IRIS AperTO

UNIVERSITÀ DEGLI STUDI DI TORINO

AperTO - Archivio Istituzionale Open Access dell'Università di Torino

h-Bernstein basis functions over a triangular domain

This is a pre print version of the following article:

Original Citation:

Availability:

This version is available http://hdl.handle.net/2318/1738971 since 2025-01-14T14:16:47Z

Published version:

DOI:10.1016/j.cagd.2020.101849

Terms of use:

Open Access

Anyone can freely access the full text of works made available as "Open Access". Works made available under a Creative Commons license can be used according to the terms and conditions of said license. Use of all other works requires consent of the right holder (author or publisher) if not exempted from copyright protection by the applicable law.

(Article begins on next page)

15 November 2026

h-Bernstein basis functions over a triangular domain

P. Lamberti, M. Lamnii, S. Remogna, D. Sbibih*

Abstract

In this paper, we introduce and study new h-Bernstein basis functions over a triangular domain. In particular, after defining the h-Bernstein polynomial functions of degree n, we prove their algebraic and geometric properties, such as partition of unity and degree elevation and we show that they form a basis for the space of polynomials of total degree less than or equal to n on a triangle. Then, we propose the h-de Casteljau algorithm and we prove the Marsden identity.
Keywords: h-Bernstein basis; h-Bernstein Bézier triangular patch; Marsden identity

1 Introduction

A fundamental tool in Computer Aided Geometric Design (CAGD) to handle curves and surfaces is the Bernstein-Bézier technique (see e.g. [1, 2, 6, 7]), which is a geometrically intuitive and meaningful technique, leading to constructive numerically robust algorithms. It is used not only in CAGD, but also in various areas of numerical analysis, computational geometry, approximation theory and other fields.

The classical Bernstein polynomials were introduced in 1912, while Bézier curves and surfaces were studied in late 1950's and early 1960's by P. de Casteljau and P. Bézier. Regarding the different kinds of Bézier surfaces, it is worthwhile to remark that in the late sixties, S. Coons and P. Bézier have developed rectangular surface patches, dominating the growing field of CAGD, although, historically, triangular patches were already considered in the late fifties by P. de Casteljau. He had then realized that what is known as Bézier curves admitted a symmetric formulation in terms of barycentric coordinates, that can be immediately generalized to surfaces, yielding triangular patches. Therefore, this fact is reflected by the mathematical statement that the triangular patches are a more "natural" generalization of Bézier curves than are tensor product patches.

Moreover, even though Bernstein polynomials are used in CAGD as an exact representation for curves and surfaces, in practice it is not possible to manufacture them exactly as specified. Manufacturing always has tolerances on what can actually be fabricated. Therefore, in the literature, the univariate h-Bernstein polynomials have been introduced (see [3, 4, 11, 12]), in order to build Bézier curves and surfaces within such tolerances during design by taking values of h near zero, that is, by putting a tolerance on the value of h. More recently, in [10] the authors investigate algorithms and identities for the h-Bernstein bases and h-Bézier curves by introducing a new form of the blossom, the h-blossom, specifically adapted to analyzing them. Furthermore, the extension to the bivariate case has been considered in [5] in the tensor product setting. In particular, the author extends the definition of h-blossoming for polynomials in one variable to $(h1, h2)$-blossoming for polynomials in two variables, generalizes identities and algorithms from [10] and studies tensor product $(h1, h2)$-Bézier surfaces.

Therefore, taking into account all the above remarks, in this paper we define and study h-Bernstein basis functions over a triangular domain. In particular, in Section 2 we define the h-Bernstein polynomial functions of degree n and in Section 3 we prove

*University of Torino, Department of Mathematics, Torino, Italy (paola.lamberti@unito.it, sara.remogna@unito.it) University Mohammed First, Faculty of Sciences, LANO Laboratory, Oujda, Morocco (m_lamnii1@yahoo.fr, sbibih@yahoo.fr)

their algebraic and geometric properties: partition of unity, degree elevation, basis for the space of polynomials of total degree less than or equal to n on a triangle. Then, in Section 4 we propose the h-de Casteljau algorithm and in Section 5 we prove the Marsden identity. We also provide some graphical examples of h-Bernstein Bézier patches on triangles, highlighting the effect of changing the h-parameter.

2 h-Bernstein basis functions on a triangle

The classical Bernstein basis functions over a triangular domain have been widely studied both in Approximation Theory and in Computer Aided Geometric Design. The idea behind these functions is to define the barycentric coordinates of any point $V := (x, y)$ in $\mathbb{R}^2$ with respect to any nondegenerate triangle $T := (V_1, V_2, V_3)$, where $V_i := (x_i, y_i),\ i = 1, 2, 3$, are the vertices of T. More precisely, these barycentric coordinates, denoted by (τ_1, τ_2, τ_3), satisfy the following system:

$$V = \sum_{i=1}^{3} \tau_i\ V_i, \quad \text{with } \sum_{i=1}^{3} \tau_i = 1,$$

i.e. they can be viewed as the solution of the following linear system

$$\begin{aligned} \tau_1\ x_1 + \tau_2\ x_2 + \tau_3\ x_3 &= x \\ \tau_1\ y_1 + \tau_2\ y_2 + \tau_3\ y_3 &= y \\ \tau_1 + \tau_2 + \tau_3 &= 1. \end{aligned}$$

Using Cramer's rule, this equation has the unique solution

$$\tau_1 = \frac{\beta_1}{\delta}, \qquad \tau_2 = \frac{\beta_2}{\delta}, \qquad \tau_3 = \frac{\beta_3}{\delta},$$

where δ is a nonzero constant and $\beta_i := \beta_i(x, y),\ i = 1, 2, 3$, are the linear functions given by:

$$\delta := \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix}, \ \beta_1 := \begin{vmatrix} x & x_2 & x_3 \\ y & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix}, \ \beta_2 := \begin{vmatrix} x_1 & x & x_3 \\ y_1 & y & y_3 \\ 1 & 1 & 1 \end{vmatrix}, \ \beta_3 := \begin{vmatrix} x_1 & x_2 & x \\ y_1 & y_2 & y \\ 1 & 1 & 1 \end{vmatrix}.$$

In the following, for any non-negative integer n and a multi-index $\alpha := (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3$, we denote $|\alpha| := \alpha_1+\alpha_2+\alpha_3$, $\binom{n}{\alpha} := \frac{n!}{\alpha_1!\alpha_2!\alpha_3!}$, $\epsilon^1 := (1, 0, 0)$, $\epsilon^2 := (0, 1, 0)$,$\epsilon^3 := (0, 0, 1)$.

Now, for a real number h and the above multi-index α, we define the h-Bernstein polynomial functions over the triangle T as follows

$$B_\alpha^n(\tau; T, h) := \begin{cases} \binom{n}{\alpha} \dfrac{\displaystyle\prod_{k=0}^{\alpha_1-1} (\delta\tau_1 + kh) \prod_{k=0}^{\alpha_2-1} (\delta\tau_2 + kh) \prod_{k=0}^{\alpha_3-1} (\delta\tau_3 + kh)}{\displaystyle\prod_{k=0}^{n-1} (\delta + kh)}, & \text{for } n \geq 1 \text{ and } |\alpha| = n, \\ 1, & \text{for } n = 0 \text{ and } |\alpha| = 0. \end{cases} \tag{2.1}$$

It is clear that the h-Bernstein polynomial functions are positive over the triangle T for all $h \geq 0$.

The h-Bernstein polynomials $B_\alpha^n(\tau; T, h), |\alpha| = n$, are the natural generalization of the univariate h-Bernstein polynomials, introduced first by Stancu [11, 12] and then by Goldman [3] on $[0, 1]$, and generalized to any interval $[a, b]$ by Simeonov et al. [10]. They are defined by:

$$B_k^n(t, [a, b], h) = \binom{n}{k} \frac{\displaystyle\prod_{i=0}^{k-1} (t - a + i\ h) \prod_{i=0}^{n-k-1} (b - t + i\ h)}{\displaystyle\prod_{i=0}^{n-1} (b - a + i\ h)}, \quad k = 0, \ldots, n. \tag{2.2}$$

Indeed, in the univariate case, the barycentric coordinates of any $t \in \mathbb{R}$ can be written as solution of the following linear system

$$\begin{aligned} \tau_1 b + \tau_2 a &= t \\ \tau_1 + \tau_2 &= 1. \end{aligned}$$

In this case, we have $\delta = b - a$, $\tau_1 = \frac{t-a}{b-a}$ and $\tau_2 = \frac{b-t}{b-a}$. Using the fact that for any multi-index $(\alpha_1, \alpha_2) \in \mathbb{N}^2$ satisfying $\alpha_1 + \alpha_2 = n$, there exists an integer $0 \leq k \leq n$ such that $(\alpha_1, \alpha_2) = (k, n-k)$, then we obtain (2.2).

