AN ELABORATION OF THE LOTMAN'S SEMIOSPHERE THEORETICAL CONSTRUCT FOR MATHEMATICS EDUCATION:
analysis of the Chinese Mathematics Lesson Study Cultural Transposition within the Italian

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AN ELABORATION OF THE LOTMAN'S SEMIOSPHERE THEORETICAL CONSTRUCT FOR MATHEMATICS EDUCATION:
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INTRODUCTION

The main aim of this research is to show that the Lotman's Semiosphere is the right framework to analyse how the Chinese Lesson Study is effectively "culturally transposed" into the Italian School context, when Italian teachers are involved in its practices. To face this issue seven research questions are formulated (p. 2, 30, and 36) and then refined and condensed into a few basic ones (p. 83). To answer them, significant theoretical constructs are introduced (until p. 81), then some teaching experiments about the introduction of Lesson Study into Italian schools are described and some excerpts from them are analysed through the new proposed theoretical lens. The analysis corroborates the main aim of the research, and this issue is summarized in the *Conclusions*.

Premise: the enhancement of teachers' professional development is a cultural challenge

Gallimore (1996), looking back at sixty years of research in the United States, argues that changes in teaching and learning practices are a challenge. He attributes this resistance to change to the fact that "we are dealing with cultural issues", and not just psychological, pedagogical (*ibid*., p. 230) or disciplinary issues. Furthermore, as defined by Michael Eraut (1977, p. 10), teacher [professional] development is "that *natural process* of professional growth in which a teacher *gradually* acquires confidence, gains new perspectives, increases in knowledge, discovers new methods, and takes on new roles" (my emphasis in the text). A *natural* process: that is, a "semi-conscious" process, strongly conditioned by cultural and social aspects and policies (or "sociosystemic factors", in Jaworski's terminology (2004)), which characterise a context and are often taken for granted by people belonging to the same cultural sphere. It is precisely to these "sociosystemic factors" that Jaworski, by presenting precise paradigmatic examples, as far as she claims, attributes the complexity of mathematics teachers' teaching and professional development (for more details see Jaworski, 2004).

The existing literature sensitive to sociosystemic factors and to the cultural dimension in the field of mathematics teacher professional development is expanding. Three research studies are given here as examples; they are distinguished on the basis of what they pinpoint as aspects belonging to the cultural dimension. Guala and Boero (2017) refer to the cultural aspects of mathematics by circumscribing them as "epistemological, historical and anthropological [aspects] (p. 209)". Stylianides and Delaney in Part II of their volume (2011 – *Understanding the Cultural Context of Mathematical Knowledge in Teaching*) outline three cultural loci, i.e. the "different, but complementary, aspects of the cultural embedding of mathematical knowledge for/in teaching" (p. 182): (a) the national educational system, (b) the diverse teacher education programmes, (c) the different "economy" and political requirement (e.g. the designed tools used to audit to evaluate the knowledge). Jaworski (2004), instead, lists the following as situation-sensitive aspects: "physical conditions, authority structures, attitudes, teacher-pupil relationships, text books, examinations, and time" (p. 18). These three examples have been chosen without any claim to exhaustiveness, but the choice has fallen on these because they are paradigmatic examples of studies on academic courses for prospective teachers (Guala & Boero, 2017), of meta-analytical studies on existing research in Mathematics Education with respect to teacher professional development (Stylianides

& Delaney, 2011) and finally of studies on the professional development of in-service teachers (Jaworski, 2004). Three different areas of research but all related to the teacher professional development in mathematics, here presented in this order from the most recent to the oldest in a chronological sense.

Sensitivity to cultural aspects in Mathematics Education research, and in particular with respect to teacher professional development, is therefore growing, even if in all three of the strands described it remains unclear what is meant by the whole set of terms linked to the cultural dimension. There seems to be a need for a global framework in which concepts such as "cultural aspects", "sociosystemic factors", "cultural dimension", "culture" can be accommodated. It seems that a more operational definition of culture than those to be found in the current literature is needed. Besides this, and specifying that this study is intentionally limited to the field of professional development of mathematics teachers, my claim is that Lotman's Semiosphere could provide a global framework and help to provide less local answers to the following research questions:

RQ1. What methodology, what theoretical framework, could be used to analyse cultural conditions and constraints in mathematics teachers' professional development practices?

RQ2. How do cultural elements affect the professional development of mathematics teachers (as individuals and as a community)?

RQ3. How can the "culturally sensitive" understanding of teachers' critical reflection be improved?

The chosen theoretical framework for defining and studying the professional development of mathematics teachers is described below to begin to draw the boundaries of my research. Situated within the framework of Mezirow's Transformative Learning, indeed, teachers' professional development is defined as a research practice and a dynamic collaborative process in action, embedded in a cultural dimension. And since I believe in a political – i.e. critical and emancipatory – vision of Mathematics Education, Kemmis' Critical Refection is proposed as frame of reference for mathematics teacher education practices.

The Transformative Learning

Mezirow's Transformative Learning (well summarised, explicated and schematised in Kitchenham's review, 2008) is a theory of adult development, adult education, and adult learning. The author defines adult learning as a "complex and multifaceted" learning (*ibid*., p. 104) and, basing his studies on the importance of Kuhn's paradigms, on Freire's idea of conscientization and on Habermas' domains of learning, he comes to describe four possible learning processes:

1. *Elaborating existing frames of reference*. "The first learning process, learning within meaning schemes, involves learners working with what they already know by expanding on, complementing, and revising their present systems of knowledge" (*ibid*., p.111).
2. *Learning new frames of reference*. "The second learning process […] is learning new meaning schemes that are compatible with existing schemes within the learners' meaning perspectives" (p.112).

3. *Transforming habits of mind*. The third "is learning through meaning transformation. […], the learner encounters a problem or anomaly that cannot be resolved through present meaning schemes or through learning new meaning schemes; the resolution comes through a redefinition of the problem. Transformation occurs by critical self-reflection of the assumptions that supported the meaning scheme or perspective in use" (p. 112).
4. *Transforming points of view.* This latter learning process – which only came about following a revision of the theory in 2000 by Mezirow himself – occur "by trying on another's point of view" (Mezirow, 2000, p. 21, in Kitchenham, 2008, p. 118).

The latter learning process differs from the third in that people can change their points of view by sharing someone else's point of view. It is not possible, however, do that for someone else's habit of mind. To illustrate this, an example taken from a Lesson Study experience carried out in Piossasco (TO), in a Primary School, is presented. This extract of discourse takes place between Nicoletta, an expert teacher who has been collaborating for years with the Nucleo di Ricerca Didattica[1] (NdRD) of Genova, and her Lesson Study group.

> **Nicoletta**: [...] I was thinking about this stuff here [she points to the Bartolini Bussi and Ramploud (2018) book] that we are facing: it is a great opportunity that is beyond the specific characteristics of our school. I mean, **the difficulty I see**, which I think is everywhere, but I see it here because I work here, **is the fact of having the willingness to dedicate time**. Then, "**we don't have time!** We don't have time to plan everything". But yesterday morning I was talking to some colleagues who at 8 a.m. – I arrive at 7.30 a.m. to do things because it's convenient for me to arrive at 7.30 a.m. instead of taking my notebooks home, and then I never manage to do it because **one colleague comes and says: "ah but you did that thing with the straws?** Ah well, it's nice to do it like that, but **tell my colleague to do it too**. Why do you do it this way, or that way?..." **No.** We cannot talk to each other about work like that. [...] but **we have to think about it, we have to talk about it**. **It's not** that **in 30 seconds**... then I go down to the canteen with the children and **while I'm giving the parmesan cheese**: "So tell them how you did it...". I mean, I thought about it for three or four hours, I discussed it with him [Ezio] because it was a mathematical thing [...]. This also undermines our professionalism [...].
>
> Lesson Study Piossasco first meeting
> 16th January 2019, 4.45 p.m.

Teacher A (the colleague Nicoletta refers to) can share the point of view that straws can replace the mere algorithm of addition and subtraction. However, this does not mean that he has adopted Teacher B's (i.e. Nicoletta's and NdRD's) position on the role of artefacts in the classroom (habit of mind). Teacher B believes that the use of artefacts in the classroom, the awareness of their semiotic potential and the diting semiotic activity are fundamental to managing the richness of children's responses to tasks and to directing them towards the mathematical meaning-making.

[1] Nucleo di Ricerca Didattica (or Educational Research Team) are research groups in Mathematics Education composed of professional mathematicians (or researchers in Mathematics Education) and motivated teachers working together during the school year, in periodic meetings (2 or 3 per month), and existing since the beginning of the 1970s all over Italy. (For more details see Arzarello & Bartolini Bussi, 1998, pp. 246-247; see also here page 69).

Teacher A could duplicate the use of alternative artefacts (point of view), such as straws, he cannot try on teacher B's belief system underlying such use (habit of mind).
In this excerpt, in fact, it is noticeable how Nicoletta underlines that the mere description of the activity "with straws", a specific artefact used and studied by this NdRD, is not enough, but it is necessary to go into details and implement a transformation of meanings.

Coming back to Mezirow's classification, its peculiarity lies in the definition of the process, featured in the last two learning processes, that according to the author distinguishes deep learning: the transformation. Mezirow, in fact, based on the emancipatory learning domain of Habermas (1971), comes to describe what he calls the "perspective transformation":

> the emancipatory process of becoming critically aware of how and why the structure of psycho-cultural assumptions has come to constrain the way we see ourselves and our relationships, reconstituting this structure to permit a more inclusive and discriminating integration of experience and acting upon these new understandings (Mezirow, 1981, p. 6, in Kitchenham, 2008, p.109).

That is, an active and constructive critical reflection process. According to Mezirow, in fact, Critical Reflection is the central element of perspective transformation. "In other words, if a learner rationalised a new point of view without dealing with the deep feelings that accompanied the original meaning scheme or perspective, perspective transformation could not occur. […] if teachers did not reconcile the deep feelings or had points of view subjected on them, they would learn without questioning the veracity or utility of the information" (Kitchenham, 2008, pp.112-113). However, and here Mezirow explicitly echoes Vygotsky, in order for learning and meaning to be relevant for the learner, a phase of "critical discourse with others" (*ibid.*, p.113) is essential. The author distinguishes three types of reflection (*ibid.*, p.114):

1. *Content reflection* "involves thinking back to what was done and, therefore, might involve a transformation of a meaning scheme".
2. *Process reflection* "causes a person to consider the aetiology of actions and whether there are other factors yet to be unveiled; this form of reflection might also transform meaning schemes".
3. *Premise reflection* "requires the person to see the larger view of what is operating within his or her value system, for instance, and could transform a meaning perspective rather than a meaning scheme".

While the first two constitute what Mezirow names "straightforward reflection" or the act of "intentional assessment" of one's actions, the process of premise reflection is a critical reflection, i.e. it "not only involves the nature and consequence of one's actions but also includes the related circumstances of their origin" (*ibid.*, p.114).

The Critical Reflection

Mezirow's studies therefore seem to support claims of Peter Gates – "one thing we do not seem to learn from experience is how little we learn from our experiences" – and of T.S. Eliot – "one may

have the experience, but miss the meaning" – echoed in Jaworski (2006), who defines the reflection as a "the missing link between experiencing and learning from that experience" (p. 38). As evidence of this, Stephen Kemmis (1985) also defines reflection as "a process of transformation of the determinate "raw material" of our experiences (given by history and culture, and mediated through the situations in which we live) into determinate products (understandings, commitments, actions), a transformation effected by our determinate labor (our thinking about the relationship between thought and action, and the relationship between the individual and society), using determinate means of production (communication, decision-making and action)" (p. 148).
Therefore, the reflection is:

(a) action-oriented and historically embedded,
(b) a social process,
(c) a political process.

According to Kemmis, in fact, reflection is not "quiet and personal" but rather acquires meaning in relation to the (historical and social) context in which it is situated and, in order to fully understand the meaning, it is necessary to study how it orients subsequent action, its product: the "praxis (informed, committed action), the most eloquent and socially significant form of human action" (*ibid.*, p.141). Moreover, Kemmis, like Mezirow, quotes Habermas (1971) and affirms that, as a political process, the "reflection [in collaboration with others] and the fruits of reflection locate us in the historical struggle for human *emancipation*, whether implicitly or explicitly" (Kemmis, 1985, p. 147). For this reason, the author considers reflection as an emancipatory, research process that "must to be studied and analysed in action", in particular through the "*spiral of self-reflection*", consisting of cycles of: planning action (on the basis of reflection); implementing plans in action (praxis); observing or monitoring processes, conditions and consequences of action; and evaluating actions in the light of the collected evidence (returning to reflection) as a basis for replanning and further action (*ibid.*, p. 156). It is a research: collaborative, as it is based on communication with others, in the sense of Habermas (1984); critical and situated, as "we do not pause to reflect in a vacuum", but rather reflecting critically means activating a process of *meta-thinking* "(thinking about thinking) in which we consider the relationship between our thoughts and action *in a particular context*" (Kemmis, 1958, p. 141, my emphasis in the text).

Since teachers' professional development is a research practice and a dynamic collaborative process in action embedded in a cultural dimension, and learning is the development of a situated "critical discourse with others", how can it be studied? Semiotics, and especially Semiotic of Culture can propose itself as a comprehensive method of analysis. Indeed, texts are the object of study of semiotics. Over time, the concept of text has gradually been redefined, to the point of taking into consideration not only written representations, but "any carrier of integral ('textual') meaning" (Uspenskij *et al.*, 1998, p. 38), "read" in its own continuity and not as a discrete collection of signs (Lotman, 1975). The discourses of mathematics teachers are the texts analysed in this my research work. The proposal of semiotics as a framework of analysis is not innovative, but well established in research in Mathematics Education. The present work is situated in an ecological, systemic context to offer new theoretical and analytical tools

THE SEMIOTIC OF CULTURE

Semiotics of Culture is a research field within semiotics that attempts to define culture from semiotic perspective and as creation of signs and a way of giving meaning to everything around. The research field is of particular interest for the Tartu–Moscow Semiotic School, of which Lotman, Torop's predecessor, is the main exponent. Geertz, American, is instead the main exponent of interpretative anthropology. He reformulated the idea of the interpretation of cultures.

The contributions of Lotman and Geertz will be the innovative focus presented in my research since they lead the way to the idea of Semiosphere. In order to understand why Semiotics Culture can offer itself as an exhaustive method of analysis of the processes of reflection and professional development of mathematics teachers, it is worth going back over the historical and epistemological reasons that have led scholars to develop theoretical concepts and analytical tools suitable for a study of the critical reflection of mathematics teachers and their professional development from a semiotic point of view. An excursus is presented both of the influences of the scholars who founded and outlined Lotman's theoretical thought, and of the role and influence that semiotic studies have had and are having within research in Mathematics Education.

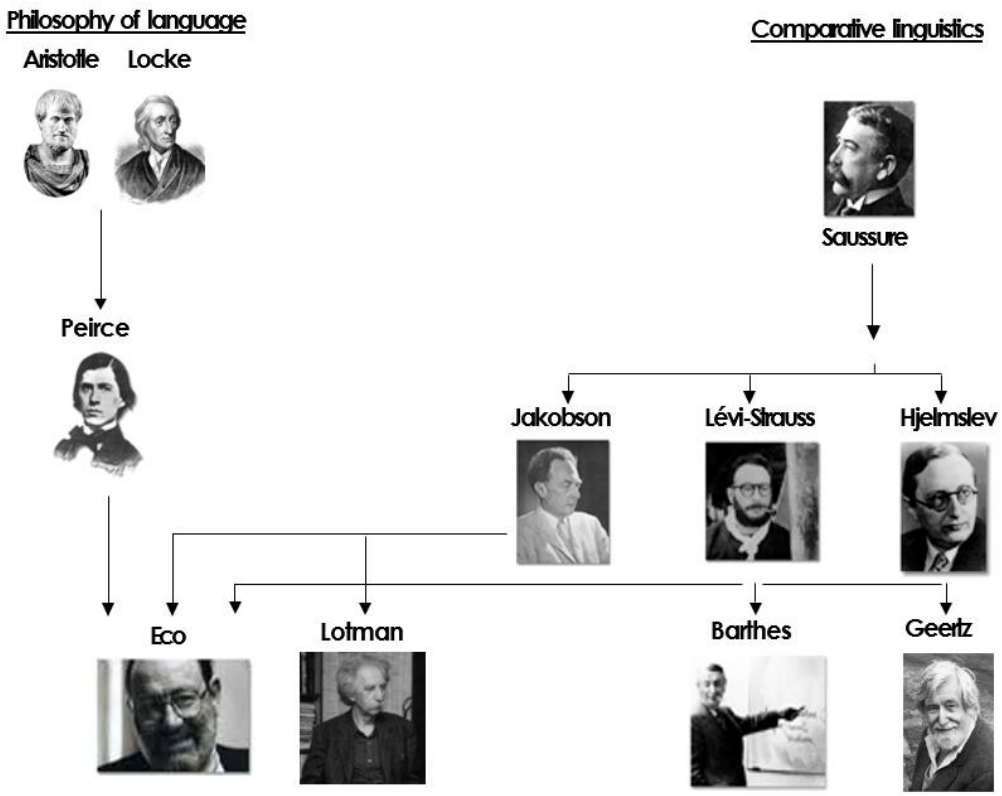


Figure 1. 1 The Semiotics of Culture' "genealogical tree".

A diachronic excursus from the origins of Semiotics to the Semiotics of Culture

Semiotic knowledge has its origins in Aristotle's *De Interpretatione* (Greek: *Περὶ Ἑρμηνείας*, *Peri Hermeneias*; English: *On Interpretation*), among the earliest philosophical works in the Western tradition to deal with the relationship between language and logic in a comprehensive, explicit, and formal way. The philosophy of language, with all its foundational path up to Locke in the 1600s, is therefore the semiotic knowledge's cornerstone. However, this is only one of the two taproots of the discipline's genealogical tree (see Figure 1. 1). Alongside this philosophical origin of the discipline, whose most important exponent is Peirce, there is also a linguistic origin, with Saussure.

Charles Sanders Peirce (1839-1914) – a mathematician, philosopher, and United States scholar is considered the founder of modern *semiotics*. His most significant contribution is the proposal of the *inferential model*, in opposition to nominalism and Cartesian intuitionism – i.e., the assumption that part of the knowledge of external reality is direct and immediate. Peirce (1992) rejects the assumption that there are indubitable and self-evident truths per se, that is, there is no knowledge that in order to be identified does not require other information already stored in the memory of the cognitive apparatus that must processes it. We have, according to Peirce, no ability to distinguish pure intuitions from mediated knowledge. If no knowledge is intuitive, then every act of cognition is mediated (Pisanty & Zijno, 2009). Mediation is enacted through *semiosis*, the object of study of semiotics: a process of continuous reformulation of the meanings of signs (Morris, 1938; 1946). For Peirce, in fact, a sign has three dimensions: a physical quality, the capacity to express an object and the capacity to necessarily produce another sign. So, semiosis is a process involving three factors: (a) a sign (the *representamen*), (b) an object, and (c) an *interpretant*, always co-present (Peirce, 1998). A knowledge based on the concept of a sign is a mediated knowledge, the only possible knowledge. It is therefore mediated by an inferential process that selects only certain properties of the external stimulus and formulates a perceptual judgment that occurs at an almost automatic level (perception = interpretation by inference = series of complex conceptual operations). Peirce's inferential model is a type of reasoning through which it is possible to derive a phenomenon from another phenomenon, that is, to make a hypothesis from an interpretation. He puts forward the idea that interpretation as both the action and the effect of interpreting. This, as expressed by Marcos-Marín (2018), means that interpretation can be active (action) or a result, i.e. passive (effect). When the speaker actively interprets, he is giving his own order to reality, whereas, when his interpretation is passive, he is receiving and accepting the order of reality that others have given.

