Compactness and existence results for quasilinear elliptic problems with singular or vanishing potentials

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Abstract

Given ≥ 3, $1 < p < N$, two measurable functions $V(r) \geq 0$, $K(r) > 0$, and a continuous function $A(r) > 0$ ($r > 0$), we study the quasilinear elliptic equation

$$-\mathrm{div}\left(A(|x|)|\nabla u|^{p-2}\nabla u\right)u + V(|x|)\,|u|^{p-2}u = K(|x|)f(u) \quad \text{in } \mathbb{R}^N.$$

We find existence of nonegative solutions by the application of variational methods, for which we have to study the compactness of the embedding of a suitable function space X into the sum of Lebesgue spaces $L_K^{q_1} + L_K^{q_2}$, and thus into L_K^q $(= L_K^q + L_K^q)$ as a particular case. Our results do not require any compatibility between how the potentials A, V and K behave at the origin and at infinity, and essentially rely on power type estimates of the relative growth of V and K, not of the potentials separately. The nonlinearity f has a double-power behavior, whose standard example is $f(t) = \min\{t^{q_1-1}, t^{q_2-1}\}$, recovering the usual case of a single-power behavior when $q_1 = q_2$.

Keywords. Weighted Sobolev spaces, compact embeddings, quasilinear elliptic PDEs, unbounded or decaying potentials

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1 Introduction

In this paper we pursue the work we made in papers [2–5, 7, 9], where we studied embedding and compactness results for weighted Sobolev spaces. These results then made possible to get existence and multiplicity results, by variational methods, for several kinds of elliptic equations in $\mathbb{R}^N$.

In the present paper we face quasilinear elliptic equations in presence of a radial potential on the derivatives, that is, equations of the following kind

$$-\mathrm{div}\left(A(|x|)|\nabla u|^{p-2}\nabla u\right)u + V(|x|)\,|u|^{p-2}u = K(|x|)f(u) \quad \text{in } \mathbb{R}^N \tag{1.1}$$

where $1 < p < N$, $f : \mathbb{R} \to \mathbb{R}$ is a continuous nonlinearity satisfying $f(0) = 0$, and V, A, K are given potentials. We study such equation by variational methods, so we introduce a suitable functional space X (see section 2) and we say that $u \in X$ is a *weak solution* to (1.1) if

$$\int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u \cdot \nabla h\,dx + \int_{\mathbb{R}^N} V(|x|)\,|u|^{p-2}uh\,dx = \int_{\mathbb{R}^N} K(|x|)\,f(u)\,h\,dx \quad \text{for all } h \in X. \tag{1.2}$$

These solutions are (at least formally) critical points of the Euler functional

$$I(u) := \frac{1}{p}\|u\|^p - \int_{\mathbb{R}^N} K(|x|)\,F(u)\,dx, \tag{1.3}$$

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where $F(t) := \int_0^t f(s)\,ds$ and $||\cdot||$ is the norm on X (see section 2 below). Then the problem of existence is easily solved if $A \equiv 1$, V does not vanish at infinity, K is bounded and $f(t) = t^{q-1}$, because standard embeddings theorems for X are available (for suitable q's). As we let V and K to vanish, or to go to infinity, as $|x| \to 0$ or $|x| \to +\infty$, and we introduce the potential A on the derivatives, the usual embeddings theorems for Sobolev spaces are not available anymore, and new embedding theorems need to be proved. This has been done in several papers: see e.g. the references in [4, 5, 9] for a bibliography concerning the usual Laplace equation, [1, 6, 8, 10–12, 14, 15] for equations involving the p-laplacian, and [7, 13] and the references therein for problems with a potential A on the derivatives.

The main novelty of our approach (in [3–5, 7] and in the present paper) is two-folded. Firstly, we look for embeddings of X not into a single Lebesgue space L_K^q but into a sum of Lebesgue spaces $L_K^{q_1} + L_K^{q_2}$. This allows to study separately the behaviour of the potentials V, K at 0 and ∞, and to assume independent set of hypotheses about these behaviours. Secondly, we assume hypotheses not on V and K separately but on their ratio, so allowing asymptotic behaviors of general kind for the two potentials.

Thanks to this second novelty we obtain embedding results, and thus existence results for equation (1.1), for more general kinds of potentials than the power type ones (cf. Example 7.2 below), which are essentially the only ones considered in the existing literature (cf. [13]). Moreover, thanks to the first novelty, we get new results also for power type potentials (cf. Example 7.2 below).

This paper is organized as follows. In Section 2 we introduce the hypotheses on A, V, K and the function spaces D_A and X in which we will work. In Section 3 we state a general result concerning the embedding properties of X into $L_K^{q_1} + L_K^{q_2}$ (Theorem 3.1) and some explicit conditions ensuring that the embedding is compact (Theorems 3.2 and 3.3). The general result is proved in Section 4, the explicit conditions in Section 5. In Section 6 we apply our embedding results to get existence of non negative solutions for (1.1). In section 7 we give some examples to explain the novelty of our results.

Notations. We end this introductory section by collecting some notations used in the paper.

• $\mathbb{R}_+ = (0, +\infty) = \{x \in \mathbb{R} : x > 0\}$.

• For every $R > 0$, we set $B_R = \{x \in \mathbb{R}^N : |x| < r\}$.

• ω_N is the $(N-1)-$dimensional measure of the unit sphere $\partial B_1 = \{x \in \mathbb{R}^N : |x| = 1\}$.

• For any subset $A \subseteq \mathbb{R}^N$, we denote $A^c := \mathbb{R}^N \setminus A$. If A is Lebesgue measurable, $|A|$ stands for its measure.

• $C_c^\infty(\Omega)$ is the space of the infinitely differentiable real functions with compact support in the open set $\Omega \subseteq \mathbb{R}^N$. If Ω has radial symmetry, $C_{c,r}^\infty(\Omega)$ is the subspace of $C_c^\infty(\Omega)$ made of radial functions.

• For any measurable set $A \subseteq \mathbb{R}^N$, $L^q(A)$ and $L_{\rm loc}^q(A)$ are the usual real Lebesgue spaces. If $\rho : A \to \mathbb{R}_+$ is a measurable function, then $L^p(A, \rho(z)\,dz)$ is the real Lebesgue space with respect to the measure $\rho(z)\,dz$ (dz stands for the Lebesgue measure on $\mathbb{R}^N$). In particular, if $K : \mathbb{R}_+ \to \mathbb{R}_+$ is measurable, we denote $L_K^q(A) := L^q(A, K(|x|)\,dx)$.

• $p' := p/(p-1)$ is the Hölder-conjugate exponent of p.

2 Hypotheses and preliminary results

Throughout this paper we assume $N \geq 3$ and $1 < p < N$. We will make use of the following hypotheses on A, V, K:

($\mathbf{A}$) $A : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and there exist real numbers $p - N < a_0, a_\infty \leq p$ and $c_0, c_\infty > 0$ such that:

$$c_0 \leq \liminf_{r\to 0^+} \frac{A(r)}{r^{a_0}} \leq \limsup_{r\to 0^+} \frac{A(r)}{r^{a_0}} < +\infty,$$

$$c_\infty \leq \liminf_{r\to +\infty} \frac{A(r)}{r^{a_\infty}} \leq \limsup_{r\to +\infty} \frac{A(r)}{r^{a_\infty}} < +\infty.$$

($\mathbf{V}$) $V : \mathbb{R}_+ \to [0, +\infty)$ is measurable and such that $V \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$;

($\mathbf{K}$) $K : \mathbb{R}_+ \to \mathbb{R}_+$ is measurable and such that $K \in L^s_{\mathrm{loc}}(\mathbb{R}_+)$ for some $s > 1$.

Remark 2.1. It is easy to check that the hypothesis ($\mathbf{A}$) implies that, for each $R > 0$, there exist $C_0 = C_0(R) > 0$ and $C_\infty = C_\infty(R) > 0$ such that

$$A(|x|) \geq C_0|x|^{a_0} \quad \text{for all } 0 < |x| \leq R, \tag{2.1}$$

$$A(|x|) \geq C_\infty|x|^{a_\infty} \quad \text{for all } |x| \geq R. \tag{2.2}$$

We now introduce the space functions in which we will work. These are the following:

- D_A is the closure of $C^\infty_{\mathrm{c},r}(\mathbb{R}^N)$ with respect to the norm $||u||_A := \left(\int_{\mathbb{R}^N} A(|x|)|\nabla u|^p dx\right)^{1/p}$ (see also Definition 2.5 below),
- $X := D_A \cap L^p(\mathbb{R}^N, V(|x|)dx)$ with norm $||u|| := \left(||u||^p_A + ||u||^p_{L^p(\mathbb{R}^N, V(|x|)dx)}\right)^{1/p}$.

The rest of this section is devoted to elucidate the characteristics of the functions in D_A. In particular we prove some relevant pointwise estimates and embedding results. To be precise, we define

$$S_A := \left\{ u \in C^\infty_{\mathrm{c},r}(\mathbb{R}^N) \;\Big|\; \int_{\mathbb{R}^N} A(|x|)|\nabla u|^p dx < +\infty \right\}.$$

S_A is a linear subspace of $C^\infty_{\mathrm{c},r}(\mathbb{R}^N)$ and $||u||_A = \left(\int_{\mathbb{R}^N} A(|x|)|\nabla u|^p dx\right)^{1/p}$ is a norm on it. The next lemmas gives the relevant pointwise estimates for the functions in S_A. In all this paper, for any radial function u, with a little abuse of notations, we will write $u(x) = u(|x|) = u(r)$ if $r = |x|$.

