Linear fractional transformations and non-linear leaping convergents of some continued fractions

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Continued fractions, which are classical objects in number theory, represent any real number α_0 by means of a sequence a_i of integers:

$$\alpha_0 = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}}.$$

The integers a_i, called the *partial quotients*, can be found by

$$\begin{cases} a_k = [\alpha_k] & \text{(that is, the integer part of } \alpha_k) \\ \alpha_{k+1} = \cfrac{1}{\alpha_k - a_k} & \text{if } \alpha_k \text{ is not an integer} \end{cases} \qquad \text{for } k = 0, 1, 2, \ldots.$$

The *n-th convergent* of α_0 is the rational number

$$\frac{p_n}{q_n} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}}.$$

For short, we will write

$$\alpha_0 = [a_0, a_1, a_2, \ldots, a_n, \ldots], \qquad \frac{p_n}{q_n} = [a_0, a_1, \ldots, a_n]$$

and we call the *n*-th *tail* of α_0 the continued fraction

$$t_n = [a_{n+1}, a_{n+2}, \ldots].$$

Let us consider a linear fractional transformation $\sigma(x) = (Ax + B)/(Cx + D)$ with $A, B, C, D \in \mathbb{Z}$. The determinant of σ is $\det \sigma := AD - BC$. It is well known that α_0 and $\sigma(\alpha_0)$ have the same tails, for n sufficiently large, if and only if $|\det \sigma| = 1$ (see for example [30]). In this paper, we show a relation between the tail of $\sigma(\alpha_0)$ and the tail of α_0 when $|\det \sigma| = 2$ and the partial quotients of α_0 have some specific behaviors. When the value of $|\det \sigma|$ increases, some numerical experiments show that the previous relations are very hard to find. Hence, in the general case, that is, when $|\det \sigma| = d$ with $d \neq 0, \pm 1$, it is very hard to predict the behaviour of $\sigma(\alpha_0)$. The proofs in this paper are greatly based on the fact that σ has determinant ± 2, in fact the main idea is to write the matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ as an unimodular matrix multiplied by auxiliary matrices that commutate with the $R - L$ matrices used by Raney [29] for representing continued fractions, like in proposition 2.1. When $|\det \sigma| > 2$, similar relations are more hard to give and we think that other ways should be used for approaching the problem for general values of determinants. However, the results of this paper could help in finding a solution to the general case.

The Gosper algorithm is a useful tool for evaluating a linear fractional transformation of a given continued fraction and it can be also used for determining sums and products of continued fractions. The original algorithm was presented in [9, Item 101B]. Some descriptions about the Gosper algorithm can be also found in [7, pp. 347–352], [10], [22] and [31]. Another standard algorithm, based on the representation of a continued fraction as a product of powers of particular matrices, is due to Raney [29]. These algorithms allow us to obtain the partial quotients of $\sigma(\alpha_0)$ starting from the partial quotients of α_0; however they are not suitable for determining general relations, since they would need to use the values of the partial quotients of α_0. Some authors have studied the linear fractional transformation of continued fractions with bounded partial quotients [21], [28]. In [23], the author presented an algorithm for determining the continued fraction expansion of a linear fractional transformation of power series. See also [1], [3].

The results obtained in this paper can also be used to determine the linear fractional transformation of certain Hurwitzian and Tasoevian continued fractions, as well as studying their leaping convergents.

Definition 1. *A* Hurwitz continued fraction *is a quasi-periodic continued fraction of the kind*

$$[a_0, \ldots, a_r, \overline{f_1(k), \ldots, f_s(k)}]_{k=h}^{+\infty} = [a_0, \ldots, a_r, f_1(h), \ldots, f_s(h), f_1(h+1), \ldots, f_s(h+1), \ldots],$$

for certain integers r, s, h, where $f_1(k), \ldots f_s(k)$ are polynomials in k with rational coefficients.

Definition 2. *A* Tasoev continued fraction *is a quasi-periodic continued fraction of the kind*

$$[a_0, \ldots, a_r, \overline{g_1(k), \ldots, g_s(k)}]_{k=h}^{+\infty} = [a_0, \ldots, a_r, g_1(h), \ldots, g_s(h), g_1(h+1), \ldots, g_s(h+1), \ldots],$$

for certain integers r, s, h, where $g_1(k), \ldots g_s(k)$ are exponentials in k with rational coefficients.

A famous example of Hurwitz continued fraction is this expansion of e:

$$e = [2, 1, 2, 1, 1, 4, 1, \ldots] = [2, \overline{1, 2k, 1}]_{k=1}^{+\infty}.$$

There are other many known Hurwitz continued fractions that represent *e-type* numbers like

$$e^{1/m} = [1, \overline{m(2k-1) - 1, 1, 1}]_{k=1}^{+\infty} \quad \text{and} \quad \frac{e-1}{e+1} = [0, \overline{4k+2}]_{k=0}^{+\infty}.$$

Some numbers expressed by means of the tangent function also have a quasi-periodic continued fraction expansion, for example

$$\tan 1 = [1, \overline{2k-1, 1}]_{k=1}^{+\infty}$$

and

$$\sqrt{\frac{m}{n}} \tan \frac{1}{\sqrt{mn}} = [0, n-1, \overline{1, (4k-1)m - 2, 1, (4k+1)n - 2}]_{k=1}^{+\infty}.$$

The properties of Hurwitz continued fractions is a classical research field; see for example [2], [24], [25], [27], [32]. Many recent studies have dealt with their leaping convergents. Usually, given the convergents p_i/q_i of a continued fraction, the leaping convergents are p_{ri+j}/q_{ri+j}, with $r \geq 2$ and $0 \leq j \leq r-1$ fixed integers, for $i = 0, 1, 2, \ldots$. In the case of Hurwitz continued fractions, the leaping step r is usually (but not necessarily) chosen to be equal to the length of the period.

In [4], the author found a recurrence formula for the leaping convergents p_{3i+1}/q_{3i+1} of the continued fraction of e. A similar result was found by Komatsu [12] for the leaping convergents p_{3i}/q_{3i} of $e^{1/m}$. In [17], Komatsu extended these results to more general Hurwitz continued fractions, and in [5] a recurrence formula was also found for non-regular Hurwitz continued fractions. In [18], Komatsu proved other recurrence formulas and closed forms for leaping convergents of generalized tanh-type Hurwitz continued fractions. In several works, the goal was to find closed forms for the convergents of some Hurwitz and Tasoev continued fractions and study their leaping convergents; see [11], [13], [15], [16], [19], and [20]. As a consequence of our results, we find relations involving leaping convergents of some Hurwitz and Tasoev continued fractions having steps provided by nonlinear functions.