Peirce (1992) therefore elaborates trichotomies; there can be no semiotics unless there is a three-way relationship. For Peirce a sign, or *representamen*, is something that stands to someone (*interpreter*, or his actual idea or thought) for something (object) under some reference or capacity (the capacities of expression and production seen above), generating a new sign (*interpretant*). The object, according to Peirce, is divided into two types: *dynamic* and *immediate*. The former is something real, the object that is at the origin of the representation but not that which is communicated, since it is necessarily outside the sign. The dynamic object is the object as it is in

reality, in its dynamism that determines its non-representability. The latter, the immediate object, is what is transmitted and is already part of the sign. Then the sign stands to someone, that is, it creates in that person's mind an equivalent or perhaps more developed sign, which is called the *interpretant* of the first sign. The sign is therefore the product of a logical process that starting from the recording of a perceptible event stimulates the inferential activity of an interpreter to arrive at the formulation of an explanatory hypothesis of the phenomenon in question.
The Peircean semiosis is therefore an infinite movement around the *dynamic object* (e.g., "the cylinder, a limit object that does not bear any of the imperfection that a material object will have (Presmeg, Radford, Roth, & Kadunz, 2016, p. 2)") that is accomplished through successive steps: each *immediate object* (new real, visible object/thing) becomes in turn a representamen, a sign, which through a new interpreter generates a new immediate object and so on. This *unlimited semiosis* is therefore a game of referral between one sign and another (to deepen see also Nöth, 1990). For Peirce, everything and every thought is a sign.

Contemporary to Peirce is the Swiss **Ferdinand de Saussure** (1857-1913), linguist, dissatisfied with the methods of comparative linguistics of the 19th century, and father of modern linguistics and *semiology*. He hopes for the foundation of a unified field, which considers language as a system of signs among others, but the most important of all:

> Language is a system of signs that express ideas, and is therefore comparable to a system of writing, the alphabet of deaf-mutes, symbolic rites, polite formulas, military signals, etc. But it is the most important of all these systems. *A science that studies the life of signs within society* is conceivable; it would be a part of social psychology and consequently of general psychology; I shall call it *semiology* (from Greek *sēmeîon* 'sign'). Semiology would show what constitutes signs, what laws govern them. Since the science does not yet exist, no one can say what it would be; but it has a right to existence, a place staked out in advance. Linguistics is only a part of the general science of semiology; the laws discovered by semiology will be applicable to linguistics, and the latter will circumscribe a well-defined area within the mass of anthropological facts (De Saussure, 1959, p. 16, my emphasis in the text).

In comparison to the Peircean inferential model, de Saussure elaborates the *model of equivalence*. To provide an insight into the difference between the two models, the distinction between the two terms used by the ancient Greeks to refer to natural signs (*sēmêia*) and linguistic signs, which Aristotle called symbols (*sýmbola*), can be a starting point. To the former refers the model of inference (if *p* than *q*), to the latter that of equivalence (A≡B). It is no coincidence that the word "symbol" originally referred to the two parts of a split medallion, used as proof of recognition of a previously established relationship (Manetti, 1987, in Pisanty & Zijno, 2009). Since one part presupposes the other, the expression "symbol", exactly as for "signs", takes on the meaning of "that which *stands for* something else". However, the fact that Aristotle treats language in terms of a symbol leads to think about a possible specificity of referral. In fact, in the case of the *sign*, the two terms of reference are not always reciprocal: a first term can refer to a second but not

necessarily vice versa. In the case of the *symbol*, on the other hand, the two terms are perfectly reciprocals.

Nevertheless, Saussure opposes the Aristotelian conception of an ordered world ready to be labeled[2]. According to the Swiss scholar, there are no concepts free from words. In his vision, before the appearance of language, thought is an indistinct mass. This mass is then defined by the acquisition of ever new words. Although we use signs to refer to things in the external world, according to Saussure, these utterances occur because we are able to fill them with meanings, even before referring them to these entities. In his opinion, this explains why we can give meaning to sentences whose referent is not tangibly present at that particular moment. Another aspect contrary to Aristotelian tradition is the fact that, for Saussure, words are not symbols that correspond to the objects of the world around us (the referents), but signs consisting of two parts – which in turn can be seen in symbolic relationship.

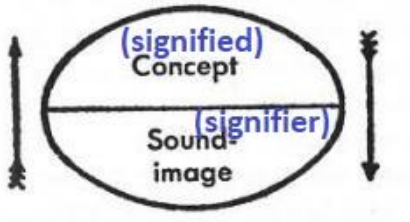


Figure 1. 2 A drawing to represent the sign for Saussure (De Saussure, 1959, p.66).

According to de Saussure the sign is a *two-sided psychic entity* (De Saussure, 1959, p.66; for more on the term *psychic* see also Roth, 2016b, in Presmeg, *et al*., 2016, p. 5). In fact, it is necessary to keep the word *sign* to indicate the total and replace the concept (what the word represents) and the acoustic image (material part of the word, visual or sound) with the words *signified* and *signifier* respectively (de Saussure, 1959). The signifier is the surface of the sign, regardless of the meaning it conveys. It is that part of the sign to which we refer when we say that "cylinder", for example, is an eight-letter word. The meaning, on the other hand, is the conceptual part of the sign and corresponds to the idea. The mental representation of the word "cylinder". These two elements, though opposites, presuppose each other (de Saussure, 1959, p. 65). In a way, Saussure denies the definition of sign as "something that stands for something else": the something (signifier) and the something else (signified) are two faces of the sign (Figure 1. 2), which appear simultaneously and

[2] In *De Interpretatione*, Aristotle describes the relationship between words and external objects on the basis of a third element: thoughts or "affections of the soul". The result is a three-term scheme (depicted with a semiotic triangle, a diagram of the process of signification). Words refer to things through thoughts. While the relationship between words and thoughts is conventional and unmotivated (which is why different languages express the same thoughts in different ways), states of mind are the same for all people, and are provoked by things in a motivated relationship. Barbarians do not speak and understand Greek because they have not learned the convention that associates words and thoughts. But, according to the Aristotelian hypothesis, both share the same ideas or thoughts. The logical organization of the world, in this way, precedes language (Pisanty & Zijno, 2009).

do not depend on the external object. That is – in de Saussure words – "I mean that it is unmotivated, i.e. arbitrary in that it actually has no natural connection with the signified (1959, p. 69)". Saussure's revolutionary move thus consists in severing the relationship between sign (*signifier + signified*) and the object of the external world (*referent*). Signs are possible due to the fact that before referring signifiers to objects of the external world we are able to attribute signified to them. This vertical referral existing within the sign (this spatial reference is used to refer to the drawing in Figure 1. 2), in the light of recent developments in the field of neuroscience, could be now assumed to involve the relationship between signifier and the neural network it activates, and for this I refer to the work of Rizzolatti, Fogassi and Gallese (2010) and Gallese and Guerra (2015) on *mirror neurons*, and to Jeff Hawkins' studies (2021) on brain theory and the neocortex.

Figure 1. 3 An internet-meme exemplifying the vertical reference existing within the sign leading up to the triggered neural network.

To better understand what this is all about, an explicatory example could be what happens when text and image like the ones in Figure 1. 3 are visualised. There is no reproduction of any sound. The impressions of the corresponding sounds and actions are called to mind: in neurological terms, these impressions would correspond to a certain neural configuration. It is as if we simulated in our head the actions we would perform if we were in the situation of seeing the scene of that film, heard the music emitted by the succession of scenes and, in so doing, activated the mnestic trace of the sounds and actions that would arise from that succession of scene-actions. Then, viewing scenes from films would consequently activate the neural network, allowing our bodies to mentally simulate the displayed scenes as if we were experiencing them first-hand. And so on. In this way, the mnestic trace activated in the neocortex allows us to perceive the sign text-image.

For Saussure, signifier and signified are types and not occurrences. The type is something never representable, because in this case it would always be a particular occurrence. According to Saussure, one always thinks of particular occurrences of the object, even if it does not really exist, and the occurrences vary depending on the context in which the word is uttered, on the network of knowledge presupposed by the communicative exchange. This happens despite the fact that there is a kind of socially crystallised signified that regulates the specific senses that a word – as Saussure

is primarily concerned with linguistic signs – can assume. This is why a distinction is made between *langue* and *parole*. The former is the social part of language, logically prior to specific occurrences, while the latter refers to concrete, unique, and unrepeatable linguistic acts. In this sense signs are arbitrary. The nexus between signifier and signified arises rather from a standardization: a practical habit that is embedded in a social environment that systematically repeats that practice. The nexus arises from the sedimentation of the practice in which it is possible to participate even without knowing the standardized rule or the rule to be standardized, since it can be reconstructed inferentially, by observing the behaviour of others. As Radford reminds us:

> The main problem for Saussure was that of the understanding of the *langue*, which he distinguished from language and from *parole*, a distinction based on the opposition between the social and the subjective. For Saussure, *parole* is of a subjective order, while *langue* is of a social order (2006, p. 34).

Saussure's work is thus a search for how signs signify within social life.
A brief remark is called for here: it should be noted that, by focusing exclusively on the internal mechanisms of a restricted category of highly standardised artificial signs (such as the words of language), Saussure's line of studies excludes from the domain of semiotics those signs that are not based on culturally shared conventions. On the other hand, taking Peirce's logico-cognitive definition into account, the ensuing category of signs is much broader. It also includes natural signs, gestures, humans themselves – and treats language signs (as well as all other artificial signs) as a specific subcategory of signs in general. Indeed, according to Peirce, the equivalence model can be retranslated into the terms of the inferential model (but not vice versa) in order to give rise to a unified theory of the sign: *Semiotics*. In fact, it is the general circulation of signs that matters, which are slowly shared and define a series of habits, references and sets of signs. It is the task of the philosopher, according to Peirce, to understand their functioning. So, the difference between Semiotics and Semiology can now be clarified:

- Semiotics studies signs and their meaning, singularly or in combination with others.
- Semiology, as interpreted by Saussure, refers to communication, to the socially and culturally mediated message underlying the use of certain signs and symbols. But above all, it refers to the social part, i.e. how much the context also influences that meaning.

In any case, because of what was said earlier following Peirce, the term Semiotics can be chosen to refer to the unified theory of signs.

After this necessary remark, it is possible to return to the study of Saussure's conception, in which so the sign is the emerging tip of a network of relations. As such, the Saussurian sign presupposes an underlying system of signification – a language, a code – without which the sign would not be able to communicate anything. In this sense we can say that Saussure develops the fundamental principles of what will be called *structuralism*.
Starting from these premises, structural semiotics quits the narrow perspective of the isolated sign, characteristic of the Peircean view, to focus on the study of codes, understood as systems of rules

that correlate the overall system of signifiers to the overall system of signified. Further reading continues with what Radford wrote in reference to this Saussurian idea:

> […] signs signify insofar as they are elements of a system. That is, the sign has meaning when it is in relation to other signs. It is because of this system that the sign is a sign. Saussure offers the analogy with the game of chess. The horse, for example, represents nothing, as a material piece: «In its pure materiality, outside its square and the other conditions of the game, the horse represents nothing for the player» (op. cit., p. 153). This material object is not converted into a real and concrete element until it has the value conferred on it by the rules of the game. The same happens with signs (Radford, 2006, p. 35).

Saussure's structuralist tradition will be followed by other scholars, such as Hjelmslev, Jakobson, Lévi-Strauss and then Barthes. This leads to the lowest part of the genealogical tree of the Semiotics of Culture (Figure 1. 1), where there are Lotman, Eco and Geertz: the most contemporary exponents of this science.

On the relevance of Semiotics (and Semiotic of Culture) in Mathematics Education

In the early 1990s, semiotics has gained the attention of scholars and researchers in Mathematics Education as well. According to the authors of the ICME-13 monograph *Signs of Signification* (Presmeg, Radford, Roth, & Kadunz, 2018), where an overview of the large body of literature related to this topic can be found, it is owed to David Kirshner and James A. Whitson, within the PME-NA conference in 1994, that the potential of semiotics (initially looking at Saussure) has been realized for Mathematics Education research. This is in spite of the fact that other colleagues, educators and psychologists, had already shown interest for the relevance of semiotics in mathematics, especially in connection with linguistics: see for this Radford's (2006) introduction to the special issue of Relime (Revista Latinoamericana de Matemática Educativa) focused on this topic.

The growing interest aroused by semiotics in the field of Mathematics Education is purely justified not only because the "mathematics relies on an intensive use of different kinds of signs" (Radford, 2001 p. 1), but also for manifold reasons – the following is a summary of three of the main reasons, all of them intimately interconnected, for a semiotic theorisation of Mathematics Education:

- the role that signs play in cognition;
- the role of semiotics in the interpretation and construction of meanings;
- the existence of sign systems and the fact that they compose a certain unit in which they function and support each other, i.e. a culture (Uspenskij *et al*., 1998, in Rebane, 2013).

Peirce's originality lies in recognising that, since thought is a form of language, the interpreter itself is but a sign, which in turn refers to an object and so to another interpreter (Presmeg *et al*., 2018). The fundamental consequence of the Peircean theory is that everything has/is a semiotic function and, more precisely, everything can serve as a sign, object or interpreter depending on the context in which it is placed. Logic is but the science of interpreting signs. Hence human logical

habit develops through unlimited interpretation, and human is in turn a sign in a universe of signs: he/she coincides with his/her language.

> For, as the fact that every thought is a sign, taken in conjunction with the fact that life is a train of thought, proves that man is a sign; so, that every thought is an *external* sign, proves that man [sic] is an external sign (Peirce, 1958, CP 5.314).

Thus there is, for Peirce, neither a definitive interpretation nor a final object. Semiosis – whose dimensions are semantics (analysis of the relations of signs to the objects they refer to), pragmatics (analysis of the relations of signs to interpreters) and syntactics (analysis of the relations of signs to each other) – is unlimited in both directions of the chains of equality and signification. It is precisely this potential incompleteness that, according to Otte and colleagues (2019), makes sense of a semiotic theorisation in Mathematics Education. Hence, semiotic approaches assume "that human knowledge is always incomplete and that the form and content of mathematical theories cannot be definitive (*ibid*., p. 24)." To argue this claim, the authors present Frege's analysis and reading of what they call "the most important diagram for mathematics", namely A = B.

> It can be interpreted in two essentially different ways: as a relation between two signs representing the same object or as a relation between two objects possessing a common property. […] Frege (1848 -1925) opted for the first interpretation. […] Frege had identified sense or meaning with the way the references are given, such that concepts or functions must necessarily refer to an extension (*ibid*., p. 28).

In fact, his approach implies the assumption of an axiom of extensionality, for which two sets are equal if and only if they contain the same elements, even if they have been defined in a different way, that is even if they are intensionally different. For example, the sets A, i.e. the difference between the square of a natural number and that of its preceding number, and $B = \{n \in \mathbb{N} n > 0 : 2n - 1\}$ are extensionally equal, but intensionally different. A and B are two signs for the same object, i.e. the set (of odd numbers) with its elements. However, during dynamic teaching-learning processes, A and B cannot be said to be equal. They bring with them different conceptualisations. In these processes A and B also play the role of "two objects possessing a common property". Otte and colleagues proceed: "different concepts [A and B] help to establish different types of relationships and thus influence the development of Knowledge different ways. In fact, with respect to cognitive growth, as well as to the foundations of knowledge, it seems relevant, or even essential, to know which definition is chosen, what perspective is taken or how a problem is represented (*ibid*., p. 29)."

These two visions of the semiotic approach, however, cannot be said to be contradictory, as extensional and intensional perspectives constantly alternate in the teaching-learning processes of mathematics. Furthermore, it can be argued that for Frege signs are essential in cognition, he and Leibniz recognised signs as *helpers* of thinking (Radford, 1998, 2001), that is, as elements through which thought expresses itself. This is one possible way of theorising the *cognition-sign* relationship, where, in Radford's words, "The language should be as clear as possible, starting with the simplest terms in order for the ideas to be conveniently dressed and exposed. The classic

itinerary is to start with signs for operations, signs for variables, propositions, predicates, rules for construction of sentences, etc. (Radford, 2001, p. 2)." Another opposite way is to look at the sign (and language) as the origin of cognition. Here we have the structuralist tradition, with – as we have seen – exponents such as Lévi-Strauss, de Saussure, Jakobson and the Prague school, and also Hjelmslev and Barthes. For the structuralist tradition, every object of study is made up of a structure, that is, an organic and global whole, whose elements do not have an autonomous functional value but assume it in the oppositional and differential relations of each element with respect to all the others in the whole. Structures are the set of signifiers with which we interpret the world and relate to it. Hence all human activities, including cognition, are constructions since they are mediated by signs within structures. The possibility provided by structuralism of gazing at the world through structures also offers the notions of *synchronicity* and *diachronicity*, particularly useful in clarifying ways of looking at the processes involved in teaching and learning mathematics. An example of a theoretical approach to Mathematics Education using these notions in addition to Fried (2007; 2008), but overcoming – in the vein of Jakobson – the Saussurian diachrony-synchrony antinomy, is Arzarello (2006)'s Semiotic Bundle:

> The semiotic bundle dynamics can be analyzed in two different and complementary ways. The first one is *synchronic analysis*, which considers the relationships among different semiotic resources simultaneously activated by the subjects at a certain moment. The second is *diachronic analysis*, which focuses on the evolution of signs activated by the subjects in successive moments (in short or long periods of time). Together, synchronic and diachronic analysis allow us to foreground the roles that the different types of signs (gestures, speech, inscriptions) play in students' cognitive processes. Considering semiotic bundles, we can fully grasp the evolution of learning processes and the role of gestures therein (Arzarello, Paola, Robutti & Sabena, 2009, pp. 100-101, italics in original).

Furthermore, as can be seen from the words of Arzarello and collegues, in this approach the role that communication – and in turn semiotics about it – plays in cognition is essential.

Using Radford (2001) it is possible to propose a spectrum of possible conceptualisations of the communication-cognition interaction. This spectrum does not claim to present irreconcilable and mutually exclusive voices, rather attempts to paint a picture of the possible interpretations of this interaction over the studies of different scholars. All of them might be of potential interest, depending on the study to be carried out, however, the position I adopt mainly reflects the third approach. A first approach, adopted for instance by Processing Information Theory (Estes, 2014), sees thought as independent of communication, which is not irrelevant in cognitive development, but definitely left in the background. Here, the ego is solitary, and communication takes place through language, which is based on only two elements: the speaker and the content of the message, the listener being a passive participant. The second approach emerges from the Saussurian tradition, which considers the communicative act between sender and receiver as isolated, and as the basic element and model of every semiotic act. There is further the approach whereby cognition arises from acts of communication, which shape the way we think. Here discourse, communication, originates thought. Both Vygotsky and Bakhtin can be located here,

and the Semiotic Bundle approach – which also differs in this from the Saussure approach – is based on a socio-cultural perspective, rooted in Vygotsky's thought.
Vygotsky and Bakhtin can be pointed to as two other major cornerstones of the semiotic approach in Mathematics Education.

