Some new results on orthogonal polynomials for Laguerre type exponential weights

Published version:

DOI:10.1007/s10474-018-0841-8

Giuseppe Mastroianni, Incoronata Notarangelo, László Szili and Péter Vértesi. Some new results on orthogonal polynomials for Laguerre type exponential weights. *Acta Mathematica Hungarica, 155 (2018), 466–478.* DOI: 10.1007/s10474-018-0841-8

SOME NEW RESULTS ON ORTHOGONAL POLYNOMIALS FOR LAGUERRE TYPE EXPONENTIAL WEIGHTS

G. MASTROIANNI[1],*, I. NOTARANGELO[1],**, L. SZILI[2],*** AND P. VÉRTESI[3],***

[1]DEPARTMENT OF MATHEMATICS, COMPUTER SCIENCES AND ECONOMICS, UNIVERSITY OF BASILICATA, VIA DELL'ATENEO LUCANO 10, 85100 POTENZA, ITALY

[2]DEPARTMENT OF NUMERICAL ANALYSIS, LORÁND EÖTVÖS UNIVERSITY, H-1117 BUDAPEST, PÁZMÁNY P. SÉTÁNY I/C, HUNGARY
DEPARTMENT OF DIFFERENTIAL EQUATIONS, BUDAPEST UNIVERSITY OF TECHNOLOGY AND ECONOMICS, H-1111 BUDAPEST, EGRY JÓZSEF U. 1., HUNGARY

[3]ALFRÉD RÉNYI INSTITUTE OF MATHEMATICS, H-1364 BUDAPEST, P.O.B. 127, HUNGARY

Abstract. In this paper we prove some results on the root-distances and the weighted Lebesgue function corresponding to orthogonal polynomials for Laguerre type exponential weights.

1. Introduction. Notations. Preliminaries

1.1. In the paper [1] and [2] Eli Levin and Doron Lubinsky investigated certain Laguerre type orthogonal polynomials. Using their results and our papers [3], [4], [5] we state further relations for the root-distances. In addition, we obtain a lower estimation for the weighted Lebesgue function of the weighted Lagrange interpolation with respect to arbitrary point systems.

1.2. (For Sections 1.1–1.4 cf. [1] and [2].)
Let

$$I = [0, d), \tag{1.1}$$

where $0 < d \le \infty$. Let $Q : I \to [0, \infty)$ be continuous, and

$$W = \exp(-Q) \tag{1.2}$$

1991 *Mathematics Subject Classification.* 40C05, 40D05, 40G05, 41A05, 41A10.
Key words and phrases. Orthogonal polynomials, exponential weights, Laguerre weights, weighted interpolation, root-distances, weighted Lebesgue function.
*The first author was partially supported by University of Basilicata (local funds).
**The second author was partially supported by University of Basilicata (local funds) and by National Group of Computing Science INdAM–GNCS.
***Research supported by the Hungarian National Scientific Research Foundation (OTKA), No. K115804.

be such that all moments $\int_I x^n W(x)\,dx$, $n \geq 0$, exists. We call W an exponential weight on I. For (a fixed) $\rho > -\frac{1}{2}$, we set

$$W_\rho(x) := x^\rho W(x), \quad x \in I.$$

The orthonormal polynomial of degree n for W^2 is denoted by $p_n\left(W^2, x\right)$ or $p_n(x)$. That for W_ρ^2 is denoted by $p_n\left(W_\rho^2, x\right)$ or $p_{n,\rho}(x)$. So

$$\int_I p_{n,\rho}(x) p_{m,\rho}(x) x^{2\rho} W^2(x)\,dx = \delta_{n,m}$$

and

$$p_{n,\rho}(x) = \gamma_{n,\rho} x^n + \cdots,$$

where $\gamma_{n,\rho} = \gamma_n\left(W_\rho^2\right) > 0$.

We denote the zeros of $p_{n,\rho}$ by $U_n\left(W_\rho^2\right) = \left\{x_{kn} = x_{kn}\left(W_\rho^2\right)\right\}$, where

$$x_{nn} < x_{n-1,n} < \cdots < x_{2n} < x_{1n}.$$

As in [1] and [2] we define an *even* weight W^* corresponding to the one-sided weight W. Given I and W as in (1.1) and (1.2), let

$$I^* := \left(-\sqrt{d}, \sqrt{d}\right)$$

and for $x \in I^*$

$$\begin{aligned} Q^*(x) &:= Q\left(x^2\right), \\ W^*(x) &:= \exp\left(-Q^*(x)\right). \end{aligned} \tag{1.3}$$

We say that $f : I \to (0, \infty)$ is *quasi-increasing*, if there exists $C > 0$ such that

$$f(x) \leq C\, f(y), \quad 0 < x < y < d.$$

The notation

$$f(x) \sim g(x)$$

means that there are positive contants C_1, C_2 such that for the relevant range of x

$$C_1 \leq f(x)/g(x) \leq C_2.$$

Similar notation is used for sequences and sequences of functions.