2   Linear fractional transformations of some continued fractions

In this section we study the relations between the continued fraction expansions of α_0 and $\sigma(\alpha_0)$, when $|\det \sigma| = 2$. We deal with infinite continued fractions of four different types.

Definition 3.   *1) The continued fraction*

$$[e_0, e_1, e_2, \ldots, e_i, \ldots],$$

where e_i, $i = 0, 1, \ldots$, are even positive integers, is called type 1 (CF1).

2) The continued fraction

$$[d_0, d_1, d_2, \ldots, d_i, \ldots],$$

where d_i, $i = 0, 1, \ldots$, are odd positive integers, is called type 2 (CF2).

3) The continued fraction

$$[d_0, e_1, d_2, e_3 \ldots, d_{2i}, e_{2i+1}, \ldots],$$

where d_{2i}, e_{2i+1}, $i = 0, 1, \ldots$, are odd and even positive integers, respectively, is called type 3 (CF3).

4) The continued fraction

$$[e_0, d_1, e_2, d_3 \ldots, e_{2i}, d_{2i+1} \ldots],$$

where e_{2i}, d_{2i+1}, $i = 0, 1, \ldots$, are even and odd positive integers, respectively, is called type 4 (CF4).

In the following, we will take d_0 and e_0 to be positive integers without loss of generality. Now we give a classification of linear fractional transformations σ with integer coefficients and $|\det \sigma| = 2$.

Proposition 2.1. *Let σ be a linear fractional transformation, i.e., $\sigma(x) = \frac{Ax+B}{Cx+D}$, where x is a real number and $A, B, C, D \in \mathbb{Z}$. Let S be the associated matrix*

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

If $|\det \sigma| = |\det S| = 2$, then

1) $S = TM$,

2) $S = TMR$ or

3) $S = TMRJ$,

where

$$M = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad R = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and T is a certain unimodular matrix with integer entries.

Proof. We observe that only one of the following situations can hold:

- The entries of the first row of S have the same parity and the entries of the second row have the same parity. In this case

$$S = TM \quad \text{with} \quad T = \begin{pmatrix} \frac{A+B}{2} & \frac{A-B}{2} \\ \frac{C+D}{2} & \frac{C-D}{2} \end{pmatrix}.$$

- In the first column of S there is at least one odd entry and the remaining entries are all even. In this case

$$S = TMR \quad \text{with} \quad T = \begin{pmatrix} \frac{B}{2} & \frac{2A-B}{2} \\ \frac{D}{2} & \frac{2C-D}{2} \end{pmatrix}.$$

- In the second column of S there is at least one odd entry and the remaining entries are all even. In this case

$$S = TMRJ \quad \text{with} \quad T = \begin{pmatrix} \frac{A}{2} & \frac{2B-A}{2} \\ \frac{C}{2} & \frac{2D-C}{2} \end{pmatrix}.$$

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Other possibilities for S are forbidden by the condition $|\det(S)| = 2$. $\qquad\qquad\square$

Lemma 2.2. *Let us consider the four matrices*

$$M = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad R = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then the following equalities hold:

a)

$$JR^h = L^h J, \quad JL^h = R^h J,$$

$$\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} L^h = R^{2h} \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} R^h = L^{2h} \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$$

for any integer h;

b)

$$MR^h = RL^{\frac{h-1}{2}} \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} R^h = L^{\frac{h-1}{2}} \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} L^h = R^{\frac{h-1}{2}} LM, \quad \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix} L^h = RLR^{\frac{h-1}{2}} \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$$

for any odd integer h;

c)

$$MR^h = RL^{\frac{h-2}{2}} \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} R^h = L^{\frac{h}{2}} \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix} L^h = RLR^{\frac{h-2}{2}} LM$$

for any even integer h.

Proof. All these identities are straightforward to check. $\qquad\qquad\square$

Now, we are ready to describe the continued fraction expansion of $\sigma(\alpha_0)$, when α_0 is of type CF1, CF2, CF3 or CF4. Let us recall that a continued fraction $[a_0, a_1, a_2, a_3, \dots]$ is equivalent to the matrix product

$$R^{a_0} L^{a_1} R^{a_2} L^{a_3} \dots;$$

see [29]. We will use this fact, Proposition 2.1, and Lemma 2.2 for proving the following theorem.

Theorem 2.3. *1) If α_0 is a continued fraction of type CF1, then it is equivalent to one of the following matrix products:*

$$T \prod_{k=1}^{+\infty} \left(RL^{\frac{e_{k-1}-2}{2}} RLR^{\frac{e_k-2}{2}} L \right), \tag{2.1}$$

$$TR \prod_{k=1}^{+\infty} \left(L^{\frac{e_{2k-2}}{2}} R^{2e_{2k-1}} \right), \tag{2.2}$$

$$TR^{2e_0+1} \prod_{k=1}^{+\infty} \left(L^{\frac{e_{2k-1}}{2}} R^{e_{2k}} \right). \tag{2.3}$$

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2) If α_0 is a continued fraction of type CF2, then it is equivalent to one of the following matrix products:

$$T \prod_{k=1}^{+\infty} \left(RL^{\frac{d_{3k-3}-1}{2}} R^{2d_{3k-2}} L^{\frac{d_{3k-1}-1}{2}} RLR^{\frac{d_{3k}-1}{2}} L^{2d_{3k+1}} R^{\frac{d_{3k+2}-1}{2}} L \right), \qquad (2.4)$$

$$TRL^{\frac{d_0-1}{2}} R \prod_{k=1}^{+\infty} \left(LR^{\frac{d_{3k-2}-1}{2}} L^{2d_{3k-1}} R^{\frac{d_{3k}-1}{2}} LRL^{\frac{d_{3k+1}-1}{2}} R^{2d_{3k+2}} L^{\frac{d_{3k+3}-1}{2}} R \right), \quad (2.5)$$

$$TR^{2d_0+1} L^{\frac{d_1-1}{2}} R \prod_{k=1}^{+\infty} \left(LR^{\frac{d_{3k-1}-1}{2}} L^{2d_{3k}} R^{\frac{d_{3k+1}-1}{2}} LRL^{\frac{d_{3k+2}-1}{2}} R^{2d_{3k+3}} L^{\frac{d_{3k+4}-1}{2}} R \right). \qquad (2.6)$$