Lev Semenovich Vygotsky (1896-1934) – is an early 20th century Soviet psychologist, known for his work on psychological development in children, notably on special education (Vygotsky, 1993). The Vygotskian approach is grounded in the mono-logical dialectical[3] developmental and activity-based philosophy of Hegel (Kaidalov, 2018). In fact, according to Vygotsky, individual is defined through consciousness, which in turn develops through the activity shared mediation, in a socio-historical approach. Thus, "[i]n human consciousness, cognition (mediation by tools) meets discourse (social mediation by signs) […] [and] the key social nature of consciousness occurs through mutual understanding (Vygotsky, 1987)" (Matusov, 2011, p. 101).

Vygotsky's thought is long predominant in the international tradition in Mathematics Education (e.g., Boero, Pedemonte & Robotti, 1997; Radford, 1998; Bartolini Bussi & Mariotti, 1999; Wertsch, 2007; Roth, 2012). Here the sign is not simply a differential part of a system of structures (Saussure) or a means of thinking (Peirce), but, above all, a means of transforming and mediating the psychic and cognitive functions of the individual. Vygotsky's research is based on the study of the psychological consequences of human activity, granting activity great importance from a psychological-cognitive point of view. In activity, the individual is an active agent, but at the same time, the social and physical environment is not irrelevant. Vygotsky rejects the "simplistic idea that the social environment is the cognitively innocuous exterior scene where human actions are achieved" (Radford, 1998, p. 3). He affirms the importance of socio-cultural context in that development takes place through the use of those tools that are available at a particular time in a particular place. Vygotsky (1987) notes that humans use artefacts (tools - technological and concrete) and signs (psychological tools produced during activity) respectively to achieve otherwise unattainable goals, and to support mental activities. The former are outward-oriented, while the latter are inward-oriented. In contrast to other psychological approaches that clearly differentiate between the two, the Vygotskian perspective asserts an analogy between them, which is increasingly reinforced over time, as Vygotsky's thought moves from a purely psychological theory to one strongly rooted in semiotics and a cultural-historical perspective (Werstch, 2007). Indeed, in later

[3] Hegel's Absolute Idealism proposes consciousness as a unity of thought and being, albeit not as an abstract unity (as in Fichte and Schelling), but achieved through a dynamic, dialectical, and historical process, absolutely intentional. Reason becomes conscious of itself as the only Absolute Reality. In this, however, Hegel eschews an undifferentiated unity – seen as the "night in which all cows are black" (Hegel, 1807, *Phänomenologie des Geistes - The Phenomenology of Spirit*), i.e. an illusion (A=A) in which objects that are factually distinct are considered equal –, but he proposes a Union mediated by a dialectical process between a thesis and an antithesis. In this process, both the thesis and antithesis are overcome but preserved in a synthesis that Hegel calls the "Absolute Spirit".

Vygotsky's works, we read about this analogy between tools and signs (psychological tools) and the decisive role they play in cognition:

> [The use of psychological tools] introduces several new functions connected with the use of the given tool and with its control; abolishes and makes unnecessary several natural processes, whose work is accomplished by the tool; and alters the course and individual features (the intensity, duration, sequence, etc.) of all the mental processes that enter into the composition of the instrumental act, replacing some functions with others (i.e., it re-creates and reorganises the whole structure of behaviour just as a technical tool re-creates the whole structure of labour operations) (Vygotsky, 1981, pp. 139-140).

The analogy between signs and artefacts is rooted in the function of *mediation* that both have in performing an activity (Bartolini Bussi & Mariotti, 2008; Radford, 1998). This function is one of the cornerstones of Vygotsky's contributions. Daniels (2015) opens his article by reporting what Vygotsky defines as the concept of mediation: "the process through which the social and the individual mutually shape each other" (*ibid*., p. 34). Although mediation is at first conceived by Vygotsky as purely explicit (Werstch, 2007), in which a stimulus means is only intentionally introduced into an activity, later Vygotsky also embraces the possibility of an "implicit mediation":

> [I]mplicit mediation typically does not need to be artificially and intentionally introduced into ongoing action. Instead, it is part of an already ongoing communicative stream that is brought into contact with other forms of action. Indeed, one of the properties that characterizes implicit mediation is that it involves signs, especially natural language, whose primary function is communication. In contrast to the case for explicit mediation, these signs are not purposefully introduced into human action, and they do not initially emerge for the purpose of organizing it. Instead, they are part of a pre-existing, independent stream of communicative action that becomes integrated with other forms of goal-directed behavior (Wertsch, 2007, p. 180-181).

In this way, "Vygotsky links the development of consciousness to semiosis, and specifically to linguistic semiosis, and thus links the specifically human aspects of our practical and mental life to socio-historical contexts" (Hasan, 2005, p.136)". Therefore, the individual cannot develop without the context, without sign, without mediation, without the Other, others, and development is the utmost purpose of the individual in his or her social nature. This triggers a dynamic transformation, moving from the social to the individual planes of development through the zone of the *Zone of Proximal Development*[4], process that Vygotsky calls the *genetic general law of cultural development*:

> Any function in the child's cultural development appears twice or in two planes. First it appears on the social plane, and then on the psychological plane. First it appears between people as an

[4] *Zone of Proximal Development* (ZPD): "the distance between the actual development level as determined by independent problem solving and the level of potential development as determined through problem solving under adult guidance or in collaboration with more capable peer" (Vygotsky, 1978, p. 86).

> interpsychological category, and then within the child as an intrapsychological category (Vygotsky, 1981, p. 163).

The psychological process of transformation of social interactions by the individual is called *internalisation* (Vygotsky & Luria, 1994) and mental functions, such as memory, problem solving, and awareness (Matusov, 2011), are a result of the genetic development of internalisation. People alone are limited and incomplete, they need each other, in their becoming – the object of study of Vygotsky – to achieve their goals (and therefore not in their being). For this reason, the term *development* is used, and in it the mediational role of signs and communication cannot but play a decisive role, so much so that it is what characterises human activity:

> The internalization of cultural forms of behavior involves the reconstruction of psychological activity on the basis of sign operations (Vygotsky, 1978, p. 57).

Therefore, communication is not disinterested but goal-oriented in a culturally identified sense within goal-directed activities. Consequently, in Vygotsky's theoretical vision, social relations are profoundly instrumental: his whole approach is essentially developmental and functional. I and others interact to reach our acme: to use Hegel's words – the Absolute Spirit, or for Vygotsky – "the inner speech of the educated Western (middle-class) adult equipped with the scientific concept" (Matusov, 2011, p. 115). Indeed, education and development lead to the attainment of this absolute mono-consciousness. Thus, from a methodological point of view, Vygotsky cannot but criticise his contemporaries, Montessori and Piaget, "for viewing children as self-contained rather than as participants of sociocultural and historical practices" (Matusov, 2011, p. 113). He writes for example:

> Piaget has already been criticized by Stern for his failure sufficiently to take into account the importance of the social situation and milieu. Whether the child's talk is more egocentric or more social depends not only on his age but also on the surrounding condition. […] in Montessori kindergartens, where children simply play with each other, the coefficient of egocentric speech is higher than that in German kindergartens, where there is more group activity (Vygotsky, 1986, pp. 55-56).

and

> Reading and writing must be something the child needs. Here we have the most vivid example of the basic contradiction that appears in teaching of writing not only in Montessori's school but in most other schools as well, namely, that writing is taught as a motor skill and not as complex cultural activity (Vygotsky, 1978, p. 118).

Or:

> The development of thought is, to Piaget, a story of the gradual socialization of deeply intimate, personal, autistic mental states. Even social speech is represented as following, not preceding, egocentric speech. […] The primary function of speech, in both children and adults, is communication, social contact. The earliest speech of the child is therefore essentially social.

> At first it is global and multifunctional; later its functions become differentiated. At a certain age the social speech of the child is quite sharply divided into egocentric and communicative speech. (We prefer to use the term *communicative* for the form of speech that Piaget calls *socialized*, as though it had been something else before becoming social. From our point of view, the two forms, communicative and egocentric, are both social, though their functions differ.) Egocentric speech emerges when the child transfers social, collaborative forms of behavior to the sphere of inner-personal psychic functions. […] Egocentric speech […] is the highly important genetic link in the transition from vocal to inner speech, […] Thus our schema of development – first social, the egocentric, then inner speech – contrasts both with the traditional behaviorist schema – vocal speech, whisper, inner speech – and with Piaget's sequence – from nonverbal autistic thought through egocentric thought and speech to socialized speech and logical thinking. […] In our conception, the true direction of the development of thinking is not from the individual to the social, but from the social to the individual (Vygotsky, 1986, pp. 34-36, italics in original).

This marks what Gherlone (2015) identifies as the "breaking point between the Western and Russian cultures", i.e. the transition from a vision of language – which is a place of recognition of the other, of encounter and socialization – as alien, source of anxiety and loneliness, to a friendly and familiar language. Moving even further in this direction, abandoning the mono-logical view of communication, are the post-Vygotskian theoretical views[5], as presented in Radford (1998,

[5] A *dialectical materialist approach* is referenced here, in the wake of Vygotsky's dialectical Hegelian heritage[3] but overcoming its idealism. In a materialist approach (see the writings of Friedrich Engels and Karl Marx), the material world is the only real world. To it belongs the individual, who perceives through his senses. Consciousness and thought are products of the brain as a material organ of the body and therefore they too belong to the material world. Hence, in a dialectical materialist approach, signs and artefacts are (external) material means of encounter and awareness, bearers of human [joint] labor, i.e. they have a functional role. Signs are a fundamental part of mathematical activity, but they do not represent knowledge – *representational* approach, nor do they mediate it – *mediational* approach (Radford, 2012; Presmeg *et al*., 2016). Consequently, within a dialectical materialist approach, activity is not "series of actions that an individual performs in the attainment of his or her goal. […] [It] does not merely mean to do something. […] [Activity] is the endless process through which individuals inscribe themselves in society (Presmeg *et al*., 2016, pp. 15-16)." Activity is termed *joint labor* in Radford's *theory of objectification* (Radford, 2021). "[It] is an attempt to understand learning not as the result of the individual student's deeds (as in individualist accounts of learning) but as a cultural-historical situated processes of knowing and becoming (Presmeg *et al*., 2016, p. 16)." Sinclair and colleagues (2020) also expressly state that they intend to complement sociocultural approaches, "which privilege language and culture as sources of meaning, goals and values, with bodily and material sources of meaning as well (*ibid*., p. 1472)", i.e. with a *socio-material approach*.

> These sources of meaning may involve such things as the actual touching of fingers on the screen or the rhythmic enunciation of words or the visceral attachments to mathematical

2001, 2015; 2021), and the Bakhtinian *dialogical* approach (in Greek διαλογικός – from διά: "through", and λόγος: "discourse", "thought", "mind"), in which Vygotsky precious heritage does not however only lie in the background. Communication is thus still collaboration in the sense of openly shared meaning. However, the negotiation of meanings is a much more complex process, in which conflicts and misunderstandings are possible, and not just a neat process of mutual reinforcement of meanings, where individuals are involved in improving each other for the better, as conceived by Vygotsky (Van Oers, 2001).

Mikhail Mikhailovich Bakhtin (1895-1975) – is an early 20th century Russian philosopher, critic, and literary theorist, contemporary of Vygotsky. Both are involved in the development of the socio-cultural paradigm within the Western social sciences, which is why the two approaches can be seen in continuity, despite their differences (Matusov, 2011).
The mainly philosophical Bakhtinian approach can be defined as pluralist, dialogical and discourse-based.

> The dialogic nature of consciousness, the dialogic nature of human life itself. The single adequate form for verbally expressing authentic human life is the *open-ended* dialogue. Life by its very nature is *dialogic*. To live means to participate in dialogue: to ask questions, to heed, to respond, to agree, and so forth. In this dialogue a person participates *wholly and throughout* his whole life: with his eyes, lips, hands, soul, spirit, with his whole body and deeds. He invests his entire self in discourse, and this discourse enters into the dialogic fabric of human life, into the world symposium.
>
> Reified (materializing, objectified) images are profoundly inadequate for life and for discourse. A reified model of the world is now being replaced by a dialogic model. Every thought and every life merge in the open-ended dialogue. Also impermissible is any materialization of the word: its nature is also dialogic.
>
> Dialectics is the abstract product of dialogue.
>
> The definition of voice. This includes height, range, timbre, aesthetic category (lyric, dramatic, etc.). It also includes a person's worldview and fate. A person enters into dialogue as an *integral* voice. He participates in it not only with his thoughts, but with his fate and with his entire individuality (Bakhtin, 1984, p. 293, my emphasis in the text).

> meanings. Such meanings are not merely subordinate to discourse, and therefore should not always be buried away by mediation of consensus (Sinclair *et al*., 2020, p. 1472).

The socio-material approach (Orlikowski, 2007) recognises the interaction between subject and object, social and material, as a recursive intertwining where both co-configure each other and continuously define and redefine themselves in terms of identity. So, the object is not a *pre-formed substance* but a *performed relation* in ongoing, situated practice. Between subject and object there is asymmetry, albeit minimal (more recent studies on artificial intelligence may show a decreasing degree of asymmetry). The subject is active, intent on conditioning, the object is conditioned passively, without will. However, this does not exclude the influence of the objects on subjects.

According to the Russian scholar, life is a continuous dialogue and human consciousness is defined through the relationship with the other, but it is not transparent – as it is for Vygotsky, namely developed through a joint and intentional labor towards development. Interaction, for Bakhtin, involves any facet of the individual, is realised through the whole body, but does not necessarily lead to consciousness of the other. There is not necessarily mutual understanding nor agreement. This approach can therefore be termed *anti-developmental*. There is however (a) mutual responsibility, (b) inter-addressivity and (c) inter-problematicity (Matusov, 2011). That is, *discourse* – Bakhtin's unit of analysis – is guided by:

(a: responsibility) what Sperber and Wilson (1986) call the *principle of relevance*, i.e. the assumption among participants in the dialogue that the exchanged messages are as relevant and appropriate to the context as possible;

(b: inter-addressivity) a genuine interest in the other, neither instrumental nor predetermined or goal-oriented;

(c: inter-problematicity) challenging and stimulating issues for participants: problematic aspects may not be the same for all participants, but there are some for everyone.

Furthermore, here, the concept of *genre* is a central tool in Bakhtin's work: genres organise discourse. They empower communities to grasp connections within texts, to grasp the meanings of texts, genres enable discourse. However, genre is not an encyclopaedia or a thresaurus, it is a social tool:

> It is a style of speaking embodied in a community's cultural inheritance, which is passed to members of that community in the same way as grammar is passed on. A genre is not so much a strict and fixed social norm, but it is a generic system of changing variants and possible utterances that fit into a community's practices; it is some kind of arena or forge where new variants of utterances are created and valued, that contribute to the essential polyphony and dissonances of meaning and discourse. […] for Bakhtin the speech genre intrinsically links the interlocutors to each other, despite their possible differences in expertise (or their asymmetry in positions) (Van Oers, 2001, pp. 69-70).

Van Oers (2001) tell us that Bakhtin calls *sign community* the "institution of persons", which establish their community on the basis of forms of signs "conditioned above all by the social organization of the participants involved and also by the immediate conditions of their interaction" (Voloshinov, 1929, in Van Oers, 2001). This provides an insight into Bakhtin's conception (and revolution) of semiotics. He criticises the classical approach to semiotics and linguistics. Interactions, according to him, are not analysable purely in linguistic terms, but as discourses and practices that take place with historical and social value. Classical semiotics, according to Bakhtin, reifies (objectifies, crystallises, stops) the image and the word, while dialogue is always open-ended and keeps the word alive. Although external objectification (i.e. definition, causal or genetic explanation, external description) is inevitable, for Bakhtin it cannot be the whole truth, simply monological. In this way he criticises Western thought that turns subjects into objects (scientific, social, psychological, etc.): in his view, nothing can be completely embedded in the external world's norms, nothing can be trivially reified. Reifying would be an excessive discounting of

responsibility. Reified signs do not fit the dialogical model; therefore, reification is inadmissible. Words and signs are in fact intersubjective social entities (Bakhtin, 1975, p. 396) and it is in signs that we encounter otherness. Therefore, in Van Oers' more modern words, we can say that: "people's utterances in a communication process are not only regulated by the processes that occur in direct interaction, but also *by the historically developed style of communicating in that particular community of practice*" (Van Oers, 2001, p. 68, italics in original). Thereby Bakhtin reaches "the concept of the *other* as a plenitude of meaning" (Gherlone, 2015, p. 8, my emphasis in the text), in which the asymmetry (not in Bakhtin's words, although he is in this sense referring to a dualism) or the *gap* existing in mutual communication is not only a necessary condition for dialogue, but is also its result, the creation of dialogical meaning and a continuous process of generative, vital exchange (Matusov, 2011):

> […] Bakhtin posits a dualistic universe of *permanent* dialogue. Life in language is in fact dependent upon the preservation of a gap. Two speakers must not, and never do, completely understand each other; they must remain only partially satisfied with each other's replies, because the continuation of dialogue is in large part dependent on neither party knowing exactly what the other means. Thus, true communication never makes languages sound the same, never erases boundaries, never pretends to a perfect fit. In a fragment written near the end of his life, Bakhtin in fact compared understanding itself to a sort of obligatorily imperfect translation:
>
> > Understanding cannot be understood as emotional empathy, or as the placing of oneself in another's place (the loss of one's own place). This is required only for the peripheral aspects of understanding. Understanding cannot be understood as translation from someone else's language into one's own language.
>
> The ideal here is contiguity without fusion. Equivalence, too, is a threshold phenomenon (Emerson, 1984, p. xxxii-xxxiii, italics in original).

For Bakhtin, translation is never a betrayal, but in its broadest sense, the crossing of linguistic boundaries is perhaps the most fundamental human act. Bakhtin is referring not only to the normative national languages, but also to the multiple "languages" that coexist within a single culture or community. In other words, the semiotics is supposed to analyse the layers of language that interpenetrate and overlap but do not exclude each other, drawing words into different fields. Unlike a dictionary, a living discourse is constantly in motion, rebelling against its own rules, and semiotics must be up to it.

Providing an interpretation of asymmetries between participants and between (semiotic) resources, with no recourse to the sender-receiver model, still presents a challenge for educational researchers. In Mathematics Education in particular, the differences between sender (e.g. teachers) and receiver (e.g. students) concerning competence and authority are often considered unbridgeable, but this is just one instance manifest for all to see. Ethnomathematics (D'Ambrosio, 2006; Barton, 2012; Salas, Godino, & Oliveras, 2015) can also be seen in this interpretation of asymmetries, or even comparative cross-cultural studies (Stigler, Gallimore, & Hiebert, 2000;

Leung, 2001; Clarke, 2003; Jablonka, Andrews, Clarke, & Xenofontos, 2018). Thus, in the last 30 years (Bishop, 1991; Saxe, 1991) advances in Mathematics Education have intensified the call for a discursive approach in which Mathematics is, furthermore, conceived as a cultural activity based on the socio-cultural practices of a community (Lerman, 2002; Sierpinska, 2005; Kim, Ferrini-Mundy & Sfard, 2012; Andresen & Dahl, 2020). As a result, researchers in Mathematics Education are particularly attentive to exploring the links between community discourse and the continuous processes in producing shared mathematical cognition (see, for example, Van Oers, 2001). However, the integration between the discursive approach and the attention to cultural and semiotic aspects is patchy.