Throughout, $C, C_1, C_2, c, c_1, c_2, \ldots$ denotes positive constants independent of n, x, t and polynomials P of degree at most n. We write $C = C(\lambda)$, $C \neq C(\lambda)$ to indicate dependence on or independence of, a parameter λ. The same symbol does not necessarily denote the same constant in different occurrences. We denote the set of polynomials of degree $\leq n$ by $\mathcal{P}_n$.

1.3. Following is our class of weights:

Definition 1.1. Let $W = e^{-Q}$ where $Q : I \to [0, \infty)$ satisfies the following properties:

(a) $\sqrt{x}Q'(x)$ is continuous in I, with limit 0 at 0 and $Q(0) = 0$;

(b) Q'' exists in $(0, d)$, while $Q^{*\prime\prime}$ is positive in $\left(0, \sqrt{d}\right)$;

(c)

$$\lim_{x \to d-} Q(x) = \infty.$$

(d) The function

$$T(x) := \frac{xQ'(x)}{Q(x)}, \quad x \in (0, d) \tag{1.4}$$

is quasi-increasing in $(0, d)$, with

$$T(x) \geq \Lambda > \tfrac{1}{2}, \quad x \in (0, d). \tag{1.5}$$

(e) There exists $C_1 > 0$ such that

$$\frac{|Q''(x)|}{Q'(x)} \leq C_1 \frac{Q'(x)}{Q(x)}, \quad \text{a.e} \ \ x \in (0, d). \tag{1.6}$$

There exists a compact subinterval J of I^*, and $C_2 > 0$ such that

$$\frac{Q^{*\prime\prime}(x)}{|Q^{*\prime}(x)|} \geq C_2 \frac{|Q^{*\prime}(x)|}{Q^*(x)}, \quad \text{a.e.} \ \ x \in I^* \setminus J. \tag{1.7}$$

If the weight W satisfies (a)–(e), then we write $W \in \mathcal{L}(C^2+)$.

Examples (cf. [1] and [2]):

$$Q(x) = x^{\alpha}, \quad x \in [0, +\infty), \quad \alpha > \tfrac{1}{2}; \tag{1.8}$$

$$Q(x) = \exp_k(x^{\alpha}) - \exp_k(0), \quad x \in [0, +\infty), \quad \alpha > \tfrac{1}{2}, \quad k \geq 0, \tag{1.9}$$

where $\exp_0(x) := x$ and for $k \geq 1$

$$\exp_k(x) = \underbrace{\exp(\exp(\exp \cdots \exp(x)))}_{k \text{ times}}$$

is the kth iterated exponential.

An example on the finite interval $I = [0, 1)$ is

$$Q(x) = \exp_k\big((1 - x)^{-\alpha}\big) - \exp_k(1), \quad x \in [0, 1), \tag{1.10}$$

where $\alpha > 0$ and $k \geq 0$.

1.4. One of the important quantities we need is the Mhaskar–Rakhmanov–Saff number for the weight W_ρ denoted by $a_t = a_t(Q)$, defined for $t > 0$ as the positive root of the equation

$$t = \frac{1}{\pi} \int_0^1 \frac{a_t\, u\, Q'(a_t\, u)}{\sqrt{u\,(1-u)}}\, du.$$

If $xQ'(x)$ is strictly increasing and continuous, with limits 0 and $+\infty$ at 0 and d respectively, a_t is uniquely defined. Moreover, a_t is an increasing function of $t \in (0, +\infty)$, with

$$\lim_{t \to +\infty} a_t = d.$$

Let us introduce the notation

$$T_n := T(a_n) \qquad (n \in \mathbf{N}). \tag{1.11}$$

Since $T(x) \geq \Lambda > 1/2$, $x \in (0, d)$ (see (1.5)), thus we have

$$\lim_{n \to +\infty} (n\, T_n) = +\infty \tag{1.12}$$

From the important relation

$$Q(a_t) \sim \frac{t}{\sqrt{T(a_t)}},$$

which holds uniformly for $t > 0$ (see [1, (1.27) and Lemma 3.1]), using the condition $\lim_{x \to d-} Q(x) = +\infty$ (see Definition (c)) it follows that

$$\lim_{n \to +\infty} \frac{n}{\sqrt{T_n}} = +\infty. \tag{1.13}$$

In the sequel, we shall denote the positive Mhaskar–Rakhmanov–Saff number for the weight W^* by $a_t^* = a_t^*\big(Q^*\big)$, $t > 0$. Thus a_t^* is defined as the positive root of the equation