3 Algebraic and geometric properties of the h-Bernstein polynomial functions

As an immediate consequence of the definition of $B_\alpha^n(\tau; T, h)$ in (2.1), we obtain the following recurrence relation:

$$\begin{aligned} B_\alpha^n(\tau; T, h) = &\frac{\delta\ \tau_1 + (\alpha_1 - 1)\ h}{\delta + (n-1)\ h} B_{\alpha-\epsilon^1}^{n-1}(\tau; T, h) + \frac{\delta\ \tau_2 + (\alpha_2 - 1)\ h}{\delta + (n-1)\ h} B_{\alpha-\epsilon^2}^{n-1}(\tau; T, h) \\ &+ \frac{\delta\ \tau_3 + (\alpha_3 - 1)\ h}{\delta + (n-1)\ h} B_{\alpha-\epsilon^3}^{n-1}(\tau; T, h). \end{aligned} \tag{3.1}$$

Moreover, as shown in the following result, some classical bases on a triangle are obtained as particular cases of the bivariate h-Bernstein polynomial functions.

Lemma 1. *Let $T = (V_1, V_2, V_3)$ be a non-degenerate triangle with barycentric coordinates (τ_1, τ_2, τ_3). For any $\alpha \in \mathbb{N}^3$ such that $|\alpha| = n$, we have*

$$B_\alpha^n(\tau, T, 0) = B_\alpha^n(\tau, T) \quad \text{and} \quad B_\alpha^n(\tau, T, -\frac{\delta}{n}) = L_\alpha^n(\tau, T), \tag{3.2}$$

where $B_\alpha^n(\tau, T)$ are the classical Bernstein polynomials on T, and

$$L_\alpha^n(\tau, T) = \frac{\prod_{k=0}^{\alpha_1 - 1} (\tau_1 - \frac{k}{n})\ \prod_{k=0}^{\alpha_2 - 1} (\tau_2 - \frac{k}{n})\ \prod_{k=0}^{\alpha_3 - 1} (\tau_3 - \frac{k}{n})}{\prod_{k=0}^{\alpha_1 - 1} (\alpha_1 - \frac{k}{n})\ \prod_{k=0}^{\alpha_2 - 1} (\alpha_2 - \frac{k}{n})\ \prod_{k=0}^{\alpha_3 - 1} (\alpha_3 - \frac{k}{n})}$$

are the Lagrange polynomial satisfying $L_\alpha^n(\frac{\beta}{n}, T)$ equal to 1 if $\beta = \alpha$ and 0 otherwise.

Proof. It is easy to verify the first equality in (3.2). To prove the second one, we show by a simple computation that the right-hand side and the left-hand side are both equal to

$$\frac{n^n}{\alpha_1! \alpha_2! \alpha_3!} \prod_{k=0}^{\alpha_1 - 1} (\tau_1 - \frac{k}{n}) \prod_{k=0}^{\alpha_2 - 1} (\tau_2 - \frac{k}{n}) \prod_{k=0}^{\alpha_3 - 1} (\tau_3 - \frac{k}{n}).$$

□

In addition to the above results, we show that the h-Bernstein polynomial functions defined by (2.1) are non-negative for all $h \geq 0$, form a partition of unity, satisfy the property of degree elevation, and provide a basis for all polynomials of total degree less than or equal to n on the triangle T.
A proof of the the following result, see Remark 1, can be obtained by a simple application of the h-de Casteljau algorithm described in Section 4.

Proposition 2. *The h-Bernstein polynomial functions, defined by (2.1) satisfy the partition of unity, i.e.*

$$\sum_{|\alpha|=n} B_\alpha^n(\tau; T, h) = 1.$$

Now, we show how to write a polynomial

$$b^n(\tau;T,h) := \sum_{|\alpha|=n} b_\alpha \; B^n_\alpha(\tau;T,h) \tag{3.3}$$

of degree n on T as an h-Bernstein polynomial of degree $n+1$, where $\{b_\alpha,\ |\alpha| = n\}$ denote the Bernstein Bézier (BB-) coefficients.

Proposition 3. *The h-Bernstein Bézier patch $b^n(\tau;T,h)$ of degree n can be written in terms of h-Bernstein polynomial functions of degree $n+1$ as follows:*

$$b^n(\tau;T,h) = \sum_{|\mu|=n+1} b^{(1)}_\mu \; B^{n+1}_\mu(\tau;T,h)$$

where $b^{(1)}_\mu := \dfrac{1}{n+1}\sum_{k=1}^{3} \mu_k \; b_{\mu-\epsilon^k}, \quad |\mu| = n+1.$

Proof. The h-Bernstein Bézier patch

$$b^n(\tau;T,h) = \sum_{|\alpha|=n} b_\alpha \; B^n_\alpha(\tau;T,h) \tag{3.4}$$

is of degree n. By multiplying the right hand side of (3.4) by $1 = \frac{\delta+n\ h}{\delta+n\ h}$ and by using the fact that $|\tau| = 1$ and $|\alpha| = n$, we get

$$\begin{aligned} b^n(\tau;T,h) &= \sum_{|\alpha|=n} b_\alpha \; B^n_\alpha(\tau;T,h)\frac{\delta+n\ h}{\delta+n\ h} \\ &= \sum_{|\alpha|=n} \frac{b_\alpha}{\delta+n\ h} \; B^n_\alpha(\tau;T,h)(\delta\ \tau_1+\delta\ \tau_2+\delta\ \tau_3+\alpha_1\ h+\alpha_2\ h+\alpha_3\ h). \end{aligned}$$

By expanding the h-Bernstein polynomials as product of factors, we show that

$$b^n(\tau;T,h) = \sum_{|\alpha|=n} b_\alpha \binom{n}{\alpha} \frac{\prod_{k=0}^{\alpha_1}\left(\delta\ \tau_1+k\ h\right)\prod_{k=0}^{\alpha_2-1}\left(\delta\ \tau_2+k\ h\right)\prod_{k=0}^{\alpha_3-1}\left(\delta\ \tau_3+k\ h\right)}{\prod_{k=0}^{n}\left(\delta+k\ h\right)} \tag{3.5}$$

$$+ \sum_{|\alpha|=n} b_\alpha \binom{n}{\alpha} \frac{\prod_{k=0}^{\alpha_1-1}\left(\delta\ \tau_1+k\ h\right)\prod_{k=0}^{\alpha_2}\left(\delta\ \tau_2+k\ h\right)\prod_{k=0}^{\alpha_3-1}\left(\delta\ \tau_3+k\ h\right)}{\prod_{k=0}^{n}\left(\delta+k\ h\right)} \tag{3.6}$$

$$+ \sum_{|\alpha|=n} b_\alpha \binom{n}{\alpha} \frac{\prod_{k=0}^{\alpha_1-1}\left(\delta\ \tau_1+k\ h\right)\prod_{k=0}^{\alpha_2-1}\left(\delta\ \tau_2+k\ h\right)\prod_{k=0}^{\alpha_3}\left(\delta\ \tau_3+k\ h\right)}{\prod_{k=0}^{n}\left(\delta+k\ h\right)}. \tag{3.7}$$

For any multi-index $\alpha' = (\alpha'_1, \alpha'_2, \alpha'_3)$ such that $|\alpha'| = n+1$, the term

$$\binom{n+1}{\alpha'} \prod_{k=0}^{\alpha'_1-1}\left(\delta\ \tau_1+k\ h\right) \prod_{k=0}^{\alpha'_2-1}\left(\delta\ \tau_2+k\ h\right) \prod_{k=0}^{\alpha'_3-1}\left(\delta\ \tau_3+k\ h\right)$$

appears in (3.5) with the coefficient $\frac{\alpha'_1}{n+1} b_{\alpha'-\epsilon^1}$, in (3.6) with the coefficient $\frac{\alpha'_2}{n+1} b_{\alpha'-\epsilon^2}$ and in (3.7) with the coefficient $\frac{\alpha'_3}{n+1} b_{\alpha'-\epsilon^3}$. Then, by summation of these coefficients we obtain $b^{(1)}_{\alpha'} = \dfrac{1}{n+1}\sum_{k=1}^{3} \alpha'_k \; b_{\alpha'-\epsilon^k}$.

□

As a direct application of the above property, we get the following representations.

