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Francesca Boccuni
Andrea Sereni *Editors*

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Chapter 16
Computability, Finiteness and the Standard Model of Arithmetic

Massimiliano Carrara, Matteo Plebani and Enrico Martino

Abstract This paper investigates the question of how we manage to single out the natural number structure as the intended interpretation of our arithmetical language. Horsten (2012) submits that the reference of our arithmetical vocabulary is determined by our knowledge of some principles of arithmetic on the one hand, and by our computational abilities on the other. We argue against such a view and we submit an alternative answer. We single out the structure of natural numbers through our intuition of the *absolute* notion of finiteness.

Keywords Computational structuralism · Finiteness · Standard model of arithmetic

16.1 Introduction

Claiming that arithmetic is about the natural number structure (N) may sound trite. But since every first-order version of arithmetic has models that are not isomorphic to each other, it is natural to wonder how we manage to single out a class of isomorphic models, i.e. the natural number structure, as the intended interpretation of our arithmetical language.

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According to computational structuralism, the position defended by Horsten (2012) (see also Halbach and Horsten 2005), there are two aspects of our arithmetical practice that contribute to fix the natural number structure as its intended interpretation (Horsten 2012, p. 280):

(I) A *know-that component*: our endorsement of some arithmetical principles, as expressed for instance in Peano Arithmetic (PA).

(II) A *know-how component*: our use of an algorithm of computation to calculate sums.

The major novelty of the proposal lies in step (II), whose point, however, is very easy to be misunderstood. In particular, it might not be always clear exactly which question computational structuralists are addressing. A careful scrutiny of their proposal is thus called for.

16.2 The Question

The question **Q** we are going to address has been nicely phrased by Horsten (2012, p. 276):

(Q) *How do we* manage *to single out the structure of the natural numbers as the* interpretation *of our arithmetical vocabulary?*

Horsten pinpoints that **Q** is not to be understood as the *skeptical* question whether we succeed in singling out the natural number structure as the target of our arithmetical practice. **Q** *presupposes* that we do succeed in that, and just asks *how* this is possible. In Horsten's words:

> We could, with Skolem or Putnam [...] deny that arithmetic has an intended interpretation that is unique up to isomorphism [Putnam 1980]. But I will do the opposite. I will presume that we can isolate the natural-number structure in our referential practice. So I will have no qualms about referring to the natural numbers and operations on them in my philosophical account. The question addressed in this article is how we have managed to refer to the natural-number structure: *how has our reference to the natural-number structure come about?* (Horsten 2012, p. 277)

We think, however, that the relevant question is: what is the standard model of PA? What characterizes it among the infinite plurality of models? Once the peculiar feature of such a model has been characterized, no further question arises about how to refer to it: we refer to it as to the model so characterized. On the other hand, the mere presupposition that we are able to single out a certain privileged model is obviously of no help for identifying it.

16.3 The Computational Structuralist Answer

In a nutshell, Horsten's answer to **Q** is:

"(Thesis 2) The reference of our arithmetical vocabulary is determined by:

(I) our principles of arithmetic together with
(II) our use of an algorithm of computation." (Horsten 2012, p. 280, numeration added)

From (I) and (II) we can reach the conclusion that N is the intended interpretation of our arithmetical vocabulary in the following way: First, our endorsement of some principles of arithmetic, as codified, say, in the axioms of Peano Arithmetic, singles out a class of models (those satisfying the axioms) as possible interpretations of our mathematical practice (element I). This is not enough to rule out non-standard models, of course. But we notice that there is a further aspect in our arithmetical practice: we *use* an algorithm to compute sums (element II). Also this aspect of our arithmetical practice should be reflected in the interpretation of our arithmetical vocabulary: the function associated to '$+$' in the model must be computable.

By the **Church-Turing Thesis**, if a function is intuitively computable, it is Turing-computable. And by **Tennenbaum's theorem,** all models of PA, in which the addition function is Turing-computable, are isomorphic to N (Tennenbaum 1959). So it follows by the **Church-Turing Thesis** and **Tennenbaum's theorem** that all models of PA, in which the addition function is computable, are isomorphic to N (see Horsten 2012, p. 280).

Notice that Horsten makes clear that the notion of computability relevant here is a pre-theoretical one. It is the kind of intuitive notion that we try to model mathematically using notions like Turing computability, but it is not itself a theoretical notion.[1] It is important to stress this point in order to understand Horsten's own position. One version of the so-called 'just more theory' objection (Putnam 1980) applies to the suggestion that the defining feature of the standard model is that in such model addition is a computable function. The objection is that, adding to the axioms of PA the translation into first-order language of the claim that addition is recursive, one still obtains a theory that has non-standard models. One thing Horsten says in reply to this objection (Horsten 2012, p. 287) is that all that the existence of non-standard models shows in this case is that the intuitive notion of computability cannot be completely formalized; but this does not make it an illegitimate notion: the fact that invisible gases cannot be seen is no reason to believe there are no invisible gases, to use a nice example from Liggins (2012). Moreover, Horsten's appeal to an intuitive, pre-theoretical notion is also important for the dialectic of this paper, because we

