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Towards open–closed string duality:

Closed Strings as Open String Fields

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Abstract

We establish a translation dictionary between open and closed strings, starting from open string field theory. Under this correspondence, (off–shell) level–matched closed string states are represented by star algebra projectors in open string field theory. Particular attention is paid to the zero mode sector, which is indispensable in order to generate closed string states with momentum. As an outcome of our identification, we show that boundary states, which in closed string theory represent D–branes, correspond to the identity string field in the open string side. It is to be remarked that closed string theory D–branes are thus given by an infinite superposition of star algebra projectors.

Keywords: String Field Theory, Open–closed string duality

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1 Introduction

The duality between open and closed strings has been a well–known topic since the very beginning of string theory. The AdS/CFT correspondence has revived the interest on this subject: it is a sort of limiting case in which the open string side of the correspondence is represented by a conformal gauge theory. More recently A. Sen, [1], has suggested that open string theory might be able to describe all the closed string physics, at least in a background where D–branes are present. In this sense Witten's open string field theory should be a privileged ground to check this idea. For open string field theory is of course formulated in terms of open strings degrees of freedom, but there is ample evidence that tachyon condensation leads to a new vacuum and that this new vacuum is the closed string one.

In a recent paper [2] a remarkable correspondence was pointed out (in the context of the AdS/CFT duality) between N=4 SYM states in 4D and star algebra projectors, or, more appropriately, family of them and star algebra projectors in SFT (which can be interpreted also as VSFT solutions). Upon taking a coarse graining limit, the former give rise to the geometry of supergravity solutions (the 1/2 BPS solutions of [3]). Although the correspondence is imperfect due to the lack of supersymmetry on the SFT side, it is very suggestive, because it implies that supergravity solutions can be constructed out of open string bricks. Logically one expects that closed string modes should be expressible in terms of open string degrees of freedom. Following Sen's suggestion and this example we set out in this paper to tackle the problem of writing out an explicit relation between open and closed string modes, a sort of dictionary to translate from the open string to the closed string language. Our proposal can be summarized as in the title: (perturbative) closed string modes are SFT projectors. More precisely: momentum and level–matched off–shell closed string states are in one–to–one correspondence with star algebra projectors in SFT. In this paper we start also to verify the validity of this dictionary. One very interesting outcome is the proof that (if we neglect localization, which is taken care of by the zero mode sector) a boundary state describing a D–brane in the closed string language under this correspondence gets translated into the open string identity state. This implies in particular that such boundary states are superpositions of infinite many star algebra projectors

Before we proceed to expound our paper it is necessary for us to make a short digression to comment the state of affairs in SFT, which is actually very fluid and promising. Recently M. Schnabl [4, 5] has found an exact analytic solution to the SFT equation of motion, which corresponds to a vacuum without perturbative open strings modes, see also [6, 7, 8]. More solutions of this type have been found in [9]. These prove the first two Sen's conjectures [10]. For the third one, existence of lower dimensional brane solutions, more work is needed. The existence of such solutions was shown in the past in the context of the VSFT [11], a simplified (and singular) version of Witten's open SFT, which is likely to give the correct response at least for static solutions and it may be that to any such VSFT solution there correspond an analytic SFT solution à la Schnabl. We recall that the solutions to the VSFT equation of motion are star algebra projectors (at least for the matter part). In this paper the basic objects are precisely star algebra projectors. Since the star product is the same in SFT and in VSFT, star algebra projectors are well defined objects in SFT, even without reference to VSFT. This is the sense in which they will be considered in this paper, namely as objects pertaining to SFT. In this regard a warning is in order: generally speaking the (perturbative) closed string states we introduce in this paper are star projectors, that is D–branes in the VSFT interpretation, while D–branes (or boundary states) in closed string theory correspond to infinite superposition of the latter.

The paper is organized as follows. Section 2 and 3 are essentially preliminary and contain mostly well–known results about split string field theory and the comma vertex algebra, although elaborated the way we need. In section 4 we establish the conjectured correspondence between zero momentum closed string states and star algebra projectors. In section 5 we associate to such states a momentum eigenfunction and conclude that the resulting (off–shell) closed string states are again star algebra projectors. In section 6 we show that a boundary state describing a D–brane in the closed string language gets translated into the open string identity state, and suggest an interpretation of this fact. As a test of our construction we also show how to compute the closed string exchange between two boundary states using standard star algebra manipulations. Finally, section 7 is devoted to a discussion of our results and of the questions they raise.

2 Preliminary algebras

The SFT action is

$$\mathcal{S}(\Psi) = -\left(\frac{1}{2}\langle\Psi|Q_B|\Psi\rangle + \frac{1}{3}\langle\Psi|\Psi * \Psi\rangle\right) \tag{1}$$

where Q_B is the open string BRST charge.

We will consider star algebra projectors, that is states that satisfy the equation

$$\Psi * \Psi = \Psi$$

Since the star product factorizes into matter and ghost part it is natural to make for projectors the following factorized ansatz

$$\Psi = \Psi_m \otimes \Psi_g \tag{2}$$

where Ψ_g and Ψ_m depend purely on ghost and matter degrees of freedom, respectively. The projector equation splits into

$$\Psi_g = \Psi_g *_g \Psi_g \tag{3}$$

$$\Psi_m = \Psi_m *_m \Psi_m \tag{4}$$

where $*_g$ and $*_m$ refer to the star product involving only the ghost and matter part.

We will concentrate on the matter part, eq.(4). The $*_m$ product in the operator formalism is defined as follows

$$_{123}\langle V_3|\Psi_1\rangle_1|\Psi_2\rangle_2 =_3 \langle \Psi_1 *_m \Psi_2|, \tag{5}$$

see [12, 13, 14, 15] for the definition of the three string vertex $_{123}\langle V_3|$. The basic ingredient in this definition are the matrices of vertex coefficients V_{nm}^{rs}, $r, s = 1, 2, 3,\ n, m = 1, \ldots, \infty$.

The following developments are based on the sliver solution.

$$|\Xi\rangle = \mathcal{N} e^{-\frac{1}{2}a^\dagger \cdot S \cdot a^\dagger}|0\rangle, \qquad a^\dagger \cdot S \cdot a^\dagger = \sum_{n,m=1}^{\infty} a_n^{\mu\dagger} S_{nm} a_m^{\nu\dagger}\eta_{\mu\nu} \tag{6}$$

where $S = CT$ and

$$T = \frac{1}{2X}(1 + X - \sqrt{(1+3X)(1-X)}) \tag{7}$$

with $X = CV^{11}$, where $C_{nm} = (-1)^n \delta_{nm}$ is the twist matrix. The normalization constant $\mathcal{N} = \left(\sqrt{\det(1+T)(1-X)}\right)^D$ is formally vanishing and needs to be regularized. It has been showed in other papers how this and related problems could be dealt with, [16, 17]. Our basic projector will have the form of the sliver along the the space–time directions.

In SFT there are two preferred ways to split the set of open string oscillators in two distinct sets which could mimic the two sets of holomorphic and anti–holomorphic closed string oscillators: the even–odd and left–right splitting. The former consists in splitting according to the eigenvalues of the twist matrix C; one can quickly verify, however, that it does not fit our purposes. The latter is based on the separation between the left and right modes of the open string. This turns out to be a better chance.

The construction that follows is based on split string field theory and the so–called comma vertex algebra, [18, 19, 20, 21, 22, 23, 24]. Many formulas we use in this section can be traced back to ref. [24].

Let us introduce the left and right Fock space projector ρ_L and ρ_R:

$$\rho_L^2 = \rho_L, \qquad \rho_R^2 = \rho_R, \qquad \rho_L + \rho_R = 1 \tag{8}$$

with the properties

$$\rho_L = \rho_L^t = C\rho_R C, \qquad \rho_R = \rho_R^t = C\rho_L C \tag{9}$$

They turn out to project onto the left and right hand part of the string respectively. Next we define the operators

$$s^\mu = \omega(a^\mu + Sa^{\mu\,\dagger}) = (a^\mu + Sa^{\mu\,\dagger})\omega, \qquad \omega = \frac{1}{\sqrt{1-S^2}} \tag{10}$$

and the conjugate ones, where the labels n, m running from 1 to $+\infty$ are understood. Using the algebra of open string creation and annihilation operators these operators can be shown to satisfy

$$[s_m^\mu, s_n^{\nu\dagger}] = \delta_{nm}\eta^{\mu\nu} \tag{11}$$

while the other commutators vanish. Moreover, understanding the Lorentz indexes,

$$s_n|\Xi\rangle = \mathcal{N} e^{-\frac{1}{2}a^\dagger S a^\dagger}\omega(a - Sa^\dagger + Sa^\dagger)|0\rangle = 0 \tag{12}$$

Therefore the combinations s_n represent Bogoliubov transformations, which map the Fock space based on the initial vacuum $|0\rangle$ to a new Fock space in which the role of vacuum is played by the sliver.

Now we introduce the vector ξ such that

$$\rho_L \xi = \xi, \qquad \rho_R \xi = 0 \tag{13}$$

As a consequence

$$\rho_R C\xi = C\xi, \qquad \rho_L C\xi = 0$$

There exists a complete basis ξ_n $(n = 1, 2, ...)$ that satisfy these conditions and are orthonormal in the sense that

$$\langle \xi_n | \frac{1}{1 - T^2} | \xi_m \rangle = \delta_{nm} \tag{14}$$

see, for instance, [16].

Let us define, for any ξ,

$$\xi^L = \frac{1}{\sqrt{1 - S^2}} \xi, \qquad \xi^R = -\frac{1}{\sqrt{1 - S^2}} C\xi \tag{15}$$

In this way we have two complementary bases ξ_n^L and ξ_n^R. They are complementary in the sense that

$$\sum_{n=1}^{\infty} \Big(\xi_n^L(k) \xi_n^L(k') + \xi_n^R(k) \xi_n^R(k') \Big) = \delta(k, k') \tag{16}$$

Notice that

$$\xi_n^R(k) = \xi_n^L(-k), \qquad \text{while} \qquad \xi_n^R(-k) = \xi_n^L(k) = 0. \tag{17}$$

We can project ξ_n^L and ξ_n^R on the ordinary $\mathrm{v}_n(k)$ basis of eigenvector of the continuous spectrum, [25] (see Appendix) and define the coefficients

$$b_{nl} = \langle \xi_n^L | \mathrm{v}_l \rangle, \qquad \tilde{b}_{nl} = \langle \xi_n^R | \mathrm{v}_l \rangle$$

Using the latter we can introduce

$$\beta_m^\mu = \sum_{l=1}^{\infty} b_{ml} s_l^\mu, \qquad \tilde{\beta}_m^\mu = -\sum_{l=1}^{\infty} \tilde{b}_{ml} s_l^\mu \tag{18}$$

with the respective hermitian conjugates. The reason for the minus sign in the second definition above will become clear shortly. These operators satisfy the algebra

$$[\beta_m^\mu, \beta_n^{\nu\dagger}] = \delta_{m,n} \eta^{\mu\nu}$$
$$[\tilde{\beta}_m^\mu, \tilde{\beta}_n^{\nu\dagger}] = \delta_{m,n} \eta^{\mu\nu}$$

while all the other commutators vanish.