3) If α_0 is a continued fraction of type CF3, then it is equivalent to one of the following matrix products:

$$T \prod_{k=1}^{+\infty} \left(RL^{\frac{d_{4k-4}-1}{2}} R^{2e_{4k-3}} L^{\frac{d_{4k-2}-1}{2}} RLR^{\frac{e_{4k-1}-2}{2}} L \right),$$

$$TRL^{\frac{d_0-1}{2}} RLR^{\frac{e_1-2}{2}} L \prod_{k=1}^{+\infty} \left(RL^{\frac{d_{4k-2}-1}{2}} R^{2e_{4k-1}} L^{\frac{d_{4k}-1}{2}} RLR^{\frac{e_{4k+1}-2}{2}} L \right),$$

$$TR^{2d_0+1} \prod_{k=1}^{+\infty} \left(L^{\frac{e_{2k-1}}{2}} R^{2d_{2k}} \right).$$

4) If α_0 is a continued fraction of type CF4, then it is equivalent to one of the following matrix products:

$$TRL^{\frac{e_0-2}{2}} R \prod_{k=1}^{+\infty} \left(LR^{\frac{d_{4k-3}-1}{2}} L^{2e_{4k-2}} R^{\frac{d_{4k-1}-1}{2}} LRL^{\frac{e_{4k}-2}{2}} R \right),$$

$$TR \prod_{k=1}^{+\infty} \left(L^{\frac{e_{2k-2}}{2}} R^{2d_{2k-1}} \right), \qquad (2.7)$$

$$TR^{2e_0+1} L^{\frac{d_1-1}{2}} RLR^{\frac{e_2-2}{2}} L \prod_{k=1}^{+\infty} \left(RL^{\frac{d_{4k-1}-1}{2}} R^{2e_{4k}} L^{\frac{d_{4k+1}-1}{2}} RLR^{\frac{e_{4k+2}-2}{2}} L \right).$$

Proof. We prove the theorem for continued fractions of type CF1 and we give a sketch of the proof for continued fractions of type CF2. The results concerning continued fractions of type CF3 and CF4 can be easily proved with the same techniques. To prove (2.1), we consider the infinite product

$$MR^{e_0} L^{e_1} R^{e_2} L^{e_3} \ldots$$

and we use the first relation in Lemma 2.2 part c), obtaining

$$RL^{\frac{e_0-2}{2}} \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix} L^{e_1} R^{e_2} L^{e_3} \ldots.$$

Using the third relation in Lemma 2.2 part c), we have

$$RL^{\frac{e_0-2}{2}} RLR^{\frac{e_1-2}{2}} LMR^{e_2} L^{e_3} \ldots.$$

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Now, we may iterate the applications of these two matrix relations, so that the matrix M vanishes on the right, and we find the product on the right of T in (2.1). Now, we deal with (2.2) and consider the infinite product

$$MR^{e_0+1}L^{e_1}R^{e_2}L^{e_3}\ldots.$$

Applying the first relation in Lemma 2.2 part b), since $e_0 + 1$ is an odd integer, we get

$$RL^{\frac{e_0}{2}}\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}L^{e_1}R^{e_2}L^{e_3}\ldots.$$

Thanks to the third relation in Lemma 2.2 part a), we have

$$RL^{\frac{e_0}{2}}R^{2e_1}\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}R^{e_2}L^{e_3}\ldots,$$

then, using the second relation in Lemma 2.2 part c), we find

$$RL^{\frac{e_0}{2}}R^{2e_1}L^{\frac{e_2}{2}}\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}L^{e_3}\ldots.$$

Finally, we can apply the last two relations so that the matrix $\left(\begin{smallmatrix} 0 & 2 \\ 1 & 0 \end{smallmatrix}\right)$ vanishes on the right, obtaining the product on the right of T in (2.2). For a proof of (2.3), we consider the infinite product

$$MRJR^{e_0}L^{e_1}R^{e_2}L^{e_3}\ldots.$$

Then we start using the first two relations in Lemma 2.2 part a), so that the matrix J vanishes on the right and we find

$$MRL^{e_0}R^{e_1}L^{e_2}R^{e_3}L^{e_4}\ldots.$$

Consequently, we apply the first relation in Lemma 2.2 part b), which gives

$$R\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}L^{e_0}R^{e_1}L^{e_2}R^{e_3}L^{e_4}\ldots.$$

So we can infinitely iterate the third relation in Lemma 2.2 part a) and the second relation in Lemma 2.2 part c), so that the matrix $\left(\begin{smallmatrix} 0 & 2 \\ 1 & 0 \end{smallmatrix}\right)$ vanishes on the right and we retrieve the product on the right of T in (2.3). In the case of contniued fractions of type CF2, for proving (2.4), we take into account that

$$MR^{d_0}L^{d_1}\cdots = RL^{\frac{d_0-1}{2}}\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}R^{d_2}L^{d_3}\cdots = RL^{\frac{d_0-1}{2}}R^{2d_1}\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}R^{d_2}L^{d_3}\cdots =$$

$$= RL^{\frac{d_0-1}{2}}R^{2d_1}L^{\frac{d_2-1}{2}}\begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix}L^{d_3}R^{d_4}\cdots = RL^{\frac{d_0-1}{2}}R^{2d_1}L^{\frac{d_2-1}{2}}RLR^{\frac{d_3-1}{2}}\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}R^{d_4}L^{d_5}\cdots =$$

$$= RL^{\frac{d_0-1}{2}}R^{2d_1}L^{\frac{d_2-1}{2}}RLR^{\frac{d_3-1}{2}}L^{2d_4}\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}L^{d_5}R^{d_6}\cdots =$$

$$= RL^{\frac{d_0-1}{2}}R^{2d_1}L^{\frac{d_2-1}{2}}RLR^{\frac{d_3-1}{2}}L^{2d_4}R^{\frac{d_5-1}{2}}LMR^{d_6}L^{d_7}\cdots.$$

Similarly, for the purpose of proving (2.5), we observe that

$$MR^{d_0+1}L^{d_1}\cdots = RL^{\frac{d_0-1}{2}}\begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix}L^{d_1}R^{d_2}\cdots = RL^{\frac{d_0-1}{2}}RLR^{\frac{d_1-1}{2}}\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}R^{d_2}L^{d_3}\cdots =$$

$$= RL^{\frac{d_0-1}{2}} RLR^{\frac{d_1-1}{2}} L^{2d_2} \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} L^{d_3} R^{d_4} \cdots = RL^{\frac{d_0-1}{2}} RLR^{\frac{d_1-1}{2}} L^{2d_2} R^{\frac{d_3-1}{2}} LMR^{d_4} L^{d_5} \cdots .$$