Corollary 4. *The following identities hold:*

$$x = \sum_{|\alpha|=n} \frac{1}{n} (\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3)\ B_\alpha^n(\tau; T, h), \tag{3.8}$$

$$y = \sum_{|\alpha|=n} \frac{1}{n} (\alpha_1 y_1 + \alpha_2 y_2 + \alpha_3 y_3)\ B_\alpha^n(\tau; T, h). \tag{3.9}$$

Proof. The proof can be done by induction on n. Indeed, for $n = 1$ we have

$$x = \tau_1\ x_1 + \tau_2\ x_2 + \tau_3\ x_3 = \sum_{|\alpha|=1} (\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3)\ B_\alpha^1(\tau; T, h),$$

Assume now that the equality (3.8) holds for the degree n, then we have

$$x = \sum_{|\alpha|=n} \frac{1}{n} (\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3)\ B_\alpha^n(\tau; T, h).$$

Let $c_\alpha := \frac{1}{n} (\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3)\,,\ |\alpha| = n$, denote the coefficients in the last formula. Using the degree elevation formula, we deduce that

$$x = \sum_{|\alpha|=n+1} c_\alpha^{(1)}\ B_\alpha^{n+1}(\tau; T, h),$$

where $c_\alpha^{(1)} := \dfrac{1}{n+1} \displaystyle\sum_{k=1}^{3} \alpha_k\ c_{\alpha-\epsilon^k}, \quad |\alpha| = n+1$. By expanding the coefficient $c_\alpha^{(1)}$, we obtain

$$\begin{aligned}
(n+1)c_\alpha^{(1)} &= \frac{\alpha_1}{n} ((\alpha_1 - 1)x_1 + \alpha_2 x_2 + \alpha_3 x_3) + \frac{\alpha_2}{n} (\alpha_1 x_1 + (\alpha_2 - 1)x_2 + \alpha_3 x_3) \\
&+ \frac{\alpha_3}{n} (\alpha_1 x_1 + \alpha_2 x_2 + (\alpha_3 - 1)x_3) \\
&= \frac{\alpha_1 x_1}{n} (\alpha_1 - 1 + \alpha_2 + \alpha_3) + \frac{\alpha_2 x_2}{n} (\alpha_2 - 1 + \alpha_1 + \alpha_3) \\
&+ \frac{\alpha_3 x_3}{n} (\alpha_3 - 1 + \alpha_1 + \alpha_2) \\
&= \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3.
\end{aligned}$$

This completes the proof. The same argument can be used to prove (3.9).

□

The degree elevation property is used to prove the following result.

Proposition 5. *The family of the h-Bernstein polynomials* $\mathfrak{B}_h^n = \{B_\alpha^n(\tau; T, h),\ |\alpha| = n\}$ *forms a basis for* $\mathbb{P}_n(T)$*: space of all polynomials of total degree less than or equal to n on* T*.*

Proof. It is simple to see that the cardinal number of the set $\mathfrak{B}_h^n$ is equal to the dimension $\binom{n+2}{2}$ of the space $\mathbb{P}_n(T)$. Then, it suffices to prove that all the polynomials $x^\nu y^\mu,\ 0 \leq \nu + \mu \leq n$, are in the space spanned by the family $\mathfrak{B}_h^n$, i.e. $\mathfrak{B}_h^n$ is a generating family of the space $\mathbb{P}_n(T)$.

Since the h-Bernstein polynomial functions form a partition of unity, 1 is in the space spanned by $\mathfrak{B}_h^n$. From (3.8) and (3.9) both x and y are also in this space. We now proceed by induction on the degree n. Suppose that the proposition holds for $n-1$, then there exist coefficients $c_\alpha(h),\ |\alpha| = n-1$, which depend only on h such that

$$x^{\nu-1} y^\mu = \sum_{|\alpha|=n-1} c_\alpha(h)\ B_\alpha^{n-1}(\tau; T, h).$$

Using the fact that $x = \tau_1\ x_1 + \tau_2\ x_2 + \tau_3\ x_3$, we obtain

$$x^\nu y^\mu = (\tau_1\ x_1 + \tau_2\ x_2 + \tau_3\ x_3) \sum_{|\alpha|=n-1} c_\alpha(h)\ B_\alpha^{n-1}(\tau;T,h).$$

Therefore,

$$\begin{aligned} x^\nu y^\mu &= \sum_{|\alpha|=n-1} \frac{x_1}{\delta}\ c_\alpha(h)\ \delta\ \tau_1\ B_\alpha^{n-1}(\tau;T,h)\ +\ \sum_{|\alpha|=n-1} \frac{x_2}{\delta}\ c_\alpha(h)\ \delta\ \tau_2\ B_\alpha^{n-1}(\tau;T,h) \\ &+ \sum_{|\alpha|=n-1} \frac{x_3}{\delta}\ c_\alpha(h)\ \delta\ \tau_3\ B_\alpha^{n-1}(\tau;T,h), \end{aligned}$$

which implies that

$$\begin{aligned} x^\nu y^\mu &= \sum_{|\alpha|=n-1} \frac{x_1}{\delta}\ c_\alpha(h)\ \big(\delta\ \tau_1 + \alpha_1 h\big)\ B_\alpha^{n-1}(\tau;T,h)\ -\ \sum_{|\alpha|=n-1} \frac{\alpha_1\ x_1\ h}{\delta}\ c_\alpha(h)\ B_\alpha^{n-1}(\tau;T,h) \\ &+ \sum_{|\alpha|=n-1} \frac{x_2}{\delta}\ c_\alpha(h)\ \big(\delta\ \tau_2 + \alpha_2 h\big)\ B_\alpha^{n-1}(\tau;T,h)\ -\ \sum_{|\alpha|=n-1} \frac{\alpha_2\ x_2\ h}{\delta}\ c_\alpha(h)\ B_\alpha^{n-1}(\tau;T,h) \\ &+ \sum_{|\alpha|=n-1} \frac{x_3}{\delta}\ c_\alpha(h)\ \big(\delta\ \tau_3 + \alpha_3 h\big)\ B_\alpha^{n-1}(\tau;T,h)\ -\ \sum_{|\alpha|=n-1} \frac{\alpha_3\ x_3\ h}{\delta}\ c_\alpha(h)\ B_\alpha^{n-1}(\tau;T,h). \end{aligned}$$

Hence, we deduce that

$$\begin{aligned} x^\nu y^\mu &= \sum_{|\alpha|=n-1} d^1_{\alpha+\epsilon^1}(h)\ B^n_{\alpha+\epsilon^1}(\tau;T,h)\ +\ \sum_{|\alpha|=n-1} d^1_{\alpha+\epsilon^2}(h)\ B^n_{\alpha+\epsilon^2}(\tau;T,h) \\ &+ \sum_{|\alpha|=n-1} d^1_{\alpha+\epsilon^3}(h)\ B^n_{\alpha+\epsilon^3}(\tau;T,h) - \sum_{|\alpha|=n-1} d^2_\alpha(h)\ B_\alpha^{n-1}(\tau;T,h) \end{aligned}$$

where

$$d^1_{\alpha+\epsilon^i}(h) := \frac{(\alpha_i+1)x_i}{n\ \delta}\big(\delta + (n-1)h\big)\ c_\alpha(h), \quad i = 1,2,3$$

and

$$d^2_\alpha(h) := \frac{h}{\delta}(\alpha_1\ x_1\ + \alpha_2\ x_2\ + \alpha_3\ x_3)\ c_\alpha(h).$$

By interchanging the indices we get

$$x^\nu y^\mu = \sum_{|\alpha|=n} d^1_\alpha(h)\ B^n_\alpha(\tau;T,h)\ -\ \sum_{|\alpha|=n-1} d^2_\alpha(h)\ B_\alpha^{n-1}(\tau;T,h).$$

Now, with the help of the degree elevation property, there exist coefficients $d^3_\alpha(h),\ |\alpha| = n$, such that:

$$\sum_{|\alpha|=n-1} d^2_\alpha(h)\ B_\alpha^{n-1}(\tau;T,h) = \sum_{|\alpha|=n} d^3_\alpha(h)\ B^n_\alpha(\tau;T,h).$$

Then, by combining the above equalities, we finally get

$$x^\nu y^\mu = \sum_{|\alpha|=n} d_\alpha(h)\ B^n_\alpha(\tau;T,h) \quad \text{with} \quad d_\alpha(h) := d^1_\alpha(h) - d^3_\alpha(h).$$

□

In Figs. 1, 2, 3, we give the graphs of some cubic h-Bernstein basis functions where we show the effect of the shape parameter h.

