Liftings for ultra-modulation spaces, and one-parameter groups of Gevrey-type pseudo-differential operators

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LIFTINGS FOR ULTRA-MODULATION SPACES, AND ONE-PARAMETER GROUPS OF GEVREY TYPE PSEUDO-DIFFERENTIAL OPERATORS

AHMED ABDELJAWAD, SANDRO CORIASCO, AND JOACHIM TOFT

Abstract. We deduce one-parameter group properties for pseudo-differential operators $\operatorname{Op}(a)$, where a belongs to the class $\Gamma_*^{(\omega_0)}$ of certain Gevrey symbols. We use this to show that there are pseudo-differential operators $\operatorname{Op}(a)$ and $\operatorname{Op}(b)$ which are inverses to each others, where $a \in \Gamma_*^{(\omega_0)}$ and $b \in \Gamma_*^{(1/\omega_0)}$.

We apply these results to deduce lifting property for modulation spaces and construct explicit isomorpisms between them. For each weight functions ω, ω_0 moderated by GRS submultiplicative weights, we prove that the Toeplitz operator (or localization operator) $\operatorname{Tp}(\omega_0)$ is an isomorphism from $M^{p,q}_{(\omega)}$ to $M^{p,q}_{(\omega/\omega_0)}$ for every $p, q \in (0, \infty]$.

0. Introduction

The topological vector spaces V_1 and V_2 are said to possess lifting property if there exists a "convenient" homeomorphisms (a lifting) between them. For example, for any weight ω on $\mathbf{R}^d$, $p \in (0, \infty]$ and $s \in \mathbf{R}$ the mappings $f \mapsto \omega \cdot f$ and $f \mapsto (1-\Delta)^{s/2} f$ are homeomorphic from the (weighted) Lebesgue space $L^p_{(\omega)}$ and the Sobolev space H^p_s, respectively, into $L^p = H^p_0$, with inverses $f \mapsto \omega^{-1} \cdot f$ and $f \mapsto (1-\Delta)^{-s/2} f$, respectively. (Cf. [34] and Section 1 for notations.) Hence, these spaces possess lifting properties.

It is often uncomplicated to deduce lifting properties between (quasi-)Banach spaces of functions and distributions, if the definition of their norms only differs by a multiplicative weight on the involved distributions, or on their Fourier transforms, which is the case in the previous homeomorphisms. Here note that multiplications on the Fourier transform side are linked to questions on differentiation of the involved elements. A more complicated situation appears when there are some kind of interactions between multiplication and differentiation in the definition of the involved vector spaces.

An example where such interactions occur concerns the extended family of Sobolev spaces, introduced by Bony and Chemin in [3] (see also [38]). More precisely, let ω, ω_0 be suitable weight functions and g a suitable Riemannian metric, which are defined on the phase space $W \simeq T^*\mathbf{R}^d \simeq \mathbf{R}^{2d}$. Then Bony and Chemin introduced in [3] the generalised Sobolev space $H(\omega, g)$ which fits the Hörmander-Weyl calculus well in the sense that $H(1, g) = L^2$, and if a belongs to the Hörmander class $S(\omega_0, g)$, then the Weyl operator $\operatorname{Op}^w(a)$ with symbol a is continuous from $H(\omega_0\omega, g)$ to $H(\omega, g)$. Moreover, they form group algebras, from which it follows that to each such weight ω_0, there exist symbols a and b such that

$$\operatorname{Op}^w(a) \circ \operatorname{Op}^w(b) = \operatorname{Op}^w(b) \circ \operatorname{Op}^w(a) = I, \quad a \in S(\omega_0, g),\ b \in S(1/\omega_0, g). \qquad (0.1)$$

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Here I is the identity operator on $\mathscr{S}'$. In particular, by the continuity properties of $\operatorname{Op}^w(a)$ it follows that $H(\omega_0\omega,g)$ and $H(\omega,g)$ possess lifting properties with the homeomorphism $\operatorname{Op}^w(a)$, and with $\operatorname{Op}^w(b)$ as its inverse.

The existence of a and b in (0.1) is a consequence of solution properties of the evolution equation

$$(\partial_t a)(t,\,\cdot\,) = (b+\log\vartheta)\# a(t,\,\cdot\,), \qquad a(0,\,\cdot\,) = a_0 \in S(\omega,g),\ \vartheta\in S(\vartheta,g), \tag{0.2}$$

which involve the Weyl product $\#$ and a fixed element $b\in S(1,g)$. It is proved that (0.2) has a unique solution $a(t,\,\cdot\,)$ which belongs to $S(\omega\vartheta^t,g)$ (cf. [3, Theorem 6.4] or [38, Theorem 2.6.15]). The existence of a and b in (0.1) will follow by choosing $\omega = a_0 = 1$, $t=1$ and $\vartheta=\omega_0$.

If g is the constant euclidean metric on the phase space $\mathbf{R}^{2d}$, then $S(\omega_0,g)$ equals $S^{(\omega_0)}(\mathbf{R}^{2d})$, the set of all smooth symbols a which satisfies $|\partial^\alpha a|\lesssim\omega_0$. We notice that also for such simple choices of g, (0.1) above leads to lifting properties that are not trivial. In fact, let ω and ω_0 be polynomially moderate weight on the phase space, and let $\mathscr{B}$ be a suitable translation invariant BF-space. Then it is observed in [30] that the continuity results for pseudo-differential operators on modulation spaces in [54,56] imply that $\operatorname{Op}^w(a)$ in (0.1) is continuous and bijective from $M(\omega_0\omega,\mathscr{B})$ to $M(\omega,\mathscr{B})$ with continuous inverse $\operatorname{Op}^w(b)$. In particular, by choosing $\mathscr{B}$ to be the mixed norm space $L^{p,q}(\mathbf{R}^{2d})$ of Lebesgue type, then $M(\omega,\mathscr{B})$ is equal to the classical modulation space $M^{p,q}_{(\omega)}$, introduced by Feichtinger in [15]. Consequently, $\operatorname{Op}^w(a)$ above lifts $M^{p,q}_{(\omega_0\omega)}$ into $M^{p,q}_{(\omega)}$.

An important class of operators in quantum mechanics and time-frequency analysis concerns Toeplitz, or localisation operators. The main issue in [30, 31] is to show that the Toeplitz operator $\operatorname{Tp}(\omega_0)$ lifts $M^{p,q}_{(\omega_0\omega)}$ into $M^{p,q}_{(\omega)}$ for suitable ω_0. The assumptions on ω_0 in [30] is that it should be polynomially moderate and satisfies $\omega_0\in S^{(\omega_0)}$. In [31], the assumptions are different compared to [30]. On one hand the growth and decay conditions on ω_0 is relaxed compared to [30] in the sense that it is only assumed that ω_0 should be moderated by a so-called GRS-weight, which is allowed to grow subexponentially. On the other hand, in [31] it is required that ω_0 is *radial symmetric in each phase shift*, i. e. ω_0 should satisfy

$$\omega_0(x_1,\dots,x_d,\xi_1,\dots,\xi_d) = \vartheta(r_1,\dots,r_d), \qquad r_j = |(x_j,\xi_j)|, \tag{0.3}$$

for some weight ϑ.

The approaches in [30, 31] are also different. In [31], the lifting properties for $\operatorname{Tp}(\omega_0)$ are reached by using the links between modulation spaces and Bargmann-Foch spaces in combination of suitable estimates for a sort of generalised gamma-functions. The approach in [30] relies on corresponding lifting properties for pseudo-differential operators, as follows:

(1) $\operatorname{Tp}(\omega_0) = \operatorname{Op}^w(c)$ for some $c\in S^{(\omega_0)}$;

(2) by the definitions, it follows by straightforward computations that if $\vartheta = \omega_0^{\frac{1}{2}}$, then $\operatorname{Tp}(\omega_0)$ is a homeomorphism from $M^{p,q}_{(\vartheta)}$ to $M^{p,q}_{(1/\vartheta)}$;

(3) combining (0.1) with Wiener's lemma for $(S^{(1)},\#)$ to ensure that the inverse of $\operatorname{Tp}(\omega_0)$ in (2) is a pseudo-differential operator $\operatorname{Op}^w(b)$ with the symbol b in $S^{(1/\omega_0)}$;

(4) by (1), (3) and duality,

$$T_1\equiv \operatorname{Op}^w(b)\circ\operatorname{Tp}(\omega_0) \quad\text{and}\quad T_2\equiv \operatorname{Tp}(\omega_0)\circ\operatorname{Op}^w(b)$$

are both the identity operator on $\mathscr{S}'(\mathbf{R}^d)$, since T_1 is the identity operator on $M^{p,q}_{(\vartheta)}$, T_2 is the identity operator on $M^{p,q}_{(1/\vartheta)}$, and $\mathscr{S}\subseteq M^{p,q}_{(\vartheta)}\cap M^{p,q}_{(1/\vartheta)}$.

(5) by (4), $T_1 = T_2 = \operatorname{Op}^w(1)$ is the identity operator on each $M^{p,q}_{(\omega)}$. Since

$$\operatorname{Tp}(\omega_0) = \operatorname{Op}^w(c) \, : \, M^{p,q}_{(\omega_0\omega)} \to M^{p,q}_{(\omega)} \quad \text{and} \quad \operatorname{Op}^w(b) \, : \, M^{p,q}_{(\omega)} \to M^{p,q}_{(\omega_0\omega)}$$

are continuous (cf. [54, 56]) and inverses to each other, it follows that they are homeomorphisms.

In the first part of the paper we deduce an analog of (0.1) for the Gevrey type symbol classes $\Gamma_s^{(\omega_0)}$ and $\Gamma_{0,s}^{(\omega_0)}$ of orders $s \ge 1$, the set of all $a \in C^\infty$ such that

$$|\partial^\alpha a(X)| \lesssim h^{|\alpha|}\alpha!^s\omega(X) \tag{0.4}$$

for some $h > 0$, respectively for every $h > 0$, considered in [5]. That is, in Section 3 we show that there exist symbols a and b such that

$$\operatorname{Op}^w(a) \circ \operatorname{Op}^w(b) = \operatorname{Op}^w(b) \circ \operatorname{Op}^w(a) = I, \quad a \in \Gamma_s^{(\omega_0)}, \ b \in \Gamma_s^{(1/\omega_0)}, \tag{0.5}$$

and similarly when $\Gamma_s^{(\omega_0)}$ and $\Gamma_s^{(1/\omega_0)}$ are replaced by $\Gamma_{0,s}^{(\omega_0)}$ and $\Gamma_{0,s}^{(1/\omega_0)}$, respectively.

As in [3], (0.5) is obtained by proving that the evolution equation

$$(\partial_t a)(t, \, \cdot \,) = (b + \log\vartheta)\#a(t, \, \cdot \,), \qquad a(0, \, \cdot \,) = a_0 \in \Gamma_s^{(\omega)}, \ \vartheta \in \Gamma_s^{(\vartheta)}, \tag{0.6}$$

analogous to (0.2), has a unique solution $a(t, \, \cdot \,)$ which belongs to $\Gamma_s^{(\omega\vartheta^t)}$ (and similarly when the $\Gamma_s^{(\omega)}$-spaces are replaced by corresponding $\Gamma_{0,s}^{(\omega)}$-spaces), given in Section 3.

In Sections 4 and 5 we use the framework of [30] in combination with (0.5) to extend the lifting properties in [30] in such ways that the involved weights are allowed to belong to $\mathscr P_{E,s}^0$ or in $\mathscr P_{E,s}$ instead of the smaller set $\mathscr P$ which is the assumption in [30].

Our main result, which is similar to [30, Theorem 0.1], can be stated as follows.

Theorem 0.1. *Let $s \ge 1$, $\omega, \omega_0 \in \mathscr P_{E,s}^0(\mathbf R^{2d})$, $p, q \in (0, \infty]$ and let $\phi \in \mathcal S_s(\mathbf R^d)$. Then the Toeplitz operator $\operatorname{Tp}_\phi(\omega_0)$ is an isomorphism from $M^{p,q}_{(\omega)}(\mathbf R^{2d})$ onto $M^{p,q}_{(\omega/\omega_0)}(\mathbf R^{2d})$.*

For general ω_0 it is clear that $\Gamma_{0,s}^{(\omega_0)} \subseteq \Gamma_s^{(\omega_0)} \subseteq S^{(\omega_0)}$. On the other hand, for the weights ω_1, ω_2 and ω_3 in $\Gamma_{0,s}^{(\omega_1)}$, $\Gamma_s^{(\omega_2)}$ and $S^{(\omega_3)}$ we always assume that they belong to $\mathscr P_{E,s}(\mathbf R^{2d})$, $\mathscr P_{E,s}^0(\mathbf R^{2d})$ and $\mathscr P(\mathbf R^{2d})$, respectively. That is, they should satisfy

$$\omega_1(X+Y) \lesssim \omega_1(X)e^{r_1|Y|^{\frac 1s}}, \quad \omega_2(X+Y) \lesssim \omega_2(X)e^{r_2|Y|^{\frac 1s}},$$

$$\text{and} \quad \omega_3(X+Y) \lesssim \omega_3(X)(1+|Y|)^N,$$

for some $r_1 > 0$ and $N > 0$, and every $r_2 > 0$. From these relations we might have

$$\Gamma_{0,s}^{(\omega_1)} \bigcap \Gamma_s^{(\omega_2)} \nsubseteq S^{(\omega_0)}, \quad \Gamma_{0,s}^{(\omega_1)} \bigcap S^{(\omega_0)} \nsubseteq \Gamma_s^{(\omega_2)} \quad \text{and} \quad \Gamma_s^{(\omega_2)} \bigcap S^{(\omega_0)} \nsubseteq \Gamma_{0,s}^{(\omega_1)}.$$

We note that, in contrast to [30, 31], our lifting properties also hold for modulation spaces which may fail to be Banach spaces, since p and q in Theorem 0.1 are allowed to be smaller than 1.

We establish several results related to Theorem 0.1. Firstly, the window function ϕ may be chosen in certain modulation spaces that are much larger than the Gelfand-Shilov space $\mathcal S_s$. Secondly, the theorem holds for a more general family of modulation spaces that includes the classical ones.

It is expected that the results in Sections 4 and 5 will be more applicable compared to corresponding results in [30] because the restrictions on the weights are significantly relaxed. Beside the extension of the interval for p, q, the annoying condition (0.3) in [31] is completely removed in Theorem 0.1. Consequently, Theorem 0.1 alone strictly improves the corresponding results in [31] (cf. e. g. [31, Theorem

4.3]). Summing up, our lifting results in Sections 4 and 5 extend the lifting results in [30,31].

The paper is organised as follows. In Section 1 we introduce some notation, and discuss modulation spaces and Gelfand-Shilov spaces of functions and distributions, and pseudo-differential calculus. In Section 2 we introduce and discuss basic properties for confinements of symbols in $\Gamma_s^{(\omega_0)}$ and in $\Gamma_{0,s}^{(\omega_0)}$. These considerations are related to the discussions in [3,38], but here adapted to symbols that possess Gevrey regularity, e. g. when the symbols belong to $\Gamma_s^{(\omega_0)}$ or $\Gamma_{0,s}^{(\omega_0)}$.

In contrast to the classical Hörmander symbol classes $S^r_{1,0}$ and the SG-classes $\mathrm{SG}^{m,\mu}_{1,1}$, techniques on asymptotic expansions are absent for symbols in the classes $\Gamma_s^{(\omega_0)}$ or in $\Gamma_{0,s}^{(\omega_0)}$, and might be absent for symbols in the general Hörmander class $S(m,g)$. The approach with confinements is, roughly speaking, a sort of stand-in of these absent asymptotic expansion techniques.

In Section 3 we show that (0.6) has a unique solution with the requested properties, which leads to (0.5). In Sections 4 and 5 we use the results from Section 3 to deduce lifting properties for modulation spaces under pseudo-differential operators and Toeplitz operators with symbols in $\Gamma_s^{(\omega_0)}$, $\Gamma_{0,s}^{(\omega_0)}$ or in suitable modulation spaces.

Finally we show some examples on applications of our results in Section 6. In Examples 6.5 and 6.8 we consider operators $\mathrm{Op}(\omega_0)$ and $\mathrm{Tp}(\omega_0)$, where ω_0 may be of the form

$$\omega_0(x,\xi) = \big((1+|x|^{2t})^{\frac{r}{2t}} + (1+|\xi|^{2\tau})^{\frac{\rho}{2\tau}} \big)^{r_0}.$$

Here $t,\tau \ge 1$ are integers and $r,\rho > 0$ are real. By Theorem 0.1 it follows that $\mathrm{Tp}(\omega_0)$ is a homeomorphism from $M^{p,q}_{(\omega)}$ to $M^{p,q}_{(\omega/\omega_0)}$. If in addition $r_0 = 1$, then We show that this result in combination with Fredholm theory can be used to deduce that the same homeomorphism property is true for $\mathrm{Op}(\omega_0)$.

An other consequence of Example 6.8 is that if $r_0 \cdot \max(r,\rho) < 1$ and $a = e^{\omega_0^{\frac{1}{r_0}}}$, then $\mathrm{Op}(a)$ from $M^{p,q}_{(\omega)}$ to $M^{p,q}_{(\omega/e^{\omega_0})}$ has index zero.

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1. Preliminaries

In this section we recall some basic facts on modulation spaces, Gelfand-Shilov spaces of functions and distributions and pseudo-differential calculus (cf. [14–19,21, 25,29,34,35,38,43,48,50,53–57]).

1.1. **Weight functions.**

A *weight* on $\mathbf{R}^d$ is a positive function $\omega \in L^\infty_{loc}(\mathbf{R}^d)$ such that $1/\omega \in L^\infty_{loc}(\mathbf{R}^d)$. If ω and v are weights on $\mathbf{R}^d$, then ω is called *moderate* or *v-moderate*, if

$$\omega(x+y) \le C\omega(x)v(y), \quad x,y \in \mathbf{R}^d, \tag{1.1}$$

for some constant $C \ge 1$. The set of all moderate weights on $\mathbf{R}^d$ is denoted by $\mathscr{P}_E(\mathbf{R}^d)$. We notice that if the weight v is even and (1.1) is fulfilled with $\omega = v$ and $C = 1$, and that $v(x) \ge c$ for some $c \in (0,1]$, then $v_1(x) = c^{-1}v(x)$ satisfies the same properties, as well as $v_1(x) \ge 1$.

The weight v on $\mathbf{R}^d$ is called *submultiplicative*, if it is even, bounded from below by 1 and (1.1) holds for $\omega = v$ and $C = 1$. From now on, v always denote a submultiplicative weight if nothing else is stated. In particular, if (1.1) holds and v is submultiplicative, then it follows by straightforward computations that

$$\begin{aligned} C^{-1}\frac{\omega(x)}{v(y)} \le \omega(x+y) \le C\omega(x)v(y), \\ v(x+y) \le v(x)v(y) \quad \text{and} \quad v(x) = v(-x) \ge 1, \quad x,y \in \mathbf{R}^d. \end{aligned} \tag{1.2}$$

If ω is a moderate weight on $\mathbf{R}^d$, then by [58] and above, there is a submultiplicative weight v on $\mathbf{R}^d$ such that (1.1) and (1.2) hold (see also [25, 53, 54]). Moreover if v is submultiplicative on $\mathbf{R}^d$, then

$$1 \lesssim v(x) \lesssim e^{r|x|} \tag{1.3}$$

for some constant $r > 0$ (cf. [28]). Here and in what follows we write $A(\theta) \lesssim B(\theta)$, $\theta \in \Omega$, if there is a constant $c > 0$ such that $A(\theta) \le cB(\theta)$ for all $\theta \in \Omega$. In particular, if ω is moderate, then

$$\omega(x+y) \lesssim \omega(x)e^{r|y|} \quad \text{and} \quad e^{-r|x|} \le \omega(x) \lesssim e^{r|x|}, \quad x,y \in \mathbf{R}^d \tag{1.4}$$

for some $r > 0$.

Next we introduce suitable subclasses of $\mathscr{P}_E$.

Definition 1.1. Let $s > 0$. The set $\mathscr{P}_{E,s}(\mathbf{R}^d)$ ($\mathscr{P}^0_{E,s}(\mathbf{R}^d)$) consists of all $\omega \in \mathscr{P}_E(\mathbf{R}^d)$ such that

$$\omega(x+y) \lesssim \omega(x)e^{r|y|^{\frac{1}{s}}}, \quad x,y \in \mathbf{R}^d; \tag{1.5}$$

holds for some (every) $r > 0$.

By (1.4) it follows that $\mathscr{P}^0_{E,s_1} = \mathscr{P}_{E,s_2} = \mathscr{P}_E$ when $s_1 < 1$ and $s_2 \le 1$. For convenience we set $\mathscr{P}^0_E(\mathbf{R}^d) = \mathscr{P}^0_{E,1}(\mathbf{R}^d)$.

1.2. **Gelfand-Shilov spaces.** Let $\mathscr{F}$ be the Fourier transform given by

$$(\mathscr{F}f)(\xi) = \widehat{f}(\xi) \equiv (2\pi)^{-\frac{d}{2}} \int_{\mathbf{R}^d} f(x)e^{-i\langle x,\xi\rangle}\, dx \tag{1.6}$$

when $f \in L^1(\mathbf{R}^d)$. Here $\langle\, \cdot \,, \cdot \,\rangle$ denotes the usual scalar product on $\mathbf{R}^d$.

Let $h, s, s_0, \sigma, \sigma_0 \in \mathbf{R}_+$, and let $\mathcal{S}^\sigma_{s,h}(\mathbf{R}^d)$ be the set of all $f \in C^\infty(\mathbf{R}^d)$ such that

$$\|f\|_{\mathcal{S}^\sigma_{s,h}} \equiv \sup \frac{|x^\beta \partial^\alpha f(x)|}{h^{|\alpha+\beta|}\alpha!^\sigma \beta!^s}$$

is finite. Here the supremum is taken over all $\alpha, \beta \in \mathbf{N}^d$ and $x \in \mathbf{R}^d$.

Obviously $\mathcal{S}^\sigma_{s,h}(\mathbf{R}^d)$ is a Banach space which increases as h, s and σ increase, and is contained in $\mathscr{S}(\mathbf{R}^d)$, the set of Schwartz functions on $\mathbf{R}^d$. If in addition $s + \sigma > 1$ and $s_0 + \sigma_0 \ge 1$

$$\mathcal{S}^\sigma_{s,h}(\mathbf{R}^d) \quad \text{and} \quad \bigcup_{h>0} \mathcal{S}^{\sigma_0}_{s_0,h}(\mathbf{R}^d)$$

are dense in $\mathscr{S}(\mathbf{R}^d)$. Hence, the dual $(\mathcal{S}^\sigma_{s,h})'(\mathbf{R}^d)$ of $\mathcal{S}^\sigma_{s,h}(\mathbf{R}^d)$ is a Banach space which contains $\mathscr{S}'(\mathbf{R}^d)$.

The Gelfand-Shilov spaces $\mathcal{S}^\sigma_s(\mathbf{R}^d)$ and $\Sigma^\sigma_s(\mathbf{R}^d)$ of Roumieu respective Beurling type of order (s,σ) are the inductive and projective limits, respectively, of $\mathcal{S}^\sigma_{s,h}(\mathbf{R}^d)$ with respect to h. This implies that

$$\mathcal{S}^\sigma_s(\mathbf{R}^d) = \bigcup_{h>0} \mathcal{S}^\sigma_{s,h}(\mathbf{R}^d) \quad \text{and} \quad \Sigma^\sigma_s(\mathbf{R}^d) = \bigcap_{h>0} \mathcal{S}^\sigma_{s,h}(\mathbf{R}^d), \tag{1.7}$$

and that the topology for $\mathcal{S}^\sigma_s(\mathbf{R}^d)$ is the strongest possible one such that each inclusion map from $\mathcal{S}^\sigma_{s,h}(\mathbf{R}^d)$ to $\mathcal{S}^\sigma_s(\mathbf{R}^d)$ is continuous, see also Remark 1.4 below.

The Gelfand-Shilov distribution spaces $(\mathcal{S}_s^\sigma)'(\mathbf{R}^d)$ and $(\Sigma_s^\sigma)'(\mathbf{R}^d)$ are the projective and inductive limit respectively of $(\mathcal{S}_{s,h}^\sigma)'(\mathbf{R}^d)$. Hence

$$(\mathcal{S}_s^\sigma)'(\mathbf{R}^d) = \bigcap_{h>0} (\mathcal{S}_{s,h}^\sigma)'(\mathbf{R}^d) \quad \text{and} \quad (\Sigma_s^\sigma)'(\mathbf{R}^d) = \bigcup_{h>0} (\mathcal{S}_{s,h}^\sigma)'(\mathbf{R}^d). \tag{1.7$'$}$$

We have that $(\mathcal{S}_s^\sigma)'$ and $(\Sigma_s^\sigma)'$ are the topological duals of $\mathcal{S}_s^\sigma$ and Σ_s^σ, respectively (see [24,42]).

We also set $\mathcal{S}_s = \mathcal{S}_s^s$ and $\Sigma_s = \Sigma_s^s$, and similarly for their distribution spaces.

The classes $\mathcal{S}_s^\sigma(\mathbf{R}^d)$ and related generalizations were widely studied, and used in the applications to partial differential equations, see for example [2,4,9,12,32,39,40,43]. We recall the following characterisations of $\mathcal{S}_s^\sigma(\mathbf{R}^d)$.

Proposition 1.2. *Let $s,\sigma > 0$, $p \in [1,\infty]$ and let $f \in \mathscr{S}(\mathbf{R}^d)$. Then the following conditions are equivalent:*

(1) *$f \in \mathcal{S}_s^\sigma(\mathbf{R}^d)$ ($f \in \Sigma_s^\sigma(\mathbf{R}^d)$);*
(2) *for some (every) $h > 0$ it holds*

$$\|x^\alpha f\|_{L^p} \lesssim h^{|\alpha|}\alpha!^s \quad \textit{and} \quad \|\xi^\beta \widehat{f}\|_{L^p} \lesssim h^{|\beta|}\beta!^\sigma, \quad \alpha,\beta \in \mathbf{N}^d;$$

(3) *for some (every) $h > 0$ it holds*

$$\|x^\alpha f\|_{L^p} \lesssim h^{|\alpha|}\alpha!^s \quad \textit{and} \quad \|\partial^\beta f\|_{L^p} \lesssim h^{|\beta|}\beta!^\sigma, \quad \alpha,\beta \in \mathbf{N}^d;$$

(4) *for some (every) $h > 0$ it holds*

$$\|x^\alpha \partial^\beta f(x)\|_{L^p} \lesssim h^{|\alpha+\beta|}\alpha!^s\,\beta!^\sigma, \quad \alpha,\beta \in \mathbf{N}^d;$$

(5) *for some (every) $h,r > 0$ it holds*

$$\|e^{r|\,\cdot\,|^{\frac{1}{s}}}\partial^\alpha f\|_{L^p} \lesssim h^{|\alpha|}(\alpha!)^\sigma\, \alpha \in \mathbf{N}^d;$$

(6) *for some (every) $r > 0$ it holds*

$$\|f \cdot e^{r|\,\cdot\,|^{\frac{1}{s}}}\|_{L^p} < \infty \quad \textit{and} \quad \|\widehat{f} \cdot e^{r|\,\cdot\,|^{\frac{1}{\sigma}}}\|_{L^p} < \infty.$$

Remark 1.3. Any of the conditions (2)–(6) in Proposition 1.2 induce the same topology for $\mathcal{S}_s^\sigma(\mathbf{R}^d)$ and $\Sigma_s^\sigma(\mathbf{R}^d)$.

Remark 1.4. Let $s,\sigma > 0$. Then $\Sigma_s^\sigma(\mathbf{R}^d)$ is a Fréchet space with seminorms $\|\cdot\|_{\mathcal{S}_{s,h}^\sigma}$, $h > 0$. Moreover, $\mathcal{S}_s^\sigma(\mathbf{R}^d) \neq \{0\}$ if and only if $s+\sigma \geq 1$, and $\Sigma_s^\sigma(\mathbf{R}^d) \neq \{0\}$ if and only if $s+\sigma \geq 1$ and $(s,\sigma) \neq (\frac{1}{2},\frac{1}{2})$. If $\varepsilon > 0$ and $s+\sigma \geq 1$, then

$$\Sigma_s^\sigma(\mathbf{R}^d) \subseteq \mathcal{S}_s^\sigma(\mathbf{R}^d) \subseteq \Sigma_{s+\varepsilon}^{\sigma+\varepsilon}(\mathbf{R}^d) \subseteq \mathscr{S}(\mathbf{R}^d) \subseteq \mathscr{S}'(\mathbf{R}^d) \subseteq (\Sigma_{s+\varepsilon}^{\sigma+\varepsilon})'(\mathbf{R}^d) \subseteq (\mathcal{S}_s^\sigma)'(\mathbf{R}^d),$$

and if in addition $(s,\sigma) \neq (\frac{1}{2},\frac{1}{2})$, then

$$(\mathcal{S}_s^\sigma)'(\mathbf{R}^d) \subseteq (\Sigma_s^\sigma)'(\mathbf{R}^d).$$

The Gelfand-Shilov spaces are invariant and possess convenient mapping properties under several basic transformations, e.g. under translations, dilations and (partial) Fourier transformations.