In an overarching Bakhtinian and post-Vygotskian vision, "we do not inhabit a mere concrete, material world, but a world full of meaning, and that meaning belongs to the order of signs" (Voloshinov, in Radford, 1998, p. 7). At the same time, however, signs do not dwell in a world made up only of ideas and abstraction either. Therefore, there is a (non-physical, but relational) meeting place, which is the sign, between the biological organism and the external world. It goes beyond physiological aspects, beyond symbolic aspects, and so its products cannot be analysed as things, but understood and interpreted as signs. Figure 1. 4 shows a topological representation of what Voloshinov calls *the territory of signs*, a space in which the work of interpretation is carried out.

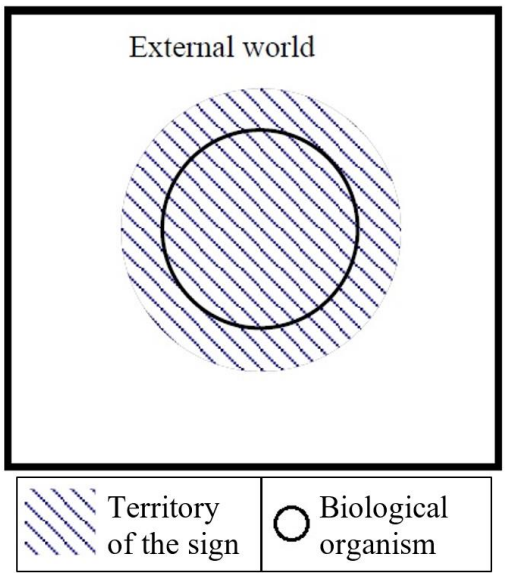


Figure 1. 4 A topological representation of Voloshinov's Territory of the sign (Radford, 1998, p. 9).

This territory is made up of relations and proves that it is impossible for signs to merely "stand for something else", since signs cannot but be embodied in their reality and in their webs of relations. The sign and its signified are not in a sole relationship of substitution, they are already embedded in a cultural system. Hence, not everything is a sign (at least not always and not directly), but nothing exists outside a semiotic system. Radford call it a *cultural semiotic system*. It is what "make available varied sources for meaning-making through specific social signifying practices"

(Radford, 1998, p. 13). What differentiates a semiotic cultural system from the territory of the sign is that the first not only makes the production and understanding of a sign inextricably linked to the situation in which the sign is implemented, but becomes a structural element of the activity, i.e. it provides "the basis of the generation of modes of knowing (or *épistèmes*, to use Foucault's expression)" (Radford, 1998, p.15), the basis for cognition.

My example to highlight the double function of the *cultural semiotic system* echoes the example presented several times by Radford (1998, 2001), but it stems from an explorative experience carried out with prospective primary school teachers, namely students pursuing the 5-year postgraduate degree in Childhood and Primary Teachers Education at the University of Turin.

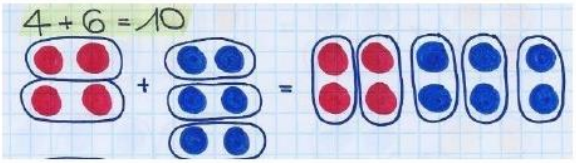
A. and S. are two of these students. During a lecture of *Fundamentals and Didactics of Mathematics* first-year course, A. asks: "I would like to know if the arguments I have made are correct, clear and complete." and she presents the protocol in Figure 1. 5. Then S. intervenes and asks: "But are the argumentations made by A. proofs? Does the exercise end like this, or do I necessarily have to use the algebraic formulation? And so, is my protocol [Figure 1. 6] even correct?"

(See student's protocols in the next pages.)

SUMS OF ODD AND EVEN

1. THE SUM OF TWO EVEN NUMBERS IS AN EVEN NUMBER

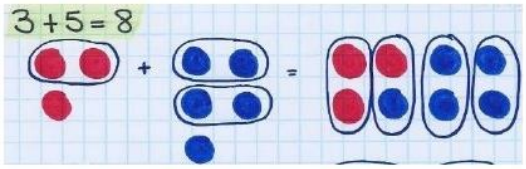
Numerical explorations:
4+6=10
8+12=20
This statement is true.
If I were to present this statement in a class, I would use graphic representations.I could, for example, represent even numbers as repeated sums of 2.
Example: 4+6=10



2. THE SUM OF TWO ODD NUMBERS IS AN EVEN NUMBER

Numerical explorations:
3+5=8
7+9=16
This statement is true.
If I were to present this statement in a class, I would use graphic representations.
I could, for example, represent odd numbers as repeated sums of 2 to which 1 is added.

Example: 3+5=8



3. THE SUM OF AN EVEN NUMBER AND AN ODD NUMBER IS AN ODD NUMBER

Numerical explorations:
4+5=9
8+9=17
This statement is true.
If I were to present this statement in a class, I would use graphic representations.
I could for example represent the even number as repeated sums of 2 and the odd numberas repeated sums of 2 to which 1 is added.
Example: 4+5=9

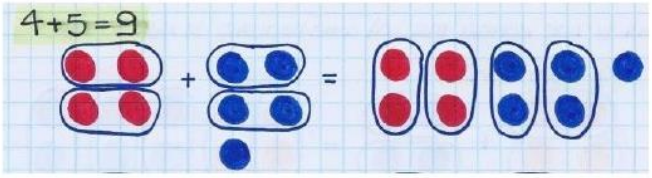


Figure 1. 5 The A.'s protocol.

CONSECUTIVE NUMBERS

a) Given seven consecutive numbers, define their sum. Then explain how it is possible to get this result.
b) If instead of seven consecutive numbers only four are considered, how many and in what ways could the sum of these numbers be known?

a) Exploratory examples:

1+2+3+4+5+6+7=28
2+3+4+5+6+7+8=35
3+4+5+6+7+8+9=42

These sums can also be seen as multiplications of each central number, for example the 4 in the first sum, by the number of addends, namely 7:

4x7=28
5x7=35
6x7=42

Demonstration of regularity:
with n I identify the central number (for example the 4); with (n+1) its following (the 5); with (n+2) the 6; with (n+3) the 7; with (n-1) the 3; with (n-2) the 2; with (n-3) the 1.

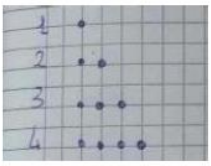
(n-3)+(n-2)+(n-1)+n+(n+1)+(n+2)+(n+3) =
= n+n+n+n+n+n+n-3-2-1+1+2+3 = 7n

It is a repeated addition, and I can see it as a multiplication of any number n ∈ ℕ x the number of addends (7).

b) We start with exploratory examples:

1+2+3+4=10
2+3+4+5=14

Since there are four consecutive numbers considered, i.e. an even number, we do not have a central number, so we can display the sums as follows:



the number 1 is n n ∈ ℕ
the number 2 is n+1
the number 3 is n+2
the number 4 is n+3

We generalize:
n+(n+1)+(n+2)+(n+3) =
= n+n+n+1+2+3 =
= 4n+6

We can consider the addition of four consecutive numbers as a multiplication of the first number, which I identify with n, by 4 (the number of the addends), to which I add 6 = (1+2+3)

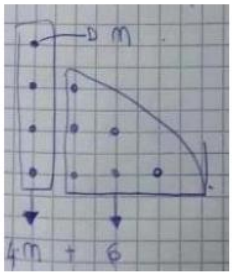


Figure 1. 6 The S.'s protocol.

S.'s question is not obvious at all. The central idea in A.'s proof is the property of even numbers as those that can be repeated additions of 2 (i.e., multiples of 2, thus including the zero). The proof would then be in considering A.'s graphical representations not as examples but rather as generalisations. It does not matter how many dots are drawn, but whether they satisfy the property of grouping them by 2, or if 1 is left over. S.'s protocol tries to go in this direction, to avoid the example but see the generalisation in the drawing, as can be seen from the final representation in which S. tries to graphically represent $4n$+6.

These argumentations, however, do not satisfy a third student C., who argues as follows:

"It puzzles me. It's not a proof, it's a way of graphically showing what is happening. What A. writes would seem to be just a proof that 4+5 makes 9. You would need to at least add a few lines of commentary to be able to take it as a proof for every n. In my opinion it might be said that it does not meet the formal definition of proof, as it does not generalise. It is not true that writing two statements ensures that the two are connected. Where is the guarantee that the example in the sketch is related to the statements above it? And how do I know how they are connected? Mathematics is a set of tautologies, to make a proof is to make the tautologies explicit. To show all the tautologies that are used. If there is any connection or explanation, it should be written down because anything is trivial, but without pointing it out it is not obvious. Because, a person who does not *see* it as trivial must still be able to grasp it. So, I don't see these two protocols as proofs. However, starting from S.'s last sketch, I would suggest making a series of *observations* like these (he refers to Figure 1. 7), link them with S. last sketch and *see*, write down and *display* the conclusions: then yes, even a graphic representation can replace the algebraic representation in a proof. But the links leading to the generalisation ought to be explicit. It is therefore no longer a question of counting, but of *looking at the form*. Once you've *seen* the rectangle and the triangle, it's done!" [italics on C.'s vocal emphasis]."

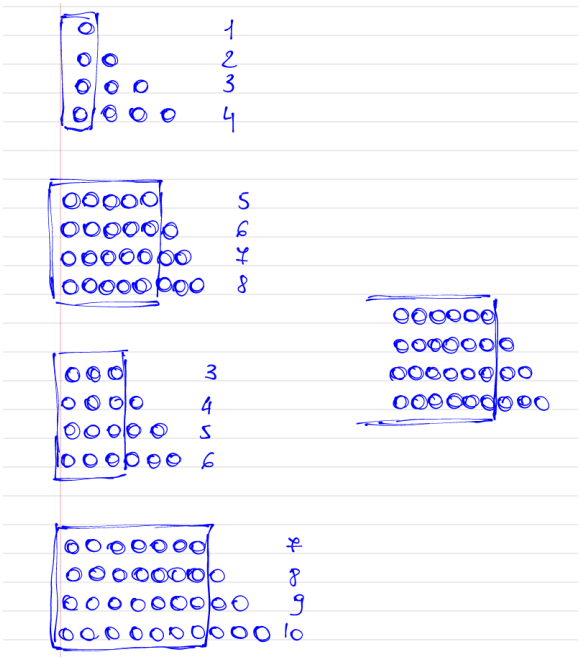


Figure 1. 7 C.'s suggestion.

The whole episode is excerpted from an online lecture, delivered via the Webex platform and through the exchange of digital resources between the participants.

Students' mode of proving cannot be understood within the field of Mathematics itself. Their mode of considering proof and reasoning is transposed also from other fields to Mathematics and then to Mathematics Education. Origins of this discussion lie in the art of dialectics, in a Socratic and Aristotelian tradition, in which deductive, inductive, and abductive methods of reasoning are distinguished, and from which Peirce (1866) draws his classification of signs, respectively:

- symbol: a sign that is such independently of its similarity or relation to the object, it is deduced;
- index: a sign that interacts with an object through a causal relation and is indicative, a possible specific case, of the totality or rule from which it can be traced;
- icon: a sign that resembles something and might trigger an intuition.

Peirce would refer to the drawings of the three students as iconic signs. It is not by chance that C. strongly states that: "What A. writes would seem to be just a proof that 4+5 makes 9." General conclusions may not be stated from a specific observation. Euclid's effort to re-discuss what could be mathematically accepted as a proof was and is crucial in this respect. Despite the fact that, according to Peirce, abduction is the only form of reasoning that can improve our knowledge, i.e., that enables us to conjecture new ideas, to foresee, to guess – in fact, the etymology of abduction goes back to the Latin verb *abducĕre* «to move away», composed of *ab* «from» and *ducĕre* «to lead», thus alluding to a sense of transformation and change –, deductive reasoning remains the only one that, from true premises, leads to always certain conclusions (Mariotti, 2006; Hanna & de Villiers, 2012). A. and S. both start their investigations of the problem by using exploratory examples. In order to hypothesise a conclusion, they need to "see" it. Examples activate abductive thinking. A. and S., in fact, "see" the conclusion. Their sketches, although are only proofs of specific cases, provide to A. the certainty of a proof. While S. feels, due to a formal and institutional need (perhaps due to a didactic contract?), that she has to challenge this certainty: "But are the argumentations made by A. proofs?" S., in fact, juxtaposes drawing with algebraic representation, ideally identified by her as the only mode of proof acceptable to "mathematicians", someone who is alien to her and who imposes a demand on her: "Does the exercise end like this, or *do I necessarily have to* use the algebraic formulation?". S. naively recognises in the sketches an abductive reasoning, which, like induction, does not hold in itself its logical validity and needs to be corroborated by empirical evidence, which will in any case only be expressed in terms of probability. However, the necessity of an empirical proof to be managed in terms of probability, the uncertainty, the "seeing" a result by means of graphic semiotic resources but having its proof expressed in an algebraic representation, are all factors that when gathered together are hardly managed by S.. C. tries to approach S. with "a series of *observations*", a collection of examples, which would like to help his colleague towards a generalisation and therefore the desired proof realised through graphic representations. C., in fact, seeks to stimulate in S. the use of signs that are no longer iconic, but rather symbolic. According to C. the few lines written by S. and linked to her last sketch, in which S. describes n as the first number and therefore the first line of her

drawing, are not enough. It remains a possibly indexical sign, since that line is for now represented by a single dot, and it is only that dot that is indicated with n, both there and in the previous drawing. However, C. concludes: "then yes, even a graphic representation can replace the algebraic representation in a proof." Namely, there is acceptance of a graphic proof as long as the link leading to generation is explicit. The sign must explicitly go from iconic to symbolic. Not only does it have to be symbolic for the sender, it must also become so for the receiver.
The self-same sign, in fact, can mean different things and change its value depending on who is seeing it and how he/she sees it. That is, because each gaze is based on a different cultural semiotic system.

> The specific way to learn to see something as something else in a very particular way among the vast arsenal of perceptual and conceptual possibilities is, indeed, one of the effects of a cultural semiotic system and the practices that such a system legitimizes (Radford, 2001, p. 6).

This, however, is not to be read as a justification, a pass to pure interpretation. Radford, in fact, does not state that sign interpretation is a cultural semiotic system, rather he claims that it is an effect of the system and of practices legitimised by the system. Hence, in order to achieve a proper semiotic interpretation, learning to read and describe the cultural semiotic system becomes necessary in understanding where is immersed what we are looking at. Along the lines of Radford's work (1998, 2001), going beyond the mere description of the sign-object substitution relation, what we are looking at is the cultural semiotic system of the prospective teachers. Some observations on the cultural semiotic system of C., S. and A. are then possible.
First of all, the three prospective teachers' competence on the handling of deductive reasoning is lacking, yet they are consciously in an institutional context that requires them to move from empirical justification to deductive reasoning (and in this also to be able to support their students). So, S. and A. show *generic examples*, as described by Mason and Pimm (1984, in Lesseig, 2016), namely "a specific example that is presented in such a way as to communicate generality across cases" (p. 260). Indeed, the Western thought in which prospective teachers are embedded is wedded to a Euclidean tradition, in which theorising is a precondition for scientific knowledge and the realm of sensible objects is not recognised as a reliable source of knowledge. This unlike previous schools of thought that embraced the manipulation of objects as a means of accessing cognition (Radford, 1998). The three prospective teachers are caught in what Bruno De Finetti (1974), in his speech at the C.I.I.M. (*Commissione Italiana per l'Insegnamento della Matematica* – Italian Commission for the Teaching of Mathematics) conference in Viareggio, defines as the "opposition between the 'cultural' value of a 'pure' mathematics in the sense of 'abstract', and the purely 'instrumental', 'utilitarian' value of a 'applied' mathematics to problems concerning concrete, practical, or even 'useful' things and notions!". Adopting an approach based on practical examples that precede any theorising, in order to "create, first of all, a motivation that predisposes to the acceptance of abstractions that appear justified, and thus avoid the reaction of rejection that the opposite path often produces, not entirely unjustifiably", is what De Finetti recommends as a strategy for successful teaching. The three teachers are aware of this at a theoretical level, learned in university courses, but it is not yet part of their practice.

Secondly, C., S. and A. are embedded in a cultural context in which there is a growing interest in the use of images within science and communication. As Kadunz and Yerushalmy (2015) report, from a "linguistic turn" we are now in the time of a "pictorial" or "iconic turn". Technology and the use of software and social media embeds people in a world of images. The importance of visualisation in Mathematics Education has been theorised since some time ago (Vinner, 1992; Duval, 1993; Fischbein, 1993), but the reception of theories that link the image to its conceptual meaning and consider it within its cultural context, such as Barthes' visual semiotics (Van Leeuwen, 2001), is a matter of recent interest, not only in Mathematics Education. In order to understand the meaning of what we are talking about, it is worth remembering that for Barthes the image is no longer understood as an isolated sign to be set in relation to others (mainly linguistic) within an abstract system, but as a complex of elements that are themselves signifiers – that is, a text. According to the scholar, a discourse can only be constructed on complex entities, because even if conceptuality is not part of the material image, it springs from and is part of the interpretative discursiveness that is grafted onto it. Interpretative discursiveness cannot but be a founding part of the image itself. In this sense the image becomes a semiotic resource with an epistemic power (see for instance, Bini *et al*., 2020, where mathematical memes are "mathematical statements", p. 34).

Thirdly, the insecurity of prospective teachers about the concept of causality of events and the handling of reasoning processes when faced with the degree of certainty-uncertainty of a statement should be borne in mind. The lack of a basic preparation in probability in mathematics teachers' professional development is a reality, in Italy and elsewhere (Batanero *et al*., 2011), and since those who do not know a topic may not be well inclined towards it, and do not intend to study it in depth, school have a further disadvantage: those who do not know a topic in depth rarely motivate themselves and their students to approach it. Thus, despite the fact that the study of uncertainty belongs to deductive mathematical reasoning (Anichini, 2010), recognised as institutionally necessary by C., S., and A., it remains hostile and confusing for the prospective teachers, since for them the study of uncertainty is linked to a strong mathematical theorisation and far from reality.

Of course, possible observations do not end here, but from these the meaning of the dot in S's protocol can already be grasped. For example, it is possible to read the correlation between the dot and the concluding sentence of C: "then yes, even a graphic representation can replace the algebraic representation in a proof. But the links leading to the generalisation ought to be explicit. It is therefore no longer a question of counting, but of *looking at the form*. Once you've seen the rectangle and the triangle, it's done!"

With his suggestion, C. aims to encourage his colleagues to consider the shape of the figure as a generalisation. Thus, the *rectangle* can be interpreted as denoting what in algebraic form could be denoted by $4n$. There are 4 rows of n dots, with n the number of dots representing the first number of the quatern. The triangle next to the rectangle is the addition +1+2+3, again generalised by the concept of "successor". The second, third and fourth rows show respectively the successor (+1), the successor of the successor (+1+1) and the third successor (+1+1+1) of the first number. Thus,

the rectangle, the triangle, and the concept of successor give rise to the generalisation, and it is this that can be interpreted especially thanks to the image on the right, the one in which one detail is glittering by its absence: the numbers.
This sign could not have been interpreted comprehensively without such a description of the cultural semiotic system in which it is embedded. It should be noted that the focus is on the relations existing within the cultural semiotic system, not on the single sign, which would otherwise be inaccessible in itself. The object of analysis are the semiotic networks and thus the dialogue, the text, that is engendered from the discourse between the participants, the written signs and, in this case, also the voice intonations. Observations on the cultural semiotic system and thus interpretations arise from them. What we call *cultural (semiotic) aspects* are for our purposes the networks, the relations, the objects of analysis. However, it remained obscure what is meant by "culture" and which method of analysis makes it possible to research within the cultural semiotic system and to interpret and draw conclusions about the processes of acquisition and development of cognition. From the definitions given so far of socio-cultural context (Vygotsky), of dialogue and sign community (Bakhtin), of territory of signs (Voloshinov), and of cultural semiotic system (Radford), three further questions arise:

RQ4. What is meant by culture?
RQ5. How is culture characterised in Mathematics Education?
RQ6. How to develop a semiotic-cultural analysis of the mathematics texts (discourses)?

