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Abstract

We describe the fundamental groups of ordered and unordered $k-$point sets in $\mathbb{C}^n$ generating an affine subspace of fixed dimension.

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complex space, configuration spaces,
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1 Introduction

Let M be a manifold and Σ_k be the symmetric group on k elements. The *ordered* and *unordered configuration spaces* of k distinct points in M, $\mathcal{F}_k(M) = \{(x_1, \ldots, x_k) \in M^k | x_i \neq x_j,\ i \neq j\}$ and $\mathcal{C}_k(M) = \mathcal{F}_k(M)/\Sigma_k$, have been widely studied. It is well known that for a simply connected manifold M of dimension ≥ 3, the *pure braid group* $\pi_1(\mathcal{F}_k(M))$ is trivial and the *braid group* $\pi_1(\mathcal{C}_k(M))$ is isomorphic to Σ_k, while in low dimensions there are non trivial pure braids. For example, (see [F]) the pure braid group of the plane $\mathcal{PB}_n$ has the following presentation

$$\mathcal{PB}_n = \pi_1(\mathcal{F}_n(\mathbb{C})) \cong \left\langle \alpha_{ij},\ 1 \leq i < j \leq n \,\middle|\, (YB\,3)_n, (YB\,4)_n \right\rangle,$$

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where $(YB\,3)_n$ and $(YB\,4)_n$ are the Yang-Baxter relations:

$$(YB\,3)_n\colon \quad \alpha_{ij}\alpha_{ik}\alpha_{jk} = \alpha_{ik}\alpha_{jk}\alpha_{ij} = \alpha_{jk}\alpha_{ij}\alpha_{ik},\ 1 \le i < j < k \le n,$$
$$(YB\,4)_n\colon \quad [\alpha_{kl}, \alpha_{ij}] = [\alpha_{il}, \alpha_{jk}] = [\alpha_{jl}, \alpha_{jk}^{-1}\alpha_{ik}\alpha_{jk}] = [\alpha_{jl}, \alpha_{kl}\alpha_{ik}\alpha_{kl}^{-1}] = 1,$$
$$1 \le i < j < k < l \le n,$$

while the braid group of the plane $\mathcal{B}_n$ has the well known presentation (see [A])

$$\mathcal{B}_n = \pi_1(\mathcal{C}_n(\mathbb{C})) \cong \big\langle \sigma_i,\ 1 \le i \le n-1 \mid (A)_n \big\rangle,$$

where $(A)_n$ are the classical Artin relations:

$$(A)_n : \sigma_i\sigma_j = \sigma_j\sigma_i,\ 1 \le i < j \le n-1,\ j-i \ge 2,$$
$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1},\ 1 \le i < n-1.$$

Other interesting examples are the pure braid group and the braid group of the sphere $S^2 \approx \mathbb{C}P^1$ with presentations (see [B2] and [F])

$$\pi_1(\mathcal{F}_n(\mathbb{C}P^1)) \cong \big\langle \alpha_{ij}, 1 \le i < j \le n-1 \,\big|\, (YB\,3)_{n-1}, (YB\,4)_{n-1}, D_{n-1}^2 = 1 \big\rangle$$
$$\pi_1(\mathcal{C}_n(\mathbb{C}P^1)) \cong \big\langle \sigma_i, 1 \le i \le n-1 \,\big|\, (A)_n,\ \sigma_1\sigma_2 \ldots \sigma_{n-1}^2 \ldots \sigma_2\sigma_1 = 1 \big\rangle,$$

where $D_k = \alpha_{12}(\alpha_{13}\alpha_{23})(\alpha_{14}\alpha_{24}\alpha_{34}) \cdots (\alpha_{1k}\alpha_{2k} \cdots \alpha_{k-1\ k})$.
The inclusion morphisms $\mathcal{PB}_n \to \mathcal{B}_n$ are given by (see [B2])

$$\alpha_{ij} \mapsto \sigma_{j-1}\sigma_{j-2} \ldots \sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1} \ldots \sigma_{j-1}^{-1}$$

and due to these inclusions, we can identify the pure braid D_n with Δ_n^2, the square of the fundamental Garside braid ([G]). In a recent paper ([BS]) Berceanu and the second author introduced new configuration spaces. They stratify the classical configuration spaces $\mathcal{F}_k(\mathbb{C}P^n)$ (resp. $\mathcal{C}_k(\mathbb{C}P^n)$) with complex submanifolds $\mathcal{F}_k^i(\mathbb{C}P^n)$ (resp. $\mathcal{C}_k^i(\mathbb{C}P^n)$) defined as the ordered (resp. unordered) configuration spaces of all k points in $\mathbb{C}P^n$ generating a projective subspace of dimension i. Then they compute the fundamental groups $\pi_1(\mathcal{F}_k^i(\mathbb{C}P^n))$ and $\pi_1(\mathcal{C}_k^i(\mathbb{C}P^n))$, proving that the former are trivial and the latter are isomorphic to Σ_k except when $i = 1$ providing, in this last case, a presentation for both $\pi_1(\mathcal{F}_k^1(\mathbb{C}P^n))$ and $\pi_1(\mathcal{C}_k^1(\mathbb{C}P^n))$ similar to those of the braid groups of the sphere. In this paper we apply the same technique to the affine case, i.e. to $\mathcal{F}_k(\mathbb{C}^n)$ and $\mathcal{C}_k(\mathbb{C}^n)$, showing that the situation is similar except in one case. More precisely we prove that, if $\mathcal{F}_k^{i,n} = \mathcal{F}_k^i(\mathbb{C}^n)$ and $\mathcal{C}_k^{i,n} = \mathcal{C}_k^i(\mathbb{C}^n)$ denote, respectively, the ordered and unordered configuration spaces of all k points in $\mathbb{C}^n$ generating an affine subspace of dimension i, then the following theorem holds:

Theorem 1.1. *The spaces* $\mathcal{F}_k^{i,n}$ *are simply connected except for* $i = 1$ *or* $i = n = k - 1$*. In these cases*

1. $\pi_1(\mathcal{F}_k^{1,1}) = \mathcal{PB}_k$,
2. $\pi_1(\mathcal{F}_k^{1,n}) = \mathcal{PB}_k / < D_k >$ *when* $n > 1$,
3. $\pi_1(\mathcal{F}_{n+1}^{n,n}) = \mathbb{Z}$ *for all* $n \geq 1$.

The fundamental group of $\mathcal{C}_k^{i,n}$ *is isomorphic to the symmetric group* Σ_k *except for* $i = 1$ *or* $i = n = k - 1$*. In these cases:*

1. $\pi_1(\mathcal{C}_k^{1,1}) = \mathcal{B}_k$,
2. $\pi_1(\mathcal{C}_k^{1,n}) = \mathcal{B}_k / < \Delta_k^2 >$ *when* $n > 1$,
3. $\pi_1(\mathcal{C}_{n+1}^{n,n}) = \mathcal{B}_{n+1} / < {\sigma_1}^2 = {\sigma_2}^2 = \cdots = {\sigma_n}^2 >$ *for all* $n \geq 1$.