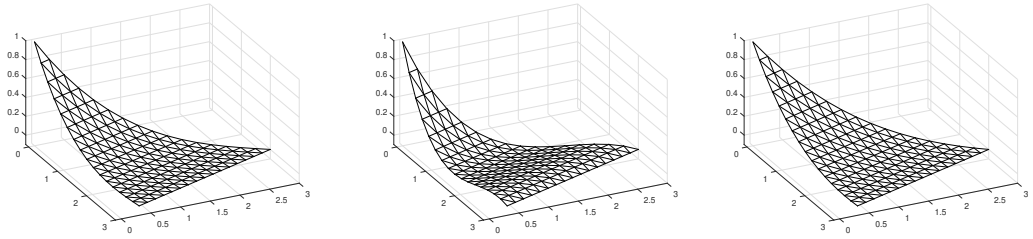


Figure 1: The effect of changing the h-parameter on the cubic h-Bernstein polynomial $B^n_{300}(\tau;T,h)$. From left to right: $h=0$, $h=-0.5$ and $h=2.5$.

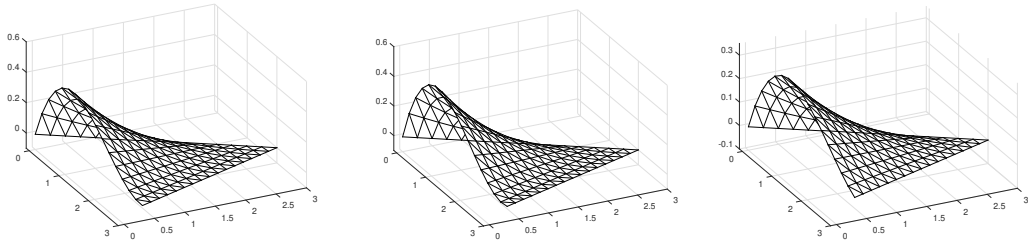


Figure 2: The effect of changing the h-parameter on the cubic h-Bernstein polynomial $B^n_{210}(\tau;T,h)$. From left to right: $h=0$, $h=-0.5$ and $h=2.5$.

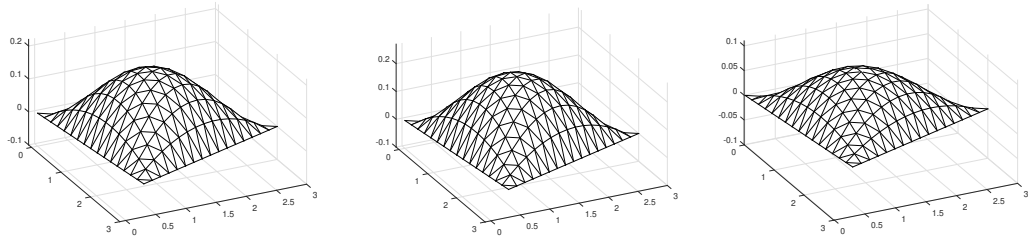


Figure 3: The effect of changing the h-parameter on the cubic h-Bernstein polynomial $B^n_{111}(\tau;T,h)$. From left to right: $h=0$, $h=-0.5$ and $h=2.5$.

4 h-Bézier patches over triangles and h-de Casteljau algorithm

The (h_1,h_2) Bézier patches over a rectangle was introduced in [5] by using tensor products of the univariate h-Bernstein basis. This technique is easy as a generalization to multivariate case, but it is limited in applications. Our aim in this section is to extend the notion of h-Bézier curves [10] to h-Bernstein Bézier patches over triangles. More precisely, we define the h-Bernstein Bézier (h-BB) patch of degree n as a surface parameterized by the variables x and y such that the point $V=(x,y)$ ranges over a non-degenerate triangle T in the plane, with vertices V_1, V_2 and V_3. The computation of the value of this surface at any point V in T can be done by the associated h-de Casteljau algorithm.

Let Q be a parameterized surface defined by

$$Q(\tau) := \sum_{|\alpha|=n} P_\alpha \; B^n_\alpha(\tau;T,h), \tag{4.1}$$

where $\tau=(\tau_1,\tau_2,\tau_3)$ are the barycentric coordinates of a point V with respect to the triangle T and $P_\alpha \in \mathbb{R}^3$, $|\alpha|=n$, are the $\binom{n+2}{2}$ control points of the h-BB patch Q.

The following result describes the h-de Casteljau algorithm, shown in Fig. 4, for $n=3$.

Proposition 6. *Let* $Q(\tau) = \sum_{|\alpha|=n} P_\alpha \; B_\alpha^n(\tau; T, h)$ *be an h-BB patch defined on a triangle* T*. Then, for any point* V *with barycentric coordinates* τ *in* T*, the value* $Q(\tau)$ *can be computed recursively as follows:*

$$Q(\tau) = \sum_{|\alpha|=n-r} P_\alpha^r(\tau) \; B_\alpha^{n-r}(\tau; T, h), \tag{4.2}$$

where $P_\alpha^0(\tau) := P_\alpha$*, and for* $r = 1, \ldots, n$*,* $|\alpha| = n - r$*, where* $P_\alpha^0(\tau) := P_\alpha$*, and for* $r = 1, \ldots, n$*,* $|\alpha| = n - r$

$$P_\alpha^r(\tau) := \frac{\delta\tau_1 + \alpha_1 h}{\delta + (n-r)h} \; P_{\alpha+\varepsilon^1}^{r-1}(\tau) + \frac{\delta\tau_2 + \alpha_2 h}{\delta + (n-r)h} \; P_{\alpha+\varepsilon^2}^{r-1}(\tau) + \frac{\delta\tau_3 + \alpha_3 h}{\delta + (n-r)h} \; P_{\alpha+\varepsilon^3}^{r-1}(\tau).$$

Then $P_{(0,0,0)}^n(\tau) := Q(\tau)$*.*

Proof. The proof can be easily done by using the recurrence relation property. □

Remark 1. *As a simple consequence of the h -de Casteljau algorithm, we easily get a proof of Proposition 2. Indeed, we can write:*

$$\sum_{|\alpha|=n} B_\alpha^n(\tau; T, h) = \sum_{|\alpha|=n} P_\alpha^0(\tau) B_\alpha^n(\tau; T, h)$$

where $P_\alpha^0(\tau) = 1, \; |\alpha| = n$*.*
Using the h -de Casteljau algorithm, we obtain $P_\alpha^r(\tau) = 1$ *for any multi-index* α *such that* $|\alpha| = n - r, \quad r = 1, \ldots, n$*. Then, we finally get*

$$\sum_{|\alpha|=n} B_\alpha^n(\tau; T, h) = \sum_{|\alpha|=0} P_\alpha^n(\tau) B_\alpha^0(\tau; T, h) = P_{(0,0,0)}^n(\tau) = 1.$$

In Fig. 5 we give the graph of an h-BB patch of the form (3.3) defined on a triangle, showing the effect of changing the shape parameter h.

In Figs. 6 and 7 we give the graphs of h-BB patches of the form (4.1), defined on a triangular domain, showing the effect of changing the shape parameter h.

5 Polynomial representation and Marsden identity

In the literature, see e.g. [8, 9], the blossoming or polar form has been showed as powerful technique for easily analyzing and proving different properties of Bézier and B-spline curves and surfaces. In the classical case, blossoming has been successfully used to establish the Marsden identity over triangles. The fundamental idea in this approach (see [8, 9]) is that, for any degree n, the polynomial function p has a unique symmetric n-affine polynomial $B[p](u_1, \ldots, u_n)$ that satisfies the following three properties:

1. Symmetry:
   $B[p](u_1, \ldots, u_n) = B[p](u_{\pi(1)}, \ldots, u_{\pi(n)})$ for any permutation π of the integers $1, \ldots, n$.