Definition(s?) of Culture

I propose to the reader a metaphorical journey through History to trace the (Western) concept of culture. Such a journey is required in order to understand current assumptions about what are considered "cultural data", "cultural zones", or "cultural aspects". Since the adjective "cultural" is imbued with significance shaped over the course of History and societies, we trace the nodal points that have led contemporary thinkers to understand culture as "essentially semiotic" (Geertz, 1973). Our journey begins with the agricultural metaphor of culture, it literally follows the sea routes across the ocean, and takes shape with the anthropologist E. B. Tylor (1832-1917). Then it places culture in the external, in the non-organic, passing through the rejected stratification with the archaeological discoveries. Finally, as a fish, we are caught in C. Geertz's (1926-2006) semiotic webs. In fact, it is our opinion that the two definitions, by Tylor and Geertz, if read in relation to their context (which we can no longer do without) are for us neatly relevant in the same line of time development.

Throughout the course of history, numerous historical (e.g., Washburn, 1959; Leroi-Gourhan, 1964), linguistic (e.g., Jakobson, 1963), semiotic (e.g. Eco, 1968), and anthropological (e.g., Kroeber & Kluckhohn, 1952) studies – and even more, related disciplines are multifarious and their boundaries are fuzzy (Avruch, 1998; Remotti, 2011; Spencer-Oatey, 2012; Boroch, 2013; Arciniegas Rodríguez & Pérez Peña, 2014) – have made it possible to construct an increasingly broad spectrum of the meaning of the term *culture*. However, each definition reflects the intention and origin context from which it arises. These intentions and contexts, consequently, from time to

time limit the concept, highlighting some of its characteristics. Therefore, our argument is that it is not possible to provide a single definition of culture, or at least it is inconvenient or fruitless, but rather requires picking each time a context-specific choice. Nonetheless, we deem it crucial to trace a possible path of construction of the meaning of the word culture over the centuries. The importance of this lies in allowing a better understanding of the definition(s?) of culture which shall later be pick out as suitable for our study.

The notion of culture is originally based on the agricultural metaphor stemming from classical Latin, in particular by Cicero: from the verb *cŏlĕre*, which mainly means "to cultivate", "to work", "to care for", "to inhabit". "At the basis there is the idea of a modifying intervention, immediately transmitted by the gesture of those who settle in a place to live and therefore transform it [...] (Remotti, 2011, p. 3)". Since Cicero, who in the *Tusculanae disputationes* (liber secundus - par.13) states "cultura animi philosophia est", that is "philosophy is education of the soul", in the history of Western thought[6] the cultivation, the care of the soul, the *cultura animi* recurs frequently when talking about the "exercise whose purpose is the order and welfare of the republic of sciences (Kant, 1781, in Remotti 2011, p. 4)". In this way, History constructs a first vision, a classical sense, of the term *culture*: something inconsistent with *customs*[7] and everyday vices, it elevates the man who "cultivates his own soul" to the ideal and abstract status of a citizen of a "republic of the sciences" – Remotti (2011, p. 4) points out: "an abstract society free from local and temporal constraints [...] without customs […]".

The meaning attached to the agricultural metaphor in some sense persists throughout history, "linked with terms and concepts such as *civilised*, *well educated*, *refined*, *cultured*", and "associated with the results of such refinement – a society's art, literature, music, and so on" (Spencer-Oatey, 2012, p. 15, my emphasis in the text). It exists today and can be found in the late 18th century. Matthew Arnold (1889), in the preface to his work *Culture and Anarchy*, writes that the whole scope of his essay is "to recommend culture" (p. viii) as it is "the study of perfection" (p. xi). In Arnold, as in the above-mentioned scholars, an aesthetic meaning of culture is conveyed:

> […] culture being a pursuit of our total perfection by means of getting to know [...] the best which has been thought and said in the world, and, through this knowledge, turning a stream of fresh and free thought upon our stock notions and habits, which we now follow staunchly but mechanically, vainly imagining that there is a virtue in following them staunchly which makes up for the mischief of following them mechanically. This, and this alone, is the scope of the following essay. And the culture we recommend is, above all, an inward operation (Arnold, 1889, p. viii).

[6] It should be emphasised that this recurrence and meaning belong exclusively to Western thought, and therefore cannot be generalised.

[7] In the Italian language the words *custom* (practices oft repeated by a multitude of people) and *costume* (set of clothes/traditional dress and outfit) are spelled the same way: "costume", and it is countable.

In Arnold's words, however, an element of innovation comes into play: the signs of time, the here and now, are considered. Culture is "the great help to get out of our present difficulties" (Arnold, 1889, p. viii), it is active in the present. Culture is no longer – as mentioned above – an aid "for an abstract society free from local and temporal constraints"; and so, as Remotti (2011, p. 4) adds, "[i]t can probably be argued that the modern conception of culture is a refutation of this claim to universality". Indeed, within the Age of Discovery, or the Age of Exploration, (i.e., the early Modern Period) – approximately from the 15th century to the 18th century in European history, an ethnographic enlargement of the concept of culture takes hold. Sea-faring European nations explore regions across the globe. The voyages and encounters with unknown peoples give rise to historical and philosophical reflections on the concept of humanity. Thus, from the writings of François-Marie Arouet (known by his nom de plume Voltaire) and Johann Gottfried Herder, the concept of culture could not fail to appear as divergent, alternative, if not contrasting, to the concept of *cultura animi*, which Cicero, Kant, Arnold, but also Hegel, Descartes and Bacon used. It is precisely because of its new content now made up of utensils and costume, and because its boundaries now coincide with those of the whole of humanity (Remotti, 2011). We may now speak of *cultura humanitatis*. So, although ethnocentric and not yet anthropological, the eighteenth-century thinkers claimed a broadening of the contents and boundaries of the concept of culture while maintaining the agricultural metaphor.

History thus leads to the first organic definition of culture, expressed by Tylor in 1871 in the opening lines of his work *Primitive Culture*:

> Culture or Civilization, taken in its wide ethnographic sense, is that complex whole which includes knowledge, belief, art, morals, law, custom, and any other capabilities and habits acquired by man as a member of society (1871, p. 1).

The unavoidability of the ethnocentric dimension is made explicit by Tylor [he writes: "in its wide ethnographic sense"]. This dimension extends the boundaries of the notion of culture and views it as a quality possessed by all people in all social groups. However, it is not yet anthropological. Indeed, from the second half of the 19th century, Cultural Anthropology took its first steps as an autonomous discipline of study. Scholars perceive the need to react to the previous tradition, to transcend the "high products" of the intellect, to establish scientific rather than aesthetic bases for culture (Dei, 2012). In particular, it is Franz Boas (1911) who dismisses value judgements of high and low culture and reacts to the evolutionism still present in Tylor. Boas emphasises the uniqueness of many and varied cultures (Spencer-Oatey, 2012). However, it was only after the Second World War that the world community of anthropologists recognises the world as composed of an irreducible plurality of cultures, understood as autonomous, distinct entities of equal dignity, non-hierarchically classifiable and in some respects non-commensurable (Dei, 2012).

Coming back to Tylor's definition, a general character (neither global[8] nor universal) emerges, which overcomes the exclusivity and partiality of the classical notion of culture [a "complex

[8] In the sense of Foucault (1969, in Lorusso, 2010, pp. 10-11).

whole" versus "an abstract society free from local and temporal constraints [...] without customs"] and initiates the basis for its modern vision as an integrated system - to be discussed later. To these are added the explicitness of the acquired (not genetically transmitted) ["acquired by man"] and contextual character ["man as a member of society"]. Moreover, in a strategic position within his definition, Tylor places the term "custom". The ethnographic origins of the concept of *cultura humanitatis* have indeed led to identifying as its contents those aspects of human behaviour that are endowed with *variable regularity* "from place to place and time to time, that is, between society and society and, within the same society, among different moments in its history (Remotti, 2011, p. 8)". This allows order to be brought to what had previously seemed a confusing thicket; modeling human behaviour and merging the concept of culture with that of custom/costume or habit[9]. This fusion highlights the external nature of culture: as Remotti puts it, "[f]rom its most rudimentary and primitive manifestations, culture is configured as a set of forms and processes that lie *between* human organisms and the outside world [...], constituting an extension *outwards*, or rather *into* the outside world – beyond the boundaries of organisms –, of both physical and mental potentialities and faculties"[10] (*ibid*., p. 11, my emphasis in the text). Examples include utensils and clothing, but alongside the technological examples – coming in leaps and bounds at the beginning of the 20th century – language also emerges.

In further support of the external character/outward features of culture, and of language as a cultural tool, we refer to Kroeber (1917) who emphasises the naked birth of man – i.e., devoid of customs and utensils, and the existence of a dichotomy between what is learned (cultural level) and what is inherited. Specifically, he equates what is inherited with what is organic (organic level). In this way Kreober goes further. With Kreober, exteriority becomes extraneousness. Culture is outside, above, the inner organic level. Using the paradigmatic example of language, he clarifies as follows: "[human] language is nonhereditary-as much so as […] the hole which, in conformity to fashion, he may or may not bore in his ears. It is not that speech is mental and facial proportions physical; the distinction that has meaning and use is that human language is non-hereditary and social, eye-color and nose-shape hereditary and [therefore] organic (*ibidem*, p. 171)". Language is thus historically regarded as proof of the external character of culture, being "the most overtly symbolic part of the cultural universe" (Remotti, 2011, p. 12).

Kroeber's conception of the extraneousness of culture is coupled with a stratigraphic vision of human reality. Clifford Geertz in fact writes:

> Man was a hierarchically stratified animal, a sort of evolutionary deposit, in whose definition each level – organic, psychological, social, and cultural – had an assigned and incontestable

[9] Consider here the double meaning of the term, both as a regular routine and as a piece of clothing.

[10] My translation. Original text in Italian: "[f]in dalle sue manifestazioni più rudimentali e primitive, la cultura si configura come un insieme di forme e processi che si collocano *tra* gli organismi umani e il mondo esterno […], costituiscono un prolungamento *verso* l'esterno, anzi *nell*'esterno – aldilà dei confini degli organismi –, di potenzialità e facoltà sia fisiche che mentali".

> place. To see what he really was, we had to superimpose findings from the various relevant sciences – anthropology, sociology, psychology, biology – upon one another like so many patterns in a *moiré*, and when that was done, the cardinal importance of the cultural level, the only one distinctive to man, would naturally appear, as would what it had to tell us, in its own right, about what he really was. For the eighteenth-century image of man as the naked reasoner that appeared when he took his cultural costumes off, the anthropology of the late nineteenth and early twentieth centuries substituted the image of man as the transfigured animal that appeared when he put them on (1973, p. 38).

In Geertz's words, the theoretical perspective that, in 1920s, retains the character of the exteriority of customs but forgoes its superficiality is manifested. Paleoanthropological discoveries allow scholars to differentiate between biological (organic) development and cultural evolution: hominids had some form of culture despite the fact that their brains were smaller than ours (Leroi-Gourhan, 1964; Geertz, 1973). The rejection of the temporal and diachronic order (first the brain and then the culture development) also implies the rejection of the hierarchical and synchronic order (stratigraphic vision). Such developments in Cultural Anthropology mean that costumes do not conceal or cover man, nor do they complement his biological development, but are an integral part of it. Returning to Geertz's words:

> there is no way to state in any precise and testable way the interlevel relationships that are conceived to hold (p. 42). [...] we need to look for systematic relationships among diverse phenomena, not for substantive identities among similar ones. And to do that with any effectiveness, we need to replace the "stratigraphic" conception of the relations between the various aspects of human existence with a synthetic one; that is, one in which biological, psychological, sociological, and cultural factors can be treated as variables within unitary systems of analysis (1973, p. 44).

Culture is therefore a system. It is an organised and coherent whole – to also echo Tylor, in which the components are interconnected, so much so that, as Ferraro (1998, in Spencer-Oatey, 2012, p. 15) notes: "a change in one part of the system is likely to produce concomitant changes in other parts of the system. […] culture changes beget other culture changes." Here, thus, is Geertz's definition:

> an historically transmitted pattern of meanings embodied in symbols, a system of inherited conceptions expressed in symbolic forms by means of which men communicate, perpetuate, and develop their knowledge about and attitudes toward (1973, p. 89).

Yet before this, in introducing his idea of an interpretative anthropology, Geertz feels the need to make explicit the consonance of culture with semiotics:

> *The concept of culture* I espouse, and whose utility the essays blow attempt to demonstrate, *is essentially a semiotic one*. Believing, with Max Weber, that man is an animal suspended in *webs of significance he himself has span*. I take culture to be those webs, and the analysis of it

to be therefore not an experimental science in search of law but an interpretative one in search of meaning (Geertz, 1973, p. 5, my emphasis in the text).

As Lorusso (2010) notes, some salient features of Geertz's approach already emerge from these two short excerpts of his work – we list those most relevant to our study:

- the idea of *webs of significance* (i.e. systemic, albeit flexible, organisations that define spaces). The organisation of culture is constitutively reticular, there is relatedness but no hierarchy. There is no isolation of elements endowed with an identity in themselves, but inter-definition and reciprocal relations, in function of which each element takes on meaning. Meaning, in short, is neither an essence nor an intrinsic property of objects, but a function that changes by virtue of the relationships in which it is embedded, and which can therefore never be understood in isolation but always and only within the context.
- the idea that culture lives in an *externalised* dimension and is therefore not something purely ideal or mental. Culture is something that comes out of the *animi* of people, communicable, socially shared. To understand, read the words of Anna Maria Lorusso:

  > In both Eco and Geertz, culture is something symbolic and externalised: a *common sense* of which we are unaware, but which nonetheless manifests itself outside our inner selves and "ensnares" us, taking us in its meshes without giving us any chance of getting out. There is no intuitionism or "introspectionism" possible (pardon the ugly term) in their study of culture. The meaning always derives from a cultural already-said, which it is a matter of tracing. (Lorusso, 2010, pp. 28-29, my emphasis in the text).

Although the spectrum is much wider, and some of the relationships between culture and other concepts and disciplines – such as nature, sociology – are far from exhausted, we choose not to go further. The fundamental nuts and bolts to our thesis, with respect to the definition of culture, are in place. We are interested, in fact, first of all to state that every moment of learning mathematics and, in particular, of professional development of mathematics teachers is contextual and therefore, according to what Tylor says, "acquired by man as a member of society". Only when the fact that (in our analysis, transformative) learning is contextual is taken for granted do we go on to investigate how the subject constructs the mathematical world and to do this means working within an interpretative meta-dimension of the "pattern of meanings embodied in symbols" (in Geertz's words). Let us then consider, in a next step, an interpretative semiotic dimension of research in mathematics education, in particular the analysis of teachers' professional development.

A theory of culture – beyond phenomenology to implement the hermeneutics of reality, i.e. to interpret the sign

While Saussure had written about observing "*a science that studies the life of signs within society*" (see previous paragraph), Cultural Semiotics observes semiosis *within culture*. That is, it observes what the sign represents for a given culture from two points of view: those who produce culture and those who observe that culture (emic-etic dichotomy). Thus, there is no true semiotics that is not a semiotics of culture. Umberto Eco (1968) states that semiotics is *necessarily* a theory

of culture. Semiotics of Culture, as a discipline, tries to go further than others applied semiotics. Through the text – of whatever nature it may be, not only the written text, but any portion of reality carrying cultural content, any creation that derives from the cultural subject: a dress, the body, a gaze, the paraverbal language –, we try to understand the cultural system that is behind that text and that has created that text. Moreover, texts give rise to culture. Texts allow culture to communicate with the outside, to self-represent, to translate itself into objects, into practices, into representations that the analyst can somehow see and interpret. Texts are the places where culture is modelled. Thus, culture becomes *textualised*. With the Semiotics of Culture, we do not only dwell on the characteristics of a purely linguistic semiotics of the text, as in Saussure's studies, but we go further towards that practice of interpretation that culture always absolutely demands of us as analysts.

The Semiotics of Culture, as the study of text, is thus proposed as frame of reference in which to consider the mathematics teachers' professional development practices. In fact, the discourses of mathematics teachers are cultural texts. Here then arises the last research question:

RQ7. Which theoretical lens(es), embedded in the Semiotic of Culture as study of text, allows such an analysis to be achieved?

LOTMAN'S SEMIOSPHERE

The Russian semiologist Jurij Michajlovič Lotman (1922 -1993) is the leader of the well-known school of Tartu (or Tartu-Moscow), in Estonia. In his 1984 essay (translated into Italian in 1985 and English in 1989) entitled *The Semiosphere* (Lotman, 1989), Lotman characterises his approach in terms of a substantial and explicit reversal of perspective with respect to both the two main Western semiotic traditions, that of Peirce which "takes the concept of a sign as the primary element of any semiotic system" (*ibid*., p. 42), and that of Saussure and the Prague school which "takes the antimony between language and speech (text) as its basis" (*ibid*., p. 42). He writes:

> […] despite the differences between these approaches, they have one essential characteristic in common: they take the simplest, atomic element as a basis and examine everything that follows from the standpoint of similarity to it. Thus, in the first case [Peirce], an isolated sign is taken as a basis for analysis, and all subsequent semiotic phenomena are regarded as sequences of signs. The second viewpoint [Saussurre] found particular expression in the endeavor to postulate the discrete communicative act, the exchange of a message between the addressant and addressee, as the primary element and a model of any semiotic act. As a result, an individual act of sign exchange began to be regarded as a model of a natural language, and models of natural languages as universal semiotic models, and an effort was made to interpret semiotics itself as the extension of linguistic methods to objects that had not been included in traditional linguistics. […] This approach corresponded to a rule of scientific thinking – moving from the simple to the complex – and definitely justified itself initially. However, it also contained a latent danger: heuristic expediency (convenience of analysis) begins to be perceived as an ontological property of the object, to which a structure is ascribed that moves

> from simple and clearly defined atomic elements to gradually more complicated elements. The complex object is reduced to the sum of simple objects. The past 20 years of semiotic research enable us to look at many things differently (Lotman, 1989, p. 42).