$$t = \frac{1}{\pi} \int_{-a_t^*}^{a_t^*} \frac{x\, Q^{*\prime}(x)}{\sqrt{a_t^{*2} - x^2}}\, dx = \frac{2}{\pi} \int_0^1 \frac{a_t^*\, u\, Q^{*\prime}\,(a_t^*\, u)}{\sqrt{1-u^2}}\, du = \frac{2}{\pi} \int_0^1 \frac{{a_t^*}^2\, v\, Q'\,({a_t^*}^2\, v)}{\sqrt{v(1-v)}}\, dv,$$

whence ([1, p. 211])

$$a_t = {a_{2t}^*}^2 \qquad (t > 0).$$

1.5. Let $W \in \mathcal{L}(C^2+)$, W^* is given by (1.3), $m = 2n$ and $y_{k+1,m} < y_{km}$ ($n \in \mathbf{N}$), where

$$y_{km} = y_{km}\left(W^{*2}\right) = a_m^* t_{km} = a_m^* \cos \vartheta_{km}, \quad k = 1, 2, \ldots, n$$

are the positive roots of the orthonormal polynomial $p_m\left(W^{*2}\right)$. We define

$$y_{0m} = a_m^* = a_m^* t_{0m}, \ \vartheta_{0m} = 0 \quad \text{and} \quad y_{n+1,m} = t_{n+1,m} = 0, \ \ \vartheta_{n+1,m} = \pi/2.$$

By reformulating [5, Theorem 2.1] and using [2, §2], [1, (2.7)–(2.9)], the Hermite-type roots y_{km}'s satisfy

Theorem 1.1. *We have if $W \in \mathcal{L}(C^2+)$ then with $m = 2n \in \mathbf{N}$*

(i) $\vartheta_{km} - \vartheta_{k-1,m} \sim \dfrac{1}{(nT_n)^{1/3} k^{2/3}}, \qquad 1 \le k \le \dfrac{c_1 n}{\sqrt{T_n}},$

(i1) $t_{k-1,m} - t_{km} \sim \dfrac{1}{(nT_n)^{2/3} k^{1/3}}, \qquad 1 \le k \le \dfrac{c_1 n}{\sqrt{T_n}},$

(ii) $\vartheta_{k+1,m} - \vartheta_{km} \sim \dfrac{1}{n}, \qquad \dfrac{c_1 n}{\sqrt{T_n}} \le k \le n,$

(ii1) $t_{km} - t_{k+1,m} \sim \dfrac{k}{n^2}, \qquad \dfrac{c_1 n}{\sqrt{T_n}} \le k \le n.$

2. New relations

2.1. Our first relations on the root distances of the Laguerre type roots

$$x_{kn}\left(W_\rho^2\right) = x_{kn} = a_n u_{kn} = a_n \cos \gamma_{kn} \qquad (k = 1, 2, \ldots, n; \ n \in \mathbf{N})$$

are as follows

Theorem 2.1. *If $W \in \mathcal{L}(C^2+)$ and $x_{0n} = a_n$, $x_{n+1,n} = 0$, then with $n \in \mathbf{N}$*

(a) $\gamma_{kn} - \gamma_{k-1,n} \sim \dfrac{1}{(nT_n)^{1/3} k^{2/3}}, \qquad 1 \le k \le \dfrac{c_2 n}{\sqrt{T_n}},$

(a1) $u_{k-1,n} - u_{kn} \sim \dfrac{1}{(nT_n)^{2/3} k^{1/3}}, \qquad 1 \le k \le \dfrac{c_2 n}{\sqrt{T_n}},$

(b) $\gamma_{k+1,n} - \gamma_{kn} \sim \dfrac{n-k+1}{n^2}, \qquad \dfrac{c_2 n}{\sqrt{T_n}} \le k \le n,$

(b1) $u_{kn} - u_{k+1,n} \sim \dfrac{(n-k+1)k}{n^3}, \qquad \dfrac{c_2 n}{\sqrt{T_n}} \le k \le n.$

Remark. Theorem 2.1 shows that the order of the distances *does not depend* on ρ (cf. (3.1)).

2.2. As an application of the above theorem we prove a lower estimation on the *weighted Lebesgue function* of the weighted Lagrange interpolation on *arbitrary* point systems.

We need the following definitions.