Our paper begins by defining a geometric fibration that connects the spaces $\mathcal{F}_k^{i,n}$ to the affine grasmannian manifolds $Graff^i(\mathbb{C}^n)$. In Section 3 we compute the fundamental groups for two special cases: points on a line $\mathcal{F}_k^{1,n}$ and points in general position $\mathcal{F}_k^{k-1,n}$. Then, in Section 4, we describe an open cover of $\mathcal{F}_k^{n,n}$ and, using a Van-Kampen argument, we prove the main result for the ordered configuration spaces. In Section 5 we prove the main result for the unordered configuration spaces.

2 Geometric fibrations on the affine grassmannian manifold

We consider $\mathbb{C}^n$ with its affine structure. If $p_1, \ldots, p_k \in \mathbb{C}^n$ we write $< p_1, \ldots, p_k >$ for the affine subspace generated by $p_1, \ldots, p_k$. We stratify the configuration spaces $\mathcal{F}_k(\mathbb{C}^n)$ with complex submanifolds as follows:

$$\mathcal{F}_k(\mathbb{C}^n) = \coprod_{i=0}^{n} \mathcal{F}_k^{i,n} ,$$

where $\mathcal{F}_k^{i,n}$ is the ordered configuration space of all k distinct points $p_1, \ldots, p_k$ in $\mathbb{C}^n$ such that the dimension $\dim < p_1, \ldots, p_k >= i$.

Remark 2.1. *The following easy facts hold:*

1. $\mathcal{F}_k^{i,n} \neq \emptyset$ *if and only if* $i \leq \min(k+1, n)$*; so, in order to get a non empty set,* $i = 0$ *forces* $k = 1$*, and* $\mathcal{F}_1^{0,n} = \mathbb{C}^n$.
2. $\mathcal{F}_k^{1,1} = \mathcal{F}_k(\mathbb{C}),\ \mathcal{F}_2^{1,n} = \mathcal{F}_2(\mathbb{C}^n)$*;*
3. *the adjacency of the strata is given by*

$$\overline{\mathcal{F}_k^{i,n}} = \mathcal{F}_k^{1,n} \coprod \cdots \coprod \mathcal{F}_k^{i,n}.$$

By the above remark, it follows that the case $k = 1$ is trivial, so from now on we will consider $k > 1$ (and hence $i > 0$).

For $i \leq n$, let $Graff^i(\mathbb{C}^n)$ be the affine grassmannian manifold parametrizing i-dimensional affine subspaces of $\mathbb{C}^n$.
We recall that the map $Graff^i(\mathbb{C}^n) \to Gr^i(\mathbb{C}^n)$ which sends an affine subspace to its direction, exibits $Graff^i(\mathbb{C}^n)$ as a vector bundle over the ordinary grassmannian manifold $Gr^i(\mathbb{C}^n)$ with fiber of dimension $n - i$. Hence, $\dim Graff^i(\mathbb{C}^n) = (i+1)(n-i)$ and it has the same homotopy groups as $Gr^i(\mathbb{C}^n)$. In particular, affine grassmannian manifolds are simply connnected and $\pi_2(Graff^i(\mathbb{C}^n)) \cong \mathbb{Z}$ if $i < n$ (and trivial if $i = n$). We can also identify a generator for $\pi_2(Graff^i(\mathbb{C}^n))$ given by the map

$$g : (D^2, S^1) \to (Graff^i(\mathbb{C}^n), L_1), \quad g(z) = L_z$$

where L_z is the linear subspace of $\mathbb{C}^n$ given by the equations

$$(1 - |z|)X_1 - zX_2 = X_{i+2} = \cdots = X_n = 0 \ .$$

Affine grasmannian manifolds are related to the spaces $\mathcal{F}_k^{i,n}$ through the following fibrations.

Proposition 2.2. *The projection*

$$\gamma : \mathcal{F}_k^{i,n} \to Graff^i(\mathbb{C}^n)$$

given by

$$(x_1, \ldots, x_k) \mapsto < x_1, x_2, \ldots, x_k >$$

is a locally trivial fibration with fiber $\mathcal{F}_k^{i,i}$.

Proof. Take $V_0 \in Graff^i(\mathbb{C}^n)$ and choose $L_0 \in Gr^{n-i}(\mathbb{C}^n)$ such that L_0 intersects V_0 in one point and define $\mathcal{U}_{L_0}$, an open neighborhood of V_0, by

$$\mathcal{U}_{L_0} = \{V \in Graff^i(\mathbb{C}^n)|\ L_0 \text{ intersects } V \text{ in one point}\}.$$

For $V \in \mathcal{U}_{L_0}$, define the affine isomorphism

$$\varphi_V : V \to V_0,\ \varphi_V(x) = (L_0 + x) \cap V_0.$$

The local trivialization is given by the homeomorphism

$$f : \gamma^{-1}(\mathcal{U}_{L_0}) \to \mathcal{U}_{L_0} \times \mathcal{F}_k^{i,i}(V_0)$$
$$y = (y_1, \ldots, y_k) \mapsto \big(\gamma(y), (\varphi_{\gamma(y)}(y_1), \ldots, \varphi_{\gamma(y)}(y_k))\big)$$

making the following diagram commute (where $\mathcal{F}_k^{i,i}(V_0) = \mathcal{F}_k^{i,i}$ upon choosing a coordinate system in V_0)

$$\begin{array}{ccc} \gamma^{-1}(\mathcal{U}_{L_0}) & \xrightarrow{\ f\ } & \mathcal{U}_{L_0} \times \mathcal{F}_k^{i,i} \\ & {\scriptstyle \gamma} \searrow \quad \swarrow {\scriptstyle pr_1} & \\ & \mathcal{U}_{L_0} & \end{array}$$

□

Corollary 2.3. *The complex dimensions of the strata are given by*

$$\dim(\mathcal{F}_k^{i,n}) = \dim(\mathcal{F}_k^{i,i}) + \dim(Graff^i(\mathbb{C}^n)) = ki + (i+1)(n-i).$$

Proof. $\mathcal{F}_k^{i,i}$ is a Zariski open subset in $(\mathbb{C}^i)^k$ for $k \geq i+1$. □