Lemma 2.2. *Assume the hypothesis* ($\mathbf{A}$). *Fix $R_0 > 0$. Then there exists a constant $C = C(N, R_0, p, a_\infty) > 0$ such that for all $u \in S_A$ one has*

$$|u(x)| \leq C\, |x|^{-\frac{N+a_\infty-p}{p}} \left(\int_{B^c_{R_0}} A(|x|)\, |\nabla u|^p\, dx\right)^{1/p} \quad \text{for } |x| \geq R_0. \tag{2.3}$$

Proof. Assume $u \in S_A$. For $|x| = r \geq R_0$ we have

$$-u(r) = \int_r^\infty u'(s)ds.$$

Using the hypothesis $(\mathbf{A})$ and Hölder inequality, we obtain

$$\begin{aligned}
|u(r)| &\leq \int_r^\infty |u'(s)|ds = \int_r^\infty |u'(s)| s^{\frac{N+a_\infty-1}{p}} s^{-\frac{N+a_\infty-1}{p}} ds \\
&\leq \left(\int_r^\infty |u'(s)|^p s^{N-1} s^{a_\infty} ds \right)^{\frac{1}{p}} \left(\int_r^\infty s^{-\frac{N+a_\infty-1}{p-1}} ds \right)^{\frac{p-1}{p}} \\
&= (\omega_N)^{-\frac{1}{p}} \left(\int_{B_r^c} |x|^{a_\infty} |\nabla u|^p dx \right)^{\frac{1}{p}} \left(\int_r^\infty s^{-\frac{N+a_\infty-1}{p-1}} ds \right)^{\frac{p-1}{p}} \\
&\leq C_\infty^{-1/p} \omega_N^{-\frac{1}{p}} \left(\frac{p-1}{a_\infty + N - p} \right)^{\frac{p-1}{p}} \left(\int_{B_r^c} A(|x|) |\nabla u|^p dx \right)^{\frac{1}{p}} r^{-\frac{N+a_\infty-p}{p}},
\end{aligned}$$

where $C_\infty = C_\infty(R_0)$ is the constant in 2.2. Hence the thesis follows. □

Lemma 2.3. *Assume the hypothesis* $(\mathbf{A})$. *Fix* $R_0 > 0$. *Then there exists a constant* $C = C(N, R_0, p, a_0) > 0$ *such that for all* $u \in S_A \cap C_{c,r}^\infty(B_{R_0})$ *one has*

$$|u(x)| \leq C\, |x|^{-\frac{N+a_0-p}{p}} \left(\int_{B_{R_0}} A(|x|)\, |\nabla u|^p\, dx \right)^{1/p} \quad \text{for } 0 < |x| < R_0. \tag{2.4}$$

Proof. Let $u \in S_A \cap C_{c,r}^\infty(B_{R_0})$ and take $|x| = r < R_0$. Since $u(R_0) = 0$, we have

$$-u(r) = u(R_0) - u(r) = \int_r^{R_0} u'(s)ds.$$

The same arguments of Lemma 2.2 yield

$$\begin{aligned}
|u(r)| &\leq \int_r^{R_0} |u'(s)|ds \leq \left(\int_r^{R_0} |u'(s)|^p s^{N-1} s^{a_0} ds \right)^{\frac{1}{p}} \left(\int_r^{R_0} s^{-\frac{N+a_0-1}{p-1}} ds \right)^{\frac{p-1}{p}} \\
&\leq (\omega_N)^{-\frac{1}{p}} \left(\int_{B_{R_0}} |x|^{a_0} |\nabla u|^p dx \right)^{\frac{1}{p}} \left(\int_r^{R_0} s^{-\frac{N+a_0-1}{p-1}} ds \right)^{\frac{p-1}{p}} \\
&\leq \omega_N^{-\frac{1}{p}} C_0^{-1/p} \left(\frac{p-1}{N + a_0 - p} \right)^{p-1/p} \left(\int_{B_{R_0}} A(|x|)\, |\nabla u|^p\, dx \right)^{1/p} r^{-\frac{N+a_0-p}{p}},
\end{aligned}$$

where $C_0 = C_0(R_0)$ is the constant in 2.1. Then the thesis ensues. □

Lemma 2.4. *Assume the hypothesis* $(\mathbf{A})$. *Fix* $R_0 > 0$. *Then there exists a constant* $C = C(N, R_0, p, a_0, a_\infty) > 0$ *such that for all* $u \in S_A$ *one has*

$$|u(x)| \leq C\, |x|^{-\frac{N+a_0-p}{p}} \left(\int_{B_{R_0+1}} A(|x|)\, |\nabla u|^p\, dx + \int_{B_{R_0}^c} A(|x|) |\nabla u|^p dx \right)^{1/p} \quad \text{for } 0 < |x| < R_0.$$

Proof. Let $u \in S_A$. Take a radial function $\rho \in C^\infty_{c,r}(\mathbb{R}^N)$ such that $\rho(x) \in [0,1]$, $\rho \equiv 1$ in B_{R_0} and $\rho(x) \equiv 0$ if $|x| \geq R_0 + 1/2$. Hence $\rho\, u \in C^\infty_{c,r}(B_{R_0+1})$, so that we can apply Lemma 2.3 (in the ball B_{R_0+1}) and get

$$|\rho(x)u(x)| \leq C|x|^{-\frac{N+a_0-p}{p}} \left(\int_{B_{R_0+1}} A(|x|)|\nabla(\rho u)|^p dx \right)^{1/p}.$$

If $|x| < R_0$ we have $\rho(x) = 1$ and hence

$$|u(x)| \leq C|x|^{-\frac{N+a_0-p}{p}} \left(\int_{B_{R_0+1}} A(|x|)|\nabla(\rho u)|^p dx \right)^{1/p}.$$

We also have

$$|\nabla(\rho u)|^p \leq (\rho|\nabla u| + |u||\nabla \rho|)^p \leq C_p \left(\rho^p |\nabla u|^p + |u|^p |\nabla \rho|^p\right)$$

and hence, for $x \in B_{R_0}$,

$$|u(x)| \leq C|x|^{-\frac{N+a_0-p}{p}} C_p^{1/p} \left(\int_{B_{R_0+1}} A(|x|)|\nabla u|^p dx + \int_{B_{R_0+1}} A(|x|)|u|^p\, |\nabla \rho|^p dx \right)^{1/p}.$$

Moreover

$$\int_{B_{R_0+1}} A(|x|)|u|^p\, |\nabla \rho|^p dx \leq C_1 \int_{B_{R_0+1} \setminus B_{R_0}} A(|x|)|u|^p\, dx$$

where the constant $C_1 = \max |\nabla \rho|^p$ depends only on ρ, hence on R_0. We can now apply Lemma 2.2 in the domain $B^c_{R_0}$, and we get

$$|u|^p(y) \leq C|y|^{-N-a_\infty+p} \int_{B^c_{R_0}} A(|x|)|\nabla u|^p dx \quad \text{for } |y| > R_0.$$

Recalling that A is bounded in $B_{R_0+1} \setminus B_{R_0}$, for $y \in B_{R_0+1} \setminus B_{R_0}$ and $C_2 = \max \left\{ A(|y|) \,\middle|\, y \in B_{R_0+1} \setminus B_{R_0} \right\}$ we get

$$A(|y|)|u|^p(y) \leq C_2\, C|y|^{-N-a_\infty+p} \int_{B^c_{R_0}} A(|x|)|\nabla u|^p dx$$

and hence, integrating w.r.t. $y \in B_{R_0+1} \setminus B_{R_0}$, we obtain

$$\int_{B_{R_0+1} \setminus B_{R_0}} A(|y|)|u|^p(y) dy \leq C_3 \int_{B^c_{R_0}} A(|x|)|\nabla u|^p dx$$

where $C_3 = C_3(N, a_\infty, R_0, p)$. Pasting all together, for $|x| < R_0$ we get

$$\begin{aligned} |u(x)| &\leq C|x|^{-\frac{N+a_0-p}{p}} C_p^{1/p} \left(\int_{B_{R_0+1}} A(|x|)|\nabla u|^p dx + \int_{B_{R_0+1}} A(|x|)|u|^p\, |\nabla \rho|^p dx \right)^{1/p} \\ &\leq C_4 |x|^{-\frac{N+a_0-p}{p}} \left(\int_{B_{R_0+1}} A(|x|)|\nabla u|^p dx + \int_{B^c_{R_0}} A(|x|)\, |\nabla u|^p dx \right)^{1/p}, \end{aligned}$$

where $C_4 = C_4(N, R_0, p, a_0, a_\infty)$. Hence the thesis. □

We can now give a precise definition of D_A.

Definition 2.5. D_A is the completion of S_A with respect to the norm $|| \cdot ||_A$.

The pointwise estimates of the previous lemmas imply the following proposition, which gives the main properties of D_A. The proof is a simple exercise in funtional analysis and we skip it.

Lemma 2.6. *Assume the hypothesis* $(\mathbf{A})$. *Then the following properties hold.*

(i) *If $u \in D_A$, then u has weak derivatives $D_i u$ in the open set $\Omega = \mathbb{R}^N \backslash \{0\}$ ($i = 1, ..., N$) and one has $D_i u \in L^p_{loc}(\Omega)$.*

(ii) *If $u \in D_A$, then $\int_{\mathbb{R}^N} A(|x|)|\nabla u|^p \, dx < +\infty$ and $||u||_A = \left(\int_{\mathbb{R}^N} A(|x|)|\nabla u|^p \, dx\right)^{1/p}$ is a norm on D_A. With this norm, D_A is a Banach space.*

(iii) *For any $R_0 > 0$, there exists a constant $M = M(N, R_0, a_0, a_\infty) > 0$ such that for all $u \in D_A$ one has*

$$|u(x)| \leq M|x|^{-\frac{N+a_0-p}{p}} \, ||u||_A, \quad \textit{for a.e. } x \in B_{R_0},$$

$$|u(x)| \leq M|x|^{-\frac{N+a_\infty-p}{p}} \, ||u||_A, \quad \textit{for a.e. } x \in B^c_{R_0}.$$

We now prove some embedding properties of the space D_A. To this aim, we define the exponents p_0, p_∞ as follows:

$$p_0 := \frac{pN}{N + a_0 - p}, \quad p_\infty := \frac{pN}{N + a_\infty - p}.$$

Recall that $p - N < a_0, \, a_\infty \leq p$ and notice that, from the hypotheses, we have $p_0, p_\infty \geq p$.

Lemma 2.7. *Assume the hypothesis* $(\mathbf{A})$. *For every $R > 0$ we have the continuous embeddings*

$$D_A \hookrightarrow L^{p_0}(B_R) \quad \textit{and} \quad D_A \hookrightarrow L^{p_\infty}(B^c_R).$$

Proof. Let $u \in S_A$ (the general case follows by density). Fix $R > 0$ and denote by C any positive constant only dependent on N, p, a_0, a_∞ and R. We first prove the embedding in $L^{p_\infty}(B^c_R)$, so we estimate the norm of u in such a space. With an integration by parts, we get

$$\int_{B^c_R} |u(x)|^{p_\infty} dx = \omega_N \int_R^{+\infty} r^{N-1} |u(r)|^{p_\infty} dr \leq \frac{\omega_N p_\infty}{N} \int_R^{+\infty} r^N |u(r)|^{p_\infty - 1} \, |u'(r)| dr.$$