Finally, for a proof of (2.6), we have

$$MRJR^{d_0} L^{d_1} R^{d_2} L^{d_3} R^{d_4} L^{d_5} \cdots = MRL^{d_0} R^{d_1} L^{d_2} R^{d_3} L^{d_4} R^{d_5} \cdots =$$

$$= R \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} L^{d_0} R^{d_1} \cdots = R^{2d_0+1} \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} R^{d_1} \cdots = R^{2d_0+1} L^{\frac{d_1-1}{2}} \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix} L^{d_2} R^{d_3} \cdots =$$

$$= R^{2d_0+1} L^{\frac{d_1-1}{2}} RLR^{\frac{d_2-1}{2}} \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} R^{d_3} \cdots = R^{2d_0+1} L^{\frac{d_1-1}{2}} RLR^{\frac{d_2-1}{2}} L^{2d_3} \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} L^{d_4} \cdots =$$

$$= R^{2d_0+1} L^{\frac{d_1-1}{2}} RLR^{\frac{d_2-1}{2}} L^{2d_3} R^{\frac{d_4-1}{2}} LMR^{d_5} L^{d_6} \cdots .$$

$\square$

Since T is an unimodular matrix, for n sufficiently large, we find an explicit expression for the tails of $\sigma(\alpha_0)$, where α_0 is a continued fraction of type CF1, CF2, CF3 or CF4. We will describe these tails in the next corollary.

Corollary 2.4. • *If α_0 is a continued fraction of type CF1 and $e_{k-1} \geq 4$ for all $k \geq k_0 \geq 1$, then the tail of $\sigma(\alpha_0)$ is*

$$\left[\frac{e_{k-1}-2}{2}, 1, 1 \right]_{k=k_0}^{+\infty} ; \tag{2.8}$$

otherwise it is

$$\left[\frac{e_{2k-2}}{2}, 2e_{2k-1} \right]_{k=k_0}^{+\infty} , \tag{2.9}$$

$$\left[\frac{e_{2k-1}}{2}, 2e_{2k} \right]_{k=k_0}^{+\infty} . \tag{2.10}$$

• *If α_0 is a continued fraction of type CF2 and $d_{3k-3} \geq 3$ and $d_{3k-1} \geq 3$ for all $k \geq k_0 \geq 1$, then the tail of $\sigma(\alpha_0)$ is*

$$\left[\frac{d_{3k-3}-1}{2}, 2d_{3k-2}, \frac{d_{3k-1}-1}{2}, 1, 1 \right]_{k=k_0}^{+\infty} ; \tag{2.11}$$

if $d_{3k-2} \geq 3$ and $d_{3k} \geq 3$ for all $k \geq k_0 \geq 1$, then it is

$$\left[\frac{d_{3k-2}-1}{2}, 2d_{3k-1}, \frac{d_{3k}-1}{2}, 1, 1 \right]_{k=k_0}^{+\infty} ; \tag{2.12}$$

if $d_{3k-1} \geq 3$ and $d_{3k+1} \geq 3$ for all $k \geq k_0 \geq 1$, then it is

$$\left[\frac{d_{3k-1}-1}{2}, 2d_{3k}, \frac{d_{3k+1}-1}{2}, 1, 1 \right]_{k=k_0}^{+\infty} . \tag{2.13}$$

• *If α_0 is a continued fraction of type CF3 and $d_{4k-4} \geq 3$, $d_{4k-2} \geq 3$ and $e_{4k-1} \geq 4$ for all $k \geq k_0 \geq 1$, then the tail of $\sigma(\alpha_0)$ is*

$$\left[\frac{d_{4k-4}-1}{2}, 2e_{4k-3}, \frac{d_{4k-2}-1}{2}, 1, 1, \frac{e_{4k-1}-2}{2}, 1, 1 \right]_{k=k_0}^{+\infty} ; \tag{2.14}$$

if $d_{4k-2} \geq 3$, $d_{4k} \geq 3$ and $e_{4k+1} \geq 4$ for all $k \geq k_0 \geq 1$, then it is

$$\left[\frac{d_{4k-2} - 1}{2}, 2e_{4k-1}, \frac{d_{4k} - 1}{2}, 1, 1, \frac{e_{4k+1} - 2}{2}, 1, 1\right]_{k=k_0}^{+\infty} ; \qquad (2.15)$$

otherwise it is

$$\left[\frac{e_{2k-1}}{2}, 2d_{2k}\right]_{k=k_0}^{+\infty} . \qquad (2.16)$$

- *If α_0 is a continued fraction of type CF4 and $d_{4k-3} \geq 3$, $d_{4k-1} \geq 3$ and $e_{4k} \geq 4$ for all $k \geq k_0 \geq 1$, then the tail of $\sigma(\alpha_0)$ is*

$$\left[\frac{d_{4k-3} - 1}{2}, 2e_{4k-2}, \frac{d_{4k-1} - 1}{2}, 1, 1, \frac{e_{4k} - 2}{2}, 1, 1\right]_{k=k_0}^{+\infty} ; \qquad (2.17)$$

if $d_{4k-1} \geq 3$, $d_{4k+1} \geq 3$ and $e_{4k+2} \geq 4$ for all $k \geq k_0 \geq 1$, then

$$\left[\frac{e_{2k-2}}{2}, 2d_{2k-1}\right]_{k=k_0}^{+\infty} ,$$

$$\left[\frac{d_{4k-1} - 1}{2}, 2e_{4k}, \frac{d_{4k+1} - 1}{2}, 1, 1, \frac{e_{4k+2} - 2}{2}, 1, 1\right]_{k=k_0}^{+\infty} . \qquad (2.18)$$

Proof. The proof is straightforward, thanks to the matrix representations described in Theorem 2.3. $\qquad\qquad\square$

Remark 2.5. When T has nonnegative entries and $\det T = 1$, then T can be also written as a product of powers of R and L matrices, with exponents given by nonnegative integers; see [29]. Thus, in this case, by Theorem 2.3, the relations for the tail of $\sigma(\alpha_0)$ hold for $n = 1$.

3 Relations for the convergents

Let x be a continued fraction and $H_n = u_n/v_n$ its convergents. Given $B_n = \sigma(H_n)$ and $\beta = \sigma(x)$, we clearly have

$$\lim_{n \to +\infty} B_n = \beta.$$

In general, the B_n's do not coincide with the convergents U_n/V_n of β. However, when $|\det \sigma| = 2$ and x is a continued fraction of type CF1, CF2, CF3 or CF4, we have $B_n = U_t/V_t$ for certain values of n and t defined by nonlinear functions. We start proving a theorem which highlights nonlinear recurrence relations involving the sequences $\{U_t\}_{t \geq 0}$ and $\{V_t\}_{t \geq 0}$.

