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# Perturbation Theory Treatment of Spin-Orbit Coupling. I: Double Perturbation Theory Based on a Single-Reference Initial Approximation

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We develop a perturbation theory for solving the many-body Dirac equation within a given relativistic effective-core potential approximation. Starting from a scalar-relativistic unrestricted Hartree-Fock solution (SR UHF) we carry out a double perturbation expansion in terms of spin-orbit coupling (SOC) and the electron fluctuation potential. Computationally convenient energy expressions are derived through fourth-order in SOC, second-order in the electron fluctuation potential, and a total of third-order in the coupling between the two. Illustrative calculations on the halogen series of neutral and singly positive diatomic molecules show that the perturbation expansion is well-converged by taking into account only the leading (non-vanishing) term at each order of the electron fluctuation potential. Our perturbation theory approach provides a computationally attractive alternative to a two-component self-consistent field treatment of SOC. In addition, it includes coupling with the fluctuation potential through third-order, and can be extended (in principle) to multi-reference calculations, when necessary for both closed- and open-shell cases, using quasi-degenerate perturbation theory.

Keywords:

## I. INTRODUCTION

The treatment of relativistic effects in solids and molecular systems is usually done, nowadays, starting from a self-consistent field (SCF) treatment. This can be achieved using two- or four-component representations of the Dirac equation. Some of the most popular two-component variational approaches include the zeroth- (and higher-) order regular approximations (ZORA),<sup>1-5</sup> the Douglas Kroll Hess family of methods,<sup>6-17</sup> and methods based on the eXact-2-component (X2C) approach.<sup>18-28</sup> Four component variational treatments are also in use, both for molecules<sup>29,30</sup> and solids.<sup>31</sup>

The computational bottleneck in performing variational relativistic calculations is the need to explicitly diagonalize the secular equations in a large two- or four-component spinor basis (2c-SCF or 4c-SCF). “Diagonalization-free” methods have also been discussed, but usually necessitate a small number of diagonalizations.<sup>32-34</sup> Diagonalization cost can be greatly reduced by exploitation of double-group symmetry, if the corresponding irreducible representations (IRREPs) can be built. While algorithms exist for constructing the IRREPs of the double group, they are generally limited to the treatment of states that preserve time-reversal symmetry (the so-called Kramer’s restricted, KR, variant of the theory), but could also be extended to Kramer’s unrestricted theory.<sup>29,35-37</sup> KR theory is best suited to treat closed-shell electronic configurations although, even in that context, the symmetries are usually not exploited for diagonalization due, in part, to a lack of efficient algebraic routines. As for open-shell systems, a single-

reference 2c-SCF or 4c-SCF is insufficient for treating those cases with strong multi-reference character. Attempts for a multi-reference generalization of such self-consistent treatments lead to approaches that are either prohibitively expensive or lack the property of size extensivity.<sup>30,38-45</sup> Both cases represent important obstacles for the application of multi-reference approaches to extended systems.

Perturbation theory represents an alternative to the SCF treatment of relativistic effects. Some spin-orbit configuration-interaction (CI) algorithms include part of the SOC effects both in the CI diagonalization and by a perturbation theory treatment. As a matter of fact, the CIPSO algorithm of Teichteil, Pelissier and Spiegelmann<sup>46</sup> is part of a class of algorithms sometimes referred as  $CI^{LS}+SO$ , in which SOC is partly included through perturbation theory in a target space determined from the correlated scalar relativistic (SR) wave function and its quasi-degenerate complement. Double group CI methods are somewhat similar in spirit, with one important difference being that the diagonalization of the full Hamiltonian is achieved in a target space that may be enlarged by configurations which are selected by a SOC criterion.<sup>47,48</sup>

Some success in the perturbation theory treatment of relativistic effects has been found from methods based on the two-component Pauli or Breit-Pauli equations.<sup>49-58</sup> A possibly more accurate alternative, however, is to directly solve a suitable approximation for the four-component Dirac equation by a perturbation expansion, following the direct perturbation theory (DPT) developed independently by Rutkowski,<sup>59-61</sup> and by Kutzelnigg and co-workers.<sup>62</sup> Most total energy DPT calculations reported thus far involve the perturbation theory treatment of

both SR (generally more important for the total energy) as well as SOC effects. DPT has been applied in the context of density-functional theory (DFT) calculations,<sup>63</sup> Hartree-Fock calculations<sup>64–66</sup> and with coupled cluster wave functions.<sup>67</sup> The extension of DPT to open-shell systems by means of quasi-degenerate perturbation theory (QDPT) has also been discussed.<sup>68–72</sup> Coincidentally, QDPT also allows for some otherwise non-dynamical electron correlation to be included in a size extensive way.<sup>73–76</sup>

An alternative to the traditional two- or four-component representations of the Dirac equation is the use of relativistic effective core potentials (RECP).<sup>77–80</sup> In fact, the RECP approximation is probably the most popular approach to treat relativistic effects (albeit, only at the SR level) and is implemented in most first-principles programs.<sup>77,81–97</sup> The SR RECP approach represents a very convenient means to treat relativistic effects, because it is no more expensive than a one-component non-relativistic (NR) calculation. However, SOC effects are not included. If they were, the RECP approach could lead to a computational method that is more accurate than the costlier four-component Dirac-Coulomb approach.<sup>77,98</sup> The main drawback of the RECP methodology is the use of the frozen-core approximation, which can be inappropriate for calculating properties that directly involve core orbitals (e.g. nuclear magnetic resonance chemical shifts, Mössbauer isomer shifts and X-ray spectroscopies).<sup>77</sup>

Given the widespread use of the SR RECP approach, it is desirable to develop a means of including SOC effects perturbatively. We intend to provide the necessary theory for such treatments in a series of papers. The first paper of this series (part I), presented here, provides an appropriate (non-iterative) double perturbation theory for solving the many-body Dirac equation in a given RECP approximation, starting from the corresponding SR single-reference unrestricted Hartree-Fock (UHF) solution. Part II will present an iterative single-reference approach, based on the coupled-perturbed Hartree-Fock/Kohn-Sham (CPHF/CPKS) spin-current density functional perturbation theory formalism.<sup>99</sup> Further additions to the series will consider multi-reference generalizations of the theory, based on QDPT, as well as the treatment of periodic systems. This series complements work by some of the present authors to provide a program for two-component spin-current DFT calculations on periodic systems within the CRYSTAL code.<sup>100–105</sup>

In this paper, we present a formalism for solving the many-body Dirac equation starting from the SR UHF solution. Our formalism is based on a double-perturbation treatment of SOC and the electron fluctuation potential. Computational results are provided for the halogen series of diatomic molecules (F<sub>2</sub>, Cl<sub>2</sub>, Br<sub>2</sub>, I<sub>2</sub> and At<sub>2</sub>). Although the formalism could be used to calculate the density matrix and related properties, as well as orbital energy levels, we limit our considerations in this first pa-

per to the convergence of the perturbation series for the energy. The illustrative calculations show that the perturbation series in SOC — both zeroth order and first order in the electron fluctuation potential — converge rapidly to the expected values. Thus, our treatment provides a convenient starting point for the extension to include, for example, ordinary Møller-Plesset second-order correlation energy (second order in the electron fluctuation potential and zeroth order in SOC) and higher order coupling between SOC and the electron fluctuation potential. Detailed formulas are provided through total order three for this purpose. Moreover, a brief discussion is included regarding cases where the application of QDPT may become necessary.

## II. FORMALISM

### A. Statement of the Problem

Our treatment is formulated within the RECP approximation. Thus, the many-electron problem is partitioned into one involving only the core electrons and one describing valence-valence and core-valence interactions. We assume that the core electron problem has already been solved within a sufficiently accurate approximation to the many-electron four-component Dirac equation (e.g. Dirac-Coulomb, Dirac-Coulomb-Breit, or generally any other variant, possibly including higher-order corrections from quantum electrodynamics) and that the parameters of the RECP have been extracted. The reader is referred to excellent reviews of the RECP method for more explicit details on the procedure for extraction of the RECP operators.<sup>77,78,97,106–109</sup>

Our task is to solve the remaining problem involving the valence-valence and core-valence interactions. The starting point is the many-electron time-independent Dirac equation in the RECP approximation:

$$\hat{H}|\psi_I\rangle = E_I|\psi_I\rangle, \quad (1)$$

in which  $E_I$  are the energies of the stationary states  $|\psi_I\rangle$  labelled by the index  $I$ . Our approach could, in principle, be used to treat excited states, but here we limit our discussion to approximating the ground-state  $|\psi\rangle$ . In general, Eq. (1) can be written in terms of a two- or four-component representation. Although some RECP calculations have been performed within an explicitly relativistic four-component representation of the valence Hamiltonian  $\hat{H}$ , these calculations have not found appreciable differences with respect to a computationally less demanding two-component *ansatz*.<sup>77,110–114</sup> We, therefore, write  $\hat{H}$  in a two-component basis wherein all explicitly relativistic effects are accounted for by the SR  $\hat{U}^{AREP}$  (the averaged relativistic effective potential) plus the spin-dependent parts  $\hat{h}_{SO}$  of the effective-core potential:

$$\hat{H} = \mathbb{I}_2 \odot \left[ \hat{H}_{NR} + \hat{U}^{AREP} \right] + \hat{h}_{SO}. \quad (2)$$

Here  $\mathbb{I}_2$  is the  $2 \times 2$  identity matrix and  $\hat{H}_{NR}$  is the many-electron NR valence Schrödinger Hamiltonian. The operator  $\odot$  ensures that the direct product  $\otimes$  with  $\mathbb{I}_2$  is taken once for the one-electron part and twice for the two-electron part of the Hamiltonian. All spin-dependent relativistic effects (SOC, spin-other-orbit coupling, spin-spin coupling, etc...) are included in  $\hat{h}_{SO}$  (sometimes called the spin-orbit relativistic effective potential, SOREP). The two-electron scalar and spin-dependent relativistic effects are included in the core-valence interactions, but are assumed to be negligible for the valence-valence interactions.  $\hat{U}^{AREP}$  and  $\hat{h}_{SO}$  can, in turn, be written as a sum of one-electron operators (labelled by the index  $i$ ):

$$\hat{U}^{AREP} = \sum_i^{val} \hat{U}_i^{AREP}, \quad (3a)$$

$$\hat{h}_{SO} = \sum_i^{val} \hat{h}_{SO,i}, \quad (3b)$$

where *val* indicates that the index  $i$  runs over the set of valence electron coordinates including both space and spin. The operators  $\hat{h}_{SO,i}$  may be expressed in terms of the components of the one-electron spin  $\hat{S}_i$  and angular momentum  $\hat{L}_i$  operators as follows:<sup>100</sup>

$$\hat{h}_{SO,i} = \hat{\zeta}_i \left( \hat{L}_{z,i} \hat{S}_{z,i} + \frac{1}{2} \hat{L}_{+,i} \hat{S}_{-,i} + \frac{1}{2} \hat{L}_{-,i} \hat{S}_{+,i} \right), \quad (4)$$

in which  $\hat{\zeta}_i$  is a linear combination of radial and angular operators that depend on the parameters obtained from the RECP fitting,  $\hat{S}_{z,i}$  and  $\hat{L}_{z,i}$  are the  $z$ -components of the one-electron spin and angular-momentum operators (the product  $\hat{L}_{z,i} \hat{S}_{z,i}$  is pure imaginary) and  $\hat{S}_{\pm,i}$  and  $\hat{L}_{\pm,i}$  are the one-electron spin and angular momentum ladder operators.

Our choice for  $\hat{H}^{(0,0)}$  is the analog of the usual Møller-Plesset sum of one-electron Fock operators:<sup>115,116</sup>

$$\hat{H}^{(0,0)} = \sum_i^{val} \hat{F}_i = \sum_i^{val} \left[ \hat{h}_i + \hat{U}_i^{AREP} + (\hat{C}_i - \hat{K}_i) \right], \quad (5)$$

in which  $\hat{h}_i$  contains the NR electronic kinetic energy and electron-nuclear attraction terms, while  $\hat{C}_i$  and  $\hat{K}_i$  are the usual Coulomb and exchange operators. Practically,  $\hat{F}_i$  is determined by the self-consistent solution of the one-component SR unrestricted Hartree-Fock-Roothaan equations:

$$\hat{F}_m |m\rangle = \epsilon_m |m\rangle, \quad (6)$$

where the  $|m\rangle$  are singly-occupied orbitals. In Eq. (6) the spin index is implicit; when desirable to make it explicit, we write:

$$\hat{F}_m^\sigma |m, \sigma\rangle = \epsilon_m^\sigma |m, \sigma\rangle, \quad (7)$$

in which  $\sigma = \alpha$  or  $\beta$ . We note that  $|m, \sigma\rangle$  can be expanded in the atomic orbital (AO)  $|\nu\rangle$  basis, (here a set of atom-centered Gaussian functions), as follows:

$$|m, \sigma\rangle = \sum_\nu c_{\nu m}^\sigma |\nu\rangle \otimes |\sigma\rangle. \quad (8)$$

With the above definition of  $\hat{H}^{(0,0)}$ , the first-order Hamiltonians associated with SOC and the electron fluctuation potential, i.e.  $\hat{H}^{(1,0)}$  and  $\hat{H}^{(0,1)}$ , become:

$$\hat{H}^{(1,0)} = \hat{h}_{SO}, \quad (9a)$$

$$\begin{aligned} \hat{H}^{(0,1)} &= \frac{1}{2} \sum_{j \neq i}^{val} \frac{1}{r_{ij}} - \sum_i^{val} (\hat{C}_i - \hat{K}_i) \\ &= \hat{V}_{ee} - (\hat{C} - \hat{K}), \end{aligned} \quad (9b)$$

We solve Eq. (1) with  $\hat{H}$  given by the sum of Hamiltonians in Eqs. (5), (9a) and (9b), by expanding the valence energy and wavefunction, including core-valence interactions, in a conventional Dalgarno double perturbation series:<sup>117</sup>

$$\begin{aligned} \hat{H} |\psi_I\rangle &= \left[ \hat{H}^{(0,0)} + \lambda \hat{H}^{(1,0)} + \mu \hat{H}^{(0,1)} \right] \left[ |\psi_I^{(0,0)}\rangle + \lambda |\psi_I^{(1,0)}\rangle + \mu |\psi_I^{(0,1)}\rangle + \lambda^2 |\psi_I^{(2,0)}\rangle + \mu^2 |\psi_I^{(0,2)}\rangle + \lambda \mu |\psi_I^{(1,1)}\rangle + \dots \right] \\ &= \left[ E_I^{(0,0)} + \lambda E_I^{(1,0)} + \mu E_I^{(0,1)} + \lambda^2 E_I^{(2,0)} + \mu^2 E_I^{(0,2)} + \lambda \mu E_I^{(1,1)} + \dots \right] \\ &\quad \times \left[ |\psi_I^{(0,0)}\rangle + \lambda |\psi_I^{(1,0)}\rangle + \mu |\psi_I^{(0,1)}\rangle + \lambda^2 |\psi_I^{(2,0)}\rangle + \mu^2 |\psi_I^{(0,2)}\rangle + \lambda \mu |\psi_I^{(1,1)}\rangle + \dots \right], \end{aligned} \quad (10)$$

The energy terms of zeroth order in SOC and arbitrary order  $M = 0, 1, 2, \dots$  in the electron fluctuation poten-

tial,  $E^{(0,M)}$ , are elements of the well-known SR MP series. The energy terms of arbitrary order  $N = 0, 1, 2, \dots$

in SOC and zeroth order in the electron fluctuation potential,  $E^{(N,0)}$ , represent pure SOC contributions to the energy. All other terms couple SOC with the electron fluctuation potential. A key issue, of course, is the convergence properties of the various energy sub-series.

## B. Energy Contributions

In accordance with standard MP theory, the sum of the terms  $E^{(0,0)}$  and  $E^{(0,1)}$  give us the SR UHF energy:

$$E_{UHF} = E^{(0,0)} + E^{(0,1)}. \quad (11a)$$

with:

$$E^{(0,0)} = \langle \psi^{(0,0)} | \hat{H}^{(0,0)} | \psi^{(0,0)} \rangle = \sum_m^{occ} \epsilon_m, \quad (11b)$$

where *occ* denotes the set of occupied SR UHF valence orbitals; similarly *virt* will later denote the set of virtual (or unoccupied) orbitals. For the MP1 term, we have:

$$\begin{aligned} E^{(0,1)} &= \langle \psi^{(0,0)} | \hat{H}^{(0,1)} | \psi^{(0,0)} \rangle \\ &= -\frac{1}{2} \sum_{mn}^{occ} [(mm|nn) - (mn|nm)], \quad (11c) \end{aligned}$$

wherein the bi-electronic integrals are written in Mulliken notation.