2. Multiaffine:

$$\begin{aligned} B[p](u_1, \ldots, au + (1-a)v, \ldots, u_n) \quad &= \quad aB[p](u_1, \ldots, u, \ldots, u_n) + \\ &\quad (1-a)B[p](u_1, \ldots, v, \ldots, u_n) \end{aligned}$$

3. Diagonal:
   $B[p](u, \ldots, u) = p(u)$.

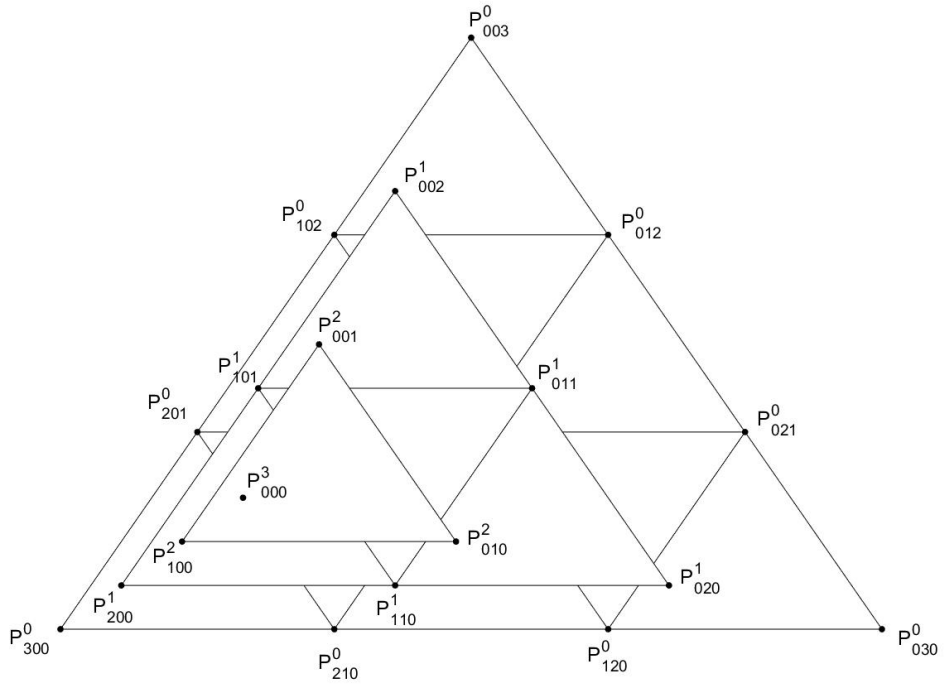


Figure 4: The triangular scheme illustrating the h-de Casteljau algorithm for $n = 3$.

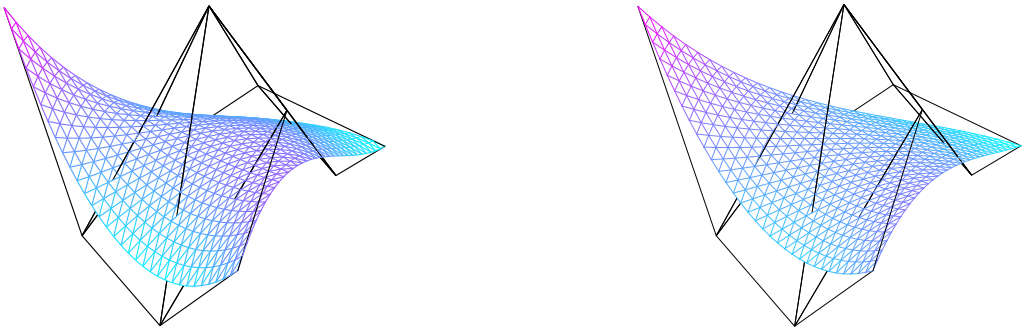


Figure 5: The effect of changing the h-parameter on the cubic h-BB patch. From left to right: $h = 0$ and $h = 2.5$. The BB-coefficients are $b_{300} = 2.9$, $b_{030} = 1.5$, $b_{003} = 0.5$, $b_{210} = 0.5$, $b_{120} = 0$, $b_{021} = 2.8$, $b_{012} = 1$, $b_{102} = 1.5$, $b_{201} = 0.5$, $b_{111} = 3.5$.

As shown in [8], the Bézier ordinates of the polynomial p related to the triangle T can be obtained by evaluating the blossom of the polynomial p at the vertices of the triangle T. More precisely, if $p \in \mathbb{P}_n(T)$ we have

$$p(\tau) = \sum_{|\alpha|=n} B[p]\big(\underbrace{V_1, \ldots, V_1}_{\alpha_1}, \underbrace{V_2, \ldots, V_2}_{\alpha_2}, \underbrace{V_3, \ldots, V_3}_{\alpha_3} \big) B^n_\alpha(\tau; T, 0), \qquad (5.1)$$

where $B^n_\alpha(\tau; T, 0)$ are the classical Bernstein basis polynomials on T.

In order to define an equivalent formula of (5.1) for the representation of polynomials in terms of the h-Bernstein basis functions, we need the following notation.

For p in $\mathbb{P}_n(T)$ and a multi-index α we set:

$$c_\alpha(p) := B[p]\big(\underbrace{V_1, \ldots, V_1}_{\alpha_1}, \underbrace{V_2, \ldots, V_2}_{\alpha_2}, \underbrace{V_3, \ldots, V_3}_{\alpha_3} \big).$$

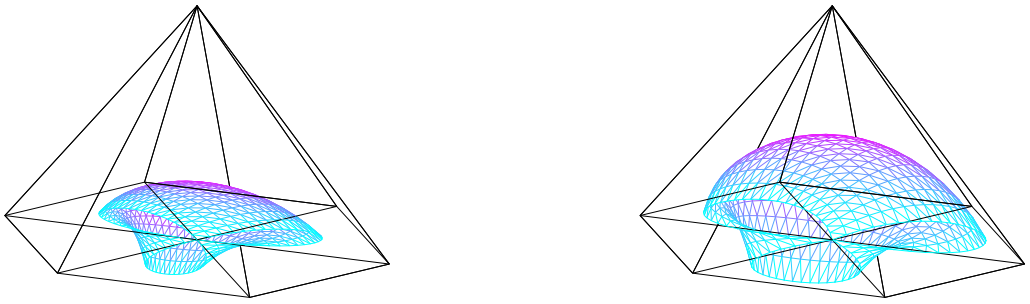


Figure 6: The effect of changing the h-parameter on the cubic h-BB patches. From left to right: $h = 0$, and $h = -0.5$. The controls points are $P_{300} = P_{030} = P_{003} = (0, 0, 0)$, $P_{210} = (-1, -1, 0)$, $P_{120} = (0, -1, 0)$, $P_{021} = (1, 0, 0)$, $P_{012} = (1, 1, 0)$, $P_{102} = (0, 1, 0)$, $P_{201} = (-1, 0, 0)$, $P_{111} = (0, 0, 1)$.

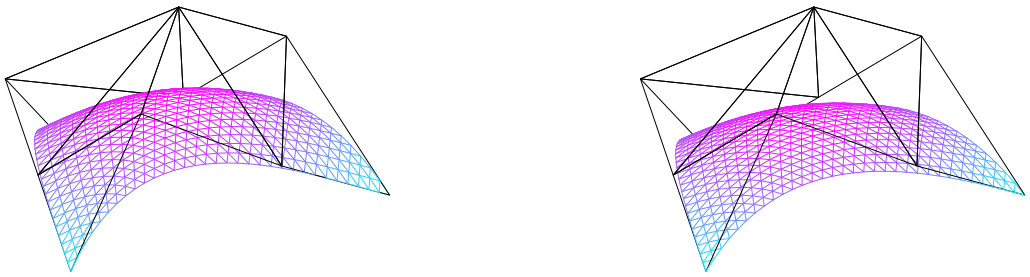


Figure 7: The effect of changing the h-parameter on the cubic h-BB patches. From left to right: $h = 0$, and $h = 1.33$. The controls points are $P_{300} = (0, 0, 0)$, $P_{030} = (5, 1, 0)$, $P_{003} = (2, 4, 0)$, $P_{210} = (1, 0, 2)$, $P_{120} = (3, 0, 1)$, $P_{021} = (4, 2, 2)$, $P_{012} = (3, 3, 1)$, $P_{102} = (0, 2, 2)$, $P_{201} = (0, 1, 1)$, $P_{111} = (2, 1, 3)$.

We also set $\mathbf{B_h^n} := \left\{B_\alpha^n(\tau; T, h)\right\}_{|\alpha|=n}$ the line vector composed of the elements of the h-Bernstein basis stored in lexicographical order, i.e. $B_\alpha^n(\tau; T, h)$ comes before $B_\mu^n(\tau; T, h)$ provided that $\alpha_1 > \mu_1$, or if $\alpha_1 = \mu_1$, then $\alpha_2 > \mu_2$. For example, for $n = 3$ the order is: $B_{300}^3(\tau; T, h)$, $B_{210}^3(\tau; T, h)$, $B_{201}^3(\tau; T, h)$, $B_{120}^3(\tau; T, h)$, $B_{111}^3(\tau; T, h)$, $B_{102}^3(\tau; T, h)$, $B_{030}^3(\tau; T, h)$, $B_{021}^3(\tau; T, h)$, $B_{012}^3(\tau; T, h)$, $B_{003}^3(\tau; T, h)$.