The canonical embedding

$$\mathbb{C}^m \longrightarrow \mathbb{C}^n, \quad \{z_0, \ldots, z_m\} \mapsto \{z_0, \ldots, z_m, 0, \ldots, 0\}$$

induces, for $i \leq m$, the following commutative diagram of fibrations

$$\begin{array}{ccccc} \mathcal{F}_k^{i,i} & \longrightarrow & \mathcal{F}_k^{i,m} & \longrightarrow & Graff^i(\mathbb{C}^m) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{F}_k^{i,i} & \longrightarrow & \mathcal{F}_k^{i,n} & \longrightarrow & Graff^i(\mathbb{C}^n) \end{array}$$

which gives rise, for $i < m$, to the commutative diagram of homotopy groups

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & \mathbb{Z} & \longrightarrow & \pi_1(\mathcal{F}_k^{i,i}) & \longrightarrow & \pi_1(\mathcal{F}_k^{i,m}) & \longrightarrow & 1 \\ & & \downarrow \cong & & \downarrow \cong & & \downarrow & & \\ \cdots & \longrightarrow & \mathbb{Z} & \longrightarrow & \pi_1(\mathcal{F}_k^{i,i}) & \longrightarrow & \pi_1(\mathcal{F}_k^{i,n}) & \longrightarrow & 1 \end{array}$$

where the leftmost and central vertical homomorphisms are isomorphisms. Then, also the rightmost vertical homomorphisms are isomorphisms, and we have

$$\pi_1(\mathcal{F}_k^{i,n}) \cong \pi_1(\mathcal{F}_k^{i,m}) \cong \pi_1(\mathcal{F}_k^{i,i+1}) \text{ for } i < m \leq n. \tag{1}$$

Thus, in order to compute $\pi_1(\mathcal{F}_k^{i,n})$ we can restrict to the case $k \geq n$ (note that $k > i$), computing the fundamental groups $\pi_1(\mathcal{F}_k^{i,i+1})$, and for this we can use the homotopy exact sequence of the fibration from Proposition 2.2, which leads us to compute the fundamental groups $\pi_1(\mathcal{F}_k^{i,i})$. This is equivalent, simplifying notations, to compute $\pi_1(\mathcal{F}_k^{n,n})$ when $k \geq n+1$.
We begin by studying two special cases, points on a line and points in general position.

3 Special cases

The case $i = 1$, points on a line.

By remark 2.1 the space $\mathcal{F}_k^{1,1} = \mathcal{F}_k(\mathbb{C})$ for all $k \geq 2$ and the fibration in Proposition 2.2 gives rise to the exact sequence

$$\mathbb{Z} = \pi_2(Graff^1(\mathbb{C}^2)) \xrightarrow{\delta_*} \mathcal{PB}_n = \pi_1(\mathcal{F}_k(\mathbb{C})) \to \pi_1(\mathcal{F}_k^{1,2}) \to 1\ . \tag{2}$$

It follows that $\pi_1(\mathcal{F}_k^{1,2}) \cong \mathcal{PB}_n/\mathrm{Im}\delta_*$. Since $\pi_2(Graff^1(\mathbb{C}^2)) = \mathbb{Z}$, we need to know the image of a generator of this group in $\mathcal{PB}_n$. Taking as generator the map

$$g : (D^2, S^1) \to (Graff^1(\mathbb{C}^2), L_1), \quad g(z) = L_z,$$

where L_z is the line of equation $(1 - |z|)X_1 = zX_2$, we chose the lifting

$$\tilde{g} : (D^2, S^1) \to (\mathcal{F}_k^{1,2}, \mathcal{F}_k(L_1))$$

$$\tilde{g}(z) = \big((z, 1-|z|), 2(z, 1-|z|), \ldots, k(z, 1-|z|)\big)$$

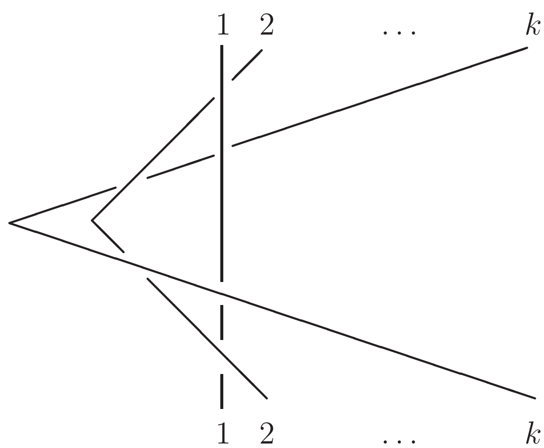
whose restriction to S^1 gives the map

$$\gamma : S^1 \longrightarrow \mathcal{F}_k(L_1) = \mathcal{F}_k(\mathbb{C})$$

$$\gamma(z) = ((z, 0), (2z, 0), \ldots, (kz, 0))$$

Lemma 3.1. *(see [BS]) The homotopy class of the map γ corresponds to the following pure braid in $\pi_1(\mathcal{F}_k(\mathbb{C}))$:*

$$[\gamma] = \alpha_{12}(\alpha_{13}\alpha_{23}) \ldots (\alpha_{1k}\alpha_{2k} \ldots \alpha_{k-1,k}) = D_k \ .$$



From the above Lemma and the exact sequence in (2) we get that the image in $\pi_1(\mathcal{F}_k(\mathbb{C}))$ of the generator of $\pi_2(Graff^1(\mathbb{C}^2))$ is D_k and the following theorem is proved.

Theorem 3.2. *For $n > 1$, the fundamental group of the configuration space of k distinct points in $\mathbb{C}^n$ lying on a line has the following presentation (not depending on n)*

$$\pi_1(\mathcal{F}_k^{1,n}) = \big\langle \alpha_{ij},\ 1 \le i < j \le k \ \big|\ (YB3)_k, (YB4)_k,\ D_k = 1 \big\rangle \ .$$

The case $k = i+1$, points in general position.

Lemma 3.3. *For $1 < k \leq n+1$, the projection*

$$p : \mathcal{F}_k^{k-1,n} \longrightarrow \mathcal{F}_{k-1}^{k-2,n}, \quad (x_1, \ldots, x_k) \mapsto (x_1, \ldots, x_{k-1})$$

is a locally trivial fibration with fiber $\mathbb{C}^n \setminus \mathbb{C}^{k-2}$

Proof. Take $(x_1^0, \ldots, x_{k-1}^0) \in \mathcal{F}_{k-1}^{k-2,n}$ and fix $x_k^0, \ldots, x_{n+1}^0 \in \mathbb{C}^n$ such that $< x_1^0, \ldots, x_{n+1}^0 >= \mathbb{C}^n$ (that is $< x_k^0, \ldots, x_{n+1}^0 >$ and $< x_1^0, \ldots, x_{k-1}^0 >$ are skew subspaces). Define the open neighbourhood $\mathcal{U}$ of $(x_1^0, \ldots, x_{k-1}^0)$ by

$$\mathcal{U} = \{(x_1, \ldots, x_{k-1}) \in \mathcal{F}_{k-1}^{k-2,n} | \ < x_1, \ldots, x_{k-1}, x_k^0, \ldots, x_{n+1}^0 >= \mathbb{C}^n\}.$$