Then, by several applications of Hölder inequality and using the pointwise estimates of Lemma 2.6, we get

$$\int_R^{+\infty} r^N |u(r)|^{p_\infty - 1} \, |u'(r)| dr = \int_R^{+\infty} r^{\frac{N-1}{p}} |u'(r)| r^{\frac{a_\infty}{p}} \, r^{-\frac{a_\infty}{p}} \, r^{\frac{Np-N+1}{p}} |u(r)|^{p_\infty - 1} dr$$

$$\leq \left(\int_R^{+\infty} r^{N-1} |u'(r)|^p r^{a_\infty} dr\right)^{1/p} \left(\int_R^{+\infty} r^{\frac{Np-N+1-a_\infty}{p-1}} |u(r)|^{(p_\infty - 1)\frac{p}{p-1}} dr\right)^{\frac{p-1}{p}}$$

$$\leq C \left(\int_R^{+\infty} r^{N-1} A(r) |u'(r)|^p dr\right)^{1/p} \left(\int_R^{+\infty} r^{N-1} |u(r)|^{p_\infty} \, r^{\frac{p-a_\infty}{p-1}} \, |u(r)|^{\frac{p_\infty - p}{p-1}} dr\right)^{\frac{p-1}{p}}$$

$$\leq C\left(\int_{B_R^c} A(|x|)\,|\nabla u(x)|^p dx\right)^{1/p}\times$$

$$\times\left(\int_R^{+\infty} r^{N-1}|u(r)|^{p_\infty} r^{\frac{p-a_\infty}{p-1}}\, r^{-\frac{N+a_\infty-p}{p}\frac{p_\infty-p}{p-1}}\left(\int_{B_R^c} A(|x|)\,|\nabla u(x)|^p dx\right)^{\frac{p_\infty-p}{p(p-1)}}\right)^{\frac{p-1}{p}}$$

$$\leq C\left(\int_{B_R^c} A(|x|)\,|\nabla u(x)|^p dx\right)^{\frac{p_\infty}{p^2}}\left(\int_{B_R^c} |u(x)|^{p_\infty} dx\right)^{\frac{p-1}{p}}$$

Notice that we have $\frac{p-a_\infty}{p-1}-\frac{N+a_\infty-p}{p}\frac{p_\infty-p}{p-1}=0$, from the definition of p_∞.
From the previous computations, we get

$$\int_{B_R^c} |u(x)|^{p_\infty} dx \leq C\left(\int_{B_R^c} A(|x|)\,|\nabla u(x)|^p dx\right)^{\frac{p_\infty}{p^2}}\left(\int_{B_R^c} |u(x)|^{p_\infty} dx\right)^{\frac{p-1}{p}}$$

and hence

$$\left(\int_{B_R^c} |u(x)|^{p_\infty} dx\right)^{1/p}\leq C\left(\int_{B_R^c} A(|x|)\,|\nabla u(x)|^p dx\right)^{\frac{p_\infty}{p^2}},$$

that is

$$\left(\int_{B_R^c} |u(x)|^{p_\infty} dx\right)^{\frac{1}{p_\infty}}\leq C\left(\int_{B_R^c} A(|x|)\,|\nabla u(x)|^p dx\right)^{\frac{1}{p}}\leq C||u||_A,$$

which is the thesis. In order to prove the embedding in $L^{p_0}(B_R)$, we use an argument similar to the one of Lemma 2.4. So we fix a cut-off function ρ as we did there (with R instead of R_0). For $u\in S_A$ we set $v=\rho u\in C^\infty_{c,r}(B_{R+1})$. Arguing as for the previous case, we get

$$\int_{B_{R+1}} |v(x)|^{p_0} dx=\omega_N\int_0^{R+1} r^{N-1}|v(r)|^{p_0} dr\leq\frac{p_0\omega_N}{N}\int_0^{R+1} r^N|v(r)|^{p_0-1}|v'(r)|dr=$$

$$=\frac{p_0\omega_N}{N}\int_0^{R+1} r^{\frac{N-1}{p}}|v'(r)|r^{\frac{a_0}{p}}r^{-\frac{a_0}{p}}r^{\frac{Np-N+1}{p}}|v(r)|^{p_0-1}dr$$

$$\leq\frac{p_0\omega_N}{N}\left(\int_0^{R+1} r^{N-1}|v'(r)|^p r^{a_0}dr\right)^{1/p}\left(\int_0^{R+1} r^{\frac{Np-N+1-a_0}{p-1}}|v(r)|^{(p_0-1)\frac{p}{p-1}}dr\right)^{\frac{p-1}{p}}$$

$$\leq\frac{p_0\omega_N}{N}\left(\int_0^{R+1} r^{N-1}A(r)|v'(r)|^p dr\right)^{1/p}\left(\int_0^{R+1} r^{N-1}|v(r)|^{p_0} r^{\frac{p-a_0}{p-1}}|v(r)|^{\frac{p_0-p}{p-1}}dr\right)^{\frac{p-1}{p}}$$

$$\leq C\left(\int_{B_{R+1}} A(|x|)|\nabla v(x)|^p dx\right)^{1/p}\times$$

$$\times\left(\int_0^{R+1} r^{N-1}|v(r)|^{p_0} r^{\frac{p-a_0}{p-1}} r^{-\frac{N+a_0-p}{p}\frac{p_0-p}{p-1}}\left(\int_{B_{R+1}} A(|x|)\,|\nabla v(x)|^p dx\right)^{\frac{p_0-p}{p(p-1)}}\right)^{\frac{p-1}{p}}$$

$$= C\left(\int_{B_{R+1}} A(|x|)|\nabla v(x)|^p dx\right)^{\frac{p_0}{p^2}}\left(\int_{B_{R+1}} |v(x)|^{p_0} dx\right)^{\frac{p-1}{p}},$$

Notice that $\frac{p-a_0}{p-1} - \frac{N+a_0-p}{p}\frac{p_0-p}{p-1} = 0$, by the definition of p_0. From these computations we easily deduce, as before, that

$$\left(\int_{B_{R+1}} |v(x)|^{p_0} dx\right)^{\frac{1}{p_0}} \leq C\left(\int_{B_{R+1}} A(|x|)|\nabla v(x)|^p dx\right)^{\frac{1}{p}}.$$

Now, as in Lemma 2.4, we use the fact that A is continuous and strictly positive on the compact set $\overline{B_{R+1}\backslash B_R}$, and we get

$$\int_{B_{R+1}} A(|x|)|\nabla \rho u|^p dx \leq C\int_{B_{R+1}} A(|x|)\rho^p|\nabla u|^p dx + C\int_{B_{R+1}\backslash B_R} A(|x|)|u|^p|\nabla\rho|^p dx \leq$$

$$\leq C||u||_A^p + C\int_{B_R^c} A(|x|)|\nabla u|^p dx\left(\int_{B_{R+1}\backslash B_R} x^{-N-a_\infty+p} dx\right)^{1/p} \leq C||u||_A^p.$$

Hence we get

$$\left(\int_{B_{R+1}} |\rho u|^{p_0} dx\right)^{1/p_0} \leq C||u||_A$$

and therefore

$$\left(\int_{B_R} |u|^{p_0} dx\right)^{1/p_0} = \left(\int_{B_R} |\rho u|^{p_0} dx\right)^{1/p_0} \leq \left(\int_{B_{R+1}} |\rho u|^{p_0} dx\right)^{1/p_0} \leq C||u||_A,$$

which is the thesis. □

The following lemma gives another embedding result that we will use.

Lemma 2.8. *Assume the hypothesis* $(\mathbf{A})$ *and fix* $0 < r < R$. *Then the embedding*

$$D_A \hookrightarrow L^p(B_R \setminus \overline{B}_r)$$

is continuous and compact.

Proof. Set $E := B_R \setminus \overline{B}_r$ for brevity. The continuity of the embedding easily derives from (2.3), by integrating over the set E. As to compactness, let $\{u_n\}_n$ be a bounded sequence in D_A. By continuity of the embedding we obtain that also $\{||u_n||_{L^p(E)}\}_n$ is bounded. Moreover, as A is continuous and strictly positive on the compact set $\overline{E}$, we have

$$\int_E |\nabla u_n|^p dx \leq C\int_E A(|x|)|\nabla u_n|^p dx \leq C_1.$$

Thus $\{u_n\}_n$ is bounded also in the space $W^{1,p}(E)$. Thanks to Rellich's Theorem, $\{u_n\}_n$ has a convergent subsequence in $L^p(E)$, and this gives the thesis. □

3 Compactness results for the space X

In this section we state the main compactness results of this paper, concerning the space X. Recall that we define such a space as

$$X := D_A \cap L^p(\mathbb{R}^N, V(|x|)dx)$$

endowed with the norm $||u|| := \left(||u||_A^p + ||u||^p_{L^p(\mathbb{R}^N, V(|x|)dx)} \right)^{1/p}$, with respect to which X is a Banach space. The compactness results that we state here will be proved in sections 4 and 5.

Given A, V and K as in $(\mathbf{A})$, $(\mathbf{V})$ and $(\mathbf{K})$, we define the following functions of $R > 0$ and $q > 1$:

$$\mathcal{S}_0(q, R) := \sup_{u \in X,\, \|u\|=1} \int_{B_R} K(|x|)\, |u|^q \, dx, \tag{3.1}$$

$$\mathcal{S}_\infty(q, R) := \sup_{u \in X,\, \|u\|=1} \int_{\mathbb{R}^N \setminus B_R} K(|x|)\, |u|^q \, dx. \tag{3.2}$$

Clearly $\mathcal{S}_0(q, \cdot)$ is nondecreasing, $\mathcal{S}_\infty(q, \cdot)$ is nonincreasing and both of them can be infinite at some R.

Our first result concerns the embedding properties of X into the sum space

$$L^{q_1}_K + L^{q_2}_K := \left\{ u_1 + u_2 : u_1 \in L^{q_1}_K\left(\mathbb{R}^N\right),\, u_2 \in L^{q_2}_K\left(\mathbb{R}^N\right) \right\}, \quad 1 < q_i < \infty.$$

We recall from [6] that such a space can be characterized as the set of measurable mappings $u : \mathbb{R}^N \to \mathbb{R}$ for which there exists a measurable set $E \subseteq \mathbb{R}^N$ such that $u \in L^{q_1}_K(E) \cap L^{q_2}_K(E^c)$. It is a Banach space with respect to the norm

$$\|u\|_{L^{q_1}_K + L^{q_2}_K} := \inf_{u_1 + u_2 = u} \max\left\{ \|u_1\|_{L^{q_1}_K(\mathbb{R}^N)}, \|u_2\|_{L^{q_2}_K(\mathbb{R}^N)} \right\}$$

and the continuous embedding $L^q_K \hookrightarrow L^{q_1}_K + L^{q_2}_K$ holds for all $q \in [\min\{q_1, q_2\}, \max\{q_1, q_2\}]$. The assumptions of our result are quite general but not so easy to check, so more handy conditions ensuring these general assumptions will be provided by the next results.

Theorem 3.1. *Let $1 < p < N$, let A, V and K be as in $(\mathbf{A})$, $(\mathbf{V})$ and $(\mathbf{K})$, and let $q_1, q_2 > 1$.*

(i) If

$$\mathcal{S}_0(q_1, R_1) < \infty \quad \text{and} \quad \mathcal{S}_\infty(q_2, R_2) < \infty \quad \text{for some } R_1, R_2 > 0, \tag{$\mathcal{S}'_{q_1,q_2}$}$$

then X is continuously embedded into $L^{q_1}_K(\mathbb{R}^N) + L^{q_2}_K(\mathbb{R}^N)$.

