. Soc. Transl., Ser. 2, **93** (1970), 45–59.
27. A. Morando, *L^p-regularity for a class of pseudo-differential operators in* $\mathbb{R}^n$, J. Partial Differential Equations **18** (2005), 241–262.

28. F. Nicola, L. Rodino, *Global pseudo-differential calculus on Euclidean spaces*, Birkhäuser, Basel, 2010.
29. S. Pilipović, *Generalization of Zemanian spaces of generalized functions which have orthonormal series expansions*, SIAM J. Math. Anal. **17** (1986), 477–484.
30. S. Pilipović, *Tempered ultradistributions.* Boll. Unione Mat. Ital. VII. Ser. B **2** (1988), 235–251.
31. S. Pilipović, N. Teofanov, *Pseudodifferential operators on ultramodulation spaces*, J. Funct. Anal. **208** (2004), 194–228.
32. M. Reed, B. Simon, *Methods of modern mathematical physics* Vol 1. Academic Press, San Diego Ca., 1975.
33. R.T. Seeley, *Integro-differential operators on vector boundes*, Trans. Am. Math. Soc. **117** (1965), 167–204.
34. R.T. Seeley, *Eigenfunction expansions of analytic functions*, Proc. Am. Math. Soc. **21** (1969), 734–738.
35. M. Shubin, *Pseudodifferential operators and spectral theory.* Springer Series in Soviet Mathematics, Springer Verlag, Berlin, 1987.
36. J. Toft, *Multiplication properties in Gelfand-Shilov pseudo-differential calculus.* In: Molahajlo, S., Pilipović, S., Toft, J., Wong, M.W. (eds.) Pseudo-Differential Operators, Generalized Functions and Asymptotics, Operator Theory: Advances and Applications, Birkhüser, Basel, **231** (2013), 117–172.
37. J. Toft, *Images of function and distribution spaces under the Bargmann transform*, J. Pseudo-Differ. Oper. Appl. DOI 10.1007/s11868-016-0165-9 (2016).
38. J. Vindas, D. Vučković, *Eigenfunction expansions of ultradifferentiable functions and ultradistributions in* $\mathbb{R}^n$, J. Pseudo-Differ. Oper. Appl. **7** (2016) n. 4, 519–531.
39. M.W. Wong, *The heat equation for the Hermite operator on the Heisenberg group.* Hokkaido Math. J. **34** (2005), 393–404.

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