If $Z = \{z_{kn}\}$ is an interpolatory matrix on I that is

$$0 \le z_{nn} < z_{n-1,n} < \cdots < z_{2n} < z_{1n} < d, \quad n \in \mathbf{N},$$

then for $f \in C(W_\rho, I)$, $W \in \mathcal{L}(C^2+)$, where

$$C(W_\rho, I) := \Big\{ f;\ f \text{ is continuous on } (0,d) \text{ and } \lim_{\substack{x\to 0+\\ x\to d-}} f(x)\,W_\rho(x) = 0 \Big\},$$

we investigate the *weighted Lagrange interpolation* defined by

$$L_n\big(f, W_\rho, Z, x\big) = \sum_{k=1}^{n} f(z_{kn})\,W_\rho(z_{kn})\,g_{kn}(W_\rho, Z, x), \quad n \in \mathbf{N}. \tag{2.1}$$

Above

$$g_{kn}(W_\rho, Z, x) = \frac{W_\rho(x)}{W_\rho(z_{kn})}\,\ell_{kn}(Z, x), \quad 1 \le k \le n, \tag{2.2}$$

$$\ell_k(x) = \ell_{kn}(Z, x) = \frac{\omega_n(Z, x)}{\omega_n'(Z, z_{kn})(x - z_{kn})}, \quad 1 \le k \le n, \tag{2.3}$$

and

$$\omega_n(Z, x) = c_n \prod_{i=1}^{n} (x - z_{in}), \quad n \in \mathbf{N}. \tag{2.4}$$

The polynomials ℓ_k of degree exactly $n-1$ (that is $\ell_k \in \mathcal{P}_{n-1} \setminus \mathcal{P}_{n-2}$) are the fundamental functions of the (usual) Lagrange interpolation while functions g_k are the fundamental functions of the weighted Lagrange interpolation.

The classical Lebesgue estimation now has the form

$$\big|L_n(f, W_\rho, Z, x) - f(x)\,W_\rho(x)\big| \le \big\{\lambda_n\big(W_\rho, Z, x\big) + 1\big\} E_{n-1}(f, W_\rho),$$

where the *weighted Lebesgue function* is

$$\lambda_n\big(W_\rho, Z, x\big) := \sum_{k=1}^{n} \big|g_{kn}\,(W_\rho, Z, x)\big|, \quad x \in \mathbf{R},\ n \in \mathbf{N} \tag{2.5}$$

and

$$E_{n-1}(f, W_\rho) := \inf_{p \in \mathcal{P}_{n-1}} \big\|(f - p)\,W_\rho\big\|, \quad n \in \mathbf{N},$$

where $\|\cdot\|$ is the maximum norm on I.

Moreover, the *weighted Lebesgue constant* is

$$\Lambda_n(W_\rho, Z) := \|\lambda_n(W_\rho, Z, x)\|.$$

We state (cf. [5] and its references).

Theorem 2.2. *Let $W \in \mathcal{L}(C^2+)$ and let $0 < \varepsilon < 1$ be fixed. Then for any fixed interpolatory matrix $Z \subset I$ there exists set $H_n = H_n(W_\rho, \varepsilon, Z)$ with $|H_n| \le \varepsilon$ such that*

$$\lambda_n(W_\rho, Z, x) > \frac{1}{3840}\varepsilon \log n \quad \text{if} \quad x \in [0,\ a_n(W_\rho)] \setminus H_n \tag{2.6}$$

whenever $n \ge n_1$.

2.3. We give another application of Theorem 2.1.

Let $\{z_{kn}\} = U_n\left(W_\rho^2\right) = \left\{x_{kn}\left(W_\rho^2\right)\right\}$ with

$$g_{kn}\left(W_\rho, U_n\left(W_\rho^2\right); x\right) = \frac{W_\rho(x)}{W_\rho\left(x_{kn}\left(W_\rho^2\right)\right)}\,\ell_{kn}\left(U_n\left(W_\rho^2\right), x\right), \quad 1 \le k \le n. \tag{2.7}$$

Then

Corollary 2.3. *The weighted Lebesgue constants satisfy*

$$\Lambda_n\left(W_\rho, U_n\left(W_\rho^2\right)\right) \sim n^{1/2}, \qquad n \in \mathbf{N}. \tag{2.8}$$

Now let

$$\{z_{kn}\} = U_n\left(W_\rho^2\right) \cup \left\{x_{0n}\left(W_\rho^2\right), x_{n+1,n}\left(W_\rho^2\right)\right\} = V_n\left(W_\rho^2\right).$$

Then (cf. [8, Theorem 1])

Corollary 2.4. *We have*

$$\Lambda_n\left(W_\rho, V_n\left(W_\rho^2\right)\right) \sim \log n, \qquad n \in \mathbf{N}. \tag{2.9}$$

Above

$$\Lambda_n\left(W_\rho, V_n\left(W_\rho^2\right)\right) = \left\|\sum_{k=0}^{n+1}\left|g_{kn}\left(W_\rho, V_n\left(W_\rho^2\right); x\right)\right|\right\|,$$

where $\ell_{kn}\left(V_n, x\right)$ (in $g_{kn}(V_n, x)$) is based on the $n+2$ nodes $\left\{x_{kn}\left(W_\rho^2\right),\ 0 \le k \le n+1\right\}$.