(ii) If

$$\lim_{R \to 0^+} \mathcal{S}_0(q_1, R) = \lim_{R \to +\infty} \mathcal{S}_\infty(q_2, R) = 0, \tag{$\mathcal{S}''_{q_1,q_2}$}$$

then X is compactly embedded into $L^{q_1}_K(\mathbb{R}^N) + L^{q_2}_K(\mathbb{R}^N)$.

Observe that, of course, $(\mathcal{S}''_{q_1,q_2})$ implies $(\mathcal{S}'_{q_1,q_2})$. Moreover, these assumptions can hold with $q_1 = q_2 = q$ and therefore Theorem 3.1 also concerns the embedding properties of X into L^q_K, $1 < q < \infty$.

We now look for explicit conditions on V and K implying $(\mathcal{S}''_{q_1,q_2})$ for some q_1 and q_2. More precisely, in Theorem 3.2 we will find a range of exponents q_1 such that $\lim_{R\to 0^+} \mathcal{S}_0(q_1, R) = 0$, while in Theorem 3.3 we will do the same for exponents q_2 such that $\lim_{R\to +\infty} \mathcal{S}_\infty(q_2, R) = 0$.

For $\alpha \in \mathbb{R}$, $\beta \in [0,1]$, $a > p - N$, we define two functions $\alpha^*(a,\beta)$ and $q^*(a,\alpha,\beta)$ by setting

$$\alpha^*(a,\beta) := \max\left\{ p\beta - 1 - \frac{p-1}{p}N - a\beta + \frac{a}{p}, -(1-\beta)N \right\} =$$

$$\begin{cases} p\beta - 1 - \frac{p-1}{p}N - a\beta + \frac{a}{p} & \text{if } 0 \leq \beta \leq \frac{1}{p} \\ -(1-\beta)N & \text{if } \frac{1}{p} \leq \beta \leq 1 \end{cases}$$

and

$$q^*(a,\alpha,\beta) := p\frac{\alpha - p\beta + N + a\beta}{N - p + a}.$$

Theorem 3.2. *Let A, V, K be as in* $(\mathbf{A})$, $(\mathbf{V})$, $(\mathbf{K})$. *Assume that there exists $R_1 > 0$ such that $V(r) < +\infty$ almost everywhere in $(0, R_1)$ and*

$$\operatorname*{ess\,sup}_{r\in(0,R_1)} \frac{K(r)}{r^{\alpha_0} V(r)^{\beta_0}} < +\infty \quad \textit{for some } 0 \leq \beta_0 \leq 1 \textit{ and } \alpha_0 > \alpha^*(a_0,\beta_0). \tag{3.3}$$

Then $\lim\limits_{R\to 0^+} \mathcal{S}_0(q_1, R) = 0$ *for every $q_1 \in \mathbb{R}$ such that*

$$\max\{1, p\beta_0\} < q_1 < q^*(a_0, \alpha_0, \beta_0). \tag{3.4}$$

Theorem 3.3. *Let A, V, K be as in* $(\mathbf{A})$, $(\mathbf{V})$, $(\mathbf{K})$. *Assume that there exists $R_2 > 0$ such that $V(r) < +\infty$ for almost every $r > R_2$ and*

$$\operatorname*{ess\,sup}_{r>R_2} \frac{K(r)}{r^{\alpha_\infty} V(r)^{\beta_\infty}} < +\infty \quad \textit{for some } 0 \leq \beta_\infty \leq 1 \textit{ and } \alpha_\infty \in \mathbb{R}. \tag{3.5}$$

Then $\lim\limits_{R\to +\infty} \mathcal{S}_\infty(q_2, R) = 0$ *for every $q_2 \in \mathbb{R}$ such that*

$$q_2 > \max\{1, p\beta_\infty, q^*(a_\infty, \alpha_\infty, \beta_\infty)\}. \tag{3.6}$$

We observe explicitly that for every a, α, β as above one has

$$\max\{1, p\beta, q^*(a,\alpha,\beta)\} = \begin{cases} q^*(a,\alpha,\beta) & \text{if } \alpha \geq \alpha^*(a,\beta) \\ \max\{1, p\beta\} & \text{if } \alpha \leq \alpha^*(a,\beta) \end{cases}.$$

Remark 3.4.

1. We mean $V(r)^0 = 1$ for every r (even if $V(r) = 0$). In particular, if $V(r) = 0$ for almost every $r > R_2$, then Theorem 3.3 can be applied with $\beta_\infty = 0$ and assumption (3.5) means

$$\operatorname*{ess\,sup}_{r>R_2} \frac{K(r)}{r^{\alpha_\infty}} < +\infty \quad \text{for some } \alpha_\infty \in \mathbb{R}.$$

   Similarly for Theorem 3.2 and assumption (3.3), if $V(r) = 0$ for almost every $r \in (0, R_1)$.

2. The inequality $\max\{1, p\beta_0\} < q^*(a_0, \alpha_0, \beta_0)$ is equivalent to $\alpha_0 > \alpha^*(a_0, \beta_0)$. Then, in (3.4), such inequality is automatically true and does not ask for further conditions on α_0 and β_0.

3. The assumptions of Theorems 3.2 and 3.3 may hold for different pairs (α_0, β_0), $(\alpha_\infty, \beta_\infty)$ (assuming p and a_0 fixed). In this case, of course, one chooses them in order to get the ranges for q_1, q_2 as large as possible. For instance, assume that $a_0 \leq p$ and V is not singular at the origin, i.e., V is essentially bounded in a neighbourhood of 0. If condition (3.3) holds true for a pair (α_0, β_0), then (3.3) also holds for all pairs (α_0', β_0') such that $\alpha_0' > \alpha_0$ and $\beta_0' < \beta_0$. Therefore, since $\max\{1, p\beta\}$ is nondecreasing in β and $q^*(a, \alpha, \beta)$ is increasing in α and decreasing in β (because $a_0 \leq p$) , it is convenient to choose $\beta_0 = 0$ and the best interval where one can take q_1 is $1 < q_1 < q^*(a_0, \overline{\alpha}, 0)$ with $\overline{\alpha} := \sup\left\{\alpha_0 : \operatorname{ess\,sup}_{r\in(0,R_1)} K(r)/r^{\alpha_0} < +\infty\right\}$ (here we mean $q^*(a_0, +\infty, 0) = +\infty$).

4 Proof of Theorem 3.1

Assume as usual $N \geq 3$ and $1 < p < N$, and let A, V and K be as in $(\mathbf{A})$, $(\mathbf{V})$ and $(\mathbf{K})$. Recall from assumption $(\mathbf{K})$ that $K \in L^s_{\rm loc}((0, +\infty))$ for some $s > 1$.

Lemma 4.1. *Let $R > r > 0$ and $1 < q < \infty$. Then there exist $\tilde{C} = \tilde{C}(N, p, r, R, q, s) > 0$ and $l = l(p, q, s) > 0$ such that $q - lp > 0$ and $\forall u \in X$ one has*

$$\int_{B_R\setminus B_r} K(|x|)\, |u|^q\, dx \leq \tilde{C}\, \|K(|\cdot|)\|_{L^s(B_R\setminus B_r)} \|u\|^{q-lp} \left(\int_{B_R\setminus B_r} |u|^p\, dx\right)^l. \tag{4.1}$$

Notice that, in the second part of Lemma 4.1, $s > \frac{Np}{N(p-1)+p-a_+}$ implies $\tilde{q} > 1$.

Proof. Let $u \in X$ and fix $t \in (1, s)$ such that $t'q > p$ (where $t' = t/(t-1)$). Then, by Hölder inequality and the pointwise estimates of Section 2, we have

$$\begin{aligned}
&\int_{B_R\setminus B_r} K(|x|)\, |u|^q\, dx \\
&\leq \left(\int_{B_R\setminus B_r} K(|x|)^t\, dx\right)^{\frac{1}{t}} \left(\int_{B_R\setminus B_r} |u|^{t'q}\, dx\right)^{\frac{1}{t'}} \\
&\leq |B_R \setminus B_r|^{\frac{1}{t}-\frac{1}{s}} \|K(|\cdot|)\|_{L^s(B_R\setminus B_r)} \left(\int_{B_R\setminus B_r} |u|^{t'q-p}\, |u|^p\, dx\right)^{\frac{1}{t'}} \\
&\leq |B_R \setminus B_r|^{\frac{1}{t}-\frac{1}{s}} \|K(|\cdot|)\|_{L^s(B_R\setminus B_r)} \left(\frac{C\,\|u\|}{r^{\frac{N-p+a_0}{p}}}\right)^{q-p/t'} \left(\int_{B_R\setminus B_r} |u|^p\, dx\right)^{\frac{1}{t'}}.
\end{aligned}$$

This proves (4.1). ∎

We now prove Theorem 3.1. Recall the definitions (3.1)-(3.2) of the functions $\mathcal{S}_0$ and $\mathcal{S}_\infty$, and the following result from [6] concerning convergence in the sum of Lebesgue spaces.

Proposition 4.2 ([6, Proposition 2.7])**.** *Let* $\{u_n\} \subseteq L_K^{p_1} + L_K^{p_2}$ *be a sequence such that* $\forall \varepsilon > 0$ *there exist* $n_\varepsilon > 0$ *and a sequence of measurable sets* $E_{\varepsilon,n} \subseteq \mathbb{R}^N$ *satisfying*

$$\forall n > n_\varepsilon, \quad \int_{E_{\varepsilon,n}} K(|x|) |u_n|^{p_1} dx + \int_{E_{\varepsilon,n}^c} K(|x|) |u_n|^{p_2} dx < \varepsilon. \tag{4.2}$$

Then $u_n \to 0$ *in* $L_K^{p_1} + L_K^{p_2}$*.*

Proof of Theorem 3.1. We prove each part of the theorem separately.

(i) By the monotonicity of $\mathcal{S}_0$ and $\mathcal{S}_\infty$, it is not restrictive to assume $R_1 < R_2$ in hypothesis $\left(\mathcal{S}'_{q_1,q_2}\right)$. In order to prove the continuous embedding, let $u \in X$, $u \neq 0$. Then we have

$$\int_{B_{R_1}} K(|x|) |u|^{q_1} dx = \|u\|^{q_1} \int_{B_{R_1}} K(|x|) \frac{|u|^{q_1}}{\|u\|^{q_1}} dx \leq \|u\|^{q_1} \mathcal{S}_0(q_1, R_1) \tag{4.3}$$

and, similarly,

$$\int_{B_{R_2}^c} K(|x|) |u|^{q_2} dx \leq \|u\|^{q_2} \mathcal{S}_\infty(q_2, R_2). \tag{4.4}$$

We now use (4.1) of Lemma 4.1 and Lemma 2.8 to deduce that there exists a constant $C_1 > 0$, independent from u, such that

$$\int_{B_{R_2} \setminus B_{R_1}} K(|x|) |u|^{q_1} dx \leq C_1 \|u\|^{q_1}. \tag{4.5}$$

Hence $u \in L_K^{q_1}(B_{R_2}) \cap L_K^{q_2}(B_{R_2}^c)$ and thus $u \in L_K^{q_1} + L_K^{q_2}$. Moreover, if $u_n \to 0$ in X, then, using (4.3), (4.4) and (4.5), we get

$$\int_{B_{R_2}} K(|x|) |u_n|^{q_1} dx + \int_{B_{R_2}^c} K(|x|) |u_n|^{q_2} dx = o(1)_{n\to\infty},$$

which means $u_n \to 0$ in $L_K^{q_1} + L_K^{q_2}$ by Proposition 4.2.