For $(x_1, \ldots, x_{k-1}) \in \mathcal{U}$, there exists a unique affine isomorphism $T_{(x_1,\ldots,x_{k-1})}$: $\mathbb{C}^n \longrightarrow \mathbb{C}^n$, which depends continuously on $(x_1, \ldots, x_{k-1})$, such that

$$T_{(x_1,\ldots,x_{k-1})}(x_i^0) = (x_i) \text{ for } i = 1, \ldots, k-1$$

and

$$T_{(x_1,\ldots,x_{k-1})}(x_i^0) = (x_i^0) \text{ for } i = k, \ldots, n+1 \ .$$

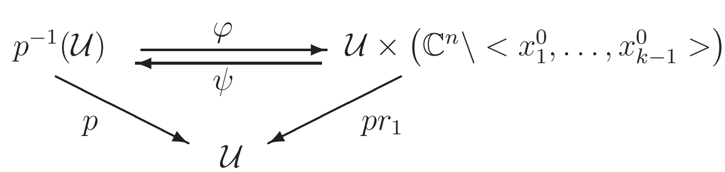
We can define the homeomorphisms φ, ψ by :

$$p^{-1}(\mathcal{U}) \underset{\psi}{\overset{\varphi}{\rightleftarrows}} \mathcal{U} \times \left(\mathbb{C}^n \setminus < x_1^0, \ldots, x_{k-1}^0 > \right)$$

$$\varphi(x_1, \ldots, x_{k-1}, x) = ((x_1, \ldots, x_{k-1}), T_{(x_1,\ldots,x_{k-1})}^{-1}(x))$$

$$\psi((x_1, \ldots, x_{k-1}), y) = (x_1, \ldots, x_{k-1}, T_{(x_1,\ldots,x_{k-1})}(y))$$

satisfying $pr_1 \circ \varphi = p$.



□

As $\mathbb{C}^n \setminus \mathbb{C}^{k-2}$ is simply connected when $n > k-1$ and $k > 1$, we have

$$\pi_1(\mathcal{F}_k^{k-1,n}) \cong \pi_1(\mathcal{F}_{k-1}^{k-2,n}) \cong \pi_1(\mathcal{F}_2^{1,n}) = \pi_1(\mathcal{F}_2(\mathbb{C}^n)) \cong \pi_1(\mathcal{F}_1^{0,n}) = \pi_1(\mathbb{C}^n) = 0,$$

in particular $\pi_1(\mathcal{F}_n^{n-1,n}) = 0$. Moreover, since $\mathbb{C}^n \setminus \mathbb{C}^{k-2}$ is homotopically equivalent to an odd dimensional (real) sphere $S^{2(n-k)-1}$, its second homotopy group vanish and we have

$$\pi_2(\mathcal{F}_{k+1}^{k,n}) \cong \pi_2(\mathcal{F}_k^{k-1,n}) \cong \pi_2(\mathcal{F}_1^{0,n}) = \pi_2(\mathbb{C}^n) = 0.$$

in particular $\pi_2(\mathcal{F}_n^{n-1,n}) = 0$.
In the case $k = n+1$, $\mathbb{C}^n \setminus \mathbb{C}^{n-1}$ is homotopically equivalent to $\mathbb{C}^*$, and we obtain the exact sequence:

$$\pi_2(\mathcal{F}_n^{n-1,n}) \to \mathbb{Z} \to \pi_1(\mathcal{F}_{n+1}^{n,n}) \to \pi_1(\mathcal{F}_n^{n-1,n}) \to 0.$$

By the above remarks, the leftmost and rightmost groups are trivial, so we have that $\pi_1(\mathcal{F}_{n+1}^{n,n})$ is infinite cyclic.
We have proven the following

Theorem 3.4. *For $n \geq 1$, the configuration space of k distinct points in $\mathbb{C}^n$ in general position $\mathcal{F}_k^{k-1,n}$ is simply connected except for $k = n+1$ in which case $\pi_1(\mathcal{F}_{n+1}^{n,n}) = \mathbb{Z}$.*

We can also identify a generator for $\pi_1(\mathcal{F}_{n+1}^{n,n})$ via the map

$$h : S^1 \to \mathcal{F}_{n+1}^{n,n} \quad h(z) = (0, e_1, \ldots e_{n-1}, ze_n), \tag{3}$$

where $e_1, \ldots e_n$ is the canonical basis for $\mathbb{C}^n$ (i.e. a loop that goes around the hyperplane $< 0, e_1, \ldots e_{n-1} >$).

4 The general case

From now on we will consider $n, i > 1$.
Let us recall that, by Proposition 2.2 and equation (1), in order to compute the fundamental group of the general case $\mathcal{F}_k^{i,n}$, we need to compute $\pi_1(\mathcal{F}_k^{n,n})$ when $k \geq n+1$. To do this, we will cover $\mathcal{F}_k^{n,n}$ by open sets with an infinite cyclic fundamental group and then we will apply the Van-Kampen theorem to them.

4.1 A *good* cover

Let $\mathcal{A} = (A_1, \ldots, A_p)$ be a sequence of subsets of $\{1, \ldots, k\}$ and the integers $d_1, \ldots, d_p$ given by $d_j = |A_j| - 1,\ \ j = 1, \ldots, p$. Let us define

$$\mathcal{F}_k^{\mathcal{A},n} = \{(x_1, \ldots, x_k) \in \mathcal{F}_k(\mathbb{C}^n) \big| \dim < x_i >_{i \in A_j} = d_j \text{ for } j = 1, \ldots, p\}.$$

Example 4.1. *The following easy facts hold:*

1. *If* $\mathcal{A} = \{A_1\},\ A_1 = \{1, \ldots, k\}$, *then* $\mathcal{F}_k^{\mathcal{A},n} = \mathcal{F}_k^{k-1,n}$;
2. *if all* A_i *have cardinality* $|A_i| \leq 2$, *then* $\mathcal{F}_k^{\mathcal{A},n} = \mathcal{F}_k(\mathbb{C}^n)$;
3. *if* $p \geq 2$ *and* $|A_p| \leq 2$, *then* $\mathcal{F}_k^{(A_1,\ldots,A_p),n} = \mathcal{F}_k^{(A_1,\ldots,A_{p-1}),n}$;
4. *if* $p \geq 2$ *and* $A_p \subseteq A_1$, *then* $\mathcal{F}_k^{(A_1,\ldots,A_p),n} = \mathcal{F}_k^{(A_1,\ldots,A_{p-1}),n}$;
5. $\bigcup_{j \geq i} \mathcal{F}_k^{j,n} = \bigcup_{\mathcal{A}=\{A\}, A \in \binom{\{1,\ldots,k\}}{i+1}} \mathcal{F}_k^{\mathcal{A},n}$.