It must be remarked that the definition of $\beta_n, \tilde{\beta}_n$ depends on the ξ_n basis we use. This entails a $O(\infty)$ 'gauge' freedom in the choice of these operators.

These $\beta, \tilde{\beta}$ operators are natural candidates as closed string creation and annihilation operators. For the same reason it is natural to interpret the sliver $|\Xi\rangle$ as the closed string vacuum $|0_c\rangle$.

3 Properties of the β and $\tilde{\beta}$ operators

The operators β_n and $\tilde{\beta}_n$ and their conjugates are characterized by a Heisenberg algebra isomorphic to the algebra of closed string creation and annihilation operators. The zero mode oscillators have not been introduced yet. This will be done in the following section. Ignoring for the time being the zero modes, in this section we would like delve into the properties of the $\beta, \tilde{\beta}$ operators.

Let us consider the identity

$$\sum_n \beta_n^{\mu\dagger} \tilde{\beta}_n^{\nu\dagger} \eta_{\mu\nu} = -\sum_n \langle s^{\mu\dagger} | \xi_n^L \rangle \langle \xi_n^R | s^{\nu\dagger} \rangle \eta_{\mu\nu}$$
$$= \sum_n \langle s^{\mu\dagger} | \xi_n^L \rangle \langle \xi_n^L | C s^{\nu\dagger} \rangle \eta_{\mu\nu} = \frac{1}{2} \sum_{k=1}^{\infty} s_k^{\mu\dagger} C_{kl} s_l^{\nu\dagger} \eta_{\mu\nu}$$

The factor of $\frac{1}{2}$ comes from the fact that ξ_n is a complete basis for the left ξ's. We have to consider also the other half made of $C\xi_n$, which gives the same contribution, see (16). Hence the factor of $\frac{1}{2}$. The - signs come from the definition (18) and from the property (17).

Motivated by the above isomorphism we denote $|\Xi\rangle$ by $|0_c\rangle$ (the 'closed string vacuum') when β operators are applied to it. We have the following identity

$$e^{-\sum_n \beta_n^{\mu\dagger} \tilde{\beta}_n^{\nu\dagger} \eta_{\mu\nu}} |0_c\rangle = e^{-\frac{1}{2} \sum_{k=1}^{\infty} s_k^{\mu\dagger} C_{kl} s_l^{\nu\dagger} \eta_{\mu\nu}} |\Xi\rangle \sim e^{-\frac{1}{2} \sum_{k=1}^{\infty} a_k^{\mu\dagger} C_{kl} a_l^{\nu\dagger} \eta_{\mu\nu}} |0\rangle \tag{19}$$

where $|0\rangle$ is the original open string vacuum. The last step of the proof can be found for instance in [26], the equality holds up to a constant.

The LHS is proportional to the boundary state in closed string theory, the right hand side is the identity state in open string field theory. The boundary state represents a D25–brane in the closed string language. The identity state represents absence of interaction in the open string theory language: the identity state leaves any string state invariant under star multiplication, and the star multiplication represents string interaction. An interpretation of this identification will be presented at the end of section 5.

3.1 Closed string oscillators as $*$-algebra multiplication operators

A useful way of defining the β and $\tilde{\beta}$ operators is as multiplication operators in the $*$-algebra of string states. This is what we wish to discuss now. At the same time we would like to introduce the issue of Lorentz covariance. Let us begin by considering a particular set of half string states, containing only one closed string excitation, say l. A basis for such states is given by

$$\Lambda_l{}^{\mu_1 \ldots \mu_m, \nu_1 \ldots \nu_n} = \frac{(-1)^n}{\sqrt{n! m!}} \beta_l^{\mu_1 \dagger} \ldots \beta_l^{\mu_m \dagger} \tilde{\beta}_l^{\nu_1 \dagger} \ldots \tilde{\beta}_l^{\nu_n \dagger} |\Xi\rangle \tag{20}$$

It is easy to prove that these states form the following subalgebra (indexes are lowered with the Minkowski metric)

$$\Lambda_l{}^{\mu_1 \ldots \mu_n, \nu_1 \ldots \nu_m} * \Lambda_{l \rho_1 \ldots \rho_p}{}^{\sigma_1 \ldots \sigma_q} = \delta_{mp} \, \hat{\delta}^{\nu_1 \ldots \nu_m}_{\rho_1 \ldots \rho_m} \, \Lambda_l{}^{\mu_1 \ldots \mu_n, \sigma_1 \ldots \sigma_q} \tag{21}$$

where we have used the symmetrized delta

$$\hat{\delta}^{\mu_1\ldots\mu_n}_{\nu_1\ldots\nu_n} = \frac{1}{n!}\sum_{\sigma(1\ldots n)} \delta^{\mu_{\sigma(1)}}_{\nu_1}\ldots\,\delta^{\mu_{\sigma(n)}}_{\nu_n}$$

Note that in this new representation the labels (n,m) are naturally interpreted as two independent (left/right) spin quantities (number of symmetric indexes). It is easy to prove the following identities

$$\beta^{\mu}_{l}\Lambda_l{}^{\mu_1\ldots\mu_m,\nu_1\ldots\nu_n} = \sqrt{m}\,\eta^{\mu(\mu_1}\,\Lambda_l{}^{\mu_2\ldots\mu_m),\nu_1\ldots\nu_n} \tag{22}$$

$$\beta^{\mu\dagger}_{l}\Lambda_l{}^{\mu_1\ldots\mu_m,\nu_1\ldots\nu_n} = \sqrt{m+1}\,\Lambda_l{}^{\mu\mu_1\ldots\mu_m,\nu_1\ldots\nu_n} \tag{23}$$

$$\tilde{\beta}^{\nu}_{l}\Lambda_l{}^{\mu_1\ldots\mu_m,\nu_1\ldots\nu_n} = \sqrt{n}\,\Lambda_l{}^{\mu_1\ldots\mu_m,(\nu_1\ldots\nu_{n-1}}\eta^{\nu_n)\nu} \tag{24}$$

$$\tilde{\beta}^{\nu\dagger}_{l}\Lambda_l{}^{\mu_1\ldots\mu_m,\nu_1\ldots\nu_n} = \sqrt{n+1}\,\Lambda_l{}^{\mu_1\ldots\mu_m,\nu_1\ldots\nu_n\nu} \tag{25}$$

When symmetrizing indexes the normalization is always understood to be defined, for any tensor $T^{\mu_1\ldots\mu_n}$, by

$$T^{(\mu_1\ldots\mu_n)} = \frac{1}{n!}\sum_{\sigma} T^{\mu_{\sigma(1)}\ldots\mu_{\sigma(n)}}$$

An important subset of these states is formed by the fully traced ones

$$\Lambda_{l,n} = \Lambda_l{}^{\mu_1\ldots\mu_n}{}_{\mu_1\ldots\mu_n}$$

It is interesting to consider the sum of all these states

$$\begin{aligned}\sum_{n=0}^{\infty}\Lambda_{l,n} &= \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\beta_l^{\mu_1\,\dagger}\ldots\,\beta_l^{\mu_n\,\dagger}\tilde{\beta}^{\dagger}_{l\;\mu_1}\ldots\,\tilde{\beta}^{\dagger}_{l\;\mu_n}|\Xi\rangle \\ &= \sum_{n}\frac{(-1)^n}{n!}\left(\beta^{\dagger}_l\cdot\tilde{\beta}^{\dagger}_l\right)^n|\Xi\rangle = e^{-\beta^{\dagger}_l\cdot\tilde{\beta}^{\dagger}_l}|\Xi\rangle = |I_l\rangle\end{aligned} \tag{26}$$

The reason for the latter notation is that $|I_l\rangle$ acts as the identity while $*$-multiplying the states (20).

As in [24], let us introduce the string fields

$$\begin{aligned}A^{\mu}_{l} &= \beta^{\mu}_{l}|I_l\rangle = -\tilde{\beta}^{\mu\,\dagger}_{l}|I_l\rangle = \sum_{n=0}^{\infty}\sqrt{n}\,\eta^{\mu\mu_1}\Lambda_l{}^{\mu_2\ldots\mu_n}{}_{\mu_1\ldots\mu_n} \\ A^{\mu,\dagger}_{l} &= \beta^{\mu\,\dagger}_{l}|I_l\rangle = -\tilde{\beta}^{\mu}_{l}|I_l\rangle = \sum_{n=0}^{\infty}\sqrt{n+1}\,\Lambda_l{}^{\mu\mu_1\ldots\mu_n}{}_{\mu_1\ldots\mu_n}\end{aligned} \tag{27}$$

They obey[4]

$$[A^{\mu}_{l}, A^{\nu,\dagger}_{r}]_* = \delta_{lr}\eta^{\mu\nu}|I_l\rangle \tag{28}$$

[4]Note that the first part of (27) is nothing but the overlap condition for the Boundary state along the Neumann directions.

The action of these string field under left/right $*$-multiplication on string fields of the type 20 is as follows

$$A_l^{\mu} * \Lambda_l{}^{\mu_1\ldots\mu_m,\,\nu_1\ldots\nu_n} = \sqrt{m}\,\eta^{\mu(\mu_1}\,\Lambda_l{}^{\mu_2\ldots\mu_m),\,\nu_1\ldots\nu_n} \tag{29}$$

$$A_l^{\mu\,\dagger} * \Lambda_l{}^{\mu_1\ldots\mu_m,\,\nu_1\ldots\nu_n} = \sqrt{m+1}\,\Lambda_l{}^{\mu\mu_1\ldots\mu_m,\,\nu_1\ldots\nu_n} \tag{30}$$

$$\Lambda_l{}^{\mu_1\ldots\mu_m,\,\nu_1\ldots\nu_n} * A_l^{\nu\,\dagger} = \sqrt{n}\,\Lambda_l{}^{\mu_1\ldots\mu_m,\,(\nu_1\ldots\nu_{n-1}}\,\eta^{\nu_n)\,\nu} \tag{31}$$

$$\Lambda_l{}^{\mu_1\ldots\mu_m,\,\nu_1\ldots\nu_n} * A_l^{\nu} = \sqrt{n+1}\,\Lambda_l{}^{\mu_1\ldots\mu_m,\,\nu_1\ldots\nu_n\nu} \tag{32}$$

Comparing (29) with (22) we see that the left $*$-multiplication by A_l^μ corresponds to applying β_l^μ to the string field. Similarly the left $*$-multiplication by $A_l^{\mu\,\dagger}$ corresponds to applying $\beta_l^{\mu\,\dagger}$, the right $*$-multiplication by $A_l^{\nu\,\dagger}$ corresponds to applying $\tilde{\beta}_l^\nu$ and the right $*$-multiplication by A_l^μ corresponds to applying $\tilde{\beta}_l^{\nu\,\dagger}$. We recall that A and $A^\dagger$ should not be confused with creation and annihilation operators, they are just string fields.