(ii) Assume hypothesis $\left(\mathcal{S}''_{q_1,q_2}\right)$. Let $\varepsilon > 0$ and let $u_n \rightharpoonup 0$ in X. Then $\{\|u_n\|\}_n$ is bounded and, arguing as for (4.3) and (4.4), we can take $r_\varepsilon > 0$ and $R_\varepsilon > r_\varepsilon$ such that for all n one has

$$\int_{B_{r_\varepsilon}} K(|x|) |u_n|^{q_1} dx \leq \left(\|u_n\|^{q_1}\right) \mathcal{S}_0(q_1, r_\varepsilon) \leq \left(\sup_n \|u_n\|^{q_1}\right) \mathcal{S}_0(q_1, r_\varepsilon) < \frac{\varepsilon}{3}$$

and

$$\int_{B_{R_\varepsilon}^c} K(|x|) |u_n|^{q_2} dx \leq \left(\sup_n \|u_n\|^{q_2}\right) \mathcal{S}_\infty(q_2, R_\varepsilon) < \frac{\varepsilon}{3}.$$

Using (4.1) of Lemma 4.1 and the boundedness of $\{\|u_n\|\}$ again, we infer that there exist two constants $C_2, l > 0$, independent from n, such that

$$\int_{B_{R_\varepsilon} \setminus B_{r_\varepsilon}} K(|x|) |u_n|^{q_1} dx \leq C_2 \left(\int_{B_{R_\varepsilon} \setminus B_{r_\varepsilon}} |u_n|^p dx\right)^l,$$

where

$$\int_{B_{R_\varepsilon} \setminus B_{r_\varepsilon}} |u_n|^p dx \to 0 \quad \text{as } n \to \infty \quad (\varepsilon \text{ fixed})$$

thanks to Lemma 2.8. Therefore we obtain

$$\int_{B_{R_\varepsilon}} K(|x|) |u_n|^{q_1} dx + \int_{B_{R_\varepsilon}^c} K(|x|) |u_n|^{q_2} dx < \varepsilon$$

for all n sufficiently large, which means $u_n \to 0$ in $L_K^{q_1} + L_K^{q_2}$ (Proposition 4.2). This concludes the proof of part (ii). $\square$

5 Proof of Theorems 3.2 and 3.3

Assume as usual $N \geq 3$ and $1 < p < N$, and let A, V and K be as in $(\mathbf{A})$, $(\mathbf{V})$ and $(\mathbf{K})$.

Lemma 5.1. *Let $R_0 > 0$ and assume that $V(r) < +\infty$ almost everywhere in B_{R_0} and*

$$\Lambda := \operatorname*{ess\,sup}_{x \in B_{R_0}} \frac{K(|x|)}{|x|^{\alpha} V(|x|)^{\beta}} < +\infty \quad \text{for some } 0 \leq \beta \leq 1 \text{ and } \alpha \in \mathbb{R}.$$

Let $u \in X$ and assume that there exist $\nu \in \mathbb{R}$ and $m > 0$ such that

$$|u(x)| \leq \frac{m}{|x|^{\nu}} \quad \text{almost everywhere in } B_{R_0}.$$

Then there exists a constant $C = C(N, R_0, a_0, a_\infty, \beta) > 0$ such that $\forall R \in (0, R_0)$ and $\forall q > \max\{1, p\beta\}$, one has

$$\int_{B_R} K(|x|) |u|^q \, dx$$
$$\leq \begin{cases} \Lambda m^{q-1} C \left(\int_{B_R} |x|^{\frac{\alpha - \nu(q-1)}{N(p-1)+p(1-p\beta+a_0\beta)-a_0} pN} \, dx \right)^{\frac{N(p-1)+p(1-p\beta+a_0\beta)-a_0}{pN}} \|u\| & \text{if } 0 \leq \beta \leq \frac{1}{p} \\ \Lambda m^{q-p\beta} \left(\int_{B_R} |x|^{\frac{\alpha-\nu(q-p\beta)}{1-\beta}} \, dx \right)^{1-\beta} \|u\|^{p\beta} & \text{if } \frac{1}{p} < \beta < 1 \\ \Lambda m^{q-p} \left(\int_{B_R} |x|^{\frac{p}{p-1}(\alpha - \nu(q-p))} V(|x|) |u|^p \, dx \right)^{\frac{p-1}{p}} \|u\| & \text{if } \beta = 1. \end{cases}$$

Proof. We distinguish several cases, where we will use Hölder inequality many times, without explicitly noting it.

Case $\beta = 0$.

We apply Hölder inequality with exponents $p_0 = \frac{Np}{N-p+a_0}$, $(p_0)' = \frac{Np}{N(p-1)-a_0+p}$, and we use Lemma 2.7. We have

$$\begin{aligned} \frac{1}{\Lambda} \int_{B_R} K(|x|) |u|^q \, dx &\leq \int_{B_R} |x|^{\alpha} |u|^{q-1} |u| \, dx \\ &\leq \left(\int_{B_R} \left(|x|^{\alpha} |u|^{q-1} \right)^{\frac{pN}{N(p-1)+p-a_0}} dx \right)^{\frac{N(p-1)+p-a_0}{pN}} \left(\int_{B_R} |u|^{p_0} \, dx \right)^{\frac{1}{p_0}} \\ &\leq m^{q-1} C \left(\int_{B_R} |x|^{\frac{\alpha-\nu(q-1)}{N(p-1)+p-a_0} pN} \, dx \right)^{\frac{N(p-1)+p-a_0}{pN}} \|u\| . \end{aligned}$$

Case $0 < \beta < 1/p$.

One has $\frac{1}{\beta} > 1$ and $\frac{1-\beta}{1-p\beta} p_0 > 1$, with Hölder conjugate exponents $\left(\frac{1}{\beta}\right)' = \frac{1}{1-\beta}$ and $\left(\frac{1-\beta}{1-p\beta} p_0\right)' = \frac{pN(1-\beta)}{N(p-1)+p(1-p\beta+a_0\beta)-a_0}$. Then we get

$$\begin{aligned} &\frac{1}{\Lambda} \int_{B_R} K(|x|) |u|^q \, dx \\ &\leq \int_{B_R} |x|^{\alpha} V(|x|)^{\beta} |u|^q \, dx = \int_{B_R} |x|^{\alpha} |u|^{q-1} |u|^{1-p\beta} V(|x|)^{\beta} |u|^{p\beta} \, dx \end{aligned}$$

$$
\begin{aligned}
&\leq \left(\int_{B_R}\left(|x|^{\alpha}|u|^{q-1}|u|^{1-p\beta}\right)^{\frac{1}{1-\beta}}dx\right)^{1-\beta}\left(\int_{B_R}V(|x|)|u|^p\,dx\right)^{\beta}\\
&\leq \left(\left(\left(\int_{B_R}\left(|x|^{\frac{\alpha}{1-\beta}}|u|^{\frac{q-1}{1-\beta}}\right)^{\left(\frac{1-\beta}{1-p\beta}p_0\right)'}dx\right)^{\frac{1}{\left(\frac{1-\beta}{1-p\beta}p_0\right)'}}\left(\int_{B_R}|u|^{p_0}\,dx\right)^{\frac{1-p\beta}{(1-\beta)p_0}}\right)^{1-\beta}\|u\|^{p\beta}\\
&\leq m^{q-1}C\left(\left(\left(\int_{B_R}\left(|x|^{\frac{\alpha}{1-\beta}-\nu\frac{q-1}{1-\beta}}\right)^{\left(\frac{1-\beta}{1-p\beta}p_0\right)'}dx\right)^{\frac{1}{\left(\frac{1-\beta}{1-\beta}p_0\right)'}}C\|u\|^{\frac{1-p\beta}{1-\beta}}\right)^{1-\beta}\|u\|^{p\beta}\\
&= m^{q-1}C\left(\int_{B_R}|x|^{\frac{\alpha-\nu(q-1)}{N(p-1)+p(1-p\beta+a_0\beta)-a_0}pN}\,dx\right)^{\frac{N(p-1)+p(1-p\beta+a_0\beta)-a_0}{pN}}C\|u\|.
\end{aligned}
$$

Case $\beta=\frac{1}{p}$.

We have

$$
\begin{aligned}
\frac{1}{\Lambda}\int_{B_R}K(|x|)|u|^q\,dx &\leq \int_{B_R}|x|^{\alpha}|u|^{q-1}V(|x|)^{\frac{1}{p}}\,||\,dx\\
u &\leq \left(\int_{B_R}|x|^{\alpha\frac{p}{p-1}}|u|^{(q-1)\frac{p}{p-1}}\,dx\right)^{\frac{p-1}{p}}\left(\int_{B_R}V(|x|)|u|^p\,dx\right)^{\frac{1}{p}}\\
&\leq m^{q-1}\left(\int_{B_R}|x|^{(\alpha-\nu(q-1))\frac{p}{p-1}}\,dx\right)^{\frac{p-1}{p}}\|u\|.
\end{aligned}
$$

Case $1/p<\beta<1$.