Also according to Vygotsky (1986), a look at the results of previous research on thought and language shows that all the theories transmitted since ancient times, vary between two extremes: from an identification or fusion of thought and language, to their absolute disjunction and separation. In all these theories the problem of the relationship between thought and language loses its meaning: if they are the same thing, no relationship can arise between them, or, if they are two separate elements, they are two distinct processes. The scholar considers it more appropriate to adopt another type of analysis: *by unity*. By the term "unity" Vygotsky indicates a product of analysis which, unlike the elements, retains all the fundamental properties of the whole and which cannot be further divided without losing them – not the chemical composition of water, but its molecules and their behaviour are the key to understanding the properties of water. The unity of verbal thought, which according to the author corresponds to these requirements, is the meaning of the word. In the meaning of the word, thought and language are united in verbal thought.
Lotman thus argues that traditional scientific approaches have led to considering *the whole* as the sum of its parts in an ontological way, while division into parts is only a heuristic necessity. In fact, no part taken separately can really function: "they function only when they are immersed in a semiotic continuum filled with semiotic structures of different types and with different levels of organization" (*ibid*., pp. 42-43). Lotman call this continuum a *Semiosphere*, by analogy with the concept of a biosphere introduced by Vernadskij.
As follows, Vernadskij (1945) describes the concept of *biosphere*, i.e. "the domain of life":

> "Living matter" is the totality of living organisms. It is but a scientific empirical generalization of empirically indisputable facts known to all, observable easily and with precision. [Instead, t]he concept of "life" always steps outside the boundaries of the concept of "living matter"; it enters the realm of philosophy, folklore, religion, and the arts. All that is left outside the notion of "living matter".
>
> In the thick of life today, intense and complex as it is, a person practically forgets that he, and all of mankind, from which he is inseparable, are inseparably connected with the biosphere [...]. Hitherto neither historians, scientists in the humanities, nor, to a certain extent, even biologists have consciously taken into account the laws of the nature of the biosphere—the envelope of Earth, which is the only place where life can exist. Man is elementally indivisible from the biosphere. And this inseparability is only now beginning to become precisely clear to us. In reality, no living organism exists in a free state on Earth. All of these organisms are inseparably and continuously connected—first and foremost by feeding and breathing—with their material-energetic environment.
>
> The outstanding Petersburg academician Caspar Wolf (1733-1794), who dedicated his whole life to Russia, expressed this brilliantly in his book [...]. Unlike the majority of biologists of his day, he relied upon Newton, rather than Descartes.

> Mankind, as living matter, is inseparably connected with the material-energetic processes of a specific geological envelope of the Earth—its *biosphere*. Mankind cannot be physically independent of the biosphere for a single minute (Vernadskij, 1945, p. 8, italics in original).

The organicist metaphor receives an illustrious recognition in those years, so much so that, in addition to the concept of the biosphere, Vladimir Ivanovič Vernadskij, Edouard Le Roy and Pierre Teilhard De Chardin introduce the new concept of the *noosphere*:

> Mankind [sic] taken as a whole is becoming a mighty geological force. There arises the problem of the *reconstruction of the biosphere in the interests of freely thinking humanity as a single totality*. This new state of the biosphere, which we approach without our noticing, is the *noosphere* (Vernadskij, 1945, p. 11, italics in original).

Since noosphere is the future living environment of the humankind, created in mutual agreement and on rational principles, a necessity immediately follows from this definition: how to understand the noosphere? The answer of the Tartu-Moscow school is unambiguous: semiotics must assist humankind in understanding both history and future. Ivanov in fact stresses both the scientific as well as the social value of semiotics and defines the main task of semiotics: "to describe the semiosphere without which the noosphere is inconceivable. Semiotics must help us orient ourselves in history" (Ivanov, 1998, in Torop, 2005, p. 160).
Thus, Lotman defines his new concept of the semiosphere in comparison to Vernadskij's noosphere:

> We should caution against confusing the term noosphere, introduced by V. I. Vernadskii, with the concept of semiosphere, which is our contribution. The noosphere is a specific stage in the development of the biosphere, a stage associated with the rational activity of man. […] The biosphere is situated on the surface of our planet and comprises the totality of living matter; it processes the radiant energy of the sun into chemical and physical energy, which in turn is directed toward reprocessing the "inert," nonliving material of our planet. The noosphere is formed when human reason acquires a dominant role in this process. Whereas the noosphere has a material and spatial existence that embraces part of our planet, the space of the semiosphere is abstract in nature. However, this by no means implies that the concept of space is used here in a metaphorical sense. It is a specific sphere, with the same attributes that are ascribed to a closed space. The realization of communicative processes and the elaboration of new information are possible only within this space (Lotman, 1989, p. 43).

This is an indication of Lotman's very contemporary interest in close collaboration between the sciences. The organicist metaphor helps to conceive of the Semiosphere as a single large environment and to distinguish it from other macrosemiotic visions, which think of the globality of meaning in terms of a network of infinite referrals.
The semiotic universe is no longer a totality of individual texts and languages mutually closed to one another, but a single mechanism or organism where not some specific text but the semiosphere appear primary. The single text (or element), with respect to the semiosphere as a whole, can assume the role of a fragment from which to reconstruct the whole, but, Lotman (1989) warns, the

reconstruction necessary to decode a text always leads, in reality, to the creation of a new language. The idea of the continuous reformulation of meaning is a recurrent feature of Lotman's thought, which accepts the structural lesson (of the Saussurian tradition) accompanying it with great attention to the dynamic and therefore also diachronic (not only synchronic) aspects of the phenomena studied.

Reading a text is like watching one of the hundreds of reflections of an object in a mirror fragment. The object that is reflected in a mirror is in fact also reflected in any fragment of it, which thus appears to be part of the mirror and at the same time similar to it (Lotman, 1989). Between the parts, however, there must be not only a relationship of similarity but also some difference, which makes the dialogicality of the system possible, just as in the communicative exchange the presence of two similar and at the same time different partners is necessary. Each element of the semiosphere is therefore a partner in the dialogue, while the semiosphere as a whole is the space of the dialogue, its condition of possibility.

This take Lotman to states that: "The semiosphere is the semiotic space outside which even the mere existence of semiosis is impossible (*ibid*., p. 44)."

> Just as we cannot obtain a calf by gluing together veal cutlets, but can obtain veal cutlets by cutting up a calf, so we do not obtain a semiotic universe by summing up particular semiotic acts. On the contrary, only the existence of such a universe, the semiosphere, makes each symbolic act a reality (*ibid*., p. 44).

The term semiosphere, however, can be used either in a "global" sense, i.e. the whole space of signification, or to identify a local and specific aspect of it, i.e. *that* specific semiotic space. Hence two main features characterise the Semiosphere: the need for a *topological organization* and delimitation, and the characteristic of *structural uneveness*.

The semiosphere is always surrounded by an extra-systematic space or one belonging to another semiotic sphere, therefore it must manifest a form of homogeneity and individuality or semiotic personality, a characteristic trait of a whole, of a collective term. A key concept in this sense is that of *boundary*. It is, however, a "porous" boundary, which, like the membrane of a cell, is permeable, and from a cultural point of view should be thought of as a place of continuous translation processes:

> a semiotic boundary is the sum of bilingual translator "filters"; passage through these "filters" translates a text into a different language (or languages) *outside* that particular semiosphere. The "bounded" nature of the semiosphere is manifested in the fact that it cannot possess contiguity with […] nontexts. For such texts to acquire a reality for it, the semiosphere must translate them into one of the languages of its own internal space, or "semioticize" nonsemiotic facts. Thus, the points along the boundary of the semiosphere may be likened to sensory receptors that translate external stimuli into the language of our nervous system, or to transmission units adapting the particular semiosphere to the world external to it (*ibid*., pp. 44-45).

The border thus unites two different semiotic spheres from the point of view of its immanent functioning but divides them from the point of view of their mutual self-description: being self-conscious in the cultural semiotic relationship means in fact being aware of one's own specificity, of one's own opposition to other spheres. The borderlands are those in which the most accelerated semiotic processes develop, which precisely because of their dynamism should then replace those momentarily at the centre. As the border is necessary for the semiosphere, it needs an "unorganized" external environment, and when this is lacking, it creates it, as witnessed by the traditional opposition between barbarians and civilization (Lotman, 1990). However, the essential understanding is that "a 'nonsemiotic' space can in fact be the space of another semiotics. What from within a particular culture may look like an external nonsemiotic world may be its semiotic periphery to an outside observer. Thus, where the boundary of a particular culture is drawn depends upon the position of the observer" (Lotman, 1989, pp. 48-49).

Now, the second feature of the Semiosphere, that is a mandatory internal irregularity and *unevenness* (неравномерность, Andrews, 2009), increases the complexity of understanding the topological organisation of the Semiosphere:

> A semiotic space is characterized by the presence of nuclear structures (more often several) with an explicit organization and a more amorphous semiotic world, which gravitates toward the periphery, and in which the nuclear structures are immersed (Lotman, 1989, p. 49).

There occur a number of what Lotman (1990) refers to as 'invasions', variously effecting the internal structure of the semiosphere. In any synchronic section – such as the taking of a photograph – of the semiosphere there are different languages at different stage of development in conflict, and some texts immersed in languages not their own, with the codes to decipher them entirely absent: "[the semiosphere] would not have a single coding structure but a set of connected but different systems" (*ibid.*, 125). To detail this heterogeneity of the semiosphere, Lotman uses the metaphor of a museum hall:

> imagine a museum hall where exhibits from different periods are on display, along with inscriptions in known and unknown languages, and instructions for decoding them; besides there are the explanations composed by the museum staff, plans for tours and rules for the behaviour of the visitors. Imagine also in this hall tour-leaders and the visitors and imagine all this as a single mechanism (which in a certain sense it is). This is an image of the semiosphere (Lotman, 1990, pp. 126-127).

Then, however, it should be remembered that all elements of the semiosphere are in dynamic, not static, correlations whose terms are constantly changing. In short, elements that refer to subjectivities that are also very distant from each other and scarcely translatable, but which are placed in a reciprocal relationship of dynamic tension, come to generate new traits and ways of meaning. A double tendency thus characterises the semiosphere: that of the elaboration of metastructural self-descriptions, and that of continuous reformulation and "reshuffling". While the former increases the rigidity of internal structures and slows down the development of the

semiosphere, the latter can lead towards what Lotman (1992) calls *explosions*, i.e. creative phenomena, dynamic tensions between highly conflicting moments. Throughout the semiosphere, however, zones of differing evolutionary speed are observed.

Here it is possible to introduce the issue of 'modelling systems' or – as nowadays[11] it could be said – 'languages' that shape reality. In Lotman, the so-called natural language is defined as a "primary modelling system", while (all) the other sign systems are defined as "secondary". To be honest, this differentiation is a consequence of principles of opportunity, due to the fact that natural language was for a long time the most studied language and it is linked to its "exemplarity": to its power to convey the "intuitive sense of structuralism", the mechanism of structuring reality by languages, thanks to its "evident systematicity" (Lotman & Uspenskij, 1973). The centrality of the natural (human) language is acknowledged by many scholars. For example, Pier Luigi Ferrari (2021) who states that although the language of mathematics is multimodal and includes several semiotic systems, within learning processes "a special role belongs to [natural human] languages" (Ferrari, 2021, p. 8); along with Ruqaiya Hasan who writes:

> [natural] language is not the only system for making meanings: semiotics is by definition multi-modal because there are many other kinds of sign systems such as gestures and expressions, pictures, tables and graphs, even the clothes we wear, the kind of houses we live in, the ways in which we organize our material environment and so on — all of these are interpreted by social beings as having some meaning or other. But amongst the different systems for making meaning, human language has turned out to be the most versatile: it is the most capable of conveying different sorts of meanings. Not only is it able to refer to concrete phenomena that are around us here and now, but it can also recall the past, and describe the future; it is capable of creating abstract concepts and structures which refer to phenomena that exist either in memory and/or in the world of imagination — the five human senses may not be able to make any contact with some phenomena but so far as language is concerned, it has no problem referring to them. It would seem that language is not bound by any physical constraints of time and space; logic or truth: the domain of the operation of language knows no other limit than human imagination. What is more, we can use language to describe any and all systems for making meaning, including language itself; it is, however, impossible to use any other semiotic system for discussing even the major attributes of language. All this appears to furnish a cogent argument for maintaining that amongst the semiotic systems, [natural] language offers much the widest scope, and clearly the possibilities of the mediation of meaning depends on the power of the meaning-making system. (Hasan, 2011, pp. 54-55)

[11] The term *language* nowadays is used in several ways, to designate the human capacity to learn and use at least one language but also, more generically, to indicate a family of codes. The latter meaning has given rise to expressions such as 'sign language', 'bee language', 'language of mathematics' and so on. The expression *semiotic system* is almost equivalent but less ambiguous. A difficulty is that in English the same word ('language') is often used to indicate both a language and a generic family of codes. (Ferrari, 2021)

However, it is Lotman's own practice of analysis that gives dignity to *each* language: in fact, he goes from literature texts to the study of the most classic artistic text and then up to that of the city and architecture (Lotman, 1998). Therefore, the dominant principle is that of *textualisation*, understood as the "appropriative translation of reality", which, filtered through languages, is transformed into *text*. Hence there is an extended textuality that ends up encompassing all cultural forms (Lotman, 1989). As already seen, the single text assumes the role of remembrance, reference, from which the whole can be reconstructed. So, the mechanism of creation of what is "one's own" takes place through the transformation of one's own memory and of the external texts received, and it enacts the double tension between homogenisation and differentiation that is for Lotman one of the structural paradoxes of the semiosphere.

The characteristics of unevenness, delimitation and textualisation provide a glimpse into the structure of the semiosphere, and in Lotman's words is written: "the structure of the semiosphere is *asymmetrical*" (Lotman, 1990, p. 127).

> Asymmetry finds expression in the currents of internal translations with which the whole density of the semiosphere is permeated. Translation is a primary mechanism of consciousness. To express something in another language is a way of understanding it. And since in the majority of cases the different languages of the semiosphere are semiotically asymmetrical, i.e. they do not have mutual semantic correspondences, then the whole semiosphere can be regarded as a generator of information. Asymmetry is apparent in the relationship between the centre of the semiosphere and its periphery. At the centre of the semiosphere are formed the most developed and structurally organized languages, and in first place the natural language of that culture. [...] The fact is that the semiosphere, besides the structurally organized language, is crowded with partial languages, languages which can serve only certain cultural functions, as well as language-like, half-formed systems which can be bearers of semiosis if they are included in the semiotic context. Compare the latter with a stone or a strangely twisted tree-stump which can function as work of art if it is treated as one. An object will take on the function ascribed to it (Lotman, 1990, p. 128).

What Lotman is suggesting, therefore, is that the conflicting texts should not be underestimated, but rather firmly embraced. It is they that reveal the asymmetry and thus generate meaning, but only if considered within their own continuum, i.e. as embedded in the semiosphere. With the concept of asymmetry, according to Lotman, it is evident that the value of communication does not lie in what is shared at the outset, but in the possibility of bringing into confrontation the respective diversities, memories and languages that are not shared. The paradox is that the value of communication lies in what makes it difficult, even impossible, precisely because it is in this situation that the need to "translate the untranslatable" is imposed, thus generating new culture (Lotman, 1945). Culture itself, although it offers the means of potential simplification, presents itself as an internal mechanism that produces complexity.

The semiosphere, as we have already seen, needs to control its internal heterogeneity and, to do so, it must be able to create a unitary image of itself, a coherent self-description. Therefore, in

order to describe itself, the semiosphere resorts to a dynamic spatial modelling, i.e. subject to continuous modification, and in any case dependent on the position of the observer. The dimensions that distinguish the models, schematised as in Figure 1. 8, are:

- *comprehensiveness*, i.e. the types of partition of universal space (does the semiosphere occupy all of universal space or is it only a part of it?);
- the *dimension of universal space* (does universal space have known and, therefore, controllable boundaries or does it evolve and potentially expand infinitely?);
- *orientation* (within the overall space, in which direction is the semiosphere oriented?).

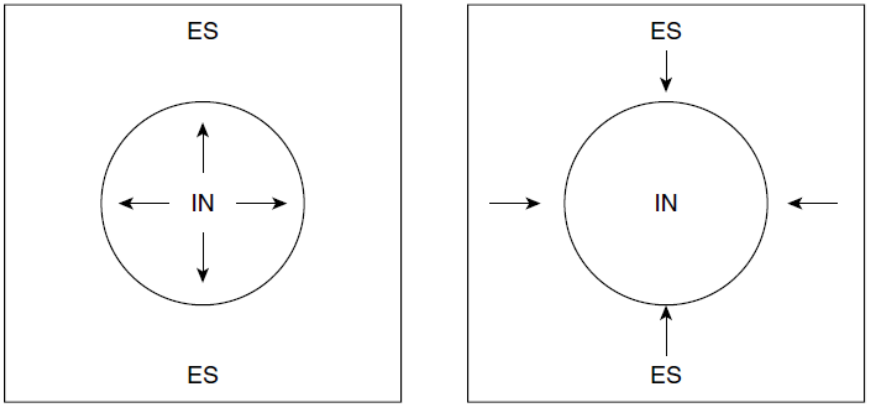


Figure 1. 8 The spatial dynamics of semiosphere (Lotman & Uspenskij, 1969 [2003]).

Although Lotman's classification is not ended at this stage, it is no longer useful for the present work of analysis to introduce others. Rather, my research requires a networking and a contextualisation of the Semiotics of Culture theoretical framework within research in Mathematics Education.

In the following paragraph a proposal of the conceptualisation of the ecological dimension of the educational system is presented. It is not the only possible proposal, but it is selected for three reasons: first of all because ATD offers some theoretical tools to identify the conditions that enable to go beyond the narrow space of the classroom and properly study what is in it. Hatano and Inagaki (1998) consider "the notion of [teaching] 'practice' as a link between culture […] and the larger cultural contexts (*ibid*., p. 80)". Such a consideration affects teachers professional development practices as well. In this sense, practices can be described as *praxeologies*. *Praxeology*, according to Chevallard (1985), is the know-how (praxis) and the know-why (logos – the discourses that justify the know-how) related to a task. *Praxis* is made by two components: the *task* and the *technique*(*s*) to solve the task. *Logos* is made by two components as well: *technology* (in this context, the discourse that justifies the technique) and *theory* that supports the discourse (Chevallard, 1985). Chevallard (2002) suggests that praxeologies are not only influenced by the teacher's decision, but at a higher level by the society in which the teacher and the student are immersed. Secondly, Chevallard's concept of *didactic transposition* paves the way to the concept of *cultural transposition*, which is a further milestone of my research work. Thirdly, in a

large majority of the studies in the Western literature the Lesson Study, i.e. the practice being analysed in my research, is framed in this theoretical context and therefore it is not negligible Clivaz, 2015; Rasmussen, 2016; Miyakawa & Winsløw, 2019; Huang, Huang, & Bosch, 2021; or also many still unpublished studies by young researchers from the WALS[12] community).

The ATD: didactic transposition and levels of didactic codeterminacy

The systemic aspects (Font, 2002, pp.143-156) of teaching-learning processes are the subject of important theoretical elaborations in the literature in Mathematics Education, among which the *Anthropological Theory of Didactic* (ATD) of Yves Chevallard. The ATD recognises the institutional and ecological dimension of mathematical knowledge as one of its main focuses. Forcing the researcher to direct his attention to the human activities dealing with mathematics (e.g., communicating mathematics and not only solving problems) is a major achievement of the anthropological approach. Its central argument is that "TAD places mathematical activity [...] within the set of human activities and social institutions" (Chevallard, 1999). The initial construction that Chevallard (2019) makes of his "theory of knowledge", within which didactics can be situated, is based not so much on the naive realist[13] notion of the *meaning* of an object (of knowledge, in general; mathematics, in particular) as on that of *rapport à l'objet*, of relationship with/to the object. Central to all this is the person who relates to the object (or the institution, as a set of persons), and not the object itself:

> An object exists when a person X (or an institution I) recognises this object as existing for him/her/them. More precisely, the object O shall be said to exist for X (respectively for I) if there exists an object, represented by R(X,O) (respectively R(I,O)) and called a *personal relation* from X to O (respectively an *institutional relation* from I to O) (Chevallard, 1992, p. 9).