3. Proofs

3.1. **Proof of Theorem 2.1.** First a *basic observation*: Using [5, Theorem 2.1] and [2, Theorem 1.4] we have

$$x_{kn}\left(W_\rho^2\right)-x_{k+1,n}\left(W_\rho^2\right)\sim x_{kn}\left(W^2_{-\frac{1}{4}}\right)-x_{k+1,n}\left(W^2_{-\frac{1}{4}}\right),\quad k=1,2,\dots,n-1, \tag{3.1}$$

that means it is enough to prove the relations when $\rho=-\frac{1}{4}$, which we suppose from now on.

The relation

$$p_n\left(W^2_{-\frac{1}{4}},x\right)=p_m\left(W^{*2},y\right),\quad x=y^2\in[0,d),\quad m=2n.$$

(cf. [1, (1.7)]) shows that

$$x_{kn}=x_{kn}\left(W^2_{-\frac{1}{4}}\right)=y_{km}^2,\quad k=1,2,\dots,n;\ m=2n. \tag{3.2}$$

Our main tool is the formula

$$\begin{aligned}x_{kn}-x_{k+1,n}&=y_{km}^2-y_{k+1,m}^2=(y_{km}+y_{k+1,m})(y_{km}-y_{k+1,m})\sim\\&\sim y_{km}\left(y_{km}-y_{k+1,m}\right),\quad 0\le k\le n,\end{aligned} \tag{3.3}$$

using that

$$y_{km}\le y_{km}+y_{k+1,m}\le 2y_{km}.$$

To get (a1) we write by (i1)

$$\begin{aligned}1-t_{km}&=\sum_{s=0}^{k-1}(t_{sm}-t_{s+1,m})\sim\frac{1}{(nT_n)^{2/3}}\sum_{s=1}^{k-1}s^{-1/3}\sim\\&\sim\left(\frac{k}{nT_n}\right)^{2/3}\le\frac{c_3}{T_n},\qquad 1\le k\le\frac{c_1n}{\sqrt{T_n}},\end{aligned} \tag{3.4}$$

i.e. in (3.4), $t_{km}\ge\frac{1}{2}$ supposing that $c_3/T_n\le 1/2$ (say). (This can be attained by a proper $c_1>0$ using that $T_n>\frac{1}{2}$ (cf. (1.5)). Summarising, we get with a proper $c_1>0$

$$x_{kn}-x_{k+1,n}\sim a_m^*\frac{a_m^*}{(nT_n)^{2/3}k^{1/3}},\qquad 1\le k\le\frac{c_2n}{\sqrt{T_n}}, \tag{3.5}$$

whence $(a1)$ is immediate (cf. (3.3), $(a_m^*)^2=a_n$ (see [1, (2.6)]) and (i1)).

To get (b1) we have by (3.4) and (ii1)

$$
\begin{aligned}
t_{km} &= \sum_{s=1}^{n-k+1} \left(t_{n-s+1,m} - t_{n-s+2,m}\right) \sim \\
&\sim \sum_{s=k}^{n} \frac{s}{n^2} \sim \frac{(n+1)^2 - k^2}{n^2} = \frac{(n+k+1)(n-k+1)}{n^2} \sim \frac{n-k+1}{n},
\end{aligned}
\tag{3.6}
$$

whence by (3.3) and (ii1)

$$
x_{kn} - x_{k+1,n} \sim (a_m^*)^2 \, \frac{n-k+1}{n} \, \frac{k}{n^2} = a_n \frac{(n-k+1)k}{n^3}, \qquad \frac{c_1 n}{\sqrt{T_n}} \le k \le n,
\tag{3.7}
$$

whence (b1) is immediate.

Using that $\rho = -\frac{1}{4}$ by (3.2) and

$$
(a_m^* t_{km})^2 = y_{km}^2 = x_{kn} = a_n u_{kn}
$$

we get $1 - t_{km}^2 = 1 - u_{kn}$, which means

$$
\sin^2 \vartheta_{km} = 2 \sin^2 \frac{\gamma_{kn}}{2}, \qquad 1 \le k \le n.
\tag{3.8}
$$

By (3.8) $\sin \vartheta_{km} = \sqrt{2} \sin \frac{\gamma_{kn}}{2}$ whence

$$
\begin{aligned}
\sin \vartheta_{k+1} - \sin \vartheta_k &= 2 \sin \frac{\vartheta_{k+1} - \vartheta_k}{2} \cdot \cos \frac{\vartheta_{k+1} + \vartheta_k}{2} = \\
= \sqrt{2} \left(\sin \frac{\gamma_{k+1}}{2} - \sin \frac{\gamma_k}{2} \right) &= \sqrt{2} \cdot 2 \cdot \sin \frac{\gamma_{k+1} - \gamma_k}{4} \cdot \cos \frac{\gamma_{k+1} + \gamma_k}{4}.
\end{aligned}
$$