Lemma 4.2. *For* $A = \{1, \ldots, j+1\}$, $j \leq n$, *and* $k > j$ *the map*

$$P_A : \mathcal{F}_k^{(A),n} \to \mathcal{F}_{j+1}^{j,n},\ \ (x_1, \ldots, x_k) \mapsto (x_1, \ldots, x_{j+1})$$

is a locally trivial fibration with fiber $\mathcal{F}_{k-j-1}(\mathbb{C}^n \setminus \{0, e_1, \ldots, e_j\})$.

Proof. Fix $(x_1, \ldots, x_{j+1}) \in \mathcal{F}_{j+1}^{j,n}$ and choose $z_{j+2}, \ldots, z_{n+1} \in \mathbb{C}^n$ such that $< x_1, \ldots, x_{j+1}, z_{j+2}, \ldots, z_{n+1} >= \mathbb{C}^n$.
Define the neighborhood $\mathcal{U}$ of $(x_1, \ldots, x_{j+1})$ by

$$\mathcal{U} = \{(y_1, \ldots, y_{j+1}) \in \mathcal{F}_{j+1}^{j,n} |\ < y_1, \ldots, y_{j+1}, z_{j+2}, \ldots, z_{n+1} >= \mathbb{C}^n\}\ .$$

There exists a unique affine isomorphism $F_y : \mathbb{C}^n \to \mathbb{C}^n$, which depends continuously on $y = (y_1, \ldots, y_{j+1})$, such that

$$\begin{array}{l} F_y(x_i) = y_i,\ \ i = 1, \ldots, j+1 \\ F_y(z_i) = z_i,\ \ i = j+2, \ldots, n+1 \end{array}$$

and this gives a local trivialization

$$f : P_A^{-1}(\mathcal{U}) \to \mathcal{U} \times \mathcal{F}_{k-j-1}(\mathbb{C}^n \setminus \{x_1, \ldots, x_{j+1}\})$$

$$(y_1, \ldots, y_k) \mapsto \big((y_1, \ldots, y_{j+1}), F_y^{-1}(y_{j+2}), \ldots, F_y^{-1}(y_k)\big)$$

which satisfies $pr_1 \circ f = P_A$. □

Let us remark that P_A is the identity map if $k = j+1$ and the fibration is (globally) trivial if $j = n$ since $\mathcal{U} = \mathcal{F}^{n,n}_{n+1}$; in this last case $\pi_1(\mathcal{F}^{(A),n}_k) = \mathbb{Z}$ (recall that we are considering $n > 1$).
Let $\mathcal{A} = (A_1, \ldots, A_p)$ be a p-uple of subsets of cardinalities $|A_j| = d_j + 1$, $j = 1, \ldots, p$. For any given integer $h \in \{1, \ldots, k\}$, we define a new p-uple $\mathcal{A}' = (A'_1, \ldots, A'_p)$ and integers $d'_1, \ldots, d'_p$ as follows:

$$A'_j = \begin{cases} A_j, \text{ if } h \notin A_j \\ A_j \setminus \{h\}, \text{ if } h \in A_j \end{cases} \quad , \quad d'_j = \begin{cases} d_j, \text{ if } h \notin A_j \\ d_j - 1, \text{ if } h \in A_j \end{cases} .$$

The following Lemma holds.

Lemma 4.3. *The map*

$$p_h : \mathcal{F}^{\mathcal{A},n}_k \to \mathcal{F}^{\mathcal{A}',n}_{k-1}, \ (x_1, \ldots, x_k) \mapsto (x_1, \ldots, \widehat{x_h}, \ldots, x_k)$$

has local sections with path-connected fibers.

Proof. Let us suppose that $h = k$ and $k \in (A_1 \cap \ldots \cap A_l) \setminus (A_{l+1} \cup \ldots \cup A_p)$. Then the fiber of the map $p_k : \mathcal{F}^{\mathcal{A},n}_k \to \mathcal{F}^{\mathcal{A}',n}_{k-1}$ is

$$p_k^{-1}(x_1, \ldots, x_{k-1}) \approx \mathbb{C}^n \setminus \big(L'_1 \cup \ldots \cup L'_l \cup \{x_1, \ldots, x_{k-1}\}\big)$$

where $L'_j =< x_i >_{i \in A'_j}$. Even in the case when $\dim L_j = n$, we have $\dim L'_j < n$, hence the fiber is path-connected and nonempty. Fix a base point $x = (x_1, \ldots, x_{k-1}) \in \mathcal{F}^{\mathcal{A}',n}_{k-1}$ and choose $x_k \in \mathbb{C}^n \setminus (L'_1 \cup \ldots \cup L'_l \cup \{x_1, \ldots, x_{k-1}\})$. There are neighborhoods $W_j \subset Graff^{d'_j}(\mathbb{C}^n)$ of L'_j $(j = 1, \ldots, l)$ such that $x_k \notin L''_j$ if $L''_j \in W_j$; we take a constant local section

$$s : W = g^{-1}\big((\mathbb{C}^n \setminus \{x_k\})^{k-1} \times \prod_{i=1}^{l} W_i\big) \to \mathcal{F}^{\mathcal{A},n}_k$$

$$(y_1, \ldots, y_{k-1}) \mapsto (y_1, \ldots, y_{k-1}, x_k),$$

where the continuous map g is given by:

$$g : \mathcal{F}^{\mathcal{A}',n}_{k-1} \to (\mathbb{C}^n)^{k-1} \times Graff^{d'_1}(\mathbb{C}^n) \times \ldots \times Graff^{d'_l}(\mathbb{C}^n)$$

$$(y_1, \ldots, y_{k-1}) \mapsto (y_1, \ldots, y_{k-1}, L''_1, \ldots, L''_l),$$

and $L''_j =< y_i >_{i \in A'_j}$ for $j = 1, \ldots, l$. $\square$

Proposition 4.4. *The space $\mathcal{F}_k^{\mathcal{A},n}$ is path-connected.*

Proof. Use induction on p and $d_1 + d_2 + \ldots + d_p$. If $p = 1$, use Lemma 4.2 and the space $\mathcal{F}_{j+1}^{j,n}$ which is path-connected. If A_p is not included in A_1 and $d_p \geq 3$, delete a point in $A_p \setminus A_1$ and use Lemma 4.3 and the fact that if C is not empty and path-connected and $p : B \to C$ is a surjective continuous map with local sections such that $p^{-1}(y)$ is path-connected for all $y \in C$, then B is path-connected (see [BS]). If $A_p \subset A_1$ or $d_p \leq 2$, use Example 4.1, (3) and (4). □

Let $e_1, \ldots, e_n$ be the canonical basis of $\mathbb{C}^n$ and

$$M_h = \{(x_1, \ldots, x_h) \in \mathcal{F}_h(\mathbb{C}^n \setminus \{0, e_1, \ldots, e_n\}) |\ x_1 \notin < e_1, \ldots, e_n >\},$$

the following Lemma holds.