We can now define the half string number operator as follow

$$N_l = A_l^{\mu\dagger} * A_{l,\,\mu} \tag{33}$$

Its left/right action computes the left/right spin of a string field for given l

$$\begin{aligned} N_l * \Lambda_l{}^{\mu_1\ldots\mu_n,\,\nu_1\ldots\nu_m} &= n\,\Lambda_l{}^{\mu_1\ldots\mu_n,\,\nu_1\ldots\nu_m} \\ \Lambda_l{}^{\mu_1\ldots\mu_n,\,\nu_1\ldots\nu_m} * N_l &= m\,\Lambda_l{}^{\mu_1\ldots\mu_n,\,\nu_1\ldots\nu_m} \end{aligned}$$

As we have already noticed, the split left/right structure we are dealing with is very reminiscent of the holomorphic/antiholomorphic structure one encounters in the first quantization of the closed string. This correspondence can be made more precise. Let us define

$$\mathcal{A}_{l,\,\mu} = \sqrt{l}\,A_{l,\,\mu}, \qquad \mathcal{A}^\dagger_{l,\,\mu} = \sqrt{l}\,A^\dagger_{l,\,\mu} \tag{34}$$

We get

$$\left[\mathcal{A}_{l,\,\mu},\,\mathcal{A}^\dagger_{l,\,\nu}\right]_* = l\,\eta_{\mu\nu}\,|I_l\rangle$$

The extension of the above to multiple half string excitations is straightforward. We define sequences of natural numbers $\mathbf{n} = n_1, n_2, \ldots$, where the label l in n_l corresponds to the oscillator type. For every type l half string oscillator we will have a collection of symmetric Lorentz indexes $\mu_1^l, \mu_2^l, \ldots, \mu_{n_l}^l$. Then for any two sequences $\mathbf{n}$ and $\mathbf{m}$ we define (generalizing (20)) the states:

$$\Lambda^{\{\mu_1\ldots\mu_{\mathbf{n}}\},\,\{\nu_1\ldots\nu_{\mathbf{m}}\}} = \prod_{l,r=1}^{\infty} \frac{(-1)^{m_r}}{\sqrt{n_l!m_r!}}\,\beta_l^{\mu_1^l\,\dagger}\ldots\beta_l^{\mu_{n_l}^l\,\dagger}\,\tilde{\beta}_r^{\nu_1^r\,\dagger}\ldots\tilde{\beta}_r^{\nu_{m_r}^r\,\dagger}|\Xi\rangle \tag{35}$$

The complete star algebra is then

$$\Lambda^{\{\mu_1^l\ldots\mu_{n_l}^l\},\,\{\nu_1^l\ldots\nu_{m_l}^l\}} * \Lambda_{\{\rho_1^l\ldots\rho_{p_l}^l\}}{}^{\{\sigma_1^l\ldots\sigma_{q_l}^l\}} = \prod_l \delta_{m_l,p_l}\,\hat{\delta}^{\nu_1^l\ldots\nu_{m_l}^l}_{\rho_1^l\ldots\rho_{p_l}^l}\,\Lambda^{\{\mu_1^l\ldots\mu_{n_l}^l\},\,\{\sigma_1^l\ldots\sigma_{q_l}^l\}} \tag{36}$$

This algebra contains a lot of orthogonal projectors, the simplest ones being given by Lorentz traces of left/right symmetric states

$$\begin{aligned} \mathrm{tr}(\Lambda_{\mathbf{n}}) &= \Lambda_{\{\mu_1^l \dots \mu_{n_l}^l\}}{}^{\{\mu_1^l \dots \mu_{n_l}^l\}} \\ \mathrm{tr}(\Lambda_{\mathbf{n}}) * \mathrm{tr}(\Lambda_{\mathbf{m}}) &= \delta_{\mathbf{nm}}\, \mathrm{tr}(\Lambda_{\mathbf{n}}) \end{aligned}$$

The sum of these states is the identity string field

$$\begin{aligned} \sum_{\mathbf{n}=\mathbf{0}}^{\infty} \mathrm{tr}(\Lambda_{\mathbf{n}}) &= \sum_{n_1} \dots \sum_{n_\infty} \prod_l \frac{(-1)^{n_l}}{n_l!} \beta_l^{\mu_1^l\,\dagger} \dots \beta_l^{\mu_{n_l}^l\,\dagger} \tilde{\beta}^{\dagger}_{l\,\mu_1^l} \dots \tilde{\beta}^{\dagger}_{l\,\mu_{n_l}^l} |\Xi\rangle \\ &= \prod_l \sum_{n_l} \frac{(-1)^{n_l}}{n_l!} \left(\beta_l^\dagger \cdot \tilde{\beta}_l^\dagger\right)^{n_l} |\Xi\rangle = e^{-\sum_l \beta_l^\dagger \cdot \tilde{\beta}_l^\dagger} |\Xi\rangle \\ &= e^{-\sum_l \beta_l^\dagger \cdot C \beta_l^\dagger} |\Xi\rangle = e^{-\frac{1}{2} s^\dagger \cdot C s^\dagger} |\Xi\rangle = |I\rangle \end{aligned} \tag{37}$$

This sum is nothing but the tensor product of the states $|I_l\rangle$ for all l. Using the latter identity it is immediate to generalize the definitions of the states A and $A^\dagger$ by replacing $|I_l\rangle$ with $|I\rangle$. The only change is

$$\left[A_{l,\,\mu},\, A^\dagger_{r,\,\nu}\right]_* = \delta_{lr}\, \eta_{\mu\nu}\, |I\rangle$$

The left/right action of this operators is

$$A_l^{\mu} * \Lambda_r^{\mu_1 \dots \mu_m,\, \nu_1 \dots \nu_n} = \delta_{lr} \sqrt{m}\, \eta^{\mu(\mu_1}\, \Lambda_l{}^{\mu_2 \dots \mu_m),\, \nu_1 \dots \nu_n} \tag{38}$$

$$A_l^{\mu\,\dagger} * \Lambda_r^{\mu_1 \dots \mu_m,\, \nu_1 \dots \nu_n} = \delta_{lr} \sqrt{m+1}\, \Lambda_l{}^{\mu\mu_1 \dots \mu_m,\, \nu_1 \dots \nu_n} \tag{39}$$

$$\Lambda_r^{\mu_1 \dots \mu_m,\, \nu_1 \dots \nu_n} * A_l^{\nu\,\dagger} = \delta_{lr} \sqrt{n}\, \Lambda_l{}^{\mu_1 \dots \mu_m,\, (\nu_1 \dots \nu_{n-1}}\, \eta^{\nu_n)\,\nu} \tag{40}$$

$$\Lambda_r^{\mu_1 \dots \mu_m,\, \nu_1 \dots \nu_n} * A_l^{\nu} = \delta_{lr} \sqrt{n+1}\, \Lambda_l{}^{\mu_1 \dots \mu_m,\, \nu_1 \dots \nu_n \nu} \tag{41}$$

To fit the closed string formalism we define the operators $\mathcal{A}_{l,\,\mu}$ as above, so that

$$\left[\mathcal{A}_{l,\,\mu},\, \mathcal{A}^\dagger_{r,\,\nu}\right]_* = l\, \delta_{lr}\, \eta_{\mu\nu}\, |I\rangle$$

Moreover we define the level ($*$-multiplication) operator

$$\mathcal{N} = \sum_{l=1}^{\infty} \mathcal{A}^\dagger_{l\,\mu} * \mathcal{A}_l^{\mu} \tag{42}$$

which counts the level (in the sense of closed string theory) in the holomorphic sector when acting on the left and in the antiholomorphic sector when acting on the right. We can therefore define closed string α operators in the traditional sense by means of the left/right actions on generic string states ψ of the type (35):

$$\begin{aligned} \alpha_{l,\,\mu}\, \psi &= \mathcal{A}_{l,\,\mu} * \psi \\ \alpha^\dagger_{l,\,\mu}\, \psi &= \mathcal{A}^\dagger_{l,\,\mu} * \psi \\ \tilde{\alpha}_{l,\,\mu}\, \psi &= \psi * \mathcal{A}^\dagger_{l,\,\mu} \\ \tilde{\alpha}^\dagger_{l,\,\mu}\, \psi &= \psi * \mathcal{A}_{l,\,\mu} \end{aligned}$$

As expected we get

$$\begin{aligned}
\left[\alpha_{l,\,\mu}, \alpha^\dagger_{r,\,\nu}\right] &= l\,\delta_{lr}\eta_{\mu\nu} \\
\left[\tilde{\alpha}_{l,\,\mu}, \tilde{\alpha}^\dagger_{r,\,\nu}\right] &= l\,\delta_{lr}\eta_{\mu\nu} \\
[\alpha, \tilde{\alpha}] &= 0
\end{aligned}$$

From this one can directly define the holomorphic and anti-holomorphic Virasoro algebras of the closed string as *open* star subalgebras.

To conclude this discussion, it is worth noting that, even if the above construction is actually dependent on the "gauge" choice of the oscillators basis $(\beta_l, \tilde{\beta}_l)$, still one can define the (closed string) level from a given string field ψ in the $O(\infty)$ star rotation invariant way (see [21])

$$n_L(\psi) = \frac{\langle \Omega\psi | \mathcal{N} * \psi\rangle}{\langle \Omega\psi|\psi\rangle} \tag{43}$$

$$n_R(\psi) = \frac{\langle \Omega\psi | \psi * \mathcal{N}\rangle}{\langle \Omega\psi|\psi\rangle} \tag{44}$$

where Ω is the twist transformation which, combined with *bpz*, gives

$$bpz|\Omega\Lambda^{\{\mu_1\ldots\mu_\mathbf{n}\},\,\{\nu_1\ldots\nu_\mathbf{m}\}}\rangle = \langle\Lambda^{\{\nu_1\ldots\nu_\mathbf{m}\},\,\{\mu_1\ldots\mu_\mathbf{n}\}}| \tag{45}$$

See also the comments at the end of section 6 for more on this issue.

3.2 Representation in terms of Laguerre polynomials

It is important to clarify the open string nature of the $\beta, \tilde{\beta}$ operators. When applied to the vacuum they turn out to be very well known objects, which have already made their appearance in SFT. The corresponding states give rise to an algebra defined by means of Laguerre polynomials.