ATD object of study is indeed a ternary relation: the *didactic system*=(students, teacher, mathematical knowledge) (Chevallard, 1989; 2019), which cannot be understood except in relation to the (external) environment that surrounds it, i.e. the society.

Mathematical *knowledge*, despite the fact that the term 'knowledge' is uncountable in English, is far from being univocally determined. Several authors have argued on this issue, enabling important theoretical reflections to flourish (Shulman, 1986; Chen, 2002; Rovegno, Chen &

[12] The World Association of Lesson Studies https://www.walsnet.org/

[13] To clarify this difference Kutschera's words are quoted: "On this [realist semantic] interpretation the meaning of a linguistic expression does not depend on its use in concrete situations, but the use is determined by the meaning, so that a sharp separation between semantics and pragmatics is possible" (Kutschera, 1975, pp. 19-20). Applying the ontological assumptions of realist semantics to mathematics, a Platonic approach to mathematical objects is necessarily derived. They have a real existence that does not depend on human beings, since they belong to an ideal domain. In realist semantics, 'to know' from a mathematical point of view means 'to discover' entities and their relations. This approach implies an absolutism of mathematical knowledge as a system of truths that are certain, eternal, unchangeable by human experience, since they are prior to it or, at least, foreign to it and independent of it.

Todorovich, 2003; Ball, Thames & Phells, 2008; Rowland & Ruthven, 2011; Carrillo, Climent, Contreras, & Muñoz-Catalán, 2013; Ribeiro, Mellone & Jakobsen, 2013; Schoenfeld, 2013; Cusi & Malara, 2015; Di Martino, Mellone, Minichini, & Ribeiro, 2017). For Chevallard the non-univocality of the term 'knowledge' is evident when attempting to define what the teaching problem is, that is, what affects teaching practices, what is part of teachers' concerns.
The teaching problem is indeed what affect teachers in moving from what Chevallard calls the *knowledge to be taught* to what is called the *taught knowledge*. The first one is initially just a project, something to be done. The second is knowledge implemented during the teaching process. Between these two pieces of knowledge there is no one-to-one correspondence: from the same curriculum proposal or description different activities can be implemented in the classroom.

Teachers' practice is to wonder *how* to move from knowledge to be taught to the taught knowledge. Simon and Tzur (1999) theorise it as an open and complex concept:

> Our use of the term *teachers' practice* indicates not only everything teachers do that contributes to their teaching (planning, assessing, interacting with students) but also everything teachers think about, know, and believe about what they do. In addition, teachers' intuitions, skills, values, and feelings about what they do are part of their practice. Thus, we see a teacher's practice as a conglomerate that cannot be understood by looking at parts split off from the whole (i.e., looking only at beliefs or methods of questioning or mathematical knowledge) (Simon & Tzur, 1999, pp. 253-254).

Therefore, teachers' practice is also to wonder how to obtain what Chevallard calls the *learned knowledge*: a new entity that does not necessarily coincide with the taught knowledge.
The whole process, between the three pieces of knowledge mentioned above, is what ATD defines the *internal didactic transposition* (see Figure 1. 10). It occurs mainly within the educational system, that is in the schools, in the classrooms (meaning by class the teacher(s)/student(s) system) and in the groups of students within the classrooms. But the knowledge to be taught comes from a system surrounding the educational system, a kind of membrane, the *noosphere* (see above, p. 38, the discussion about this notion in Vernadskij). For Chevallard it is an intermediate area, a filter, playing the role of a buffer between the educational system and the external society (see Figure 1. 9). It is a space that Chevallard (1982) re-contextualises within the Mathematics Education field of research. The noosphere is "the sphere of those who think" about the educational system. Starting from Homer, noûs (νοῦς) – contraction of the Ionic analogue νόος, (nóos) – is a term that in ancient Greek indicates the faculty of understanding an event or someone's intentions, the mental faculty and therefore the intellect. Vladimir I. Vernadskij, Pierre Teilhard De Chardin, and Edouard Le Roy are the three scholars (respectively a biologist, a Jesuit theologian, and a philosopher-theologian) that, at the beginning of the 20th century, theorise the noosphere as the whole, the complex unity, of human's conceptions, ideas, feelings, and emotions, in analogy with the very concept of biosphere. In order to understand the idea underpinning the conceptualisation carried out by the three scholars, it is useful to bear in mind, for the sake of clarity, Popper's theory of the three worlds, which defines World 1 as the set of physical things, World 2 as the interiority

of each individual with his wealth of knowledge and experience, and World 3 as the set of human concepts and labours. World 3 corresponds to the noosphere itself, to the extent that the products of individual minds are poured into the collective heritage. If, absurdly, the many Worlds 2 were completely isolated from each other, the noosphere would not exist at all (Mantovani, 2009). The noosphere is then a complex unit insofar as individual products are able to interact, thus making the emergence of unpredictable novelties possible. The noosphere is not only a filter from the external society to the educational system, rather the noosphere activity is also a stimulus for a social impact of the taught knowledge.

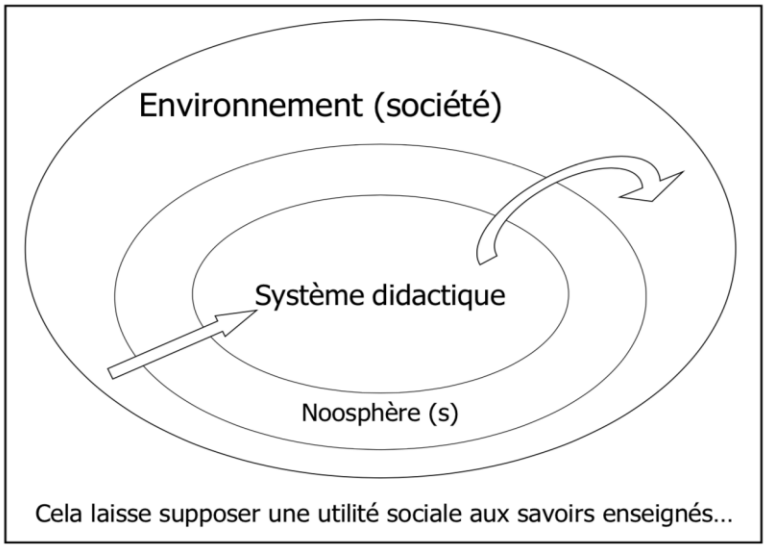


Figure 1. 9 Social constraints influencing the transposition
(Becù-Robinault, 2009, p. 10 adapted from Chevallard, 1982, p. 9).

Within the educational system the knowledge to be taught is something known, established and provided. Whereas when researching, especially in ATD, thus operating within the noosphere, another perspective is used. The piece of knowledge involved in the teaching-learning process is questioned. A new entity is considered which does not usually participate in the formulation of the teaching problem, the scholarly knowledge: the knowledge produced by scholars, those who best know/should know the knowledge to be taught, and who use it. Scholars are those who provide the epistemological legitimacy of the knowledge that is taught at school. "Any human activity […] has its own scholars, held 'to know best' than the rest of the people in the little world where they belong" (Chevallard, 2019, p. 76). This is particularly accentuated among the Italian academic tradition in which aspects of general pedagogy are traditionally separated from the problems of teaching science and mathematics. Such a task in our context surely falls to mathematicians. It is a challenge for the universities, and experts in the discipline of mathematics have to be in charge of it. Chevallard then makes an important assumption: what is taught in school is something that generally comes from outside (at least except when school and scholarly institutions are the same, i.e., when we learn from scholars directly). In general, what is taught in school exists outside the school, it is not an invention of the school. Scholarly institution legitimates and guarantees the authenticity of the taught knowledge. At this stage, the enlargement of the ATD unit of analysis is

evident: the *external didactic transposition* is added, considering the relationship between the knowledge to be taught and the scholarly knowledge. If the internal didactic transposition is related to the teaching problem, the external didactic transposition is related to the curriculum problem, that is, what is to be taught and how (see again Figure 1. 10).

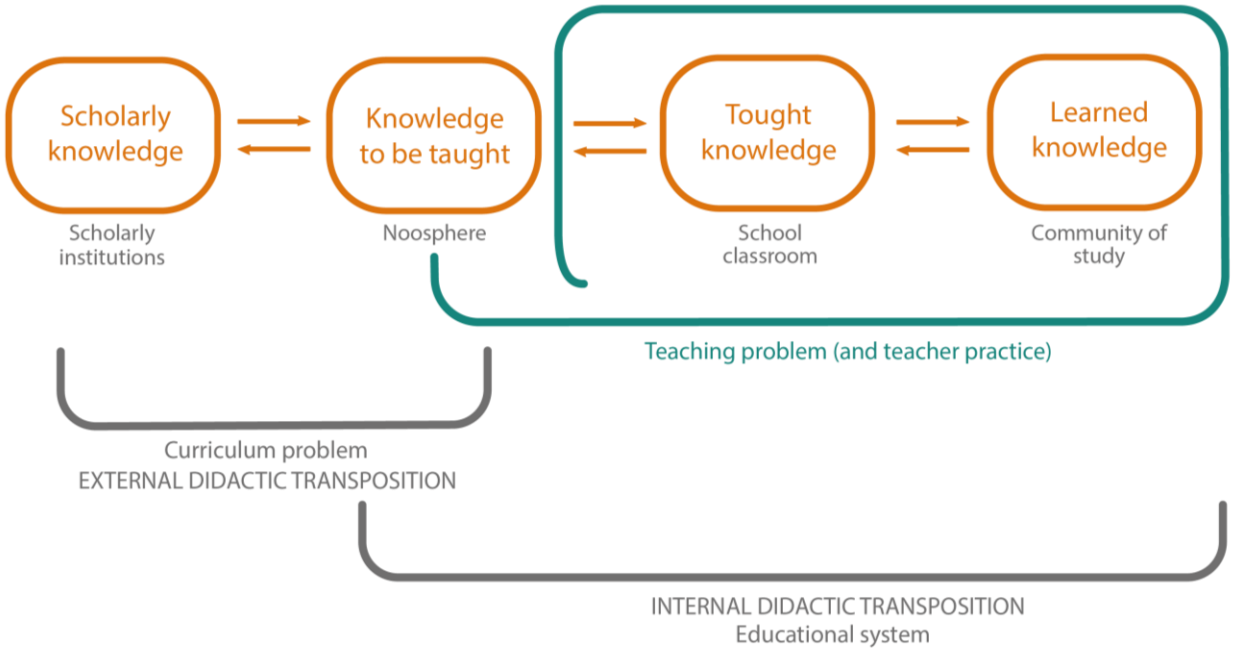


Figure 1. 10 The process of Didactic Transposition.

Questions that can interrogate scholarly institutions, the noosphere or the entire external society are for example those proposed by Marianna Bosch in the section dedicated to Yves Chevallard, module 2, within the new ICMI website dedicated to the innovative ICMI AMOR project[14]:

- How has the status of "Proportionality" varied in scholarly knowledge from the 17th to the 21st century? What differences appear between mathematicians and other scientists today?
- What were the main elements of the "theory of ratios and proportions"? How did they appear in the Arithmetic and Algebra old textbooks? What was the role played by quantities in this ancient mathematical organization?
- How did the Modern Math reform modify this content? What new links were introduced? What old connections were vanished?
- What elements of the "theory of ratios and proportions" have been reintroduced in the new curriculum and how? How do the old and new mathematical contents coexist? What redundancies, incoherencies, disconnections, etc. might appear? What is the role played by quantities in current school mathematical organizations?

[14] https://www.mathunion.org/icmi/2009-hans-freudenthal-award

Nevertheless, there are also other examples of issues that are questioning researchers in Mathematics Education concerning didactic transposition with respect to the (external) environment. Some paradigmatic examples are presented.
Bill Barton (2008), who benefits from knowing the Māori language, tells of his experience in 1987 of curating an edition of the Māori Language Dictionary of Mathematics (
Figure 1. **11**). The Māori primary school, in fact, is bilingual: all subjects, including mathematics, are taught in both English and Māori. The issue then arises of finding a suitable translation in the Māori language for the statistical terms "discrete" and "continuous". A Māori elder is not satisfied with the transliteration of the terms *konitinu* for "continuous" and *tihikiriti* for "discrete". He goes for some existing words, such as *ikeike* (height), and *tae* (score), but these terms are not representative for mathematicians. Then metaphors are proposed, many are discarded, until when the terms *rere* for "continuous" and *arawhata* for "discrete" are proposed.

> Those of us in the room with only a little Māori understand the common meanings of these words as "flying" and "ladder". It does not seem good enough for us. But the eyes of the good Māori speakers light up. They know that these words as a pair refer to the way a stream flows, either smoothly without a break, or in a series of little waterfalls over rocks. This mirrors the way that continuous data is information taken from a smooth stream of possible measurements, and discrete data is information that can only have particular values. Yes. New technical vocabulary is born (Barton, 2008, pp. 1-2).

Probably not many Italian (or English-speaker) teachers would have thought of the flow of the river and of the waterfall to describe statistical concepts; however, they are significant in the Māori cultural context. Who knows in other contexts. So, says Barton (2008, p. 6), "I became curious about the way that mathematical ideas are presented differently in other languages".

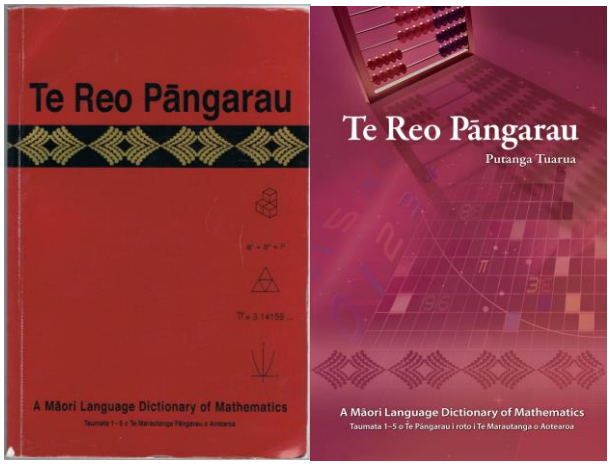


Figure 1. 11 First and second edition of Te Reo Pāngarau: a Māori language dictionary of mathematics, covering levels 1 to 5 of the pāngarau (i.e., mathematics) curriculum.[15]

[15] https://kupengahao.co.nz/product/te-reo-pangarau/
https://kohingarauemi.tki.org.nz/Catalogue-Items/Te-Reo-Pangarau-A-Maori-Language-Dictionary-of-Mathematics

The second paradigmatic example is Kim, Ferrini-Mundy and Sfard's (2012) study on the conceptualisation of the term *infinity* between English and Korean speaking university students. The authors address a sensitive topic for two countries that are both highly developed. In Korean, there is a word for colloquial idea of infinity and one for formal infinity. Since words are basic tools with which one "tinkers", it is important to be attentive to the use of words in teaching-learning in mathematics. In the Korean context, authors note, the linguistic gap is reflected in a gap in mathematical discourse (Sfard, 2008 – see p. 56). Discourse in the early stages of learning rests on the first two levels of mathematical discourse (see Table 1. 1). Only with the introduction and use of the second term – formal, and inspired by a Chinese term, i.e. foreign – the higher levels of mathematical discourses on infinity can be achieved. In the Korean context, the challenge is therefore to carry the discourse developed in the lower levels into the higher ones. Even if this carriage occurs, in a sense the gap persists: it is the word itself that changes. In the context of English speakers, on the other hand, the opposite problem occurs: linguistic continuity exists, the term used to denote infinity is always the same, but it is precisely the continuity that poses a challenge. Students' discourse tends to be anchored at lower levels failing to evolve to higher levels. Once the word is associated with that kind of infinity, with that epistemic concept, expanding the meaning of that word may be a struggle. The student may face difficulties in using the same word in a more advanced mathematical discourse in which he/she speaks about infinity in a different way. Therefore, the considerations of Kim and colleagues are as follows:

> Mathematics, known as a "universal language" or lingua franca of the modern world, is widely believed to be independent of the language in which it is practiced. And yet, as has already been noted by mathematics education researchers, things may be not as simple as that. [...] the claim that mathematics is language-dependent is a part and parcel of the growingly popular ethnomathematical stance (Ascher & D'Ambrosio, 1994), according to which mathematics is sensitive to cultural idiosyncrasies, including those related to language. [...] language may impact the learning of mathematics (Kim *et al*., 2012, p. 86).

Table 1. 1 Model of development of English infinity discourse (ID_E): the different levels and the corresponding Korean words (adapted from Kim *et al*., 2012, pp. 90-94).

level	main theme	visual mediator	Korean words	
			formal	informal
ID_E0*	Numbers (finite phenomena)	n		
ID_E1	Infinite processes Counting without end	…	mu-su (= none number)	kkeut up-eum (= end without)
ID_E2	Infinity as a limit	∞	mu-han (= none bound)	//
ID_E3	Infinity as an object in itself Set cardinality	$\aleph_0$		

*ID_E0: English infinity discourse, level 0

As a third paradigmatic example, the research work of Geoffrey Saxe – anthropologist, researcher at the University of California, Berkeley –, carried out with the Oksapmin community in Papua New Guinea between 1978 and 2001, cannot be overlooked. Saxe (2014) is interested in the interaction between cultural and cognitive processes. A case study includes Saxe's encounter with a grade 4 child from the Oksapmin school in 1980. The child adapts the Oksapmin 27-body part counting system (see Figure 1. 12) to solve arithmetic problems.

The child is posed the following problem: *You have 16 pigs; you give seven of them away; how many are left?* The child starts solving the problem from the "ear on the other side" (i.e., the cardinal number 16). He indicates that this ear (16) is "given away", and that it corresponds to the thumb (1); then the "eye on the other side" (15) is also given away, and it corresponds to the index finger (2); the nose (14) is given away, and it corresponds to the middle finger (3) ... The child advances to the shoulder (10), it is given away and corresponds to the forearm (7), leaving the biceps (9), i.e. the solution of the problem. Saxe (2014) calls this approach a *double enumeration*. It is not taught; this approach emerged when Oksapmin children struggled to make sense of the mathematics taught in school in the post-colonial approach to education.

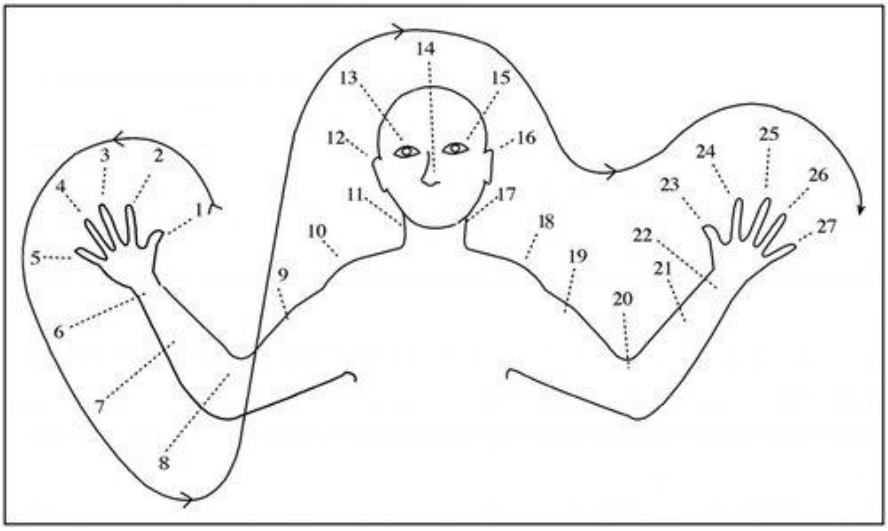


Figure 1. 12 Oksapmin 27-body part count system (Saxe & Esmonde, 2005, p. 181).