### 1. Terms of Order Zero in the Electron Fluctuation Potential

Expressions for the contributions  $E^{(N,0)}$  ( $N = 1, 2, 3, 4$ ) in terms of the many-electron wavefunctions  $|\psi_I^{(0,0)}\rangle$  of Eq. (10) are readily obtained from ordinary non-degenerate (see further below) RSPT. In first order:

$$E^{(1,0)} = \langle \psi^{(0,0)} | \hat{h}_{SO} | \psi^{(0,0)} \rangle, \quad (12)$$

and in second order:

$$E^{(2,0)} = \langle \psi^{(0,0)} | \hat{h}_{SO} \hat{R}^{(0,0)} \hat{h}_{SO} | \psi^{(0,0)} \rangle, \quad (13)$$

in which the non-degenerate RSPT resolvent operator  $\hat{R}^{(0,0)}$  is written as:

$$\hat{R}^{(0,0)} = \sum_I' |\psi_I^{(0,0)}\rangle \left( E^{(0,0)} - E_I^{(0,0)} \right)^{-1} \langle \psi_I^{(0,0)} | \quad (14)$$

where the sum is over all configurations spanned by the SR UHF valence orbitals of the (0,0) problem and the prime over the summation indicates that the ground state is excluded. The occurrence of the energy denominators  $E^{(0,0)} - E_I^{(0,0)}$  in Eq. (13), and in higher order terms of the series, implies that the perturbation theory expansion may not converge if degenerate and/or quasi-degenerate configurations interact with the ground state, as is the

case, for example, in many open-shell systems. An adequate treatment of such cases would require, in general, a QDPT approach, which is postponed until later papers of this series. The corresponding third and fourth order expressions can be found in Appendix A, along with a demonstration of how these expressions can be reduced to a computationally manageable form in terms of one-electron SR UHF orbitals.

Here we simply quote the final orbital formulas from Appendix A. In first order

$$E^{(1,0)} = \sum_m^{occ} \langle m | \hat{h}_{SO} | m \rangle = 0. \quad (15)$$

An explanation for why  $E^{(1,0)}$  vanishes is provided in the discussion associated with Eq. (A1) of Appendix A. In order to write the energy contributions of higher order, it is convenient to introduce the following complex-Hermitian matrix

$$G_{mp}^{\sigma\sigma'(1,0)} = \langle m, \sigma | \hat{h}_{SO} | p, \sigma' \rangle, \quad (16a)$$

and the following complex-anti-Hermitian matrix:

$$U_{mp}^{\sigma\sigma'(1,0)} = \frac{G_{mp}^{\sigma\sigma'(1,0)}}{\epsilon_p^{\sigma'} - \epsilon_m^{\sigma}}. \quad (16b)$$

In terms of these matrices we obtain, in second order:

$$E^{(2,0)} = \sum_{\sigma, \sigma'}^{occ} \sum_m^{occ} \sum_p^{virt} G_{mp}^{\sigma\sigma'(1,0)} U_{pm}^{\sigma'\sigma(1,0)}, \quad (17)$$

in third order:

$$\begin{aligned} E^{(3,0)} &= \sum_{\sigma, \sigma', \sigma''} \left[ \sum_{mn}^{occ} \sum_p^{virt} U_{mp}^{\sigma\sigma'(1,0)} U_{pn}^{\sigma'\sigma''(1,0)} G_{nm}^{\sigma''\sigma(1,0)} \right. \\ &\quad \left. - \sum_m^{occ} \sum_{pq}^{virt} U_{mp}^{\sigma\sigma'(1,0)} G_{pq}^{\sigma'\sigma''(1,0)} U_{qm}^{\sigma''\sigma(1,0)} \right], \quad (18) \end{aligned}$$

and in fourth order:

$$\begin{aligned} E^{(4,0)} &= \sum_{\sigma, \sigma', \sigma'', \sigma'''} \\ &\times \left[ 2 \sum_{mn}^{occ} \sum_{pq}^{virt} \frac{U_{mp}^{\sigma\sigma'(1,0)} G_{pq}^{\sigma'\sigma''(1,0)} G_{nm}^{\sigma''\sigma'''\sigma(1,0)} U_{qn}^{\sigma'''\sigma''(1,0)}}{\epsilon_m^{\sigma} - \epsilon_q^{\sigma''}} \right. \\ &- \sum_m^{occ} \sum_{pqr}^{virt} \frac{U_{mp}^{\sigma\sigma'(1,0)} G_{pq}^{\sigma'\sigma''(1,0)} G_{qr}^{\sigma''\sigma'''\sigma(1,0)} U_{rm}^{\sigma'''\sigma(1,0)}}{\epsilon_m^{\sigma} - \epsilon_q^{\sigma''}} \\ &- \sum_{mno}^{occ} \sum_p^{virt} \frac{U_{mp}^{\sigma\sigma'(1,0)} G_{nm}^{\sigma''\sigma(1,0)} G_{on}^{\sigma'''\sigma''(1,0)} U_{po}^{\sigma'\sigma'''\sigma(1,0)}}{\epsilon_n^{\sigma''} - \epsilon_p^{\sigma'}} \\ &\left. + \sum_{mn}^{occ} \sum_{pq}^{virt} \frac{U_{mp}^{\sigma\sigma'(1,0)} G_{pn}^{\sigma'\sigma''(1,0)} U_{qm}^{\sigma''\sigma(1,0)} G_{nq}^{\sigma'''\sigma''(1,0)}}{\epsilon_n^{\sigma''} - \epsilon_p^{\sigma'}} \right] \quad (19) \end{aligned}$$

The energy denominators in Eqs. (17)-(19) make explicit the possibility, discussed above, that near degeneracies due, for example, to a small HOMO-LUMO gap may occasion the need for a QDPT treatment.

## 2. Terms of Order One in the Electron Fluctuation Potential

We now proceed to the energy contributions  $E^{(N,1)}$  through third order in SOC. The detailed derivations can be found in Appendix B. For the case  $N = 1$ :

$$E^{(1,1)} = 2\mathcal{R} \left[ \langle \psi^{(0,0)} | \hat{h}_{SO} | \psi^{(0,1)} \rangle \right] = 0, \quad (20)$$

where  $\mathcal{R}$  denotes the real part of the argument. Here  $E^{(1,1)}$  vanishes since, according to Brillouin's theorem,  $|\psi^{(0,1)}\rangle$  consists only of doubly-excited configurations, whereas  $\hat{h}_{SO}$ , given by Eq. (3), consists only of mono-electronic operators. For  $N = 2$ :

$$E^{(2,1)} = 2\mathcal{R} \left[ \langle \psi^{(0,0)} | \hat{H}^{(0,1)} \hat{R}^{(0,0)} \hat{h}_{SO} \hat{R}^{(0,0)} \hat{h}_{SO} | \psi^{(0,0)} \rangle \right] \\ + \langle \psi^{(0,0)} | \hat{h}_{SO} \hat{R}^{(0,0)} \left( \hat{H}^{(0,1)} - E^{(0,1)} \right) \hat{R}^{(0,0)} \hat{h}_{SO} | \psi^{(0,0)} \rangle \quad (21)$$

and for  $N = 3$ :

$$E^{(3,1)} = 2\mathcal{R} \left[ \langle \psi^{(0,0)} | \hat{h}_{SO} \hat{R}^{(0,0)} \left( H^{(0,1)} - E^{(0,1)} \right) \right. \\ \times \left. \hat{R}^{(0,0)} \hat{h}_{SO} \hat{R}^{(0,0)} \hat{h}_{SO} | \psi^{(0,0)} \rangle \right] \\ + 2\mathcal{R} \left[ \langle \psi^{(0,0)} | \hat{H}^{(0,1)} \hat{R}^{(0,0)} \hat{h}_{SO} \hat{R}^{(0,0)} \right. \\ \times \left. \hat{h}_{SO} \hat{R}^{(0,0)} \hat{h}_{SO} | \psi^{(0,0)} \rangle \right]. \quad (22)$$

The terms  $E^{(2,1)}$  and  $E^{(3,1)}$  can be conveniently re-expressed in the AO basis (see discussion associated with Eqs. (B10)-(B22b) of Appendix B). For  $E^{(2,1)}$  we obtain:

$$E^{(2,1)} = \sum_{\sigma, \sigma'} \mathcal{R} \left[ \sum_{\omega\nu} \left[ P_{\omega\nu}^{\sigma\sigma(1,0)} \right]^* \sum_{\tau\gamma} M_{\gamma\tau}^{\sigma'\sigma'(1,0)} (\omega\nu|\tau\gamma) \right] \\ - \sum_{\sigma, \sigma'} \mathcal{R} \left[ \sum_{\omega\nu} \left[ P_{\omega\nu}^{\sigma\sigma'(1,0)} \right]^* \sum_{\tau\gamma} M_{\gamma\tau}^{\sigma\sigma'(1,0)} (\omega\gamma|\tau\nu) \right] \quad (23)$$

where  $P_{\omega\nu}^{\sigma\sigma'(1,0)}$  are elements of the complex-Hermitian first order perturbed-density matrix and  $M_{\gamma\tau}^{\sigma\sigma'(1,0)}$  are elements of a complex non-Hermitian matrix (for definition see Eqs. (B15) and (B17) Appendix B). Both of these matrices are defined in terms of the orbital coefficients  $c_{\nu m}^{\sigma}$  of Eq. (8) and the matrix of first order orbital rotations  $\mathbf{U}^{\sigma\sigma'(1,0)}$ , whose elements are defined in Eq. (16b).

For  $E^{(3,1)}$  we obtain:

$$E^{(3,1)} = 2 \sum_{\sigma\sigma'} \mathcal{R} \left[ \sum_{\omega\nu} \left[ P_{\omega\nu}^{\sigma\sigma(1,0)} \right]^* \sum_{\tau\gamma} M_{\gamma\tau}^{\sigma'\sigma'(2,0)} (\omega\nu|\tau\gamma) \right] - 2 \sum_{\sigma\sigma'} \mathcal{R} \left[ \sum_{\omega\nu} \left[ P_{\omega\nu}^{\sigma\sigma'(1,0)} \right]^* \sum_{\tau\gamma} M_{\gamma\tau}^{\sigma\sigma'(2,0)} (\omega\gamma|\tau\nu) \right] \\ + 4 \sum_{\sigma\sigma'} \mathcal{R} \left[ \sum_{\omega\nu} M_{\nu\omega}^{\sigma\sigma(1,0)} \sum_{\tau\gamma} \left[ O_{\gamma\tau}^{\sigma'\sigma'(2,0)} - V_{\gamma\tau}^{\sigma'\sigma'(2,0)} \right] (\omega\nu|\tau\gamma) \right] \\ - 4 \sum_{\sigma\sigma'} \mathcal{R} \left[ \sum_{\omega\nu} M_{\nu\omega}^{\sigma'\sigma(1,0)} \sum_{\tau\gamma} \left[ O_{\gamma\tau}^{\sigma\sigma'(2,0)} - V_{\gamma\tau}^{\sigma\sigma'(2,0)} \right] (\omega\gamma|\tau\nu) \right]. \quad (24)$$

where the  $M_{\gamma\tau}^{\sigma\sigma'(2,0)}$ ,  $O_{\gamma\tau}^{\sigma\sigma'(2,0)}$  and  $V_{\gamma\tau}^{\sigma\sigma'(2,0)}$  are elements of non-Hermitian matrices that are defined in terms of the occupied-virtual, occupied-occupied and virtual-virtual blocks, respectively, of the second order matrix of orbital rotations  $\mathbf{U}^{\sigma\sigma'(2,0)}$  (see Eqs. (B21a)-(B22b) of Appendix B).

## 3. Terms of Order Two in the Electron Fluctuation Potential

Finally, we consider terms that are second order in the fluctuation potential and through first order in SOC. Although these terms are not calculated here, it is of inter-

est to establish that they can both be obtained without significant increase in computational effort.

For  $E^{(0,2)}$ , we have the usual sum-over-states expression obtained, as before, from ordinary non-degenerate RSPT:

$$E^{(0,2)} = \langle \psi^{(0,0)} | \hat{H}^{(0,1)} \hat{R}^{(0,0)} \hat{H}^{(0,1)} | \psi^{(0,0)} \rangle, \quad (25)$$

On the other hand, for the  $E^{(1,2)}$  term a derivation is provided in the discussion associated with Eq. (C1) of

Appendix C, which yields:

$$\begin{aligned}
E^{(1,2)} &= 2\mathcal{R} \left[ \langle \psi^{(0,0)} | \hat{h}_{SO} \hat{R}^{(0,0)} \right. \\
&\quad \times \left( \hat{H}^{(0,1)} - E^{(0,1)} \right) \hat{R}^{(0,0)} \hat{H}^{(0,1)} | \psi^{(0,0)} \rangle \\
&\quad \left. + \langle \psi^{(0,0)} | \hat{H}^{(0,1)} \hat{R}^{(0,0)} \hat{h}_{SO} \hat{R}^{(0,0)} \hat{H}^{(0,1)} | \psi^{(0,0)} \rangle \right] \quad (26)
\end{aligned}$$

Writing the configurations  $|\psi_I^{(0,0)}\rangle$  in Eq. (25) as doubly-excited Slater determinants  $|\psi_{mn}^{pq}\rangle$  and evaluating matrix elements using the Slater-Condon rules, we obtain the well-known SR MP2 energy formula:<sup>115</sup>

$$E^{(0,2)} = \frac{1}{4} \sum_{mn}^{occ} \sum_{pq}^{virt} \frac{|(mp|nq) - (mq|np)|^2}{\epsilon_m + \epsilon_n - \epsilon_q - \epsilon_p}. \quad (27)$$

The corresponding expression for  $E^{(1,2)}$  in Eq. (26) is derived in the discussion associated with Eqs. (C2)-(C6) of Appendix C. To write our working expression for  $E^{(1,2)}$  in a compact way, it is expedient to introduce the following four-index tensors:

$$A_{m,n,p,q} = \frac{[(mp|nq) - (mq|np)]}{(\epsilon_m + \epsilon_n - \epsilon_p - \epsilon_q)}, \quad (28a)$$

$$B_{m,n,q,r} = \sum_p^{virt} A_{m,n,p,q} \langle p | \hat{h}_{SO} | r \rangle, \quad (28b)$$

$$D_{n,o,p,q} = \sum_m^{occ} A_{m,n,p,q} \langle o | \hat{h}_{SO} | m \rangle, \quad (28c)$$

$$I_{n,p,q,r} = - \sum_m^{occ} A_{m,n,p,q} U_{mr}^{(1,0)}, \quad (28d)$$

$$J_{m,n,o,q} = - \sum_p^{virt} A_{m,n,p,q} U_{op}^{(1,0)}, \quad (28e)$$

$$K_{n,p,q,r} = (rp|nq) - (rq|np), \quad (28f)$$

and:

$$L_{m,n,o,q} = (no|mq) - (nq|mo). \quad (28g)$$

With these tensors, we obtain:

$$\begin{aligned}
E^{(1,2)} &= \sum_n^{occ} \sum_{pqr}^{virt} \mathcal{R} [I_{n,p,q,r} K_{n,p,q,r}] \\
&\quad + \sum_{mno}^{occ} \sum_q^{virt} \mathcal{R} [J_{m,n,o,q} L_{m,n,o,q}] \\
&\quad + \frac{1}{2} \sum_{mn}^{occ} \sum_{qr}^{virt} B_{m,n,q,r} A_{m,n,r,q} \\
&\quad - \frac{1}{2} \sum_{no}^{occ} \sum_{pq}^{virt} D_{n,o,p,q} A_{o,n,p,q}. \quad (29)
\end{aligned}$$

### C. Computational Cost for Calculation of $E^{(1,2)}$

From Eq. (27), a determination of  $E^{(0,2)}$  involves calculating  $N_{occ}^2 \times N_{virt}^2$  four-index bi-electronic integrals,

where there are  $N_{occ}$  occupied orbitals, and  $N_{virt}$  virtual orbitals. If the bi-electronic integrals are calculated and stored in an AO basis, it is well-known that they can be transformed to molecular orbitals (MOs) one index at a time for a cost approximately proportional to  $N_{occ} N_{AO}^4$ . The integrals in the molecular orbital basis (including those involving three virtual orbital indices) can also be used to calculate  $E^{(1,2)}$  from Eq.(29) for a modest additional cost, as shown below.