Theorem 3.1. *Using the notation of Corollary 2.4, let us consider $p_0 = k_0 - 1$ and define the functions*

$$g(p) = p_0 + p + 2\left\lfloor\frac{p}{3}\right\rfloor, \quad h(p) = p_0 + 2p - 1 + \sin\left(\tfrac{1}{2}(p+1)\pi\right),$$

$$v(p) = 2^{\sin\left(\left(p+1+\lfloor\frac{p+2}{3}\rfloor\right)\frac{\pi}{2}\right)}, \quad z(p) = 2^{\sin\left(\frac{1}{2}(p+2)\pi\right)},$$

for $p \geq 2$. Then the sequences $\{U_t\}_{t \geq 0}$ and $\{V_t\}_{t \geq 0}$ satisfy one of the following nonlinear recurrence relations:

1)

$$U_{s(p)} = e_{l_1(p)}U_{s(p-1)} + U_{s(p-2)}, \quad V_{s(p)} = e_{l_1(p)}V_{s(p-1)} + V_{s(p-2)}, \qquad (3.1)$$

when x is a continued fraction of type CF1, with tail given by 2.8, where $s(p) = p_0 + 3p$ and $l_1(p) = k_0 + p - 2$;

2)

$$U_{g(p)} = G(p)U_{g(p-1)} + v(p)U_{g(p-2)}, \quad V_{g(p)} = G(p)V_{g(p-1)} + v(p)V_{g(p-2)}, \qquad (3.2)$$

when x is a continued fraction of type CF2 expansion, where

$$G(p) = d_{l_2(p)}2^{\sin\left(\left(p+2+\lfloor\frac{p+1}{3}\rfloor\right)\frac{\pi}{2}\right)}$$

and

$$l_2(p) = \begin{cases} 3k_0 + p - 4 & \text{for tail (2.11)} \\ 3k_0 + p - 3 & \text{for tail (2.12)} \\ 3k_0 + p - 2 & \text{for tail (2.13)}; \end{cases}$$

3)

$$U_{h(p)} = H(p)U_{h(p-1)} + z(p)U_{h(p-2)}, \quad V_{h(p)} = H(p)V_{h(p-1)} + z(p)V_{h(p-2)}, \qquad (3.3)$$

when x is a continued fraction of type CF3 or CF4, where

$$H(p) = a_{l_{3,4}(p)}2^{\sin\left(\left(p+2+\lfloor\frac{p+2}{4}\rfloor-\lfloor\frac{p}{4}\rfloor\right)\frac{\pi}{2}\right)}$$

and

$$l_{3,4}(p) = \begin{cases} 4k_0 + p - 5 & \text{for tail (2.14)} \\ 4k_0 + p - 3 & \text{for tail (2.15)} \\ 4k_0 + p - 4 & \text{for tail (2.17)} \\ 4k_0 + p - 2 & \text{for tail (2.18)}, \end{cases}$$

with $a_{l_{3,4}(p)} = e_{l_{3,4}(p)}$ if p is even and $a_{l_{3,4}(p)} = d_{l_{3,4}(p)}$ if p is odd.

Proof. 1. The following equalities clearly hold, for all $p \geq 2$:

$$U_{p_0+3p} = U_{p_0+3p-1} + U_{p_0+3p-2}, \quad U_{p_0+3p-1} = U_{p_0+3p-2} + U_{p_0+3p-3}, \qquad (3.4)$$

$$U_{p_0+3p-2} = \tfrac{1}{2}(e_{k_0+p-2} - 2)U_{p_0+3p-3} + U_{p_0+3p-4}, \qquad (3.5)$$

$$U_{p_0+3p-3} = U_{p_0+3p-4} + U_{p_0+3p-5}, \quad U_{p_0+3p-4} = U_{p_0+3p-5} + U_{p_0+3p-6}. \qquad (3.6)$$

From (3.4) and (3.6), we find

$$2U_{p_0+3p-4} = U_{p_0+3p-3} + U_{p_0+3p-6}, \qquad (3.7)$$

$$U_{p_0+3p} = U_{p_0+3p-3} + 2U_{p_0+3p-2}. \qquad (3.8)$$

Finally, using (3.5) and (3.7) in (3.8), we obtain the first recurrence of (3.1); the second one can be proved similarly.

2. We deal with three different possibilities: $p = 3m$, $p = 3m + 1$ or $p = 3m + 2$, where m is a positive integer. First we observe that

$$g(3m) = p_0 + 5m, \quad g(3m + 1) = p_0 + 5m + 1, \quad g(3m + 2) = p_0 + 5m + 2,$$

$$g(3m - 1) = p_0 + 5m - 3, \quad g(3m - 2) = p_0 + 5m - 4,$$

$$v(3m) = 2, \quad v(3m + 1) = \tfrac{1}{2}, \quad v(3m + 2) = 1,$$

$$G(3m) = d_{l_2(3m)}, \quad G(3m + 1) = \tfrac{1}{2}d_{l_2(3m+1)}, \quad G(3m + 2) = 2d_{l_2(3m+2)}.$$

Since $\sigma(x)$ has one of the tails (2.11), (2.12) or (2.13) when x is a continued fraction of type CF2, we find the following relations:

$$U_{p_0+5m-1} = U_{p_0+5m-2} + U_{p_0+5m-3}, \quad U_{p_0+5m-2} = \tfrac{1}{2}(d_{l_2(3m)} - 1)U_{p_0+5m-3} + U_{p_0+5m-4},$$

$$U_{p_0+5m} = U_{p_0+5m-1} + U_{p_0+5m-2}, \quad U_{p_0+5m+1} = \tfrac{1}{2}(d_{l_2(3m+1)} - 1)U_{p_0+5m} + U_{p_0+5m-1},$$

$$\tag{3.9}$$

from which we have

$$U_{g(3m)} = U_{p_0+5m} = d_{l_2(3m)}U_{p_0+5m-3} + 2U_{p_0+5m-4}$$
$$= G(3m)U_{g(3m-1)} + v(3m)U_{g(3m-2)}, \quad \text{(3.10)}$$

$$U_{p_0+5m-1} = \tfrac{1}{2}(U_{p_0+5m} + U_{p_0+5m-3}). \tag{3.11}$$

Using (3.11) and (3.9), we get

$$U_{g(3m+1)} = U_{p_0+5m+1} = \tfrac{1}{2}\left(d_{l_2(3m+1)}U_{p_0+5m} + U_{p_0+5m-3}\right)$$
$$= G(3m + 1)U_{g(3m)} + v(3m + 1)U_{g(3m-1)}.$$

Moreover, from

$$U_{g(3m+2)} = U_{p_0+5m+2} = 2d_{l_2(3m+2)}U_{p_0+5m} + U_{p_0+5m+1}$$
$$= G(3m + 2)U_{g(3m+1)} + v(3m + 2)U_{g(3m)},$$

the proof of the first recurrence in (3.2) is complete. The second recurrence can be proved similarly.