The Fourier transform $\mathscr{F}$ on $\mathscr{S}(\mathbf{R}^d)$ as well as any partial Fourier transform, extend uniquely to homeomorphisms on $\mathscr{S}'(\mathbf{R}^d)$, $\mathcal{S}_s'(\mathbf{R}^d)$ and $\Sigma_s'(\mathbf{R}^d)$, and restrict to homeomorphisms on $\mathcal{S}_s(\mathbf{R}^d)$ and $\Sigma_s(\mathbf{R}^d)$, and to a unitary operators on $L^2(\mathbf{R}^d)$.

We also recall some mapping properties of Gelfand-Shilov spaces under short-time Fourier transforms. Let $\phi \in \mathscr{S}(\mathbf{R}^d)$ be fixed. For every $f \in \mathscr{S}'(\mathbf{R}^d)$, the *short-time Fourier transform* $V_\phi f$ is the distribution on $\mathbf{R}^{2d}$ defined by the formula

$$(V_\phi f)(x,\xi) = \mathscr{F}(f\,\overline{\phi(\,\cdot\,-x)})(\xi) = (2\pi)^{-\frac{d}{2}}(f,\phi(\,\cdot\,-x)e^{i\langle\,\cdot\,,\xi\rangle}). \tag{1.8}$$

We recall that if $T(f,\phi) \equiv V_\phi f$ when $f,\phi \in \mathcal{S}_{1/2}(\mathbf{R}^d)$, then T is uniquely extendable to sequentially continuous mappings

$$
\begin{aligned}
T\ &: \mathcal{S}_s'(\mathbf{R}^d) \times \mathcal{S}_s(\mathbf{R}^d) \to \mathcal{S}_s'(\mathbf{R}^{2d}) \bigcap C^\infty(\mathbf{R}^{2d}),\\
T\ &: \mathcal{S}_s'(\mathbf{R}^d) \times \mathcal{S}_s'(\mathbf{R}^d) \to \mathcal{S}_s'(\mathbf{R}^{2d}),
\end{aligned}
$$

and similarly when $\mathcal{S}_s$ and $\mathcal{S}_s'$ are replaced by Σ_s and Σ_s', respectively, or by $\mathscr{S}$ and $\mathscr{S}'$, respectively (cf. [10, 58]). We also note that $V_\phi f$ takes the form

$$
V_\phi f(x,\xi) = (2\pi)^{-\frac{d}{2}} \int_{\mathbf{R}^d} f(y)\overline{\phi(y-x)} e^{-i\langle y,\xi\rangle}\, dy \tag{1.8$'$}
$$

when $f \in L^p_{(\omega)}(\mathbf{R}^d)$ for some $\omega \in \mathscr{P}_E(\mathbf{R}^d)$, $\phi \in \Sigma_1(\mathbf{R}^d)$ and $p \ge 1$. Here $L^p_{(\omega)}(\mathbf{R}^d)$, when $p \in (0,\infty]$ and $\omega \in \mathscr{P}_E(\mathbf{R}^d)$, is the set of all $f \in L^p_{loc}(\mathbf{R}^d)$ such that $\|f\|_{L^p_{(\omega)}} \equiv \|f\cdot\omega\|_{L^p}$ is finite.

1.3. **Suitable function classes with Gelfand-Shilov regularity.** The next result shows that for any $\omega \in \mathscr{P}_E(\mathbf{R}^d)$ one can find an equivalent weight ω_0 which satisfies suitable Gevrey regularity.

Proposition 1.5. *Let $\omega \in \mathscr{P}_E(\mathbf{R}^d)$ and $s > 0$. Then there is an $\omega_0 \in \mathscr{P}_E(\mathbf{R}^d) \cap C^\infty(\mathbf{R}^d)$ such that the following is true:*

(1) $\omega_0 \asymp \omega$;
(2) $|\partial^\alpha \omega_0(x)| \lesssim h^{|\alpha|}\alpha!^s \omega_0(x) \asymp h^{|\alpha|}\alpha!^s\omega(x)$ *for every* $h > 0$.

Proof. We may assume that $s < 1$. It suffices to prove that (2) should hold for some $h > 0$. Let $\phi_0 \in \Sigma^s_{1-s}(\mathbf{R}^d) \setminus \{0\}$, and let $\phi = |\phi_0|^2$. Then $\phi \in \Sigma^s_{1-s}(\mathbf{R}^d)$, giving that

$$
|\partial^\alpha \phi(x)| \lesssim h^{|\alpha|} e^{-r|x|^{\frac{1}{1-s}}} \alpha!^s,
$$

for every $h > 0$ and $r > 0$. Now let $\omega_0 = \omega * \phi$.

We have

$$
\begin{aligned}
|\partial^\alpha \omega_0(x)| &= \left| \int_{\mathbf{R}^d} \omega(y)(\partial^\alpha\phi)(x-y)\, dy \right|\\
&\lesssim h^{|\alpha|}\alpha!^s \int_{\mathbf{R}^d} \omega(y) e^{-r|x-y|^{\frac{1}{1-s}}}\, dy\\
&\lesssim h^{|\alpha|}\alpha!^s \int_{\mathbf{R}^d} \omega(x+(y-x)) e^{-r|x-y|^{\frac{1}{1-s}}}\, dy\\
&\lesssim h^{|\alpha|}\alpha!^s \omega(x) \int_{\mathbf{R}^d} e^{-\frac{r}{2}|x-y|^{\frac{1}{1-s}}}\, dy \asymp h^{|\alpha|}\alpha!^s\omega(x),
\end{aligned}
$$

where the last inequality follows from (1.4) and the fact that ϕ is bounded by a super exponential function. This gives the first part of (2).

The equivalences in (1) follows in the same way as in [58]. More precisely, by (1.4) we have

$$
\begin{aligned}
\omega_0(x) = \int_{\mathbf{R}^d} \omega(y)\phi(x-y)\, dy &= \int_{\mathbf{R}^d} \omega(x+(y-x))\phi(x-y)\, dy\\
&\lesssim \omega(x) \int_{\mathbf{R}^d} e^{r|x-y|}\phi(x-y)\, dy \asymp \omega(x).
\end{aligned}
$$

In the same way, (1.4) gives

$$\omega_0(x) = \int_{\mathbf{R}^d} \omega(y)\phi(x-y)\,dy = \int_{\mathbf{R}^d} \omega(x+(y-x))\phi(x-y)\,dy$$
$$\gtrsim \omega(x)\int_{\mathbf{R}^d} e^{-r|x-y|}\phi(x-y)\,dy \asymp \omega(x),$$

and (1) as well as the second part of (2) follow. □

A weight ω_0 which satisfies Proposition 1.5 (2) is called *elliptic* or *s-elliptic*.

Important classes of Gevrey type symbols is the following.

Definition 1.6. Let $s \ge 0$ and $\omega \in \mathscr{P}_E(\mathbf{R}^d)$. The class $\Gamma_s^{(\omega)}(\mathbf{R}^d)$ $(\Gamma_{0,s}^{(\omega)}(\mathbf{R}^d))$ consists of all $f \in C^\infty(\mathbf{R}^d)$ such that

$$|D^\alpha f(x)| \lesssim h^{|\alpha|}\alpha!^s\omega(x), \quad x \in \mathbf{R}^d,$$

for *some* $h>0$ (for *every* $h>0$).

Evidently, by Proposition 1.5 it follows that if $s<1$, then the family of symbol classes in Definition 1.6 does not (strictly) increase when the assumption $\omega \in \mathscr{P}_E(\mathbf{R}^{2d})$ is replaced by $\omega \in \mathscr{P}_{E,s}(\mathbf{R}^{2d})$ or by $\omega \in \mathscr{P}_{E,s}^0(\mathbf{R}^{2d})$.

By similar arguments as in the proof of Proposition 1.5 we get the following analogy of Proposition 2.3.16 in [37]. The details are left for the reader.

Proposition 1.7. *Let $s>1/2$, $\omega \in \mathscr{P}_E(\mathbf{R}^{2d})$, and $\phi \in \Sigma_s(\mathbf{R}^{2d})$. Then $\omega * \phi$ belongs to $\Gamma_{0,s}^{(\omega)}$.*

The following definition is motivated by Lemma 2.6.13 in [37].

Definition 1.8. Let $s \ge 1$, $\omega \in \mathscr{P}_E(\mathbf{R}^d)$ and $\vartheta_0 = 1 + |\log \omega|$. Then a is called *comparable* to ω with respect to $s \ge 1$ if

(1) $\|a - \log\omega\|_{L^\infty} < \infty$;
(2) $a \in \Gamma_s^{(\vartheta_0)}(\mathbf{R}^d)$ and $\partial^\alpha a \in \Gamma_s^{(1)}(\mathbf{R}^d)$, when $|\alpha|=1$.

Proposition 1.9. *Let $\omega, v \in \mathscr{P}_E(\mathbf{R}^d)$ be such that v is submultiplicative and* (1.1) *holds for some $C \ge 1$. Also let*

$$v_1(x) \equiv 1 + |\log v(x)| \quad \text{and} \quad \omega_1(x) \equiv 1 + |\log\omega(x)|.$$

Then v_1 is submultiplicative and ω_1 is v_1-moderate, and (1.1) *holds with* $1+\log C \ge 1$, *ω_1 and v_1 in place of $C \ge 1$, ω and v, respectively.*

Proof. If $\omega(x+y) \ge 1$, then the second inequality in (1.2) and the fact that $\log C \ge 0$ give

$$\begin{aligned}\omega_1(x+y) &= 1 + \log\omega(x+y)\\ &\le 1 + \log C + \log\omega(x) + \log v(y)\\ &\le (1+\log C)(1+|\log\omega(x)|)\,(1+\log v(y))\\ &\le (1+\log C)\,\omega_1(x)\,v_1(y).\end{aligned}$$

If instead $\omega(x+y) \le 1$, then the first inequality in (1.2) gives

$$\begin{aligned}\omega_1(x+y) &= 1 - \log\omega(x+y)\\ &\le 1 + \log C - \log\omega(x) + \log v(y)\\ &\le (1+\log C)(1+|\log\omega(x)|)\,(1+\log v(y))\\ &\le (1+\log C)\,\omega_1(x)\,v_1(y),\end{aligned}$$

giving that ω_1 is v_1-moderate with the searched constants.

By choosing $\omega = v$ and $C=1$, we deduce the submultiplicativity for v_1, and the result follows. □

Lemma 1.10. *Let $s \ge 1$, $\omega \in \mathscr{P}_E(\mathbf{R}^d)$ and $\vartheta_0 = 1 + |\log \omega|$. Then the following is true:*

(1) *there exists an elliptic weight $\omega_0 \in \mathscr{P}_E(\mathbf{R}^d) \cap \Gamma_s^{(\omega)}(\mathbf{R}^d)$ such that*
$$\omega \asymp \omega_0, \quad \log \omega_0 \in \Gamma_s^{(\vartheta_0)}(\mathbf{R}^d) \quad \text{and} \quad 1 + |\log \omega_0| \in \mathscr{P}_E(\mathbf{R}^d) \cap \Gamma_s^{(\vartheta_0)}(\mathbf{R}^d);$$

(2) *there exists an element a which is comparable to ω_0 with respect to s.*

Proof. The assertion (1) follows by letting ω_0 be the same as in Proposition 1.5, and (2) follows by letting $a = \log \omega_0$ and using the ellipticity of ω_0. □

1.4. **Modulation spaces.** Before giving the definition of modulation spaces we recall the definition of quasi-Banach spaces. A functional $f \mapsto \|f\|_{\mathscr{B}}$ on a (complex) vector space $\mathscr{B}$ is called a quasi-norm of order $r \in (0,1]$, or an r-norm, if $\|f\|_{\mathscr{B}} \ge 0$ for all $f \in \mathscr{B}$ with equality only for $f = 0$,

$$\|f+g\|_{\mathscr{B}} \le 2^{\frac{1}{r}-1}(\|f\|_{\mathscr{B}} + \|g\|_{\mathscr{B}}) \quad f, g \in \mathscr{B}, \tag{1.9}$$

and

$$\|c \cdot f\|_{\mathscr{B}} = |c| \cdot \|f\|_{\mathscr{B}} \qquad f \in \mathscr{B},\ c \in \mathbf{C}. \tag{1.10}$$

By Aoki and Rolewić in [1, 45] it follows that there is an equivalent quasi-norm to the previous one which additionally satisfies

$$\|f+g\|_{\mathscr{B}}^r \le \|f\|_{\mathscr{B}}^r + \|g\|_{\mathscr{B}}^r \qquad f, g \in \mathscr{B}. \tag{1.11}$$

From now on we suppose that the quasi-norm of $\mathscr{B}$ has been chosen such that both (1.9) and (1.11) hold true.

The space $\mathscr{B}$ above is called a quasi-Banach space or an r-Banach space, if the topology is defined by $\|\cdot\|_{\mathscr{B}}$, and that $\mathscr{B}$ is complete under this topology.

Let $\phi \in \Sigma_1(\mathbf{R}^d) \setminus 0$, $p, q \in (0,\infty]$ and $\omega \in \mathscr{P}_E(\mathbf{R}^{2d})$ be fixed. Then the *modulation space* $M^{p,q}_{(\omega)}(\mathbf{R}^d)$ consists of all $f \in \Sigma_1'(\mathbf{R}^d)$ such that

$$\|f\|_{M^{p,q}_{(\omega)}} \equiv \Big(\int_{\mathbf{R}^d} \Big(\int_{\mathbf{R}^d} |V_\phi f(x,\xi)\omega(x,\xi)|^p \, dx \Big)^{q/p} d\xi \Big)^{1/q} < \infty \tag{1.12}$$

(with the obvious modifications when $p = \infty$ and/or $q = \infty$). Evidently, $\|f\|_{M^{p,q}_{(\omega)}}$ is given by

$$\|f\|_{M^{p,q}_{(\omega)}} \equiv \|H_{f,\omega,p}\|_{L^q}, \quad H_{f,\omega,p}(\xi) = \|V_\phi f(\,\cdot\,,\xi)\omega(\,\cdot\,,\xi)\|_{L^p} \tag{1.13}$$

We set $M^p_{(\omega)} = M^{p,p}_{(\omega)}$, and if $\omega = 1$, then we set $M^{p,q} = M^{p,q}_{(\omega)}$ and $M^p = M^p_{(\omega)}$.

The following proposition is a consequence of well-known facts in [15, 23, 25, 58, 60]. Here and in what follows, we let p' denotes the conjugate exponent of p, i. e.

$$p' = \begin{cases} \infty & \text{when } p \in (0,1] \\ \dfrac{p}{p-1} & \text{when } p \in (1,\infty) \\ 1 & \text{when } p = \infty\,. \end{cases}$$

Proposition 1.11. *Let $p, q, p_j, q_j, r \in (0,\infty]$ be such that $r \le \min(1,p,q)$, $j = 1,2$, let $\omega, \omega_1, \omega_2, v \in \mathscr{P}_E(\mathbf{R}^{2d})$ be such that ω is v-moderate, $\phi \in M^r_{(v)}(\mathbf{R}^d) \setminus 0$, and let $f \in \Sigma_1'(\mathbf{R}^d)$. Then the following is true:*

(1) *$f \in M^{p,q}_{(\omega)}(\mathbf{R}^d)$ if and only if (1.12) holds, i. e. $M^{p,q}_{(\omega)}(\mathbf{R}^d)$ is independent of the choice of ϕ. Moreover, $M^{p,q}_{(\omega)}$ is an r-Banach space under the r-norm in (1.12), and different choices of ϕ give rise to equivalent r-norms. If in addition $p, q \ge 1$, then $M^{p,q}_{(\omega)}(\mathbf{R}^d)$ is a Banach space;*

(2) *if* $p_1 \le p_2$, $q_1 \le q_2$ *and* $\omega_2 \lesssim \omega_1$, *then*

$$\Sigma_1(\mathbf{R}^d) \subseteq M^{p_1,q_1}_{(\omega_1)}(\mathbf{R}^d) \subseteq M^{p_2,q_2}_{(\omega_2)}(\mathbf{R}^d) \subseteq \Sigma_1'(\mathbf{R}^d).$$

Remark 1.12. For modulation spaces of the form $M^{p,q}_{(\omega)}$ with fixed $p,q \in [1,\infty]$ the norm equivalence in Proposition 1.11(1) can be extended to a larger class of windows. In fact, assume that $\omega, v \in \mathscr{P}_E(\mathbf{R}^{2d})$ with ω being v-moderate and

$$1 \le r \le \min(p,p',q,q')\,.$$

Let $\phi \in M^r_{(v)}(\mathbf{R}^d) \setminus \{0\}$. Then a Gelfand-Shilov distribution $f \in \Sigma_1'(\mathbf{R}^d)$ belongs to $M^{p,q}_{(\omega)}(\mathbf{R}^d)$, if and only if $V_\phi f \in L^{p,q}_{(\omega)}(\mathbf{R}^{2d})$. Furthermore, different choices of $\phi \in M^r_{(v)}(\mathbf{R}^d) \setminus \{0\}$ in $\|V_\phi f\|_{L^{p,q}_{(\omega)}}$ give rise to equivalent norms. (Cf. Theorem 2.6 in [59].)

Remark 1.13. Let $s \ge 1$, $\omega_1 \in \mathscr{P}^0_{E,s}(\mathbf{R}^{2d})$, $\omega \in \mathscr{P}_{E,s}(\mathbf{R}^{2d})$ $p,q \in (0,\infty]$. Then it follows from [58, Theorem 3.9] that the first and last inclusions in Proposition 1.11 (2) can be refined into:

$$\mathcal{S}_s(\mathbf{R}^d) \subseteq M^{p,q}_{(\omega_1)}(\mathbf{R}^d) \subseteq \mathcal{S}_s'(\mathbf{R}^d) \quad \text{and} \quad \Sigma_s(\mathbf{R}^d) \subseteq M^{p,q}_{(\omega_2)}(\mathbf{R}^d) \subseteq \Sigma_s'(\mathbf{R}^d).$$

In essential parts of our analyses in Sections 4 and 5 it is convenient to use symplectic formulations of modulation spaces with functions and distributions defined on the phase spaces $\mathbf{R}^{2d}$. They are defined in the same way as the modulation spaces above, except that the short-time Fourier transforms in (1.8) are replaced by symplectic analogies in the definition of modulation space norms.

In fact, let σ be the standard symplectic form on $\mathbf{R}^{2d}$, i. e. it should satisfy

$$\sigma(X,Y) = \langle y,\xi\rangle - \langle x,\eta\rangle, \qquad X=(x,\xi)\in\mathbf{R}^{2d},\ Y=(y,\eta)\in\mathbf{R}^{2d}. \tag{1.14}$$

(Here observe the difference between the notation σ for the symplectic form in (1.14), and the positive number σ used as parameter for the Gelfand-Shilov spaces, e. g. in Subsections 1.2 and 1.3.) If

$$\{e_1,\dots,e_d,\varepsilon_1,\dots,\varepsilon_d\} \tag{1.15}$$

is the standard basis of $\mathbf{R}^{2d}$, then

$$\sigma(e_j,e_k)=0, \quad \sigma(e_j,\varepsilon_k) = -\delta_{j,k}, \quad \text{and} \quad \sigma(\varepsilon_j,\varepsilon_k)=0 \tag{1.16}$$

when $j,k\in\{1,\dots,d\}$. More generally, a basis (1.15) of $\mathbf{R}^{2d}$ which satisfies (1.16) is called a symplectic basis of $\mathbf{R}^{2d}$ to the symplectic form σ. Evidently, the standard basis of $\mathbf{R}^{2d}$ is a symplectic basis, and is sometimes called the standard symplectic basis of $\mathbf{R}^{2d}$.

Let $\phi \in \Sigma_1(\mathbf{R}^{2d}) \setminus 0$. Then the *symplectic Fourier transform* and *symplectic short-time Fourier transform* of $a \in L^1(\mathbf{R}^{2d})$ are defined by the formulae

$$(\mathscr{F}_\sigma a)(X) = \pi^{-d}\int_{\mathbf{R}^{2d}} a(Z)e^{2i\sigma(X,Z)}\,dZ \tag{1.17}$$

and

$$(\mathcal{V}_\phi a)(X,Y) = \pi^{-d}\int_{\mathbf{R}^{2d}} a(Z)\overline{\phi(Z-Y)}e^{2i\sigma(X,Z)}\,dZ. \tag{1.18}$$

By straight-forward computations, using Fourier's inversion formula, it follows that $\mathscr{F}_\sigma = T\circ(\mathscr{F}\otimes(\mathscr{F}^{-1}))$, when $(Ta)(x,\xi)=a(\xi,x)$ and

$$(\mathcal{V}_\phi a)(X,Y) = 2^d(V_\phi a)(x,\xi,-2\eta,2y), \quad X=(x,\xi)\in\mathbf{R}^{2d},\ Y=(y,\eta)\in\mathbf{R}^{2d}. \tag{1.19}$$

In particular, all continuity and extension properties valid for the usual Fourier transform and short-time Fourier transform carry over to their symplectic relatives.

For example, $\mathscr{F}_\sigma$ is continuous on $\mathcal{S}_s(\mathbf{R}^{2d})$, and extends uniquely to a homeomorphism on $\mathcal{S}_s'(\mathbf{R}^{2d})$, and to a unitary map on $L^2(\mathbf{R}^{2d})$, since similar facts hold for $\mathscr{F}$. By straight-forward computations it also follows that $\mathscr{F}_\sigma^2$ is the identity operator on such spaces.

For any $p,q\in(0,\infty]$, $\omega\in\mathscr{P}_E(\mathbf{R}^{2d}\times\mathbf{R}^{2d})$ and $a\in\Sigma_1'(\mathbf{R}^{2d})$, let $\|a\|_{\mathcal{M}^{p,q}_{(\omega)}}$ be defined by (1.13) after $V_\phi f$ is replaced by $\mathcal{V}_\phi a$. Then the *symplectic modulation space* $\mathcal{M}^{p,q}_{(\omega)}(\mathbf{R}^{2d})$ consists of all $a\in\Sigma_1'(\mathbf{R}^{2d})$ such that $\|a\|_{\mathcal{M}^{p,q}_{(\omega)}}$ is finite.

By (1.19) it follows that

$$\mathcal{M}^{p,q}_{(\omega)}(\mathbf{R}^{2d})=M^{p,q}_{(\omega_0)}(\mathbf{R}^{2d})\quad\text{when}\quad\omega(x,\xi,y,\eta)=\omega_0(x,\xi,-2\eta,2y).$$

Hence, the symplectic modulation spaces are merely other ways to formulate the modulation spaces considered in the first part of the subsection.

1.5. **A broader family of modulation spaces.** In Section 2 we consider mapping properties for pseudo-differential operators when acting on a broad class of modulation spaces which are defined by imposing (quasi-)norm conditions on the involved short-time Fourier transforms of the forms given in the following definition. (Cf. [14–19, 21].)

Definition 1.14. Let $\mathscr{B}\subseteq L^r_{loc}(\mathbf{R}^d)$ be a quasi-Banach space of order $r\in(0,1]$, and let $v\in\mathscr{P}_E(\mathbf{R}^d)$. Then $\mathscr{B}$ is called a *translation invariant Quasi-Banach Function space on* $\mathbf{R}^d$, or *invariant QBF space on* $\mathbf{R}^d$, if the following conditions are fulfilled:

(1) if $x\in\mathbf{R}^d$ and $f\in\mathscr{B}$, then $f(\,\cdot\,-x)\in\mathscr{B}$, and

$$\|f(\,\cdot\,-x)\|_{\mathscr{B}}\lesssim v(x)\|f\|_{\mathscr{B}};\tag{1.20}$$

(2) if $f,g\in L^r_{loc}(\mathbf{R}^d)$ satisfy $g\in\mathscr{B}$ and $|f|\le|g|$, then $f\in\mathscr{B}$ and

$$\|f\|_{\mathscr{B}}\lesssim\|g\|_{\mathscr{B}}.$$

It follows from (2) in Definition 1.14 that if $f\in\mathscr{B}$ and $h\in L^\infty$, then $f\cdot h\in\mathscr{B}$, and

$$\|f\cdot h\|_{\mathscr{B}}\lesssim\|f\|_{\mathscr{B}}\|h\|_{L^\infty}.\tag{1.21}$$

If $r=1$, then $\mathscr{B}$ in Definition 1.14 is a Banach space, and the condition (2) means that a translation invariant QBF-space is a solid BF-space in the sense of (A.3) in [18]. The space $\mathscr{B}$ in Definition 1.14 is called an *invariant BF-space* (with respect to v) if $r=1$, and Minkowski's inequality holds true, i. e. $f*\varphi\in\mathscr{B}$ when $f\in\mathscr{B}$ and $\varphi\in\Sigma_1(\mathbf{R}^d)$, and

$$\|f*\varphi\|_{\mathscr{B}}\lesssim\|f\|_{\mathscr{B}}\|\varphi\|_{L^1_{(v)}},\qquad f\in\mathscr{B},\ \varphi\in\Sigma_1(\mathbf{R}^d).\tag{1.22}$$

Example 1.15. Assume that $p,q\in[1,\infty]$, and let $L^{p,q}_1(\mathbf{R}^{2d})$ be the set of all $f\in L^1_{loc}(\mathbf{R}^{2d})$ such that

$$\|f\|_{L^{p,q}_1}\equiv\Big(\int_{\mathbf{R}^d}\Big(\int_{\mathbf{R}^d}|f(x,\xi)|^p\,dx\Big)^{q/p}d\xi\Big)^{1/q}$$

if finite. Also let $L^{p,q}_2(\mathbf{R}^{2d})$ be the set of all $f\in L^1_{loc}(\mathbf{R}^{2d})$ such that

$$\|f\|_{L^{p,q}_2}\equiv\Big(\int_{\mathbf{R}^d}\Big(\int_{\mathbf{R}^d}|f(x,\xi)|^q\,d\xi\Big)^{p/q}dx\Big)^{1/p}$$

is finite. Then it follows that $L^{p,q}_1$ and $L^{p,q}_2$ are translation invariant BF-spaces with respect to $v=1$.

Definition 1.16. Assume that $\mathscr{B}$ is a translation invariant QBF-space on $\mathbf{R}^{2d}$, $\omega \in \mathscr{P}_E(\mathbf{R}^{2d})$, and that $\phi \in \Sigma_1(\mathbf{R}^d) \setminus 0$. Then the modulation space $M(\omega, \mathscr{B})$ consists of all $f \in \Sigma_1'(\mathbf{R}^d)$ such that

$$\|f\|_{M(\omega,\mathscr{B})} \equiv \|V_\phi f\, \omega\|_{\mathscr{B}}$$

is finite.

Obviously, we have $M^{p,q}_{(\omega)}(\mathbf{R}^d) = M(\omega, \mathscr{B})$ when $\mathscr{B}$ is equal to $L^{p,q}_1(\mathbf{R}^{2d})$ in Example 1.15. It follows that many properties which are valid for the classical modulation spaces also hold for the spaces of the form $M(\omega, \mathscr{B})$. For example we have the following proposition, which shows that the definition of $M(\omega, \mathscr{B})$ is independent of the choice of ϕ when $\mathscr{B}$ is a Banach space. We omit the proof since the completeness assertions follows from [41], and the other parts follow by similar arguments as in the proof of Proposition 11.3.2 in [25]. (See also [41] for topological aspects of $M(\omega, \mathscr{B})$.)

Proposition 1.17. *Let $\mathscr{B}$ be an invariant BF-space with respect to $v_0 \in \mathscr{P}_E(\mathbf{R}^{2d})$ for $j = 1, 2$. Also let $\omega, v \in \mathscr{P}_E(\mathbf{R}^{2d})$ be such that ω is v-moderate, $M(\omega, \mathscr{B})$ is the same as in Definition 1.16, and let $\phi \in M^1_{(v_0 v)}(\mathbf{R}^d) \setminus 0$ and $f \in \Sigma_1'(\mathbf{R}^d)$. Then $M(\omega, \mathscr{B})$ is a Banach space, and $f \in M(\omega, \mathscr{B})$ if and only if $V_\phi f\, \omega \in \mathscr{B}$, and different choices of ϕ gives rise to equivalent norms in $M(\omega, \mathscr{B})$.*

We refer to [14–19, 21, 23, 25, 46, 60] for more facts about modulation spaces.

For translation invariant BF-spaces we make the following observation.

Proposition 1.18. *Assume that $v \in \mathscr{P}_E(\mathbf{R}^d)$, and that $\mathscr{B}$ is an invariant BF-space with respect to v such that* (1.22) *holds true. Then the convolution mapping $(\varphi, f) \mapsto \varphi * f$ from $C_0^\infty(\mathbf{R}^d) \times \mathscr{B}$ to $\mathscr{B}$ extends uniquely to a continuous mapping from $L^1_{(v)}(\mathbf{R}^d) \times \mathscr{B}$ to $\mathscr{B}$, and* (1.22) *holds true for any $f \in \mathscr{B}$ and $\varphi \in L^1_{(v)}(\mathbf{R}^d)$.*

The result is a straightforward consequence of (1.22) and the fact that Σ_1 is dense in $L^1_{(v)}$.