The tensor  $\mathbf{A}$  of Eq. (28) is constructed, using the bi-electronic integrals in the MO basis, for a cost that is proportional to  $N_{occ}^2 \times N_{virt}^2$ . Then, in order to include SOC in the third line of Eq. (29), for example, we need to calculate the tensor  $\mathbf{B}$  of Eq. (28b) which involves a computational cost proportional to  $N_{occ}^2 \times N_{virt}^3$ . Similarly, for the fourth line of Eq. (29) we require the tensor  $\mathbf{D}$  of Eq. (28c) which may be obtained at a lesser cost proportional to  $N_{occ}^3 \times N_{virt}^2$ . Then, the above tensors can be used to calculate both energy contributions as:

$$\frac{1}{2} \sum_{mn}^{occ} \sum_{qr}^{virt} B_{m,n,q,r} A_{m,n,r,q}, \quad (30)$$

for the term in the third line of Eq. (29), and:

$$-\frac{1}{2} \sum_{no}^{occ} \sum_{pq}^{virt} D_{n,o,p,q} A_{o,n,p,q}, \quad (31)$$

for the term in the fourth line of Eq. (29). Once the energy contributions in the third and fourth lines of Eq. (29) are calculated, we can concentrate on the energy contributions in the first and second lines. At this point, the tensors  $\mathbf{B}$  and  $\mathbf{D}$  can be deallocated. Then, in order to include the first line of Eq. (29), we can calculate the tensor  $\mathbf{I}$  of Eq. (28d) which is an  $N_{occ}^2 \times N_{virt}^3$  step. Similarly, for the second line of Eq. (29) we require the tensor  $\mathbf{J}$  of Eq. (28e) - an  $N_{occ}^3 \times N_{virt}^2$  step. At this point, the tensor  $\mathbf{A}$  may be deallocated to free up space for the tensors  $\mathbf{K}$  and  $\mathbf{L}$  of Eqs. (28f) and (28g). Finally, the above tensors can be used to calculate both remaining energy contributions as:

$$\sum_n^{occ} \sum_{pqr}^{virt} \mathcal{R} [I_{n,p,q,r} K_{n,p,q,r}], \quad (32)$$

for the term in the first line of Eq. (29), and:

$$\sum_{mno}^{occ} \sum_q^{virt} \mathcal{R} [J_{m,n,o,q} L_{m,n,o,q}], \quad (33)$$

for the term in the second line of Eq. (29). In summary, the total computational cost of calculating both  $E^{(0,2)}$  and  $E^{(1,2)}$  involves only a modest increase over that of an SR MP2 calculation. As in the case of the latter, the computational bottleneck is the time needed for transforming the bielectronic integrals from the AO to molecular orbital basis — a cost that is approximately proportional to  $N_{occ} N_{AO}^4$ .

### III. COMPUTATIONAL DETAILS

The zeroth-order SR UHF calculations were carried out with a developer's version of the CRYSTAL17 code.<sup>101</sup> These were performed with the RECPs and associated basis sets from the libraries available in Refs. 118 and 119. The experimental bond lengths of 1.42, 2.00, 2.28, 2.67 and 3.00 Å were used for the molecules F<sub>2</sub>, Cl<sub>2</sub>, Br<sub>2</sub>, I<sub>2</sub>, respectively whereas for At<sub>2</sub>, the bond length was estimated from the trend along the series.<sup>120</sup> All SCF procedures were converged to a total energy difference of less than  $1 \times 10^{-12}$  Hartree atomic units ( $E_h$ ). After the last SCF cycle, the SR UHF eigenvectors and eigenvalues were used to calculate the various perturbation theory energy contributions of section II. These were compared with reference energies determined from our previously existing two-component SCF (2c-SCF) implementation in the CRYSTAL code as described in section IV.

### IV. COMPUTATIONAL RESULTS FOR THE HALOGEN SERIES DIATOMIC MOLECULES

TABLE I: SOC perturbation theory contributions to the SV energy (in  $E_h$ ) calculated for the diatomic molecules of the halogen series. Our results were obtained with the fully relativistic shape-consistent RECPs and associated basis sets of the Columbus group available at Ref. 119, with 7 electron included in the valence space. See text for definition of individual entries.

	F <sub>2</sub>	Cl <sub>2</sub>	Br <sub>2</sub>	I <sub>2</sub>
$E^{(2,0)}$	$-4.09 \times 10^{-05}$	$-1.60 \times 10^{-04}$	$-3.60 \times 10^{-03}$	$-2.07 \times 10^{-02}$
$E^{(3,0)}$	$-6.98 \times 10^{-08}$	$-2.16 \times 10^{-07}$	$-2.08 \times 10^{-05}$	$-7.19 \times 10^{-04}$
$E^{(4,0)}$	$-3.02 \times 10^{-10}$	$+1.59 \times 10^{-09}$	$+1.47 \times 10^{-06}$	$-6.10 \times 10^{-05}$
$E^{(2,1)}$	$-1.19 \times 10^{-05}$	$-6.08 \times 10^{-05}$	$-1.57 \times 10^{-03}$	$-7.91 \times 10^{-03}$
$E^{(3,1)}$	$-2.82 \times 10^{-08}$	$-6.38 \times 10^{-08}$	$+7.41 \times 10^{-08}$	$-2.09 \times 10^{-04}$
PT $_{\infty}$	$-4.10 \times 10^{-05}$	$-1.61 \times 10^{-04}$	$-3.62 \times 10^{-03}$	$-2.15 \times 10^{-02}$
PT $_{\infty 1}$	$-5.30 \times 10^{-05}$	$-2.22 \times 10^{-04}$	$-5.19 \times 10^{-03}$	$-2.96 \times 10^{-02}$
SV	$-5.30 \times 10^{-05}$	$-2.21 \times 10^{-04}$	$-5.18 \times 10^{-03}$	$-2.95 \times 10^{-02}$
$\Delta_{21}$	$+9.55 \times 10^{-08}$	$+1.98 \times 10^{-07}$	$+5.65 \times 10^{-06}$	$+8.24 \times 10^{-04}$
$\Delta_{\infty 1}$	$-2.81 \times 10^{-09}$	$-2.96 \times 10^{-07}$	$-1.36 \times 10^{-05}$	$-1.64 \times 10^{-04}$

We investigated the convergence of the perturbation series by calculating the terms  $E^{(N,M)}$  for values up to order four in SOC and order one in the electron fluctuation potential ( $N = 0, 1, 2, 3, 4$  and  $M = 0, 1$ ). In analogy to Eq. (11a), we may partition the HF energy obtained

after one iteration of the SCF procedure  $E_{SV}$  into terms of zeroth and first order in the fluctuation potential (i.e.  $M = 0, 1$ ):

$$E_{SV} = E^{(0)} + E^{(1)} \quad (34a)$$

TABLE II: Same as Table I, but using instead the energy-consistent RECPs and associated basis sets of the Stuttgart group, available at Ref. 118, with 7 electrons in the valence space. The asterisk denotes a calculation on the I<sub>2</sub> molecule using the larger valence basis set of Ref. 121

	Br <sub>2</sub>	I <sub>2</sub>	I <sub>2</sub> *	At <sub>2</sub>
$E^{(2,0)}$	$-2.86 \times 10^{-03}$	$-3.86 \times 10^{-03}$	$-3.59 \times 10^{-03}$	$-3.70 \times 10^{-02}$
$E^{(3,0)}$	$+4.17 \times 10^{-05}$	$-2.32 \times 10^{-05}$	$-1.68 \times 10^{-05}$	$-5.59 \times 10^{-04}$
$E^{(4,0)}$	$-4.39 \times 10^{-06}$	$+1.72 \times 10^{-06}$	$+1.90 \times 10^{-06}$	$+1.91 \times 10^{-04}$
$E^{(2,1)}$	$-2.98 \times 10^{-04}$	$-1.63 \times 10^{-03}$	$-1.58 \times 10^{-03}$	$-1.74 \times 10^{-02}$
$E^{(3,1)}$	$-6.14 \times 10^{-06}$	$+6.41 \times 10^{-06}$	$+7.72 \times 10^{-06}$	$+5.49 \times 10^{-04}$
PT $_{\infty}$	$-2.82 \times 10^{-03}$	$-3.88 \times 10^{-03}$	$-3.61 \times 10^{-03}$	$-3.74 \times 10^{-02}$
PT $_{\infty 1}$	$-3.12 \times 10^{-03}$	$-5.50 \times 10^{-03}$	$-5.18 \times 10^{-03}$	$-5.42 \times 10^{-02}$
SV	$-3.12 \times 10^{-03}$	$-5.49 \times 10^{-03}$	$-5.17 \times 10^{-03}$	$-5.35 \times 10^{-02}$
$\Delta_{21}$	$-3.29 \times 10^{-05}$	$+1.28 \times 10^{-07}$	$-8.31 \times 10^{-06}$	$-9.04 \times 10^{-04}$
$\Delta_{\infty 1}$	$-1.68 \times 10^{-06}$	$-1.49 \times 10^{-05}$	$-1.55 \times 10^{-05}$	$-7.24 \times 10^{-04}$

TABLE III: Same as Tables I and II above, but this time using a smaller core shape-consistent RECP from the Columbus group, in which 17 electrons are treated explicitly in the valence space.

	Br <sub>2</sub>	I <sub>2</sub>	At <sub>2</sub>
$E^{(2,0)}$	$-5.73 \times 10^{-03}$	$-2.83 \times 10^{-02}$	$-2.40 \times 10^{-01}$
$E^{(3,0)}$	$-2.78 \times 10^{-05}$	$-9.04 \times 10^{-04}$	$-1.69 \times 10^{-02}$
$E^{(4,0)}$	$+1.45 \times 10^{-06}$	$-8.28 \times 10^{-05}$	$+2.03 \times 10^{-03}$
$E^{(2,1)}$	$-1.83 \times 10^{-03}$	$-8.80 \times 10^{-03}$	$-8.62 \times 10^{-02}$
$E^{(3,1)}$	$-2.54 \times 10^{-06}$	$-2.48 \times 10^{-04}$	$-3.53 \times 10^{-03}$
PT $_{\infty}$	$-5.75 \times 10^{-03}$	$-2.93 \times 10^{-02}$	$-2.54 \times 10^{-01}$
PT $_{\infty 1}$	$-7.59 \times 10^{-03}$	$-3.83 \times 10^{-02}$	$-3.44 \times 10^{-01}$
SV	$-7.57 \times 10^{-03}$	$-3.82 \times 10^{-02}$	$-3.32 \times 10^{-01}$
$\Delta_{21}$	$+1.40 \times 10^{-05}$	$+1.09 \times 10^{-03}$	$+5.84 \times 10^{-03}$
$\Delta_{\infty 1}$	$-1.49 \times 10^{-05}$	$-1.49 \times 10^{-04}$	$-1.25 \times 10^{-02}$

Expanding both the MP0  $E^{(0)}$  and MP1  $E^{(1)}$  terms of Eq. (34a) in orders of SOC, we obtain a double-perturbation series for  $E_{SV}$ :

$$E_{SV} = E^{(0,0)} + E^{(1,0)} + E^{(2,0)} + E^{(3,0)} + \dots + E^{(0,1)} + E^{(1,1)} + E^{(2,1)} + E^{(3,1)} + \dots \quad (34b)$$



Thus, according to Eq. (34b) a calculation of all terms  $E^{(N,M)}$  ( $N = 0, 1, 2, 3, 4, \dots$  and  $M = 0, 1$ ) up to order one in the fluctuation potential corresponds to a perturbation theory expansion of the second-variational (SV) approach to SOC in solid-state physics.<sup>122–124</sup> The reference SV energy  $E_{\text{SV}}$  is simply calculated by using the zeroth-order SR UHF density-matrix as a starting guess for performing one iteration of the 2c-SCF.

A comparison of the perturbation theory approach with an SV calculation, performed with our 2c-SCF implementation — also in the CRYSTAL code — 100,101,103–105 is given in Table I for the members  $\text{F}_2$ ,  $\text{Cl}_2$ ,  $\text{Br}_2$  and  $\text{I}_2$  of the halogen diatomic molecule series. This set of molecules was chosen because of the availability of many sets of RECPs and because of the large contribution of SOC to their energy. The results in Table I were obtained from large-core shape-consistent RECPs,<sup>125–129</sup> with 7 electrons in the valence space for each atom. The pure SOC contribution to the energy is reported for  $E^{(2,0)}$ ,  $E^{(3,0)}$ , and  $E^{(4,0)}$  as well as for the sum  $E^{(2,0)} + E^{(3,0)} + E^{(4,0)}$  ... extrapolated to infinite order, using an extrapolation formula from Ref. 130 (PT $\infty$ ). The addition of  $E^{(2,1)}$  and  $E^{(3,1)}$  to PT $\infty$  is denoted in the table as PT $\infty 1$ . The quantities  $\Delta_{21}$  and  $\Delta_{\infty 1}$  in the last row represent the difference between  $E^{(2,0)} + E^{(2,1)}$  and SV ( $= E_{\text{SV}} - E_{\text{UHF}}$ ), and between PT $\infty + E^{(2,1)} + E^{(3,1)}$  and SV, respectively.

It is clear from the first three rows of the table that the individual perturbation theory energy contributions —  $E^{(2,0)}$ ,  $E^{(3,0)}$ ,  $E^{(4,0)}$  ... — are monotonically decreasing (in absolute value), such that the pure SOC perturbation series is rapidly convergent for all systems. Moreover, the reported differences in the last row of the table are very small (about two to four orders of magnitude smaller than the energy of the SV approach). We note also that  $E^{(3,1)}$  is always much smaller than  $E^{(2,1)}$ . In fact, a simple calculation through  $E^{(2,0)}$  and  $E^{(2,1)}$  is sufficient to adequately reproduce the SV energy. Thus, the coupling between SOC and the fluctuation potential is well described using only the leading (non-vanishing) term for order  $M = 0, 1$  of the fluctuation potential. We speculate that this is likely to extend to  $M = 2$ , so that a first order treatment in SOC would satisfactorily describe the coupling with correlation through MP2.

In Table II, we report similar calculations (again with large-core potentials, including 7 electrons in the valence space of each atom), this time using RECPs and associated basis sets generated by the energy-consistent method from Ref. 118. These RECPs are available for the systems  $\text{Br}_2$ ,  $\text{I}_2$  and  $\text{At}_2$ . We see that the difference reported in the last row of Table II is now two to three orders of magnitude smaller than the SV energy for the systems  $\text{Br}_2$  and  $\text{I}_2$ . For the most challenging system,  $\text{At}_2$ , the difference is  $-7.24 \times 10^{-04} E_h$ , which represents an error of 1.4% in approximating the SV energy. The difference  $\Delta_{21}$  is  $-9.04 \times 10^{-04} E_h$ ; it represents an error of 1.7%. The energy contributions  $E^{(2,0)}$ ,  $E^{(3,0)}$ ,  $E^{(4,0)}$  ..., as well as  $E^{(2,1)}$ ,  $E^{(3,1)}$  ... are

again monotonically (and rapidly) decreasing. Thus, the perturbation series through order two in SOC and order one in the electron fluctuation potential again provides an excellent approximation for the SV energy.

Table III gives results for the systems  $\text{Br}_2$ ,  $\text{I}_2$  and  $\text{At}_2$ , now using a smaller core shape-consistent potential, with 17 electrons included in the valence space of each atom. Again, the differences between perturbation theory and the SV approach are two to three orders of magnitude smaller than the SV energy for  $\text{Br}_2$  and  $\text{I}_2$ . Once more  $\text{At}_2$  shows a larger discrepancy (about 3.8% for  $\Delta_{\infty 1}$  or 1.8% for  $\Delta_{21}$ ). Thus, the small-core RECPs behave in the same way as the large core potentials.

TABLE IV: Same as table I, but now for the open-shell radicals, in which one electron from the highest occupied molecular orbital has been removed.

	$\text{F}_2^+$	$\text{Cl}_2^+$	$\text{Br}_2^+$	$\text{I}_2^+$
$E^{(2,0)}$	$-4.45 \times 10^{-05}$	$-1.90 \times 10^{-04}$	$-4.35 \times 10^{-03}$	$-2.43 \times 10^{-02}$
$E^{(3,0)}$	$-7.99 \times 10^{-08}$	$-3.27 \times 10^{-07}$	$-3.13 \times 10^{-05}$	$-8.98 \times 10^{-04}$
$E^{(4,0)}$	$-3.09 \times 10^{-10}$	$+4.66 \times 10^{-09}$	$+3.62 \times 10^{-06}$	$-1.82 \times 10^{-05}$
$E^{(2,1)}$	$-1.93 \times 10^{-05}$	$-1.05 \times 10^{-04}$	$-2.68 \times 10^{-03}$	$-1.33 \times 10^{-02}$
$E^{(3,1)}$	$-2.49 \times 10^{-08}$	$+2.75 \times 10^{-07}$	$+5.30 \times 10^{-05}$	$+3.20 \times 10^{-04}$
PT $\infty$	$-4.45 \times 10^{-05}$	$-1.90 \times 10^{-04}$	$-4.38 \times 10^{-03}$	$-2.52 \times 10^{-02}$
PT $\infty 1$	$-6.38 \times 10^{-05}$	$-2.95 \times 10^{-04}$	$-7.00 \times 10^{-03}$	$-3.81 \times 10^{-02}$
SV	$-6.38 \times 10^{-05}$	$-2.94 \times 10^{-04}$	$-6.94 \times 10^{-03}$	$-3.75 \times 10^{-02}$
$\Delta_{21}$	$+5.58 \times 10^{-08}$	$-7.10 \times 10^{-07}$	$-8.36 \times 10^{-05}$	$-4.28 \times 10^{-05}$
$\Delta_{\infty 1}$	$-4.93 \times 10^{-08}$	$-7.57 \times 10^{-07}$	$-5.84 \times 10^{-05}$	$-6.39 \times 10^{-04}$

Table IV provides results for systems that are identical to those reported in Table I, except that now one electron has been removed from the highest occupied molecular orbital, such that they represent open-shell electronic configurations. The reported energy differences between perturbation theory and the SV value remain small as in the case of the closed-shell configurations in Table I. In the worst case ( $\text{I}_2^+$ ), the perturbation theory approach overestimates the SV energy by 1.7% (for  $\Delta_{\infty 1}$ ). This difference is reduced to 0.1% for  $\Delta_{21}$ . As for the closed-shell systems, the perturbation series  $E^{(2,0)}$ ,  $E^{(3,0)}$ ,  $E^{(4,0)}$  ... and  $E^{(2,1)}$ ,  $E^{(3,1)}$  ... converge rapidly. A more than adequate treatment is obtained, again, simply by calculating  $E^{(2,0)}$  and  $E^{(2,1)}$ , since the terms  $E^{(3,0)}$ ,  $E^{(4,0)}$  and  $E^{(3,1)}$  are around two to three orders of magnitude smaller than the SV energy.