3. The proof of recurrences (3.3) is similar, considering the four different possibilities $p = 4m$, $p = 4m + 1$, $p = 4m + 2$ and $p = 4m + 3$, where m is a positive integer.

□

Now we are ready to prove when the B_n's are equal to some convergents of β.

Theorem 3.2. *Let σ be a linear fractional transformation with $|\det \sigma| = 2$ and let x be a continued fraction. Given $\{u_t/v_t\}_{t\geq 0}$ the convergents of x, $\{U_t/V_t\}_{t\geq 0}$ the convergents of $\sigma(x)$, and $B_t = \sigma(u_t/v_t)$ for all $t \geq 0$, we have that*

- *if x is a continued fraction of type CF1 (with tail (2.8)), CF2, CF3 (with tail (2.14) or (2.15)), or CF4 (with tail (2.17) or (2.18)), then one of the following equalities hold*

$$\frac{U_{s(p)}}{V_{s(p)}} = B_{l_1(p)}, \tag{3.12}$$

$$\frac{U_{g(p)}}{V_{g(p)}} = B_{l_2(p)}, \tag{3.13}$$

$$\frac{U_{h(p)}}{V_{h(p)}} = B_{l_{3,4}(p)} \tag{3.14}$$

for any $p \geq 3$, where g, s, h, l_1, l_2, $l_{3,4}$ are the functions defined in Theorem 3.1;

- *if x is a continued fraction expansion of type CF1 (with tails (2.9) or (2.10)), CF3 (with tail (2.16)), or CF4 (with tail (2.7)), we have*

$$\frac{U_p}{V_p} = B_p \tag{3.15}$$

for any p sufficiently large.

Proof. We will only prove equality (3.13), since the proofs for (3.12), (3.14), and (3.15) can be obtained similarly. Let $N_{l_2(p)}$ and $D_{l_2(p)}$ be the numerator and the denominator of $B_{l_2(p)}$, respectively. Since $B_{l_2(p)} = \sigma(H_{l_2(p)})$, we have

$$N_{l_2(p)} = d_{l_2(p)} N_{l_2(p-1)} + N_{l_2(p-2)}, \quad D_{l_2(p)} = d_{l_2(p)} D_{l_2(p-1)} + D_{l_2(p-2)},$$

Now, we prove by induction that, if $p = 3m$ or $p = 3m + 2$, then

$$U_{g(p)} = N_{l_2(p)}, \quad V_{g(p)} = D_{l_2(p)}; \tag{3.16}$$

if $p = 3m + 1$, then

$$U_{g(p)} = \frac{N_{l_2(p)}}{2}, \quad V_{g(p)} = \frac{D_{l_2(p)}}{2}, \tag{3.17}$$

with m positive integer. The inductive bases for (3.16) and (3.17) are the equalities

$$U_{g(3)} = U_{p_0+5} = N_{l_2(3)}, \quad V_{g(3)} = V_{p_0+5} = D_{l_2(3)},$$

$$U_{g(5)} = U_{p_0+7} = N_{l_2(5)}, \quad V_{g(5)} = V_{p_0+7} = D_{l_2(5)},$$

$$U_{g(4)} = U_{p_0+6} = \tfrac{1}{2} N_{l_2(4)}, \quad V_{g(4)} = V_{p_0+6} = \tfrac{1}{2} D_{l_2(4)}.$$

These relations can be proved thanks to the matrix equalities

$$\begin{pmatrix} N_{l_2(4)} & N_{l_2(3)} \\ D_{l_2(4)} & D_{l_2(3)} \end{pmatrix} = SR^{d_1} \ldots L^{d_{l_2(4)}} = \begin{pmatrix} U_{p_0+5} & U_{p_0+6} \\ V_{p_0+5} & V_{p_0+6} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \quad p_0 \text{ even,}$$

$$\begin{pmatrix} N_{l_2(4)} & N_{l_2(5)} \\ D_{l_2(4)} & D_{l_2(5)} \end{pmatrix} = SR^{d_0} \ldots R^{d_{l_2(5)}} = \begin{pmatrix} U_{p_0+7} & U_{p_0+6} \\ V_{p_0+7} & V_{p_0+6} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \quad p_0 \text{ even,}$$

$$\begin{pmatrix} N_{l_2(3)} & N_{l_2(4)} \\ D_{l_2(3)} & D_{l_2(4)} \end{pmatrix} = SR^{d_0} \ldots R^{d_{l_2(4)}} = \begin{pmatrix} U_{p_0+6} & U_{p_0+5} \\ V_{p_0+6} & V_{p_0+5} \end{pmatrix} \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} \quad p_0 \text{ odd,}$$

$$\begin{pmatrix} N_{l_2(5)} & N_{l_2(4)} \\ D_{l_2(5)} & D_{l_2(4)} \end{pmatrix} = SR^{d_0} \ldots L^{d_{l_2(5)}} = \begin{pmatrix} U_{p_0+6} & U_{p_0+7} \\ V_{p_0+6} & V_{p_0+7} \end{pmatrix} \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} \quad p_0 \text{ odd.}$$

In fact, by definition of p_0, we know that p_0 and $l_2(4)$ must have different parity, moreover, for p_0 even, we have

$$SR^{d_0} \dots L^{d_{l_2(4)}} = R^{a_0} \dots R^{a_{p_0}} L^{\frac{d_{l_2(1)}-1}{2}} R^{2l_2(2)} L^{\frac{d_{l_2(3)}-1}{2}} RLR^{\frac{d_{l_2(4)}-1}{2}} \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix},$$

$$SR^{d_0} \dots R^{d_{l_2(5)}} = R^{a_0} \dots R^{a_{p_0}} L^{\frac{d_{l_2(1)}-1}{2}} R^{2l_2(2)} L^{\frac{d_{l_2(3)}-1}{2}} RLR^{\frac{d_{l_2(4)}-1}{2}} L^{2d_{l_2(5)}} \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix},$$

while, for p_0 odd,

$$SR^{d_0} \dots R^{d_{l_2(4)}} = R^{a_0} \dots L^{a_{p_0}} R^{\frac{d_{l_2(1)}-1}{2}} L^{2l_2(2)} R^{\frac{d_{l_2(3)}-1}{2}} LRL^{\frac{d_{l_2(4)}-1}{2}} \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix},$$

$$SR^{d_0} \dots L^{d_{l_2(5)}} = R^{a_0} \dots L^{a_{p_0}} R^{\frac{d_{l_2(1)}-1}{2}} L^{2l_2(2)} R^{\frac{d_{l_2(3)}-1}{2}} LRL^{\frac{d_{l_2(4)}-1}{2}} R^{2d_{l_2(5)}} \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}.$$