The quasi-Banach space $\mathscr{B}$ above is usually a mixed quasi-normed Lebesgue space, given as follows. Let E be the ordered basis $\{e_1, \dots, e_d\}$ of $\mathbf{R}^d$. Then the ordered basis $E' = \{e_1', \dots, e_d'\}$ (the dual basis of E) satisfies

$$\langle e_j, e_k' \rangle = 2\pi \delta_{jk} \quad \text{for every} \quad j, k = 1, \dots, d.$$

The corresponding parallelepiped, lattice, dual parallelepiped and dual lattice are given by

$$\kappa(E) = \{\, x_1 e_1 + \dots + x_d e_d \,;\, (x_1, \dots, x_d) \in \mathbf{R}^d,\ 0 \le x_k \le 1,\ k = 1, \dots, d \,\},$$

$$\Lambda_E = \{\, j_1 e_1 + \dots + j_d e_d \,;\, (j_1, \dots, j_d) \in \mathbf{Z}^d \,\},$$

$$\kappa(E') = \{\, \xi_1 e_1' + \dots + \xi_d e_d' \,;\, (\xi_1, \dots, \xi_d) \in \mathbf{R}^d,\ 0 \le \xi_k \le 1,\ k = 1, \dots, d \,\},$$

and

$$\Lambda_E' = \Lambda_{E'} = \{\, \iota_1 e_1' + \dots + \iota_d e_d' \,;\, (\iota_1, \dots, \iota_d) \in \mathbf{Z}^d \,\},$$

respectively. Note here that the Fourier analysis with respect to general biorthogonal bases has recently been developed in [47].

We observe that there is a matrix T_E such that $e_1, \dots, e_d$ and $e_1', \dots, e_d'$ are the images of the standard basis under T_E and $T_{E'} = 2\pi (T_E^{-1})^t$, respectively.

In the following we let

$$\max \boldsymbol{q} = \max(q_1, \dots, q_d) \quad \text{and} \quad \min \boldsymbol{q} = \min(q_1, \dots, q_d)$$

when $\boldsymbol{q} = (q_1, \dots, q_d) \in (0, \infty]^d$.

Definition 1.19. Let E be an ordered basis of $\mathbf{R}^d$, $\boldsymbol{p} = (p_1, \dots, p_d) \in (0, \infty]^d$ and $r = \min(1, \boldsymbol{p})$. If $f \in L^r_{loc}(\mathbf{R}^d)$, then $\|f\|_{L^{\boldsymbol{p}}_E}$ is defined by

$$\|f\|_{L^{\boldsymbol{p}}_E} \equiv \|g_{d-1}\|_{L^{p_d}(\mathbf{R})}$$

where $g_k(z_k)$, $z_k \in \mathbf{R}^{d-k}$, $k = 0, \dots, d-1$, are inductively defined as

$$g_0(x_1, \dots, x_d) \equiv |f(x_1 e_1 + \dots + x_d e_d)|,$$

and

$$g_k(z_k) \equiv \|g_{k-1}(\,\cdot\,, z_k)\|_{L^{p_k}(\mathbf{R})}, \quad k = 1, \dots, d-1.$$

The space $L^{\boldsymbol{p}}_E(\mathbf{R}^d)$ consists of all $f \in L^r_{loc}(\mathbf{R}^d)$ such that $\|f\|_{L^{\boldsymbol{p}}_E}$ is finite, and is called *E-split Lebesgue space (with respect to $\boldsymbol{p}$)*.

For the next definition we recall that $\sigma(X, Y)$ denotes the standard symplectic form on the phase space (cf. (1.14)).

Definition 1.20. Let $E = \{e_1, \dots, e_{2d}\}$ be an ordered basis of $\mathbf{R}^{2d}$ and let $E_0 = \{e_1, \dots, e_d\}$. Then E_0 is called a *phase split* of E, if

$$\sigma(e_j, e_k) = 0, \quad \sigma(e_j, e_{d+k}) = -2\pi\delta_{j,k}, \quad \text{and} \quad \sigma(e_{d+j}, e_{d+k}) = 0$$

when $j, k \in \{1, \dots, d\}$.

If (1.15) is the standard basis of $\mathbf{R}^{2d}$ and $e_{d+j} = 2\pi\varepsilon_j$ for $j \in \{1, \dots, d\}$, then (1.16) shows that $\{e_1, \dots, e_d\}$ is a phase split of $\{e_1, \dots, e_{2d}\}$.

The following definition takes care of our most common QBF-spaces.

Definition 1.21. The space $\mathscr{B}$ is called a *normal* QBF-space (on $\mathbf{R}^{2d}$) if it is either an invariant BF-space on $\mathbf{R}^{2d}$ or $\mathscr{B} = L^{\boldsymbol{p}}_E(\mathbf{R}^{2d})$ for some $\boldsymbol{p} \in (0, \infty]^{2d}$ and phase split basis E of $\mathbf{R}^{2d}$.

1.6. **Pseudo-differential operators.** We use the notation $\mathbf{M}(d, \Omega)$ for the set of $d \times d$-matrices with entries in the set Ω. Let $s \ge 1/2$, $a \in \mathcal{S}_s(\mathbf{R}^{2d})$, and $A \in \mathbf{M}(d, \mathbf{R})$ be fixed. Then, the pseudo-differential operator $\operatorname{Op}_A(a)$ is the linear and continuous operator on $\mathcal{S}_s(\mathbf{R}^d)$ given by

$$(\operatorname{Op}_A(a)f)(x) = (2\pi)^{-d} \iint_{\mathbf{R}^{2d}} a(x - A(x-y), \xi)\, f(y)\, e^{i\langle x-y, \xi\rangle}\, dy d\xi \tag{1.23}$$

when $f \in \mathcal{S}_s(\mathbf{R}^d)$. For general $a \in \mathcal{S}_s'(\mathbf{R}^{2d})$, the pseudo-differential operator $\operatorname{Op}_A(a)$ is defined as the continuous operator from $\mathcal{S}_s(\mathbf{R}^d)$ to $\mathcal{S}_s'(\mathbf{R}^d)$ with distribution kernel given by

$$K_{a,A}(x, y) = (2\pi)^{-\frac{d}{2}} (\mathscr{F}_2^{-1} a)(x - A(x-y), x - y). \tag{1.24}$$

Here $\mathscr{F}_2 F$ is the partial Fourier transform of $F(x, y) \in \mathcal{S}_s'(\mathbf{R}^{2d})$ with respect to the y variable. This definition makes sense, since the mappings

$$\mathscr{F}_2 \quad \text{and} \quad F(x, y) \mapsto F(x - A(x-y), y - x) \tag{1.25}$$

are homeomorphisms on $\mathcal{S}_s'(\mathbf{R}^{2d})$. In particular, the map $a \mapsto K_{a,A}$ is a homeomorphism on $\mathcal{S}_s'(\mathbf{R}^{2d})$.

If $A = \frac{1}{2} \cdot I$, then $\operatorname{Op}_A(a)$ is the Weyl quantization $\operatorname{Op}^w(a)$ of a. If instead $A = 0$, then $\operatorname{Op}_A(a)$ equals the normal or Kohn-Nirenberg representation $\operatorname{Op}(a) = a(x, D)$.

Remark 1.22. For any $K \in \mathcal{S}_s'(\mathbf{R}^{d_2+d_1})$, let T_K be the linear and continuous mapping from $\mathcal{S}_s(\mathbf{R}^{d_1})$ to $\mathcal{S}_s'(\mathbf{R}^{d_2})$, defined by the formula

$$(T_K f, g)_{L^2(\mathbf{R}^{d_2})} = (K, g \otimes \overline{f})_{L^2(\mathbf{R}^{d_2+d_1})}. \tag{1.26}$$

It is well-known that the Schwartz kernel theorem also holds in the context of Gelfand-Shilov spaces (see e. g. [7, 36, 44, 51]).

In fact, let $\mathcal{L}(V_1, V_2)$ be the set of linear continuous mappings from the topological vector space V_1 to the topological vector space V_2. Moreover, if V_j are quasi-Banach spaces, then $\| \cdot \|_{\mathcal{L}(V_1,V_2)}$ denotes the quasi-norm in $\mathcal{L}(V_1, V_2)$. We also set $\mathcal{L}(V) = \mathcal{L}(V, V)$.

If $A \in \mathbf{M}(d, \mathbf{R})$, then the mappings $K \mapsto T_K$ and $a \mapsto \operatorname{Op}_A(a)$ are bijective from $\mathcal{S}_s'(\mathbf{R}^{2d})$ to $\mathcal{L}(\mathcal{S}_s(\mathbf{R}^d), \mathcal{S}_s'(\mathbf{R}^d))$. Similar facts hold true if $\mathcal{S}_s$ and $\mathcal{S}_s'$ are replaced by Σ_s and Σ_s', respectively (or by $\mathscr{S}$ and $\mathscr{S}'$, respectively).

As a consequence of Remark 1.22 it follows that for each $a_1 \in \mathcal{S}_s'(\mathbf{R}^{2d})$ and $A_1, A_2 \in \mathbf{M}(d, \mathbf{R})$, there is a unique $a_2 \in \mathcal{S}_s'(\mathbf{R}^{2d})$ such that $\operatorname{Op}_{A_1}(a_1) = \operatorname{Op}_{A_2}(a_2)$. The relation between a_1 and a_2 is given by

$$\operatorname{Op}_{A_1}(a_1) = \operatorname{Op}_{A_2}(a_2) \qquad \Leftrightarrow \qquad a_2(x,\xi) = e^{i\langle (A_1-A_2)D_\xi, D_x\rangle} a_1(x,\xi). \tag{1.27}$$

(Cf. [34].) Note here that the right-hand side makes sense, since it is equivalent to $\widehat{a}_2(\xi, x) = e^{i(A_1-A_2)\langle x,\xi\rangle}\widehat{a}_1(\xi, x)$, and that the map $a \mapsto e^{i\langle Ax,\xi\rangle}a$ is continuous on $\mathcal{S}_s'$ when $A \in \mathbf{M}(d, \mathbf{R})$. (Cf. [5, 6, 65].)

Let $A \in \mathbf{M}(d, \mathbf{R})$ and $a \in \mathcal{S}_s'(\mathbf{R}^{2d})$ be fixed. Then a is called a rank-one element with respect to A, if the corresponding pseudo-differential operator is of rank-one, i. e.

$$\operatorname{Op}_A(a)f = (f, f_2)f_1, \qquad f \in \mathcal{S}_s(\mathbf{R}^d), \tag{1.28}$$

for some $f_1, f_2 \in \mathcal{S}_s'(\mathbf{R}^d)$. By straightforward computations it follows that (1.28) is fulfilled, if and only if $a = (2\pi)^{\frac{d}{2}} W^A_{f_1,f_2}$, where $W^A_{f_1,f_2}$ it the A-Wigner distribution defined by the formula

$$W^A_{f_1,f_2}(x,\xi) \equiv \mathscr{F}(f_1(x + A\,\cdot)\overline{f_2(x - (I - A)\,\cdot)})(\xi), \tag{1.29}$$

which takes the form

$$W^A_{f_1,f_2}(x,\xi) = (2\pi)^{-\frac{d}{2}} \int_{\mathbf{R}^d} f_1(x + Ay)\overline{f_2(x - (I - A)y)}e^{-i\langle y,\xi\rangle}\, dy,$$

when $f_1, f_2 \in \mathcal{S}_s(\mathbf{R}^d)$. Here $I \in \mathbf{M}(d, \mathbf{R})$ is the identity matrix. By combining these facts with (1.27) it follows that

$$W^{A_2}_{f_1,f_2} = e^{i\langle (A_1-A_2)D_\xi,D_x\rangle} W^{A_1}_{f_1,f_2}, \tag{1.30}$$

for each $f_1, f_2 \in \mathcal{S}_s'(\mathbf{R}^d)$ and $A_1, A_2 \in \mathbf{M}(d, \mathbf{R})$. Since the Weyl case is particularly important, we set $W^A_{f_1,f_2} = W_{f_1,f_2}$ when $A = \frac{1}{2}I$, i. e. W_{f_1,f_2} is the usual (cross-)Wigner distribution of f_1 and f_2.

For future references we note the link

$$(\operatorname{Op}_A(a)f, g)_{L^2(\mathbf{R}^d)} = (2\pi)^{-\frac{d}{2}}(a, W^A_{g,f})_{L^2(\mathbf{R}^{2d})}, \qquad a \in \mathcal{S}_s'(\mathbf{R}^{2d}) \quad \text{and} \quad f, g \in \mathcal{S}_s(\mathbf{R}^d) \tag{1.31}$$

between pseudo-differential operators and Wigner distributions, which follows by straightforward computations (see also e. g. [61]).

Next we discuss the Weyl product, the twisted convolution and related objects. Let $s \geq 1/2$ and let $a, b \in \mathcal{S}_s'(\mathbf{R}^{2d})$. Then the Weyl product $a\#b$ between a and b is the function or distribution which fulfills $\operatorname{Op}^w(a\#b) = \operatorname{Op}^w(a) \circ \operatorname{Op}^w(b)$, provided the right-hand side makes sense as a continuous operator from $\mathcal{S}_s(\mathbf{R}^d)$ to $\mathcal{S}_s'(\mathbf{R}^d)$. More generally, if $A \in \mathbf{M}(d, \mathbf{R})$, then the product $\#_A$ is defined by the formula

$$\operatorname{Op}_A(a\#_A b) = \operatorname{Op}_A(a) \circ \operatorname{Op}_A(b), \tag{1.32}$$

provided the right-hand side makes sense as a continuous operator from $\mathcal{S}_s(\mathbf{R}^d)$ to $\mathcal{S}_s'(\mathbf{R}^d)$, in which case a and b are called *suitable* or *admissible*.

The Weyl product can also, in a convenient way, be expressed in terms of the twisted convolution and the symplectic Fourier transform (cf. (1.17)). Let $s \ge 1/2$ and $a, b \in \mathcal{S}_s(\mathbf{R}^{2d})$. Then the *twisted convolution* of a and b is defined by the formula

$$(a *_\sigma b)(X) = \left(\frac{2}{\pi}\right)^{\frac{d}{2}} \int_{\mathbf{R}^{2d}} a(X-Z)b(Z)e^{2i\sigma(X,Z)}\, dZ. \tag{1.33}$$

The definition of $*_\sigma$ extends in different ways. For example, it extends to a continuous multiplication on $L^p(\mathbf{R}^{2d})$ when $p \in [1,2]$, and to a continuous map from $\mathcal{S}_s'(\mathbf{R}^{2d}) \times \mathcal{S}_s(\mathbf{R}^{2d})$ to $\mathcal{S}_s'(\mathbf{R}^{2d})$. If $a, b \in \mathcal{S}_s'(\mathbf{R}^{2d})$, then $a\# b$ makes sense if and only if $a *_\sigma (\mathscr{F}_\sigma b)$ makes sense, and then

$$a\# b = (2\pi)^{-\frac{d}{2}} a *_\sigma (\mathscr{F}_\sigma b). \tag{1.34}$$

We also remark that for the twisted convolution we have

$$\mathscr{F}_\sigma(a *_\sigma b) = (\mathscr{F}_\sigma a) *_\sigma b = \check{a} *_\sigma (\mathscr{F}_\sigma b), \tag{1.35}$$

where $\check{a}(X) = a(-X)$ (cf. [52, 57, 59]). A combination of (1.34) and (1.35) gives

$$\mathscr{F}_\sigma(a\# b) = (2\pi)^{-\frac{d}{2}} (\mathscr{F}_\sigma a) *_\sigma (\mathscr{F}_\sigma b). \tag{1.36}$$

Definition 1.23. Let

$$\|a\|_{s_\infty^w} \equiv \|\operatorname{Op}^w(a)\|_{\mathcal{L}(L^2(\mathbf{R}^d))}, \quad a \in \mathscr{S}'(\mathbf{R}^{2d}).$$

The set $s_\infty^w(\mathbf{R}^{2d})$ consists of all $a \in \mathscr{S}'(\mathbf{R}^{2d})$ such that $\operatorname{Op}^w(a)$ is linear and continuous on $L^2(\mathbf{R}^d)$, or equivalently, the set of all $a \in \mathscr{S}'(\mathbf{R}^{2d})$ such that $\|a\|_{s_\infty^w}$ is finite.

Remark 1.24. By the last part of Remark 1.22 it follows that the map $a \mapsto \operatorname{Op}^w(a)$ is an isometric bijection from $s_\infty^w(\mathbf{R}^{2d})$ to the set of linear continuous operators on $L^2(\mathbf{R}^d)$.

Remark 1.25. We remark that the relations in this subsection hold true after $\mathcal{S}_s$, $\mathcal{S}_s'$ and $s \ge \frac{1}{2}$ are replaced by Σ_s, Σ_s' and $s > \frac{1}{2}$ respectively, in each place.

Next we recall some algebraic properties and characterisations of $\Gamma_s^{(\omega)}(\mathbf{R}^{2d})$ and $\Gamma_{0,s}^{(\omega)}(\mathbf{R}^{2d})$ from the introduction, and begin with the following. We refer to [5] for its proof.

Proposition 1.26. *Let $s \ge 1$, $\omega_j \in \mathscr{P}_{E,s}^0(\mathbf{R}^{2d})$, $A_j \in \mathbf{M}(d, \mathbf{R})$ for $j = 0, 1, 2$, and let $\omega_{0,r}(X,Y) = \omega_0(X)e^{-r|Y|^{\frac{1}{s}}}$ when $r > 0$. Then the following is true:*

1. *If $a_1, a_2 \in \Sigma_s'(\mathbf{R}^{2d})$ satisfy $\operatorname{Op}_{A_1}(a_1) = \operatorname{Op}_{A_2}(a_2)$, then $a_1 \in \Gamma_s^{(\omega_0)}(\mathbf{R}^{2d})$ if and only if $a_2 \in \Gamma_s^{(\omega_0)}(\mathbf{R}^{2d})$;*
2. $\Gamma_s^{(\omega_1)} \# \Gamma_s^{(\omega_2)} \subseteq \Gamma_s^{(\omega_1\omega_2)}$;
3. $\Gamma_s^{(\omega_0)} = \bigcup_{r>0} M^{\infty,1}_{(1/\omega_{0,r})} = \bigcup_{r\ge 0} \mathcal{M}^{\infty,1}_{(1/\omega_{0,r})}$.

Proposition 1.27. *Let $s \ge 1$, $\omega_j \in \mathscr{P}_{E,s}(\mathbf{R}^{2d})$, $A_j \in \mathbf{M}(d, \mathbf{R})$ for $j = 0, 1, 2$, and let $\omega_{0,r}(X,Y) = \omega_0(X)e^{-r|Y|^{\frac{1}{s}}}$ when $r > 0$. Then the following is true:*

1. *If $a_1, a_2 \in \Sigma_s'(\mathbf{R}^{2d})$ satisfy $\operatorname{Op}_{A_1}(a_1) = \operatorname{Op}_{A_2}(a_2)$, then $a_1 \in \Gamma_{0,s}^{(\omega_0)}(\mathbf{R}^{2d})$ if and only if $a_2 \in \Gamma_{0,s}^{(\omega_0)}(\mathbf{R}^{2d})$;*
2. $\Gamma_{0,s}^{(\omega_1)} \# \Gamma_{0,s}^{(\omega_2)} \subseteq \Gamma_{0,s}^{(\omega_1\omega_2)}$;
3. $\Gamma_{0,s}^{(\omega_0)} = \bigcap_{r>0} M^{\infty,1}_{(1/\omega_{0,r})} = \bigcap_{r\ge 0} \mathcal{M}^{\infty,1}_{(1/\omega_{0,r})}$.

In time-frequency analysis one also considers mapping properties for pseudo-differential operators between modulation spaces or with symbols in modulation spaces. Especially we need the following two results, where the first one is a generalisation of [49, Theorem 2.1] by Tachizawa, see also [43, Theorem 2], and the second one is a weighted version of [25, Theorem 14.5.2]. We refer to [62] for the proof of the first two propositions and to [60, 61] for the proof of the third one.

Proposition 1.28. *Assume that* $A \in \mathbf{M}(d,\mathbf{R})$*,* $s \ge 1$*,* $\omega,\omega_0 \in \mathscr{P}^0_{E,s}(\mathbf{R}^{2d})$*,* $a \in \Gamma^{(\omega)}_s(\mathbf{R}^{2d})$*, and that* $\mathscr{B}$ *is an invariant BF-space on* $\mathbf{R}^{2d}$ *of Beurling type. Then* $\operatorname{Op}_A(a)$ *is continuous from* $M(\omega_0\omega,\mathscr{B})$ *to* $M(\omega_0,\mathscr{B})$*, and also continuous on* $\mathcal{S}_s(\mathbf{R}^d)$ *and on* $\mathcal{S}_s'(\mathbf{R}^d)$*.*

Proposition 1.29. *Assume that* $A \in \mathbf{M}(d,\mathbf{R})$*,* $s \ge 1$*,* $\omega,\omega_0 \in \mathscr{P}_{E,s}(\mathbf{R}^{2d})$*,* $a \in \Gamma^{(\omega)}_{0,s}(\mathbf{R}^{2d})$*, and that* $\mathscr{B}$ *is an invariant BF-space on* $\mathbf{R}^{2d}$ *of Roumieu type. Then* $\operatorname{Op}_A(a)$ *is continuous from* $M(\omega_0\omega,\mathscr{B})$ *to* $M(\omega_0,\mathscr{B})$*, and also continuous on* $\Sigma_s(\mathbf{R}^d)$ *and on* $\Sigma_s'(\mathbf{R}^d)$*.*

Proposition 1.30. *Assume that* $p,q \in (0,\infty]$*,* $r \le \min(p,q,1)$*,* $\omega \in \mathscr{P}_E(\mathbf{R}^{2d} \oplus \mathbf{R}^{2d})$ *and* $\omega_1,\omega_2 \in \mathscr{P}_E(\mathbf{R}^{2d})$ *satisfy*

$$\frac{\omega_2(X-Y)}{\omega_1(X+Y)} \le C\omega(X,Y), \qquad X,Y \in \mathbf{R}^{2d}, \tag{1.37}$$

for some constant C*. If* $a \in \mathcal{M}^{\infty,r}_{(\omega)}(\mathbf{R}^{2d})$*, then* $\operatorname{Op}^w(a)$ *extends uniquely to a continuous map from* $M^{p,q}_{(\omega_1)}(\mathbf{R}^d)$ *to* $M^{p,q}_{(\omega_2)}(\mathbf{R}^d)$*.*

Finally we need the following result concerning mapping properties of modulation spaces under the Weyl product. The result is a special case of [8, Theorem 2.1] (see also [13, Theorem 0.3]).

Proposition 1.31. *Assume that* $\omega_j \in \mathscr{P}_E(\mathbf{R}^{2d} \oplus \mathbf{R}^{2d})$ *for* $j = 0,1,2$ *satisfy*

$$\omega_0(X,Y) \le C\omega_1(X-Y+Z,Z)\omega_2(X+Z,Y-Z), \tag{1.38}$$

for some constant $C > 0$ *independent of* $X,Y,Z \in \mathbf{R}^{2d}$*, and let* $r \in (0,1]$*. Then the map* $(a,b) \mapsto a\# b$ *from* $\Sigma_1(\mathbf{R}^{2d}) \times \Sigma_1(\mathbf{R}^{2d})$ *to* $\Sigma_1(\mathbf{R}^{2d})$ *extends uniquely to a continuous mapping from* $\mathcal{M}^{\infty,r}_{(\omega_1)}(\mathbf{R}^{2d}) \times \mathcal{M}^{\infty,r}_{(\omega_2)}(\mathbf{R}^{2d})$ *to* $\mathcal{M}^{\infty,r}_{(\omega_0)}(\mathbf{R}^{2d})$*.*

We remark that the conditions (1.37) and (1.38) need to be reformulated in awkward or inconvenient ways, if the symplectic modulation spaces are replaced by ordinary modulation spaces in Propositions 1.30 and 1.31. Similar facts hold true for several results in Sections 4 and 5.

1.7. **The Wiener Algebra Property.** As a further crucial tool in our study of the isomorphism property of Toeplitz operators we need to combine these continuity results with convenient invertibility properties. The so-called Wiener algebra property of certain symbol classes asserts that the inversion of a pseudo-differential operator preserves the symbol class and is often referred to as the spectral invariance of a symbol class.

Proposition 1.32. *Let* $A \in \mathbf{M}(d,\mathbf{R})$*. Then the following is true:*

1. *If* $s > 1$*,* $a \in \Gamma^{(1)}_{0,s}(\mathbf{R}^{2d})$ *and* $\operatorname{Op}_A(a)$ *is invertible on* $L^2(\mathbf{R}^d)$*, then* $\operatorname{Op}_A(a)^{-1} = \operatorname{Op}_A(b)$ *for some* $b \in \Gamma^{(1)}_{0,s}(\mathbf{R}^{2d})$*.*
2. *If* $s \ge 1$*,* $a \in \Gamma^{(1)}_s(\mathbf{R}^{2d})$ *and* $\operatorname{Op}_A(a)$ *is invertible on* $L^2(\mathbf{R}^d)$*, then* $\operatorname{Op}_A(a)^{-1} = \operatorname{Op}_A(b)$ *for some* $b \in \Gamma^{(1)}_s(\mathbf{R}^{2d})$*.*

(3) *If* $s \ge 1$, $v_0 \in \mathscr{P}_{E,s}^0(\mathbf{R}^{2d})$ *is submultiplicative,* $v(X,Y) \equiv v_0(Y)$, $X,Y \in \mathbf{R}^{2d}$, $a \in M_{(v)}^{\infty,1}(\mathbf{R}^{2d})$ *and* $\operatorname{Op}_A(a)$ *is invertible on* $L^2(\mathbf{R}^d)$*, then* $\operatorname{Op}_A(a)^{-1} = \operatorname{Op}_A(b)$*, for some* $b \in M_{(v)}^{\infty,1}(\mathbf{R}^{2d})$.

Proof. The results follows essentially from [26, Corollary 5.5] or [27]. Suppose $s > 1$, $a \in \Gamma_s^{(1)}(\mathbf{R}^{2d})$, $\operatorname{Op}_A(a)$ is invertible on $L^2(\mathbf{R}^d)$, and let $v_r(X,Y) = e^{r|Y|^{\frac{1}{s}}}$ when $r \ge 0$. Then $a \in M_{(v_r)}^{\infty,1}(\mathbf{R}^{2d})$ for some $r > 0$. By [26, Corollary 5.5], $\operatorname{Op}(M_{(v_r)}^{\infty,1}(\mathbf{R}^{2d}))$ is a Wiener algebra, giving that $\operatorname{Op}(a)^{-1} = \operatorname{Op}(b)$ for some $b \in M_{(v_r)}^{\infty,1}(\mathbf{R}^{2d}) \subseteq \Gamma_s^{(1)}(\mathbf{R}^{2d})$. This gives (2) in the case $s > 1$.

If instead $s = 1$, then by [20, Theorem 4.4] there is an $r_0 > 0$ such that $\operatorname{Op}(a)^{-1} = \operatorname{Op}(b)$ for some $b \in M_{(v_{r_0})}^{\infty,1}(\mathbf{R}^{2d}) \subseteq \Gamma_1^{(1)}(\mathbf{R}^{2d})$, and (2) follows for general $s \ge 1$.

By similar arguments, (1) and (3) follow. The details are left for the reader. □

Remark 1.33. Let $A \in \mathbf{M}(d,\mathbf{R})$. Then it follows from Proposition 1.32 (3) that if $s > 1$, $v_0 \in \mathscr{P}_{E,s}(\mathbf{R}^{2d})$ is submultiplicative, $v(X,Y) \equiv v_0(Y)$, $X,Y \in \mathbf{R}^{2d}$, $a \in M_{(v)}^{\infty,1}(\mathbf{R}^{2d})$ and $\operatorname{Op}_A(a)$ is invertible on $L^2(\mathbf{R}^d)$, then $\operatorname{Op}_A(a)^{-1} = \operatorname{Op}_A(b)$, for some $b \in M_{(v)}^{\infty,1}(\mathbf{R}^{2d})$.

1.8. **Toeplitz Operators.** Let $a \in \Sigma_1(\mathbf{R}^{2d})$ and $\phi \in \Sigma_1(\mathbf{R}^d)$. Then the Toeplitz operator $\operatorname{Tp}_\phi(a)$ (with symbol a and window ϕ) is defined by the formula

$$(\operatorname{Tp}_\phi(a)f_1, f_2)_{L^2(\mathbf{R}^d)} = (a\, V_\phi f_1, V_\phi f_2)_{L^2(\mathbf{R}^{2d})}\,, \tag{1.39}$$

when $f_1, f_2 \in \Sigma_1(\mathbf{R}^d)$. Obviously, $\operatorname{Tp}_\phi(a)$ is well-defined and extends uniquely to a continuous operator from $\Sigma_1'(\mathbf{R}^d)$ to $\Sigma_1(\mathbf{R}^d)$.

The definition of Toeplitz operators can be extended to more general classes of windows and symbols by using appropriate estimates for the short-time Fourier transforms in (1.39).