Lemma 4.5. *The map*

$$p_h : M_h \to (\mathbb{C}^n)^* \setminus < e_1, \ldots, e_n >$$

sending $(x_1, \ldots, x_h) \mapsto x_1$, is a locally trivial fibration with fiber $\mathcal{F}_{h-1}(\mathbb{C}^n \setminus \{0, e_1, \ldots, e_n, e_1 + \cdots + e_n\})$.

Proof. Let $G : B^m \to \mathbb{R}^m$ be the homeomorphism from the open unit m-ball to $\mathbb{R}^m$ given by $G(x) = \frac{x}{1-|x|}$, (whose inverse is the map $G^{-1}(y) = \frac{y}{1+|y|}$). For $x \in B^m$ let $\tilde{G}_x = G^{-1} \circ \tau_{-G(x)} \circ G$ be an homeomorphism of B^m, where $\tau_v : \mathbb{R}^n \to \mathbb{R}^n$ is the translation by v. $\tilde{G}_x$ sends x to 0 and can be extended to a homeomorphism of the closure $\overline{B^m}$, by requiring it to be the identity on the $m-1$-sphere (the exact formula for $\tilde{G}_x(y)$ is $\frac{(1-|x|)y-(1-|y|)x}{(1-|x|)(1-|y|)+|(1-|x|)y-(1-|y|)x|}$). We can further extend it to an homomorphism G_x of $\mathbb{R}^m$ by setting $G_x(y) = y$ if $|y| > 1$. Notice that G_x depends continuously on x.
Let $\bar{x} \in (\mathbb{C}^n)^* \setminus < e_1, \ldots, e_n >$, fix an open complex ball B in $(\mathbb{C}^n)^* \setminus < e_1, \ldots, e_n >$ centered at $\bar{x}$ and an affine isomorphism H of $\mathbb{C}^n$ sending B to the open real $2n$-ball B^{2n}. For $x \in B$, define the homeomorphism F_x of $\mathbb{C}^n$ $F_x = H^{-1} \circ G_{H(x)} \circ H$ which sends x to $\bar{x}$, is the identity outside of B and depends continuously on x. The result follows from the continuous map

$$F : p_h^{-1}(B) \to B \times p_h^{-1}(\bar{x})$$

$$F(x, x_2, \ldots, x_h) = (x, (\bar{x}, F_x(x_2), \ldots, F_x(x_h)))$$

(whose inverse is the map $F^{-1} : B \times p_h^{-1}(\bar{x}) \to p_h^{-1}(B)$, $F^{-1}(x, (\bar{x}, x_2, \ldots, x_h)) = (x, F_x^{-1}(x_2), \ldots, F_x^{-1}(x_h)))$.
The fiber $p_h^{-1}(\bar{x})$ is homeomorphic to $\mathcal{F}_{h-1}(\mathbb{C}^n \setminus \{0, e_1, \ldots, e_n, e_1 + \cdots + e_n\})$ via an homeomorphism of $\mathbb{C}^n$ which fixes $0, e_1, \ldots, e_n$ and sends $\bar{x}$ to the sum $e_1 + \ldots + e_n$. □

Thus we have, since $n \geq 2$, $\pi_1(M_h) = \mathbb{Z}$, and we can choose as generator the map $S^1 \to M_h$ sending $z \mapsto (z(e_1 + \cdots + e_n), x_2, \ldots, x_h)$ with $x_2, \ldots, x_h$ of sufficient high norm (i.e. a loop that goes round the hyperplane $< e_1, \ldots, e_n >$).

Lemma 4.6. *For* $A = \{1, \ldots, n+1\}$, $B = \{2, \ldots, n+2\}$, *and* $k > n+1$ *the map*

$$P_{A,B} : \mathcal{F}_k^{(A,B),n} \to \mathcal{F}_{n+1}^{n,n}, \ \ (x_1, \ldots, x_k) \mapsto (x_1, \ldots, x_{n+1})$$

is a trivial fibration with fiber M_{k-n-1}

Proof. For $x = (x_1, \ldots, x_{n+1}) \in \mathcal{F}_{n+1}^{n,n}$ let F_x be the affine isomorphism of $\mathbb{C}^n$ such that $F_x(0) = x_1$, $F_x(e_i) = x_{i+1}$, for $i = 1, \ldots, n$. The map

$$\mathcal{F}_{n+1}^{n,n} \times M_{k-n-1} \to \mathcal{F}_k^{(A,B),n}$$

sending

$$((x_1, \ldots, x_{n+1}), (x_{n+2}, \ldots, x_k)) \mapsto (x_1, \ldots, x_{n+1}, F_x(x_{n+2}), \ldots, F_x(x_k))$$

gives the result. □

4.2 Computation of the fundamental group

From Lemma 4.6 it follows that $\pi_1(\mathcal{F}_k^{(A,B),n}) = \mathbb{Z} \times \mathbb{Z}$ and that it has two generators: $((z+1)(e_1 + \ldots + e_n), e_1, \ldots, e_n, e_1 + \ldots + e_n, x_{n+3}, \ldots, x_k)$ and $(0, e_1, \ldots, e_n, z(e_1 + \ldots + e_n), x_{n+3}, \ldots, x_k)$, where $x_{n+3}, \ldots, x_k$ are chosen *far enough* to be different from the first $n+2$ points. The first generator is the one coming from the base, the second is the one from the fiber of the fibration $P_{A,B}$.
Note that using the map

$$P'_{A,B} : \mathcal{F}_k^{(A,B),n} \to \mathcal{F}_{n+1}^{n,n}, \ \ (x_1, \ldots, x_k) \mapsto (x_2, \ldots, x_{n+2})$$

we obtain the same result and the generator coming from the base here is the one coming from the fiber above and vice versa.