Let $M_{n,h}$ be the transition matrix from the classical Bernstein basis $\mathfrak{B}_0^n$ to the new basis $\mathfrak{B}_h^n$, i.e.

$$\mathbf{B_h^n} = \mathbf{B_0^n} M_{n,h} \tag{5.2}$$

The matrix $M_{n,h}$ is nonsingular and it can be computed column by column. More specifically, the entries in the column of the matrix $M_{n,h}$ that correspond to a fixed index α can be determined by expanding $B_\alpha^n(\tau; T, h)$ and using the degree elevation property. Indeed, we have

$$\begin{aligned} B_\alpha^n(\tau; T, h) &= \binom{n}{\alpha} \frac{\prod_{k=0}^{\alpha_1 - 1} (\delta\ \tau_1 + k\ h)\ \prod_{k=0}^{\alpha_2 - 1} (\delta\ \tau_2 + k\ h)\ \prod_{k=0}^{\alpha_3 - 1} (\delta\ \tau_3 + k\ h)}{\prod_{k=0}^{n-1}\ (\delta + k\ h)} \\ &= \frac{\delta^n}{\prod_{k=0}^{n-1}\ (\delta + k\ h)} \Big[B_\alpha^n(\tau; T, 0) + \sum_{m=1}^{n-1} \delta^{m-n} h^m \sum_{|\mu|=m} a_{m,\mu}\ B_\mu^m(\tau; T, 0) \Big], \end{aligned}$$

where $a_{m,\mu},\ m = 1, , n-1,\ |\mu| = m$, are real numbers. Such numbers can be computed explicitly by using binomial formulae.

It should be noted that the last equality can be easily proved by induction. Moreover, by classical degree elevation property, we know that for each $\alpha \in \mathbb{N}^3$, there exists a set

$a_{\alpha,\mu}(h)$ of polynomials in variable h with real coefficients such that:

$$B_\alpha^n(\tau;T,h) = \frac{\delta^n}{\prod_{k=0}^{n-1}\ (\delta + k\ h)} \sum_{|\mu|=n} a_{\alpha,\mu}(h)\ B_\mu^n(\tau;T,0). \tag{5.3}$$

In the sequel of this section, we consider the coefficients $c_\alpha(p)$ of a polynomial p written in the B-form (5.1) as components of a column vector $\mathbf{C}_p$.

The following result provides an important tool to explicitly express any polynomial p of total degree less than or equal to n in terms of the h-Bernstein basis functions.

Proposition 7. *(Marsden identity) For any polynomial $p \in \mathbb{P}_n(T)$, we have:*

$$p(\tau) = \sum_{|\alpha|=n} c_\alpha(p,h)\ B_\alpha^n(\tau;T,h), \tag{5.4}$$

where each coefficient $c_\alpha(p,h)$ is a polynomial in the variable h that corresponds to an index α in a column vector $\mathbf{C}_p(h)$ obtained by $\mathbf{C}_p(h) = M_{n,h}^{-1}\ \mathbf{C}_p$.

Proof. By writing the polynomial p in terms of both basis, i.e.

$$p(\tau) = \sum_{|\alpha|=n} c_\alpha(p)\ B_\alpha^n(\tau;T,0) = \sum_{|\alpha|=n} c_\alpha(p,h)\ B_\alpha^n(\tau;T,h),$$

and by using the matrix forms

$$p(\tau) = \mathbf{B_0^n} \mathbf{C}_p^T = \mathbf{B_h^n} \mathbf{C}_p(h)^T,$$

we get, from (5.2), the announced result. □

In order to illustrate the above identity, we give in the following examples the explicit representations of the monomials of the space $\mathbb{P}_n(T)$ for $n = 1, 2, 3$.

Example 1.

For $n = 1$ there is nothing to prove. Indeed, we have $B_\alpha^1(\tau;T,h) = B_\alpha^1(\tau;T,0),\ |\alpha| = 1$, then

$$p(\tau) = \sum_{|\alpha|=1} c_\alpha(p)\ B_\alpha^1(\tau;T,h),$$

from which we deduce

$$\begin{aligned} 1 &= \sum_{|\alpha|=1} B_\alpha^1(\tau;T,h), \quad x = \sum_{|\alpha|=1} \left(\alpha_1\ x_1 + \alpha_2\ x_2 + \alpha_3\ x_3\right)\ B_\alpha^1(\tau;T,h), \\ y &= \sum_{|\alpha|=1} \left(\alpha_1\ y_1 + \alpha_2\ y_2 + \alpha_3\ y_3\right)\ B_\alpha^1(\tau;T,h). \end{aligned}$$

Example 2.

For $n = 2$, we can write

$$p(\tau) = \sum_{|\alpha|=2} c_\alpha(p,h)\ B_\alpha^2(\tau;T,h) = \sum_{|\alpha|=2} c_\alpha(p)\ B_\alpha^2(\tau;T,0),$$

or more explicitly

$$\begin{aligned} p(\tau) &= c_{(2,0,0)}(p,h)\ \frac{\delta\ \tau_1\ \left(\delta\ \tau_1 +\ h\right)}{\delta\ \left(\delta +\ h\right)} + 2\ c_{(1,1,0)}(p,h)\ \frac{\delta\ \tau_1\ \delta\ \tau_2}{\delta\ \left(\delta +\ h\right)} \\ &+ c_{(0,2,0)}(p,h)\ \frac{\delta\ \tau_2\ \left(\delta\ \tau_2 +\ h\right)}{\delta\ \left(\delta +\ h\right)} + 2\ c_{(0,1,1)}(p,h)\ \frac{\delta\ \tau_2\ \delta\ \tau_3}{\delta\ \left(\delta +\ h\right)} \end{aligned}$$

$$+ \quad c_{(0,0,2)}(p,h)\ \frac{\delta\ \tau_3\ \big(\delta\ \tau_3 + \ h\big)}{\delta\ \big(\delta + \ h\big)} + 2\ c_{(1,0,1)}(p,h)\ \frac{\delta\ \tau_3\ \delta\ \tau_1}{\delta\ \big(\delta + \ h\big)}.$$

Then

$$\begin{aligned}
p(\tau) &= \frac{\delta}{\delta + \ h}\Big[c_{(2,0,0)}(p,h)\ B^2_{(2,0,0)}(\tau;T,0) + c_{(1,1,0)}(p,h)\ B^2_{(1,1,0)}(\tau;T,0) \\
&+ c_{(0,2,0)}(p,h)\ B^2_{(0,2,0)}(\tau;T,0) + c_{(0,1,1)}(p,h)\ B^2_{(0,1,1)}(\tau;T,0) \\
&+ c_{(0,0,2)}(p,h)\ B^2_{(0,0,2)}(\tau;T,0) + c_{(1,0,1)}(p,h)\ B^2_{(1,0,1)}(\tau;T,0)\Big] \\
&+ \frac{h}{\delta + \ h}\Big[c_{(2,0,0)}(p,h)\ B^1_{(1,0,0)}(\tau;T,0) + c_{(0,2,0)}(p,h)\ B^1_{(0,1,0)}(\tau;T,0) \\
&+ c_{(0,0,2)}(p,h)\ B^1_{(0,0,1)}(\tau;T,0)\Big].
\end{aligned}$$

Let us denote by q the linear polynomial defined by

$$\begin{aligned}
q(\tau) &:= c_{(2,0,0)}(p,h)\ B^1_{(1,0,0)}(\tau;T,0) + c_{(0,2,0)}(p,h)\ B^1_{(0,1,0)}(\tau;T,0) \\
&+ c_{(0,0,2)}(p,h)\ B^1_{(0,0,1)}(\tau;T,0).
\end{aligned}$$

It can also be written as

$$q(\tau) := \sum_{|\alpha|=1} b_\alpha(p,h)\ B^1_\alpha(\tau;T,0), \quad \text{where} \quad b_\alpha(p,h) = \sum_{i=1}^{3} \alpha_i\ c_{\alpha+\epsilon^i}(p,h)$$

Using the degree elevation formula, we get

$$\begin{aligned}
q(\tau) &= c_{(2,0,0)}(p,h)\ B^2_{(2,0,0)}(\tau;T,0) + \frac{1}{2}\big(c_{(2,0,0)}(p,h)\ + c_{(0,2,0)}(p,h)\big)\ B^2_{(1,1,0)}(\tau;T,0) \\
&+ c_{(0,2,0)}(p,h)\ B^2_{(0,2,0)}(\tau;T,0) + \frac{1}{2}\big(c_{(0,2,0)}(p,h)\ + c_{(0,0,2)}(p,h)\big)\ B^2_{(0,1,1)}(\tau;T,0) \\
&+ c_{(0,0,2)}(p,h)\ B^2_{(0,0,2)}(\tau;T,0) + \frac{1}{2}\big(c_{(0,0,2)}(p,h)\ + c_{(2,0,0)}(p,h)\big)\ B^2_{(1,0,1)}(\tau;T,0).
\end{aligned}$$