Let us consider a particular state of the type (35):

$$\frac{1}{\sqrt{n!m!}}(\beta_k^\dagger)^n(-\tilde{\beta}_l^\dagger)^m|0_c\rangle$$

where, for simplicity, we have dropped the Lorentz index μ. Written out explicitly in open string language the state takes the form

$$\begin{aligned}
&(\beta_k^\dagger)^n(-\tilde{\beta}_l^\dagger)^m|0_c\rangle = \langle\xi_l C\omega^2(a^\dagger + Sa)\rangle^m\langle\xi_k\omega^2(a^\dagger + Sa)\rangle^n e^{-\frac{1}{2}a^\dagger S a^\dagger}|0\rangle \\
&= e^{-\frac{1}{2}a^\dagger S a^\dagger}\langle\xi_l C\omega^2((1-S^2)a^\dagger + Sa)\rangle^m\langle\xi_k\omega^2((1-S^2)a^\dagger + Sa)\rangle^n|0\rangle \\
&= e^{-\frac{1}{2}a^\dagger S a^\dagger}\sum_{p=0}^{m}\begin{pmatrix} m \\ p\end{pmatrix}\langle\xi_l C a^\dagger\rangle^{m-p}\langle\xi_l C\omega^2 S a\rangle^p\langle\xi_k a^\dagger\rangle^n|0\rangle = *
\end{aligned}$$

This is due to the fact that the contractions implicit in the reordering of the above terms all vanish

$$\langle\xi_k C\frac{T}{1-T^2}\xi_k\rangle = 0, \qquad \langle\xi_l C\frac{T}{1-T^2}\xi_l\rangle = 0$$

In the next passages we will also need

$$\langle \xi_k C \frac{T}{1-T^2} \xi_l \rangle = 0$$

and the definition

$$\kappa_{kl} = \langle \xi_k \frac{T}{1-T^2} \xi_l \rangle \tag{46}$$

Proceeding with the algebraic manipulations, and assuming from now on $n \leq m$,

$$\begin{aligned}
* &= e^{-\frac{1}{2}a^\dagger S a^\dagger} \sum_{p=0}^{n} \begin{pmatrix} m \\ p \end{pmatrix} \langle \xi_l C a^\dagger \rangle^{m-p} \langle \xi_k a^\dagger \rangle^{n-p} \frac{n!}{(n-p)!} (\kappa_{kl})^p |0\rangle \\
&= e^{-\frac{1}{2}a^\dagger S a^\dagger} \sum_{p=0}^{n} \frac{m!n!}{(m-k)!(n-k)!k!} \langle \xi_l C a^\dagger \rangle^{m-n} \langle \xi_l C a^\dagger \rangle^{n-p} \langle \xi_k a^\dagger \rangle^{n-p} \frac{n!}{(n-p)!} (\kappa_{kl})^p |0\rangle \\
&= (\kappa_{kl})^n \, n! \, Y_l^{m-n} L_n^{m-n}(-\frac{X_k Y_l}{\kappa_{kl}}) |\Xi\rangle
\end{aligned} \tag{47}$$

where $X_k = \langle \xi_k a^\dagger \rangle$, $Y_l = \langle \xi_l C a^\dagger \rangle$ and

$$L_n^{m-n}(z) = \sum_{p=0}^{m} \begin{pmatrix} m \\ n-p \end{pmatrix} \frac{(-z)^p}{p!}, \qquad n \leq m$$

Therefore, eventually

$$\frac{1}{\sqrt{n!m!}} (\beta_k^\dagger)^n (-\tilde{\beta}_l^\dagger)^m |0_c\rangle = \sqrt{\frac{n!}{m!}} (\kappa_{kl})^n Y_l^{m-n} L_n^{m-n}(-\frac{X_k Y_l}{\kappa_{kl}}) |\Xi\rangle \tag{48}$$

For $n = m$ these states have already appeared in the literature. They have been interpreted as D–brane solutions of vacuum SFT, [27, 28].

4 Closed string states: zero momentum

Let us return to the main problem. We have seen so far a correspondence between star algebra operators and closed string creation and annihilation operators. The relevant question is now: what are the (open string) string fields that correspond to closed string Fock states created under the above correspondence? By closed string states we mean both off-shell and on-shell states. For instance a graviton state with momentum k in closed string theory is given by

$$\theta_{\mu\nu} \alpha_1^{\mu\dagger} \alpha_1^{\nu\dagger} |0_c, k\rangle \tag{49}$$

where $|0_c, k\rangle$ is the closed string vacuum with momentum k, and $\theta_{\mu\nu}$ is the polarization. This state is on-shell when $k^2 = 0$ and $\theta_{\mu\nu} k^\nu = \theta_{\mu\nu} k^\mu = 0$. When the latter conditions are not satisfied the graviton is off-shell. Off-shell states are not so generic as one might think, they must satisfy precise conditions: they must have definite momentum (i.e. the holomorphic and antiholomorphic momenta must be equal) and they must be level–matched. Usually in dealing with closed strings, these two conditions are so obvious that they are understood, but, as we shall see, under the correspondence with open strings, they become significant and select a very precise class of string fields, the projectors. In this and the next sections we will concentrate on *off–shell* closed string states. In the present section, to start with, we consider only zero momentum states. Non-zero momentum states will be introduced in the next section.

It is evident from the above that there is a correspondence between (zero momentum) states in the Fock space of the closed string theory and open string fields of the type (35). The question is: what are the string fields that correspond to off–shell states in the closed string theory?

To start with we (formally) define Virasoro generators $L_n, \tilde{L}_n$ using the $\beta, \tilde{\beta}$ operators in the usual way. Then using L_0 and $\tilde{L}_0$ we define the mass operator and the level matching condition by means of

$$N_L = \sum_{n=1}^{\infty} n\, \beta_n^{\dagger} \cdot \beta_n, \qquad N_R = \sum_{n=1}^{\infty} n\, \tilde{\beta}_n^{\dagger} \cdot \tilde{\beta}_n, \tag{50}$$

Off-shell states are characterized in particular by the condition $N_R = N_L = N$, where the number N specifies the level of the state. They are in general combination of monomials of β and $\tilde{\beta}$ applied to the vacuum with arbitrary coefficients. The statement we wish to prove is the following:

Closed string Fock space states of given level, satisfying the level matching condition, can always be decomposed into combinations of states of the type (35) that are $*$*-algebra projectors.* Loosely speaking, level–matched states of the closed string Fock space come from star algebra projectors.

The relevant states of the Fock space must form representations of the Lorentz group. However this is not a significant issue here since the corresponding tensors are saturated with arbitrary polarizations. Let us write out the states (35) in the more explicit form

$$\sim \beta_1^{\mu_1^1\dagger} \beta_1^{\mu_2^1\dagger} \ldots \beta_1^{\mu_{n_1}^1\dagger} \ldots \beta_l^{\mu_1^l\dagger} \beta_l^{\mu_2^l\dagger} \ldots \beta_l^{\mu_{n_l}^l\dagger} \tilde{\beta}_1^{\nu_1^1\dagger} \tilde{\beta}_1^{\nu_2^1\dagger} \ldots \tilde{\beta}_1^{\nu_{m_1}^1\dagger} \tilde{\beta}_r^{\nu_1^r\dagger} \ldots \tilde{\beta}_r^{\mu_2^r\dagger} \ldots \tilde{\beta}_r^{\mu_{n_r}^r\dagger} |\Xi\rangle \tag{51}$$

We can rewrite this in the form

$$\begin{aligned}\sim (\beta_1^{0\dagger})^{n_{0,1}} (\beta_1^{1\dagger})^{n_{1,1}} \ldots (\beta_1^{25\dagger})^{n_{25,1}} (\beta_2^{0\dagger})^{n_{0,2}} \ldots (\beta_2^{25\dagger})^{n_{25,2}} \ldots (\beta_l^{0\dagger})^{n_{0,l}} \ldots (\beta_l^{25\dagger})^{n_{25,l}} \\ (\tilde{\beta}_1^{0\dagger})^{m_{0,1}} \ldots (\tilde{\beta}_1^{25\dagger})^{m_{25,1}} \ldots (\tilde{\beta}_r^{0\dagger})^{m_{0,r}} \ldots (\tilde{\beta}_r^{25\dagger})^{m_{25,r}} |\Xi\rangle\end{aligned} \tag{52}$$

It is evident that the first family of terms coincides with the second one provided that $\sum_\mu n_{\mu,i} = n_i$, $i = 1, \ldots, l$ and $\sum_\nu m_{\nu,j} = m_j$ with $j = 1, \ldots, r$. In the sequel we preferably use the form (52).

A state belonging to the closed string Hilbert space, even though it is not on–shell, satisfies the level matching condition, i.e. $N_L = \sum_{i=1}^{l} i n_i$ and $N_R = \sum_{j=1}^{r} j m_j$ coincide, $N_L = N_R = N$. It is a combination with arbitrary coefficients of all the states of the type (35) satisfying this condition. However, to start with, let us ignore the complication of the Lorentz indexes and drop the index μ altogether (i.e. we pretend there is only one space–time direction). In this case the states can be fully identified by the symbols $\Lambda_{\mathbf{n},\mathbf{m}}$, because they are completely specified if we know the two sequences $\mathbf{n}$ and $\mathbf{m}$

$$\Lambda_{\mathbf{n},\mathbf{m}} = \prod_{l,r=1}^{\infty} \frac{(-1)^{m_r}}{\sqrt{n_l! m_r!}} (\beta_l^\dagger)^{n_l} (\tilde{\beta}_r^\dagger)^{m_r} |\Xi\rangle \tag{53}$$

One has

$$\Lambda_{\mathbf{n},\mathbf{m}} * \Lambda_{\mathbf{p},\mathbf{q}} = \delta_{\mathbf{m},\mathbf{p}} \Lambda_{\mathbf{n},\mathbf{q}} \tag{54}$$

where $\delta_{\mathbf{m},\mathbf{p}} = \prod_{l,r} \delta(m_l, p_r)$. An off–shell closed string state will therefore be represented by a superposition of states $\Lambda_{\mathbf{n},\mathbf{m}}$ in the closed string Fock space. Setting $N_L = N_R = N$, there will be a leading state with $\mathbf{n} = \mathbf{m} = (0, ..., 0, 1, 0, ...)$ with one single non-vanishing entry equal to 1 in the N–th position (which corresponds to one single operator $\beta_N^\dagger$ and one single operator $\tilde{\beta}_N^\dagger$ of highest order applied to the vacuum), which we denote simply by $\Lambda_{N,N}$. We refer to the full set of states as the family $\mathcal{F}_{(N,N)}$. We have now the problem of dealing with all the other states $\Lambda_{\mathbf{n},\mathbf{m}}$ in the family different from $\Lambda_{N,N}$. To this end we recall that any sequence $\mathbf{n}$ naturally represents a partition of N (i.e. $(n_1, n_2, ...)$ is read as the partition such that $\sum_i in_i = N$, the sequence corresponding to the partition that contaisn only N itself will be denoted by $\mathbf{n}_N$). In order to be able to deal with all these possibilities, we introduce a partial ordering among the sequences $\mathbf{n}$: we say that $\mathbf{n} \geq \mathbf{n}'$ iff the rightmost nonzero n_i and $n'_{i'}$ in $\mathbf{n}$ and $\mathbf{n}'$, respectively, are such that $i \geq i'$ and, if $i = i'$, $n_i \geq n'_i$, and, if also $n_i = n'_i$ the second rightmost numbers in $\mathbf{n}$ and $\mathbf{n}'$ are to be considered, and so on. Among the states of the family let us pick those of the form $\Lambda_{\mathbf{n},\mathbf{n}}$, among which is $\Lambda_{N,N}$, and call them *principal*. All the other states are *descendants*. Given a principal state we can define a subfamily as follows: it contains the principal state $\Lambda_{\mathbf{n},\mathbf{n}}$ as well as $\Lambda_{\mathbf{n},\mathbf{n}_d}$ and $\Lambda_{\mathbf{n}_d,\mathbf{n}}$, where $\mathbf{n}_d$ is any sequence $\leq \mathbf{n}$ but not identical to it. This will be referred to as the $(\mathbf{n}, \mathbf{n})$ subfamily. All the subfamilies are naturally ordered, the highest one being (N, N).