One has $\frac{p-1}{p\beta-1}>1$, with Hölder conjugate exponent $\left(\frac{p-1}{p\beta-1}\right)'=\frac{p-1}{p(1-\beta)}$. Then

$$
\begin{aligned}
&\frac{1}{\Lambda}\int_{B_R}K(|x|)|u|^q\,dx\\
&\leq \int_{B_R}|x|^{\alpha}V(|x|)^{\beta}|u|^q\,dx=\int_{B_R}|x|^{\alpha}V(|x|)^{\frac{p\beta-1}{p}}|u|^{q-1}V(|x|)^{\frac{1}{p}}|u|\,dx\\
&\leq \left(\int_{B_R}|x|^{\alpha\frac{p}{p-1}}V(|x|)^{\frac{p\beta-1}{p-1}}|u|^{(q-1)\frac{p}{p-1}}\,dx\right)^{\frac{p-1}{p}}\left(\int_{B_R}V(|x|)|u|^p\,dx\right)^{\frac{1}{p}}\\
&\leq \left(\int_{B_R}|x|^{\alpha\frac{p}{p-1}}|u|^{(q-1)\frac{p}{p-1}-p\frac{p\beta-1}{p-1}}V(|x|)^{\frac{p\beta-1}{p-1}}|u|^{p\frac{p\beta-1}{p-1}}\,dx\right)^{\frac{p-1}{p}}\|u\|\\
&\leq \left(\left(\int_{B_R}|x|^{\frac{\alpha}{1-\beta}}|u|^{\frac{q-p\beta}{1-\beta}}\,dx\right)^{\frac{p}{p-1}(1-\beta)}\left(\int_{B_R}V(|x|)|u|^p\,dx\right)^{\frac{p\beta-1}{p-1}}\right)^{\frac{p-1}{p}}\|u\|\\
&\leq m^{q-p\beta}\left(\int_{B_R}|x|^{\frac{\alpha}{1-\beta}-\nu\frac{q-p\beta}{1-\beta}}\,dx\right)^{1-\beta}\left(\int_{B_R}V(|x|)|u|^p\,dx\right)^{\frac{p\beta-1}{p}}\|u\|\\
&\leq m^{q-p\beta}\left(\int_{B_R}|x|^{\frac{\alpha-\nu(q-p\beta)}{1-\beta}}\,dx\right)^{1-\beta}\|u\|^{p\beta-1}\|u\|.
\end{aligned}
$$

Case $\beta=1$.

Assumption $q>\max\{1,p\beta\}$ means $q>p$ and thus we have

$$
\frac{1}{\Lambda}\int_{B_R}K(|x|)|u|^q\,dx
$$

$$\leq \int_{B_R} |x|^{\alpha} V(|x|) |u|^q \, dx = \int_{B_R} |x|^{\alpha} V(|x|)^{\frac{p-1}{p}} |u|^{q-1} V(|x|)^{\frac{1}{p}} |u| \, dx$$
$$\leq \left(\int_{B_R} |x|^{\alpha \frac{p}{p-1}} V(|x|) |u|^{(q-1)\frac{p}{p-1}} \, dx \right)^{\frac{p-1}{p}} \left(\int_{B_R} V(|x|) |u|^p \, dx \right)^{\frac{1}{p}}$$
$$\leq \left(\int_{B_R} |x|^{\alpha \frac{p}{p-1}} |u|^{(q-1)\frac{p}{p-1} - p} V(|x|) |u|^p \, dx \right)^{\frac{p-1}{p}} \|u\|$$
$$\leq m^{q-p} \left(\int_{B_R} |x|^{\frac{p}{p-1}(\alpha - \nu(q-p))} V(|x|) |u|^p \, dx \right)^{\frac{p-1}{p}} \|u\| .$$

□

The following lemma is analogous to the previous one, so we skip its proof for brevity.

Lemma 5.2. *Let $R_0 > 0$ and assume that $V(r) < +\infty$ almost everywhere in $B_{R_0^c}$ and*

$$\Lambda := \operatorname*{ess\,sup}_{x \in B_{R_0^c}} \frac{K(|x|)}{|x|^{\alpha} V(|x|)^{\beta}} < +\infty \quad \text{for some } 0 \leq \beta \leq 1 \text{ and } \alpha \in \mathbb{R}.$$

Let $u \in X$ and assume that there exist $\nu \in \mathbb{R}$ and $m > 0$ such that

$$|u(x)| \leq \frac{m}{|x|^{\nu}} \quad \text{almost everywhere on } B_{R_0^c}.$$

Then there exists a constant $C = C(N, R_0, a_0, a_\infty, \beta) > 0$ such that $\forall R > R_0$ and $\forall q > \max\{1, p\beta\}$, one has

$$\int_{B_R^c} K(|x|) |u|^{q-1} |h| \, dx$$
$$\leq \begin{cases} \Lambda m^{q-1} C \left(\int_{B_R^c} |x|^{\frac{\alpha - \nu(q-1)}{N(p-1) + p(1 - p\beta + a_\infty \beta) - a_\infty} pN} \, dx \right)^{\frac{N(p-1) + p(1 - p\beta + a_\infty \beta) - a_\infty}{pN}} \|u\| & \text{if } 0 \leq \beta \leq \frac{1}{p} \\ \Lambda m^{q-p\beta} \left(\int_{B_R^c} |x|^{\frac{\alpha - \nu(q - p\beta)}{1-\beta}} \, dx \right)^{1-\beta} \|u\|^{p\beta} & \text{if } \frac{1}{p} < \beta < 1 \\ \Lambda m^{q-p} \left(\int_{B_R^c} |x|^{\frac{p}{p-1}(\alpha - \nu(q-p))} V(|x|) |u|^p \, dx \right)^{\frac{p-1}{p}} \|u\| & \text{if } \beta = 1. \end{cases}$$

We can now prove Theorems 3.2 and 3.3.

Proof of Theorem 3.2. Assume the hypotheses of the theorem and let $u \in X$ be such that $\|u\| = 1$. Let $0 < R < R_1$. We will denote by C any positive constant which does not depend on u and R.
Recalling the pointwise estimates of Lemma 2.6 and the fact that

$$\operatorname*{ess\,sup}_{x \in B_R} \frac{K(|x|)}{|x|^{\alpha_0} V(|x|)^{\beta_0}} \leq \operatorname*{ess\,sup}_{r \in (0, R_1)} \frac{K(r)}{r^{\alpha_0} V(r)^{\beta_0}} < +\infty,$$

we can apply Lemma 5.1 with $R_0 = R_1$, $\alpha = \alpha_0$, $\beta = \beta_0$, $m = M\|u\| = M$ and $\nu = \frac{N-p+a_0}{p}$.
If $0 \leq \beta_0 \leq 1/p$ we get

$$\int_{B_R} K(|x|) |u|^{q_1} \, dx \leq C \left(\int_{B_R} |x|^{\frac{\alpha_0 - \nu(q_1 - 1)}{N(p-1) - a_0 + p(1 - p\beta_0 + a_0 \beta_0)} pN} \, dx \right)^{\frac{N(p-1) - a_0 + p(1 - p\beta_0 + a_0\beta_0)}{pN}}$$

$$\leq C\left(\int_0^R r^{\frac{\alpha_0-\nu(q_1-1)}{N(p-1)-a_0+p(1-p\beta_0+a_0\beta_0)}pN+N-1}dr\right)^{\frac{N(p-1)-a_0+p(1-p\beta_0+a_0\beta_0)}{pN}}.$$

Notice now that

$$\frac{\alpha_0-\nu\left(q_1-1\right)}{N(p-1)-a_0+p\left(1-p\beta_0+a_0\beta_0\right)}pN+N=$$

$$=\frac{N}{N(p-1)-a_0+p(1-p\beta_0+a_0\beta_0)}\left[p(N+\alpha_0-p\beta_0+\alpha_0\beta_0)-(N+a_0-p)q_1\right]=$$

$$=\frac{N(N+a_0-p)}{N(p-1)-a_0+p(1-p\beta_0+a_0\beta_0)}\left[q^*(a_0,\alpha_0,\beta_0)-q_1\right]>0,$$

thanks to the hypotheses. Hence we deduce

$$\int_{B_R}K\left(|x|\right)|u|^{q_1}\,dx\leq CR^{\frac{N+a_0-p}{p}[q^*(a_0,\alpha_0,\beta_0)-q_1]}.$$

On the other hand, if $1/p<\beta_0<1$ we have

$$\int_{B_R}K\left(|x|\right)|u|^{q_1}\,dx\;\leq\;C\left(\int_{B_R}|x|^{\frac{\alpha_0-\nu(q_1-p\beta_0)}{1-\beta_0}}\,dx\right)^{1-\beta_0}$$

$$\leq\;C\left(\int_0^R r^{\frac{\alpha_0-\nu(q_1-p\beta_0)}{1-\beta_0}+N-1}dr\right)^{1-\beta_0},$$

where

$$\frac{\alpha_0-\nu(q_1-p\beta_0)}{1-\beta_0}+N=\frac{p\alpha_0-(N+\alpha_0-p)q_1-p\beta_0}{p(1-\beta_0)}+N=$$

$$\frac{p\alpha_0-(N+a_0-p)q_1+Np\beta_0+pa_0\beta_0-p^2\beta_0+Np-Np\beta_0}{p(1-\beta_0)}=$$

$$\frac{p(\alpha_0-p\beta_0+N+a_0\beta_0)-(N+a_0-p)q_1}{p(1-\beta_0)}=\frac{N+a_0-p}{p(1-\beta_0)}\left[q^*(a_0,\alpha_0,\beta_0)-q_1\right]>0.$$

Hence we get

$$\int_{B_R}K\left(|x|\right)|u|^{q_1}\,dx\leq CR^{\frac{N+a_0-p}{p}[q^*(a_0,\alpha_0,\beta_0)-q_1]}.$$

Finally, if $\beta_0=1$, we obtain

$$\int_{B_R}K\left(|x|\right)|u|^{q_1}\,dx\leq C\left(\int_{B_R}|x|^{\frac{p}{p-1}(\alpha_0-\nu(q_1-p))}\,V\left(|x|\right)|u|^p dx\right)^{\frac{p-1}{p}},$$

where

$$\alpha_0-\nu(q_1-p)=\alpha_0-\frac{N+a_0-p}{p}(q_1-p)=\frac{1}{p}\left(p\alpha_0-(N+a_0-p)\right)q_1+pN+pa_0-p^2\right)=$$

$$\frac{1}{p}\left(p\left(\alpha_0-p+N+a_0\right)-\left(N+a_0-p\right)q_1\right)=\frac{N+a_0-p}{p}\left(q^*(a_0,\alpha_0,1)-q_1\right)>0.$$

Hence we get

$$\int_{B_R} K\left(|x|\right)|u|^{q_1}\,dx\le$$

$$\le C\left(R^{\frac{N+a_0-p}{p-1}\left(q^*(a_0,\alpha_0,1)-q_1\right)}\int_{B_R} V\left(|x|\right)|u|^p\,dx\right)^{\frac{p-1}{p}}\le CR^{\frac{N+a_0-p}{p}\left(q^*(a_0,\alpha_0,1)-q_1\right)}.$$

So, in any case, we deduce $\mathcal{S}_0\left(q_1,R\right)\le CR^{\delta}$ for some $\delta=\delta\left(N,p,\alpha_0,\beta_0,q_1\right)>0$ and this concludes the proof. □

Proof of Theorem 3.3. Assume the hypotheses of the theorem and let $u\in X$ be such that $\|u\|=1$. Let $R>R_2$. We will denote by C any positive constant which does not depend on u and R.