We state two possible ways of extending (1.39). The first result follows from [11, Corollary 4.2] and its proof, and the second result is a special case of [63, Theorem 3.1]. We also set

$$\omega_{0,t}(X,Y) = v_1(2Y)^{t-1}\omega_0(X) \qquad \text{for } X,Y \in \mathbf{R}^{2d}\,. \tag{1.40}$$

Proposition 1.34. *Let* $0 \le t \le 1$, $p,q \in [1,\infty]$, *and* $\omega, \omega_0, v_1, v_0 \in \mathscr{P}_E(\mathbf{R}^{2d})$ *be such that* v_0 *and* v_1 *are submultiplicative,* ω_0 *is* v_0*-moderate and* ω *is* v_1*-moderate. Set*

$$v = v_1^t v_0 \quad \text{and} \quad \vartheta = \omega_0^{1/2}\,,$$

and let $\omega_{0,t}$ *be as in* (1.40)*. Then the following is true:*

(1) *The definition of* $(a,\phi) \mapsto \operatorname{Tp}_\phi(a)$ *from* $\Sigma_1(\mathbf{R}^{2d}) \times \Sigma_1(\mathbf{R}^d)$ *to* $\mathcal{L}(\Sigma_1(\mathbf{R}^d), \Sigma_1'(\mathbf{R}^d))$ *extends uniquely to a continuous map from* $\mathcal{M}_{(1/\omega_{0,t})}^{\infty}(\mathbf{R}^{2d}) \times M_{(v)}^1(\mathbf{R}^d)$ *to* $\mathcal{L}(\mathscr{S}(\mathbf{R}^d), \mathscr{S}'(\mathbf{R}^d))$.

(2) *If* $\phi \in M_{(v)}^1(\mathbf{R}^d)$ *and* $a \in \mathcal{M}_{(1/\omega_{0,t})}^{\infty}(\mathbf{R}^{2d})$*, then* $\operatorname{Tp}_\phi(a)$ *extends uniquely to a continuous map from* $M_{(\vartheta\omega)}^{p,q}(\mathbf{R}^d)$ *to* $M_{(\omega/\vartheta)}^{p,q}(\mathbf{R}^d)$.

Proposition 1.35. *Let* $\omega, \omega_1, \omega_2, v \in \mathscr{P}_E(\mathbf{R}^{2d})$ *be such that* ω_1 *is* v*-moderate,* ω_2 *is* v*-moderate and* $\omega = \omega_1/\omega_2$*. Then the following is true:*

(1) *The mapping* $(a,\phi) \mapsto \operatorname{Tp}_\phi(a)$ *extends uniquely to a continuous map from* $L_{(\omega)}^{\infty}(\mathbf{R}^{2d}) \times M_{(v)}^2(\mathbf{R}^d)$ *to* $\mathcal{L}(\Sigma_1(\mathbf{R}^d), \Sigma_1'(\mathbf{R}^d))$.

(2) *If* $\phi \in M_{(v)}^2(\mathbf{R}^d)$ *and* $a \in L_{(1/\omega)}^{\infty}(\mathbf{R}^{2d})$*, then* $\operatorname{Tp}_\phi(a)$ *extends uniquely to a continuous operator from* $M_{(\omega_1)}^2(\mathbf{R}^d)$ *to* $M_{(\omega_2)}^2(\mathbf{R}^d)$.

The symbol of a Toeplitz operator with respect to the A representation (1.23) and (1.24) is the convolution between the Toeplitz symbol and an A-Wigner distribution. More precisely, if $a \in \Sigma_1(\mathbf{R}^{2d})$ and $\phi \in \Sigma_1(\mathbf{R}^d)$, then

$$\operatorname{Tp}_\phi(a) = (2\pi)^{-\frac{d}{2}} \operatorname{Op}_A(a * W^A_{\phi,\phi}), \qquad A \in \mathbf{M}(d,\mathbf{R}). \tag{1.41}$$

The formula (1.41) has appeared frequently in the literature in the Weyl case (cf. e. g. [11,22,30,52,61] and the references therein). For general A the formula follows by a straight-forward combination of the Weyl case, (1.27), (1.30) and

$$e^{i\langle AD_\xi,D_x\rangle}(a*b) = a*(e^{i\langle AD_\xi,D_x\rangle}*b)$$

for suitable a and b, which follows from Fourier's inversion formula.

Our analysis of Toeplitz operators is based on the pseudo-differential operator representation given by (1.41), and remark that similar interpretations might be difficult or impossible to make in the framework of (1.39). (See the end of Section 2 and Remark 4.8 in [30].)

2. Confinement of the symbol classes $\Gamma_s^{(\omega)}(\mathbf{R}^d)$ and $\Gamma_{0,s}^{(\omega)}(\mathbf{R}^d)$

In this section we introduce and discuss basic properties for confinements for symbols in $\Gamma_s^{(\omega_0)}$ and in $\Gamma_{0,s}^{(\omega_0)}$. These considerations are related to the discussions in [3, 38], but are here adapted to symbols that possess Gevrey regularity. In particular, this requires the deduction of various types of delicate estimates on compositions of symbols that are confined in certain ways.

2.1. **Estimates of translated and localised Weyl products.** In what follows we let $a_Y = a(\,\cdot\, - Y)$ when $a \in \mathcal{S}_{1/2}'(\mathbf{R}^{2d})$ and $Y \in \mathbf{R}^{2d}$, and in analogous ways, b_Y, ϕ_Y, φ_Y, ψ_Y etc. are defined when $b,\phi,\varphi,\psi \in \mathcal{S}_{1/2}'(\mathbf{R}^{2d})$. For admissible a and b we have

$$(a\#b)_Y = a_Y\#b_Y, \tag{2.1}$$

which follows by straight-forward computations. We also recall that if $\varphi \in \mathcal{S}_s(\mathbf{R}^{2d})$, then there are functions $\phi,\psi \in \mathcal{S}_s(\mathbf{R}^{2d})$ such that $\varphi = \phi\#\psi$. The same is true if $\mathcal{S}_s$ is replaced by Σ_s or by $\mathscr{S}$ (cf. [7, 64]). In particular, by choosing φ such that $\int_{\mathbf{R}^{2d}} \varphi(X)\,dX = 1$, (2.1) gives the following.

Proposition 2.1. *Let $s \ge \frac{1}{2}$. Then there are $\phi,\psi \in \mathcal{S}_s(\mathbf{R}^{2d})$ such that*

$$\int_{\mathbf{R}^{2d}} \psi_Y\#\phi_Y\,dY = 1. \tag{2.2}$$

For independent translations in Weyl products we have the following.

Proposition 2.2. *Let $s \ge \frac{1}{2}$ and let $\phi,\psi \in \mathcal{S}_s(\mathbf{R}^{2d})$. Then*

$$(\phi_Y\#\psi_Z)(X) = \Psi(X-Y,X-Z) \tag{2.3}$$

for some $\Psi \in \mathcal{S}_s(\mathbf{R}^{2d}\times\mathbf{R}^{2d})$. The same holds true with Σ_s or $\mathscr{S}$ in place of $\mathcal{S}_s$.

Proof. We only prove the result when $\phi,\psi \in \mathcal{S}_s(\mathbf{R}^{2d})$. The other cases follow by similar arguments and are left for the reader.

We have

$$\begin{aligned}(\phi_Y\#\psi_Z)(X) &= \pi^{-d}\int_{\mathbf{R}^{2d}} \phi(X-Y-Y_1)\widehat{\psi}(Y_1)e^{2i\sigma(Y_1,Z)}e^{2i\sigma(X,Y_1)}\,dY_1\\ &= \pi^{-d}\int_{\mathbf{R}^{2d}} \phi((X-Y)-Y_1)\widehat{\psi}(Y_1)e^{2i\sigma(X-Z,Y_1)}\,dY_1 = \Psi(X-Y,X-Z),\end{aligned}$$

where

$$\Psi(X,Z) = \pi^{-d}\int_{\mathbf{R}^{2d}} \phi(X-Y_1)\widehat{\psi}(Y_1)e^{2i\sigma(Z,Y_1)}\,dY_1.$$

We note that
$$\Psi = (\mathscr{F}_{\sigma,2} \circ T)(\phi \otimes \widehat{\psi}),$$
where $(T\Phi)(X,Z) = \Phi(X-Z,Z)$ when $\Phi \in \mathcal{S}_s(\mathbf{R}^{2d} \times \mathbf{R}^{2d})$, and $\mathscr{F}_{\sigma,2}\Phi$ is the partial symplectic Fourier transform of $\Phi(X,Z)$ with respect to the Z variable. Since $(\phi,\psi) \mapsto \phi\otimes\widehat{\psi}$ is continuous from $\mathcal{S}_s(\mathbf{R}^{2d})\times\mathcal{S}_s(\mathbf{R}^{2d})$ to $\mathcal{S}_s(\mathbf{R}^{2d}\times\mathbf{R}^{2d})$, and T and $\mathscr{F}_{\sigma,2}\Phi$ are continuous on $\mathcal{S}_s(\mathbf{R}^{2d}\times\mathbf{R}^{2d})$, it follows that $\Psi \in \mathcal{S}_s(\mathbf{R}^{2d}\times\mathbf{R}^{2d})$. □

Since Ψ in Proposition 2.2 belongs to similar types of spaces as ϕ and ψ, it follows that estimates of the form
$$|D^\alpha \Psi(X,Y)| \lesssim h^{|\alpha|}\alpha!^s e^{-(|X|^{\frac{1}{s}}+|Y|^{\frac{1}{s}})/h}$$
hold true. In particular, the following is an immediate consequence of Proposition 2.2 and some standard manipulations in Gelfand-Shilov theory.

Corollary 2.3. *Let $s \ge \frac{1}{2}$. If $\phi,\psi \in \mathcal{S}_s(\mathbf{R}^{2d})$ ($\phi,\psi \in \Sigma_s(\mathbf{R}^{2d})$), then*
$$|D_X^\alpha D_Y^\beta D_Z^\gamma(\phi_Y \# \psi_Z)(X)| \lesssim h^{|\alpha+\beta+\gamma|}(\alpha!\beta!\gamma!)^s e^{-(|X-Y|^{\frac{1}{s}}+|X-Z|^{\frac{1}{s}})/h} \tag{2.4}$$
for some $h>0$ (for every $h>0$).

Proof. By Proposition 2.2, (2.3) holds for some $\Psi \in \mathcal{S}_s(\mathbf{R}^{2d}\times\mathbf{R}^{2d})$. Thus
$$\begin{aligned}
|D_X^\alpha D_Y^\beta D_Z^\gamma \Psi(X-Y,X-Z)| &= \left| D_X^\alpha \left(D_1^\beta D_2^\gamma \Psi\right)(X-Y,X-Z)\right| \\
&\le \sum_{\delta\le\alpha}\binom{\alpha}{\delta}\left|\left(D_1^{\beta+\delta}D_2^{\gamma+\alpha-\delta}\Psi\right)(X-Y,X-Z)\right| \\
&\le h^{|\alpha+\beta+\gamma|}\sum_{\delta\le\alpha}\binom{\alpha}{\delta}\left((\beta+\delta)!(\gamma+\alpha-\delta)!\right)^s e^{-r\left(|X-Y|^{\frac{1}{s}}+|X-Z|^{\frac{1}{s}}\right)}.
\end{aligned}$$
We have
$$\sum_{\delta\le\alpha}\binom{\alpha}{\delta}\left((\beta+\delta)!(\gamma+\alpha-\delta)!\right)^s \le 2^{|\alpha|}4^{s|\alpha+\beta+\gamma|}\left(\alpha!\beta!\gamma!\right)^s.$$
Indeed, by $(n+k)! \le 2^{n+k}n!k!$ we get
$$(\alpha+\beta+\gamma)! = \prod_{j=1}^d(\alpha_j+\beta_j+\gamma_j)! \le \prod_{j=1}^d 4^{\alpha_j+\beta_j+\gamma_j}\alpha_j!\beta_j!\gamma_j! = 4^{|\alpha+\beta+\gamma|}\alpha!\beta!\gamma!.$$
Thus (2.4) holds with $2\cdot 4^s h$ in place of h. □

The next result is a consequence of Theorem 4.12 in [5].

Proposition 2.4. *Let $s \ge \frac{1}{2}$ and $\vartheta \in \mathscr{P}_E(\mathbf{R}^{2d})$. Then the map $(\phi,a)\mapsto \phi\# a$ is continuous from $\Sigma_s(\mathbf{R}^{2d})\times\Gamma_s^{(\vartheta)}(\mathbf{R}^{2d})$ to $\mathcal{S}_s(\mathbf{R}^{2d})$.*

The next lemma concerns uniform estimates of the Weyl product between elements in sets
$$\{\, a_j(\,\cdot\,+Y,Y)\,;\, Y\in\mathbf{R}^{2d}\,\},\qquad j=1,2 \tag{2.5}$$
which are bounded in $\mathcal{S}_s(\mathbf{R}^{2d})$ or in $\Sigma_s(\mathbf{R}^{2d})$.

Lemma 2.5. *Let $s\ge\frac{1}{2}$. Then the following is true:*

(1) *if the sets in (2.5) are bounded in $\mathcal{S}_s(\mathbf{R}^{2d})$, then there are constants $C>0$ and $h>0$ which are independent of $Y_1,Y_2\in\mathbf{R}^{2d}$ and $\alpha,\alpha_1,\alpha_2\in\mathbf{N}^{2d}$ such that*
$$\begin{aligned}
&|((D_1^{\alpha_1}a_1)(\,\cdot\,,Y_1)\#(D_1^{\alpha_2}a_2)(\,\cdot\,,Y_2))(X)| \\
&\qquad\qquad \le Ch^{|\alpha_1+\alpha_2|}(\alpha_1!\alpha_2!)^s e^{-\frac{1}{h}\cdot(|X-Y_1|^{\frac{1}{s}}+|X-Y_2|^{\frac{1}{s}}+|Y_1-Y_2|^{\frac{1}{s}})}
\end{aligned}\tag{2.6}$$

and

$$|D_1^\alpha (a_1(\,\cdot\,,Y_1)\# a_2(\,\cdot\,,Y_2))(X)| \le Ch^{|\alpha|}\alpha!^s e^{-\frac{1}{h}\cdot(|X-Y_1|^{\frac{1}{s}}+|X-Y_2|^{\frac{1}{s}}+|Y_1-Y_2|^{\frac{1}{s}})} \tag{2.7}$$

hold;

(2) *if the sets in* (2.5) *are bounded in* $\Sigma_s(\mathbf{R}^{2d})$*, then for every* $h>0$*, there is a constant* $C>0$ *which is independent of* $Y_1,Y_2\in\mathbf{R}^{2d}$ *and* $\alpha,\alpha_1,\alpha_2\in\mathbf{N}^{2d}$ *such that* (2.6) *and* (2.7) *hold.*

Proof. We only prove (2). The assertion (1) follows by similar arguments and is left for the reader.

Let $Y=Y_1$, $Z=Y_2$, $a(X,Y)=a_1(X+Y,Y)$ and $b(X,Z)=a_2(X+Z,Z)$. Then

$$\begin{aligned}(a_1(\,\cdot\,,Y)\# a_2(\,\cdot\,,Z))(X) &= \pi^{-d}\int_{\mathbf{R}^{2d}} a((X-Y)-Y_1,Y)\mathscr{F}_\sigma(b(\,\cdot\,-Z,Z))(Y_1)e^{2i\sigma(X,Y_1)}\,dY_1\\ &= \pi^{-d}\int_{\mathbf{R}^{2d}} a((X-Y)-Y_1,Y)\mathscr{F}_\sigma(b(\,\cdot\,,Z))(Y_1)e^{2i\sigma(X-Z,Y_1)}\,dY_1\\ &= \Phi_{Y,Z}(X-Y,X-Z),\end{aligned}$$

where

$$\Phi_{Y,Z}(X_1,X_2)=\pi^{-d}\int_{\mathbf{R}^{2d}} a(X_1-Y_1,Y)\mathscr{F}_\sigma(b(\,\cdot\,,Z))(Y_1)e^{2i\sigma(X_2,Y_1)}\,dY_1.$$

We observe that

$$D_{X_1}^{\alpha_1}D_{X_2}^{\alpha_2}\Phi_{Y,Z}(X_1,X_2) = \pi^{-d}\int_{\mathbf{R}^{2d}} (D_1^{\alpha_1}a)(X_1-Y_1,Y)\mathscr{F}_\sigma((D_1^{\alpha_2}b)(\,\cdot\,,Z))(Y_1)e^{2i\sigma(X_2,Y_1)}\,dY_1. \tag{2.8}$$

which implies that the Leibnitz rule

$$\begin{aligned}D_1^\alpha(a_1(\,\cdot\,,Y)\# a_2(\,\cdot\,,Z))(X) &= \sum_{\gamma\le\alpha}\binom{\alpha}{\gamma}(D_1^{\alpha-\gamma}D_2^\gamma\Phi_{Y,Z})(X-Y,X-Z)\\ &= \sum_{\gamma\le\alpha}\binom{\alpha}{\gamma}\pi^{-d}\int_{\mathbf{R}^{2d}}(D_1^{\alpha-\gamma}a)(X_1-Y_1,Y)\mathscr{F}_\sigma((D_1^\gamma b)(\,\cdot\,,Z))(Y_1)e^{2i\sigma(X_2,Y_1)}\,dY_1\end{aligned} \tag{2.9}$$

holds. We also have

$$\Phi_{Y,Z}=(T_1\circ T_2\circ T_1)(a(\,\cdot\,,Y)\otimes b(\,\cdot\,,Z)),$$

where

$$(T_1F)(X_1,X_2)=\mathscr{F}_\sigma(F(X_1,\,\cdot\,))(X_2)\quad\text{and}\quad (T_2F)(X_1,X_2)=F(X_1-X_2,X_2),$$

for admissible F, and observe that both T_1 and T_2 are continuous mappings on $\Sigma_s(\mathbf{R}^{2d}\times\mathbf{R}^{2d})$.

By the continuity of T_1 and T_2 on Σ_s, and the boundedness of the sets in (2.5), it follows that

$$\sup_{Y,Z\in\mathbf{R}^{2d}}|D_{X_1}^{\alpha_1}D_{X_2}^{\alpha_2}\Phi_{Y,Z}(X_1,X_2)|\lesssim h^{|\alpha_1+\alpha_2|}(\alpha_1!\alpha_2!)^s e^{-\frac{1}{h}\cdot(|X_1|^{\frac{1}{s}}+|X_2|^{\frac{1}{s}})},$$

which is the same as

$$|(D_1^{\alpha_1} a_1(\,\cdot\,,Y))\#(D_1^{\alpha_2} a_2)(\,\cdot\,,Z))(X)| \lesssim h^{|\alpha_1+\alpha_2|}(\alpha_1!\alpha_2!)^s e^{-\frac{1}{h}\cdot(|X-Y|^{\frac{1}{s}}+|X-Z|^{\frac{1}{s}})}$$

for every $h>0$, where the involved constants are independent of $Y,Z\in \mathbf{R}^{2d}$. A combination of the latter estimate and the fact that

$$|X-Y|^{\frac{1}{s}}+|X-Z|^{\frac{1}{s}} \asymp |X-Y|^{\frac{1}{s}}+|X-Z|^{\frac{1}{s}}+|Y-Z|^{\frac{1}{s}},\quad X,Y,Z\in \mathbf{R}^{2d},\tag{2.10}$$

shows that (2.6) holds for every $h>0$.

By (2.6), (2.8), (2.10) and the inequality $(\alpha+\beta)!\le 2^{|\alpha+\beta|}\alpha!\beta!$ we get

$$
\begin{aligned}
&|D_1^{\alpha}(a_1(\,\cdot\,,Y)\#a_2(\,\cdot\,,Z))(X)|\\
&\qquad \le \sum_{\gamma\le\alpha}\binom{\alpha}{\gamma}|(D_1^{\alpha-\gamma}D_2^{\gamma}\Phi_{Y,Z})(X-Y,X-Z)|\\
&\qquad \lesssim h^{|\alpha|}\sum_{\gamma\le\alpha}\binom{\alpha}{\gamma}((\alpha-\gamma)!\gamma!)^s e^{-\frac{1}{h}\cdot(|X-Y|^{\frac{1}{s}}+|X-Z|^{\frac{1}{s}})+|Y-Z|^{\frac{1}{s}})}\\
&\qquad \le (2^s h)^{|\alpha|}\left(\sum_{\gamma\le\alpha}\binom{\alpha}{\gamma}\right)e^{-\frac{1}{h}\cdot(|X-Y|^{\frac{1}{s}}+|X-Z|^{\frac{1}{s}})+|Y-Z|^{\frac{1}{s}})}\\
&\qquad = (2^{s+1}h)^{|\alpha|}e^{-\frac{1}{h}\cdot(|X-Y|^{\frac{1}{s}}+|X-Z|^{\frac{1}{s}})+|Y-Z|^{\frac{1}{s}})}
\end{aligned}
$$

for every $h>0$, and the result follows. □

Remark 2.6. Let Ω_1 and Ω_2 be (countable or uncountable) index sets. By similar arguments as in the previous proof, it follows that the conclusions of Lemma 2.5 also holds when considering more general bounded subsets

$$\{\, a_{\theta,j}(\,\cdot\,+Y,Y)\,;\, Y\in\mathbf{R}^{2d},\ \theta\in\Omega_j\,\},\qquad j=1,2$$

of $\mathcal{S}_s(\mathbf{R}^{2d})$ respective $\Sigma_s(\mathbf{R}^{2d})$.

Lemma 2.7. *Let $s\ge\frac{1}{2}$, $\phi,\psi\in\Sigma_s(\mathbf{R}^{2d})$, $\omega,\vartheta\in\mathscr{P}_E(\mathbf{R}^{2d})$, $\phi_Y=\phi(\,\cdot\,-Y)$, and $\psi_Z=\psi(\,\cdot\,-Z)$. Then the following is true:*

1. *if $a\in\Gamma_s^{(\omega)}(\mathbf{R}^{2d})$ ($a\in\Gamma_{0,s}^{(\omega)}(\mathbf{R}^{2d})$), then*

$$|D_X^{\alpha}D_Y^{\beta}(\phi_Y a)(X)| \lesssim h_1^{|\alpha|}h_2^{|\beta|}(\alpha!\beta!)^s e^{-|X-Y|^{\frac{1}{s}}/h_1}\min(\omega(X),\omega(Y))\tag{2.11}$$

and

$$|D_X^{\alpha}D_Y^{\beta}(\phi_Y \# a)(X)| \lesssim h_1^{|\alpha|}h_2^{|\beta|}(\alpha!\beta!)^s e^{-|X-Y|^{\frac{1}{s}}/h_1}\min(\omega(X),\omega(Y)),\tag{2.12}$$

for some $h_1>0$ (for every $h_1>0$) and every $h_2>0$;

2. *if $a_1\in\Gamma_s^{(\omega)}(\mathbf{R}^{2d})$ and $a_2\in\Gamma_s^{(\vartheta)}(\mathbf{R}^{2d})$ ($a_1\in\Gamma_{0,s}^{(\omega)}(\mathbf{R}^{2d})$ and $a_2\in\Gamma_{0,s}^{(\vartheta)}(\mathbf{R}^{2d})$), then*

$$
\begin{aligned}
&|D_X^{\alpha}D_Y^{\beta}D_Z^{\gamma}((\phi_Y a_1)\#(\psi_Z a_2))(X)|\\
&\lesssim h_1^{|\alpha+\beta|}h_2^{|\gamma|}(\alpha!\beta!\gamma!)^s e^{-(|X-Y|^{\frac{1}{s}}+|X-Z|^{\frac{1}{s}}+|Y-Z|^{\frac{1}{s}})/h_1}\min_{X_1,X_2\in\{X,Y,Z\}}\big(\omega(X_1)\vartheta(X_2)\big),
\end{aligned}
$$

for some $h_1>0$ (for every $h_1>0$) and every $h_2>0$.

Proof. We only consider the case when $a_1\in\Gamma_{0,s}^{(\omega)}(\mathbf{R}^{2d})$ and $a_2\in\Gamma_{0,s}^{(\vartheta)}(\mathbf{R}^{2d})$. The other cases follow by similar arguments and are left for the reader.

Let

$$\Psi(X,Y)=\phi(X-Y)a(X).$$

By Leibniz rule we get

$$
\begin{aligned}
|D_X^\alpha D_Y^\beta \Psi(X,Y)| &\le \sum_{\gamma\le\alpha} \binom{\alpha}{\gamma} |\phi^{(\alpha+\beta-\gamma)}(X-Y)a^{(\gamma)}(X)|\\
&\lesssim 2^{|\alpha|} \sup_{\gamma\le\alpha} \left(h^{|\alpha+\beta|}((\alpha+\beta-\gamma)!\gamma!)^s e^{-|X-Y|^{\frac 1s}/h}\omega(X)\right)
\end{aligned}
$$

$$
\le (2^{1+s}h)^{|\alpha+\beta|}(\alpha!\beta!)^s e^{-|X-Y|^{\frac 1s}/h}\omega(X) \lesssim (2^{1+s}h)^{|\alpha+\beta|}(\alpha!\beta!)^s e^{-|X-Y|^{\frac 1s}/(2h)}\omega(Y),
$$

for every $h>0$ which is chosen small enough. Here we have used the fact that for some $r>0$

$$
\omega(X)\lesssim \omega(Y)e^{r|X-Y|}\lesssim \omega(Y)e^{|X-Y|^{\frac 1s}/(2h)},
$$

since ω is a moderate function. This gives (2.11).

Next we prove (2). Let

$$
b_{1,\beta,h}(\,\cdot\,,Y) = \frac{D_Y^\beta(\phi_Y a_1)}{h^{|\beta|}\beta!^s\omega(Y)} \quad \text{and} \quad b_{2,\gamma,h}(\,\cdot\,,Z) = \frac{D_Z^\gamma(\psi_Z a_2)}{h^{|\gamma|}\gamma!^s\vartheta(Z)}
$$

Then (1) and Remark 2.6 show that

$$
\{\, b_{1,\beta,h}(\,\cdot\,+Y,Y)\,;\, Y\in \mathbf R^{2d},\ h>0,\ \beta\in \mathbf N^{2d}\,\}
$$

and

$$
\{\, b_{2,\gamma,h}(\,\cdot\,+Z,Z)\,;\, Z\in \mathbf R^{2d},\ h>0,\ \gamma\in \mathbf N^{2d}\,\}
$$

are bounded subsets of $\Sigma_s(\mathbf R^{2d})$.

Hence, Remark 2.6 shows that

$$
|D_X^\alpha(b_{1,\beta,h}(\,\cdot\,,Y)\# b_{2,\gamma,h}(\,\cdot\,,Z))(X)| \lesssim h^{|\alpha|}\alpha!^s e^{-(|X-Y|^{\frac 1s}+|X-Z|^{\frac 1s}+|Y-Z|^{\frac 1s})/h}
$$

for every $h>0$, or equivalently,

$$
\begin{aligned}
&|D_X^\alpha D_Y^\beta D_Z^\gamma((\phi_Y a)\#(\psi_Z b))(X)|\\
&\qquad\lesssim h^{|\alpha+\beta+\gamma|}(\alpha!\beta!\gamma!)^s e^{-(|X-Y|^{\frac 1s}+|X-Z|^{\frac 1s}+|Y-Z|^{\frac 1s})/h}\omega(Y)\vartheta(Z).
\end{aligned}
$$

The assertion now follows from the latter estimate and the fact that ω and ϑ are moderate weights, giving that

$$
\omega(Y)\lesssim \omega(X)e^{|X-Y|^{\frac 1s}/(2h)}\lesssim \omega(Z)e^{(|X-Y|^{\frac 1s}+|X-Z|^{\frac 1s})/(2h)},
$$

and similarly for ϑ. □

Lemmas 2.5 and 2.7 imply the following characterisation of $\Gamma_s^{(\omega)}(\mathbf R^{2d})$.

Proposition 2.8. *Suppose $s>1/2$, $\phi\in\Sigma_s(\mathbf R^{2d})$ have non-vanishing integral, $\omega\in\mathscr P_E(\mathbf R^{2d})$, $a\in\Sigma_1'(\mathbf R^{2d})$, and let $\phi_Y=\phi(\,\cdot\,-Y)$. Then the following conditions are equivalent:*

1. *$a\in\Gamma_s^{(\omega)}$ ($a\in\Gamma_{0,s}^{(\omega)}$);*
2. *$\phi_Y a$ is smooth and satisfies (2.11) for some $h_1>0$ (for every $h_1>0$) and every $h_2>0$;*
3. *$\phi_Y\# a$ is smooth and satisfies (2.12) for some $h_1>0$ (for every $h_1>0$) and every $h_2>0$;*

(4)

$$|D_X^\alpha(\phi_Y a)(X)| \lesssim h_1^{|\alpha|}\alpha!^s e^{-|X-Y|^{\frac{1}{s}}/h_1}\min(\omega(X),\omega(Y)) \tag{2.13}$$

for some $h_1>0$ (for every $h_1>0$);

(5)

$$|D_X^\alpha(\phi_Y \# a)(X)| \lesssim h_1^{|\alpha|}\alpha!^s e^{-|X-Y|^{\frac{1}{s}}/h_1}\min(\omega(X),\omega(Y)) \tag{2.14}$$

for some $h_1>0$ (for every $h_1>0$).

Proof. By Lemmas 2.5 and 2.7, (1) implies that (2) and (3) hold, which in turn imply (4) and (5).