Now by $\cos \frac{\vartheta_{k+1} + \vartheta_k}{2} \approx \cos \vartheta_k = t_k$ and $\cos \frac{\gamma_{k+1} + \gamma_k}{4} \approx \cos \frac{\gamma_k}{2} \ge \cos \frac{\pi}{4} = \frac{\sqrt{2}}{2}$, we get

$$
(\vartheta_{k+1,m} - \vartheta_{km}) \, t_{km} \sim \gamma_{k+1} - \gamma_k, \qquad 0 \le k \le n.
\tag{3.9}
$$

(We omitted some obvious details.) By (3.9) we get (a) (or (b)) using Theorem 1.1 (i), (ii); (3.4) and (3.6). □

3.2. **Proof of Theorem 2.2.** First we formulate some other relations applied later. Let $I_n(\varepsilon) = \left[\varepsilon a_n, (1-\varepsilon) a_n\right]$, $0 < \varepsilon < 1$ fixed. Then

$$
x_{kn} - x_{k+1,n} \sim \frac{a_n}{n}, \qquad x_{kn},\ x_{k+1,n} \in I_n(\varepsilon).
\tag{3.10}
$$

Indeed, by $u_{kn} = t_{km}^2$ (again $\rho = -\frac{1}{4}$)

$$
u_k \in [\varepsilon, 1-\varepsilon] \ \text{ iff } \ t_k \in \left[\sqrt{\varepsilon}, \sqrt{1-\varepsilon}\right],
$$

whence using that

$$u_{kn} - u_{k+1,n} \sim t_{km}(t_{km} - t_{k+1,m}), \quad t_{km} \sim 1 \quad \text{and} \quad t_{km} - t_{k+1,m} \sim \frac{1}{n}$$

(cf. [5, (3.9)]) we get (3.10).

Moreover, by [2, (1.13) and (1.17)], using (3.10) and the relations $x_{kn} \sim a_n - x_{kn} \sim a_n$ whenever $x_{kn} \in I_n(\varepsilon)$, we get

$$\left|p'_{n\rho}(x_{kn}) W_\rho(x_{kn})\right| \sim \frac{n}{a_n^{3/2}}, \qquad x_{kn} \in I_n(\varepsilon). \tag{3.11}$$

Finally, we quote [2, Theorem 1.2] saying that

$$\sup_{x\in I} \left|p_{n\rho}(x)\right| W(x) \left(x + \frac{a_n}{n^2}\right)^\rho \sim \left(\frac{n}{a_n}\right)^{1/2}, \qquad x \in I. \tag{3.12}$$

Using relations (3.10)–(3.12), we can prove Theorem 2.2. as we did in [5, §3.4–3.10]. We can omit the details. □

3.3. **Proof of Corollary 2.3.** By [1, (1.15)] and [8, Lemma 1] we can restrict ourselves to the interval $[0, a_n]$.

Fix $n \in \mathbf{N}$ and the point $x \in [x_{j+1,n}, x_{j,n}] =: \triangle x_{jn}$ ($0 \le j \le n$; obviously $j = j(n)$). Even more, by [2, Theorem 1.3], we can suppose that x is "far" from the nodes, namely

$$\left|x - \left(x_{jn} + x_{j+1,n}\right)/2\right| \le \left|\triangle x_{jn}\right| /4. \tag{3.13}$$

Then by (2.5), (2.7) and [2, Theorem 1.3] we have

$$\lambda_n\left(W_\rho, U_n\left(W_\rho^2\right), x\right) \sim {\sum_{k=1}^{n}}' \frac{\left|\triangle u_{kn}\right|}{\left|u_{jn} - u_{kn}\right|} \left(\frac{u_{kn}(1-u_{kn})}{u_{jn}(1-u_{jn})}\right)^{1/4}. \tag{3.14}$$

Here and hereafter for a fixed index j we use the notation

$${\sum_{k=1}^{n}}' a(k,j) = \sum_{\substack{k=1\\k\neq j}}^{n} a(k,j)$$

Moreover we used the fact that the jth term of the sum (2.5) has the same order as the lth ones, whenever $|l - j| \le c$ (see [2, Theorem 1.3]).

Let from now

$${\sum_{k=1}^{n}}' \cdots = {\sum_{k=1}^{cn-1}}' \cdots + {\sum_{k=cn}^{n}}' \cdots =: S_2 + S_1.$$

In order to estimate S_1 and S_2, we distinguish several cases.