By replacing q by its last expression, p can be written in a matrix form as

$$p(\tau) = \mathbf{B_0^2}\ \mathbf{C}_p^T = \mathbf{B_h^2}\mathbf{C}_p(h)^T,$$

where $\mathbf{C}_p(h) := \big\{c_{(2,0,0)}(p,h),\ c_{(1,1,0)}(p,h),\ c_{(1,0,1)}(p,h),\ c_{(0,2,0)}(p,h),\ c_{(0,1,1)}(p,h),\ c_{(0,0,2)}(p,h)\big\}$ and the matrix $M_{2,h}$ is given by:

$$M_{2,h} = \frac{1}{\delta + \ h}\begin{pmatrix}
\delta + h & 0 & 0 & 0 & 0 & 0 \\
\frac{h}{2} & \delta & 0 & \frac{h}{2} & 0 & 0 \\
\frac{h}{2} & 0 & \delta & 0 & 0 & \frac{h}{2} \\
0 & 0 & 0 & \delta + h & 0 & 0 \\
0 & 0 & 0 & \frac{h}{2} & \delta & \frac{h}{2} \\
0 & 0 & 0 & 0 & 0 & \delta + h
\end{pmatrix}$$

The inverse of this matrix is:

$$M_{2,h}^{-1} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
-\frac{h}{2\delta} & \frac{h+\delta}{\delta} & 0 & -\frac{h}{2\delta} & 0 & 0 \\
-\frac{h}{2\delta} & 0 & \frac{h+\delta}{\delta} & 0 & 0 & -\frac{h}{2\delta} \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & -\frac{h}{2\delta} & \frac{h+\delta}{\delta} & -\frac{h}{2\delta} \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

The technique for the evaluation of the coefficients $c_\alpha(p,h),\ |\alpha| = 2$, at a given parameter h, is similar. Hence, it suffices to compute one coefficient, for example $c_{(1,1,0)}(p,h)$, because the other ones can be computed in a similar way.

According to the above results, by using the second row of the matrix $M_{2,h}^{-1}$, we have

$$c_{(1,1,0)}(p,h) = c_{(1,1,0)}(p) + \frac{h}{\delta}\Big[c_{(1,1,0)}(p) - \frac{1}{2}c_{(2,0,0)}(p) - \frac{1}{2}c_{(0,2,0)}(p)\Big]$$

Now, by setting $B_h[p]\big(\ \underbrace{V_1,\ldots,V_1}_{\alpha_1}\ ,\ \underbrace{V_2,\ldots,V_2}_{\alpha_2}\ ,\ \underbrace{V_3,\ldots,V_3}_{\alpha_3}\ \big) := c_\alpha(p,h)$, we get

$$B_h[p]\big(\ V_1,V_2\ \big) = B[p]\big(\ V_1,V_2\ \big) + \frac{h}{\delta}\Big[B[p]\big(\ V_1,V_2\ \big) - \frac{1}{2}B[p]\big(\ V_1,V_1\ \big) - \frac{1}{2}B[p]\big(\ V_2,V_2\ \big)\Big] \tag{5.5}$$

We now use the formula (5.5) to derive the expressions of $x^\mu y^\nu,\ 0 \le \mu+\nu \le 2$.

- If $p(x,y)=1$, we have $B[1]\big(\ V_1,V_2\ \big)=1$, then $B_h[p]\big(\ V_1,V_2\ \big)=1$. We deduce that

$$1 = \sum_{|\alpha|=2} B_\alpha^2(\tau;T,h).$$

- If $p(x,y)=x$, we have $B[x]\big(\ V_1,V_2\ \big)=\frac{1}{2}(x_1+x_2)$, then $B_h[p]\big(\ V_1,V_2\ \big)=\frac{1}{2}(x_1+x_2)$. We obtain

$$x = \sum_{|\alpha|=2} \frac{1}{2}(\alpha_1\ x_1 + \alpha_2\ x_2 + \alpha_3\ x_3)\ B_\alpha^2(\tau;T,h).$$

- If $p(x,y)=xy$, we have $B[xy]\big(\ V_1,V_2\ \big)=\frac{1}{2}(x_1y_2+y_1x_2)$, then by using the formula (5.5), we have $B_h[p]\big(\ V_1,V_2\ \big)=\frac{1}{2}(x_1y_2+y_1x_2)+\frac{h}{2\delta}\Big[x_1y_2+y_1x_2-x_1y_1-x_2y_2\Big]$. Thus, we deduce that

$$\begin{aligned}
xy &= x_1y_1\ B_{(2,0,0)}^2(\tau;T,h) + \Big[\frac{1}{2}(x_1y_2+y_1x_2) + \frac{h}{2\delta}(x_1y_2+y_1x_2-x_1y_1-x_2y_2)\Big]\ B_{(1,1,0)}^2(\tau;T,h)\\
&+ x_2y_2\ B_{(0,2,0)}^2(\tau;T,h) + \Big[\frac{1}{2}(x_2y_3+y_2x_3) + \frac{h}{2\delta}(x_2y_3+y_2x_3-x_2y_2-x_3y_3)\Big]\ B_{(0,1,1)}^2(\tau;T,h)\\
&+ x_3y_3\ B_{(0,0,2)}^2(\tau;T,h) + \Big[\frac{1}{2}(x_3y_1+y_3x_1) + \frac{h}{2\delta}(x_3y_1+y_3x_1-x_3y_3-x_1y_1)\Big]\ B_{(1,0,1)}^2(\tau;T,h).
\end{aligned}$$

- If $p(x,y)=x^2$, we have $B[x^2]\big(\ V_1,V_2\ \big)=x_1x_2$, then with the help of formula (5.5), we get $B_h[p]\big(\ V_1,V_2\ \big)=x_1x_2+\frac{h}{2\delta}\Big[2x_1x_2-x_1^2-x_2^2\Big]$. We deduce then that

$$\begin{aligned}
x^2 &= x_1^2\ B_{(2,0,0)}^2(\tau;T,h) + \Big[x_1x_2 + \frac{h}{2\delta}(2x_1x_2-x_1^2-x_2^2)\Big]\ B_{(1,1,0)}^2(\tau;T,h)\\
&+ x_2^2\ B_{(0,2,0)}^2(\tau;T,h) + \Big[x_2x_3 + \frac{h}{2\delta}(2x_2x_3-x_2^2-x_3^2)\Big]\ B_{(0,1,1)}^2(\tau;T,h)\\
&+ x_3^2\ B_{(0,0,2)}^2(\tau;T,h) + \Big[x_1x_3 + \frac{h}{2\delta}(2x_1x_3-x_1^2-x_3^2)\Big]\ B_{(1,0,1)}^2(\tau;T,h).
\end{aligned}$$

Example 3.

For $n=3$, the expression of any cubic polynomial p on a triangle T is given by

$$p(\tau) = \sum_{|\alpha|=3} c_\alpha(p,h)\ B_\alpha^3(\tau;T,h) = \sum_{|\alpha|=3} c_\alpha(p)\ B_\alpha^3(\tau;T,0).$$

Following the same procedure as in the case $n=2$, we compute all the monomials $x^\mu y^\nu,\ 0\le\mu+\nu\le 3$, of $\mathbb{P}_3(T)$. In this case, the matrix $M_{3,h}$ is given by:

$$M_{3,h} = \frac{1}{(\delta+h)(\delta+2h)} \times$$

$$\begin{pmatrix}
(h+\delta)(2h+\delta) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{3}h(4h+3\delta) & \delta(h+\delta) & 0 & h\delta & 0 & 0 & \frac{2h^2}{3} & 0 & 0 & 0 \\
\frac{1}{3}h(4h+3\delta) & 0 & \delta(h+\delta) & 0 & 0 & h\delta & 0 & 0 & 0 & \frac{2h^2}{3} \\
\frac{2h^2}{3} & h\delta & 0 & \delta(h+\delta) & 0 & 0 & \frac{1}{3}h(4h+3\delta) & 0 & 0 & 0 \\
\frac{2h^2}{3} & \frac{h\delta}{2} & \frac{h\delta}{2} & \frac{h\delta}{2} & \delta^2 & \frac{h\delta}{2} & \frac{2h^2}{3} & \frac{h\delta}{2} & \frac{h\delta}{2} & \frac{2h^2}{3} \\
\frac{2h^2}{3} & 0 & h\delta & 0 & 0 & \delta(h+\delta) & 0 & 0 & 0 & \frac{1}{3}h(4h+3\delta) \\
0 & 0 & 0 & 0 & 0 & 0 & (h+\delta)(2h+\delta) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3}h(4h+3\delta) & \delta(h+\delta) & h\delta & \frac{2h^2}{3} \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{2h^2}{3} & h\delta & \delta(h+\delta) & \frac{1}{3}h(4h+3\delta) \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & (h+\delta)(2h+\delta)
\end{pmatrix}$$