The map $P_{A,B}$ factors through the inclusion $i_A : \mathcal{F}_k^{(A,B),n} \hookrightarrow \mathcal{F}_k^{(A),n}$ followed by the map

$$P_A : \mathcal{F}_k^{(A),n} \to \mathcal{F}_{n+1}^{n,n}, \ \ (x_1, \dots, x_k) \mapsto (x_1, \dots, x_{n+1})$$

and we get the following commutative diagram of fundamental groups:

$$\begin{array}{ccccc} \pi_1(\mathcal{F}_k^{(A,B),n}) & & \xrightarrow{\ P_{A,B*}\ } & & \pi_1(\mathcal{F}_{n+1}^{n,n}) \\ & {\scriptstyle i_{A*}} \searrow & & \nearrow {\scriptstyle P_{A*}} & \\ & & \pi_1(\mathcal{F}_k^{(A),n}) & & \end{array} .$$

Since P_A induces an isomorphism on the fundamental groups, this means that i_{A*} sends the generator of $\pi_1(\mathcal{F}_k^{(A,B),n})$ coming from the fiber to 0 in $\pi_1(\mathcal{F}_{n+1}^{n,n})$. That is, the generator of $\pi_1(\mathcal{F}_k^{(B),n})$ (which is homotopically equivalent to the generator of $\pi_1(\mathcal{F}_k^{(A,B),n})$ coming from the fiber) is trivial in $\pi_1(\mathcal{F}_k^{(A),n})$ and (given the symmetry of the matter) vice versa.
Applying Van Kampen theorem, we have that $\mathcal{F}_k^{(A),n} \cup \mathcal{F}_k^{(B),n}$ is simply connected. Moreover the intersection of any number of $\mathcal{F}_k^{(A),n}$'s is path connected and the same is true for the intersection of two unions of $\mathcal{F}_k^{(A),n}$'s since the intersection $\bigcap_{A \in \binom{\{1,\dots,k\}}{n+1}} \mathcal{F}_k^{(A),n}$ is not empty.

From the last example in 4.1 with $i = n$ we have $\mathcal{F}_k^{n,n} = \bigcup_{A \in \binom{\{1,\dots,k\}}{n+1}} \mathcal{F}_k^{(A),n}$, and when $k > n+1$, we can cover it with a finite number of simply connected open sets with path connected intersections, so it is simply connected by the following

Lemma 4.7. *Let X be a topological space which has a finite open cover $U_1, \dots, U_n$ such that each U_i is simply connected, $U_i \cap U_j$ is connected for all $i, j = 1, \dots, n$ and $\bigcap_{i=1}^n U_i \neq \emptyset$. Then X is simply connected.*

Proof. By induction, let's suppose $\bigcup_{i=1}^{k-1} U_i$ is simply connected. Then, applying Van Kampen theorem to U_k and $\bigcup_{i=1}^{k-1} U_i$, we get that $\bigcup_{i=1}^{k} U_i$ is simply connected if $U_k \cap (\bigcup_{i=1}^{k-1} U_i)$ is connected. But $U_k \cap (\bigcup_{i=1}^{k-1} U_i) = \bigcup_{i=1}^{k-1} (U_k \cap U_i)$ is the union of connected sets with non empty intersection, and therefore is connected. □

Now, using the fibration in Proposition 2.2 with $n = i+1$, we obtain that $\mathcal{F}_k^{n-1,n}$ is simply connected when $k > n$.
Summing up the results for the oredered case, the following main theorem is proved

Theorem 4.8. *The spaces $\mathcal{F}_k^{i,n}$ are simply connected except*

1. $\pi_1(\mathcal{F}_k^{1,1}) = \mathcal{PB}_k$,
2. $\pi_1(\mathcal{F}_k^{1,n}) = \left\langle \alpha_{ij}\,,\, 1 \le i < j \le k \middle| (YB3)_k, (YB4)_k,\, D_k = 1 \right\rangle$ *when* $n > 1$,
3. $\pi_1(\mathcal{F}_{n+1}^{n,n}) = \mathbb{Z}$ *for all* $n \ge 1$*, with generator described in (3).*

5 The unordered case: $\mathcal{C}_k^{i,n}$

Let $\mathcal{C}_k^{i,n}$ be the unordered configuration space of all k distinct points $p_1, \ldots, p_k$ in $\mathbb{C}^n$ which generate an i-dimensional space. Then $\mathcal{C}_k^{i,n}$ is obtained quotienting $\mathcal{F}_k^{i,n}$ by the action of the symmetric group Σ_k. The map $p : \mathcal{F}_k^{i,n} \to \mathcal{C}_k^{i,n}$ is a regular covering with Σ_k as deck transformation group, so we have the exact sequence:

$$1 \to \pi_1(\mathcal{F}_k^{i,n}) \xrightarrow{p_*} \pi_1(\mathcal{C}_k^{i,n}) \xrightarrow{\tau} \Sigma_k \to 1$$

which gives immediately $\pi_1(\mathcal{C}_k^{i,n}) = \Sigma_k$ in case $\mathcal{F}_k^{i,n}$ is simply connected.
Observe that the fibration in Proposition 2.2 may be quotiented obtaining a locally trivial fibration $\mathcal{C}_k^{i,n} \to Graff^i(\mathbb{C}^n)$ with fiber $\mathcal{C}_k^{i,i}$.
This gives an exact sequence of homotopy groups which, together with the one from Proposition 2.2 and those coming from regular coverings, gives the following commutative diagram for $i < n$:

$$
\begin{array}{ccccccccc}
 & & & & 1 & & 1 & & \\
 & & & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & \mathbb{Z} & \xrightarrow{\delta_*} & \pi_1(\mathcal{F}_k^{i,i}) & \longrightarrow & \pi_1(\mathcal{F}_k^{i,n}) & \longrightarrow & 1 \\
 & & \downarrow \cong & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & \mathbb{Z} & \xrightarrow{\delta'_*} & \pi_1(\mathcal{C}_k^{i,i}) & \longrightarrow & \pi_1(\mathcal{C}_k^{i,n}) & \longrightarrow & 1 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \Sigma_k & \longrightarrow & \Sigma_k & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & &
\end{array}
$$

In case $i = 1$, $\mathcal{F}_k^{1,1} = \mathcal{F}_k(\mathbb{C})$ and $\mathcal{C}_k^{1,1} = \mathcal{C}_k(\mathbb{C})$, so $\pi_1(\mathcal{F}_k^{1,1}) = \mathcal{PB}_k$ and $\pi_1(\mathcal{C}_k^{1,1}) = \mathcal{B}_k$, and since $\mathrm{Im}\delta_* =< D_k >\subset \mathcal{PB}_k$, the left square gives $\mathrm{Im}\delta'_* =< \Delta_k^2 >\subset \mathcal{B}_k$, therefore $\pi_1(\mathcal{C}_k^{1,n}) = \mathcal{B}_k/ < \Delta_k^2 >$.

For $i = n = k-1$, we have $\pi_1(\mathcal{F}_{n+1}^{n,n}) = \mathbb{Z}$, and we can use the exact sequence of the regular covering $p : \mathcal{F}_{n+1}^{n,n} \to \mathcal{C}_{n+1}^{n,n}$ to get a presentation of $\pi_1(\mathcal{C}_{n+1}^{n,n})$.