A. First we suppose that $0 < u_{jn} \le c < 1$. Let

$$v_{Kn} = u_{n-K+1,n}, \quad |\triangle v_{Kn}| = v_{K+1,n} - v_{Kn} \qquad (0 < K \le n)$$

and $v_{Jn} = u_{n-J+1,n}$ $(0 \le J \le n)$. By (b1) of Theorem 2.1 we have ($J > K$, say)

$$|v_{Jn} - v_{Kn}| \sim \frac{1}{n^2}\sum_{s=K}^{J} s \sim \frac{|J-K|\,|J+K|}{n^2} \quad \text{and}$$

$$v_{Kn} \sim \sum_{s=1}^{K} \frac{s}{n^2} \sim \frac{K^2}{n^2}.$$

We can write as follows

$$S_1 \sim {\sum_{K=1}^{cn}}' \frac{\triangle v_{Kn}}{|v_{Jn} - v_{Kn}|}\left(\frac{v_{Kn}}{v_{Jn}}\right)^{1/4} \sim \sum_{K \le J/2} \cdots + {\sum_{J/2 \le K < 2J}}' \cdots + \sum_{K=2J}^{cn} \cdots \sim$$

$$\sim \left(\frac{n^2}{J^2}\right)^{5/4} \sum_{K=1}^{J} \frac{K}{n^2}\left(\frac{K^2}{n^2}\right)^{1/4} + \log 2J + \sum_{K=2J}^{cn} \frac{K}{n^2}\left(\frac{n^2}{K^2}\right)^{3/4}\left(\frac{n^2}{J^2}\right)^{1/4} \sim$$

$$\sim 1 + \log(2J) + \left(\frac{n}{J}\right)^{1/2} \le c\, n^{1/2}.$$

(For example, the second sum can be estimated by Theorem 2.1 as follows

$${\sum_{J/2 \le K < 2J}}' \cdots \le c {\sum_{J/2 \le K < 2J}}' \frac{K}{n^2} \cdot \frac{n^2}{|K+J|\,|K-J|} \cdot \left(\frac{K}{J}\right)^{1/4} \sim$$

$$\sim {\sum_{J/2 \le K < 2J}}' \frac{1}{|K-J|} \sim \log(2J)$$

using that now $0 < v_{Jn} \le c < 1$ and $K \sim J$.)

B. If $0 \le u_{jn} \le c$ but $|u_{kn} - u_{jn}| \ge c_1$ (where $0 < c_1 < c < 1$), using that now

$$S_1 = \sum_{\substack{k=cn \\ |u_{kn}-u_{jn}| \ge c_1}}^{n} \cdots,$$

we get by analogue consideration as before that $S_1 \le c$.

Similarly one can obtain:

C. If $0 < c_1 \le u_{jn} \le c_2 < 1$, then $S_1 \sim \log n$.

D. Let $|u_{jn}-u_{kn}|\geq c_1$. Then by arguments as in Part **A** we have

$$S_1\sim n^{1/2},\quad c_2,\quad (nT_n)^{1/6},\quad \text{if}$$
$$u_{jn}\sim n^{-2},\quad c_3\leq u_{jn}\leq c_4,\quad 1-u_{jn}\sim (nT_n)^{-2/3},\quad \text{respectively.}$$

Using that $T_n<n^{2-\varepsilon}$ (see [7, (3.38)]), we get

$$S_1\leq c\left(n^{1/2}+\left(n\cdot n^{2-\varepsilon}\right)^{1/6}\right)\leq c\,n^{1/2}.$$

E. We estimate $S_2=\sum_{k=1}^{cn}{}'\cdots$. If $u_{jn}\geq c_0$ then using Theorems 1.1 and 2.1 further the argument in [4, Part 4.7] we get that $S_2\sim (nT_n)^{1/6}$. Now let $0<u_{jn}<c_0<c_1\leq u_{kn}$, $k\leq c\,n$. Then

$$S_2\leq \sum_{k=1}^{cn}{}' \frac{|\triangle u_{kn}|}{|u_{1n}-u_{kn}|}\left(\frac{u_{kn}(1-u_{kn})}{u_{1n}(1-u_{1n})}\right)^{1/4}<n^{1/2}\sum_{k=1}^{cn}|\triangle u_{kn}|\sim n^{1/2}\qquad (\text{by } u_{1n}\sim 1).$$

Summarizing the points **A**–**E**, we obtain Corollary 2.3. □

3.4. **Proof of Corollary 2.4.** The argument is similar to the ones in Part **3.3**, so we only sketch it.