Let us start from $\Lambda_{N,N}$ and the relative subfamily. On the basis of (36), $\Lambda_{N,N}$ defines a projector. Then we consider $\Lambda_{N,\mathbf{m}_d}$ where $\mathbf{m}_d$ is any $\mathbf{m}_d \leq \mathbf{m}_N$ but not identical to $\mathbf{m}_N$ (for instance one $\mathbf{m}_d$ is $(1, 0, ..., 0, 1, 0, ...)$ where the nonzero entries are in the first and $N-1$–th position, which corresponds to the product $\tilde{\beta}_1^\dagger \tilde{\beta}_{N-1}^\dagger$). $\Lambda_{N,\mathbf{m}_d}$ is not a star–projector, but the sum $\Lambda_{N,N} + a\Lambda_{N,\mathbf{m}_d}$ is, for any arbitrary constant a. Indeed, using (54), one gets

$$(\Lambda_{N,N} + a\Lambda_{N,\mathbf{m}_d}) * (\Lambda_{N,N} + a\Lambda_{N,\mathbf{m}_d}) = (\Lambda_{N,N} + a\Lambda_{N,\mathbf{m}_d})$$

If we take a combination of the corresponding closed string states with arbitrary coefficient we see that the combination will contain, beside $\Lambda_{N,N}$, any state $\Lambda_{N,\mathbf{m}_d}$ with an arbitrary coefficient in front of it. The same can be said of the states $\Lambda_{\mathbf{n}_d,N}$, with $\mathbf{n}_d < n_N$. Now we can pass to the second subfamily and play the same game, the role of $\Lambda_{N,N}$ being played by the relevant $\Lambda_{\mathbf{n},\mathbf{n}}$. We can do the same for any subfamily and therefore exhaust the full set of states, showing that each one of them can be inserted into a distinct projector. We can claim therefore that each distinct term in the family $\mathcal{F}_{(N,N)}$ corresponds to a distinct projector.

Extending the previous proof to the states (35) is straightforward. Remembering (52) the latter can be written in terms of a multi–sequences $\mathbf{N} = (\mathbf{n}_0, \mathbf{n}_1, ..., \mathbf{n}_{D-1})$.

$$\Lambda_{\mathbf{N},\mathbf{M}} = \prod_{\mu=0}^{D-1} \prod_{l,r} \frac{(-1)^{n_{\mu,r}}}{\sqrt{n_{\mu,l}! m_{\mu,r}!}} (\beta_l^{\mu\dagger})^{n_{\mu,l}} (\tilde{\beta}_r^{\mu\dagger})^{n_{\mu,r}} |\Xi\rangle \tag{55}$$

In this new notation (36) becomes

$$\Lambda_{\mathbf{N},\mathbf{M}} * \Lambda_{\mathbf{P},\mathbf{Q}} = \delta_{\mathbf{MP}} \Lambda_{\mathbf{N},\mathbf{Q}} \tag{56}$$

where $\delta_{\mathbf{MP}} = \prod_\mu \prod_{l,r} \delta(m_{\mu,l}, p_{\mu,r})$.

What one has to do is first of all to define an ordering for the multi–sequences $\mathbf{N}$. This is easy to accomplish, starting for instance from the ordering of the $\mathbf{n}_0$ sequence, then looking at the ordering of $\mathbf{n}_1$ and so on, and then proceeding as above. Moreover $N_L = \sum_\mu \sum_i i n_{\mu,i}$ and $N_R = \sum_\mu \sum_j j m_{\mu,j}$. One can define families characterized by the properties of containing all the states with $N_L = N_R = N$. Likewise one can define principal states (having $\mathbf{N} = \mathbf{M}$) and subfamilies as above, and extend the previous proof to such generalized families.

5 Closed string states: the momentum eigenfunction

Every closed string state is constructed by tensoring a Fock space state with a momentum eigenfunction, which, in the coordinate representation, is the plane wave e^{ikx}. The momentum k comes in equal parts from the left and the right-handed sectors. The purpose of this section is to explain where this factor comes from in the open-closed correspondence of the previous sections. Once more we shall see that the origin of this factor is a star algebra projector.

To start with we remark that in the previous sections all developments were based on the sliver projector, which is translationally invariant in all directions. If we want to find a momentum dependence we have therefore to start from projectors that are not translationally invariant. To this end we will use the *lump* projector. Let us recall what it is. The lump projector was engineered to represent a lower dimensional brane (Dk-brane) in VSFT, therefore it has $(25 - k)$ transverse space directions along which translational invariance is broken. Accordingly one splits the three string vertex into the tensor product of the perpendicular part and the parallel part

$$|V_3\rangle = |V_{3,\perp}\rangle \otimes |V_{3,\parallel}\rangle \tag{57}$$

The parallel part is the same as in the sliver case while the perpendicular part is modified as follows. Following [29], we denote by $x^{\bar{\mu}}, p^{\bar{\mu}}$, $\bar{\mu} = 1, ..., k$ the coordinates and momenta in the transverse directions and introduce the canonical zero modes oscillators

$$a_0^{(r)\bar{\mu}} = \frac{1}{2}\sqrt{b}\hat{p}^{(r)\bar{\mu}} - i\frac{1}{\sqrt{b}}\hat{x}^{(r)\bar{\mu}}, \qquad a_0^{(r)\bar{\mu}\dagger} = \frac{1}{2}\sqrt{b}\hat{p}^{(r)\bar{\mu}} + i\frac{1}{\sqrt{b}}\hat{x}^{(r)\bar{\mu}}, \tag{58}$$

where b is a free parameter. Denoting by $|\Omega_b\rangle$ the oscillator vacuum ($a_0^{\bar{\mu}}|\Omega_b\rangle = 0$), in this new basis the three string vertex is given by

$$|V_{3,\perp}\rangle' = K\, e^{-E'}|\Omega_b\rangle \tag{59}$$

K being a suitable constant and

$$E' = \frac{1}{2} \sum_{r,s=1}^{3} \sum_{M,N \geq 0} a_M^{(r)\bar{\mu}\dagger} V'^{rs}_{MN} a_N^{(s)\bar{\nu}\dagger} \eta_{\bar{\mu}\bar{\nu}} \tag{60}$$

where M, N denote the couple of indexes $\{0, m\}$ and $\{0, n\}$, respectively. The coefficients V'^{rs}_{MN} are given in Appendix B of [29]. The new Neumann coefficients matrices V'^{rs} satisfy the same relations as the V^{rs} ones. In particular one can introduce the matrices $X'^{rs} = CV'^{rs}$, where $C_{NM} = (-1)^N \delta_{NM}$. The lump projector $|\Xi'_k\rangle$ has the form (6) with S along the parallel directions, while $|0\rangle$ is replaced by $|\Omega_b\rangle$ and S is replaced by S' along the perpendicular ones. Here $S' = CT'$ and T' has the same form as T eq.(7) with X replaced by X'. The normalization constant $\mathcal{N}'$ is defined in a way analogous to $\mathcal{N}$. The diagonal representation of X'^{rs} is summarized in Appendix.

We now repeat the same steps as in section 2 in order to define the operators β_N and $\tilde{\beta}_N$. We are of course interested in particular in the zero mode. Let us consider a lump projector $|\Xi'\rangle$ and concentrate on a transverse direction, say μ. We introduce, in a way analogous to section 2, left and right Fock space projectors ρ'_L and ρ'_R, with the same properties as ρ_L and ρ_R, which will not be repeated here. These operators can be diagonalized (see Appendix). Differently from the sliver case here we have both a continuous and discrete spectrum. The continuous spectrum is spanned by a real number k, $-\infty < k < +\infty$. The discrete spectrum can be written in terms of a positive real number η and by $-\eta$ (η is related to the parameter b, see Appendix). The corresponding eigenvectors are denoted $V_N(k), V_N(\eta), V_N(-\eta)$. Their completeness relation can be found in eq.(102). Using this basis, S' and $\omega' = 1/\sqrt{1 - T'^2}$, we write down the analog of formula (10). The operators ${s'_M}^{\mu}$ satisfy the Heisenberg algebra

$$[{s'_M}^{\mu}, {s'_N}^{\nu\dagger}] = \delta_{MN} \eta^{\mu\nu} \tag{61}$$

and annihilate the lump projector $|\Xi'\rangle$.

In the diagonal representation ρ'_L and ρ'_R take the following form:

$$\rho'_R = \int_0^\infty |k\rangle dk \langle k| + |\eta\rangle\langle\eta|, \qquad \rho'_L = \int_{-\infty}^0 |k\rangle dk \langle k| + |-\eta\rangle\langle -\eta|$$

where $|k\rangle, |\eta\rangle$ and $|-\eta\rangle$ form a basis such that $\langle k|V_N\rangle = V_N(k), \langle\eta|V_N\rangle = V_N(\eta)$ and $\langle -\eta|V_N\rangle = V_N(-\eta)$.

In analogy with what we did in section 2 in the sliver case, we define now vectors ξ' such that $\rho'_L \xi' = \xi'$ and $\rho'_R \xi' = 0$. There exists a complete basis of ξ'_N ($N = 0, 1, 2, \ldots$) that satisfy these conditions and are orthonormal in the sense that

$$\langle \xi'_N | \frac{1}{1 - T'^2} | \xi'_M \rangle = \delta_{NM} \tag{62}$$

Then we define

$${\xi'_N}^L = \omega' \xi'_N, \qquad {\xi'_N}^R = \omega' C \xi'_N \tag{63}$$

When projected on the continuous basis $|k\rangle$ and the discrete one $|\eta\rangle$, $|-\eta\rangle$, they give rise to a vector of functions and numbers ${\xi'_N}^L(k)$, ${\xi'_N}^L(-\eta)$ and ${\xi'_N}^R(k)$, respectively, ${\xi'_N}^R(\eta)$, which satisfy the orthogonality relations

$$\sum_{N=0}^{\infty}\Big({\xi'_N}^L(k){\xi'_N}^L(k') + {\xi'_N}^R(k){\xi'_N}^R(k')\Big) = \delta(k,k') \tag{64}$$

$$\sum_{N=0}^{\infty}\Big({\xi'_N}^L(\eta){\xi'_N}^L(\eta) + {\xi'_N}^R(\eta){\xi'_N}^R(\eta)\Big) = 1 \tag{65}$$

For later purposes it is convenient to choose the basis in such a way that

$${\xi'_0}^L(-k) = {\xi'_0}^R(k) = 0, \qquad {\xi'_n}^R(\eta) = {\xi'_n}^L(-\eta) = 0, \quad k>0, \quad n=1,2,... \tag{66}$$

This will allow us to separate the continuous from the discrete spectrum–dependent objects.