To provide results on other challenging cases for perturbation theory, in Table S1 of the electronic supporting information (ESI), we report on calculations for Selenium oxide  $\text{SeO}^{131}$  (using the experimental bond length of 2.648 Å from Ref. 132) and the positively charged homonuclear diatomic Tenessine  $\text{Ts}_2^+$  molecule (bond length of 3.00 Å). The ECP10MDF and ECP92MDFQ Stuttgart potentials were used for Se and Ts, respec-

tively along with the associated double-zeta valence basis sets,<sup>118</sup> while the basis set of Ref. 133 was used for O. Excellent agreement between the double-perturbation and SV methods is obtained in both cases (see Table S1), as the reported  $\Delta_{21}$  and  $\Delta_{\infty 1}$  values are smaller than the SV energy by around one order of magnitude for  $\text{Ts}_2^+$  and four orders of magnitude for SeO. In the case of  $\text{Ts}_2^+$ , monotonicity of the  $E^{(2,0)}, E^{(3,0)}, E^{(4,0)} \dots$  series breaks down in 4th order, because  $E^{(4,0)}$  ( $= 0.0504$ ) is slightly larger in absolute value than  $E^{(3,0)}$  ( $= -0.0459$ ). For  $\text{Ts}_2^+$ , monotonicity is restored by scaling down the SOC operator (scaling factor  $\leq 0.90$ ), to ensure that the ratio of  $|E^{(2,0)}|$  to the HOMO-LUMO gap  $E_{\text{gap}}$  is less than 3.9. Coincidentally, this criterion on  $|E^{(2,0)}|/E_{\text{gap}} < 3.9$  also ensures monotonicity for the whole positively charged homonuclear diatomic halogen series through fourth order, as shown in Figure S1, as well as Table S2.

## V. CONCLUSIONS

We have presented a double perturbation theory approach to the solution of the many-body Dirac equation in a given relativistic effective-core potential (RECP) approximation. The zeroth-order approximation is the corresponding scalar relativistic unrestricted Hartree-Fock (SR UHF) solution. This is followed by a double perturbation theory expansion in terms of the spin-orbit coupling (SOC) and electron fluctuation potential. Practical perturbation energy expressions are provided up to a total order of three with most higher-order terms included.

Tests on the neutral halogen diatomic molecules  $\text{F}_2$ ,  $\text{Cl}_2$ ,  $\text{Br}_2$ ,  $\text{I}_2$  and  $\text{At}_2$ , as well as their single positive open-shell ions, show that the proposed approach is well-converged to the second variational (SV) value through second order in SOC and first order in the fluctuation

potential. Since the double perturbation treatment is no more difficult to carry out than the SV procedure, it is a computationally attractive alternative to performing a 2c-SCF for treatment of SOC, and, in addition, represents a convenient starting point for further improvements.

A calculation of  $E^{(0,2)}$ , that is to say, the familiar SR MP2 energy — a procedure that represents an  $O(N_{\text{occ}}N_{\text{AO}}^4)$  computational problem for a calculation with  $N_{\text{AO}}$  atomic and  $N_{\text{occ}}$  occupied orbitals — would provide a computationally tractable, relativistic correlated approach. A further calculation of  $E^{(1,2)}$  involves only a modest additional cost, and would allow for a full treatment of the coupling between SOC and the fluctuation potential through total order three. Furthermore, the present approach opens the possibility of a multi-reference treatment through quasi-degenerate perturbation theory (QDPT) that could be used in those cases where it is necessary (i.e. open-shell or closed-shell systems of a strong multi-reference character).

### Supplemental Material

See Supplemental Material at URL for the derivation of Eqs. (19), (24), (B19) and (B20) of the main text of the manuscript, and for additional calculations on the systems SeO and  $\text{Ts}_2^+$ , as well as a discussion on a necessary criterion to ensure monotonic convergence of the  $E^{(2,0)}, E^{(3,0)}, E^{(4,0)} \dots$  series through fourth order.

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<sup>1</sup> van Lenthe, E.; Baerends, E.-J.; Snijders, J. G. Relativistic regular two-component Hamiltonians. *J. Chem. Phys.* **1993**, *99*, 4597–4610.

<sup>2</sup> van Lenthe, E.; Baerends, E.-J.; Snijders, J. G. Relativistic total energy using regular approximations. *J. Chem. Phys.* **1994**, *101*, 9783–9792.

<sup>3</sup> Snijders, J. G.; Sadlej, A. J. Perturbation versus variation treatment of regular relativistic Hamiltonians. *Chem. Phys. Lett.* **1996**, *252*, 51–61.

<sup>4</sup> Chang, C.; Pelissier, M.; Durand, P. Regular two-component Pauli-like effective Hamiltonians in Dirac theory. *Phys. Scr.* **1986**, *34*, 394.

<sup>5</sup> Dyall, K. G.; van Lenthe, E. Relativistic regular approximations revisited: An infinite-order relativistic approximation. *J. Chem. Phys.* **1999**, *111*, 1366–1372.

<sup>6</sup> Buenker, R. J.; Chandra, P.; Hess, B. A. Matrix representation of the relativistic kinetic energy operator: Two-component variational procedure for the treatment of

many-electron atoms and molecules. *Chem. Phys.* **1984**, *84*, 1–9.

<sup>7</sup> Almlöf, J.; Faegri Jr, K.; Grelland, H. A variational approach to relativistic effects in LCAO calculations. *Chem. Phys. Lett.* **1985**, *114*, 53–57.

<sup>8</sup> Douglas, M.; Kroll, N. M. Quantum electrodynamic corrections to the fine structure of helium. *Ann. Phys.* **1974**, *82*, 89–155.

<sup>9</sup> Nakajima, T.; Hirao, K. The higher-order Douglas–Kroll transformation. *J. Chem. Phys.* **2000**, *113*, 7786–7789.

<sup>10</sup> Wolf, A.; Reiher, M.; Hess, B. A. The generalized Douglas–Kroll transformation. *J. Chem. Phys.* **2002**, *117*, 9215–9226.

<sup>11</sup> van Wüllen, C. Relation between different variants of the generalized Douglas–Kroll transformation through sixth order. *J. Chem. Phys.* **2004**, *120*, 7307–7313.

<sup>12</sup> Reiher, M.; Wolf, A. Exact decoupling of the Dirac Hamiltonian. II. The generalized Douglas–Kroll–Hess transformation up to arbitrary order. *J. Chem. Phys.* **2004**, *121*, 10945–10956.

- <sup>13</sup> Peng, D.; Hirao, K. An arbitrary order Douglas–Kroll method with polynomial cost. *J. Chem. Phys.* **2009**, *130*, 044102.
- <sup>14</sup> Barysz, M.; Sadlej, A. J.; Snijders, J. G. Nonsingular two/one-component relativistic Hamiltonians accurate through arbitrary high order in  $\alpha^2$ . *Int. J. Q. Chem.* **1997**, *65*, 225–239.
- <sup>15</sup> Barysz, M. Systematic treatment of relativistic effects accurate through arbitrarily high order in  $\alpha^2$ . *J. Chem. Phys.* **2001**, *114*, 9315–9324.
- <sup>16</sup> Barysz, M.; Sadlej, A. J. Infinite-order two-component theory for relativistic quantum chemistry. *J. Chem. Phys.* **2002**, *116*, 2696–2704.
- <sup>17</sup> Kedziera, D.; Barysz, M.; Sadlej, A. J. Expectation values in spin-averaged Douglas–Kroll and Infinite-order relativistic methods. *Struct. Chem.* **2004**, *15*, 369–377.
- <sup>18</sup> Dyall, K. G. Interfacing relativistic and nonrelativistic methods. I. Normalized elimination of the small component in the modified Dirac equation. *J. Chem. Phys.* **1997**, *106*, 9618–9626.
- <sup>19</sup> Iliáš, M.; Jensen, H. J. A.; Kellö, V.; Roos, B. O.; Urban, M. Theoretical study of PbO and the PbO anion. *Chem. Phys. Lett.* **2005**, *408*, 210–215.
- <sup>20</sup> Kutzelnigg, W.; Liu\*, W. Quasirelativistic theory I. Theory in terms of a quasi-relativistic operator. *Mol. Phys.* **2006**, *104*, 2225–2240.
- <sup>21</sup> Liu, W.; Kutzelnigg, W. Quasirelativistic theory. II. Theory at matrix level. *J. Chem. Phys.* **2007**, *126*, 114107.
- <sup>22</sup> Liu, W.; Peng, D. Infinite-order quasirelativistic density functional method based on the exact matrix quasirelativistic theory. *J. Chem. Phys.* **2006**, *125*, 044102.
- <sup>23</sup> Liu, W.; Daoling, P. Exact two-component Hamiltonians revisited. *J. Chem. Phys.* **2009**, *131*.
- <sup>24</sup> Peng, D.; Liu, W.; Xiao, Y.; Cheng, L. Making four- and two-component relativistic density functional methods fully equivalent based on the idea of “from atoms to molecule”. *J. Chem. Phys.* **2007**, *127*, 104106.
- <sup>25</sup> Sikkema, J.; Visscher, L.; Saue, T.; Iliáš, M. The molecular mean-field approach for correlated relativistic calculations. *J. Chem. Phys.* **2009**, *131*, 124116.
- <sup>26</sup> Seino, J.; Hada, M. Examination of accuracy of electron–electron Coulomb interactions in two-component relativistic methods. *Chem. Phys. Lett.* **2008**, *461*, 327–331.
- <sup>27</sup> Seino, J.; Nakai, H. Local unitary transformation method for large-scale two-component relativistic calculations: Case for a one-electron Dirac Hamiltonian. *J. Chem. Phys.* **2012**, *136*, 244102.
- <sup>28</sup> Iliáš, M.; Saue, T. An infinite-order two-component relativistic Hamiltonian by a simple one-step transformation. *J. Chem. Phys.* **2007**, *126*, 064102.
- <sup>29</sup> Saue, T.; Jensen, H. A. Quaternion symmetry in relativistic molecular calculations: The Dirac–Hartree–Fock method. *J. Chem. Phys.* **1999**, *111*, 6211–6222.
- <sup>30</sup> Jørgen Aa. Jensen, H.; Dyall, K. G.; Saue, T.; Fægri Jr, K. Relativistic four-component multiconfigurational self-consistent-field theory for molecules: Formalism. *J. Chem. Phys.* **1996**, *104*, 4083–4097.
- <sup>31</sup> Kadek, M.; Repisky, M.; Ruud, K. All-electron fully relativistic Kohn–Sham theory for solids based on the Dirac–Coulomb Hamiltonian and Gaussian-type functions. *Phys. Rev. B* **2019**, *99*, 205103.
- <sup>32</sup> Hrdá, M.; Kulich, T.; Repiský, M.; Noga, J.; Malkina, O. L.; Malkin, V. G. Implementation of the diagonalization-free algorithm in the self-consistent field procedure within the four-component relativistic scheme. **2014**.
- <sup>33</sup> Baldes, A.; Klopper, W.; Šimunek, J.; Noga, J.; Weigend, F. Acceleration of self-consistent-field convergence by combining conventional diagonalization and a diagonalization-free procedure. *J. Comput. Chem.* **2011**, *32*, 3129–3134.
- <sup>34</sup> Vilkas, M. J.; Ishikawa, Y.; Koc, K. Quadratically convergent multiconfiguration Dirac–Fock and multireference relativistic configuration–interaction calculations for many-electron systems. *Phys. Rev. E* **1998**, *58*, 5096.
- <sup>35</sup> Peng, D.; Ma, J.; Liu, W. On the construction of Kramers paired double group symmetry functions. *Int. J. Q. Chem.* **2009**, *109*, 2149–2167.
- <sup>36</sup> Yanai, T.; Harrison, R. J.; Nakajima, T.; Ishikawa, Y.; Hirao, K. New implementation of molecular double point-group symmetry in four-component relativistic Gaussian-type spinors. *Int. J. Q. Chem.* **2007**, *107*, 1382–1389.
- <sup>37</sup> Armbruster, M. K. Quaternionic formulation of the two-component Kohn–Sham equations and efficient exploitation of point group symmetry. *J. Chem. Phys.* **2017**, *147*, 054101.
- <sup>38</sup> Fleig, T.; Olsen, J.; Marian, C. M. The generalized active space concept for the relativistic treatment of electron correlation. I. Kramers-restricted two-component configuration interaction. *J. Chem. Phys.* **2001**, *114*, 4775–4790.
- <sup>39</sup> Fleig, T.; Olsen, J.; Visscher, L. The generalized active space concept for the relativistic treatment of electron correlation. II. Large-scale configuration interaction implementation based on relativistic 2- and 4-spinors and its application. *J. Chem. Phys.* **2003**, *119*, 2963–2971.
- <sup>40</sup> Fleig, T.; Jensen, H. J. A.; Olsen, J.; Visscher, L. The generalized active space concept for the relativistic treatment of electron correlation. III. Large-scale configuration interaction and multiconfiguration self-consistent-field four-component methods with application to UO 2. *J. Chem. Phys.* **2006**, *124*, 104106.
- <sup>41</sup> Ganyushin, D.; Neese, F. A fully variational spin-orbit coupled complete active space self-consistent field approach: Application to electron paramagnetic resonance g-tensors. *J. Chem. Phys.* **2013**, *138*, 104113.
- <sup>42</sup> Fleig, T. Invited review: Relativistic wave-function based electron correlation methods. *Chem. Phys.* **2012**, *395*, 2–15.
- <sup>43</sup> Thyssen, J.; Fleig, T.; Jensen, H. J. A. A direct relativistic four-component multiconfiguration self-consistent-field method for molecules. *J. Chem. Phys.* **2008**, *129*, 034109.
- <sup>44</sup> Kim, Y. S.; Lee, Y. S. The Kramers’ restricted complete active space self-consistent-field method for two-component molecular spinors and relativistic effective core potentials including spin–orbit interactions. *J. Chem. Phys.* **2003**, *119*, 12169–12178.
- <sup>45</sup> Kim, I.; Lee, Y. S. Two-component Kramers restricted complete active space self-consistent field method with relativistic effective core potential revisited: Theory, implementation, and applications to spin-orbit splitting of lower p-block atoms. *J. Chem. Phys.* **2013**, *139*, 134115.
- <sup>46</sup> Teichteil, C.; Pelissier, M.; Spiegelmann, F. Ab initio molecular calculations including spin-orbit coupling. I. Method and atomic tests. *Chem. Phys.* **1983**, *81*, 273–282.
- <sup>47</sup> Vallet, V.; Maron, L.; Teichteil, C.; Flament, J.-P. A two-step uncontracted determinantal effective hamiltonian-based so–ci method. *J. Chem. Phys.* **2000**, *113*, 1391–