If (4) holds, then (0.4) follows by integrating (2.13) with respect to Y, and using the fact that $\int_{\mathbf{R}^{2d}}\phi_Y\,dY$ is a non-zero constant, since ϕ has non-vanishing integral. In the same way it follows that (5) leads to (0.4). Consequently, (4) as well as (5) imply (1), and the result follows. □

2.2. **A family related to $\Gamma_s^{(1)}$ and $\Gamma_{0,s}^{(1)}$.** Let $I_R=[-R,R]$ and $E^0=E^0_{h,s}=L^\infty(I_R\times\mathbf{R}^{2d};s_\infty^w(\mathbf{R}^{2d}))$, with the symbol subspace $s_\infty^w(\mathbf{R}^{2d})$ from Definition 1.23. We shall consider suitable decreasing family $\{E^n_{h,s}\}_{n=0}^\infty$ of Banach spaces. To this aim, let

$$G_n=\{(Y,T_1,\dots,T_n)\in\mathbf{R}^{2d(n+1)}\colon Y,T_j\in\mathbf{R}^{2d}\text{ with }|T_j|\le 1,\ j=1,\dots,n\},\quad n\in\mathbf{N}.$$

We define $E^n_{h,s}=E^n_{R,h,s}$, $n\ge 1$, as the set of all $a\in E^0$ such that

$$\|a\|^{(n)}=\sup_{1\le k\le n}\sup_{t\in I_R}\sup_{(Y,T_1,\dots,T_k)\in G_k}\frac{\|\langle T_1,D_X\rangle\cdots\langle T_k,D_X\rangle a(t,Y,\,\cdot\,)\|_{s_\infty^w}}{h^k(k!)^s}<\infty,$$

with the norm

$$\|a\|_{E^n_{h,s}}=\|a\|_{E^n_{R,h,s}}\equiv\max(\|a\|_{E^0},\|a\|^{(n)}).$$

We also let $E^\infty_{h,s}=E^\infty_{R,h,s}$ be the set of all

$$a\in\bigcap_{n\ge 0}E^n_{R,h,s} \tag{2.15}$$

such that

$$\|a\|_{E^\infty_{R,h,s}}\equiv\sup_{n\ge 0}\|a\|_{E^n_{R,h,s}}$$

is finite.

Lemma 2.9. *Let $n\ge 0$, $R>0$ and $s>0$. Then $E^n_{h,s}$ and $E^\infty_{h,s}$ are Banach spaces.*

Proof. Let $\{a_j\}_{j\ge 0}$ be a Cauchy sequence in $E^n_{h,s}$, $n\ge 1$. By definition, this sequence clearly has a limit $a\in E^0$, and for some $X\mapsto b_k(t,Y,T_1,\dots,T_k,X)\in s_\infty^w(\mathbf{R}^{2d})$ we have

$$\lim_{j\to\infty}\sup\frac{\|\langle T_1,D_X\rangle\cdots\langle T_k,D_X\rangle a_j(t,Y,\,\cdot\,)-b_k(t,Y,T_1,\dots,T_k,\,\cdot\,)\|_{s_\infty^w}}{h^k(k!)^s}=0,$$

where the supremum is taken over all

$$k\in\{1,\dots,n\},\quad t\in I_R\quad\text{and}\quad (Y,T_1,\dots,T_k)\in G_k.$$

We need to prove that $a\in E^n_{h,s}$, and $a_j\to a$ in $E^n_{h,s}$.

The conditions here above are equivalent to

$$\lim_{j\to\infty}\left(\sup_{t\in I_R}\sup_{Y\in\mathbf{R}^{2d}}\|a_j(t,Y,\,\cdot\,)-a(t,Y,\,\cdot\,)\|_{s_\infty^w}\right)=0 \tag{2.16}$$

and

$$\lim_{j\to\infty} \sup \frac{\|(-1)^k \langle T_1, D\rangle \cdots \langle T_k, D\rangle a_j(t,Y,\,\cdot\,) - b_k(t,Y,T_1,\dots,T_k,\,\cdot\,)\|_{s^w_\infty}}{h^k(k!)^s} = 0, \tag{2.17}$$

where the latter supremum should be taken over all

$$k \in \{1,\dots,n\}, \quad t \in I_R \quad \text{and} \quad (Y,T_1,\dots,T_k) \in G_k.$$

Since $s^w_\infty(\mathbf{R}^{2d})$ is continuously embedded in $\mathscr{S}'(\mathbf{R}^{2d})$, it follows from (2.16) and (2.17) that

$$X \mapsto (-1)^k \langle T_1, D_X\rangle \cdots \langle T_k, D_X\rangle a_j(t,Y,X)$$

has the limit

$$X \mapsto (-1)^k \langle T_1, D_X\rangle \cdots \langle T_k, D_X\rangle a(t,Y,X)$$

in $\mathscr{S}'(\mathbf{R}^{2d})$, and the limit

$$X \mapsto b_k(t,Y,T_1,\dots,T_k,X)$$

in $s^w_\infty(\mathbf{R}^{2d})$, and thereby in $\mathscr{S}'(\mathbf{R}^{2d})$, as j tends to ∞. Hence

$$b_k(t,Y,T_1,\dots,T_k,X) = (-1)^k \langle T_1, D_X\rangle \cdots \langle T_k, D_X\rangle a(t,Y,X)$$

and it follows that $E^n_{h,s}$ is a Banach space for every $h>0$, $s>0$ and integer $n\ge 0$.

If in addition $\{a_j\}_{j\ge 0}$ is a Cauchy sequence in $E^\infty_{h,s}$, then the limit a above satisfy (2.15). Since a_j stays bounded in $E^\infty_{h,s}$, it follows that a has bounded $E^\infty_{h,s}$ norm, and therefore, $E^\infty_{h,s}$ is complete and thereby a Banach space. □

The spaces $E^\infty_{h,s}$ can be related to $\Gamma^{(1)}_s$ and $\Gamma^{(1)}_{0,s}$, as the following lemma shows. The details are left for the reader.

Lemma 2.10. *Let $a \in L^\infty(I_R \times \mathbf{R}^{2d}; s^w_\infty(\mathbf{R}^{2d}))$. Then $\{a(t,Y,\,\cdot\,)\}_{t\in I_R, Y\in\mathbf{R}^{2d}}$ is a uniformly bounded family in $\Gamma^{(1)}_s(\mathbf{R}^{2d})$ ($\Gamma^{(1)}_{0,s}(\mathbf{R}^{2d})$), if and only if*

$$\|a\|_{E^\infty_{h,s}} < \infty$$

for some $h>0$ (for every $h>0$).

Later on we also need the following result of differential equations with functions depending on a real variable with values in $E^\infty_{h,s}$. The proof is omitted since the result can be considered as a part of the standard theory of ordinary differential equations of first order in Banach spaces.

Lemma 2.11. *Suppose $s\ge 0$, $n\ge 0$ be an integer, $T>0$, and let $\mathcal{K}$ be an operator from $E^n_{h,s}$ to $E^n_{h,s}$ for every $h>0$ such that*

$$\|\mathcal{K}a\|_{E^n_{h,s}} \le C\|a\|_{E^n_{h,s}}, \qquad a \in E^n_{h,s}, \tag{2.18}$$

for some constant C which only depend on $h>0$. Then

$$\frac{dc(t)}{dt} = \mathcal{K}(c(t)), \qquad c(0) \in E^n_{h,s},$$

has a unique solution $t\mapsto c(t)$ from $[-T,T]$ to $E^n_{h,s}$ which satisfies

$$\|c(t)\|_{E^n_{h,s}} \le \|c(0)\|_{E^n_{h,s}} e^{CT},$$

where C is the same as in (2.18). The same holds true with $E^\infty_{h,s}$ in place of $E^n_{h,s}$ at each occurrence.

3. One-parameter group of elliptic symbols in the classes $\Gamma_s^{(\omega)}(\mathbf{R}^d)$ and $\Gamma_{0,s}^{(\omega)}(\mathbf{R}^d)$

In this section we show that for suitable s and ω_0, there are elements $a \in \Gamma_s^{(\omega_0)}$ and $b \in \Gamma_s^{(1/\omega_0)}$ such that $a\#b = b\#a = 1$. This is essentially a consequence of Theorem 3.8, where it is proved that the evolution equation (0.6) has a unique solution $a(t,\,\cdot\,)$ which belongs to $\Gamma_s^{(\omega\vartheta^t)}$, thereby deducing needed semigroup properties for scales of pseudo-differential operators. Similar facts hold for corresponding Beurling type spaces (cf. Theorem 3.9).

First we have the following result on certain logarithms of weight functions.

Proposition 3.1. *Let $\omega \in \mathscr{P}_E(\mathbf{R}^{2d}) \cap \Gamma_{s_0}^{(\omega)}(\mathbf{R}^{2d})$, $s_0 \in (0,1]$, $v \in \mathscr{P}_E(\mathbf{R}^{2d})$, be such that ω is v-moderate, $\vartheta(X) = 1 + \log v(X)$ and let*

$$c(X,Y) = \log \frac{\omega(X+Y)}{\omega(Y)}.$$

Then,

1. *$\{c(\,\cdot\,,Y)\}_{Y\in\mathbf{R}^{2d}}$ is a uniformly bounded family in $\Gamma_s^{(\vartheta)}(\mathbf{R}^{2d})$, $s \ge 1$;*
2. *for $\alpha \neq 0$, $\{(\partial_X^\alpha c)(\,\cdot\,,Y)\}_{Y\in\mathbf{R}^{2d}}$ is a uniformly bounded family in $\Gamma_s^{(1)}(\mathbf{R}^{2d})$, $s \ge 1$.*

For the proof of Proposition 3.1 we need the following multidimensional version of the well-known Faà di Bruno formula for the derivatives of composed functions. It can be found, e.g., setting $q = p = 1$, $n = 2d$, in equations (2.3) and (2.4) in [33].

Lemma 3.2. *Let $f \in C^\infty(\mathbf{R})$ and $g \in C^\infty(\mathbf{R}^d;\mathbf{R})$. Then*

$$\frac{\partial^\alpha f(g(x))}{\alpha!} = \sum_{1\le k\le |\alpha|} \frac{f^{(k)}(g(x))}{k!} \sum_{\substack{\beta_1+\cdots+\beta_k=\alpha \\ \beta_j\neq 0,\, j=1,\dots,k}} \prod_{1\le j\le k} \frac{(\partial^{\beta_j} g)(x)}{\beta_j!}, \qquad \alpha \in \mathbf{N}^d \setminus 0. \tag{3.1}$$

We will also need the next *factorial estimate*, for expressions involving decompositions of $\alpha \in \mathbf{N}^{2d}$, $\alpha \neq 0$, into the sum of k nontrivial multi-indeces β_j, $j = 1,\dots,k$, as in (3.1), and corresponding products of (powers of) factorials.

Lemma 3.3. *Let $s_0 \in (0,1]$, $\alpha \in \mathbf{N}^{2d}$, $\alpha \neq 0$. Then, for suitable $C_0 > 0$, depending only in d,*

$$\sum_{1\le k\le |\alpha|} \frac{1}{k} \sum_{\substack{\beta_1+\cdots+\beta_k=\alpha \\ \beta_j\neq 0,\, j=1,\dots,k}} \prod_{1\le j\le k} \beta_j!^{s_0-1} \lesssim C_0^{|\alpha|}. \tag{3.2}$$

Lemma 3.3 follows from Lemma A.2 in Appendix A.

Proof of Proposition 3.1. In order to prove (1) we need to show that $c(\,\cdot\,,Y)$ satisfies $\Gamma_s^{(\vartheta)}$ estimates, uniformly with respect to $Y \in \mathbf{R}^{2d}$. By (1.2) we get

$$c(X,Y) \le \log(Cv(X)) \lesssim 1 + \log v(X) = \vartheta(X)$$

and

$$c(X,Y) \ge \log((Cv(X))^{-1}) \gtrsim -(1 + \log v(X)) = -\vartheta(X).$$

Hence, $|c(X,Y)| \lesssim \vartheta(X)$, $X \in \mathbf{R}^{2d}$. If $c(X,Y) \ge 0$, then it follows by submultiplicativity of ω, that

$$\begin{aligned} c(X,Y) &= \log\omega(Y+X) - \log\omega(Y) \lesssim \log\omega(Y) + \log v(X) - \log\omega(Y) \\ &\le \vartheta(X), \end{aligned}$$

for any $Y \in \mathbf{R}^{2d}$. Again by moderateness, when $c(X,Y) \le 0$, recall that $\omega(X+Y) \ge \frac{\omega(Y)}{v(X)}$, so that

$$c(X,Y) \gtrsim \log \frac{\omega(Y)}{v(X)} - \log \omega(Y) \ge -\log v(X) \ge -\vartheta(X),$$

and we can conclude $|c(X,Y)| \lesssim \vartheta(X)$, $X \in \mathbf{R}^{2d}$. Now, for $\alpha \in \mathbf{N}^{2d}$, $\alpha \neq 0$, (0.4) with $a = \omega$ and (3.1) give

$$\partial_X^\alpha c(X,Y) = \alpha! \sum_{1 \le k \le |\alpha|} \frac{(-1)^{k+1}}{k\,[\omega(X+Y)]^k} \sum_{\substack{\beta_1+\cdots+\beta_k=\alpha \\ \beta_j \neq 0,\, j=1,\dots,k}} \prod_{1 \le j \le k} \frac{(\partial^{\beta_j}\omega)(X+Y)}{\beta_j!},$$

and by (3.2),

$$\begin{aligned}|\partial_X^\alpha c(X,Y)| &\lesssim \alpha! \sum_{1 \le k \le |\alpha|} \frac{1}{k\,[\omega(X+Y)]^k} \sum_{\substack{\beta_1+\cdots+\beta_k=\alpha \\ \beta_j \neq 0,\, j=1,\dots,k}} \prod_{1 \le j \le k} \frac{\omega(X+Y) h^{|\beta_j|} \beta_j!^{s_0}}{\beta_j!} \\ &= h^{|\alpha|}\alpha! \sum_{1 \le k \le |\alpha|} \frac{1}{k} \sum_{\substack{\beta_1+\cdots+\beta_k=\alpha \\ \beta_j \neq 0,\, j=1,\dots,k}} \prod_{1 \le j \le k} \beta_j!^{s_0-1} \lesssim (C_0 h)^{|\alpha|} \alpha!^s,\end{aligned}$$

which gives the result. □

Proposition 3.4. *Assume $s > \frac{1}{2}$ and $\omega(X) \lesssim e^{r|X|^{\frac{1}{s}}}$ for some $r > 0$. Let $\{a(\,\cdot\,,Y)\}_{Y \in \mathbf{R}^{2d}}$ be a uniformly bounded family in $\Sigma_s(\mathbf{R}^{2d})$ and $\{c(\,\cdot\,,Z)\}_{Z \in \mathbf{R}^{2d}}$ be a bounded family in $\Gamma_s^{(\omega)}(\mathbf{R}^{2d})$. Then,*

$$\{a(\,\cdot\,,Y)\#c(\,\cdot\,,Z)\}_{Y,Z \in \mathbf{R}^{2d}} \text{ and } \{c(\,\cdot\,,Z)\#a(\,\cdot\,,Y)\}_{Y,Z \in \mathbf{R}^{2d}}$$

are bounded families in $\mathcal{S}_s(\mathbf{R}^{2d})$.

Proof. Let $\phi \in \Sigma_s$ and $a \in \Gamma_s^{(\omega)}$. By Lemma 2.7 it follows that

$$|D_X^\alpha(\phi\#a)(X)| \le C h^{|\alpha|} \alpha!^s e^{-r|X|^{\frac{1}{s}}}, \tag{3.3}$$

for some $h, r > 0$. Then (3.3) holds if and only if $\phi\#a$ belongs to $\mathcal{S}_s$. By the proof of (3.3), the constants C, h and r can be chosen to depend continuously on $\phi \in \Sigma_s(\mathbf{R}^{2d})$ and $a \in \Gamma_s^{(\omega)}(\mathbf{R}^{2d})$. Hence if Ω_1 is bounded in $\Sigma_s(\mathbf{R}^{2d})$ and Ω_2 is bounded in $\Gamma_s^{(\omega)}(\mathbf{R}^{2d})$, then it follows that $\{\phi\#a\}_{\phi\in\Omega_1, a\in\Omega_2}$ is a bounded family in $\mathcal{S}_s(\mathbf{R}^{2d})$. □

The following result can be found e. g. in [52].

Lemma 3.5. *Let $a \in \mathscr{S}'(\mathbf{R}^{2d})$. Then*

$$\|a\|_{s_\infty^w} \le C \sum_{|\alpha| \le d+1} \|\partial^\alpha a\|_{L^\infty} \tag{3.4}$$

and

$$\|a\|_{L^\infty} \le C \sum_{|\alpha| \le 2d+1} \|\partial^\alpha a\|_{s_\infty^w} \tag{3.5}$$

for some constant $C > 0$ depending on the dimension d only.

Proposition 3.6. *Let $a \in \mathscr{S}'(\mathbf{R}^{2d})$, $s \ge \frac{1}{2}$ and set $b_{\alpha\beta}(X) = \partial^\alpha(X^\beta a(X))$ when $\alpha, \beta \in \mathbf{N}^{2d}$. Then the following conditions are equivalent:*

(1) *$a \in \mathcal{S}_s(\mathbf{R}^{2d})$ ($a \in \Sigma_s(\mathbf{R}^{2d})$);*

(2) *for some $h>0$ (every $h>0$) it holds*

$$\|b_{\alpha\beta}\|_{L^\infty} \lesssim h^{|\alpha+\beta|}(\alpha!\beta!)^s, \qquad \alpha,\beta\in \mathbf{N}^{2d};$$

(3) *for some $h>0$ (every $h>0$) it holds*

$$\|b_{\alpha\beta}\|_{s^w_\infty} \lesssim h^{|\alpha+\beta|}(\alpha!\beta!)^s, \qquad \alpha,\beta\in \mathbf{N}^{2d}.$$

Proof. We only prove the result in the Roumieu case. The Beurling case follows by similar arguments and is left for the reader.

The equivalence between (1) and (2) follows from the definitions. The proof of the equivalence of (2) and (3) follows by a straightforward application of Lemma 3.5. In fact, assume that (2) holds true. Then (3.4) gives

$$\begin{aligned}\|b_{\alpha\beta}\|_{s^w_\infty} &\le C\sum_{|\gamma|\le d+1}\|\partial^\gamma b_{\alpha\beta}\|_{L^\infty} \lesssim \sum_{|\gamma|\le d+1} h^{|\alpha+\beta+\gamma|}((\alpha+\gamma)!\beta!)^s\\ &= h^{|\alpha+\beta|}(\alpha!\beta!)^s\sum_{|\gamma|\le d+1} h^{|\gamma|}\gamma!^s\left(\frac{(\alpha+\gamma)!}{\alpha!\,\gamma!}\right)^s \lesssim (2^s h)^{|\alpha+\beta|}(\alpha!\beta!)^s.\end{aligned}$$

In the last inequality we have used

$$\sum_{|\gamma|\le d+1} h^{|\gamma|}\gamma!^s\left(\frac{(\alpha+\gamma)!}{\alpha!\,\gamma!}\right)^s \le C_1\cdot 2^{s(|\alpha|+d+1)} \le C_2 2^{s|\alpha+\beta|},$$

where the constants C_1 and C_2 only depend on d and h. Hence (3) holds true, as claimed. The proof of the converse follows by similar argument, employing (3.5) instead of (3.4). □

We also need the following characterisation of $\Gamma^{(1)}_s(\mathbf{R}^{2d})$.

Proposition 3.7. *Let $a\in\mathscr{S}'(\mathbf{R}^{2d})$ and $s>0$. Then the following conditions are equivalent:*

(1) $a\in\Gamma^{(1)}_s(\mathbf{R}^{2d})$*;*

(2) *there exists $h>0$ such that*

$$\|\partial^\alpha a\|_{L^\infty(\mathbf{R}^{2d})} \lesssim h^{|\alpha|}\alpha!^s, \qquad \alpha\in\mathbf{N}^{2d};$$

(3) *there exists $h>0$ such that*

$$\|\partial^\alpha a\|_{s^w_\infty} \lesssim h^{|\alpha|}\alpha!^s, \qquad \alpha\in\mathbf{N}^{2d}; \tag{3.6}$$

(4) *there exists $h>0$ such that*

$$\|\langle T_1,D_X\rangle\cdots\langle T_m,D_X\rangle a\|_{s^w_\infty} \tag{3.7}$$

for any $T_1,\dots,T_m\in\mathbf{R}^{2d}$ such that $|T_j|\le 1$, $j=1,\dots,m$, $m\ge 1$.

Proof. The equivalence between (1) and (2) is well known. The equivalence of (2) and (3) is proved by similar arguments to the one employed in the proof of Proposition 3.6, using Lemma 3.5. It remains to prove the equivalence with (4). Assume that (3) holds true, and let

$$T_k=\sum_{l=1}^d (t_{k,l}e_l+\tau_{k,l}\varepsilon_l),$$

for the standard symplectic basis (1.15) of $\mathbf{R}^{2d}$. If we set $e_{d+l}=\varepsilon_l$, $t_{k,d+l}=\tau_{k,l}$, $l\in\{1,\dots,d\}$, and letting X_l being the coordinates for $X=(x,\xi)\in\mathbf{R}^{2d}$ with respect to this basis, then

$$\langle T_k,D_X\rangle a=\sum_{l=1}^{2d} t_{k,l}\frac{\partial a}{\partial X_l},$$

so that the symbol $\langle T_1, D_X\rangle \cdots \langle T_m, D_X\rangle a$ is in the span of symbols of the form

$$\left(\prod_{k=1}^{m} t_{k,l_k}\right)\left(\partial_{X_{1,l_1}}\cdots\partial_{X_{m,l_m}} a\right)$$

where the summation contains at most $(2d)^m$ terms. Since $|T_j|\le 1$, $j=1,\dots,m$, (3.4) gives

$$\begin{aligned}
\|\langle T_1, D_X\rangle \cdots \langle T_m, D_X\rangle a\|_{s^w_\infty} &\le (2d)^m \sup_{|\alpha|=m} \|\partial^\alpha a\|_{s^w_\infty}\\
&\lesssim \sup_{|\alpha|=m} \sum_{|\gamma|\le d+1} h^{|\alpha+\gamma|}(\alpha+\gamma)!^s\\
&= \sup_{|\alpha|=m} h^{|\alpha|}\alpha!^s \sum_{|\gamma|\le d+1} h^{|\gamma|}\gamma!^s\left(\frac{(\alpha+\gamma)!}{\alpha!\,\gamma!}\right)^s\\
&\lesssim (2^{s+1}h)^m m!^s,
\end{aligned}$$

which gives (4).

If instead (4) holds, then choosing $T_1,\dots T_{|\alpha|}$ in suitable ways, the left-hand sides of (3.6) and (3.7) agree. The assertion (3) now follows from (4) by using the inequality $|\alpha|!\le d^{|\alpha|}\alpha!$. □

The first main result of this section is the following analogy of [3, Theorem 6.4] and [38, Theorem 2.6.15] in the framework of Gevrey regularity. It deals with the existence of one-parameter groups of pseudo-differential operators, obtained as solutions to suitable evolution equations.

Theorem 3.8. *Let $s\ge 1$, $\omega,\vartheta\in \mathscr{P}^0_{E,s}(\mathbf{R}^{2d})$ be such that $\omega\in\Gamma^{(\omega)}_s(\mathbf{R}^{2d})$ and $\vartheta\in\Gamma^{(\vartheta)}_s(\mathbf{R}^{2d})$, and let $a_0\in\Gamma^{(\omega)}_s(\mathbf{R}^{2d})$, $b\in\Gamma^{(1)}_s(\mathbf{R}^{2d})$. Then, there exists a unique smooth map $(t,X)\mapsto a(t,X)\in\mathbf{C}$ such that $a(t,\,\cdot\,)\in\Gamma^{(\omega\,\vartheta^t)}_s(\mathbf{R}^{2d})$ for all $t\in\mathbf{R}$, and*

$$\begin{cases}(\partial_t a)(t,\,\cdot\,) = (b+\log\vartheta)\# a(t,\,\cdot\,)\\ a(0,\,\cdot\,)=a_0.\end{cases}\tag{3.8}$$

If in addition $\omega\equiv a_0\equiv 1$, then $a(t,X)$ also satisfies

$$\begin{cases}(\partial_t a)(t,\,\cdot\,) = a(t,\,\cdot\,)\#(b+\log\vartheta)\\ a(0,\,\cdot\,)=a_0,\end{cases}\tag{3.9}$$

and

$$a(t_1,\,\cdot\,)\# a(t_2,\,\cdot\,) = a(t_1+t_2,\,\cdot\,),\quad a(t,\,\cdot\,)\in\Gamma^{(\vartheta^t)}_s(\mathbf{R}^{2d}),\quad t,t_1,t_2\in\mathbf{R}.\tag{3.10}$$

Proof. First suppose that a solution $a(t,X)$ of (3.8) exists. Then

$$a(t,X) = a_0(X) + \int_0^t c(u,X)\,du$$

with

$$c(t,\,\cdot\,) = (b+\log\vartheta)\# a(t,\,\cdot\,)\in\Gamma^{(\omega\vartheta^t\langle\log\vartheta\rangle)}_s(\mathbf{R}^{2d}),$$

in view of Propositions 1.27 and 3.1. This implies that the map $t\mapsto a(t,\,\cdot\,)$ is C^1 from $[-R,R]$ into the symbol space

$$\Gamma^{(\omega(\vartheta+\vartheta^{-1})^R\langle\log\vartheta\rangle)}_s(\mathbf{R}^{2d}).$$

Choose $s_0<s$, and $\phi,\psi\in\mathcal{S}_{s_0}(\mathbf{R}^{2d})$ such that (2.2) holds true. Let

$$c_1(t,Y,\,\cdot\,) = \omega(Y)^{-1}\vartheta(Y)^{-t}\,\phi_Y\# a(t,\,\cdot\,).\tag{3.11}$$

By Lemma 2.7 (1) we have $t \mapsto c_1(t, Y, \cdot\,)$ is a C^1 map from $[-R, R]$ into $\mathcal{S}_s(\mathbf{R}^{2d})$, for any $Y \in \mathbf{R}^{2d}$. Moreover,

$$\partial_t c_1(t, Y, \cdot\,) = \omega(Y)^{-1}\vartheta(Y)^{-t}\, \phi_Y \# f(\,\cdot\,, Y) \# a(t, \cdot\,)$$

when

$$f(X, Y) = b(X) + \log \frac{\vartheta(X)}{\vartheta(Y)}.$$

Then,

$$(\partial_t c_1)(t, Y, \cdot\,) = \omega(Y)^{-1}\; \vartheta(Y)^{-t} \int_{\mathbf{R}^{2d}} \phi_Y \# f(\,\cdot\,, Y) \# \psi_Z \# \phi_Z \# a(t, \cdot\,)\, dZ$$

giving that

$$(\partial_t c_1)(t, Y, \cdot\,) = \int_{\mathbf{R}^{2d}} K_{Y,Z}(t, \cdot\,) \# c_1(t, Z, \cdot\,)\, dZ \tag{3.12}$$

with

$$K_{Y,Z}(t, \cdot\,) = \frac{\omega(Z)\,\vartheta(Z)^t}{\omega(Y)\,\vartheta(Y)^t} \phi_Y \# f(\,\cdot\,, Y) \# \psi_Z. \tag{3.13}$$

We also need to consider the similar situation where $f(\,\cdot\,, Y)$ is replaced by $f(\,\cdot\,, Z)$, that is

$$\partial_t c_2(t, Y, \cdot\,) = \int_{\mathbf{R}^{2d}} \widetilde{K}_{Y,Z}(t, \cdot\,) \# c_2(t, Z, \cdot\,)\, dZ, \tag{3.12$'$}$$

where

$$\widetilde{K}_{Y,Z}(t, \cdot\,) = \frac{\omega(Z)\,\vartheta(Z)^t}{\omega(Y)\,\vartheta(Y)^t} \phi_Y \# f(\,\cdot\,, Z) \# \psi_Z, \tag{3.13$'$}$$

and

$$c_2(0, Y, \cdot\,) = c_1(0, Y, \cdot\,) = \omega(Y)^{-1} \phi_Y \# a_0. \tag{3.14}$$

We consider the operators $\mathcal{K}$ and $\widetilde{\mathcal{K}}$ when acting on E^0 from Subsection 2.2, defined by

$$(\mathcal{K}a)(t, Y, X) = \int_{\mathbf{R}^{2d}} (K_{Y,Z}(t, \cdot\,) \# a(t, Z, \cdot\,))(X)\, dZ,$$

and

$$(\widetilde{\mathcal{K}}a)(t, Y, X) = \int_{\mathbf{R}^{2d}} (\widetilde{K}_{Y,Z}(t, \cdot\,) \# a(t, Z, \cdot\,))(X)\, dZ.$$

We claim that

$$\|\mathcal{K}a\|_{E^n_{h,s}} \le C(n+1)\|a\|_{E^n_{h,s}} \quad \text{and} \quad \|\widetilde{\mathcal{K}}a\|_{E^n_{h,s}} \le C(n+1)\|a\|_{E^n_{h,s}} \tag{3.15}$$

for some constant C, which is independent of h, n and s.