Now, in analogy with section 2, we define the coefficients

$$b'_{NM} = \langle {\xi'_N}^L|V_M\rangle, \qquad \tilde{b}'_{NM} = \langle {\xi'_N}^R|V_M\rangle \tag{67}$$

and the operators

$$\beta^\mu_N = \sum_{M=0}^{\infty} b'_{NM}{s'_M}^\mu, \qquad \tilde{\beta}^\mu_N = -\sum_{M=0}^{\infty}\tilde{b}'_{NM}{s'_M}^\mu \tag{68}$$

Needless to say they satisfy the algebra

$$[\beta^\mu_M, \beta^{\nu\dagger}_N] = \eta^{\mu\nu}\delta_{MN}, \qquad [\tilde{\beta}^\mu_M, \tilde{\beta}^{\nu\dagger}_N] = \eta^{\mu\nu}\delta_{MN}, \tag{69}$$

while the other commutators vanish. Here μ, ν are any two transverse directions. We remark that we have dropped the prime from the β's, in order to use a uniform notation for the closed string operators. However it should be kept in mind that the $\beta_n, \tilde{\beta}_n$ operators are different from those defined in section 2. We will return to this point later on.

We are now ready to discuss the momentum eigenstates. To start with let us define the state

$$|p,q\rangle = \frac{1}{K}\sqrt{\frac{b}{2\pi}}e^{-\frac{b}{4}(p^2+q^2)+\sqrt{b}(q\beta_0^\dagger+p\tilde{\beta}_0^\dagger)-\frac{1}{2}({\beta_0^\dagger}^2+{\tilde{\beta}_0^\dagger}{}^2)}|0'_c\rangle \tag{70}$$

where p and q are real numbers, K is the constant that appear in eq.(59), and $|0'_c\rangle$ stands for the lump $|\Xi'\rangle$. For notational simplicity we drop Lorentz indexes. They can be straightforwardly reinserted when needed. We remark the β_0 and $\tilde{\beta}_0$ are not self-adjoint, therefore they cannot be interpreted as momenta, not even as half–momenta. We define the selfadjoint half–momenta operators as

$$\hat{q} = \frac{1}{2\sqrt{b}}(\beta_0+\beta_0^\dagger), \qquad \hat{p} = \frac{1}{2\sqrt{b}}(\tilde{\beta}_0+\tilde{\beta}_0^\dagger) \tag{71}$$

It is easy to verify that the states (70) satisfy

$$\hat{p}|p,q\rangle = \frac{p}{2}|p,q\rangle, \qquad \hat{q}|p,q\rangle = \frac{q}{2}|p,q\rangle$$

Now we are going to compute the star product of two such $|p, q\rangle$ states. The formula for the star product is considerably simplified if (like in our case) the vacuum state is $|\Xi'\rangle$ instead of the ordinary open string vacuum. In fact the vertex can be written in the following shorthand form

$$\langle V_3'| = K \;_{123}\langle \Xi'| e^{-\frac{1}{2}s^{(a)} C\hat{V}^{ab} C s^{(b)}} \tag{72}$$

where $a, b = 1, 2, 3$ label the three strings and

$$C\hat{V} = \begin{pmatrix} 0 & \rho_L' & \rho_R' \\ \rho_R' & 0 & \rho_L' \\ \rho_L' & \rho_R' & 0 \end{pmatrix} \tag{73}$$

Another prescription one must introduce is the bpz transformation for the zero modes[5]

$$bpz(\beta_0) = +\tilde{\beta}_0^\dagger \tag{74}$$

The star product of two states like (70) can now be straightforwardly computed, because, due to the choice of basis (66), the zero mode calculation decouples from the rest. The only caution one must exercise is introducing a regulator since a naive calculation would bring about infinite factors. This is easily accomplished by multiplying the term $(\beta_0^{\dagger\,2} + \tilde{\beta}_0^{\dagger\,2})$ in the exponent of (70) by a parameter ϵ and eventually taking the limit $\epsilon \to 1$. The result is as follows

$$|p_1, q_1\rangle * |p_2, q_2\rangle = \lim_{\epsilon \to 1} \; C(\epsilon, q_1, p_2)\, |p_1, q_2\rangle$$

where

$$C(\epsilon, q_1, p_2) = \frac{1}{2}\sqrt{\frac{b}{\pi(1-\epsilon)}}\, e^{-\frac{b(q_1 - p_2)^2}{4(1-\epsilon)}}$$

The limit for $\epsilon \to 1$ of this expression is $\delta(q_1 - p_2)$. Therefore

$$|p_1, q_1\rangle * |p_2, q_2\rangle = \delta(q_1 - p_2)|p_1, q_2\rangle \tag{75}$$

This equation is clearly the natural generalization of equations like (21) and (54), when continuous parameters are involved (instead of discrete indexes). For this reason we say that $|p, p\rangle$ is a star algebra projector (by slightly extending this notion). We remark that this happens when the left half–momentum is equal to the right half–momentum.

We can therefore improve our description of the closed string states, by giving them a nonzero momentum in the transverse directions: we tensor the states discussed in the previous sections (constructed as in the previous sections, but out of $\beta_n^{\mu\dagger}$ and $\tilde{\beta}_n^{\mu\dagger}$ given by eq.(68)) with momentum eigenstates $|p, p\rangle$. The resulting state will have transverse momentum p, which is the eigenvalue of $\frac{1}{2\sqrt{b}}(\beta_0^\mu + \beta_0^{\mu\dagger} + \tilde{\beta}_0^\mu + \tilde{\beta}_0^{\mu\dagger})$.

What about the longitudinal momentum? In the longitudinal directions the star product is determined by the three strings coefficients V_{nm}^{ab} and setting the momenta to zero (instead

[5]The + sign in the RHS of (74) is to be traced back to the - sign in the second eq.(68).

of integrating over them, see [29]), while the corresponding (zero momentum off–shell) closed string states have been introduced in section 4. Generating a momentum eigenfunction in this context is impossible. Some kind of modification has to be introduced. This would require a rather long digression. Since longitudinal momenta do not enter in what follows we postpone dealing with this issue to another occasion.

It is instructive to complete this subject by giving further properties of the momentum eigenstates and their conjugates. To start with we get the following *bpz* product

$$\langle p,p|q,q\rangle = \frac{1}{K^2}\sqrt{\frac{b}{2\pi}}\delta(p+q) \tag{76}$$

from which we see that the normalization of $|p,p\rangle$ as a star projector differs from the normalization as a wavefunction.

In order to introduce the coordinate eigenstates let us define

$$|x,y\rangle = \frac{1}{K}\sqrt{\frac{2}{b\pi}}\, e^{-\frac{1}{2}(x^2+y^2)-\frac{2i}{\sqrt{b}}(y\beta_0^\dagger - x\tilde{\beta}_0^\dagger)+\frac{1}{2}(\beta_0^{\dagger\,2}+\tilde{\beta}_0^{\dagger\,2})}\,|0_c'\rangle \tag{77}$$

The star algebra yields

$$|x_1,y_1\rangle * |x_2,y_2\rangle = \delta(y_1-x_2)|x_1,y_2\rangle \tag{78}$$

Like before, $|x,x\rangle$ can be interpreted as a star algebra projector. However the position operators are

$$\hat{x} = i\frac{\sqrt{b}}{2}(\tilde{\beta}_0 - \tilde{\beta}_0^\dagger), \qquad \hat{y} = i\frac{\sqrt{b}}{2}(\beta_0 - \beta_0^\dagger),$$

so that

$$[\hat{x}+\hat{y}, \hat{p}+\hat{q}] = i,$$

as it must be. But we get

$$\hat{x}|x,y\rangle = -x|x,y\rangle, \qquad \hat{y}|x,y\rangle = y|x,y\rangle$$

So that the position eigenvalue, that is the eigenvalue of $\hat{x}+\hat{y}$, is $z = y-x$. Therefore $|x,x\rangle$ has position $z=0$. In order to get something meaningful we should choose, as position eigenstate $|-x,x\rangle$. This is confirmed by the following fact. When we contract $|x,y\rangle$ with the $|p,q\rangle$ we find

$$\langle p,q|x,y\rangle \sim e^{i(-px+qy)}$$

When $p=q$ and $x=y$ this becomes a constant. Therefore $|x,x\rangle$ cannot be interpreted as a position eigenstate. On the contrary $|-x,x\rangle$ works very well as a position eigenstate. But it is not a star projector.

This fact seems to translate at the level of star algebra the quantum impossibility of simultaneously describing coordinate and momentum.

6 The boundary state in the transverse directions

It is very instructive to redo the computation we did at the beginning of section 3 for transverse directions. Let ij denote transverse directions and let us consider the identity

$$\sum_n \beta_n^{i\dagger}\tilde{\beta}_n^{j\dagger}\eta_{ij} = -\sum_n \langle {s'}^{i\dagger}|{\xi'_n}^L\rangle\langle {\xi'_n}^R|{s'}^{j\dagger}\rangle\eta_{ij}$$
$$= -\sum_n \langle {s'}^{i\dagger}|{\xi'_n}^L\rangle\langle {\xi'_n}^L|C{s'}^{j\dagger}\rangle\eta_{ij} = -\frac{1}{2}\sum_{k=1}^{\infty} {s'_k}^{i\dagger}C_{kl}{s'_l}^{j\dagger}\eta_{ij}$$

The factor of $\frac{1}{2}$ comes from the fact that ξ'_n is a complete basis for the left ξ''s, as far as the continuous spectrum is concerned (see (66)). We have to consider also the other half made of $C\xi'_n$, which gives the same contribution, see (16). Hence the factor of $\frac{1}{2}$. The – sign come from the definition (68). This is not compensated anymore now by the twist properties of the basis since

$$\xi'^R = C\xi'^L, \tag{79}$$

which in turn descends from the sign change of eq.(103) in Appendix in passing from the 'sliver basis' to the 'lump basis'.

For the transverse directions we have therefore the following identity

$$e^{\sum_n \beta_n^{i\dagger}\tilde{\beta}_n^{j\dagger}\eta_{ij}}|0_c\rangle = e^{-\frac{1}{2}\sum_{k=1}^{\infty}{s'_k}^{i\dagger}C_{kl}{s'_l}^{j\dagger}\eta_{ij}}|\Xi\rangle \sim e^{-\frac{1}{2}\sum_{k=1}^{\infty}a_k^{i\dagger}C_{kl}a_l^{j\dagger}\eta_{ij}}|0\rangle \tag{80}$$

where $|0\rangle$ is the original open string vacuum.