- 1402.
- <sup>48</sup> Yabushita, S.; Zhang, Z.; Pitzer, R. M. Spin-Orbit Configuration Interaction Using the Graphical Unitary Group Approach and Relativistic Core Potential and Spin-Orbit Operators. *J. Phys. Chem. A* **1999**, *103*, 5791–5800.
- <sup>49</sup> Przybytek, M.; Cencek, W.; Komasa, J.; Lach, G.; Jeziorski, B.; Szalewicz, K. Relativistic and quantum electrodynamics effects in the helium pair potential. *Phys. Rev. Lett.* **2010**, *104*, 183003.
- <sup>50</sup> Snijders, J.; Baerends, E. A perturbation theory approach to relativistic calculations: I. Atoms. *Mol. Phys.* **1978**, *36*, 1789–1804.
- <sup>51</sup> Snijders, J.; Baerends, E.; Ros, P. A perturbation theory approach to relativistic calculations: II. Molecules. *Mol. Phys.* **1979**, *38*, 1909–1929.
- <sup>52</sup> Ziegler, T.; Snijders, J.; Baerends, E. Relativistic effects on bonding. *J. Chem. Phys.* **1981**, *74*, 1271–1284.
- <sup>53</sup> Dekock, R. L.; Baerends, E. J.; Boerrigter, P. M.; Snijders, J. G. On the nature of the first excited states of the uranyl ion. *Chem. Phys. Lett.* **1984**, *105*, 308–316.
- <sup>54</sup> DeKock, R.; Baerends, E. J.; Boerrigter, P.; Hengelmolen, R. Electronic structure and bonding of Hg (CH<sub>3</sub>)<sub>2</sub>, Hg (CN)<sub>2</sub>, Hg (CH<sub>3</sub>)(CN), Hg (CCCH<sub>3</sub>)<sub>2</sub>, and Au (PMe<sub>3</sub>)(CH<sub>3</sub>). *J. Am. Chem. Soc.* **1984**, *106*, 3387–3396.
- <sup>55</sup> Boerrigter, P.; Buijse, M.; Snijders, J. Spin-orbit interaction in the excited states of the dihalogen ions F<sup>+</sup> 2, Cl<sup>+</sup> 2 and Br<sup>+</sup> 2. *Chem. Phys.* **1987**, *111*, 47–53.
- <sup>56</sup> Boerrigter, P.; Baerends, E.; Snijders, J. A relativistic LCAO Hartree-Fock-Slater investigation of the electronic structure of the actinocenes M (COT)<sub>2</sub>, M = Th, Pa, U, Np and Pu. *Chem. Phys.* **1988**, *122*, 357–374.
- <sup>57</sup> Ziegler, T.; Baerends, E. J.; Snijders, J. G.; Ravenek, W.; Tschinke, V. Calculation of bond energies in compounds of heavy elements by a quasi-relativistic approach. *J. Phys. Chem.* **1989**, *93*, 3050–3056.
- <sup>58</sup> Li, J.; Schreckenbach, G.; Ziegler, T. A reassessment of the first metal-carbonyl dissociation energy in M (CO)<sub>4</sub> (M = Ni, Pd, Pt), M (CO)<sub>5</sub> (M = Fe, Ru, Os), and M (CO)<sub>6</sub> (M = Cr, Mo, W) by a quasirelativistic density functional method. *J. Am. Chem. Soc.* **1995**, *117*, 486–494.
- <sup>59</sup> Rutkowski, A. Relativistic perturbation theory. I. A new perturbation approach to the Dirac equation. *J. Phys. B* **1986**, *19*, 149.
- <sup>60</sup> Rutkowski, A. Relativistic perturbation theory: II. One-electron variational perturbation calculations. *J. Phys. B* **1986**, *19*, 3431.
- <sup>61</sup> Rutkowski, A. Relativistic perturbation theory. III. A new perturbation approach to the two-electron Dirac-Coulomb equation. *J. Phys. B* **1986**, *19*, 3443.
- <sup>62</sup> Kutzelnigg, W. Stationary direct perturbation theory of relativistic corrections. *Phys. Rev. A* **1996**, *54*, 1183.
- <sup>63</sup> van Wüllen, C. A relativistic Kohn-Sham density functional procedure by means of direct perturbation theory. *J. Chem. Phys.* **1995**, *103*, 3589–3599.
- <sup>64</sup> Stopkowicz, S.; Gauss, J. Direct perturbation theory in terms of energy derivatives: Fourth-order relativistic corrections at the Hartree-Fock level. *J. Chem. Phys.* **2011**, *134*, 064114.
- <sup>65</sup> Stopkowicz, S.; Gauss, J. A one-electron variant of direct perturbation theory for the treatment of scalar-relativistic effects. *Mol. Phys.* **2019**, *117*, 1242–1251.
- <sup>66</sup> Cheng, L.; Stopkowicz, S.; Gauss, J. Spin-free Dirac-Coulomb calculations augmented with a perturbative treatment of spin-orbit effects at the Hartree-Fock level. *J. Chem. Phys.* **2013**, *139*, 214114.
- <sup>67</sup> Cheng, L.; Wang, F.; Stanton, J. F.; Gauss, J. Perturbative treatment of spin-orbit-coupling within spin-free exact two-component theory using equation-of-motion coupled-cluster methods. *J. Chem. Phys.* **2018**, *148*, 044108.
- <sup>68</sup> Kutzelnigg, W.; Liu, W. Relativistic MCSCF by means of quasidegenerate direct perturbation theory. I. Theory. *J. Chem. Phys.* **2000**, *112*, 3540–3558.
- <sup>69</sup> Liu, W.; Kutzelnigg, W.; van Wüllen, C. Relativistic MCSCF by means of quasidegenerate direct perturbation theory. II. Preliminary applications. *J. Chem. Phys.* **2000**, *112*, 3559–3571.
- <sup>70</sup> Rutkowski, A.; Schwarz, W. Effective Hamiltonian for near-degenerate states in direct relativistic perturbation theory. I. Formalism. *J. Chem. Phys.* **1996**, *104*, 8546–8552.
- <sup>71</sup> Rutkowski, A.; Schwarz, W.; Kozłowski, R.; Beczek, J.; Franke, R. Effective Hamiltonian for near-degenerate states in relativistic direct perturbation theory. II. H 2+ like systems. *J. Chem. Phys.* **1998**, *109*, 2135–2143.
- <sup>72</sup> Kutzelnigg, W.; Liu, W. Matrix formulation of direct perturbation theory of relativistic effects in a kinetically balanced basis. *Chem. Phys.* **2008**, *349*, 133–146.
- <sup>73</sup> Kirtman, B. Simultaneous calculation of several interacting electronic states by generalized Van Vleck perturbation theory. *J. Chem. Phys.* **1981**, *75*, 798–808.
- <sup>74</sup> Kirtman, B. Variational form of Van Vleck degenerate perturbation theory with particular application to electronic structure problems. *J. Chem. Phys.* **1968**, *49*, 3890–3894.
- <sup>75</sup> Kirtman, B. Interchange Theorems in Degenerate Perturbation Theory. *J. Chem. Phys.* **1968**, *49*, 3895–3898.
- <sup>76</sup> de Heer, J.; Kirtman, B. Interchange Theorems, Accessible Correlation Expressions, and Physical Properties. *J. Chem. Phys.* **1970**, *53*, 4115–4116.
- <sup>77</sup> Dolg, M.; Cao, X. Relativistic pseudopotentials: their development and scope of applications. *Chem. Revs.* **2012**, *112*, 403–480.
- <sup>78</sup> Ermler, W. C.; Lee, Y. S.; Christiansen, P. A.; Pitzer, K. S. AB initio effective core potentials including relativistic effects. A procedure for the inclusion of spin-orbit coupling in molecular wavefunctions. *Chem. Phys. Lett.* **1981**, *81*, 70–74.
- <sup>79</sup> El-Kelany, K. E.; Ravoux, C.; Desmarais, J. K.; Cortona, P.; Pan, Y.; Tse, J.; Erba, A. Spin Localization, Magnetic Ordering and Electronic Properties of Strongly Correlated Ln<sub>2</sub>O<sub>3</sub> Sesquioxides (Ln = La, Ce, Pr, Nd). *Phys. Rev. B* **2018**, *97*, 245118.
- <sup>80</sup> Desmarais, J. K.; Erba, A.; Pan, Y.; Civalleri, B.; Tse, J. S. Unravelling the Electronic Structure Mechanisms Driving Isostructural Phase Transitions in Strongly Correlated EuO. *Phys. Rev. Lett.* **Under Review**.
- <sup>81</sup> Frisch, M. J. et al. Gaussian 16 Revision C.01. 2016.
- <sup>82</sup> Werner, H.-J.; Knowles, P. J.; Knizia, G.; Manby, F. R.; Schütz, M., et al. MOLPRO, version 2019.2, a package of ab initio programs. 2019; see <https://www.molpro.net/>.
- <sup>83</sup> TURBOMOLE V7.0 2015, a development of University of Karlsruhe and Forschungszentrum Karlsruhe GmbH, 1989-2007, TURBOMOLE GmbH, since 2007; available from <http://www.turbomole.com>.
- <sup>84</sup> DIRAC, a relativistic ab initio electronic struc-

- ture program, Release DIRAC19 (2019), written by A. S. P. Gomes, T. Saue, L. Visscher, H. J. Aa. Jensen, and R. Bast, with contributions from I. A. Aucar, V. Bakken, K. G. Dyall, S. Dubillard, U. Ekström, E. Eliav, T. Enevoldsen, E. Faßhauer, T. Fleig, O. Fossgaard, L. Halbert, E. D. Hedegård, B. Heimlich–Paris, T. Helgaker, J. Henriksson, M. Iliaš, Ch. R. Jacob, S. Knecht, S. Komorovský, O. Kullie, J. K. Lærdahl, C. V. Larsen, Y. S. Lee, H. S. Nataraj, M. K. Nayak, P. Norman, G. Olejniczak, J. Olsen, J. M. H. Olsen, Y. C. Park, J. K. Pedersen, M. Pernpointner, R. di Remigio, K. Ruud, P. Salek, B. Schimmelpfennig, B. Senjean, A. Shee, J. Sikkema, A. J. Thorvaldsen, J. Thyssen, J. van Stralen, M. L. Vidal, S. Villaume, O. Visser, T. Winther, and S. Yamamoto (available at <http://dx.doi.org/10.5281/zenodo.3572669>, see also <http://www.diracprogram.org>).
- <sup>85</sup> Valiev, M.; Bylaska, E. J.; Govind, N.; Kowalski, K.; Straatsma, T. P.; Van Dam, H. J.; Wang, D.; Nieplocha, J.; Apra, E.; Windus, T. L., et al. NWChem: A comprehensive and scalable open-source solution for large scale molecular simulations. *Comput. Phys. Comms.* **2010**, *181*, 1477–1489.
- <sup>86</sup> Neese, F. Software update: the ORCA program system, version 4.0. *Wiley Inter. Rrvs.: Comput. Mol. Sci.* **2018**, *8*, e1327.
- <sup>87</sup> Shao, Y.; Gan, Z.; Epifanovsky, E.; Gilbert, A. T.; Wormit, M.; Kussmann, J.; Lange, A. W.; Behn, A.; Deng, J.; Feng, X., et al. Advances in molecular quantum chemistry contained in the Q-Chem 4 program package. *Mol. Phys.* **2015**, *113*, 184–215.
- <sup>88</sup> Aquilante, F.; Autschbach, J.; Carlson, R. K.; Chibotaru, L. F.; Delcey, M. G.; De Vico, L.; Fdez. Galván, I.; Ferré, N.; Frutos, L. M.; Gagliardi, L., et al. Molcas 8: New capabilities for multiconfigurational quantum chemical calculations across the periodic table. *J. Comput. Chem.* **2016**, *37*, 506–541.
- <sup>89</sup> Dupuis, M.; Marquez, A.; Davidson, E. HONDO2004, based on HONDO-95 available from the Quantum Chemistry Program Exchange. *Indiana University* **2004**,
- <sup>90</sup> Turney, J. M.; Simmonett, A. C.; Parrish, R. M.; Hohenstein, E. G.; Evangelista, F. A.; Fermann, J. T.; Mintz, B. J.; Burns, L. A.; Wilke, J. J.; Abrams, M. L., et al. Psi4: an open-source ab initio electronic structure program. *Wiley Inter. Revs.: Comput. Mol. Sci.* **2012**, *2*, 556–565.
- <sup>91</sup> Kresse, G.; Hafner, J. *Phys. Rev. B* **1994**, *49*, 14521.
- <sup>92</sup> Giannozzi, P. et al. *J. Phys.: Condens. Matter* **2009**, *21*, 395502.
- <sup>93</sup> Dovesi, R.; Saunders, V.; Roetti, C.; Orlando, R.; Zicovich-Wilson, C.; Pascale, F.; Civalieri, B.; Doll, K.; Harrison, N.; Bush, I., et al. CRYSTAL17 User’s Manual. 2017; <http://www.crystal.unito.it>.
- <sup>94</sup> Erba, A.; Halo, M. CRYSCOR09 User’s Manual. 2010; <http://www.cryscor.unito.it>.
- <sup>95</sup> Soler, J. M.; Artacho, E.; Gale, J. D.; García, A.; Junquera, J.; Ordejón, P.; Sánchez-Portal, D. The Siesta method for ab initio order-N materials simulation. *J. Phys.: Condens. Matter* **2002**, *14*, 2745–2779.
- <sup>96</sup> <http://www.flapw.de/>.
- <sup>97</sup> Dolg, M., et al. Effective core potentials. *Modern methods and algorithms of quantum chemistry* **2000**, *3*, 507–540.
- <sup>98</sup> Weigand, A.; Cao, X.; Vallet, V.; Flament, J.-P.; Dolg, M. Multiconfiguration Dirac–Hartree–Fock Adjusted Energy-Consistent Pseudopotential for Uranium: Spin-Orbit Configuration Interaction and Fock-Space Coupled-Cluster Study of U4+ and U5+. *J. Phys. Chem. A* **2009**, *113*, 11509–11516.
- <sup>99</sup> Desmarais, J.; Erba, A.; Flament, J.; Kirtman, B. Perturbative Treatment of Spin-Orbit Coupling. Part II: Coupled-Perturbed Kohn-Sham Method. *J. Chem. Theor. Comput.: Under Review* **2021**,
- <sup>100</sup> Desmarais, J. K.; Flament, J.-P.; Erba, A. Spin-orbit coupling from a two-component self-consistent approach. I. Generalized Hartree-Fock theory. *J. Chem. Phys.* **2019**, *151*, 074107.
- <sup>101</sup> Dovesi, R.; Erba, A.; Orlando, R.; Zicovich-Wilson, C. M.; Civalieri, B.; Maschio, L.; Rérat, M.; Casassa, S.; Baima, J.; Salustro, S.; Kirtman, B. Quantum-Mechanical Condensed Matter Simulations with CRYSTAL. *WIREs Comput. Mol. Sci.* **2018**, *8*, e1360.
- <sup>102</sup> Desmarais, J.; Erba, A.; Dovesi, R. Generalization of the Periodic LCAO Approach in the CRYSTAL Code to g-type Orbitals. *Theor. Chem. Acc.* **2018**, *137*, 28.
- <sup>103</sup> Desmarais, J. K.; Flament, J.-P.; Erba, A. Fundamental Role of Fock Exchange in Relativistic Density Functional Theory. *J. Phys. Chem. Lett.* **2019**, *10*, 3580–3585.
- <sup>104</sup> Desmarais, J. K.; Flament, J.-P.; Erba, A. Spin-orbit coupling in periodic systems with broken time-reversal symmetry: Formal and computational aspects. *Phys. Rev. B* **2020**, *101*, 235142.
- <sup>105</sup> Desmarais, J. K.; Flament, J.-P.; Erba, A. Adiabatic connection in spin-current density functional theory. *Phys. Rev. B* **2020**, *102*, 235118.
- <sup>106</sup> Dolg, M. *Theor. Comput. Chem.*; 2002; Vol. 11; pp 793–862.
- <sup>107</sup> Stoll, H.; Metz, B.; Dolg, M. Relativistic energy-consistent pseudopotentials—Recent developments. *J. Comput. Chem.* **2002**, *23*, 767–778.
- <sup>108</sup> Cao, X.; Dolg, M. Pseudopotentials and modelpotentials. *Wiley Inter. Revs.: Comput. Mol. Sci.* **2011**, *1*, 200–210.
- <sup>109</sup> Schwerdtfeger, P. The pseudopotential approximation in electronic structure theory. *ChemPhysChem* **2011**, *12*, 3143–3155.
- <sup>110</sup> Dolg, M. Fully relativistic pseudopotentials for alkaline atoms: Dirac-Hartree-Fock and configuration interaction calculations of alkaline monohydrides. *Theor. Chim. Acta* **1996**, *93*, 141–156.
- <sup>111</sup> Ishikawa, Y.; Malli, G. Effective core potentials for fully relativistic Dirac–Fock calculations. *J. Chem. Phys.* **1981**, *75*, 5423–5431.
- <sup>112</sup> Datta, S. N.; Ewig, C. S.; van Wazer, J. R. Application of effective potentials to relativistic hartree–fock calculations. *Chem. Phys. Lett.* **1978**, *57*, 83–89.
- <sup>113</sup> Pyper, N. Relativistic pseudopotential theories and corrections to the Hartree-Fock method. *Mol. Phys.* **1980**, *39*, 1327–1358.
- <sup>114</sup> Pyper, N.; Marketos, P. Atomic fine-structure inversions explained as first-order relativistic corrections to the Hartree-Fock energy. *J. Phys. B* **1981**, *14*, 4469.
- <sup>115</sup> Szabo, A.; Ostlund, N. S. *Modern quantum chemistry: introduction to advanced electronic structure theory*; Courier Corporation, 2012.
- <sup>116</sup> Møller, C.; Plesset, M. S. Note on an approximation treatment for many-electron systems. *Phys. Rev.* **1934**, *46*, 618.
- <sup>117</sup> Dalgarno, A.; Stewart, A. A perturbation calculation of properties of the helium iso-electronic sequence. *Proc. R.*