[1]"The sense in which addition is algorithmically computable is not to be understood as Turing-computability or μ-recursiveness. What is operative here is the pre-theoretical practical sense of computability, which has motivated the mathematical definitions of computability. It is a practical notion of computing on strings of symbols" (Horsten 2012, p. 278).

will also appeal to an intuitive, pre-theoretical notion, more fundamental than that considered by Horsten.[2]

The two main ingredients of Horsten's answer are given by a formal theory like PA and our ability to compute sums, in the pre-theoretical sense of *computation*. In the next section we are going to argue that both these aspects rest on something more fundamental: our ability to generate the relevant syntactical entities that constitute a formal theory like PA and are the basis on which the addition algorithm works. This, in turn, rests on our ability to grasp a primitive notion of finiteness. Observe that what is relevant in our perspective is that *absolute finiteness* is presupposed for the understanding of a formal system as well of a Turing machine.

16.4 Problems with Computational Structuralism

Let us start with (I): our endorsement of some principles of arithmetic.

The knowledge of the principles of arithmetic presupposes, in turn, the knowledge of the *formal* arithmetical language L (including terms and formulas). Horsten says explicitly that the language L at issue is that of first-order Peano arithmetic PA. The problem arises: how can one understand the inductive definition of the arithmetical language?

It is often remarked that syntactical notions like numerals, formulas and so on are informally defined in terms of the notion of *finiteness* (see Field 2001, p. 338): a numeral is defined as a *finite* sequence of signs, formulas are *finite* sequences of symbols meeting certain conditions. And the idea that all these entities can be inductively defined relies on the idea that the inductive clauses describe a procedure for generating each of them in a *finite* number of steps. So the endorsement of the axioms of PA presupposes the notion of *finiteness.*

Something similar holds for (II), i.e. our use of an algorithm for computing sums. First of all, an algorithm is computable in a *finite* sequence of steps. Moreover, computation, as Horsten says, is something we perform on symbols (2012, p. 278). Our use of an algorithm for computation is possible only on the condition of understanding which objects this algorithm works on. In the case of the sum algorithm, one takes numerals as inputs and outputs. And since the sum must operate on every pair of numerals, the notion of sum presupposes the *general* notion of numeral and hence the general notion of a *finite string of signs*. This is more fundamental than the items (I) and (II).

Now, the notion of finiteness at issue is just what serves the purpose of singling out the standard model: this is characterized by the fact that every element has finitely many predecessors. The set of numerals, structured with the usual arithmetical operations, is a paradigmatic example of the standard model of arithmetic.

[2]To show that the notions of intuitive computability/finiteness are legitimate it is not enough to show that they are not illegitimate: one needs also positive reasons to accept them. We provide such reasons (for the finiteness case) in Sects. 16.5 and 16.6 (Thanks to an anonymous referee for pressing us on that).

Question **Q** can, therefore, reformulated as follows:

Q*: how can we grasp the notion of finiteness?

We will discuss this question in the next section.

16.5 The Absolute Notion of Finiteness

As it is well known, the notion of finiteness defined in set theory is relative to a model of set theory. In each model of set theory one defines the set of natural numbers and therefore a notion of finiteness. But the sets of natural numbers of two different models may be non-isomorphic. Of course, one could define the standard model of arithmetic as the structure of the natural numbers of a *standard model of set theory*, which should catch our pre-theoretical notion of finiteness. But that would lead us to the much more problematic question: what is the standard model of set theory? Similarly for the notion of finiteness defined in second order logic (see Weston 1976).

The notion of finiteness we are concerned with is a *primitive absolute* (i. e. independent of any set-theorethical model) notion that no axiomatic system is adequate to capture. It is just this inadequacy that may suggest the skeptical doubt that the alleged notion of *absolute* finiteness is illusory. We want to argue that such a doubt is untenable.

The crucial point is that one can recognize the existence of non-standard models of arithmetic only after having introduced the formal language of arithmetic and having grasped a structure where the axioms of arithmetic hold.

Now, as already stressed, the comprehension of the formal language rests on the intuition of a *finite string* of signs. This intuition is primitive and transcends any practical mastery of constructing particular strings. Such strings, which C. Parsons calls "quasi-concrete" (Parsons 1990), as types of spatial objects, are immediately given to intuition. They are, according to Hilbert, extra-logical objects presupposed by logic, objects whose intelligibility is an essential condition for human reasoning:

> As a condition for the use of logical inferences and the performance of logical operations, something must already be given to our faculty of representation, certain extra-logical concrete objects that are intuitively present as immediate experience prior to all thought. If logical inference is to be reliable, it must be possible to survey these objects completely in all their parts, and the fact that they occur, that they differ from one another, and that they follow each other, or are concatenated, is immediately given intuitively, together with the objects, as something that neither can be reduced to anything else nor requires reduction. This is the basic philosophical position that I consider requisite for mathematics and, in general, for all scientific thinking, understanding and communication. And in mathematics, in particular, what we consider is the concrete signs themselves, whose shape, according to the conception we have adopted, is immediately clear and recognizable. (Hilbert 1926, 376).