In order to prove (3.15), it is convenient to let $\mathcal{P}_k$ be the family of all subsets of $\{1, \dots, k\}$, $k \ge 1$. For each $P \in \mathcal{P}_k$, $a \in s^w_\infty(\mathbf{R}^{2d})$, we set

$$H(a, P) = \begin{cases} a & \text{when } P = \emptyset, \\ \langle T_{j_1}, D_X\rangle \cdots \langle T_{j_l}, D_X\rangle a & \text{when } P = \{j_1 < \cdots < j_l\},\ l \le k. \end{cases}$$

We shall estimate

$$\frac{\|(\langle T_1, D_X\rangle \cdots \langle T_k, D_X\rangle \mathcal{K}a)(t, Y, \cdot\,)\|_{s^w_\infty(\mathbf{R}^{2d})}}{h^k (k!)^s}$$

when $a \in E^n_{h,s}$. Since

$$
\begin{aligned}
(\langle T_1, D_X\rangle \cdots \langle T_k, D_X\rangle \mathcal{K}a)(t,Y,X)
&= \langle T_1, D_X\rangle \cdots \langle T_k, D_X\rangle \int_{\mathbf{R}^{2d}} (K_{Y,Z}(t,\,\cdot\,)\# a(t,Z,\,\cdot\,))(X)\, dZ \\
&= \sum_{P\in\mathcal{P}_k} \int_{\mathbf{R}^{2d}} (H(K_{Y,Z}(t,\,\cdot\,),P)\# H(a(t,Z,\,\cdot\,),P^c))(X)\, dZ,
\end{aligned}
$$

we find

$$
\begin{aligned}
&\frac{\|(\langle T_1, D_X\rangle \cdots \langle T_k, D_X\rangle \mathcal{K}a)(t,Y,\,\cdot\,)\|_{s^w_\infty}}{h^k (k!)^s} \\
&\le \sum_{l=0}^{k} \sum_{|P|=l} \binom{k}{l}^{-s} \int_{\mathbf{R}^{2d}} \frac{\|H(K_{Y,Z}(t,\,\cdot\,),P)\|_{s^w_\infty}}{h^l\, l!^s} \cdot \frac{\|H(a(t,Z,\,\cdot\,),P^c)\|_{s^w_\infty}}{h^{k-l}((k-l)!)^s}\, dZ \\
&\le \sum_{l=0}^{k} \sum_{|P|=l} \|a\|_{E^{k-l}_{h,s}} \binom{k}{l}^{-s} \int_{\mathbf{R}^{2d}} \frac{\|H(K_{Y,Z}(t,\,\cdot\,),P)\|_{s^w_\infty}}{h^l\, l!^s}\, dZ \\
&\lesssim \|a\|_{E^k_{h,s}} \sum_{l=0}^{k} \sum_{|P|=l} \binom{k}{l}^{-1} \int_{\mathbf{R}^{2d}} \frac{\|H(K_{Y,Z}(t,\,\cdot\,),P)\|_{s^w_\infty}}{h^l\, l!^s}\, dZ \\
&\le (k+1) D_k(Y) \|a\|_{E^k_{h,s}}, \qquad (3.16)
\end{aligned}
$$

where

$$
D_k(Y) = \sup_{l\le k} \sup_{|P|=l} \left(\int_{\mathbf{R}^{2d}} \frac{\|H(K_{Y,Z}(t,\,\cdot\,),P)\|_{s^w_\infty}}{h^l\, l!^s}\, dZ \right), \qquad (3.17)
$$

Here the third inequality in (3.16) follows from the fact that $s \ge 1$ and $\|a\|_{E^n_{h,s}}$ increases with n.

We have to estimate $D_k(Y)$ in (3.17) and study the different quantities on the right-hand side of (3.13). Since ω and ϑ belong to $\mathscr{P}^0_{E,s}$, it follows that for every $r > 0$,

$$
\begin{aligned}
\frac{\omega(Z)\,\vartheta(Z)^t}{\omega(Y)\,\vartheta(Y)^t} = \frac{\omega(Z)}{\omega(Y)} \left(\frac{\vartheta(Z)}{\vartheta(Y)}\right)^t &\lesssim e^{r|Y-Z|^{\frac{1}{s}}} \left(e^{r|Y-Z|^{\frac{1}{s}}}\right)^t \\
&= e^{r(1+t)|Y-Z|^{\frac{1}{s}}}, \quad Y,Z \in \mathbf{R}^{2d}. \qquad (3.18)
\end{aligned}
$$

For the Weyl product in (3.13) we have

$$
\begin{aligned}
\phi_Y \# f(Y,\,\cdot\,) &= \phi(\,\cdot\, - Y) \# \Big(b + \log \frac{\vartheta}{\vartheta(Y)}\Big) \\
&= \Big(\phi \# b(\,\cdot\, + Y)\Big)_Y + \phi(\,\cdot\, - Y)\# \Big(\log\frac{\vartheta}{\vartheta(Y)}\Big) \\
&= \Big(\phi \# b(\,\cdot\, + Y)\Big)_Y + \Big(\phi \# \log \frac{\vartheta(\,\cdot\, + Y)}{\vartheta(Y)}\Big)_Y .
\end{aligned}
$$

By Propositions 3.1 and 3.4,

$$
\Big\{\phi \# b(\,\cdot\, + Y)\Big\}_{Y\in\mathbf{R}^{2d}} \quad \text{and} \quad \Big\{\phi \# \log \frac{\vartheta(\,\cdot\, + Y)}{\vartheta(Y)}\Big\}_{Y\in\mathbf{R}^{2d}} \qquad (3.19)
$$

are uniformly bounded families in $\mathcal{S}_s(\mathbf{R}^{2d})$. Note that

$$
a_2(Z,X) = \psi_Z(X) \quad \Rightarrow \quad \{a_2(Z,\,\cdot\, + Z)\}_{Z\in\mathbf{R}^{2d}} = \{\psi\}_{Z\in\mathbf{R}^{2d}},
$$

which is evidently a uniformly bounded family in $\mathcal{S}_s(\mathbf{R}^{2d})$. Combining this last observation with the computations on $\phi_Y \# f(\,\cdot\,,Y)$ above, using Lemmata 2.5 and 2.7, we finally obtain

$$|D_X^\alpha(\phi_Y \# f(Y,\,\cdot\,)\#\psi_Z)(X)| \lesssim h^{|\alpha|}\alpha!^s e^{-r_0(|X-Y|^{\frac{1}{s}}+|X-Z|^{\frac{1}{s}}+|Y-Z|^{\frac{1}{s}})},\\ X,Y,Z\in\mathbf{R}^{2d},\alpha\in\mathbf{N}^{2d}, \tag{3.20}$$

for some $h,r_0>0$.

By Proposition 3.7, (3.18) and (3.20) we get for all $P\in\mathcal{P}_k$ $Y,Z\in\mathbf{R}^{2d}$ and some $r_0,h>0$ that

$$\|H(K_{Y,Z}(t,\,\cdot\,),P)\|_{s_\infty^w} \le Ch^l l!^s e^{-r_0|Y-Z|^{\frac{1}{s}}}, \quad l=|P|,$$

where C is independent of k. Hence D_k in (3.17) satisfies

$$D_k(Y)\le C_1\int_{\mathbf{R}^{2d}} e^{-r_0|Y-Z|^{\frac{1}{s}}}\,dZ = C_2,$$

for some constants C_1 and C_2 which are independent of $Y\in\mathbf{R}^{2d}$, $h>0$ and $k\ge 0$. Hence (3.16) gives

$$\|\mathcal{K}a(t,Y,\,\cdot\,)\|_{s_\infty^w}\le C\|a\|_{E_{h,s}^k},$$

and

$$\frac{\|\langle T_1,D_X\rangle\cdots\langle T_k,D_X\rangle\mathcal{K}a(t,Y,\,\cdot\,)\|_{s_\infty^w}}{h^k(k!)^s}\le C(k+1)\|a\|_{E_{h,s}^k},$$

as claimed, where C is independent of $Y\in\mathbf{R}^{2d}$, k and $h>0$.

By a completely similar argument, an analogous result can be obtained for $\widetilde{\mathcal{K}}$. In fact, by similar arguments that lead to (3.19) it follows that

$$\{b(\,\cdot\,+Z)\#\psi\}_{Z\in\mathbf{R}^{2d}} \quad\text{and}\quad \left\{\log\frac{\vartheta(\,\cdot\,+Z)}{\vartheta(Z)}\#\psi\right\}_{Z\in\mathbf{R}^{2d}}$$

are bounded in $\mathcal{S}_s(\mathbf{R}^{2d})$, given that (3.20) holds with $f(Z,\,\cdot\,)$ in place of $f(Y,\,\cdot\,)$. This gives (3.15).

We have proven that for any $T>0$, then

$$\|\mathcal{K}\|_{E_{h,s}^n\to E_{h,s}^n}\le C(n+1) \quad\text{and}\quad \|\widetilde{\mathcal{K}}\|_{E_{h,s}^n\to E_{h,s}^n}\le C(n+1), \qquad |t|\le T, \tag{3.21}$$

where C is independent of n. As a consequence, since $\omega(Y)^{-1}\phi_Y\#a_0$ belongs to $E_{h,s}^n$ for every n and with uniform bound of the norms with respect to n it follows that the equations

$$\frac{dc_1}{dt}=\mathcal{K}c_1,\quad \frac{dc_2}{dt}=\widetilde{\mathcal{K}}c_2 \qquad c_1(0)=c_2(0)=\omega(Y)^{-1}\phi_Y\#a_0 \tag{3.22}$$

have unique solutions on $[-T,T]$ belonging to $E_{h,s}^n$, in view of Lemma 2.11, and that

$$\|c_j\|_{E_{h,s}^n}\le\|c_j(0)\|_{E_{h,s}^n}e^{C(n+1)T}\le\|c_j(0)\|_{E_{h,s}^\infty}e^{C(n+1)T}, \qquad j=1,2, \tag{3.23}$$

where the constant C is the same as in (3.21) and is therefore independent of n.

This gives

$$\sup\left(\frac{\|\langle T_1,D_X\rangle\cdots\langle T_n,D_X\rangle c_j(t,Y,\,\cdot\,)\|_{s_\infty^w}}{h^n(n!)^s}\right)\le\|c_j(0)\|_{E_{h,s}^\infty}e^{C(n+1)T},$$

which is the same as

$$\sup\left(\frac{\|\langle T_1,D_X\rangle\cdots\langle T_n,D_X\rangle c_j(t,Y,\,\cdot\,)\|_{s_\infty^w}}{h_0^n(n!)^s}\right)\le\|c_j(0)\|_{E_{h,s}^\infty}e^{CT}, \qquad h_0=he^{CT}. \tag{3.24}$$

Here the supremum is taken over all $T_1,\dots,T_n,Y\in \mathbf{R}^{2d}$ such that $|T_j|\le 1$, and $t\in[-T,T]$. By taking the supremum of the left-hand side of (3.24) over all $n\ge 0$ we get

$$\|c_j\|_{E^\infty_{h_0,s}} \le \|c_j(0)\|_{E^\infty_{h,s}} e^{CT}, \qquad h_0 = he^{CT}.$$

By Proposition 2.8 it follows that $c_j(t,Y,\,\cdot\,)\in \Gamma^{(1)}_s(\mathbf{R}^{2d})$, uniformly in Y and for bounded t.

In order to prove the uniqueness of the solution a of (3.8), first we assume the existence and by what we have proven above i.e. that $c_1(t,Y,\,\cdot\,)$ in (3.11) satisfies (3.22) which implies the uniqueness of the solution of (3.8), since

$$a(t,\,\cdot\,) = \int_{\mathbf{R}^{2d}} \psi_Y\#\phi_Y\#a(t,\,\cdot\,)\,dY = \int_{\mathbf{R}^{2d}} \omega(Y)\vartheta(Y)^t\psi_Y\#c_1(t,Y,\,\cdot\,)\,dY. \tag{3.25}$$

To prove the existence of a solution of (3.8), we consider the solution $c_2(t,Y,\,\cdot\,)$ of (3.12)′ with the initial data (3.14), and we let

$$a(t,\,\cdot\,) = \int_{\mathbf{R}^{2d}} \omega(Y)\vartheta(Y)^t\psi_Y\#c_2(t,Y,\,\cdot\,)\,dY. \tag{3.26}$$

By Propositions 1.7 and 2.8, the family $\{\psi_Y\#c_2(t,Y,\,\cdot\,)\}_{Y\in\mathbf{R}^{2d}}$ belongs to $\mathcal{S}_s$ and $a(t,\,\cdot\,)$ belongs to $\Gamma^{(w\vartheta^t)}_s$. Moreover,

$$\begin{aligned}
\frac{da(t,\,\cdot\,)}{dt} &= \int_{\mathbf{R}^{2d}} \omega(Y)\vartheta(Y)^t\log\vartheta(Y)\psi_Y\#c_2(t,Y,\,\cdot\,)\,dY\\
&\qquad + \int_{\mathbf{R}^{2d}}\int_{\mathbf{R}^{2d}} \omega(Y)\vartheta(Y)^t\psi_Y\#\widetilde{K}_{Y,Z}(t,\,\cdot\,)\#c_2(t,Z,\,\cdot\,)\,dY\,dZ\\
&= \int_{\mathbf{R}^{2d}} \omega(Z)\vartheta(Z)^t\log\vartheta(Z)\psi_Z\#c_2(t,Z,\,\cdot\,)\,dZ\\
&\qquad + \int_{\mathbf{R}^{2d}}\int_{\mathbf{R}^{2d}} \omega(Z)\,\vartheta(Z)^t\psi_Y\#\phi_Y\#f(Z,\,\cdot\,)\#\psi_Z\#c_2(t,Z,\,\cdot\,)\,dY\,dZ\\
&= \int_{\mathbf{R}^{2d}} \omega(Z)\vartheta(Z)^t(b+\log\vartheta)\#\psi_Z\#c_2(t,Z,\,\cdot\,)\,dZ\\
&= (b+\log\vartheta)\#a(t,\,\cdot\,),
\end{aligned}$$

with the initial data

$$a(0,\,\cdot\,) = \int_{\mathbf{R}^{2d}} \omega(Y)\psi_Y\#(\omega(Y)^{-1}\phi_Y\#a_0)\,dY = a_0,$$

which provide a solution of (3.8).

In order to prove the last part we consider the unique solution $a(t,\,\cdot\,)$ of (3.8) with the initial data $a(0,\,\cdot\,)\equiv 1$. If $\omega\equiv 1$, then for $u\in\mathbf{R}$ the mappings

$$t\mapsto a(t+u,\,\cdot\,) \quad \text{and} \quad t\mapsto a(t,\,\cdot\,)\#a(u,\,\cdot\,)$$

are both solutions of (3.8) with value $a(u,\,\cdot\,)$ at $t=0$, and

$$a(t+u,\,\cdot\,) = a(t,\,\cdot\,)\#a(u,\,\cdot\,), \tag{3.27}$$

by the uniqueness property for the solution of (3.8).

Using (3.27) we have for all $t\in\mathbf{R}$, $a(t,\,\cdot\,)\#a(-t,\,\cdot\,)=1$. Taking the derivative we get

$$0 = \frac{d}{dt}(a(t,\,\cdot\,)\#a(-t,\,\cdot\,)) = (b+\log\vartheta)\#a(t,\,\cdot\,)\#a(-t,\,\cdot\,) - a(t,\,\cdot\,)\#(b+\log\vartheta)\#a(-t,\,\cdot\,).$$

That is $(b+\log\vartheta) = a(t,\,\cdot\,)\#(b+\log\vartheta)\#a(-t,\,\cdot\,)$, implying the commutation for the sharp product of $a(t,\,\cdot\,)$ with $(b+\log\vartheta)$, and the result follows. □

By similar argument as for the previous result we get the following. The verifications are left for the reader.

Theorem 3.9. *Let $s \ge 1$, $\omega, \vartheta \in \mathscr P_{E,s}(\mathbf R^{2d})$ be such that $\omega \in \Gamma_s^{(\omega)}(\mathbf R^{2d})$ and $\vartheta \in \Gamma_s^{(\vartheta)}(\mathbf R^{2d})$, and let $a_0 \in \Gamma_{0,s}^{(\omega)}(\mathbf R^{2d})$, $b \in \Gamma_{0,s}^{(1)}(\mathbf R^{2d})$. Then, there exists a unique smooth map $(t,X) \mapsto a(t,X) \in \mathbf C$ such that $a(t,\,\cdot\,) \in \Gamma_{0,s}^{(\omega\,\vartheta^t)}(\mathbf R^{2d})$ for all $t \in \mathbf R$, and $a(t,\,\cdot\,)$ satisfies* (3.8).

Moreover, if $\omega \equiv a_0 \equiv 1$, then $a(t,X)$ also satisfies (3.9) *and*

$$a(t_1,\,\cdot\,)\#a(t_2,\,\cdot\,) = a(t_1+t_2,\,\cdot\,), \quad a(t,\,\cdot\,) \in \Gamma_{0,s}^{(\vartheta^t)}(\mathbf R^{2d}), \quad t,t_1,t_2 \in \mathbf R.$$

4. Lifting of pseudo-differential operators and Toeplitz operators on modulation spaces

In this section we apply the group properties in Theorems 3.8 and 3.9 to deduce lifting properties of pseudo-differential operators on modulation spaces. Thereafter we combine these results by the Wiener property of certain pseudo-differential operators with symbols in suitable modulation spaces to get lifting properties for Toeplitz operators with weights as their symbols.

We begin to apply Theorems 3.8 and 3.9 to get the following. (See Definition 1.21 concerning normal QBF-spaces.)

Theorem 4.1. *Let $s \ge 1$, $\boldsymbol p \in (0,\infty]^{2d}$, $A \in \mathbf M(d,\mathbf R)$, $\omega \in \mathscr P_{E,s}^0(\mathbf R^{2d})$, and let $\mathscr B$ be a normal QBF-space on $\mathbf R^{2d}$. Then the following is true:*

(1) *There exist $a \in \Gamma_s^{(\omega)}(\mathbf R^{2d})$ and $b \in \Gamma_s^{(1/\omega)}(\mathbf R^{2d})$ such that*
$$\operatorname{Op}_A(a) \circ \operatorname{Op}_A(b) = \operatorname{Op}_A(b) \circ \operatorname{Op}_A(a) = \operatorname{Id}_{\mathcal S_s'(\mathbf R^d)}\,. \tag{4.1}$$
Furthermore, $\operatorname{Op}_A(a)$ is an isomorphism from $M(\omega_0,\mathscr B)$ onto $M(\omega_0/\omega,\mathscr B)$, for every $\omega_0 \in \mathscr P_{E,s}^0(\mathbf R^{2d})$.

(2) *Let $a \in \Gamma_s^{(\omega)}(\mathbf R^{2d})$. Then the following conditions are equivalent:*
  (i) *$\operatorname{Op}_A(a)$ is an isomorphism from $M^2_{(\omega_1)}(\mathbf R^d)$ to $M^2_{(\omega_1/\omega)}(\mathbf R^d)$ for some $\omega_1 \in \mathscr P_{E,s}^0(\mathbf R^{2d})$;*
  (ii) *$\operatorname{Op}_A(a)$ is an isomorphism from $M(\omega_2,\mathscr B)$ to $M(\omega_2/\omega,\mathscr B)$ for every $\omega_2 \in \mathscr P_{E,s}^0(\mathbf R^{2d})$ and normal QBF-space $\mathscr B$ on $\mathbf R^{2d}$.*
  Furthermore, if (i) *or* (ii) *hold, then the inverse of $\operatorname{Op}_A(a)$ is given by $\operatorname{Op}_A(b)$ for some $b \in \Gamma_s^{(1/\omega)}(\mathbf R^{2d})$, and* (4.1) *holds.*

Theorem 4.2. *Let $s > 1$, $\boldsymbol p \in (0,\infty]^{2d}$, $A \in \mathbf M(d,\mathbf R)$ and let $\omega \in \mathscr P_{E,s}(\mathbf R^{2d})$. Then the following is true:*

(1) *There exist $a \in \Gamma_{0,s}^{(\omega)}(\mathbf R^{2d})$ and $b \in \Gamma_{0,s}^{(1/\omega)}(\mathbf R^{2d})$ such that*
$$\operatorname{Op}_A(a) \circ \operatorname{Op}_A(b) = \operatorname{Op}_A(b) \circ \operatorname{Op}_A(a) = \operatorname{Id}_{\Sigma_s'(\mathbf R^d)}\,. \tag{4.2}$$
Furthermore, $\operatorname{Op}_A(a)$ is an isomorphism from $M(\omega_0,\mathscr B)$ onto $M(\omega_0/\omega,\mathscr B)$, for every $\omega_0 \in \mathscr P_{E,s}(\mathbf R^{2d})$ and normal QBF-space $\mathscr B$ on $\mathbf R^{2d}$;

(2) *Let $a \in \Gamma_{0,s}^{(\omega)}(\mathbf R^{2d})$. Then the following conditions are equivalent:*
  (i) *$\operatorname{Op}_A(a)$ is an isomorphism from $M^2_{(\omega_1)}(\mathbf R^d)$ to $M^2_{(\omega_1/\omega)}(\mathbf R^d)$ for some $\omega_1 \in \mathscr P_{E,s}(\mathbf R^{2d})$;*
  (ii) *$\operatorname{Op}_A(a)$ is an isomorphism from $M(\omega_2,\mathscr B)$ to $M(\omega_2/\omega,\mathscr B)$ for every $\omega_2 \in \mathscr P_{E,s}(\mathbf R^{2d})$ and normal QBF-space $\mathscr B$ on $\mathbf R^{2d}$.*
  Furthermore, if (i) *or* (ii) *hold, then the inverse of $\operatorname{Op}_A(a)$ is equal to $\operatorname{Op}_A(b)$ for some $b \in \Gamma_{0,s}^{(1/\omega)}(\mathbf R^{2d})$, and* (4.2) *holds.*

We only prove Theorem 4.2. Theorem 4.1 follows by similar arguments and is left for the reader.

Proof of Theorem 4.2. The existence of $a \in \Gamma_{0,s}^{(\omega)}(\mathbf{R}^{2d})$ and $b \in \Gamma_{0,s}^{(1/\omega)}(\mathbf{R}^{2d})$ such that (4.2) holds is guaranteed by Theorem 3.9. By [62, Theorems 2.5 and 2.8] it follows that

$$\operatorname{Op}_A(a) : M(\omega_0, \mathscr{B}) \quad \to M(\omega_0/\omega, \mathscr{B}) \tag{4.3}$$

and

$$\operatorname{Op}_A(b) : M(\omega_0/\omega, \mathscr{B}) \to M(\omega_0, \mathscr{B}) \tag{4.4}$$

are continuous. By (4.2) and the fact that $M(\omega_0, \mathscr{B})$ and $M(\omega_0/\omega, \mathscr{B})$ are contained in $\Sigma_s'(\mathbf{R}^{2d})$, it follows that (4.3) and (4.4) are homeomorphisms, and (1) follows.

It suffices to prove (2) in the Weyl case, $A = \frac{1}{2}I$, in view of Proposition 1.26. It is also clear that (ii) implies (i). We need to prove that (i) implies (ii).

By (1), we may find

$$a_1 \in \Gamma_{0,s}^{(\omega_1)}, \quad b_1 \in \Gamma_{0,s}^{(1/\omega_1)}, \quad a_2 \in \Gamma_{0,s}^{(\omega_1/\omega)}, \quad b_2 \in \Gamma_{0,s}^{(\omega/\omega_1)}$$

satisfying the following properties:

- $\operatorname{Op}^w(a_j)$ and $\operatorname{Op}^w(b_j)$ are inverses to each others on $\Sigma_s'(\mathbf{R}^d)$ for $j = 1, 2$;
- For arbitrary $\omega_2 \in \mathscr{P}_{E,s}(\mathbf{R}^{2d})$, the mappings

$$\begin{aligned} \operatorname{Op}^w(a_1) &: M^2_{(\omega_2)} \to M^2_{(\omega_2/\omega_1)}, \\ \operatorname{Op}^w(b_1) &: M^2_{(\omega_2)} \to M^2_{(\omega_2\omega_1)}, \\ \operatorname{Op}^w(a_2) &: M^2_{(\omega_2)} \to M^2_{(\omega_2\omega/\omega_1)}, \\ \operatorname{Op}^w(b_2) &: M^2_{(\omega_2)} \to M^2_{(\omega_2\omega_1/\omega)} \end{aligned} \tag{4.5}$$

  are isomorphisms.

In particular, $\operatorname{Op}^w(a_1)$ is an isomorphism from $M^2_{(\omega_1)}$ to L^2, and $\operatorname{Op}^w(b_1)$ is an isomorphism from L^2 to $M^2_{(\omega_1)}$.

Now set $c = a_2 \# a \# b_1$. By Proposition 1.27 the symbol c satisfies

$$c = a_2 \# a \# b_1 \in \Gamma_{0,s}^{(\omega_1/\omega)} \# \Gamma_{0,s}^{(\omega)} \# \Gamma_{0,s}^{(1/\omega_1)} \subseteq \Gamma_{0,s}^{(1)}.$$

Furthermore, $\operatorname{Op}^w(c)$ is a composition of three isomorphisms and consequently $\operatorname{Op}^w(c)$ is boundedly invertible on L^2.

By Proposition 1.32 (2), $\operatorname{Op}^w(c)^{-1} = \operatorname{Op}^w(c_1)$ for some $c_1 \in \Gamma_{0,s}^{(1)}$. Hence, by (1) it follows that $\operatorname{Op}^w(c)$ and $\operatorname{Op}^w(c_1)$ are isomorphisms on $M(\omega_2, \mathscr{B})$, for each $\omega_2 \in \mathscr{P}_{E,s}(\mathbf{R}^{2d})$ and normal QBF-space $\mathscr{B}$ on $\mathbf{R}^{2d}$. Since $\operatorname{Op}^w(c)$ and $\operatorname{Op}^w(c_1)$ are bounded on every $M(\omega, \mathscr{B})$, the factorization of the identity $\operatorname{Op}^w(c)\operatorname{Op}^w(c_1) = \operatorname{Id}$ is well-defined on every $M(\omega, \mathscr{B})$. Consequently, $\operatorname{Op}^w(c)$ is an isomorphism on $M(\omega, \mathscr{B})$.

Using the inverses of a_2 and b_1, we now find that

$$\operatorname{Op}^w(a) = \operatorname{Op}^w(b_2) \circ \operatorname{Op}^w(c) \circ \operatorname{Op}^w(a_1)$$

is a composition of isomorphisms from the domain space $M(\omega_2, \mathscr{B})$ onto the image space $M(\omega_2/\omega, \mathscr{B})$ (factoring through some intermediate spaces) for every $\omega_2 \in \mathscr{P}_{E,s}(\mathbf{R}^{2d})$ and every invariant BF-space $\mathscr{B}$. This proves the isomorphism assertions for $\operatorname{Op}^w(a)$.

Finally, the inverse of $\operatorname{Op}^w(a)$ is given by

$$\operatorname{Op}^w(b_1) \circ \operatorname{Op}^w(c_1) \circ \operatorname{Op}^w(a_2).$$

which is a Weyl operator with symbol in $\Gamma_{0,s}^{(1/\omega)}$ in view of Proposition 1.27, and the result follows. □

5. Mapping properties for Toeplitz operators

In this section we study the isomorphism properties of Toeplitz operators between modulation spaces as in [30]. We first state results for Toeplitz operators that are well-defined in the sense of (1.39) and Propositions 1.34 and 1.35. Then we state and prove more general results for Toeplitz operators that are defined only in the framework of pseudo-differential calculus.

5.1. **Lifting properties for Toeplitz operators with windows in $M^r_{(v)}$.** We start with the following result about Toeplitz operators with smooth symbols.

Theorem 5.1. *Let $s \ge 1$ $\omega, \omega_0, v \in \mathscr{P}^0_{E,s}(\mathbf{R}^{2d})$ be such that $\omega_0 \in \Gamma^{(\omega_0)}_s(\mathbf{R}^{2d})$ and that ω_0 is v-moderate, and let $\mathscr{B}$ be a normal QBF-space on $\mathbf{R}^{2d}$. If $\phi \in M^1_{(v)}(\mathbf{R}^d)$, then $\operatorname{Tp}_\phi(\omega_0)$ is an isomorphism from $M(\omega, \mathscr{B})$ to $M(\omega/\omega_0, \mathscr{B})$.*

In the next result we relax our restrictions on the weights but impose more restrictions on $\mathscr{B}$.

Theorem 5.2. *Let $s > 1$, $0 \le t \le 1$, $p, q \in [1, \infty]$, and $\omega, \omega_0, v_0, v_1 \in \mathscr{P}_{E,s}(\mathbf{R}^{2d})$ be such that ω_0 is v_0-moderate and ω is v_1-moderate. Set $v = v_1^t v_0$, $\vartheta = \omega_0^{1/2}$ and let $\omega_{0,t}$ be the same as in (1.40). If $\phi \in M^1_{(v)}(\mathbf{R}^d)$ and $\omega_0 \in \mathcal{M}^\infty_{(1/\omega_{0,t})}(\mathbf{R}^{2d})$, then $\operatorname{Tp}_\phi(\omega_0)$ is an isomorphism from $M^{p,q}_{(\vartheta\omega)}(\mathbf{R}^d)$ to $M^{p,q}_{(\omega/\vartheta)}(\mathbf{R}^d)$.*

Before the proofs we have the following consequence of Theorem 5.2 which is the Gevrey version of [30, Corollary 4.3], as well as the original searched result. It also generalize corresponding results in [31] (cf. e. g. [31, Theorem 4.3]).