- Soc. A* **1958**, *247*, 245–259.
- <sup>118</sup> <http://www.tc.uni-koeln.de/PP/clickpse.en.html>.
- <sup>119</sup> <https://lin-web.clarkson.edu/~pchristi/reps.html>.
- <sup>120</sup> Huber, K.-P. *Molecular spectra and molecular structure: IV. Constants of diatomic molecules*; Springer Science & Business Media, 2013.
- <sup>121</sup> Martin, J. M.; Sundermann, A. Correlation consistent valence basis sets for use with the Stuttgart–Dresden–Bonn relativistic effective core potentials: The atoms Ga–Kr and In–Xe. *J. Chem. Phys.* **2001**, *114*, 3408–3420.
- <sup>122</sup> Koelling, D.; Harmon, B. A technique for relativistic spin-polarised calculations. *J. Phys. C* **1977**, *10*, 3107.
- <sup>123</sup> C. Li, H. J., A.J. Freeman; Fu, C. Magnetic anisotropy in low-dimensional ferromagnetic systems: Fe monolayers on Ag (001), Au (001), and Pd (001) substrates. *Phys. Rev. B* **1990**, *42*, 5433.
- <sup>124</sup> S. Steiner, M. M., S. Khmelevskiy; Kresse, G. Calculation of the magnetic anisotropy with projected-augmented-wave methodology and the case study of disordered Fe 1- x Co x alloys. *Phys. Rev. B* **2016**, *93*, 224425.
- <sup>125</sup> Fernandez Pacios, L.; Christiansen, P. Ab initio relativistic effective potentials with spin-orbit operators. I. Li through Ar. *J. Chem. Phys.* **1985**, *82*, 2664–2671.
- <sup>126</sup> Hurley, M.; Pacios, L. F.; Christiansen, P.; Ross, R.; Ermler, W. A binitio relativistic effective potentials with spin-orbit operators. II. K through Kr. *J. Chem. Phys.* **1986**, *84*, 6840–6853.
- <sup>127</sup> LaJohn, L.; Christiansen, P.; Ross, R.; Atashroo, T.; Ermler, W. A binitio relativistic effective potentials with spin-orbit operators. III. Rb through Xe. *J. Chem. Phys.* **1987**, *87*, 2812–2824.
- <sup>128</sup> Ross, R.; Powers, J.; Atashroo, T.; Ermler, W.; LaJohn, L.; Christiansen, P. A binitio relativistic effective potentials with spin-orbit operators. IV. Cs through Rn. *J. Chem. Phys.* **1990**, *93*, 6654–6670.
- <sup>129</sup> Ermler, W.; Ross, R.; Christiansen, P. Ab initio relativistic effective potentials with spin-orbit operators. VI. Fr through Pu. *Int. J. Q. Chem.* **1991**, *40*, 829–846.
- <sup>130</sup> Weniger, E. J.; Kirtman, B. Extrapolation methods for improving the convergence of oligomer calculations to the infinite chain limit of quasi-one-dimensional stereoregular polymers. *Comput. Math. Appl.* **2003**, *45*, 189–215.
- <sup>131</sup> Mokambe, R. M.; Hicks, J. M.; Kerker, D.; Jiang, W.; Theis, D.; Chen, Z.; Khait, Y. G.; Hoffmann, M. R. GVVPT2 multireference perturbation theory study of selenium oxides. *Mol. Phys.* **2013**, *111*, 1078–1091.
- <sup>132</sup> Lide, D. R. *CRC handbook of chemistry and physics*; CRC press, 2004; Vol. 85.
- <sup>133</sup> Heifets, E.; Kotomin, E.; Bagaturyants, A.; Maier, J. Ab initio study of BiFeO<sub>3</sub>: thermodynamic stability conditions. *J. Phys. Chem. Lett.* **2015**, *6*, 2847–2851.
- <sup>134</sup> Desmarais, J. K. Development of Tools for the Study of Heavy-Element Containing Periodic Systems in the CRYSTAL Code and their Application. Ph.D. thesis, University of Saskatchewan and University of Turin, 2020.

#### Appendix A: Energy Contributions of Zeroth Order in the Electron Fluctuation Potential in Terms of SR UHF Orbitals

Since  $\hat{L}_{z,i}\hat{S}_{z,i}$  is purely imaginary, it follows from Eq. (4) that the real part of the matrix elements of  $\hat{h}_{SO}$  in-

volving orbitals of like spin (the real part of diagonal spin-block matrix elements of  $\hat{h}_{SO}$ ) vanishes:

$$\mathcal{R} \left[ \langle m, \alpha | \hat{h}_{SO} | n, \alpha \rangle \right] = 0, \quad (\text{A1})$$

On the other hand, the off-diagonal spin-block matrix elements of  $\hat{h}_{SO}$  are complex. It follows that  $E^{(1,0)}$  in Eq. (15) vanishes because  $\hat{h}_{SO}$  is Hermitian, which means that diagonal matrix elements must be real.

In second order, ordinary non-degenerate RSPT reduces to Eq. (17) since  $\hat{h}_{SO}$  is a sum of one-electron operators and only single excitation configurations contribute to the sum-over-states. Our choice for the (0,0) problem ensures that the  $E^{(0,0)} - E_I^{(0,0)}$  denominators of Eq. (13) become just a difference of SR UHF eigenvalues.

Ordinary non-degenerate RSPT, in third order, gives:

$$E^{(3,0)} = \langle \psi^{(0,0)} | \hat{h}_{SO} \hat{R}^{(0,0)} \hat{h}_{SO} \hat{R}^{(0,0)} \hat{h}_{SO} | \psi^{(0,0)} \rangle, \quad (\text{A2})$$

and in fourth order (using intermediated normalization):

$$E^{(4,0)} = \langle \psi^{(0,0)} | \hat{h}_{SO} \hat{R}^{(0,0)} \hat{h}_{SO} \hat{R}^{(0,0)} \hat{h}_{SO} \hat{R}^{(0,0)} \hat{h}_{SO} | \psi^{(0,0)} \rangle - \langle \psi^{(0,0)} | \hat{h}_{SO} \hat{R}^{(0,0)} E^{(2,0)} \hat{R}^{(0,0)} \hat{h}_{SO} | \psi^{(0,0)} \rangle. \quad (\text{A3})$$

To simplify Eq. (A2), we denote singly-excited configurations by  $|\psi_m^p\rangle$  where virtual orbital  $|p\rangle$  is substituted for occupied orbital  $|m\rangle$ ; the associated energy is  $E_m^p$ . Doubly-excited configurations will be denoted as  $|\psi_{mn}^{pq}\rangle$ . Expanding Eq. (A2) in singly-excited Slater determinants, we find

$$E^{(3,0)} = \sum_m \sum_{pq}^{occ \ virt} \frac{\langle \psi^{(0,0)} | \hat{h}_{SO} | \psi_m^p \rangle \langle \psi_m^p | \hat{h}_{SO} | \psi_n^q \rangle \langle \psi_n^q | \hat{h}_{SO} | \psi^{(0,0)} \rangle}{(E^{(0,0)} - E_m^p) (E^{(0,0)} - E_n^q)} + \sum_{mn} \sum_p^{occ \ virt} \frac{\langle \psi^{(0,0)} | \hat{h}_{SO} | \psi_m^p \rangle \langle \psi_m^p | \hat{h}_{SO} | \psi_n^p \rangle \langle \psi_n^p | \hat{h}_{SO} | \psi^{(0,0)} \rangle}{(E^{(0,0)} - E_m^p) (E^{(0,0)} - E_n^p)} \quad (\text{A4})$$

which, can be written in terms of one-electron orbitals as:

$$E^{(3,0)} = \sum_m \sum_{pq}^{occ \ virt} \frac{\langle m | \hat{h}_{SO} | p \rangle \langle p | \hat{h}_{SO} | q \rangle \langle q | \hat{h}_{SO} | m \rangle}{(\epsilon_m - \epsilon_p) (\epsilon_m - \epsilon_q)} - \sum_{mn} \sum_p^{occ \ virt} \frac{\langle m | \hat{h}_{SO} | p \rangle \langle n | \hat{h}_{SO} | m \rangle \langle p | \hat{h}_{SO} | n \rangle}{(\epsilon_m - \epsilon_p) (\epsilon_n - \epsilon_p)} \quad (\text{A5})$$

Expressing Eq. (A5) in terms of the matrices reported in Eq. (16), we obtain Eq. (18).

A similar demonstration can be carried out in fourth order, starting from Eq. (A3) to obtain Eq. (19). The full derivation is reported in the ESI.

#### Appendix B: Energy Contributions of First Order in the Electron Fluctuation Potential in Terms of SR UHF Orbitals

Eq. (20) is obtained by combining the (1,1), (1,0) and (0,1) RSPT equations. We use intermediate normalization here and throughout this paper:

$$\langle \psi^{(0,0)} | \psi^{(N,M)} \rangle = \delta_{N,0} \delta_{M,0}. \quad (\text{B1})$$

The (1,1) RSPT equation:

$$\begin{aligned} & \left[ \hat{H}^{(0,0)} - E^{(0,0)} \right] |\psi^{(1,1)}\rangle + \left[ \hat{h}_{SO} - E^{(1,0)} \right] |\psi^{(0,1)}\rangle \\ & + \left[ \hat{H}^{(0,1)} - E^{(0,1)} \right] |\psi^{(1,0)}\rangle - E^{(1,1)} |\psi^{(0,0)}\rangle = 0. \end{aligned} \quad (\text{B2})$$

may be multiplied on the left by  $\langle \psi^{(0,0)} |$  to yield:

$$E^{(1,1)} = \langle \psi^{(0,0)} | \hat{h}_{SO} | \psi^{(0,1)} \rangle + \langle \psi^{(0,0)} | \hat{H}^{(0,1)} | \psi^{(1,0)} \rangle. \quad (\text{B3})$$

Similarly, for the (1,0) RSPT equation:

$$\left[ \hat{H}^{(0,0)} - E^{(0,0)} \right] |\psi^{(1,0)}\rangle + \left[ \hat{h}_{SO} - E^{(1,0)} \right] |\psi^{(0,0)}\rangle = 0. \quad (\text{B4})$$

we multiply on the left by  $\langle \psi^{(0,1)} |$ , which gives:

$$\langle \psi^{(0,1)} | \hat{H}^{(0,0)} - E^{(0,0)} | \psi^{(1,0)} \rangle + \langle \psi^{(0,1)} | \hat{h}_{SO} | \psi^{(0,0)} \rangle = 0. \quad (\text{B5})$$

Finally, the (0,1) RSPT equation leads to:

$$\langle \psi^{(1,0)} | \hat{H}^{(0,0)} - E^{(0,0)} | \psi^{(0,1)} \rangle + \langle \psi^{(1,0)} | \hat{H}^{(0,1)} | \psi^{(0,0)} \rangle = 0. \quad (\text{B6})$$

after multiplication on the left by  $\langle \psi^{(1,0)} |$ . The sum of Eq. (B3) minus the complex conjugate-transpose of Eq. (B6) plus Eq. (B5), leads directly to Eq. (20).

Below we provide an outline of the derivation that allows us to obtain expressions for  $E^{(2,1)}$  and  $E^{(3,1)}$ . More details on the derivation can be found in section H.6 of Ref. 134. A procedure analogous to that developed in Eqs. (B2)-(B6) can be used for  $E^{(2,1)}$  by combining the (1,0), (0,1), (1,1), (2,0) and (2,1) RSPT equations to ob-

tain:

$$\begin{aligned} E^{(2,1)} &= 2\mathcal{R} \left[ \langle \psi^{(0,0)} | \hat{H}^{(0,1)} | \psi^{(2,0)} \rangle \right] \\ &+ \langle \psi^{(1,0)} | \hat{H}^{(0,1)} - E^{(0,1)} | \psi^{(1,0)} \rangle, \end{aligned} \quad (\text{B7})$$

For  $E^{(3,1)}$  we combine the (1,0), (0,1), (1,1), (2,0), (2,1), (3,0) and (3,1) RSPT equations:

$$\begin{aligned} E^{(3,1)} &= 2\mathcal{R} \left[ \langle \psi^{(1,0)} | \hat{H}^{(0,1)} - E^{(0,1)} | \psi^{(2,0)} \rangle \right] \\ &+ 2\mathcal{R} \left[ \langle \psi^{(0,1)} | \hat{h}_{SO} | \psi^{(2,0)} \rangle \right] \\ &- 2E^{(2,0)} \mathcal{R} \left[ \langle \psi^{(1,0)} | \psi^{(0,1)} \rangle \right]. \end{aligned} \quad (\text{B8})$$

The last term in Eq. (B8) vanishes since, from Brillouin's theorem,  $|\psi^{(0,1)}\rangle$  consists only of doubly-excited configurations, while  $|\psi^{(1,0)}\rangle$  consists only of singly-excited configurations because the SOC operator is a sum of mono-electronic operators. Thus, Eq. (B8) reduces to:

$$\begin{aligned} E^{(3,1)} &= 2\mathcal{R} \left[ \langle \psi^{(1,0)} | \hat{H}^{(0,1)} - E^{(0,1)} | \psi^{(2,0)} \rangle \right] \\ &+ 2\mathcal{R} \left[ \langle \psi^{(0,1)} | \hat{h}_{SO} | \psi^{(2,0)} \rangle \right]. \end{aligned} \quad (\text{B9})$$

Then, expanding  $|\psi^{(2,0)}\rangle$ ,  $|\psi^{(1,0)}\rangle$  and  $|\psi^{(0,1)}\rangle$  in Eqs. (B7) and (B9) in eigenfunctions of the (0,0) problem, we obtain Eqs. (21) and (22). Taking into account the form of the perturbation operators from Eq. (9) for the case of  $E^{(2,1)}$  leads to:

$$\begin{aligned} E^{(2,1)} &= 2 \sum_{n < m} \sum_{q < p}^{\text{occ virt}} \mathcal{R} \left[ \langle \psi^{(0,0)} | \hat{V}_{ee} | \psi_{mn}^{pq} \rangle \left\{ \frac{\langle \psi_{mn}^{pq} | \hat{h}_{SO} | \psi_m^p \rangle \langle \psi_m^p | \hat{h}_{SO} | \psi^{(0,0)} \rangle}{(E^{(0,0)} - E_{mn}^{pq}) (E^{(0,0)} - E_m^p)} + \frac{\langle \psi_{mn}^{pq} | \hat{h}_{SO} | \psi_n^p \rangle \langle \psi_n^p | \hat{h}_{SO} | \psi^{(0,0)} \rangle}{(E^{(0,0)} - E_{mn}^{pq}) (E^{(0,0)} - E_n^p)} \right. \right. \\ &+ \left. \left. \frac{\langle \psi_{mn}^{pq} | \hat{h}_{SO} | \psi_m^q \rangle \langle \psi_m^q | \hat{h}_{SO} | \psi^{(0,0)} \rangle}{(E^{(0,0)} - E_{mn}^{pq}) (E^{(0,0)} - E_m^q)} + \frac{\langle \psi_{mn}^{pq} | \hat{h}_{SO} | \psi_n^q \rangle \langle \psi_n^q | \hat{h}_{SO} | \psi^{(0,0)} \rangle}{(E^{(0,0)} - E_{mn}^{pq}) (E^{(0,0)} - E_n^q)} \right\} \right] \\ &+ \sum_{n \neq m} \sum_{q \neq p}^{\text{occ virt}} \langle \psi_m^p | \hat{V}_{ee} | \psi_n^q \rangle \frac{\langle \psi^{(0,0)} | \hat{h}_{SO} | \psi_m^p \rangle \langle \psi_n^q | \hat{h}_{SO} | \psi^{(0,0)} \rangle}{(E^{(0,0)} - E_m^p) (E^{(0,0)} - E_n^q)} + \sum_m \sum_{q \neq p}^{\text{occ virt}} \langle \psi_m^p | \hat{V}_{ee} - (\hat{C} - \hat{K}) | \psi_n^q \rangle \frac{\langle \psi^{(0,0)} | \hat{h}_{SO} | \psi_m^p \rangle \langle \psi_n^q | \hat{h}_{SO} | \psi^{(0,0)} \rangle}{(E^{(0,0)} - E_m^p) (E^{(0,0)} - E_n^q)} \\ &+ \sum_{n \neq m} \sum_p^{\text{occ virt}} \langle \psi_m^p | \hat{V}_{ee} - (\hat{C} - \hat{K}) | \psi_n^p \rangle \frac{\langle \psi^{(0,0)} | \hat{h}_{SO} | \psi_m^p \rangle \langle \psi_n^p | \hat{h}_{SO} | \psi^{(0,0)} \rangle}{(E^{(0,0)} - E_m^p) (E^{(0,0)} - E_n^p)} - E^{(0,1)} \sum_m \sum_p^{\text{occ virt}} \frac{\langle \psi^{(0,0)} | \hat{h}_{SO} | \psi_m^p \rangle \langle \psi_m^p | \hat{h}_{SO} | \psi^{(0,0)} \rangle}{(E^{(0,0)} - E_m^p) (E^{(0,0)} - E_m^p)} \\ &+ \sum_m \sum_p^{\text{occ virt}} \langle \psi_m^p | \hat{V}_{ee} - (\hat{C} - \hat{K}) | \psi_m^p \rangle \frac{\langle \psi^{(0,0)} | \hat{h}_{SO} | \psi_m^p \rangle \langle \psi_m^p | \hat{h}_{SO} | \psi^{(0,0)} \rangle}{(E^{(0,0)} - E_m^p) (E^{(0,0)} - E_m^p)}, \end{aligned} \quad (\text{B10})$$