Suppose we have Dk–brane in closed string theory, i.e. we have $25-k$ transverse directions and $k+1$ parallel ones (including time). Then the oscillator part of the corresponding boundary state in closed string theory is the tensor product of a factor like the LHS of eq.(19) and a factor given by the LHS of the above eq.(80). As one can see the RHS of the two equations takes the same form. This miracle has to be traced back to the twist properties of the 'sliver basis' and the 'lump basis'.

The identification (80) generalizes the corresponding result in section 3. But we are now in a position to offer an interpretation of it. The LHS is proportional to the boundary state in closed string theory, the right hand side is the identity state in open string field theory. The boundary state represents a Dk–brane in the closed string language. The identity state represents absence of interaction in the open string field theory language. We can interpret the above equality in the following way: closed strings are reflected by the Dk–brane (they feel it). Open string live on the Dk–brane, therefore they perceive the corresponding state as an identity state (they don't feel it).

At this stage it is also clear that one cannot speak about closed string states in absolute generality but only with respect to a given background. The closed string states we have introduced are the ones that interact with the open string excitations of a given D–brane, which is manifest in the structure of the vacuum they act upon.

The elements brought forth in this section are evidence in favor of our identification of closed string modes with open string star algebra projectors. In particular the above mentioned automatic change in boundary conditions can hardly be a mere accident.

6.1 Closed string exchange between two boundary states

As a consistency check of the identification made above, in this section we would like to reproduce the well known computation describing the closed string exchange between two D–brane, by explicitly converting closed string oscillators into star algebra inner operators. To start with let us recall the basic result (see for instance [30])[6]

$$\langle B(0)|\hat{D}|B(y^i)\rangle = V_{p+1}\, N_p\, T_p \int_0^\infty dt\, t^{-\frac{D-p-1}{2}} e^{-\frac{y^2}{2\pi\alpha' t}}\, e^{2\pi t} \prod_{n=1}^{\infty} \left(\frac{1}{1-e^{-2\pi n t}}\right)^D \tag{81}$$

where T_p is the Dp–brane tension and N_p is a proportionality constant. Introducing the matrix $B_{\mu\nu} = (\eta_{\alpha\beta}, -\delta_{ij})$, the (matter part of the) boundary state at transverse position y^i is given by

$$|B(y^i)\rangle = \frac{T_p}{2} \exp\left(-\sum_{n=1}^{\infty} \beta_n^\dagger \cdot B \cdot \tilde{\beta}_n^\dagger\right) \delta(\hat{x}^i - y^i)|p^\alpha = 0\rangle \tag{82}$$

and the closed string propagator is

$$\hat{D} = \frac{\alpha'}{4\pi} \int_{|z|\leq 1} \frac{d^2 z}{|z|^2}\, z^{L_0 - 1}\, \bar{z}^{\tilde{L}_0 - 1}$$

Let us first compute the contribution from nonzero modes by using the dictionary introduced above. The nonzero mode part is given by

$$\langle 0|\exp\left(-\sum_{n=1}^{\infty} \beta_n \cdot B \cdot \tilde{\beta}_n\right) z^N\, \bar{z}^{\tilde{N}} \exp\left(-\sum_{n=1}^{\infty} \beta_n^\dagger \cdot B \cdot \tilde{\beta}_n^\dagger\right)|0\rangle = \prod_{n=1}^{\infty} \left(\frac{1}{1-|z|^{2n}}\right)^D \tag{83}$$

where N and $\tilde{N}$ are the usual closed string holomorphic and anti–holomorphic level operators. In OSFT language these two operators are given by the left/right action of the operator defined in (42), explicitly

$$N|\psi\rangle = |\mathcal{N} * \psi\rangle, \qquad \tilde{N}|\psi\rangle = |\psi * \mathcal{N}\rangle$$

As we have already shown, the oscillator part of the boundary state gets mapped to the identity string field in the non–zero mode sector. Hence the expression (83) is proportional to

$$\langle I|z^{\mathcal{N}} * I * \bar{z}^{\mathcal{N}}\rangle = \langle z^{\mathcal{N}}|\bar{z}^{\mathcal{N}}\rangle$$

The string field level operator decomposes into the sum of all half string levels

$$\mathcal{N} = \sum_{l=1}^{\infty} \mathcal{N}_l$$

Let's now write $z^{\mathcal{N}}$ in terms of the building blocks Λ_{nm}.

$$z^{\mathcal{N}} = z^{\sum_l \mathcal{N}_l} = \otimes_l z^{\hat{\mathcal{N}}_l}$$

[6] We disregard the ghost contribution (which modifies $D \to D-2$ in the last parenthesis).

Notice that the vacuum state (the sliver or the lump, according to the direction) is actually the tensor product of the vacua for the oscillators $(\beta_l, \tilde{\beta}_l)$

$$\Xi = \otimes_l \Xi_l$$

Accordingly the star product also factorizes

$$* = \otimes_l \, (*_l)$$

This factorization obviously extends to string fields containing just on l–type of operators. To be precise

$$\begin{aligned}\mathcal{N}_l &= \hat{\mathcal{N}}_l \otimes_{r\neq l} \hat{\Xi}_r \\ \Lambda^{(l)}_{n_l,n_l} &= \hat{\Lambda}^{(l)}_{n_l,n_l} \otimes_{r\neq l} \hat{\Xi}_r\end{aligned}$$

With the above understanding it is immediate to see that

$$\hat{\mathcal{N}}_l = \sum_{n_l=0}^{\infty} l\, n_l \, \hat{\Lambda}^{(l)}_{n_l,n_l}.$$

Now we explicitly get

$$z^{\mathcal{N}} = \otimes_l \left(\sum_{n_l=0}^{\infty} z^{l\, n_l} \, \hat{\Lambda}^{(l)}_{n_l,n_l} \right)$$

with an analogous result for $\bar{z}^{\mathcal{N}}$. We can finally write (considering just 1 space-time dimension)

$$\begin{aligned}\langle z^{\mathcal{N}} | \bar{z}^{\mathcal{N}} \rangle^{(D=1)} &= \otimes_l \left(\sum_{n_l=0}^{\infty} z^{l\, n_l} \, \langle \hat{\Lambda}^{(l)}_{n_l,n_l} | \right) \otimes_{l'} \left(\sum_{m'_l=0}^{\infty} \bar{z}^{l'\, m'_l} \, | \hat{\Lambda}^{(l')}_{m'_l,m'_l} \rangle \right) \\ &= \otimes_l \left(\sum_{n_l} |z|^{2l\, n_l} \langle \hat{\Xi}_l | \hat{\Xi}_l \rangle \right) \\ &= \prod_l \left(\frac{1}{1-|z|^{2l}} \right) \otimes_l \, \langle \hat{\Xi}_l | \hat{\Xi}_l \rangle = \prod_l \left(\frac{1}{1-|z|^{2l}} \right) \, \langle \Xi | \Xi \rangle^{(D=1)} \qquad (84)\end{aligned}$$

Taking into account the total number of dimensions we finally get

$$\langle z^{\mathcal{N}} | \bar{z}^{\mathcal{N}} \rangle = \langle \Xi_p | \Xi_p \rangle \prod_{l=1}^{\infty} \left(\frac{1}{1-|z|^{2l}} \right)^D \qquad (85)$$

Here we denote by Ξ_p the sliver in the $p+1$ longitudinal direction, tensored with the lump on the remaining $D-p-1$'s.

Next let us turn to the zero mode part of (81). To reproduce it in the open string language let us concentrate on the zero mode part of (82): $\delta(\hat{x}^i - y^i)|p_\perp = 0\rangle$ (we disregard the longitudinal part, which is trivial). Therefore we have to represent, in the open string language, such states as

$$\delta(\hat{x}-y)|p\rangle = \frac{1}{2\pi}\int dq\, e^{iq(\hat{x}-y)}|p\rangle \tag{86}$$

where, once again, we have dropped all Lorentz indexes and concentrated on a single transverse direction. Now we represent $|p\rangle$ by means of the star projector $|p,p\rangle$ introduced in section 5 and $\hat{x}$ by the operator $i\frac{\sqrt{b}}{2}(\tilde{\beta}_0 - \tilde{\beta}_0^\dagger + \beta_0 - \beta_0^\dagger)$, considered in the same section. Then it is easy to verify that

$$e^{iq(\hat{x}-y)}|p\rangle = e^{-iqy}|p+q\rangle$$

and that

$$\left(\int dq\, e^{-iqx}|p+q\rangle\right) * \left(\int dq'\, e^{-iq'y}|p+q'\rangle\right) = \int dq\, e^{-iq(x+y)}|p+q\rangle \tag{87}$$

This clarifies the open string nature of the states (86). With these results at hand one can now proceed to the explicit evaluation of the LHS of (81). The calculation in the open string language now parallels exactly the one in the closed string language, and will not be repeated here, see [30]. The final result is the classic result (81), provided we make the identification

$$\langle \Xi_p|\Xi_p\rangle \sim T_p \tag{88}$$

This is expected if we keep in mind the relation with VSFT, since the sliver (or the lump) are classical solutions having an energy that reproduces the correct ratio of D–brane tensions. One should also realize that this result can be the clue to understand why the D–brane tension (as computed from the open string one loop computation or, equivalently, from closed string exchange) is actually the same as the energy density of the corresponding OSFT classical solution.

As a last remark, we would like to point out that our previous computation is invariant under star rotations (since it is just the computation of a bpz–norm) and so one can expect (in the complete theory where the ghost sector is coupled consistently) this to be a gauge invariant observable in OSFT. At the moment, however, we are not able to give a physical interpretation of what this gauge invariant object actually *is*, from a purely open string (field) theoretical point of view[7].

7 Discussion

In this paper we have put forward a translation dictionary between open and closed string theory in the framework of open string field theory. We can summarize our proposal with the slogan: closed string modes are star algebra projectors, where the star algebra is the one

[7]In perturbative open string theory the interpretation is in terms of one–loop amplitude, but here the open string degrees of freedom appear non–perturbatively, therefore the interpretation is expected to be different.

that appear in open string field theory. Our starting point has been the identification of the left and right sectors of the open string theory with the holomorphic and antiholomorphic sectors of the closed string via a Bogoliubov transform. The latter, in particular, maps the open string vacuum into the sliver string field, which is identified with the closed string vacuum. We have shown that zero momentum level–matched (off–shell) closed string states are associated under our dictionary with star algebra projectors (or families thereof) in the open string side. To associate a momentum to a given state we have to shift to the lump vacuum and to tensor the previous states by a momentum eigenstate which is itself a star algebra projector. So, altogether, we can claim that according to our dictionary, off-shell closed string states (i.e. momentum and level-matched closed string states) correspond to star algebra projectors in the open string side.