Thus, if we suppose that (3.16) and (3.17) hold for all the positive integers $3, \dots, p-1$, using the recurrence relations (3.2), we find

$$U_{g(p)} = d_{l_2(p)} U_{g(p-1)} + 2U_{g(p-2)} = d_{l_2} N_{l_2(p-1)} + N_{l_2(p-2)} = N_{l_2(p)} \quad \text{if } p = 3m,$$

$$U_{g(p)} = \tfrac{1}{2} d_{l_2(p)} U_{g(p-1)} + \tfrac{1}{2} U_{g(p-2)} = \tfrac{1}{2} d_{l_2} N_{l_2(p-1)} + \tfrac{1}{2} N_{l_2(p-2)} = \tfrac{1}{2} N_{l_2(p)} \quad \text{if } p = 3m+1,$$

$$U_{g(p)} = 2d_{l_2(p)} U_{g(p-1)} + U_{g(p-2)} = d_{l_2} N_{l_2(p-1)} + N_{l_2(p-2)} = N_{l_2(p)} \quad \text{if } p = 3m+2,$$

and the equivalent relations for $V_{g(p)}$ and $D_{l_2(p)}$ are also true. $\qquad \square$

4 Nonlinear leaping convergents of some Hurwitzian and Tasoev continued fractions

In this section we apply the previous results to some quasi-periodic continued fractions. Many well-studied Hurwitzian and Tasoevian continued fractions are continued fractions of type CF1, CF2, CF3 or CF4.

Let us consider the Hurwitz continued fraction

$$h(a, n) = \overline{[a(1+kn)]}_{k=0}^{+\infty}, \tag{4.1}$$

for a, n positive integers, whose convergents will be denoted by $H_p(a, n)$, for $p = 0, 1, \dots$. The identity

$$H_p(a, n) = \frac{\sum_{i=0}^{\lfloor \frac{p+1}{2} \rfloor} a^{p-2i+1} \binom{p-i+1}{i} \prod_{k=i}^{-i+p}(kn+1)}{\sum_{i=0}^{\lfloor \frac{p}{2} \rfloor} a^{p-2i} \binom{p-i}{i} \prod_{k=i+1}^{p-i}(kn+1)}$$

can be retrieved as a special case of Corollary 7 in [26]. Moreover, it is simple to show that

$$B_p = \sigma(H_p(a, n)) = \frac{\sum_{i=0}^{\lfloor \frac{p}{2} \rfloor} a^{p-2i} \left(Aa(in+1)\binom{p-i+1}{i} + B\binom{p-i}{i} \right) \prod_{k=i+1}^{p-i}(kn+1) + A\delta(p)}{\sum_{i=0}^{\lfloor \frac{p}{2} \rfloor} a^{p-2i} \left(Ca(in+1)\binom{p-i+1}{i} + D\binom{p-i}{i} \right) \prod_{k=i+1}^{p-i}(kn+1) + C\delta(p)},$$

for $p = 0, 1, \dots$, where $\delta(p) = \frac{1-(-1)^p}{2}$. We now observe that the continued fraction (4.1) is

- of type CF1, when a is even, and the possible tails of $\sigma(h(a,n))$ are given by

$$\left[\frac{1}{2}(a(1+n(k-1))-2),1,1\right]_{k=k_0}^{+\infty}, \tag{4.2}$$

$$\left[\frac{a}{2}(1+n(2k-2)),2a(1+(2k-1)n)\right]_{k=k_0}^{+\infty},$$

$$\left[\frac{a}{2}(1+n(2k-1)),2a(1+2kn)\right]_{k=k_0}^{+\infty};$$

- of type CF2, when a is odd and n even, and the possible tails of $\sigma(h(a,n))$ are given by

$$\left[\frac{(a(1+3n(k-1))-1)}{2},2a(1+n(3k-2)),\frac{(a(1+n(3k-1))-1)}{2},1,1\right]_{k=k_0}^{+\infty}, \tag{4.3}$$

$$\left[\frac{(a(1+n(3k-2))-1)}{2},2a(1+n(3k-1)),\frac{(a(1+3kn)-1)}{2},1,1\right]_{k=k_0}^{+\infty}, \tag{4.4}$$

$$\left[\frac{(a(1+n(3k-1))-1)}{2},2a(1+3kn),\frac{(a(1+n(3k+1))-1)}{2},1,1\right]_{k=k_0}^{+\infty}; \tag{4.5}$$

- of type CF3, when a, n are odd and greater than 3, and the possible tails of $\sigma(h(a,n))$ are given by

$$\left[\frac{a}{2}(1+n(2k-1)),2a(1+2kn)\right]_{k=k_0}^{+\infty}.$$

$$[w_1,w_2,w_3,w_4,w_5,w_6,w_7,w_8]_{k=k_0}^{+\infty},$$

where $w_4=w_5=w_7=w_8=1$ and

$$w_1=\frac{a(1+(4k-4)n)-1}{2}, \quad w_2=2a(1+(4k-3)n)$$
$$w_3=\frac{a(1+(4k-2)n)-1}{2}, \quad w_6=\frac{a(1+(4k-1)n)-1}{2}, \tag{4.6}$$

or

$$w_1=\frac{a(1+(4k-2)n)-1}{2}, \quad w_2=2a(1+(4k-1)n)$$
$$w_3=\frac{a(1+4kn)-1}{2}, \quad w_6=\frac{a(1+(4k+1)n)-1}{2}. \tag{4.7}$$

Thus, if $\{U_t/V_t\}_{t\geq 0}$ is the sequence of convergents of $\sigma(h(a,n))$, from Theorem 3.2 we have

$$\frac{U_{s(p)}}{V_{s(p)}}=B_{l_1(p)} \tag{4.8}$$

for a even and tail (4.2), with $s(p)=p_0+3p$, $l_1(p)=k_0+p-2$,

$$\frac{U_{g(p)}}{V_{g(p)}}=B_{l_2(p)}$$

for a odd and n even with

$$g(p)=p_0+p+2\left\lfloor\frac{p}{3}\right\rfloor, \quad l_2(p)=\begin{cases}3k_0+p-4 & \text{for tail (4.3)}\\ 3k_0+p-3 & \text{for tail (4.4)}\\ 3k_0+p-2 & \text{for tail (4.5)}\end{cases},$$

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and, finally,

$$\frac{U_{h(p)}}{V_{h(p)}} = B_{l_3(p)}$$

for a, n odd and greater than 3, with

$$l_3(p) = \begin{cases} 4k_0 + p - 5 & \text{for tail (4.6)} \\ 4k_0 + p - 3 & \text{for tail (4.7),} \end{cases} \qquad h(p) = p_0 + 2p - 1 + \sin\left(\tfrac{1}{2}(p+1)\pi\right).$$