Based on the Slater-Condon rules, we can evaluate all of the integrals in Eq. (B10) and express  $E^{(2,1)}$  as a sum of two terms  $E_S^{(2,1)}$  and  $E_D^{(2,1)}$ , arising, respectively, from singly-excited and doubly- (as well as singly-) ex-

cited configurations:

$$E^{(2,1)} = E_S^{(2,1)} + E_D^{(2,1)}, \quad (\text{B11})$$

where

$$E_S^{(2,1)} = \sum_{mn}^{occ} \sum_{pq}^{virt} \{ (pm|nq) - (pq|nm) \} \\ \times \frac{\langle m|\hat{h}_{SO}|p\rangle \langle q|\hat{h}_{SO}|n\rangle}{(\epsilon_m - \epsilon_p)(\epsilon_n - \epsilon_q)}, \quad (\text{B12})$$

and

$$E_D^{(2,1)} = 2 \sum_{n < m < q < p}^{occ} \sum_{pq}^{virt} \mathcal{R} \left[ \frac{(mp|nq) - (mq|np)}{\epsilon_m + \epsilon_n - \epsilon_p - \epsilon_q} \right. \\ \times \left\{ \frac{\langle q|\hat{h}_{SO}|n\rangle \langle p|\hat{h}_{SO}|m\rangle}{\epsilon_m - \epsilon_p} - \frac{\langle q|\hat{h}_{SO}|m\rangle \langle p|\hat{h}_{SO}|n\rangle}{\epsilon_n - \epsilon_p} \right. \\ \left. - \frac{\langle p|\hat{h}_{SO}|n\rangle \langle q|\hat{h}_{SO}|m\rangle}{\epsilon_m - \epsilon_q} + \frac{\langle p|\hat{h}_{SO}|m\rangle \langle q|\hat{h}_{SO}|n\rangle}{\epsilon_n - \epsilon_q} \right] \quad (\text{B13})$$

Since the contribution to  $E_D^{(2,1)}$  in Eq. (B13) from the terms  $m = n$  and  $p = q$  vanish, we can extend the summations over orbital indices to all values of the occupied and virtual sets *occ* and *virt*. Then, using the fact that

$(mp|nq) - (mq|np) = -\{(np|mq) - (nq|mp)\}$ , Eq. (B13) can be simplified to:

$$E_D^{(2,1)} = \sum_{mn}^{occ} \sum_{pq}^{virt} \mathcal{R} \left[ \{ (mp|nq) - (mq|np) \} \right. \\ \left. \times \frac{\langle q|\hat{h}_{SO}|n\rangle \langle p|\hat{h}_{SO}|m\rangle}{(\epsilon_n - \epsilon_q)(\epsilon_m - \epsilon_p)} \right]. \quad (\text{B14})$$

Eqs. (B11), (B12) and (B14) can be combined to write a computationally convenient expression for  $E^{(2,1)}$  by introducing the complex non-Hermitian matrix:

$$M_{\omega\nu}^{\sigma\sigma'(N,0)} = \sum_m^{occ} \sum_p^{virt} c_{\omega p}^{\sigma'} [c_{\nu m}^{\sigma}]^* U_{pm}^{\sigma\sigma'(N,0)}, \quad (\text{B15})$$

where the orbital coefficients  $c_{\omega p}^{\sigma}$  are defined through Eq. (8) and the elements of the matrix of orbital rotations  $\mathbf{U}^{\sigma\sigma'(N,0)}$  are defined for the case  $N = 1$  in Eq. (16b). Combining Eqs. (B11), (B12), (B14) and (B15), we obtain:

$$E^{(2,1)} = \sum_{\sigma\sigma'} \mathcal{R} \left[ \sum_{\omega\nu} \left\{ [M_{\omega\nu}^{\sigma\sigma(1,0)}]^* + M_{\nu\omega}^{\sigma\sigma(1,0)} \right\} \sum_{\tau\gamma} M_{\gamma\tau}^{\sigma'\sigma'(1,0)} (\omega\nu|\tau\gamma) \right] \\ - \sum_{\sigma\sigma'} \mathcal{R} \left[ \sum_{\omega\nu} \left\{ [M_{\omega\nu}^{\sigma\sigma'(1,0)}]^* + M_{\nu\omega}^{\sigma'\sigma(1,0)} \right\} \sum_{\tau\gamma} M_{\gamma\tau}^{\sigma\sigma'(1,0)} (\omega\gamma|\tau\nu) \right]. \quad (\text{B16})$$

Finally, in order to write Eq. (B16) in a more compact way, it proves useful to define the elements of the first order perturbed-density matrix:

$$P_{\omega\nu}^{\sigma\sigma'(1,0)} = M_{\omega\nu}^{\sigma\sigma'(1,0)} + [M_{\nu\omega}^{\sigma'\sigma(1,0)}]^*. \quad (\text{B17})$$

Substituting Eq. (B17) in Eq. (B16), we obtain Eq. (23).

A similar demonstration to that provided in Eqs. (B10)-(B13) can be developed for  $E^{(3,1)}$  starting from Eq. (22). The full derivation is provided in the ESI. We write  $E^{(3,1)}$  as a sum of two terms  $E_S^{(3,1)}$  and  $E_D^{(3,1)}$ , arising, respectively, from singly-excited and doubly- (as well as singly-) excited configurations:

$$E^{(3,1)} = E_S^{(3,1)} + E_D^{(3,1)}, \quad (\text{B18})$$

After evaluation of all the integrals employing the Slater-Condon rules, the corresponding expressions for  $E_S^{(3,1)}$

and  $E_D^{(3,1)}$  in Eq. (B18) are:

$$E_S^{(3,1)} = 2 \sum_{mn}^{occ} \sum_{pqr}^{virt} \mathcal{R} \left[ \{ (pm|nq) - (pq|nm) \} \right. \\ \left. \times \frac{\langle m|\hat{h}_{SO}|p\rangle \langle q|\hat{h}_{SO}|r\rangle \langle r|\hat{h}_{SO}|n\rangle}{(\epsilon_m - \epsilon_p)(\epsilon_n - \epsilon_q)(\epsilon_n - \epsilon_r)} \right] \\ - 2 \sum_{mno}^{occ} \sum_{pq}^{virt} \mathcal{R} \left[ \{ (pm|nq) - (pq|nm) \} \right. \\ \left. \times \frac{\langle m|\hat{h}_{SO}|p\rangle \langle o|\hat{h}_{SO}|n\rangle \langle q|\hat{h}_{SO}|o\rangle}{(\epsilon_m - \epsilon_p)(\epsilon_n - \epsilon_q)(\epsilon_o - \epsilon_q)} \right], \quad (\text{B19})$$



and:

$$\begin{aligned}
E_D^{(3,1)} &= 2 \sum_{mn}^{occ} \sum_{pqr}^{virt} \mathcal{R} \left[ \{ (rp|nq) - (rq|np) \} \right. \\
&\times \left. \frac{\langle m|\hat{h}_{SO}|r\rangle\langle q|\hat{h}_{SO}|n\rangle\langle p|\hat{h}_{SO}|m\rangle}{(\epsilon_m - \epsilon_r)(\epsilon_n - \epsilon_q)(\epsilon_m - \epsilon_p)} \right] \\
&+ 2 \sum_{mno}^{occ} \sum_{pq}^{virt} \mathcal{R} \left[ \{ (mq|no) - (mo|nq) \} \right. \\
&\times \left. \frac{\langle o|\hat{h}_{SO}|p\rangle\langle q|\hat{h}_{SON}\rangle\langle p|\hat{h}_{SO}|m\rangle}{(\epsilon_o - \epsilon_p)(\epsilon_n - \epsilon_q)(\epsilon_m - \epsilon_p)} \right] \\
&+ 2 \sum_{mn}^{occ} \sum_{pqr}^{virt} \mathcal{R} \left[ \{ (mp|nq) - (mq|np) \} \right. \\
&\times \left. \frac{\langle q|\hat{h}_{SO}|n\rangle\langle p|\hat{h}_{SO}|r\rangle\langle r|\hat{h}_{SO}|m\rangle}{(\epsilon_n - \epsilon_q)(\epsilon_m - \epsilon_p)(\epsilon_m - \epsilon_r)} \right] \\
&- 2 \sum_{mno}^{occ} \sum_{pq}^{virt} \mathcal{R} \left[ \{ (mp|nq) - (mq|np) \} \right. \\
&\times \left. \frac{\langle q|\hat{h}_{SO}|n\rangle\langle o|\hat{h}_{SO}|m\rangle\langle p|\hat{h}_{SO}|o\rangle}{(\epsilon_n - \epsilon_q)(\epsilon_m - \epsilon_p)(\epsilon_o - \epsilon_p)} \right]. \quad (B20)
\end{aligned}$$

We can represent  $E^{(3,1)}$  in the AO basis by introducing the elements of the matrix of second order orbital rotations  $\mathbf{U}^{(2,0)}$  which reads as follows for the occupied-virtual block:

$$\begin{aligned}
U_{pm}^{\sigma'\sigma(2,0)} &= \frac{1}{\epsilon_p^{\sigma'} - \epsilon_m^{\sigma}} \sum_{\sigma''} \left( \sum_n^{occ} U_{pn}^{\sigma'\sigma''(1,0)} G_{nm}^{\sigma''\sigma(1,0)} \right. \\
&- \left. \sum_q^{virt} G_{pq}^{\sigma'\sigma''(1,0)} U_{qm}^{\sigma''\sigma(1,0)} \right). \quad (B21a)
\end{aligned}$$

For the virtual-virtual blocks:

$$U_{qr}^{\sigma\sigma'(2,0)} = -\frac{1}{2} \sum_{\sigma''} \sum_n^{occ} \left[ U_{nq}^{\sigma''\sigma(1,0)} \right]^* U_{nr}^{\sigma''\sigma'(1,0)}, \quad (B21b)$$

and for the occupied-occupied blocks:

$$U_{on}^{\sigma\sigma'(2,0)} = -\frac{1}{2} \sum_{\sigma''} \sum_q^{virt} \left[ U_{qo}^{\sigma''\sigma(1,0)} \right]^* U_{qn}^{\sigma''\sigma'(1,0)}. \quad (B21c)$$

$U_{qr}^{\sigma\sigma'(2,0)}$  and  $U_{on}^{\sigma\sigma'(2,0)}$  are used to represent  $E^{(3,1)}$  in the basis of AOs through the following matrices:

$$O_{\omega\nu}^{\sigma\sigma'(2,0)} = \sum_{on}^{occ} c_{\omega o}^{\sigma} \left[ c_{\nu n}^{\sigma'} \right]^* U_{on}^{\sigma\sigma'(2,0)}, \quad (B22a)$$

and:

$$V_{\omega\nu}^{\sigma\sigma'(2,0)} = \sum_{qr}^{virt} c_{\omega q}^{\sigma} \left[ c_{\nu r}^{\sigma'} \right]^* U_{qr}^{\sigma\sigma'(2,0)}, \quad (B22b)$$

Substituting Eqs. (B21a)-(B22b) and (B15) into Eqs. (B19) and (B20), we obtain Eq. (24) (see ESI for more details).

### Appendix C: $E^{(1,2)}$ Contribution in Terms of SR UHF Orbitals

For Eq. (26), we proceed in a manner similar to that outlined in Eqs. (B2)-(B6) and combine the (0,1), (1,0), (1,1), (0,2) and (1,2) RSPT equations to find:

$$\begin{aligned}
E^{(1,2)} &= 2\mathcal{R} \left[ \langle \psi^{(1,0)} | \hat{H}^{(0,1)} - E^{(0,1)} | \psi^{(0,1)} \rangle \right] \\
&+ \langle \psi^{(0,1)} | \hat{h}_{SO} | \psi^{(0,1)} \rangle. \quad (C1)
\end{aligned}$$

Then, expanding  $|\psi^{(0,1)}\rangle$  and  $|\psi^{(1,0)}\rangle$  in the basis of configurations of the (0,0) problem we obtain Eq. (26). Only singly- and doubly-excited configurations contribute to the double sum, which leads to:

$$\begin{aligned}
E^{(1,2)} = & 2 \sum_{n < m} \sum_{q < p}^{\text{occ virt}} \mathcal{R} \left[ \frac{\langle \psi_{mn}^{pq} | \hat{V}_{ee} | \psi^{(0,0)} \rangle}{E^{(0,0)} - E_{mn}^{pq}} \left\{ \frac{\langle \psi^{(0,0)} | \hat{h}_{SO} | \psi_m^p \rangle \langle \psi_m^p | \hat{V}_{ee} - (\hat{C} - \hat{K}) | \psi_{mn}^{pq} \rangle}{E^{(0,0)} - E_m^p} + \frac{\langle \psi^{(0,0)} | \hat{h}_{SO} | \psi_n^p \rangle \langle \psi_n^p | \hat{V}_{ee} - (\hat{C} - \hat{K}) | \psi_{mn}^{pq} \rangle}{E^{(0,0)} - E_n^p} \right. \right. \\
& + \left. \frac{\langle \psi^{(0,0)} | \hat{h}_{SO} | \psi_m^q \rangle \langle \psi_m^q | \hat{V}_{ee} - (\hat{C} - \hat{K}) | \psi_{mn}^{pq} \rangle}{E^{(0,0)} - E_m^q} + \frac{\langle \psi^{(0,0)} | \hat{h}_{SO} | \psi_n^q \rangle \langle \psi_n^q | \hat{V}_{ee} - (\hat{C} - \hat{K}) | \psi_{mn}^{pq} \rangle}{E^{(0,0)} - E_n^q} \right\} \Big] \\
& + 2 \sum_{n < m} \sum_{q < p}^{\text{occ virt}} \sum_r'' \mathcal{R} \left[ \frac{\langle \psi_{mn}^{pq} | \hat{V}_{ee} | \psi^{(0,0)} \rangle}{E^{(0,0)} - E_{mn}^{pq}} \left\{ \frac{\langle \psi^{(0,0)} | \hat{h}_{SO} | \psi_m^r \rangle \langle \psi_m^r | \hat{V}_{ee} | \psi_{mn}^{pq} \rangle}{E^{(0,0)} - E_m^r} + \frac{\langle \psi^{(0,0)} | \hat{h}_{SO} | \psi_n^r \rangle \langle \psi_n^r | \hat{V}_{ee} | \psi_{mn}^{pq} \rangle}{E^{(0,0)} - E_n^r} \right\} \right] \\
& + 2 \sum_{n < m} \sum_o^{\text{occ}} \sum_{q < p}^{\text{virt}} \mathcal{R} \left[ \frac{\langle \psi_{mn}^{pq} | \hat{V}_{ee} | \psi^{(0,0)} \rangle}{E^{(0,0)} - E_{mn}^{pq}} \left\{ \frac{\langle \psi^{(0,0)} | \hat{h}_{SO} | \psi_o^p \rangle \langle \psi_o^p | \hat{V}_{ee} | \psi_{mn}^{pq} \rangle}{E^{(0,0)} - E_o^p} + \frac{\langle \psi^{(0,0)} | \hat{h}_{SO} | \psi_o^q \rangle \langle \psi_o^q | \hat{V}_{ee} | \psi_{mn}^{pq} \rangle}{E^{(0,0)} - E_o^q} \right\} \right] \\
& + \sum_{n < m} \sum_{q < p}^{\text{occ virt}} \sum_r'' \frac{\langle \psi^{(0,0)} | \hat{V}_{ee} | \psi_{mn}^{pq} \rangle \langle \psi_{mn}^{pq} | \hat{h}_{SO} | \psi_m^r \rangle \langle \psi_m^r | \hat{V}_{ee} | \psi^{(0,0)} \rangle}{(E^{(0,0)} - E_{mn}^{pq}) (E^{(0,0)} - E_m^r)} + \sum_{n < m} \sum_{r < p < q}^{\text{occ virt}} \frac{\langle \psi^{(0,0)} | \hat{V}_{ee} | \psi_{mn}^{pq} \rangle \langle \psi_{mn}^{pq} | \hat{h}_{SO} | \psi_m^r \rangle \langle \psi_m^r | \hat{V}_{ee} | \psi^{(0,0)} \rangle}{(E^{(0,0)} - E_{mn}^{pq}) (E^{(0,0)} - E_m^r)} \\
& + \sum_{n < m} \sum_o^{\text{occ}} \sum_{q < p}^{\text{virt}} \frac{\langle \psi^{(0,0)} | \hat{V}_{ee} | \psi_{mn}^{pq} \rangle \langle \psi_{mn}^{pq} | \hat{h}_{SO} | \psi_o^p \rangle \langle \psi_o^p | \hat{V}_{ee} | \psi^{(0,0)} \rangle}{(E^{(0,0)} - E_{mn}^{pq}) (E^{(0,0)} - E_o^p)} + \sum_{o < n < m} \sum_{q < p}^{\text{virt}} \frac{\langle \psi^{(0,0)} | \hat{V}_{ee} | \psi_{mn}^{pq} \rangle \langle \psi_{mn}^{pq} | \hat{h}_{SO} | \psi_o^q \rangle \langle \psi_o^q | \hat{V}_{ee} | \psi^{(0,0)} \rangle}{(E^{(0,0)} - E_{mn}^{pq}) (E^{(0,0)} - E_o^q)} \quad (\text{C2})
\end{aligned}$$