We have presented one important outcome of our proposal, by showing that the boundary state that represents a Dk–brane in the closed string language is translated into the identity state in the open string side, which is precisely the result one expects if our identification is correct. We have tested this result by explicitly showing how one can compute the closed string exchange between two boundary states by using elementary star algebra operations.

We also recall that the string states that in [2] were set in correspondence with the 1/2 BPS LLM geometries, [3], turn out to be, in the light of the present paper, infinite superpositions of closed string states of the type (48) with $n = m$. This is another element that fits the general scheme presented in this paper.

Of course this is only a beginning. Many other tests have to be carried out and many problems have to be clarified. To finish this paper we would like to make a list of the impending issues.

Ghosts. We have to complete our dictionary with the inclusion of the ghost sector. This is quite nontrivial because the analogy with the VSFT solutions in this case is not very helpful. We recall that the VSFT equation of motion for the ghost part is not a projector equation, while one can expect our ghost completion to be again related to projectors, in order to be 'bpz–dual' to the correct ghost number 3 boundary state. This implies a nontrivial modification of the ghost Neumann coefficients and ghost Fock space. We will deal with it in a separate paper.

Role of the string midpoint. In open string field theory the string midpoint plays a crucial role. In particular in VSFT, one could actually say that all the physical observables are concentrated at that point (modulo singularities). In the correspondence we have outlined in this paper the open string midpoint does not seem to play any role: the reason can be traced back to the fact that the Bogoliubov transformation is singular exactly at the midpoint, so the latter has been swept away since the very beginning. This seems to mark a basic difference between open and closed strings.

On–shell closed string states. In this paper we have considered level and momentum–matched, but off–shell, closed string states. The natural question is whether there is a simple characterization of *on–shell* closed string states (i.e. states that satisfy the full set of closed string Virasoro constraints) in terms of open string modes. So far we have not been able to find any appealing answer to this question.

Gauge freedom. As we have already remarked there is a large freedom in choosing the basis ξ_n or ξ'_n, which we introduced in section 2 and 5, respectively. In fact this freedom corresponds to an $O(\infty)$ group. Such large gauge freedom is far from surprising in a string field theory context. We have already seen that some relevant physical quantities (like the left/right levels or the closed string exchange between two D–branes) are actually independent of this choice.

The situation is more complicated when we come to other types of amplitudes. For our dictionary may allow us to calculate amplitudes between non–perturbative open string objects and perturbative closed string modes: for instance, it may allow us to compute the decay probability into the various closed string modes in the process of a D–brane decay represented by a time–dependent rolling tachyon–like solution, [31] (or, rather, by the corresponding analytic solution à la Schnabl). This would allow us to identify the tachyonic matter in a SFT context. Actually this has been the original motivation of our research. The challenge in this direction is precisely how to deal with the above large gauge freedom. Since such gauge freedom is not present in open string theory, the gauge freedom must be completely fixed. We have already started to do so by choosing the basis as in (66), which was dictated by the physical requirement that a boundary state in closed string theory should coincide with the identity state in the open SFT side. However more comparisons like this are needed between corresponding closed and open string objects in order to fix the gauge freedom completely or, at least, to an acceptable degree, which may allow us to do explicit calculations.

Closed string modes and analytic solutions of SFT. The basic objects throughout our paper have been the star projectors. They are (for the matter part) also solutions to the VSFT equations of motion. This property has not not played any role in the above. However, as was mentioned in the introduction, it may indicate that closed string modes that correspond to star projectors might in fact correspond to full analytic solutions to the SFT equation of motion. Verifying this may be crucial in understanding open–closed string duality.

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Appendix: diagonal representation of the X and X' matrices

In this Appendix we collect some results, which are necessary in the text, concerning the spectroscopy and diagonal representation of X and X' matrices.

The diagonalization of the X matrix was carried out in [25], while the same analysis for X' was accomplished in [32] and [33]. Here, for later use, we summarize the results of these references. The eigenvalues of $X = X^{11}, X_+ = X^{12}, X_- = X^{21}$ and T are given, respectively, by

$$\mu^{rs}(k) = \frac{1 - 2\,\delta_{r,s} + e^{\frac{\pi k}{2}}\,\delta_{r+1,s} + e^{-\frac{\pi k}{2}}\,\delta_{r,s+1}}{1 + 2\cosh\frac{\pi k}{2}} \tag{89}$$

$$t(k) = -e^{-\frac{\pi|k|}{2}} \tag{90}$$

where $-\infty < k < \infty$. The generating function for the eigenvectors is

$$f^{(k)}(z) = \sum_{n=1}^{\infty} v_n^{(k)} \frac{z^n}{\sqrt{n}} = \frac{1}{k}(1 - e^{-k\arctan z}) \tag{91}$$

The completeness and orthonormality equations for the eigenfunctions are as follows

$$\sum_{n=1}^{\infty} v_n^{(k)} v_n^{(k')} = \mathcal{N}(k)\delta(k-k'), \qquad \mathcal{N}(k) = \frac{2}{k}\sinh\frac{\pi k}{2}, \qquad \int_{-\infty}^{\infty} dk\, \frac{v_n^{(k)} v_m^{(k)}}{\mathcal{N}(k)} = \delta_{nm} \tag{92}$$

We define the normalized eigenvectors

$$\mathrm{v}_n(k) = \frac{v_n^{(k)}}{\sqrt{\mathcal{N}}(k)}$$

and refer to $\mathrm{v}_n(k)$ as the *sliver basis.*

The spectrum of X is continuous and lies in the interval $[-1/3, 0)$. It is doubly degenerate except at $-\frac{1}{3}$. The continuous spectrum of X' lies in the same interval, but X' in addition has a discrete spectrum. To describe it we follow [32]. We consider the decomposition

$$X'^{rs} = \frac{1}{3}(1 + \alpha^{s-r} CU' + \alpha^{r-s} U'C) \tag{93}$$

where $\alpha = e^{\frac{2\pi i}{3}}$. It is convenient to express everything in terms of CU' eigenvalues and eigenvectors. The discrete eigenvalues are denoted by ξ and $\bar{\xi}$. The matrix CU' is hermitian, unitary and commutes with $U'C$. Therefore ξ and $\bar{\xi}$ lie on the unit circle and are determined as follows, [32]. Let

$$\xi = -\frac{2 - \cosh\eta - i\sqrt{3}\sinh\eta}{1 - 2\cosh\eta} \tag{94}$$

and

$$F(\eta) = \psi\left(\frac{1}{2} + \frac{\eta}{2\pi i}\right) - \psi\left(\frac{1}{2}\right), \qquad \psi(z) = \frac{d\log\Gamma(z)}{dz} \tag{95}$$

Then the eigenvalues ξ and $\bar{\xi}$ are the solutions to

$$\Re F(\eta) = \frac{b}{4} \tag{96}$$

To each value of b there corresponds a couple of values of η with opposite sign (except for $b = 0$ which implies $\eta = 0$).

The eigenvectors $V_n^{(\xi)}$ are defined via the generating function

$$\begin{aligned} F^{(\xi)}(z) = \quad & \sum_{n=1}^{\infty} V_n^{(\xi)} \frac{z^n}{\sqrt{n}} = -\sqrt{\frac{2}{b}} V_0^{(\xi)} \left[\frac{b}{4} + \frac{\pi}{2\sqrt{3}} \frac{\xi - 1}{\xi + 1} + \log iz \right. \\ & \left. + \, e^{-2i(1+\frac{\eta}{\pi i})\arctan z} \Phi(e^{-4i \arctan z}, 1, \frac{1}{2} + \frac{\eta}{2\pi i}) \right] \end{aligned} \tag{97}$$

where $\Phi(x, 1, y) = 1/y \, {}_2F_1(1, y; y + 1; x)$, while

$$V_0^{(\xi)} = \left(\sinh \eta \frac{\partial}{\partial \eta} \left[\log \Re F(\eta) \right] \right)^{-\frac{1}{2}} \tag{98}$$

As for the continuous spectrum, it is spanned by the variable k, $-\infty < k < \infty$. The eigenvalues of CU' are given by

$$\nu(k) = -\frac{2 + \cosh \frac{\pi k}{2} + i\sqrt{3} \sinh \frac{\pi k}{2}}{1 + 2 \cosh \frac{\pi k}{2}}$$

The generating function for the eigenvectors is

$$\begin{aligned} F_c^{(k)}(z) = \sum_{n=1}^{\infty} V_n^{(k)} \frac{z^n}{\sqrt{n}} = V_0^{(k)} \sqrt{\frac{2}{b}} \left[-\frac{b}{4} - \left(\Re F_c(k) - \frac{b}{4} \right) e^{-k \arctan z} - \log iz \right. \\ \left. - \left(\frac{\pi}{2\sqrt{3}} \frac{\nu(k) - 1}{\nu(k) + 1} + \frac{2i}{k} \right) + 2i \, f^{(k)}(z) - \Phi(e^{-4i \arctan z}, 1, 1 + \frac{k}{4i}) \, e^{-4i \arctan z} \, e^{-k \arctan z} \right] \end{aligned} \tag{99}$$

where

$$F_c(k) = \psi(1 + \frac{k}{4\pi i}) - \psi(\frac{1}{2}),$$

while

$$V_0^{(k)} = \sqrt{\frac{b}{2\mathcal{N}(k)}} \left[4 + k^2 \left(\Re F_c(k) - \frac{b}{4} \right)^2 \right]^{-\frac{1}{2}} \tag{100}$$

The continuous eigenvalues of X', X'_-, X'_- and T' (for the conventional lump) are given by same formulas as for the X, X_+, X_- and T case, eqs(89,90). As for the discrete eigenvalues, they are given by the formulas

$$\begin{aligned} \mu_\xi^{rs} &= \frac{1 - 2\,\delta_{r,s} - e^{\eta}\,\delta_{r+1,s} - e^{-\eta}\,\delta_{r,s+1}}{1 - 2\cosh \eta} \\ t_\xi &= e^{-|\eta|} \end{aligned} \tag{101}$$

The eigenvectors corresponding to the continuous spectrum are $V_N(k)$ $(-\infty < k < \infty)$, while the eigenvectors of the discrete spectrum are denoted by $V_N(\eta)$ and $V_N(-\eta)$. They form a complete basis. They will be normalized so that the completeness relation takes the form

$$\int_{-\infty}^{\infty} dk\, V_N(k)V_M(k) + V_N(\eta)V_M(\eta) + V_N(-\eta)V_M(-\eta) = \delta_{NM} \tag{102}$$

We refer to V_N as the *lump basis*.

One important difference between the v_n and V_N basis is determined by the twist transformation properties (in vector notation, $\mathrm{v} = \{\mathrm{v}_n\}$, etc.)

$$C\mathrm{v}(k) = -\mathrm{v}(-k), \qquad \text{while} \qquad CV(k) = V(-k) \tag{103}$$

We have also $CV(\eta) = V(-\eta)$.

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