Corollary 5.3. *Let $s \ge 1$, $\omega, \omega_0, v_1, v_0 \in \mathscr{P}_{E,s}(\mathbf{R}^{2d})$ and that ω_0 is v_0-moderate and ω is v_1-moderate. Set $v = v_1 v_0$ and $\vartheta = \omega_0^{1/2}$. If $\phi \in M^1_{(v)}(\mathbf{R}^d)$, then $\operatorname{Tp}_\phi(\omega_0)$ is an isomorphism from $M^{p,q}_{(\vartheta\omega)}(\mathbf{R}^d)$ to $M^{p,q}_{(\omega/\vartheta)}(\mathbf{R}^d)$ simultaneously for all $p, q \in [1, \infty]$.*

Proof. Let $\omega_1 \in \mathscr{P}_{E,s}(\mathbf{R}^{2d}) \cap \Gamma^{(\omega_1)}_{0,s}(\mathbf{R}^{2d})$ be such that $C^{-1} \le \omega_1/\omega_0 \le C$, for some constant C. Hence, $\omega/\omega_0 \in L^\infty \subseteq M^\infty$. By Theorem 2.2 in [56], it follows that $\omega = \omega_1 \cdot (\omega/\omega_1)$ belongs to $M^\infty_{(\omega_2)}(\mathbf{R}^{2d})$, when $\omega_2(x, \xi, \eta, y) = 1/\omega_0(x, \xi)$. The result now follows by setting $t = 1$ and $q_0 = 1$ in Theorem 5.2. $\square$

Theorems 5.1 and 5.2 are special cases of the following results.

Theorem 5.1′. *Let $s \ge 1$, $\omega, v, v_0 \in \mathscr{P}^0_{E,s}(\mathbf{R}^{2d})$ be such that $\omega_0 \in \Gamma^{(\omega_0)}_s(\mathbf{R}^{2d})$ and that ω_0 is v-moderate, and let $\mathscr{B}$ be a normal QBF-space on $\mathbf{R}^{2d}$. If $\phi \in M^2_{(v)}(\mathbf{R}^d)$, then $\operatorname{Tp}_\phi(\omega_0)$ is an isomorphism from $M(\omega, \mathscr{B})$ to $M(\omega/\omega_0, \mathscr{B})$.*

Theorem 5.2′. *Let $s > 1$, $0 \le t \le 1$, $p, q, q_0 \in [1, \infty]$ and $\omega, \omega_0, v_0, v_1 \in \mathscr{P}_{E,s}(\mathbf{R}^{2d})$ be such that ω_0 is v_0-moderate and ω is v_1-moderate. Set $r_0 = 2q_0/(2q_0 - 1)$, $v = v_1^t v_0$, $\vartheta = \omega_0^{1/2}$ and let $\omega_{0,t}$ be the same as in (1.40). If $\phi \in M^{r_0}_{(v)}(\mathbf{R}^d)$ and $\omega_0 \in \mathcal{M}^{\infty, q_0}_{(1/\omega_{0,t})}$, then $\operatorname{Tp}_\phi(\omega_0)$ is an isomorphism from $M^{p,q}_{(\vartheta\omega)}(\mathbf{R}^d)$ to $M^{p,q}_{(\omega/\vartheta)}(\mathbf{R}^d)$.*

5.2. **Lifting properties for Toeplitz operators with smooth symbols acting on normal QBF-spaces.** By imposing stronger conditions on the window function ϕ in the previous results, we may relax the restrictions on the modulation spaces as in the following generalization of Theorem 0.1.

Theorem 0.1′. *Let $s \ge 1$, $\omega, \omega_0 \in \mathscr{P}^0_{E,s}(\mathbf{R}^{2d})$, $\boldsymbol{p} \in (0, \infty]^{2d}$, $\mathscr{B}$ be a normal QBF-space on $\mathbf{R}^{2d}$, and let $\phi \in \mathcal{S}_s(\mathbf{R}^d)$. Then the Toeplitz operator $\operatorname{Tp}_\phi(\omega_0)$ is an isomorphism from $M(\omega, \mathscr{B})$ onto $M(\omega/\omega_0, \mathscr{B})$.*

5.3. **Mapping properties for Toeplitz operators.** We postpone the proofs of these theorems after performing some preparations and deducing some results of independent interests.

Lemma 5.4. *Let $s \ge 1$, $\omega, v \in \mathscr{P}_{E,s}(\mathbf{R}^{2d})$ be such that $\vartheta = \omega^{1/2}$ is v-moderate. Assume that $\phi \in M^2_{(v)}$. Then $\operatorname{Tp}_\phi(\omega)$ is an isomorphism from $M^2_{(\vartheta)}(\mathbf{R}^d)$ onto $M^2_{(1/\vartheta)}(\mathbf{R}^d)$.*

Proof. Recall from Remark 1.12 that for $\phi \in M^2_{(v)}(\mathbf{R}^d) \setminus \{0\}$ the expression $\|V_\phi f \cdot \vartheta\|_{L^2}$ defines an equivalent norm on $M^2_{(\vartheta)}$. Thus the occurring STFTs with respect to ϕ are well defined.

Since $\operatorname{Tp}_\phi(\omega)$ is bounded from $M^2_{(\vartheta)}$ to $M^2_{(1/\vartheta)}$ by Proposition 1.35, the estimate

$$\|\operatorname{Tp}_\phi(\omega) f\|_{M^2_{(1/\vartheta)}} \lesssim \|f\|_{M^2_{(\vartheta)}} \tag{5.1}$$

holds for all $f \in M^2_{(\vartheta)}$.

Next we observe that

$$(\operatorname{Tp}_\phi(\omega) f, g)_{L^2(\mathbf{R}^d)} = (\omega V_\phi f, V_\phi g)_{L^2(\mathbf{R}^{2d})} = (f,g)_{M^{2,\phi}_{(\vartheta)}}, \tag{5.2}$$

for $f, g \in M^2_{(\vartheta)}(\mathbf{R}^d)$ and $\phi \in M^2_{(v)}(\mathbf{R}^d)$. The duality of modulation spaces (Proposition 1.11(3)) now yields the following identity:

$$\begin{aligned} \|f\|_{M^2_{(\vartheta)}} &\asymp \sup_{\|g\|_{M^2_{(\vartheta)}}=1} |(f,g)_{M^2_{(\vartheta)}}| \\ &\asymp \sup_{\|g\|_{M^2_{(\vartheta)}}=1} |(\operatorname{Tp}_\phi(\omega) f, g)_{L^2}| \asymp \|\operatorname{Tp}_\phi(\omega) f\|_{M^2_{(1/\vartheta)}}. \end{aligned} \tag{5.3}$$

A combination of (5.1) and (5.3) shows that $\|f\|_{M^2_{(\vartheta)}}$ and $\|\operatorname{Tp}_\phi(\omega) f\|_{M^2_{(1/\vartheta)}}$ are equivalent norms on $M^2_{(\vartheta)}$.

In particular, $\operatorname{Tp}_\phi(\omega)$ is one-to-one from $M^2_{(\vartheta)}$ to $M^2_{(1/\vartheta)}$ with closed range. Since $\operatorname{Tp}_\phi(\omega)$ is self-adjoint with respect to L^2, it follows by duality that $\operatorname{Tp}_\phi(\omega)$ has dense range in $M^2_{(1/\vartheta)}$. Consequently, $\operatorname{Tp}_\phi(\omega)$ is onto $M^2_{(1/\vartheta)}$. By Banach's theorem $\operatorname{Tp}_\phi(\omega)$ is an isomorphism from $M^2_{(\vartheta)}$ to $M^2_{(1/\vartheta)}$. □

We need a further generalization of Proposition 1.34 to more general classes of symbols and windows. Set

$$\omega_1(X,Y) = \frac{v_0(2Y)^{1/2} v_1(2Y)}{\omega_0(X+Y)^{1/2}\omega_0(X-Y)^{1/2}}. \tag{5.4}$$

Proposition 1.34′. *Let $s \ge 1$, $0 \le t \le 1$, $p,q,q_0 \in [1,\infty]$, and $\omega, \omega_0, v_0, v_1 \in \mathscr{P}_{E,s}(\mathbf{R}^{2d})$ be such that v_0 and v_1 are submultiplicative, ω_0 is v_0-moderate and ω is v_1-moderate. Set*

$$r_0 = 2q_0/(2q_0-1), \quad v = v_1^t v_0 \quad \text{and} \quad \vartheta = \omega_0^{1/2},$$

and let $\omega_{0,t}$ and ω_1 be as in (1.40) and (5.4). Then the following is true:

1. *The definition of $(a,\phi) \mapsto \operatorname{Tp}_\phi(a)$ from $\Sigma_s(\mathbf{R}^{2d}) \times \Sigma_s(\mathbf{R}^d)$ to $\mathcal{L}(\Sigma_s(\mathbf{R}^d), \Sigma_s'(\mathbf{R}^d))$ extends uniquely to a continuous map from $\mathcal{M}^{\infty,q_0}_{(1/\omega_{0,t})}(\mathbf{R}^{2d}) \times M^{r_0}_{(v)}(\mathbf{R}^d)$ to $\mathcal{L}(\Sigma_s(\mathbf{R}^d), \Sigma_s'(\mathbf{R}^d))$.*
2. *If $\phi \in M^{r_0}_{(v)}(\mathbf{R}^d)$ and $a \in \mathcal{M}^{\infty,q_0}_{(1/\omega_{0,t})}(\mathbf{R}^{2d})$, then $\operatorname{Tp}_\phi(a) = \operatorname{Op}^w(a_0)$ for some $a_0 \in \mathcal{M}^{\infty,1}_{(\omega_1)}(\mathbf{R}^{2d})$, and $\operatorname{Tp}_\phi(a)$ extends uniquely to a continuous map from $M^{p,q}_{(\vartheta\omega)}(\mathbf{R}^d)$ to $M^{p,q}_{(\omega/\vartheta)}(\mathbf{R}^d)$.*

For the proof we need the following result, which follows from [55, Proposition 2.1] and its proof. The proof is therefore omitted.

Lemma 5.5. *Assume that $s \ge 1$, $q_0, r_0 \in [1,\infty]$ satisfy $r_0 = 2q_0/(2q_0-1)$. Also assume that $v \in \mathscr P_{E,s}(\mathbf R^{2d})$ is submultiplicative, and that $\kappa, \kappa_0 \in \mathscr P_{E,s}(\mathbf R^{2d} \oplus \mathbf R^{2d})$ satisfy*

$$\kappa_0(X_1+X_2,Y) \le C\kappa(X_1,Y)\,v(Y+X_2)v(Y-X_2) \quad X_1,X_2,Y \in \mathbf R^{2d}, \qquad (5.5)$$

for some constant $C > 0$. Then the map $(a,\phi) \mapsto \operatorname{Tp}_\phi(a)$ from $\Sigma_s(\mathbf R^{2d}) \times \Sigma_s(\mathbf R^d)$ to $\mathcal L(\Sigma_s(\mathbf R^d), \Sigma_s'(\mathbf R^d))$ extends uniquely to a continuous mapping from $\mathcal M^{\infty,q_0}_{(\omega)}(\mathbf R^{2d}) \times M^{r_0}_{(v)}(\mathbf R^d)$ to $\mathcal L(\Sigma_s(\mathbf R^d), \Sigma_s'(\mathbf R^d))$. Furthermore, if $\phi \in M^{r_0}_{(v)}(\mathbf R^d)$ and $a \in \mathcal M^{\infty,q_0}_{(\kappa)}(\mathbf R^{2d})$, then $\operatorname{Tp}_\phi(a) = \operatorname{Op}^w(b)$ for some $b \in \mathcal M^{\infty,1}_{(\kappa_0)}$.

Proof of Proposition 1.34′. We show that the conditions on the involved parameters and weight functions satisfy the conditions of Lemma 5.5.

First we observe that

$$v_j(2Y) \le Cv_j(Y+X_2)v_j(Y-X_2), \quad j = 0,1$$

for some constant C which is independent of $X_2, Y \in \mathbf R^{2d}$, because v_0 and v_1 are submultiplicative. By (5.4) we get

$$\begin{aligned}
\omega_1(X_1+X_2,Y) &= \frac{v_0(2Y)^{1/2}v_1(2Y)}{\omega_0(X_1+X_2+Y)^{1/2}\omega_0(X_1+X_2-Y)^{1/2}} \\
&\le C_1\frac{v_0(2Y)^{1/2}v_1(2Y)v_0(X_2+Y)^{1/2}v_0(X_2-Y)^{1/2}}{\omega_0(X_1)} \\
&= C_1 v_1(2Y)^{1-t}\frac{v_0(2Y)^{1/2}v_1(2Y)^t v_0(X_2+Y)^{1/2}v_0(X_2-Y)^{1/2}}{\omega_0(X_1)} \\
&\le C_2 v_1(2Y)^{1-t}\frac{v_1(X_2+Y)^t v_1(X_2-Y)^t v_0(X_2+Y)v_0(X_2-Y)}{\omega_0(X_1)}.
\end{aligned}$$

Hence

$$\omega_1(X_1+X_2,Y) \le C\frac{v_1(2Y)^{1-t}v(X_2+Y)v(X_2-Y)}{\omega_0(X_1)}. \qquad (5.6)$$

By letting $\kappa_0 = \omega_1$ and $\kappa = 1/\omega_{0,t}$, it follows that (5.6) agrees with (5.5). The result now follows from Lemma 5.5. □

5.4. **Proofs of the lifting results for Toeplitz operators.** Theorem 5.1′ is an immediate consequences of Remark 1.13, Theorem 4.1, Lemma 5.4 and the following proposition.

Proposition 5.6. *Assume that $s \ge 1$, $\omega_0 \in \mathscr P^0_{E,s}(\mathbf R^{2d})$ be such that $\omega_0 \in \Gamma^{(\omega_0)}_s(\mathbf R^{2d})$, that $v \in \mathscr P^0_{E,s}(\mathbf R^{2d})$ is submultiplicative, and that $\omega_0^{1/2}$ is v-moderate. If $\phi \in M^2_{(v)}(\mathbf R^d)$, then $\operatorname{Tp}_\phi(\omega_0) = \operatorname{Op}^w(b)$ for some $b \in \Gamma^{(\omega_0)}_s(\mathbf R^{2d})$.*

Proof. By Propositions 1.26 and 1.27 with $t = 0$ we have $\omega_0 \in \mathcal M^{\infty,1}_{(1/\omega_{0,r_0})}(\mathbf R^{2d})$ for some $r_0 \ge 0$, where $\omega_{0,r_0}(X,Y) = \omega_0(X)e^{-r_0|Y|^{\frac1s}}$. Furthermore, by letting $v_1(Y) = e^{r_0|Y|^{\frac1s}}$, and ω_1 in (5.4) we have

$$\omega_1(X,Y) \gtrsim \frac{e^{r_0|2Y|^{\frac1s}}v(2Y)^{1/2}}{\omega_0(X+Y)^{1/2}\omega_0(X-Y)^{1/2}} \gtrsim \frac{e^{r_0|Y|^{\frac1s}}}{\omega_0(X)}.$$

Hence, Proposition 1.34′ gives $\operatorname{Tp}_\phi(\omega_0) = \operatorname{Op}^w(b)$, for some

$$b \in \mathcal M^{\infty,1}_{(1/\omega_{0,r_0})}(\mathbf R^{2d}) \subseteq \Gamma^{(\omega_0)}_s(\mathbf R^{2d}). \quad □$$

For the proof of Theorem 5.2′ we need the following two lemmas, where the first one is the Gevrey version of [30, Proposition 2.11].

Lemma 5.7. *Let $s \ge 1$, $\omega_0, v_0, v_1 \in \mathscr{P}^0_{E,s}(\mathbf{R}^{2d} \oplus \mathbf{R}^{2d})$ be such that ω_0 is v_0-moderate. Set $\vartheta = \omega_0^{1/2}$, and*

$$\begin{aligned} \omega_1(X,Y) &= \frac{v_0(2Y)^{1/2} v_1(2Y)}{\vartheta(X+Y)\vartheta(X-Y)}, \\ \omega_2(X,Y) &= \vartheta(X-Y)\vartheta(X+Y)v_1(2Y), \\ v_2(X,Y) &= v_1(2Y). \end{aligned} \tag{5.7}$$

Then

$$\Gamma_s^{(1/\vartheta)} \# \mathcal{M}^{\infty,1}_{(\omega_1)} \# \Gamma_s^{(1/\vartheta)} \subseteq \mathcal{M}^{\infty,1}_{(v_2)}, \tag{5.8}$$

$$\Gamma_s^{(1/\vartheta)} \# \mathcal{M}^{\infty,1}_{(v_2)} \# \Gamma_s^{(1/\vartheta)} \subseteq \mathcal{M}^{\infty,1}_{(\omega_2)}. \tag{5.9}$$

The same holds true with $\mathscr{P}_{E,s}$ and $\Gamma_{0,s}^{(1/\vartheta)}$ in place of $\mathscr{P}^0_{E,s}$ and $\Gamma_s^{(1/\vartheta)}$ respectively, at each occurrence.

Proof. We shall mainly follows the proof of [30, Proposition 2.11]. Since $\Gamma_s^{(1/\vartheta)} = \bigcup_{r\ge 0} \mathcal{M}^{\infty,1}_{(\vartheta_r)}$ with $\vartheta_r(X,Y) = \vartheta(X)e^{r|Y|^{\frac{1}{s}}}$ (Proposition 1.26(3)), it suffices to argue with the symbol class $\mathcal{M}^{\infty,1}_{(\vartheta_r)}$ for some sufficiently large r instead of $\Gamma_s^{(1/\vartheta)}$.

For suitable r we show that

$$\omega_3(X,Y) \lesssim \omega_1(X-Y+Z,Z)\vartheta_r(X+Z,Y-Z) \tag{5.10}$$

$$v_1(2Y) \lesssim \vartheta_r(X-Y+Z,Z)\omega_3(X+Z,Y-Z)\,, \tag{5.11}$$

where

$$\omega_3(X,Y) = \frac{v_1(2Y)\vartheta(X+Y)}{\omega_0(X-Y)}.$$

Proposition 1.31 applied to (5.10) gives that $\mathcal{M}^{\infty,1}_{(\omega_1)} \# \Gamma_s^{(1/\vartheta)} \subseteq \mathcal{M}^{\infty,1}_{(\omega_3)}$, and (5.11) implies that $\Gamma_s^{(1/\vartheta)} \# \mathcal{M}^{\infty,1}_{(\omega_3)} \subseteq \mathcal{M}^{\infty,1}_{(v_2)}$, and (5.8) holds.

Since ϑ is $v_0^{1/2}$-moderate and $v_0 \in \mathscr{P}^0_{E,s}$, we have

$$\vartheta(X-Y)^{-1} \le v_0(2Z)^{1/2}\vartheta(X-Y+2Z)^{-1} \quad \text{and} \quad \vartheta(X+Y) \le \vartheta(X+Z)e^{r|Y-Z|^{\frac{1}{s}}}$$

for suitable $r > 0$. This gives

$$\begin{aligned} \omega_3(X,Y) &\lesssim \frac{v_0(2Z)^{1/2} v_1(2Z)\vartheta(X+Z)e^{r|Y-Z|^{\frac{1}{s}}}}{\vartheta(X-Y+2Z)\vartheta(X-Y)} \\ &= \omega_1(X-Y+Z,Z)\vartheta_r(X+Z,Y-Z), \end{aligned}$$

for some $r > 0$.

We also have

$$\begin{aligned} v_1(2Y) &\lesssim \frac{\vartheta(X-Y)v_0(2Y)^{1/2}v_1(2Y)\vartheta(X+Y)}{\vartheta(X-Y)^2} \\ &\lesssim \frac{\vartheta(X-Y+Z)e^{r|Z|^{\frac{1}{s}}} v_0(2(Y-Z))^{1/2} v_1(2(Y-Z))\vartheta(X+Y)}{\vartheta(X-Y+2Z)^2} \\ &= \vartheta_r(X-Y+Z,Z)\omega_3(X+Z,Y-Z). \end{aligned}$$

The inclusion (5.9) is proved similarly. Let

$$\omega_4(X,Y) = \vartheta(X-Y)v_1(2Y)$$

be the intermediate weight. Then the inequality

$$
\begin{aligned}
\omega_4(X,Y) &=\lesssim \vartheta(X-Y+Z)e^{r|Z|^{\frac{1}{s}}}v_1(2(Y-Z)) \\
&= \vartheta_r(X-Y+Z,Z)v_2(X+Z,Y-Z)
\end{aligned}
$$

implies that $\Gamma_s^{(1/\vartheta)} \# \mathcal{M}^{\infty,1}_{(v_2)} \subseteq \mathcal{M}^{\infty,1}_{(\omega_4)}$.

Similarly we obtain

$$
\begin{aligned}
\omega_2(X,Y) &\lesssim \vartheta(X-Y)v_1(2Z)\vartheta(X+Z)e^{r|Z-Y|^{\frac{1}{s}}} \\
&= \omega_4(X-Y+Z,Z)\vartheta_r(X+Z,Y-Z),
\end{aligned}
$$

and thus $\mathcal{M}^{\infty,1}_{(\omega_4)} \# \Gamma_s^{(1/\vartheta)} \subseteq \mathcal{M}^{\infty,1}_{(\omega_2)}$.

The case $\mathscr{P}_{E,s}$ and $\Gamma_{0,s}^{(1/\vartheta)}$ in place of $\mathscr{P}^0_{E,s}$ and $\Gamma_s^{(1/\vartheta)}$ respectively, at each occurrence, is treated in similar ways and is left for the reader. □

Lemma 5.8. *Let s, ω_j, v_j and ϑ be the same as in Lemma 5.7, $j=0,1,2$. Also let $p,q \in [1,\infty]$ and $b \in \mathcal{M}^{\infty,1}_{(\omega_1)}(\mathbf{R}^{2d})$. Then the following is true:*

1. *$\operatorname{Op}^w(b)$ is continuous from $M^{p,q}_{(\vartheta)}(\mathbf{R}^d)$ to $M^{p,q}_{(1/\vartheta)}(\mathbf{R}^d)$;*
2. *if in addition $\operatorname{Op}^w(b)$ is an isomorphism from $M^2_{(\vartheta)}(\mathbf{R}^d)$ to $M^2_{(1/\vartheta)}(\mathbf{R}^d)$, then its inverse $\operatorname{Op}^w(b)^{-1}$ equals $\operatorname{Op}^w(c)$ for some $c \in \mathcal{M}^{\infty,1}_{(\omega_2)}(\mathbf{R}^{2d})$.*

Proof. The assertion (1) follows immediately from Proposition 1.30.

By Theorem 3.9, there are $a \in \Gamma_{0,s}^{(1/\vartheta)}(\mathbf{R}^{2d})$ and $a_0 \in \Gamma_{0,s}^{(\vartheta)}(\mathbf{R}^{2d})$ such that the map

$$
\operatorname{Op}^w(a) \,:\, L^2(\mathbf{R}^d) \to M^2_{(\vartheta)}(\mathbf{R}^d)
$$

is an isomorphism with inverse $\operatorname{Op}^w(a_0)$. By Propositions 1.26 and 1.27, $\operatorname{Op}^w(a)$ is also bijective from $M^2_{(1/\vartheta)}(\mathbf{R}^d)$ to $L^2(\mathbf{R}^d)$. Furthermore, by Theorem 4.2 it follows that $a \in \mathcal{M}^{\infty,1}_{(\vartheta_r)}$ when $r \ge 0$, where

$$
\vartheta_r(X,Y) = \vartheta(X)e^{r|Y|^{\frac{1}{s}}}.
$$

Let $b_0 = a\#b\#a$. From Lemma 5.7 we know that

$$
b_0 \in \mathcal{M}^{\infty,1}_{(v_2)}(\mathbf{R}^{2d}), \quad \text{where} \quad v_2(X,Y) = v_1(2Y) \tag{5.12}
$$

is submultiplicative and depends on Y only. Since $\operatorname{Op}^w(b)$ is bijective from $M^2_{(\vartheta)}$ to $M^2_{(1/\vartheta)}$ by Lemma 5.4 (2), $\operatorname{Op}^w(b_0)$ is bijective and continuous on L^2.

Since v_2 is submultiplicative and in $\mathscr{P}_{E,s}(\mathbf{R}^{2d})$, $\mathcal{M}^{\infty,1}_{(v_2)}$ is a Wiener algebra by Proposition 1.32. Therefore, the Weyl symbol c_0 of the inverse to the bijective operator $\operatorname{Op}^w(b_0)$ on L^2 belongs to $\mathcal{M}^{\infty,1}_{(v_2)}(\mathbf{R}^{2d})$.

Since

$$
\operatorname{Op}^w(c_0) = \operatorname{Op}^w(b_0)^{-1} = \operatorname{Op}^w(a)^{-1}\operatorname{Op}^w(b)^{-1}\operatorname{Op}^w(a)^{-1},
$$

we find

$$
\operatorname{Op}^w(c) = \operatorname{Op}^w(b)^{-1} = \operatorname{Op}^w(a)\operatorname{Op}^w(c_0)\operatorname{Op}^w(a),
$$

or equivalently,

$$
c = a\#c_0\#a, \quad \text{where} \quad a \in \Gamma_{0,s}^{(1/\vartheta)} \quad \text{and } c_0 \in \mathcal{M}^{\infty,1}_{(v_2)}. \tag{5.13}
$$

The definitions of the weights are chosen such that Lemma 5.7 implies that $c \in \mathcal{M}^{\infty,1}_{(\omega_2)}$, and (2) follows. □

Proof of Theorem 5.2′. First we note that the Toeplitz operator $\operatorname{Tp}_\phi(\omega_0)$ is an isomorphism from $M^2_{(\vartheta)}$ to $M^2_{(1/\vartheta)}$ in view of Lemma 5.4. With ω_1 defined in (5.4), Proposition 1.34′ implies that there exist $b \in \mathcal{M}^{\infty,1}_{(\omega_1)}$ and $c \in \mathcal{S}_s'(\mathbf{R}^{2d})$ such that

$$\operatorname{Tp}_\phi(\omega_0) = \operatorname{Op}^w(b) \quad \text{and} \quad \operatorname{Tp}_\phi(\omega_0)^{-1} = \operatorname{Op}^w(c)\,.$$

Let

$$\omega_2(X,Y) = \vartheta(X-Y)\vartheta(X+Y)v_1(2Y) \quad \text{and} \quad \omega_3(X,Y) = \frac{\vartheta(X+Y)}{\vartheta(X-Y)}. \tag{5.14}$$

By Lemma 5.8 and Proposition 1.30 it follows that $c \in \mathcal{M}^{\infty,1}_{(\omega_2)}(\mathbf{R}^{2d})$, and that the mappings

$$\operatorname{Op}^w(b) : M^{p,q}_{(\omega\vartheta)} \to M^{p,q}_{(\omega/\vartheta)} \quad \text{and} \quad \operatorname{Op}^w(c) : M^{p,q}_{(\omega/\vartheta)} \to M^{p,q}_{(\omega\vartheta)} \tag{5.15}$$

are continuous.

We have

$$\begin{aligned}
&\omega_1(X-Y+Z,Z)\omega_2(X+Z,Y-Z)\\
&= \Big(\frac{v_0(2Z)^{1/2}v_1(2Z)}{\vartheta(X-Y+2Z)\vartheta(X-Y)}\Big)\cdot\big(\vartheta(X-Y+2Z)\vartheta(X+Y)v_1(2(Y-Z))\big)\\
&= \frac{v_0(2Z)^{1/2}v_1(2Z)v_1(2(Y-Z))\,\vartheta(X+Y)}{\vartheta(X-Y)}\\
&\gtrsim \frac{\vartheta(X+Y)}{\vartheta(X-Y)} = \omega_3(X,Y)\,.
\end{aligned}$$

Therefore Proposition 1.31 shows that $b\#c \in \mathcal{M}^{\infty,1}_{(\omega_3)}$. Since $\operatorname{Op}^w(b)$ is an isomorphism from $M^2_{(\vartheta)}$ to $M^2_{(1/\vartheta)}$ with inverse $\operatorname{Op}^w(c)$, it follows that $b\#c = 1$ and that the constant symbol 1 belongs to $\mathcal{M}^{\infty,1}_{(\omega_3)}$. By similar arguments it follows that $c\#b = 1$. Therefore the identity operator $\mathrm{Id} = \operatorname{Op}^w(b)\circ\operatorname{Op}^w(c)$ on $M^{p,q}_{(\omega\vartheta)}$ factors through $M^{p,q}_{(\omega/\vartheta)}$, and thus $\operatorname{Op}^w(b) = \operatorname{Tp}_\phi(\omega_0)$ is an isomorphism from $M^{p,q}_{(\omega\vartheta)}$ to $M^{p,q}_{(\omega/\vartheta)}$ with inverse $\operatorname{Op}^w(c)$. This gives the result. □

5.5. **Specific bijective pseudo-differential and Toeplitz operators on modulation spaces.** We shall now apply the previous results to construct explicit isomorphisms between modulation spaces with different weights. These may be in the form of pseudo-differential operators or of Toeplitz operators.

The following result extends [30, Proposition 5.1] in the sense that it shows that the latter result holds after the class $\mathscr{P}(\mathbf{R}^{2d})$ have been replaced by the larger class $\mathscr{P}^0_{E,s}(\mathbf{R}^{2d})$.