Its corresponding inverse matrix $M_{3,h}^{-1}$ is given by:

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{h(h+3\delta)}{3\delta^2} & \frac{(h+\delta)^2}{\delta^2} & 0 & -\frac{h(h+\delta)}{\delta^2} & 0 & 0 & \frac{h^2}{3\delta^2} & 0 & 0 & 0 \\
-\frac{h(h+3\delta)}{3\delta^2} & 0 & \frac{(h+\delta)^2}{\delta^2} & 0 & 0 & -\frac{h(h+\delta)}{\delta^2} & 0 & 0 & 0 & \frac{h^2}{3\delta^2} \\
\frac{h^2}{3\delta^2} & -\frac{h(h+\delta)}{\delta^2} & 0 & \frac{(h+\delta)^2}{\delta^2} & 0 & 0 & -\frac{h(h+3\delta)}{3\delta^2} & 0 & 0 & 0 \\
\frac{h^2}{3\delta^2} & -\frac{h(h+\delta)}{2\delta^2} & -\frac{h(h+\delta)}{2\delta^2} & -\frac{h(h+\delta)}{2\delta^2} & \frac{(h+\delta)(2h+\delta)}{\delta^2} & -\frac{h(h+\delta)}{2\delta^2} & \frac{h^2}{3\delta^2} & -\frac{h(h+\delta)}{2\delta^2} & -\frac{h(h+\delta)}{2\delta^2} & \frac{h^2}{3\delta^2} \\
\frac{h^2}{3\delta^2} & 0 & -\frac{h(h+\delta)}{\delta^2} & 0 & 0 & \frac{(h+\delta)^2}{\delta^2} & 0 & 0 & 0 & -\frac{h(h+3\delta)}{3\delta^2} \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\frac{h(h+3\delta)}{3\delta^2} & \frac{(h+\delta)^2}{\delta^2} & -\frac{h(h+\delta)}{\delta^2} & \frac{h^2}{3\delta^2} \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{h^2}{3\delta^2} & -\frac{h(h+\delta)}{\delta^2} & \frac{(h+\delta)^2}{\delta^2} & -\frac{h(h+3\delta)}{3\delta^2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

Now using the fifth row of the above matrix, we get

$$\begin{aligned}
B_h[p]\big(\, V_1, V_2, V_3 \,\big) &= \frac{(\delta+h)(\delta+2h)}{\delta^2} B[p]\big(\, V_1, V_2, V_3 \,\big) \qquad (5.6) \\
&+ \frac{h^2}{3\delta^2}\Big[B[p]\big(\, V_1, V_1, V_1 \,\big) + B[p]\big(\, V_2, V_2, V_2 \,\big) + B[p]\big(\, V_3, V_3, V_3 \,\big)\Big] \\
&- \frac{h(\delta+h)}{2\delta^2}\Big[B[p]\big(\, V_1, V_1, V_2 \,\big) + B[p]\big(\, V_1, V_1, V_3 \,\big) + B[p]\big(\, V_1, V_2, V_2 \,\big) \\
&+ B[p]\big(\, V_2, V_2, V_3 \,\big) + B[p]\big(\, V_1, V_3, V_3 \,\big) + B[p]\big(\, V_2, V_3, V_3 \,\big)\Big].
\end{aligned}$$

With the help of the formula (5.6), we will be able to compute all the coefficients $c_\alpha(p,h),\ |\alpha| = 3$.

As example, we only give the explicit expression of the coefficient $c_{(1,1,1)}(p,h) = B_h[p]\big(V_1, V_2, V_3\big)$ for the monomial $p(x,y) = x^2y$, the rest of the coefficients can be computed in the same way.

If $p(x,y) = x^2y$ we have $B[x^2y]\big(\, V_1, V_2, V_3 \,\big) = \frac{1}{3}\big(x_1x_2y_3 + x_1x_3y_2 + x_2x_3y_1\big)$, then by using (5.6), we get

$$\begin{aligned}
B_h[x^2y]\big(\, V_1, V_2, V_3 \,\big) &= \frac{1}{3}\left(x_2x_3y_1 + x_1x_3y_2 + x_1x_2y_3\right) \\
&- \frac{h}{6\delta}\Big[x_1^2\left(y_2+y_3\right) + 2x_1\left(x_3\left(y_1-3y_2+y_3\right) + x_2\left(y_1+y_2-3y_3\right)\right) \\
&+ x_3^2\left(y_1+y_2\right) + x_2^2\left(y_1+y_3\right) + 2x_2x_3\left(-3y_1+y_2+y_3\right)\Big] \\
&- \frac{h^2}{6\delta^2}\Big[x_1^2\left(-2y_1+y_2+y_3\right) + 2x_1\left(x_3\left(y_1-2y_2+y_3\right) + x_2\left(y_1+y_2-2y_3\right)\right) \\
&+ 2x_2x_3\left(-2y_1+y_2+y_3\right) + x_2^2\left(y_1-2y_2+y_3\right) + x_3^2\left(y_1+y_2-2y_3\right)\Big].
\end{aligned}$$

6 Conclusion

In this paper, we have defined and studied h-Bernstein basis functions over a triangular domain, that are the natural extension of the univariate ones. In particular, after defining

the h-Bernstein polynomial functions of degree n, we have proved their algebraic and geometric properties, such as partition of unity and degree elevation and we have shown that they form a basis for the space of polynomials of total degree less than or equal to n on a triangle. Then, we have described the h-de Casteljau algorithm and we have proved the Marsden identity. Finally, we have also provided some graphical examples of h-Bernstein Bézier patches on triangles, highlighting the effect of changing the h-parameter.

In the near future, we are going to deeply study the h-blossoming related to the this new basis and construct its corresponding quasi-interpolants with applications to CAGD and numerical analysis problems.

Acknowledgements

This work is carried out within the framework of the agreement CNRST-CNR and was partially supported by Gruppo Nazionale per il Calcolo Scientifico (GNCS)-INdAM (Italy).

References

[1] R.E. Barnhill, Surfaces in computer aided geometric design: a survey with new results, Comput. Aided Geom. Des., 2(1985), 1-17.

[2] G. Farin, Triangular Bernstein-Bézier patches, CAGD, 3 (1986), 83-127.

[3] R. Goldman, R., Pólya's urn model and computer aided geometric design. SIAM J. Alg. Disc. Meth. 6(1985), 1-28.

[4] R. Goldman, P. Barry, Shape parameter deletion for Pólya' curves. Numer. Algorithms 1(1991), 121-137.

[5] I. Jegdić, Algoritms and identities for bivariate (h_1, h_2)-Blossoming, Inter. Jour. Appl. Math. 30 (2017), 321-343.

[6] M.J. Lai, L.L. Schumaker, Spline Functions on Triangulations, Cambridge University Press, 2007.

[7] H. Prautzsch, W. Boehm, M. Paluszny, Bézier and B-spline tecniques, Springer-Verlag, Berlin, Heidelberg, 2002.

[8] L. Ramshaw, Blossoms are polar forms. Comput. Aided Geom. Des. 6 (1989), 323-358.

[9] H. Seidel, An introduction to polar forms. IEEE Comput. Graph. Appl. 13 (1993), 38-46.

[10] P. Simeonov, V. Zafiris, R. Goldman, h-Blossoming: A new approach to algorithms and identities for h-Bernstein bases and h-Bézier curves. Comput. Aided Geom. Des. 28 (2011), 549-565.

[11] D. Stancu, Approximation of functions by a new class of linear polynomial operators, Rev. Roumaine Math. Pures Appl. 13 (1968), 1173-1194.

[12] D. Stancu, Generalized Bernstein approximating operators, in: Itinerant Seminar on Functional Equations, Approximation and Convexity, Univ. "Babes-Bolyai", Cluj-Napoca (1984), 185-192.