When a is even, $\sigma(h(a,n))$ has tail (4.2) and if $a_{p_0} = a_0 = 1$, $k_0 = 1$, we have a Hurwitzian continued fraction of the form

$$[1, \overline{\alpha k + \beta, 1, 1}]_{k=1}^{+\infty} \quad \text{with } \alpha = \frac{an}{2}, \quad \beta = \frac{a(1-n)-2}{2}.$$

Therefore, identity (4.8) becomes

$$\frac{U_{3p}}{V_{3p}} = B_{p-1}.$$

These Hurwitz continued fractions have been studied by Komatsu [14], who found the combinatorial expression for the leaping convergents U_{3p}/V_{3p}; see Corollary 1, [14]. Regarding Tasoevian continued fractions, we may consider two examples:

$$t_1(u, a) = [\overline{ua^k}]_{k=1}^{+\infty}, \tag{4.9}$$

$$t_2(u, v, a) = [\overline{ua^k, va^k}]_{k=1}^{+\infty}, \tag{4.10}$$

where a, u and v are positive integers. The continued fraction (4.9) is

- of type CF1, when u is even or a is even, with $\sigma(t_1(u,a))$ having tails, for $k_0 \geq 1$, given by $\left[\frac{1}{2}(ua^k - 2), 1, 1\right]_{k=k_0}^{+\infty}$ if we suppose that $ua \geq 4$, or $\left[\frac{1}{2}ua^{2k-1}, 2ua^{2k}\right]_{k=k_0}^{+\infty}$ and $\left[\frac{1}{2}ua^{2k}, 2ua^{2k+1}\right]_{k=k_0}^{+\infty}$;

- of type CF2, when u and a are odd, with $\sigma(t_1(u,a))$ having tails, for $k_0 \geq 1$ and $ua \geq 3$, given by

$$\left[\tfrac{1}{2}(ua^{3k-2} - 1), 2ua^{3k-1}, \tfrac{1}{2}(ua^{3k} - 1), 1, 1\right]_{k=k_0}^{+\infty},$$

$$\left[\tfrac{1}{2}(ua^{3k-1} - 1), 2ua^{3k}, \tfrac{1}{2}(ua^{3k+1} - 1), 1, 1\right]_{k=k_0}^{+\infty},$$

$$\left[\tfrac{1}{2}(ua^{3k} - 1), 2ua^{3k+1}, \tfrac{1}{2}(ua^{3k+2} - 1), 1, 1\right]_{k=k_0}^{+\infty}.$$

Thus, for $\sigma(t_1(u,a))$ one of the identities (3.12), (3.13), (3.15) holds. On the other hand, the continued fraction (4.10) is

- of type CF1, when a is even or u and v are even, with $\sigma(t_2(u,v,a))$ having tails, for $k_0 \geq 1$, given by

$$\left[\tfrac{1}{2}(ua^k - 2), 1, 1, \tfrac{1}{2}(va^k - 2), 1, 1\right]_{k=k_0}^{+\infty}$$

if we suppose that $ua \geq 4$ and $va \geq 4$, or $\left[\frac{1}{2}ua^k, 2va^k\right]_{k=k_0}^{+\infty}$ and $\left[\frac{1}{2}va^k, 2ua^{k+1}\right]_{k=k_0}^{+\infty}$;

- of type CF2, when a, u, v are odd, with $\sigma(t_2(u,v,a))$ having tails, for $k_0 \geq 1$ and $ua \geq 3$, $va \geq 3$, given by

$$\left[\tfrac{1}{2}(ua^{3k-2}-1), 2va^{3k-2}, \tfrac{1}{2}(ua^{3k-1}-1), 1, 1, \tfrac{1}{2}(ua^{3k-1}-1), 2ua^{3k}, \tfrac{1}{2}(va^{3k}-1), 1, 1\right]_{k=k_0}^{+\infty},$$

$$\left[\tfrac{1}{2}(va^{3k-2}-1), 2ua^{3k-1}, \tfrac{1}{2}(va^{3k-1}-1), 1, 1, \tfrac{1}{2}(ua^{3k}-1), 2va^{3k}, \tfrac{1}{2}(va^{3k+1}-1), 1, 1\right]_{k=k_0}^{+\infty},$$

$$\left[\tfrac{1}{2}(ua^{3k-1}-1), 2va^{3k-1}, \tfrac{1}{2}(ua^{3k}-1), 1, 1, \tfrac{1}{2}(va^{3k}-1), 2ua^{3k+1}, \tfrac{1}{2}(va^{3k+1}-1), 1, 1\right]_{k=k_0}^{+\infty};$$

- of type CF3, if a and u are odd and v is even, with $\sigma(t_2(u,v,a))$ having tails, for $k_0 \geq 1$ and $ua \geq 3$, $va \geq 4$, given by

$$\left[\tfrac{1}{2}(ua^{2k-1}-1), 2va^{2k-1}, \tfrac{1}{2}(ua^{2k}-1), 1, 1, \tfrac{1}{2}(va^{2k}-2), 1, 1\right]_{k=k_0}^{+\infty},$$

$$\left[\tfrac{1}{2}(ua^{2k}-1), 2va^{2k}, \tfrac{1}{2}(ua^{2k+1}-1), 1, 1, \tfrac{1}{2}(va^{2k+1}-2), 1, 1\right]_{k=k_0}^{+\infty},$$

$$\left[\tfrac{1}{2}va^{k}, 2ua^{k+1}\right]_{k=k_0}^{+\infty};$$

- of type CF4, if a, v are odd and u is even, with $\sigma(t_2(u,v,a))$ having tails for $k_0 \geq 1$ and $ua \geq 4$, $va \geq 3$ given by

$$\left[\tfrac{1}{2}(va^{2k-1}-1), 2ua^{2k}, \tfrac{1}{2}(va^{2k}-1), 1, 1, \tfrac{1}{2}(ua^{2k+1}-2), 1, 1\right]_{k=k_0}^{+\infty},$$

$$\left[\tfrac{1}{2}ua^{k}, 2va^{k}\right]_{k=k_0}^{+\infty},$$

$$\left[\tfrac{1}{2}(va^{2k}-1), 2ua^{2k+1}, \tfrac{1}{2}(va^{2k+1}-1), 1, 1, \tfrac{1}{2}(ua^{2k+2}-2), 1, 1\right]_{k=k_0}^{+\infty}.$$

Therefore for $\sigma(t_2(u,v,a))$ one of the identities (3.12), (3.13), (3.14), (3.15) holds.

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