Let's fix $Q = (0, e_1, \dots, e_n) \in \mathcal{F}_{n+1}^{n,n}$ and $p(Q) \in \mathcal{C}_{n+1}^{n,n}$ as base points and for $i = 1, \dots n$ define $\gamma_i : [0,\pi] \to \mathcal{F}_{n+1}^{n,n}$ to be the (open) path

$$\gamma_i(t) = (\frac{1}{2}(e^{i(t+\pi)} + 1)e_i, e_1, \dots, e_{i-1}, \frac{1}{2}(e^{it} + 1))e_i, e_{i+1} \dots, e_n)$$

(which fixes all entries except the first and the $(i+1)$-th and exchanges 0 and e_i by a half rotation in the line $< 0, e_i >$).
Then $p \circ \gamma_i$ is a closed path in $\mathcal{C}_{n+1}^{n,n}$ and we denote it's homotopy class in $\pi_1(\mathcal{C}_{n+1}^{n,n})$ by σ_i. Hence $\tau_i = \tau(\sigma_i)$ is the deck transformation corresponding to the transposition $(0, i)$ (we take Σ_{n+1} as acting on $\{0, 1, \dots, n\}$) and we get a set of generators for Σ_{n+1} satisfying the following relations

$$\tau_i^2 = \tau_i\tau_j\tau_i\tau_j^{-1}\tau_i^{-1}\tau_j^{-1} = 1 \text{ for } i, j = 1, \dots, n\,,$$

$$[\tau_1\tau_2 \cdots \tau_{i-1}\tau_i\tau_{i-1}^{-1} \cdots \tau_1^{-1}, \tau_1\tau_2 \cdots \tau_{j-1}\tau_j\tau_{j-1}^{-1} \cdots \tau_1^{-1}] = 1 \text{ for } |i-j| > 2\,.$$

If we take T, the (closed) path in $\mathcal{F}_{n+1}^{n,n}$ in which all entries are fixed except for one which goes round the hyperplane generated by the others counter-clockwise, as generator of $\pi_1(\mathcal{F}_{n+1}^{n,n})$, then $\pi_1(\mathcal{C}_{n+1}^{n,n})$ is generated by T and the

$\sigma_1, \ldots, \sigma_n$.
In order to get the relations, we must write the words σ_i^2, $\sigma_i\sigma_j\sigma_i\sigma_j^{-1}\sigma_i^{-1}\sigma_j^{-1}$ and $[\sigma_1\sigma_2\cdots\sigma_{i-1}\sigma_i\sigma_{i-1}^{-1}\cdots\sigma_1^{-1}, \sigma_1\sigma_2\cdots\sigma_{j-1}\sigma_j\sigma_{j-1}^{-1}\cdots\sigma_1^{-1}]$ as well as $\sigma_i T\sigma_i^{-1}$ as elements of Ker $\tau =$ Im p_* for all appropriate i, j.
Observe that the path $\gamma_i' : [\pi, 2\pi] \to \mathcal{F}_{n+1}^{n,n}$, defined by the same formula as γ_i, is a lifting of σ_i with starting point $(e_i, e_1, e_2, \ldots, e_{i-1}, 0, e_{i-1}, \ldots, e_n)$ and that $\gamma_i\gamma_i'$ is a closed path in $\mathcal{F}_{n+1}^{n,n}$ which is the generator T of $\pi_1(\mathcal{F}_{n+1}^{n,n})$ (as you can see by the homotopy $(\frac{\epsilon}{2}(e^{i(t+\pi)}+1)e_i, e_1, \ldots, e_{i-1}, \frac{2-\epsilon}{2}(e^{it}+\frac{\epsilon}{2-\epsilon}))e_i, e_{i+1}\ldots, e_n)$, $\epsilon \in [0,1]$, where for $\epsilon = 0$ we have the point e_i going round the hyperplane $< 0, e_1, e_2, \ldots, e_{i-1}, e_{i+1}, \ldots, e_n >$ counterclockwise).
Thus we have $p_*(T) = \sigma_i^2$ for all $i = 1, \ldots, n$ (and that Imp_* is the center of $\pi_1(\mathcal{C}_{n+1}^{n,n})$).
Moreover, it's easy to see, by lifting to $\mathcal{F}_{n+1}^{n,n}$, that the σ_i satisfy the relations

$$\sigma_i\sigma_j\sigma_i\sigma_j^{-1}\sigma_i^{-1}\sigma_j^{-1} = 1 \text{ for } i, j = 1, \ldots, n$$

and

$$[\sigma_1\sigma_2\cdots\sigma_{i-1}\sigma_i\sigma_{i-1}^{-1}\cdots\sigma_1^{-1}, \sigma_1\sigma_2\cdots\sigma_{j-1}\sigma_j\sigma_{j-1}^{-1}\cdots\sigma_1^{-1}] = 1 \text{ for } |i-j| > 2 \ .$$

We can represent a lifting of $\sigma_i' = \sigma_1\sigma_2\cdots\sigma_{i-1}\sigma_i\sigma_{i-1}^{-1}\cdots\sigma_1^{-1}$ (which gives the deck transformation corresponding to the transposition $(i, i+1)$) by a path which fixes all entries except the i-th and the $(i+1)$-th and exchanges e_i and e_{i+1} by a half rotation in the line $< e_i, e_{i+1} >$.
We can now change the set of generators by first deleting T and introducing the relations

$$\sigma_1^2 = \sigma_2^2 = \cdots = \sigma_n^2$$

and then by choosing the σ_i''s instead of the σ_i's. Then we get that the generators σ_i''s satisfy the relations

$$\sigma_i'\sigma_{i+1}'\sigma_i' = \sigma_{i+1}'\sigma_i'\sigma_{i+1}' \text{ for } i = 1, \ldots, n-1,$$

$$[\sigma_i', \sigma_j'] = 1 \text{ for } |i-j| > 2$$

and

$${\sigma_1'}^2 = {\sigma_2'}^2 = \cdots = {\sigma_n'}^2. \tag{4}$$

Namely, $\pi_1(\mathcal{C}_{n+1}^{n,n})$ is the quotient of the braid group $\mathcal{B}_{n+1}$ on $n+1$ strings by relations (4) and the following main theorem is proved.

Theorem 5.1. *The fundamental groups $\pi_1(\mathcal{C}_k^{i,n})$ are isomorphic to the symmetric group Σ_k except*

1. $\pi_1(\mathcal{C}_k^{1,1}) = \mathcal{B}_k$,
2. $\pi_1(\mathcal{C}_k^{1,n}) = \mathcal{B}_k / < \Delta_k^2 >$ *when* $n > 1$,
3. $\pi_1(\mathcal{C}_{n+1}^{n,n}) = \mathcal{B}_{n+1} / < {\sigma_1}^2 = {\sigma_2}^2 = \cdots = {\sigma_n}^2 >$ *for all* $n \geq 1$.

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