We suppose that $x\in[x_{nn},x_{1n}]$ and moreover x satisfies (3.13). Then

$$\begin{aligned}
&\lambda_n\left(W_\rho,V_n\left(W_\rho^2\right);x\right)\sim\\
&\sim\sum_{k=1}^{n}{}'\frac{|x_{jn}-x_{n+1,n}|\;|x_{jn}-x_{0n}|}{|x_{kn}-x_{n+1,n}|\;|x_{kn}-x_{0n}|}\left|\ell_{kn}\left(U_n\left(W_\rho^2\right),x\right)\right|+\sum_{k=0,n+1}\cdots\sim\\
&\sim\sum_{k=1}^{n}{}'\frac{u_{jn}(1-u_{jn})}{u_{kn}(1-u_{kn})}\frac{|\triangle u_{kn}|}{|u_{jn}-u_{kn}|}\left(\frac{u_{kn}(1-u_{kn})}{u_{jn}(1-u_{jn})}\right)^{1/4}=\\
&=\sum_{k=1}^{n}{}'\left(\frac{u_{jn}(1-u_{jn})}{u_{kn}(1-u_{kn})}\right)^{3/4}\frac{|\triangle u_{kn}|}{|u_{jn}-u_{kn}|}
\end{aligned}\tag{3.15}$$

considering that the second sum can be considered by $\sum_{k=1,n}\cdots$ (cf. [4, Part 4.8], say).

As before we write

a. Suppose $0 < u_{jn} \leq c < 1$. Then

$$S_1 \sim \sum_{K=1}^{cn}{}' \left(\frac{v_{Jn}}{v_{Kn}}\right)^{3/4} \frac{|\triangle v_{Kn}|}{|v_{Jn} - v_{Kn}|} \sim \sum_{K \leq J/2} \cdots + \sum_{J/2 \leq K \leq 2J}{}' \cdots + \sum_{K=2J}^{cn} \cdots \sim$$

$$\sim \left(\frac{n^2}{J^2}\right)^{1/4} \sum_{K=1}^{J} \frac{K}{n^2} \left(\frac{n^2}{K^2}\right)^{3/4} + \log(2J) + \sum_{K=2J}^{cn} \frac{K}{n^2} \left(\frac{J^2}{n^2}\right)^{3/4} \left(\frac{n^2}{K^2}\right)^{7/4} \sim$$

$$\sim 1 + \log(2J) + 1.$$

The cases **b.** and **c.** correspond to **B.** and **C.** and give

$$S_1 \leq c \qquad \text{and} \qquad S_1 \sim \log n, \qquad \text{respectively.}$$

d. Let $|u_{jn} - u_{kn}| \geq c_1$. Then by (3.15) and using

$$u_{jn}(1 - u_{jn}) \leq \left(\frac{u_{jn} + 1 - u_{jn}}{2}\right)^2 = \frac{1}{4},$$

we get

$$S_1 \leq c \sum_{K=1}^{cn} \frac{|\triangle v_{Kn}|}{v_{Kn}^{3/4}} \sim \sum_{K=1}^{cn} \frac{K}{n^2} \left(\frac{n^2}{K^2}\right)^{3/4} \sim \frac{1}{n^{1/2}} \sum_{K=1}^{n} \frac{1}{K^{1/2}} \leq c.$$

e. The estimation of S_2 is analogous to the Part 4.8 in [4], whence we get that

$$S_2 \sim \log n.$$

So we get (2.9). □

Acknowledgement. The authors thank to the referee for the very thorough and conscientious work.

References

[1] E. Levin and D. Lubinsky, Orthogonal polynomials for exponential weights $x^{2\rho}e^{-2Q(x)}$ on $[0, d)$, *J. of Approx. Theory*, **134** (2005), 199–256.

[2] E. Levin and D. Lubinsky, Orthogonal polynomials for exponential weights $x^{2\rho}e^{-2Q(x)}$ on $[0, d), II$, *J. of Approx. Theory*, **139** (2006), 107–143.

[3] P. Vértesi, An Erdős-type convergence process in weighted interpolation. I (Freud-type weights), *Acta Math. Hungar.*, **91** (2001), 195–215.

[4] L. Szili and P. Vértesi, An Erdős-type convergence process in weighted interpolation. II (Exponential weights on $[-1, 1]$), *Acta Math. Hungar.*, **98(1–2)** (2003), 129–162.

[5] P. Vértesi, Notes on orthogonal polynomials for exponential weights (Root-distances, weighted Lebesgue function), *Acta Math. Hungar.*, **119(4)** (2008), 381–392.
[6] G. Szegö, *Orthogonal Polynomials*, AMS Coll. Publ., Vol. 23, Providence, 1967.
[7] E. Levin and D. S. Lubinsky, *Orthogonal Polynomials for Exponential Weights*, Springer-Verlag, New York-Berlin-Heidelberg, 2001.
[8] J. Szabados, Weighted Lagrange and Hermite–Fejér interpolation on the real line, *Journal of Inequalities and Applications*, **1** (1997), 99–123.