It is clear in this passage that Hilbert's finitism is grounded on the notion of absolute finiteness. Ironically, Hilbert's finitism is often considered as formalizable in primitive recursive arithmetic, in spite of the fact that no axiomatic system can catch the intended notion of finiteness.

In virtue of this notion, we are able to regard the usual inductive definition of the formal system of arithmetic as a description of a process generating all the intended syntactical entities. Our grasp of such an ideal process, which leads to any string in a *finite* number of steps, rests on the intuition of *a finite number of steps* and of *a finite sequence of signs*.

Consider, for example, the inductive definition of a string of strokes:

(1) One stroke is a string of strokes;
(2) adding a stroke to a string of strokes one obtains a string of strokes.

To understand the above definition as a description of a generating process, one must possess a priori the intuitive notion of a finite string of strokes, presupposed by clause 2. Otherwise one has to understand the above clauses as mere axioms. In this case what the inductive definition says is only that strings of strokes are certain things satisfying the above clauses. Then the clauses are capable of non-standard interpretations and the notion of string of strokes is indeterminate.

As to the axioms of PA, the evidence for the induction schema rests in turn on the intuitive notion of finiteness. Lacking that, nothing would guarantee that any property, which holds for 0 and is inherited from any number to its successor, holds for all numbers. The intuition that any number is reachable, starting from 0, in a finite number of steps, is essential for the evidence that any number inherits the property at issue from 0. For this reason the evidence of induction rests on the notion of absolute finiteness.

Observe that absolute finiteness plays this role even when the principle of induction is expressed within an informal language. Concerning this point, it should be noted that, though the axioms of arithmetic fail to determine the standard model, their evidence presupposes the grasp of the standard model.[3]

Our claim is that the standard model is characterized by the fact that every natural number has finitely many predecessors, where the notion of finiteness here involved is absolute and primitive. This means that it cannot be *defined* in terms of more elementary notions.

Perhaps Horsten could object that the notion of finiteness can be grasped, in turn, through a training in practical computation. We do not deny the possibility that the intuition of a finite string of signs is acquired by an act of abstraction suggested by the computational practice. In this case, however, such an abstraction is already suggested by the mere practice of counting; the practice of adding, invoked by Horsten, is overwhelming.[4]

[3] Observe that a skeptical, who denies the existence of the standard model, should deny the evidence of the principle of induction.

[4] It is still worth noticing one point. Horsten thinks that every attempt to characterize the standard model by making reference to our ability to count up to every natural number works only if certain empirical conditions are met. Teaching how to count and adding that all the natural numbers can be generated by counting succeeds in excluding non-standard models only if time has the right structure (Horsten 2012, p. 283). Indeed, if time had an irregular structure, the procedure to generate natural numbers by counting would give unintended results. But the same holds for the addition algorithm,

16.6 Skepticism Reconsidered

Button and Smith (2012) argue that neither *computability* nor *finiteness* can answer skeptical challenges about our ability to isolate the standard model of arithmetic. As there are non-standard interpretations of our arithmetical vocabulary, they argue, there are non-standard interpretations of the theories formalizing our notion of computability and finiteness.

If the skeptic takes for granted that we have a well-determined arithmetical language and denies that this has a privileged interpretation, then we can reject her position since, as we saw, the standard model is exemplified by the numerals.

On the other hand, a more radical skeptic maintains that the inductive definition of formulas and terms is to be regarded not as a description of a generating process but—in turn—as an axiomatic system that fails to single out the intended structure of the syntactical entities. This position has been illustrated above with the example of the definition of a finite string of strokes.

This is a radical form of skepticism, namely skepticism about the possibility of singling out a well-determined infinite formal language. Such a position destroys the very aim of formalization to make precise the fundamental mathematical notions of formula and proof (see Field 2001, p. 338).

16.7 Conclusions

In this paper we have considered the question of how can we single out the standard model of arithmetic. We have analyzed Horsten's answer based on (1) our knowledge of some principles of arithmetic and (2) our mastery of the algorithm for computing sums. We argued that the understanding of item (1) rests on the absolute primitive notion of finiteness not formalizable by any axiomatic system. This notion is all that is required to single out the standard model of arithmetic. Besides it is essential even for understanding any infinite formal system. As to item (2), absolute finiteness transcends any computational practice. And if it is grasped through an act of extrapolation on our mastery of manipulating certain strings of signs, then it is already grasped by an act of abstraction on the mere mastering of counting.

(Footnote 4 continued)
mastery of which is, according to Horsten, one of the factors that determine the reference of our arithmetical vocabulary. For the addition algorithm to work, nature must play its part by having a space dimension structured in the right way: if space contracted when concatenating two numerals, then the result of summing m and n would not be the intended one. Anyway, we agree that the intuition of the generating process of the numerals involves a naive intuitive notion of time. For our purposes, it doesn't matter whether our naive notion of time is in agreement with the real structure of time or not.

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