Proposition 5.9. *Let $s \ge 1$, $\omega_0 \in \mathscr{P}^0_{E,s}(\mathbf{R}^{2d})$, $\mathscr{B}$ be a normal QBF-space on $\mathbf{R}^{2d}$ and let*

$$\Phi_\lambda(x,\xi) = Ce^{-(\lambda_1|x|^2+\lambda_2|\xi|^2)} \qquad \lambda = (\lambda_1,\lambda_2) \in \mathbf{R}^2_+.$$

Then the following is true:

1. *$\omega_0 * \Phi_\lambda$ belongs to $\mathscr{P}^0_{E,s}(\mathbf{R}^{2d}) \cap \Gamma^{(\omega_0)}_{0,1}$ for all $\lambda \in \mathbf{R}^2_+$ and*
$$\omega_0 * \Phi_\lambda \asymp \omega_0\,;$$
2. *If $\lambda_1\cdot\lambda_2 < 1$, then there exists $\nu \in \mathbf{R}^2_+$ and a Gauss function ϕ on $\mathbf{R}^d$ such that $\operatorname{Op}^w(\omega_0*\Phi_\lambda) = \operatorname{Tp}_\phi(\omega_0*\Phi_\nu)$ is bijective from $M(\omega,\mathscr{B})$ to $M(\omega/\omega_0,\mathscr{B})$ for all $\omega \in \mathscr{P}_{E,s}(\mathbf{R}^{2d})$;*
3. *If $\lambda_1\cdot\lambda_2 \le 1$ and in addition $\omega_0 \in \Gamma^{(\omega_0)}_s(\mathbf{R}^{2d})$, then $\operatorname{Op}^w(\omega_0*\Phi_\lambda) = \operatorname{Tp}_\phi(\omega_0)$ is bijective from $M(\omega,\mathscr{B})$ to $M(\omega/\omega_0,\mathscr{B})$ for all $\omega \in \mathscr{P}_{E,s}(\mathbf{R}^{2d})$.*

We shall follow the proof of [30, Proposition 5.1].

Proof. The assertion (1) is a straight-forward consequence of the definitions.

(2) Choose $\mu_j > \lambda_j$ such that $\mu_1 \cdot \mu_2 = 1$. Then $\Phi_\mu = cW(\phi,\phi)$ with $\phi(x) = e^{-\mu_1|x|^2/2}$ for some positive constant c, and there is another Gaussian Φ_ν such that $\Phi_\lambda = \Phi_\mu * \Phi_\nu$. Using (1.41), this factorization implies that the Weyl operator with symbol $\omega_0 * \Phi_\lambda$ is the Toeplitz operator

$$\operatorname{Op}^w(\omega_0 * \Phi_\lambda) = \operatorname{Op}^w(\omega_0 * \Phi_\nu * cW(\phi,\phi)) = c(2\pi)^{\frac{d}{2}}\operatorname{Tp}_\phi(\omega_0 * \Phi_\nu).$$

By (1) $\omega_0 * \Phi_\nu \in \mathscr{P}^0_{E,s}(\mathbf{R}^{2d}) \cap \Gamma^{(\omega_0)}_{0,1}(\mathbf{R}^{2d})$ is equivalent to ω_0. Hence Theorem 5.1′ shows that $\operatorname{Op}^w(\omega_0 * \Phi_\lambda)$ is bijective from $M(\omega,\mathscr{B})$ to $M(\omega/\omega_0,\mathscr{B})$, and (2) follows.

The assertion (3) follows from (2) in the case $\lambda_1 \cdot \lambda_2 < 1$. If $\lambda_1 \cdot \lambda_2 = 1$, then $\Phi_\lambda = cW(\phi,\phi)$ for $\phi(x) = e^{-\lambda_1|x|^2/2}$ and thus

$$\operatorname{Op}^w(\omega_0 * \Phi_\lambda) = \operatorname{Tp}^w_\phi(\omega_0)$$

is bijective from $M(\omega,\mathscr{B})$ to $M(\omega/\omega_0,\mathscr{B})$, since $\omega_0 \in \mathscr{P}^0_{E,s}(\mathbf{R}^{2d}) \cap \Gamma^{(\omega_0)}_s(\mathbf{R}^{2d})$. □

6. Gevrey modest weights and examples

In this section we give some examples on how the results in the previous sections can be applied. In Subsection 6.1 we introduce a weight class called *Gevrey modest weights* and discuss basic properties of such weights under compositions of power and exponential functions. This class include symbols to operators like Harmonic oscillator as well as weak forms of Schrödinger and Dirac propagators. In Subsection 6.2 we apply the lifting properties to show that the Toeplitz operators with Gevrey modest weights as their symbols possess lifting properties between suitable modulation spaces. We combine these results with Fredholm theory to deduce that corresponding pseudo-differential operators are Fredholm operators with index zero when acting between those modulation spaces. We also show that some of these pseudo-differential operators more generally possess similar lifting properties as corresponding Toeplitz operators.

6.1. Gevrey modest weights.

Before discussing the examples we introduce subclasses of smooth weights which satisfy suitable ellipticity conditions.

Definition 6.1. Let $\omega \in \mathscr{P}_E(\mathbf{R}^d)$ and $\vartheta \in \mathscr{P}_E(\mathbf{R}^d)$. Then ω is called *Gevrey modest* in Roumieu (Beurling) sense of order $s > 0$ (with respect to ϑ), if $\omega \in C^\infty(\mathbf{R}^d)$,

$$\lim_{|x|\to\infty}\omega(x) = \infty, \quad \lim_{|x|\to\infty}\frac{\vartheta(x)}{\omega(x)} = 0 \quad \text{and} \quad |\partial^\alpha\omega(x)| \lesssim h^{|\alpha|}\alpha!^s\vartheta(x),\ \alpha \in \mathbf{N}^d \setminus 0, \tag{6.1}$$

for some (for every) $h > 0$. If in addition ϑ in (6.1) can be chosen such that

$$\lim_{|x|\to\infty}\vartheta(x) = 0,$$

then ω is called *strongly Gevrey modest*.

Remark 6.2. If $\omega \in \mathscr{P}_E(\mathbf{R}^d)$ is *Gevrey modest* in Roumieu (Beurling) sense of order $s > 0$ with respect to $\vartheta \in \mathscr{P}_E(\mathbf{R}^d)$, then

$$\omega \in \Gamma^{(\omega)}_s(\mathbf{R}^d) \quad \text{and} \quad \partial^\alpha\omega \in \Gamma^{(\vartheta)}_s(\mathbf{R}^d) \subseteq \Gamma^{(\omega)}_s(\mathbf{R}^d), \quad \alpha \in \mathbf{N}^d \setminus 0$$

$$\big(\ \omega \in \Gamma^{(\omega)}_{0,s}(\mathbf{R}^d) \quad \text{and} \quad \partial^\alpha\omega \in \Gamma^{(\vartheta)}_{0,s}(\mathbf{R}^d) \subseteq \Gamma^{(\omega)}_{0,s}(\mathbf{R}^d), \quad \alpha \in \mathbf{N}^d \setminus 0\ \big).$$

It follows by straight-forward computations that additions and products of Gevrey modest weights are Gevrey modest, but similar facts do not in generally hold true for tensor products. On the other hand, for additions in the spirit of tensor products we have the following.

Proposition 6.3. *Let $\omega_j \in \mathscr{P}_E(\mathbf{R}^{d_j})$, $j = 1, 2$, be Gevrey modest in Roumieu (Beurling) sense of order $s > 0$. Then*

$$\omega(x_1, x_2) = \omega_1(x_1) + \omega_2(x_2)$$

is Gevrey modest in Roumieu (Beurling) sense of order $s > 0$.

Proof. We only prove the result in the Roumieu case. The Beurling case follows by similar arguments and is left for the reader. Let

$$d = d_1 + d_2, \quad \alpha = (\alpha_1 + \alpha_2) \in \mathbf{N}^{d_1+d_2} \setminus 0 \quad \text{and} \quad x = (x_1, x_2) \in \mathbf{R}^{d_1+d_2}.$$

If both α_1 and α_2 are non-zero, then $\partial^\alpha \omega = 0$, and it is obvious that (6.1) holds true for some choice of $\vartheta \in \mathscr{P}_E(\mathbf{R}^d)$. We need to consider the cases when $\alpha_1 = 0$ or $\alpha_2 = 0$.

Choose ϑ_1, ϑ_2 in such ways that (6.1) holds with ϑ_j and ω_j in place of ϑ and ω, and let

$$\vartheta(x_1, x_2) = \vartheta_1(x_1) + \vartheta_2(x_2).$$

We have

$$|\partial^\alpha_{x_j} \omega(x_1, x_2)| = |\partial^\alpha \omega_j(x_j)| \lesssim h^{|\alpha|} \alpha!^s \vartheta_j(x_j) \le h^{|\alpha|} \alpha!^s \vartheta(x).$$

The result follows if we prove the second limit in (6.1).

Since

$$\frac{\vartheta(x_1, x_2)}{\omega(x_1, x_2)} \asymp \max_{j=1,2} \left(\frac{\vartheta_j(x_j)}{\omega_1(x_1) + \omega_2(x_2)} \right)$$

it suffices to prove

$$\lim_{|x|\to\infty} \frac{\vartheta_j(x_j)}{\omega_1(x_1) + \omega_2(x_2)} = 0, \qquad j = 1, 2, \tag{6.2}$$

and by reasons of symmetry it suffices to prove this for $j = 1$. Let $\varepsilon \in (0, 1)$ and let $R_0 > 0$ be chosen such that

$$\frac{\vartheta_1(x_1)}{\omega_1(x_1)} < \varepsilon \quad \text{when} \quad |x_1| > R_0.$$

Also let

$$C = 1 + \sup_{x_1 \in \mathbf{R}^{d_1}} \frac{\vartheta_1(x_1)}{\omega_1(x_1)}, \quad m = \inf_{|x_1| \le R_0} |\omega_1(x_1)| \quad \text{and} \quad M = \sup_{|x_1| \le R_0} |\omega_1(x_1)|,$$

and choose $R > 2R_0$ such that

$$\omega_2(x_2) > \frac{CM}{\varepsilon} - m$$

when $|x_2| > R/2$.

Suppose that $|x| > R$. If $|x_1| > R_0$, then

$$\frac{\vartheta_1(x_1)}{\omega_1(x_1) + \omega_2(x_2)} \le \frac{\vartheta_1(x_1)}{\omega_1(x_1)} < \varepsilon.$$

If instead $|x_1| \le R_0$, then $|x_2| > R/2$ by the triangle inequality. This gives

$$\frac{\vartheta_1(x_1)}{\omega_1(x_1) + \omega_2(x_2)} \le \frac{CM}{m + \omega_2(x_2)} < \varepsilon.$$

Hence

$$\frac{\vartheta_1(x_1)}{\omega_1(x_1) + \omega_2(x_2)} < \varepsilon \quad \text{when} \quad |x| > R. \quad \square$$

Proposition 6.4. *Let $t>0$, $\omega \in \mathscr{P}_E(\mathbf{R}^d)\cap C^\infty(\mathbf{R}^d)$ be Gevrey modest in Roumieu (Beurling) sense of order $s\ge 1$ with respect to $\vartheta \in \mathscr{P}_E(\mathbf{R}^d)$. Then the following is true:*

(1) *ω^t is Gevrey modest in Roumieu (Beurling) sense of order $s\ge 1$ with respect to $\omega^{t-1}\vartheta$;*

(2) *if in addition ω is strongly Gevrey modest, then e^ω is Gevrey modest in Roumieu (Beurling) sense of order $s\ge 1$ with respect to $e^\omega\vartheta$.*

Proof. We only prove the result in the Roumieu case. The Beurling case follows by similar arguments and is left for the reader.

Let $\alpha \in \mathbf{N}^d\setminus 0$. By the assumptions it follows that $\omega(x)\ge c$ for some $c>0$. Let $f_1(u)=u^t$. Then,

$$f_1^{(n)}(u)=\binom{t}{n} n!\, u^{t-n},\qquad \binom{t}{n}=n!^{-1}\prod_{j=1}^n (t-j+1)$$

and we notice that $|\binom{t}{n}|\le h^n$ for some $h>0$ which is independent of n.

For an arbitrary term in Lemma 3.2 with $g=\omega$, $\beta_1,\dots,\beta_n\in\mathbf{N}^d\setminus 0$ such that $\beta_1+\cdots+\beta_n=\alpha$ we have

$$\begin{aligned}\left|\frac{f_1^{(n)}(\omega(x))}{n!}\right|\prod_{j=1}^n\frac{|\partial^{\beta_j}\omega(x)|}{\beta_j!} &\lesssim h^{|\beta_1+\cdots+\beta_n|}(\beta_1!\cdots\beta_n!)^{s-1}\binom{t}{n}\omega(x)^{t-n}\vartheta(x)^n\\ &\lesssim h_1^{|\alpha|}(\beta_1!\cdots\beta_n!)^{s-1}\omega(x)^t\left(\frac{\vartheta(x)}{\omega(x)}\right)^n\\ &\lesssim (Ch_1)^{|\alpha|}(\beta_1!\cdots\beta_n!)^{s-1}\omega(x)^{t-1}\vartheta(x),\qquad (6.3)\end{aligned}$$

for some $h_1>0$ and $C\ge 1$. Here the last inequality follows from the fact that $\omega(x)^{-1}\vartheta(x)$ is bounded. By Lemma A.1 there is a bound of number of terms in (3.1) of the form $h_2^{|\alpha|}$ for some $h_2>0$. A combination of this fact, Lemma 3.2, (6.3), $(\beta_1!\cdots\beta_n!)^s\le\alpha!^s$ and $|\alpha|!\le C^{|\alpha|}\beta_1!\cdots\beta_n!$, for some constant $C>0$ gives

$$|\partial^\alpha\omega(x)|\lesssim h_3^{|\alpha|}\sup\left(\frac{|\alpha|!}{\beta_1!\cdots\beta_n!}\right)\alpha!^s\omega(x)^{t-1}\vartheta(x)\lesssim h_4^{|\alpha|}\alpha!^s\omega(x)^{t-1}\vartheta(x),$$

for some $h_3,h_4>0$. Here the supremum is taken over all $\beta_1,\dots,\beta_n\in\mathbf{N}^d\setminus 0$ such that $\beta_1+\cdots+\beta_n=\alpha$.

We also notice that the limits in (6.1) hold true with ω^t and $\omega^{t-1}\vartheta$ in place of ω and ϑ. This gives (1).

(2) We recall that $\alpha\ne 0$. It follows from the assumptions that we may choose ϑ with limit zero at infinity. Let $f_2(u)=e^u$. Then $f_2^{(n)}(\omega(x))=e^{\omega(x)}$ and for an arbitrary term in (3.1) we have

$$\begin{aligned}\left|\frac{f_2^{(n)}(\omega(x))}{n!}\right|\prod_{j=1}^n\frac{|\partial^{\beta_j}\omega(x)|}{\beta_j!} &\lesssim h^{|\beta_1+\cdots+\beta_n|}(\beta_1!\cdots\beta_n!)^{s-1}\frac{e^{\omega(x)}}{n!}\vartheta(x)^n\\ &\lesssim h_1^{|\alpha|}\alpha!^{s-1}\frac{e^{\omega(x)}}{n!}\vartheta(x),\qquad (6.4)\end{aligned}$$

for some $h_1>0$. Here the last inequality follows from the fact that $\vartheta(x)$ tends to zero at infinity. By again using the fact that $|\alpha|!\le C^{|\alpha|}\alpha!$ for some constant $C>0$, Lemma 3.2, Lemma A.1 and (6.4) give

$$|\partial^\alpha(e^\omega)|\lesssim h^{|\alpha|}\alpha!^s e^\omega\vartheta$$

for some $h>0$.

We also notice that the limits in (6.1) hold true with e^ω and $e^\omega\vartheta$ in place of ω and ϑ. This gives the result. $\square$

Example 6.5. Let f be a homogeneous polynomial on $\mathbf{R}^d$ of degree n which is positive outside origin, $r>0$, $r_n = r-\frac{1}{n}$ and let

$$\omega_{r,f}(x) = (1+f(x))^r.$$

Then the following is true:

(1) $\omega_{r,f}$ is Gevrey modest in Roumieu sense of order $s=1$ and with respect to $\omega_{r_n,f}$;

(2) if in addition $r<\frac{1}{n}$, then $e^{\omega_{r,f}}$ is Gevrey modest in Roumieu sense of order $s=1$ and with respect to $e^{\omega_{r,f}}\omega_{r_n,f}$;

(3) if $r_j>0$, f_j be homogeneous polynomials on $\mathbf{R}^{d_j}$ of degree n_j which are positive outside origin, $j=1,2$, then

$$\omega_0(x_1,x_2) = \big(\,\omega_{r_1,f_1}(x_1)+\omega_{r_2,f_2}(x_2)\,\big)^r$$

is Gevrey modest in Roumieu sense of order $s=1$.

In fact, since f is positive outside origin, it follows that $n=2m$ is even, giving that

$$\omega_{1,f}(x) \asymp (1+|x|^2)^m.$$

This gives

$$\omega_{1,f}(x+y) \asymp (1+|x+y|^2)^m \lesssim (1+|x|^2)^m(1+|y|^2)^m \asymp \omega_{1,f}(x)\omega_{1,f}(y).$$

Hence, for some positive constant c it follows that $c\cdot\omega_{1,f}$ is submultiplicative and polynomially bounded. In particular, $\omega_{1,f}\in\mathscr{P}(\mathbf{R}^d)$.

We also have $\partial^\alpha\omega_{1,f}=0$ when $|\alpha|>n$, and

$$|\partial^\alpha\omega_{1,f}(x)| \lesssim (1+|x|^2)^{m-\frac{|\alpha|}{2}} \asymp \omega_{r_0,f}(x).$$

with $r_0 = 1-\frac{|\alpha|}{n}$. This gives (1) for $r=1$. For general $r>0$, (1) now follows from the case $r=1$ and Proposition 6.4.

The assertions (2) and (3) now follow from (1) and Propositions 6.3 and 6.4 (2).

Proposition 6.6. *Let $\omega\in\mathscr{P}_E(\mathbf{R}^d)$ be Gevrey modest in Roumieu (Beurling) sense of order $s\ge 1$ with respect to $\vartheta\in\mathscr{P}_E(\mathbf{R}^d)$ and let $\phi\in\Sigma_1(\mathbf{R}^d)$ be such that $\int\phi(x)\,dx=1$. Then*

$$\omega-\omega*\phi\in\Gamma_s^{(\vartheta)}(\mathbf{R}^d)\quad\big(\,\omega-\omega*\phi\in\Gamma_{0,s}^{(\vartheta)}(\mathbf{R}^d)\,\big)$$

Proof. Again we only prove the result in the Roumieu case. The Beurling case follows by similar arguments and is left for the reader.

Let $g=\omega-\omega*\phi$. Then

$$g(x) = \int_{\mathbf{R}^d}\left(\int_0^1 \langle\omega'(x-ty),y\rangle\phi(y)\,dt\right)dy.$$

By differentiations we get for some $h,C>0$ and submultiplicative function v on $\mathbf{R}^d$ that

$$\begin{aligned}|D^\alpha g(x)| &\le \int_{\mathbf{R}^d}\left(\int_0^1 |\langle D^\alpha(\omega')(x-ty),y\rangle\phi(y)|\,dt\right)dy\\ &\lesssim h^{|\alpha+1|}|\alpha+1|!\int_{\mathbf{R}^d}\left(\int_0^1\vartheta(x-ty)|y|\,|\phi(y)|\,dt\right)dy\\ &\le (Ch)^{|\alpha|}\alpha!\vartheta(x)\int_{\mathbf{R}^d}\left(\int_0^1 v(ty)|y|\,|\phi(y)|\,dt\right)dy \asymp (Ch)^{|\alpha|}\alpha!\vartheta(x),\end{aligned}$$

which gives the result. □

6.2. **Mapping properties for pseudo-differential operators and Toeplitz operators with Gevrey modestest weights as their symbols.** As a consequence of the previous result we have the following.

Theorem 6.7. *Let $s \ge 1$, $A \in \mathbf{M}(d, \mathbf{R})$, $\omega_0 \in \mathscr{P}^0_{E,s}(\mathbf{R}^{2d})$ be Gevrey modest in Roumieu sense of order s, $\phi \in \mathcal{S}_1(\mathbf{R}^d) \setminus 0$, $\omega \in \mathscr{P}^0_{E,s}(\mathbf{R}^{2d})$ and let $\mathscr{B}$ be a normal QBF-space on $\mathbf{R}^{2d}$. Then the following is true:*

(1) $\operatorname{Tp}_\phi(\omega_0)$ *is homeomorphic from* $M(\omega, \mathscr{B})$ *to* $M(\omega/\omega_0, \mathscr{B})$*;*

(2) *if in addition* $\mathscr{B}$ *is a Banach space, then* $\operatorname{Op}_A(\omega_0)$ *is continuous from* $M(\omega, \mathscr{B})$ *to* $M(\omega/\omega_0, \mathscr{B})$ *with index zero.*

Proof. We may assume that $\|\phi\|_{L^2} = 1$ The assertion (1) as well as the continuity assertions in (2) follows from Theorems 4.1 and 5.1′.

In order to prove that the index of $\operatorname{Op}_A(\omega_0)$ is zero we observe that

$$(2\pi)^{-\frac{d}{2}} \iint_{\mathbf{R}^{2d}} W^A_{\phi,\phi}(x,\xi)\, dx d\xi = \|\phi\|^2_{L^2} = 1$$

by a straight-forward application of Fourier's inversion formula. A combination of (1.41) and Proposition 6.6 shows that

$$\operatorname{Op}_A(\omega_0) = \operatorname{Tp}_\phi(\omega_0) + \operatorname{Op}_A(b),$$

for some $b \in \Gamma^{(\vartheta)}_s(\mathbf{R}^{2d})$.

The assertion (2) follows from (1) and Fredholm's theorem if we prove that

$$\operatorname{Op}_A(b) \,:\, M(\omega, \mathscr{B}) \to M(\omega/\omega_0, \mathscr{B}) \tag{6.5}$$

is compact.

We have

$$\operatorname{Op}^w(b) \,:\, M(\omega, \mathscr{B}) \to M(\omega/\vartheta, \mathscr{B}) \tag{6.6}$$

is continuous. Since

$$\frac{\omega/\omega_0}{\omega/\vartheta} = \frac{\vartheta}{\omega_0}$$

tends to zero at infinity, the embedding

$$\iota \,:\, M(\omega/\vartheta, \mathscr{B}) \to M(\omega/\omega_0, \mathscr{B}) \tag{6.7}$$

is compact in view of [41, Theorem 2.9]. Hence the operator in (6.5) factorizes into the continuous operator (6.6) and the compact operator (6.7), giving that (6.5) is compact. This gives the assertion. □

Example 6.8. Let f_j, r_j, $\omega_{r,f}$ and ω_0 be the same as in Example 6.5 with $d_j = d$, and let n_j be the degrees of f_j, $j = 1, 2$. Then,

$$\omega_0(x,\xi) = \big(\omega_{r_1,f_1}(x) + \omega_{r_2,f_2}(\xi)\big)^r.$$

Also let $A \in \mathbf{M}(d, \mathbf{R})$, $\phi \in \mathcal{S}_1(\mathbf{R}^d) \setminus 0$, $\omega \in \mathscr{P}^0_E(\mathbf{R}^{2d})$, $\mathscr{B}$ be a normal QBF-space on $\mathbf{R}^{2d}$ and

$$a(x,\xi) = e^{\omega_0(x,\xi)}.$$

Then the following is true:

(1) $\operatorname{Tp}_\phi(\omega_0)$ is homeomorphic from $M(\omega, \mathscr{B})$ to $M(\omega/\omega_0, \mathscr{B})$. If in addition $\mathscr{B}$ is a Banach space, then $\operatorname{Op}_A(\omega_0)$ is a continuous map from $M(\omega, \mathscr{B})$ to $M(\omega/\omega_0, \mathscr{B})$ with index zero.

The same conclusions hold true with e^{ω_0} in place of ω_0 at each occurrence, when $rn_j < 1$, $j = 1, 2$;

(2) If in addition $r=1$ and $\mathscr{B}$ is a Banach space, then $\operatorname{Op}_A(\omega_0)$ is homeomorphic from $M(\omega,\mathscr{B})$ to $M(\omega/\omega_0,\mathscr{B})$.

In fact, (1) follows from the conclusions in Example 6.5 in combination with Propositions 6.3, 6.4 and Theorem 6.7.

For $r=1$, we have

$$\operatorname{Op}_A(\omega_0)\,:\,M^2_{(\omega_0)}(\mathbf{R}^d)\to L^2(\mathbf{R}^d) \tag{6.8}$$

is injective, because

$$(\operatorname{Op}_A(\omega_0)f,f)_{L^2}=\int_{\mathbf{R}^d}\omega_{r_1,f_1}(x)|f(x)|^2\,dx+\int_{\mathbf{R}^d}\omega_{r_2,f_2}(\xi)|\widehat{f}(\xi)|^2\,d\xi>0$$

when $f\in M^2_{(\omega_0)}(\mathbf{R}^d)\setminus 0$. Since (6.8) also has index zero in view of (1), Banach's theorem shows that (6.8) is a homeomorphism. The assertion (2) now follows from Theorem 4.1.

Remark 6.9. We notice that the claim for $\operatorname{Tp}_\phi(\omega_0)$ and $\operatorname{Op}_A(\omega_0)$ in the previous example can only be proved for weights ω in the subset $\mathscr{P}$ of $\mathscr{P}_E^0$ if one should use the embedding results in [30] instead of the results in Section 1 and Subsection 5.5. The embedding results in [31] are not applicable for the situation in the example, because ω_0 is in general not radial symmetric at each phase space variable $(x_j,\xi_j)\in\mathbf{R}^2$.

Remark 6.10. As explained in the example, the map (6.8) is a homoemorphism. Even this, perhaps the most simple special case, seems not to be easy to achieve by other methods.

Appendix A. Proof of Lemma 3.3

Lemma A.1. *Let $\alpha=(\alpha_1,\dots,\alpha_d)\in\mathbf{N}^d$. Then the number of elements in the set*

$$\Omega_{k,\alpha}\equiv\{\,(\beta_1,\dots,\beta_k)\in\mathbf{N}^{kd}\,;\,\beta_1+\cdots+\beta_k=\alpha\,\} \tag{A.1}$$

is equal to

$$\prod_{j=1}^d\binom{\alpha_j+k-1}{k-1}.$$

For the proof we recall the formula

$$\sum_{j=0}^k\binom{n+j}{j}=\binom{n+k+1}{k}, \tag{A.2}$$

which follows by a standard induction argument.

Proof. Let N be the number of elements in the set (A.1), which is the searched number, and let N_j be the number of elements of the set

$$\{\,(\beta_1^0,\dots,\beta_k^0)\in\mathbf{N}^k\,;\,\beta_1^0+\cdots+\beta_k^0=\alpha_j\,\},\qquad j=1,\dots,d$$

By straight-forward computations it follows that $N=N_1\cdots N_d$, and it suffices to prove the result in the case $d=1$, and then $\alpha=\alpha_1$.

In order to prove the result for $d=1$, let $\gamma\in\mathbf{N}$, $S_k(\gamma)$ be the number of elements in $\Omega_{k,\gamma}$. Then the statement is

$$S_k(\gamma)=\binom{\gamma+k-1}{k-1}, \tag{A.3}$$

and we shall prove the statement by induction. We have that $S_k(\gamma)$ agrees with the number of possibilities to put γ elements into k boxes. If $k=1$, then there is only one possibilities, i.e. $S_1(\gamma)=1$. Suppose that the statement is true for $k\le j$,

$j \ge 1$. Prove it for $k = j+1$. If we have put γ_0 objects in box number $j+1$, then $\gamma - \gamma_0$ remains to put in the first j boxes. This implies

$$S_{j+1}(\gamma) = \sum_{\gamma_0=0}^{\gamma} S_j(\gamma - \gamma_0) = \sum_{\gamma_0=0}^{\gamma} S_j(\gamma_0).$$

Hence, by (A.2) and the induction hypothesis we get

$$S_{j+1}(\gamma) = \sum_{\gamma_0=0}^{\gamma} \binom{\gamma_0 + j - 1}{j-1} = \sum_{\gamma_0=0}^{\gamma} \binom{\gamma_0 + j - 1}{\gamma_0} = \binom{\gamma_0 + j}{\gamma_0} = \binom{\gamma_0 + j}{j},$$

and (A.3) follows for $k = j+1$. □

Lemma A.2. *Let $\alpha \in \mathbf{N}^d \setminus 0$, $s_0 \in (0,1]$, and let $\Omega_{k,\alpha}$ be the same as in* (A.1). *Then*

$$\sum_{k=1}^{|\alpha|} \frac{1}{k} \sum_{\beta \in \Omega_{k,\alpha}} (\beta!)^{s_0 - 1} \le 6^{|\alpha|}.$$

Proof. By Lemma A.1 and the fact that $s_0 - 1 < 0$ we get

$$\sum_{k=1}^{|\alpha|} \frac{1}{k} \left(\sum_{\beta \in \Omega_{k,\alpha}} \beta!^{s_0-1} \right) \le \sum_{k=1}^{|\alpha|} \left(\sum_{\beta \in \Omega_{k,\alpha}} 1 \right) = \sum_{k=1}^{|\alpha|} \left(\prod_{j=1}^{d} \binom{\alpha_j + k}{k} \right)$$

$$\le |\alpha| \prod_{j=1}^{d} 2^{2\alpha_j} = |\alpha| \cdot 4^{|\alpha|} \le 6^{|\alpha|}. \quad \square$$

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Dipartimento di Matematica "G. Peano", Universitá degli Studi di Torino
Email address: ahmed.abdeljawad@unito.it

Dipartimento di Matematica "G. Peano", Universitá degli Studi di Torino
Email address: sandro.coriasco@unito.it

Department of Mathematics, Linnæus University, Sweden
Email address: joachim.toft@lnu.se