where the symbol // on the sum over  $r$ , for example, indicates that it extends to all  $r$  that satisfy  $r \neq p, q$ . Eval-

uating all the integrals in Eq. (C2) gives:

$$\begin{aligned}
E^{(1,2)} = & \frac{1}{2} \sum_{m \neq n}^{\text{occ}} \sum_{p \neq q}^{\text{virt}} \mathcal{R} \left[ \frac{(pm|qn) - (qm|pn)}{\epsilon_m + \epsilon_n - \epsilon_p - \epsilon_q} \left\{ \frac{\langle m | \hat{h}_{SO} | p \rangle \left( [(nq|pp) - (np|pq)] - [(nq|mm) - (nm|m_q)] \right)}{\epsilon_m - \epsilon_p} \right. \right. \\
& + \frac{\langle n | \hat{h}_{SO} | p \rangle \left( [(mq|nn) - (mn|nq)] - [(mq|pp) - (mp|pq)] \right)}{\epsilon_n - \epsilon_p} \\
& + \frac{\langle m | \hat{h}_{SO} | q \rangle \left( [(np|mm) - (nm|mp)] - [(np|qq) - (nq|qp)] \right)}{\epsilon_m - \epsilon_q} + \frac{\langle n | \hat{h}_{SO} | q \rangle \left( [(mp|qq) - (mq|qp)] - [(mp|nn) - (mn|np)] \right)}{\epsilon_n - \epsilon_q} \left. \right\} \\
& + \frac{1}{2} \sum_{m \neq n}^{\text{occ}} \sum_{p \neq q}^{\text{virt}} \sum_r'' \mathcal{R} \left[ \frac{(pm|qn) - (qm|pn)}{\epsilon_m + \epsilon_n - \epsilon_p - \epsilon_q} \left\{ \frac{\langle m | \hat{h}_{SO} | r \rangle [(rp|nq) - (rq|np)]}{\epsilon_m - \epsilon_r} + \frac{\langle n | \hat{h}_{SO} | r \rangle [(mp|rq) - (mq|rp)]}{\epsilon_n - \epsilon_r} \right\} \right] \\
& + \frac{1}{2} \sum_{m \neq n}^{\text{occ}} \sum_o^{\text{occ}} \sum_{p \neq q}^{\text{virt}} \mathcal{R} \left[ \frac{(pm|qn) - (qm|pn)}{\epsilon_m + \epsilon_n - \epsilon_p - \epsilon_q} \left\{ \frac{\langle o | \hat{h}_{SO} | p \rangle [(no|mq) - (nq|mo)]}{\epsilon_o - \epsilon_p} + \frac{\langle o | \hat{h}_{SO} | q \rangle [(mo|np) - (mp|no)]}{\epsilon_o - \epsilon_q} \right\} \right] \\
& + \frac{1}{4} \sum_{m \neq n}^{\text{occ}} \sum_{p \neq r \neq q}^{\text{virt}} \frac{[(mp|nq) - (mq|np)] \langle p | \hat{h}_{SO} | r \rangle [(rm|qn) - (qm|rn)]}{(\epsilon_m + \epsilon_n - \epsilon_p - \epsilon_q) (\epsilon_m + \epsilon_n - \epsilon_r - \epsilon_q)} \\
& + \frac{1}{4} \sum_{m \neq n}^{\text{occ}} \sum_{p \neq q \neq r}^{\text{virt}} \frac{[(mp|nq) - (mq|np)] \langle q | \hat{h}_{SO} | r \rangle [(pm|rn) - (rm|pn)]}{(\epsilon_m + \epsilon_n - \epsilon_p - \epsilon_q) (\epsilon_m + \epsilon_n - \epsilon_p - \epsilon_r)} \\
& - \frac{1}{4} \sum_{m \neq o \neq n}^{\text{occ}} \sum_{p \neq q}^{\text{virt}} \frac{[(mp|nq) - (mq|np)] \langle o | \hat{h}_{SO} | m \rangle [(po|qn) - (qo|pn)]}{(\epsilon_m + \epsilon_n - \epsilon_p - \epsilon_q) (\epsilon_o + \epsilon_n - \epsilon_p - \epsilon_q)} \\
& - \frac{1}{4} \sum_{m \neq n \neq o}^{\text{occ}} \sum_{p \neq q}^{\text{virt}} \frac{[(mp|nq) - (mq|np)] \langle o | \hat{h}_{SO} | n \rangle [(pm|qo) - (qm|po)]}{(\epsilon_m + \epsilon_n - \epsilon_p - \epsilon_q) (\epsilon_m + \epsilon_o - \epsilon_p - \epsilon_q)}. \quad (\text{C3})
\end{aligned}$$

The double and triple sums in Eq. (C3) can now be extended to all values, by realizing that the cases  $m = n$ ,  $p = q$ ,  $o = m$ ,  $o = n$ ,  $r = p$ , and  $r = q$  have vanishing

contributions to  $E^{(1,2)}$ :

$$\begin{aligned}
E^{(1,2)} = & \frac{1}{2} \sum_{mn}^{occ} \sum_{pq}^{virt} \mathcal{R} \left[ \frac{(pm|qn) - (qm|pn)}{\epsilon_m + \epsilon_n - \epsilon_p - \epsilon_q} \left\{ \frac{\langle m|\hat{h}_{SO}|p\rangle \left( [(nq|pp) - (np|pq)] - [(nq|mm) - (nm|mq)] \right)}{\epsilon_m - \epsilon_p} \right. \right. \\
& + \frac{\langle n|\hat{h}_{SO}|p\rangle \left( [(mq|nn) - (mn|nq)] - [(mq|pp) - (mp|pq)] \right)}{\epsilon_n - \epsilon_p} \\
& + \left. \left. \frac{\langle m|\hat{h}_{SO}|q\rangle \left( [(np|mm) - (nm|mp)] - [(np|qq) - (nq|qp)] \right)}{\epsilon_m - \epsilon_q} + \frac{\langle n|\hat{h}_{SO}|q\rangle \left( [(mp|qq) - (mq|qp)] - [(mp|nn) - (mn|np)] \right)}{\epsilon_n - \epsilon_q} \right\} \right] \\
& + \frac{1}{2} \sum_{mn}^{occ} \sum_{pq}^{virt} \sum_r^{virt} \mathcal{R} \left[ \frac{(pm|qn) - (qm|pn)}{\epsilon_m + \epsilon_n - \epsilon_p - \epsilon_q} \left\{ \frac{\langle m|\hat{h}_{SO}|r\rangle [(rp|nq) - (rq|np)]}{\epsilon_m - \epsilon_r} + \frac{\langle n|\hat{h}_{SO}|r\rangle [(mp|rq) - (mq|rp)]}{\epsilon_n - \epsilon_r} \right\} \right] \\
& + \frac{1}{2} \sum_{mn}^{occ} \sum_o^{occ} \sum_{pq}^{virt} \mathcal{R} \left[ \frac{(pm|qn) - (qm|pn)}{\epsilon_m + \epsilon_n - \epsilon_p - \epsilon_q} \left\{ \frac{\langle o|\hat{h}_{SO}|p\rangle [(no|mq) - (nq|mo)]}{\epsilon_o - \epsilon_p} + \frac{\langle o|\hat{h}_{SO}|q\rangle [(mo|np) - (mp|no)]}{\epsilon_o - \epsilon_q} \right\} \right] \\
& + \frac{1}{4} \sum_{mn}^{occ} \sum_{prq}^{virt} \frac{[(mp|nq) - (mq|np)] \langle p|\hat{h}_{SO}|r\rangle [(rm|qn) - (qm|rn)]}{(\epsilon_m + \epsilon_n - \epsilon_p - \epsilon_q)(\epsilon_m + \epsilon_n - \epsilon_r - \epsilon_q)} \\
& + \frac{1}{4} \sum_{mn}^{occ} \sum_{pqr}^{virt} \frac{[(mp|nq) - (mq|np)] \langle q|\hat{h}_{SO}|r\rangle [(pm|rn) - (rm|pn)]}{(\epsilon_m + \epsilon_n - \epsilon_p - \epsilon_q)(\epsilon_m + \epsilon_n - \epsilon_p - \epsilon_r)} \\
& - \frac{1}{4} \sum_{mon}^{occ} \sum_{pq}^{virt} \frac{[(mp|nq) - (mq|np)] \langle o|\hat{h}_{SO}|m\rangle [(po|qn) - (qo|pn)]}{(\epsilon_m + \epsilon_n - \epsilon_p - \epsilon_q)(\epsilon_o + \epsilon_n - \epsilon_p - \epsilon_q)} \\
& - \frac{1}{4} \sum_{mno}^{occ} \sum_{pq}^{virt} \frac{[(mp|nq) - (mq|np)] \langle o|\hat{h}_{SO}|n\rangle [(pm|qo) - (qm|po)]}{(\epsilon_m + \epsilon_n - \epsilon_p - \epsilon_q)(\epsilon_m + \epsilon_o - \epsilon_p - \epsilon_q)}. \tag{C4}
\end{aligned}$$

Then, we use  $(pm|qn) - (qm|pn) = -[(pn|qm) - (qn|pm)]$  and interchange the indices  $m$  and  $n$  in: (i) the second line, (ii) the second term in the third line, (iii) the second term in the fourth line, and (iv) the last line of Eq. (C4), which yields:

$$\begin{aligned}
E^{(1,2)} = & \sum_{mn}^{occ} \sum_{pq}^{virt} \mathcal{R} \left[ \frac{(pm|qn) - (qm|pn)}{\epsilon_m + \epsilon_n - \epsilon_p - \epsilon_q} \left\{ \frac{\langle m|\hat{h}_{SO}|p\rangle \left( [(nq|pp) - (np|pq)] - [(nq|mm) - (nm|mq)] \right)}{\epsilon_m - \epsilon_p} \right. \right. \\
& + \left. \left. \frac{\langle m|\hat{h}_{SO}|q\rangle \left( [(np|mm) - (nm|mp)] - [(np|qq) - (nq|qp)] \right)}{\epsilon_m - \epsilon_q} \right\} \right] \\
& + \sum_{mn}^{occ} \sum_{pq}^{virt} \sum_r^{virt} \mathcal{R} \left[ \frac{(pm|qn) - (qm|pn)}{\epsilon_m + \epsilon_n - \epsilon_p - \epsilon_q} \left\{ \frac{\langle m|\hat{h}_{SO}|r\rangle [(rp|nq) - (rq|np)]}{\epsilon_m - \epsilon_r} \right\} \right] \\
& + \frac{1}{2} \sum_{mn}^{occ} \sum_o^{occ} \sum_{pq}^{virt} \mathcal{R} \left[ \frac{(pm|qn) - (qm|pn)}{\epsilon_m + \epsilon_n - \epsilon_p - \epsilon_q} \left\{ \frac{\langle o|\hat{h}_{SO}|p\rangle [(no|mq) - (nq|mo)]}{\epsilon_o - \epsilon_p} + \frac{\langle o|\hat{h}_{SO}|q\rangle [(mo|np) - (mp|no)]}{\epsilon_o - \epsilon_q} \right\} \right] \\
& + \frac{1}{4} \sum_{mn}^{occ} \sum_{prq}^{virt} \frac{[(mp|nq) - (mq|np)] \langle p|\hat{h}_{SO}|r\rangle [(rm|qn) - (qm|rn)]}{(\epsilon_m + \epsilon_n - \epsilon_p - \epsilon_q)(\epsilon_m + \epsilon_n - \epsilon_r - \epsilon_q)} \\
& + \frac{1}{4} \sum_{mn}^{occ} \sum_{pqr}^{virt} \frac{[(mp|nq) - (mq|np)] \langle q|\hat{h}_{SO}|r\rangle [(pm|rn) - (rm|pn)]}{(\epsilon_m + \epsilon_n - \epsilon_p - \epsilon_q)(\epsilon_m + \epsilon_n - \epsilon_p - \epsilon_r)} \\
& - \frac{1}{2} \sum_{mno}^{occ} \sum_{pq}^{virt} \frac{[(mp|nq) - (mq|np)] \langle o|\hat{h}_{SO}|m\rangle [(po|qn) - (qo|pn)]}{(\epsilon_m + \epsilon_n - \epsilon_p - \epsilon_q)(\epsilon_o + \epsilon_n - \epsilon_p - \epsilon_q)}. \tag{C5}
\end{aligned}$$

Similarly, in Eq. C5, we interchange the indices  $p$  and  $q$  in (i) line 2, (ii) second term in line 4, and (iii) the

next-to-last line to obtain:

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$$\begin{aligned}
E^{(1,2)} &= 2 \sum_{mn}^{occ} \sum_{pq}^{virt} \mathcal{R} \left[ \frac{(pm|qn) - (qm|pn)}{\epsilon_m + \epsilon_n - \epsilon_p - \epsilon_q} \left\{ \frac{\langle m|\hat{h}_{SO}|p\rangle \left( [(nq|pp) - (np|pq)] - [(nq|mm) - (nm|mq)] \right)}{\epsilon_m - \epsilon_p} \right\} \right] \\
&+ \sum_{mn}^{occ} \sum_{pq}^{virt} \sum_r'' \mathcal{R} \left[ \frac{(pm|qn) - (qm|pn)}{\epsilon_m + \epsilon_n - \epsilon_p - \epsilon_q} \left\{ \frac{\langle m|\hat{h}_{SO}|r\rangle [(rp|nq) - (rq|np)]}{\epsilon_m - \epsilon_r} \right\} \right] \\
&+ \sum_{mn}^{occ} \sum_o^{occ} \sum_{pq}^{virt} \mathcal{R} \left[ \frac{(pm|qn) - (qm|pn)}{\epsilon_m + \epsilon_n - \epsilon_p - \epsilon_q} \left\{ \frac{\langle o|\hat{h}_{SO}|p\rangle [(no|mq) - (nq|mo)]}{\epsilon_o - \epsilon_p} \right\} \right] \\
&+ \frac{1}{2} \sum_{mn}^{occ} \sum_{pqr}^{virt} \frac{[(mp|nq) - (mq|np)] \langle p|\hat{h}_{SO}|r\rangle [(rm|qn) - (qm|rn)]}{(\epsilon_m + \epsilon_n - \epsilon_p - \epsilon_q) (\epsilon_m + \epsilon_n - \epsilon_r - \epsilon_q)} \\
&- \frac{1}{2} \sum_{mno}^{occ} \sum_{pq}^{virt} \frac{[(mp|nq) - (mq|np)] \langle o|\hat{h}_{SO}|m\rangle [(po|qn) - (qo|pn)]}{(\epsilon_m + \epsilon_n - \epsilon_p - \epsilon_q) (\epsilon_o + \epsilon_n - \epsilon_p - \epsilon_q)}. \tag{C6}
\end{aligned}$$


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Finally, we recognize that the first line of Eq. (C6) is equivalent to a combination of the cases  $r = p$  and  $r = q$  in the second line, as well as the cases  $o = m$  and  $o = n$  in the third line, so that Eq. (C6) reduces to Eq. (29).