By the pointwise estimates of Lemma 2.6 and the fact that

$$\operatorname*{ess\,sup}_{x\in B_R^c}\frac{K\left(|x|\right)}{|x|^{\alpha_\infty}V\left(|x|\right)^{\beta_\infty}}\le\operatorname*{ess\,sup}_{r>R_2}\frac{K\left(r\right)}{r^{\alpha_\infty}V\left(r\right)^{\beta_\infty}}<+\infty,$$

we can apply Lemma 5.2 with $R_0=R_2$, $\alpha=\alpha_\infty$, $\beta=\beta_\infty$, $m=M\left\|u\right\|=M$ and $\nu=\frac{N-p+a_\infty}{p}$.
If $0\le\beta_\infty\le 1/p$ we get

$$\int_{B_R^c} K\left(|x|\right)|u|^{q_2}\,dx\,\le\,C\left(\int_{B_R^c}|x|^{\frac{\alpha_\infty-\nu\left(q_2-1\right)}{N(p-1)-a_\infty+p\left(1-p\beta_\infty+a_\infty\beta_\infty\right)}pN}\,dx\right)^{\frac{N(p-1)-a_\infty+p\left(1-p\beta_\infty+a_\infty\beta_\infty\right)}{pN}}.$$

Notice that we have

$$\frac{\alpha_\infty-\nu\left(q_2-1\right)}{N(p-1)-a_\infty+p\left(1-p\beta_\infty+a_\infty\beta_\infty\right)}\,pN+N=$$
$$=\frac{N\left(N+a_\infty-p\right)}{N(p-1)-a_\infty+p\left(1-p\beta_\infty+a_\infty\beta_\infty\right)}\left[q^*(a_\infty,\alpha_\infty,\beta_\infty)-q_2\right]<0,$$

thanks to the hypotheses. Hence we get

$$\int_{B_R^c} K\left(|x|\right)|u|^{q_2}\,dx\le$$

$$\le C\left(\int_R^{+\infty} r^{\frac{N\left(N+a_\infty-p\right)}{N(p-1)-a_\infty+p\left(1-p\beta_\infty+a_\infty\beta_\infty\right)}\left[q^*(a_\infty,\alpha_\infty,\beta_\infty)-q_2\right]-1}dr\right)^{\frac{N(p-1)-a_\infty+p\left(1-p\beta_\infty+a_\infty\beta_\infty\right)}{pN}}=CR^{\delta}$$

for some $\delta<0$. On the other hand, if $1/p<\beta_\infty<1$ we have

$$\int_{B_R^c} K\left(|x|\right)|u|^{q_2}\,dx\,\le\,C\left(\int_{B_R^c}|x|^{\frac{\alpha_\infty-\nu\left(q_2-p\beta_\infty\right)}{1-\beta_\infty}}\,dx\right)^{1-\beta_\infty}$$
$$\le\,C\left(\int_R^{+\infty} r^{\frac{\alpha_\infty-\nu\left(q_2-p\beta_\infty\right)}{1-\beta_\infty}+N-1}dr\right)^{1-\beta_\infty},$$

where

$$\frac{\alpha_\infty - \nu\left(q_2 - p\beta_\infty\right)}{1-\beta_\infty} + N = \frac{N + a_\infty - p}{p(1-\beta\infty)}\left(q^*(a_\infty, \alpha_\infty, \beta_\infty) - q_2\right) < 0.$$

Hence we get

$$\int_{B_R^c} K\left(|x|\right)|u|^{q_2}\,dx \le CR^{\frac{N+a_\infty-p}{p(1-\beta\infty)}\left(q^*(a_\infty,\alpha_\infty,\beta_\infty)-q_2\right)}.$$

Finally, if $\beta_\infty = 1$, we obtain

$$\int_{B_R^c} K\left(|x|\right)|u|^{q_2}\,dx \le C\left(\int_{B_R^c} |x|^{\frac{p}{p-1}\left(\left(\alpha_\infty - \nu(q_2-p)\right)\right)} V\left(|x|\right)|u|^p\,dx\right)^{\frac{p-1}{p}},$$

where

$$\alpha_\infty - \nu(q_2-p) = \frac{N+a_\infty-p}{p}\left(q^*(a_\infty,\alpha_\infty,\beta_\infty) - q_2\right) < 0.$$

Hence

$$\int_{B_R^c} K\left(|x|\right)|u|^{q_2}\,dx \le CR^{\frac{N+a_\infty-p}{p}\left(q^*(a_\infty,\alpha_\infty,\beta_\infty)-q_2\right)}\left(\int_{B_R^c} V(|x|)|u(x)|^p dx\right)^{\frac{p-1}{p}} \le$$

$$\le CR^{\frac{N+a_\infty-p}{p}\left(q^*(a_\infty,\alpha_\infty,\beta_\infty)-q_2\right)}.$$

So, in any case, we get $\mathcal{S}_\infty\left(q_2, R\right) \le CR^\delta$ for some $\delta = \delta(N, p, \alpha_\infty, \beta_\infty, q_2) < 0$, which completes the proof. □

6 Existence of solutions

Let $N \ge 3$ and $1 < p < N$. In this section we apply our embedding results to get existence of radial weak solutions to the equation

$$-\mathrm{div}\left(A(|x||\nabla u|^{p-2}\nabla u\right)u + V\left(|x|\right)|u|^{p-2}u = K(|x|)f(u) \quad \text{in } \mathbb{R}^N, \tag{6.1}$$

i.e., functions $u \in X$ such that

$$\int_{\mathbb{R}^N} A(|x|)|\nabla u|^{p-2}\nabla u\cdot\nabla h\,dx + \int_{\mathbb{R}^N} V\left(|x|\right)|u|^{p-2}uh\,dx = \int_{\mathbb{R}^N} K(|x|)f(u)h\,dx \quad \forall h \in X, \tag{6.2}$$

where A, V and K are potentials satisfying $(\mathbf{A})$, $(\mathbf{V})$ and $(\mathbf{K})$, and X and is the Banach spaces defined in Section 2.

Remark 6.1. We focus on super p-linear nonlinearities f just for simplicity, but our compactness results also allow to treat the case of sub p-linear f's. Moreover, multiplicity results can also be obtained. We leave the details to interested reader, which we refer to [3, 4] for similar results and related arguments.

As concerns our hypotheses on the nonlinearity, we require that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, we set $F(t) = \int_0^t f(s)ds$, and we assume the following conditions:

(f_1) $\exists\theta > p$ such that $0 \le \theta F(t) \le f(t)\,t$ for all $t \in \mathbb{R}$;

(f_2) $\exists t_0 > 0$ such that $F(t_0) > 0$;

(f_{q_1,q_2}) $|f(t)| \le (\text{const.}) \min\left\{|t|^{q_1-1}, |t|^{q_2-1}\right\}$ for all $t \in \mathbb{R}$.

We notice that these hypotheses imply $q_1, q_2 \ge \theta$. Also we observe that, if $q_1 \ne q_2$, the double-power growth condition (f_{q_1,q_2}) is more stringent than the more usual single-power one, since it implies $|f(t)| \le (\text{const.})|t|^{q-1}$ for $q = q_1$, $q = q_2$ and every q in between. On the other hand, we will never require $q_1 \ne q_2$ in (f_{q_1,q_2}), so that our results will also concern single-power nonlinearities as long as we can take $q_1 = q_2$ (cf. Example 7.2 below).

We set

$$I(u) := \frac{1}{p}\|u\|^p - \int_{\mathbb{R}^N} K(|x|)F(u)\,dx =$$
$$= \frac{1}{p}\int_{\mathbb{R}^N} A(|x|)|\nabla u|^p dx + \frac{1}{p}\int_{\mathbb{R}^N} V(|x|)|u|^p dx - \int_{\mathbb{R}^N} K(|x|)F(u)\,dx.$$

From the continuous embedding result of Theorem 3.1 and the results of [6] about Nemytskiĭ operators on the sum of Lebesgue spaces, we have that I is a C^1 functional on X provided that there exist $q_1, q_2 > 1$ such that (f_{q_1,q_2}) and $\left(\mathcal{S}'_{q_1,q_2}\right)$ hold. In this case, the Fréchet derivative of I at any $u \in X$ is given by

$$I'(u)\,h = \int_{\mathbb{R}^N} A(|x|)\left(|\nabla u|^{p-2}\nabla u\cdot\nabla h + V(|x|)\,|u|^{p-2}uh\right)dx - \int_{\mathbb{R}^N} K(|x|)f(u)\,h\,dx \tag{6.3}$$

for all $h \in X$, and therefore the critical points of $I : X \to \mathbb{R}$ satisfy (6.2).

Our existence result is the following.

Theorem 6.2. *Assume that there exist $q_1, q_2 > p$ such that (f_{q_1,q_2}) and $\left(\mathcal{S}''_{q_1,q_2}\right)$ hold. Then the functional $I : X \to \mathbb{R}$ has a nonnegative critical point $u \ne 0$.*

Remark 6.3. In Theorem 6.2, the assumptions on f need only to hold for $t \ge 0$. Indeed, all the hypotheses of the theorem still hold true if we replace $f(t)$ with $\chi_{\mathbb{R}_+}(t)\,f(t)$ ($\chi_{\mathbb{R}_+}$ is the characteristic function of $\mathbb{R}_+$) and this can be done without restriction since the theorem concerns nonnegative critical points.

The above result relies on assumption $\left(\mathcal{S}''_{q_1,q_2}\right)$, which is quite abstract but can be granted in concrete cases through Theorems 3.2 and 3.3, which ensure such assumption for suitable ranges of exponents q_1 and q_2 by explicit conditions on the potentials. As concerns examples of nonlinearities satisfying the hypotheses of Theorem 6.2, the simplest $f \in C(\mathbb{R};\mathbb{R})$ such that (f_{q_1,q_2}) holds is

$$f(t) = \min\left\{|t|^{q_1-2}\,t, |t|^{q_2-2}\,t\right\},$$

which also ensures (f_1) if $q_1, q_2 > p$ (with $\theta = \min\{q_1, q_2\}$). Another model example is

$$f(t) = \frac{|t|^{q_2-2}\,t}{1+|t|^{q_2-q_1}} \quad \text{with } 1 < q_1 \le q_2,$$

which ensures (f_1) if $q_1 > p$ (with $\theta = q_1$). Note that, in both these cases, also (f_2) holds true. Moreover, both of these functions f become $f(t) = |t|^{q-2}\,t$ if $q_1 = q_2 = q$.

We now prove Theorem 6.2, starting with some lemmas.

Lemma 6.4. *Assume the hypotheses of Theorem 6.2. Then there exist three constants $c_1, c_2 > 0$ such that*

$$I(u) \geq \frac{1}{p}\|u\|^p - c_1\|u\|^{q_1} - c_2\|u\|^{q_2} \qquad \text{for all } u \in X. \tag{6.4}$$

Proof. Let $i \in \{1,2\}$. By the monotonicity of $\mathcal{S}_0$ and $\mathcal{S}_\infty$, it is not restrictive to assume $R_1 < R_2$ in hypothesis $\left(\mathcal{S}'_{q_1,q_2}\right)$. Then, by lemmas 4.1 and 2.8, there exists a constant $c_{R_1,R_2} > 0$ such that for all $u \in X$ we have

$$\int_{B_{R_2}\setminus B_{R_1}} K(|x|)|u|^{q_1}\,dx \leq C_{R_1,R_2}\|u\|^{q_1}.$$

Therefore, by the hypotheses on f and the definitions of $\mathcal{S}_0$ and $\mathcal{S}_\infty$, we obtain

$$\begin{aligned}
&\left|\int_{\mathbb{R}^N} K(|x|)F(u)\,dx\right| \leq \\
&\leq C\int_{\mathbb{R}^N} K(|x|)\min\{|u|^{q_1},|u|^{q_2}\}\,dx \\
&\leq C\left(\int_{B_{R_1}} K(|x|)|u|^{q_1}\,dx + \int_{B_{R_2}^c} K(|x|)|u|^{q_2}\,dx + \int_{B_{R_2}\setminus B_{R_1}} K(|x|)|u|^{q_1}\,dx\right) \\
&\leq C\left(\|u\|^{q_1}\mathcal{S}_0(q_1,R_1) + \|u\|^{q_2}\mathcal{S}_\infty(q_2,R_2) + C_{R_1,R_2}\|u\|^{q_1}\right) \\
&= C_1\|u\|^{q_1} + C_2\|u\|^{q_2},
\end{aligned} \tag{6.5}$$

where the constants c_1 and c_2 are independent of u. This yields (6.4).

□

Lemma 6.5. *Under the assumptions of Theorem 6.2, the functional $I : X \to \mathbb{R}$ satisfies the Palais-Smale condition.*

Proof. Let $\{u_n\}$ be a sequence in X such that $\{I(u_n)\}$ is bounded and $I'(u_n) \to 0$ in X'. Hence

$$\frac{1}{p}\|u_n\|^p - \int_{\mathbb{R}^N} K(|x|)F(u_n)\,dx = O(1) \quad \text{and} \quad \|u_n\|^p - \int_{\mathbb{R}^N} K(|x|)f(u_n)\,u_n dx = o(1)\|u_n\|.$$

As f satisfies (f_1), we get

$$\frac{1}{p}\|u_n\|^p + O(1) = \int_{\mathbb{R}^N} K(|x|)F(u_n)\,dx \leq \frac{1}{\theta}\int_{\mathbb{R}^N} K(|x|)f(u_n)\,u_n dx = \frac{1}{\theta}\|u_n\|^p + o(1)\|u_n\|,$$

which implies that $\{\|u_n\|\}$ is bounded since $\theta > p$. Now, thanks to assumption $\left(\mathcal{S}''_{q_1,q_2}\right)$, we apply Theorem 3.1 to deduce the existence of $u \in X$ such that (up to a subsequence) $u_n \rightharpoonup u$ in X and $u_n \to u$ in $L_K^{q_1} + L_K^{q_2}$. Setting

$$I_1(u) := \frac{1}{p}\|u\|^p \quad \text{and} \quad I_2(u) := I_1(u) - I(u)$$

for brevity, we have that I_2 is of class C^1 on $L_K^{q_1} + L_K^{q_2}$ by [6, Proposition 3.8] and therefore we get $\|u_n\|^p = I'(u_n)\,u_n + I_2'(u_n)\,u_n = I_2'(u)\,u + o(1)$. Hence $\lim_{n\to\infty}\|u_n\|$ exists and one has $\|u\|^p \leq \lim_{n\to\infty}\|u_n\|^p$ by weak lower semicontinuity. Moreover, the convexity of $I_1 : X \to \mathbb{R}$ implies

$$I_1(u) - I_1(u_n) \geq I_1'(u_n)(u - u_n) = I'(u_n)(u - u_n) + I_2'(u_n)(u - u_n) = o(1)$$

and thus

$$\frac{1}{p}\|u\|^p = I_1(u) \geq \lim_{n\to\infty} I_1(u_n) = \frac{1}{p}\lim_{n\to\infty}\|u_n\|^p.$$

So $\|u_n\| \to \|u\|$ and one concludes that $u_n \to u$ in X by the uniform convexity of the norm. □

Proof of Theorem 6.2. Assume the hypotheses of the theorem, together with the following non-restrictive additional condition: $f(t) = 0$ for $t < 0$. We want to apply the Mountain-Pass Theorem. To this end, from (6.4) of Lemma 6.4 we deduce that, since $q_1, q_2 > p$, there exists $\rho > 0$ such that

$$\inf_{u\in X,\, \|u\|=\rho} I(u) > 0 = I(0). \tag{6.6}$$

Therefore, taking into account Lemma 6.5, we need only to check that $\exists \bar{u} \in X$ such that $\|\bar{u}\| > \rho$ and $I(\bar{u}) < 0$. To this end, from assumption (f_1) and (f_2) we infer that

$$F(t) \geq \frac{F(t_0)}{t_0^\theta} t^\theta \text{ for all } t \geq t_0.$$

We then fix a non negative function $u_0 \in C_c^\infty(\mathbb{R}^N\backslash\{0\})$ such that the set $\{x \in \mathbb{R}^N : u_0(x) \geq t_0\}$ has positive Lebesgue measure. Hence, since (f_1) and (f_2) ensure that $F(t) \geq 0$ for all t and $F(t_0) > 0$, for every $\lambda > 1$ we get

$$\begin{aligned}\int_{\mathbb{R}^N} K(|x|)F(\lambda u_0)\,dx &\geq \int_{\{\lambda u_0 \geq t_0\}} K(|x|)F(\lambda u_0)\,dx \geq \frac{\lambda^\theta}{t_0^\theta}\int_{\{\lambda u_0\geq t_0\}} K(|x|)F(t_0)\,u_0^\theta dx \\ &\geq \frac{\lambda^\theta}{t_0^\theta}\int_{\{u_0\geq t_0\}} K(|x|)F(t_0)\,u_0^\theta dx \geq \lambda^\theta \int_{\{u_0\geq t_0\}} K(|x|)F(t_0)\,dx > 0.\end{aligned}$$

Since $\theta > p$, this gives

$$\lim_{\lambda\to+\infty} I(\lambda u_0) \leq \lim_{\lambda\to+\infty}\left(\frac{\lambda^p}{p}\|u_0\|^p - \lambda^\theta\int_{\{u_0\geq t_0\}} K(|x|)F(t_0)\,dx\right) = -\infty.$$

As a conclusion, we can take $\bar{u} = \lambda u_0$ with λ sufficiently large and the Mountain-Pass Theorem provides the existence of a nonzero critical point $u \in X$ for I. Since the additional assumption $f(t) = 0$ for $t < 0$ implies $I'(u)\,u_- = -\|u_-\|^p$ (where $u_- \in X$ is the negative part of u), one concludes that $u_- = 0$, i.e., u is nonnegative. □

7 Examples

In this section we give some examples that might help to understand what is new in our results. We will make a comparison, in concrete cases, between our results and those of [13]. In that paper the authors prove some compactness theorems which are used to prove existence results for equation (1.1), where f is a power or a sum of powers. We will show some cases where the results of [13] do not apply, while our results give existence of solutions. In all our example we look for a nonlinearity defined by $f(t) = \min\{t^{q_1-1}, t^{q_2-1}\}$, and we will see how to choose $p < q_1 \leq q_2$ in such a way to get existence results for problem (1.1).

Example 7.1. Let A, V, K be as follows:

$$A(r) = \min\{r^{1/2}, r^{3/2}\}, \quad V(r) = \min\left\{1, r^{-3/2}\right\}, \quad K(r) = \max\{r^{1/2}, r^{3/2}\}.$$

Assume $3/2 < p \leq 2$. We first show that in this case the results of [13] do not apply. The embedding theorems of that paper are Theorems 3.2, 3.3 and 3.4. If we compute the coefficients q^* and q_* of [13], we easily obtain $q_* = \frac{p(2N+3)}{2N+1-2p}$, $q^* = \frac{p(2N+1)}{2N+3-2p}$ and this, together with $p \leq 2$, implies $q^* < q_*$, so that Theorem 3.2 of [13] cannot be applied, because it needs $q_* < q^*$. One easily verifies that also the hypotheses of Theorems 3.3 and 3.4 of [13] are not satisfied. To apply our results, we set $\beta_0 = \beta_\infty = 0$, $\alpha_0 = 1/2$, $\alpha_\infty = 3/2$, $a_0 = 3/2$, $a_\infty = 1/2$. Note that condition $a_0, a_\infty \in (p-N, p]$ is satisfied. We apply Theorems 3.2 and 3.3, and we compute

$$q^*(a_0, \alpha_0, \beta_0) = \frac{p(2N+1)}{2N+3-2p}, \quad q_*(a_\infty, \alpha_\infty, \beta_\infty) = \frac{p(2N+3)}{2N+1-2p}.$$

Notice that these are the same value obtained above, following [13]. Notice also that $q^*(a_0, \alpha_0, \beta_0) > p$ is equivalent to $p > 1$. Applying our results, we deduce that if we take q_1, q_2 such that

$$p < q_1 < \frac{p(2N+1)}{2N+3-2p} < \frac{p(2N+3)}{2N+1-2p} < q_2,$$

then $\left(\mathcal{S}''_{q_1,q_2}\right)$ holds and we get an existence result for the equation (1.1) with any nonlinearity satisfying (f_{q_1,q_2}) .

Example 7.2. Assume $N \geq 4$ and choose the functions A, V, K as follows:

$$A(r) = \max\left\{r^{-2}, r^{-1}\right\}, \quad V(r) = e^{2r}, \quad K(r) = e^r.$$

In this case the results of [13] do not apply because of the exponential growth of the potential K. Assume $1 < p < N-2$. In order to apply Theorems 3.2 and 3.3, we can choose $a_0 = -2$, $a_\infty = -1$, $\beta_0 = \alpha_0 = \alpha_\infty = 0$, $\beta_\infty = 1/2$. Notice that condition $a_0, a_\infty \in (p-N, p]$ is satisfied. We get

$$q^*(a_0, \alpha_0, \beta_0) = \frac{pN}{N-p-2} \quad \text{and} \quad q^*(a_\infty, \alpha_\infty, \beta_\infty) = \frac{pN}{N-p-1},$$

where we have $p < \frac{pN}{N-p-1} < \frac{pN}{N-p-2}$. Then $\left(\mathcal{S}''_{q_1,q_2}\right)$ holds and we get an existence result for the equation (1.1) with any nonlinearity satisfying (f_{q_1,q_2}) provided that

$$p < q_1 < \frac{pN}{N-p-2} \quad \text{and} \quad q_2 > \frac{pN}{N-p-1}.$$

In particular we can take a power nonlinearity $f(t) = t^{q-1}$ for $q \in \left(\frac{pN}{N-p-1}, \frac{pN}{N-p-2}\right)$.

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