This last example also provides an opportunity to reflect on the fact that, in the recent past, major school reforms in mathematics have been achieved in many countries of the world. Changes have taken place at all levels of mathematics in the school education system. It is not only epistemological aspects (such as the advent of the Bourbakist movement, which aimed to base all mathematics on set theory and mathematical structures - algebraic, order, and topological. To do mathematics is to grasp the invariant in all situations.), or cognitive aspects (for example with Piaget, who states that actions produce thought, conceptualisation. The scheme allows, through actions, to grasp the structure of things. In everything in the world there is already a structure, it is a question of grasping it.), but also historico-political aspects (such as the so-called "Sputnik effect", i.e. the need to keep up with the Russians) that have had a major impact on mathematics education. At the OECD conference in 1959, the contrasts between taught mathematics and

research mathematics become apparent for the first time on an international level. Dieudonné utters the famous cry: "A bas Euclid. Mort au triangle", signaling the outdatedness of Hellenistic geometry teaching and, more generally, of all traditional teaching. Then the studies and theories of behaviourism, the development of cognitive sciences and constructivism have a strong impact on the pedagogical approaches advocated in the reforms of mathematics curricula in the second half of the 20th century (Ciarrapico & Berni, 2017). But in reaction to this change, it is born the "Back to basics" movement: "a returning to the view that success in mathematics meant being able to compute accurately and quickly" (National Research Council, 2001, p. 115), and the focus on problem solving is established. It is then the broad international comparative studies on student achievement, such as the Trends in International Mathematics and Science Study (TIMSS) and the Programme for International Student Assessment (PISA), that attract much political and media attention today. From the ICMI Study 24 Discussion Document it reads:

> In recent years the internationalisation and globalisation of the economy, universality of technological development and related needs for new skills and knowledge play the role of strong motivations for curriculum reforms that have brought calls for unified standards for mathematics in school (ICMI, 2017, p. 3).

The stress is mainly on promoting:

- specific and more or less sophisticated ways of Thinking, i.e. Critical thinking (Paul & Elder, 2008) and High Order thinking (Thompson, 2011);
- Inquiry-Based Mathematical Education (Bybee, 2002; Rocard *et al.*, 2007; Dorier & Maaß, 2014; D'Acunto *et al.*, 2018);
- Quantitative literacy (Steen, NCED, 2001).

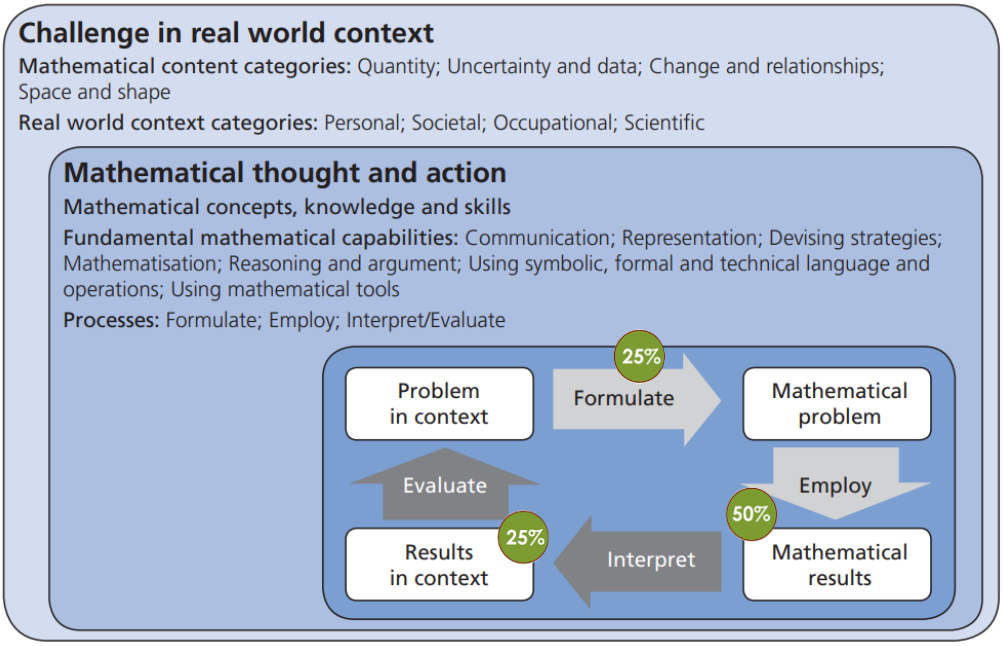


Figure 1. 13 PISA2012's model of mathematical literacy
(freely adapted from OECD, 2013, p. 26 fig. 1.1 and p. 38 table 1.1).

Comparing the PISA2012 framework (OECD, 2013) with the new PISA2022 mathematics framework it is evident how the virtuous problem-solving (mathematical modelling) cycle is now substantially modified by the addition of the mathematical reasoning process.
In PISA2012, the cycle is designed with a distribution of the percentage of weight, which is suggested to be given to the processes of formulating situations mathematically (25%), of using mathematical concepts and reasoning (50%) and of interpreting and evaluating mathematical results (25%), as expressed in Figure 1. 13. Within the PISA2022 framework, the previously absent "reasoning" item is added (see Figure 1. 14) and the profound relationship of the cycle to the needs of a globalised world is made explicit. This is without neglecting the changes promoted by spurs from the world community of researchers in pure and applied mathematics and in mathematics education, for example, with the STEM programmes for mathematics.

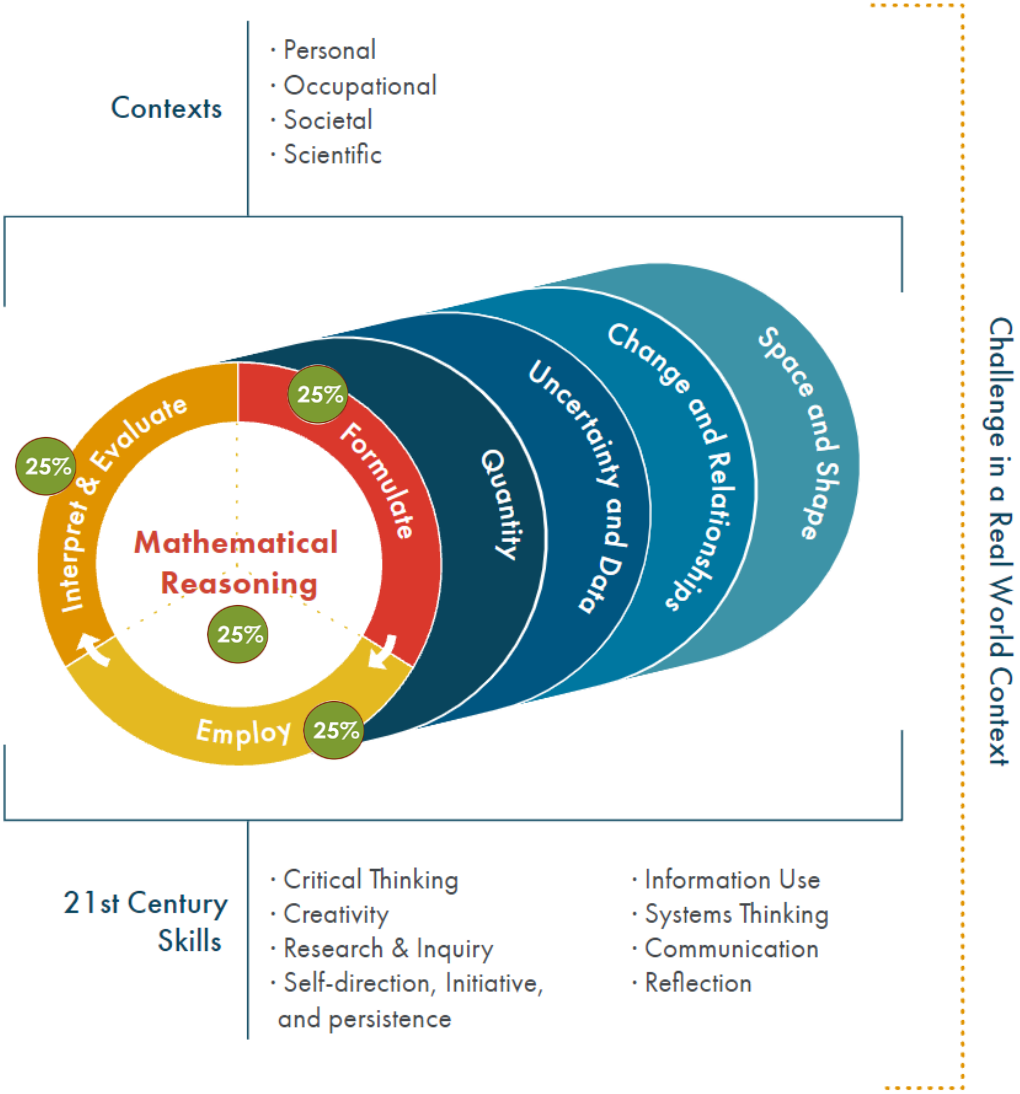


Figure 1. 14 PISA2022's model of mathematical literacy (freely adapted from interactive and multi-language web page, and from OECD, 2018, p. 10 fig. 2 and p. 33 table 1).

At this point, a delicate issue deserves to be taken into account, which is posed internationally as an explicit (e.g. EMF 2015 and ICMI study 24) or implicit (e.g. in WALS) research topic: many "universalistic" proposals apparently ignore local cultures, and consequently students' and teachers' cultures.

> Universal knowledge is not taught directly but undergoes transformations at different levels of the didactic transposition chain. […] To what extent are cultural roots taken into account in these transformations? (EMF, 2015, p. iv, my translation)

In order to be able to comprehensively answer to the above-mentioned research problems, a broader field of analysis than mere cultural transposition scale is required to delve into the study of the (external) environment.

The ATD, like other systemic theories – e.g., with due differences, the OSA (Godino, Batanero & Font, 2019), which advocates also the prescriptive character of didactics research –, aims to gear its research development also towards an "ecological approach". Under this term (ecology), French researchers refer to the study of conditions and constraints of teaching and learning processes (Chevallard, 2007). It is a way of problematizing educational reality: "we do not think in terms of what should be done, for instance how to better teach or to better learn, but in terms of possibilities, what can be done and how and mainly what cannot be done and why". As Marianna Bosch states (module 5, ICMI AMOR project[14]): "Before trying to change our educational reality, it is critical to better know the conditions and constraints that can explain the current state of things. This requires researchers to adopt a broad perspective about the didactic activities they want to describe and analyse, in order to go beyond the level of the mathematical activities or contents at stake." A key tool for the analysis of the ecology, developed within the ATD, is the *scale of level of didactic codeterminacy* (see Figure 1. 15). The specific levels relating to the structure of a given discipline are the lower levels of the scale. The higher, the *generic* levels, are common to the teaching of any discipline.

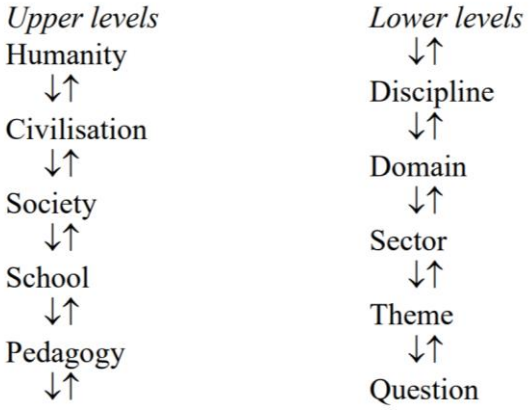


Figure 1. 15 Scale of levels of didactic co-determinacy (Florensa, Bosch, Cuadros, Gascón, 2018, p. 5).

The flow between levels reveals Chevallard's attention to cultural aspects. However, according to Bosch and Gascón (2006) but also to Florensa and collegues (2018), within the noosphere there is "a strict separation between instructional processes and the 'content' of these processes" (Florensa *et al*., 2018, p. 8), that is, using the scale of didactic co-determinacy, between the upper and the lower levels. The teaching problems are conceived as independent of the taught (or to be taught) content. Marianna Bosch (module 5, ICMI AMOR project[14]) offers this question, which might help to focus on how to deal with and perhaps overcome this division: "What methodology one can use to analyse the condition and constraints coming from the higher level of the scale?"

LEARNING, DISCOURSES, AND CONFLICTS – and their analysis in Mathematics Education

It is notoriously difficult to define concepts in a satisfactory manner, especially concepts that are as broad and abstract as the concept of *learning*. To some extent, the lack of consensus about the definition of learning should not come as a surprise. Learning has been a central topic in psychological research virtually since the inception of psychology as an independent science, however, it seems that all learning researchers carry with them some idea of what learning is (De Houwer, Barnes-Holmes, & Moors, 2013). Maybe it is because learning is a "*hypothetical construct*: it cannot be directly observed, but only inferred from observable behaviour" (Gross, 2010, p. 159).

Traditional educational studies conceptualize learning as the "acquisition" of entities such as ideas or concepts, no matter if the term "acquisition" is interpreted as passive reception or as active construction. The acquisitionist approach relies on the idea of cognitive invariants that cross cultural and situational borders. Consequently, the theories that come from acquisitionist tradition are geared toward finding and investigating what remains constant when the situation changes. And yet, as argued by many authors (e.g., Lave, 1988; Cole, 1996, Andrews, 2010, Artigue, 2008), human learning is too dynamic and too sensitive to ongoing social interactions to be fully captured in the terms of decontextualized mental schemes, built according to universal rules. The disillusionment with *acquisitionism*, although greatly precipitated by the advent of digital recording, began, in fact, prior to the advances in data-collecting techniques. Cross-cultural and cross-situational studies that had proliferated since the first decades of the 20th century systematically undermined acquisitionist claims about developmental invariants. Their results drew researchers' attention to the social and cultural contexts of learning. Learning is then defined as "a social, cultural, and historical activity (Cobb & Bowers, 1999; Polly, Allman, Casto, & Norwood, 2018; Vygotsky, 1978)" (Erbilgin & Arikan, 2021). The foundations of a new learning conceptualisation, in a mathematics field, ascribed to this change of prospective are that: mathematical knowledge is created and agreed to by a community because of a need to explain, interpret, communicate or explore (Hersh, 1979); learning is continuous, evident in every aspect of our lives, there is no one final 'knowledge' in any domain (Vollrath, 1994); and participating in activity, including a social activity or personal reflection, impacts on our knowledge, understanding and interpretation of the world, hence results in learning (Engeström, 1999; Vygotsky, 1978). According to this new prospective, Sfard define learning as "a special kind of

social interaction aimed at modification of other social interactions (Sfard, 2001, p. 25)". Thus, rather than looking for those learner's properties that can be held responsible for his/her constancy in cognition, a framework that allows to stay tuned to the interactions from which the change, the transformation, arises is needed, without rejecting the acquisition metaphor, but rather subsuming this more traditional outlook, while modifying its hidden epistemological infrastructure. Psychologists, sociologists, anthropologists, and cultural studies scholars now conceptualise developmental transformations as changes not in individuals, but rather in what and how people are doing, and they claim that patterned collective activities are developmentally prior to those of the individual. Sfard (2007) calls this a *participationist* perspective.

One basic principle of the participationist perspective is the overcoming of the thinking-communicating dualism. Although, even today, thoughts are "conveyed" or "expressed" in the act of communication, and this implies two distinct processes, that of thinking and that of communicating, with the former slightly preceding the latter and constantly feeding into it.

> Whereas acquisitionists have been working with this dualist vision of human cognition for centuries, participationists are likely to view the idea of thought conveyed in communication as but a direct result of an unhelpful objectification. With Wittgenstein (1953), they believe that "Thought is not an incorporeal process which lends life and sense to speaking, and which it would be possible to detach from speaking" (p. 108). Having accepted this claim, one can also see that it remains in force when the somewhat limiting word *speaking* is replaced with the more general term *communicating*. Consequently, thinking stops being a self-sustained process separate from and, in a sense, primary to any act of communication and becomes an act of communication in itself, although not necessarily interpersonal. To stress this fact, I propose to combine the terms *cognitive* and *communicational* into the new adjective *commognitive*. The etymology of this new word will always remind us that whatever is said with its help refers to phenomena traditionally included in the term *cognition*, as well as to those usually associated with interpersonal exchanges (Sfard, 2007, p. 570, italics in original).

And,

> Thinking is a special case of the activity of communicating. […] This is true whether the thinking is in words, in images, or in any other symbols. Our thinking is clearly a dialogical endeavor, where we inform ourselves, we argue, we ask questions, and we wait for our own response. If so, becoming a participant in mathematical discourse is tantamount to learning to *think* in a mathematical way (Sfard, 2001, p. 26, italics in original).

Cobb and collegues (2011) write about the collective learning of the classroom community in terms of the evolution of classroom mathematical practices. Lerman (2002) outlines the principles of a cultural, discursive psychology, where learning is seen as an initiation into the practices of school mathematics including learning to speak mathematically. The teacher therefore has a vital role in showing what is approved within the discourse, i.e. the accountability to the discipline.

> In the mathematics classroom, interactions should not be seen as windows on the mind but as discursive contributions that may pull others forward into their increasing participation in mathematical speaking/thinking, in their zones of proximal development (*ibid*., p. 89).

Thus, in a cultural, discursive psychological view, the students' utterances should not be interpreted in terms of their grasping or understanding certain concepts, explanations or relations, but rather that the answers are interpreted as acts of participation. This is in line with Sfard's (2008) view of learning as a combination of acquisition and participation.

In accordance with the participationist perspective, Sfard understands *discourses* as social interactions including construction of intersubjectivity and norms, that is, as the different types of communication that bring some people together while excluding some others. An example of particular type of discourse (thus thinking) is the mathematical discourse.

Sfard (2008) means by *discourse* the different types of communication that bring some people together while excluding some others. An example of particular type of discourse (thus thinking) is the mathematical discourse.

Learning mathematics may now be defined as a "development" of mathematical discourse. Mathematics discursive development is characterised by Sfard in identifying *transformations* in each of what she defines as the four discursive characteristics: the use of words characteristic of the discourse, the use of mediators, endorsed narratives, and routines (see Sfard, 2008 for further details).

Within the *commognitive framework* two types of learning exist: *object-level learning*, which expresses itself in the expansion of the existing discourse, attained through extending a vocabulary, constructing new routines, and producing new endorsed narratives – transformations that can be achieved by the students on their own, without the help of a more experienced participant; and *meta-level learning*, which involves changes in meta-rules of the discourse – for these transformations some special conditions are necessary, one of which is the fact that such learning can only take place collectively and with the support of the expert participant. Not only Sfard, but also Bartolini Bussi (1998), with her definition of *mathematical discussion*, argues in favour of a construction of mathematical discourse as theoretically understood as impossible without an experienced participant. In this line Gravemeijer (2004, p. 126) states that "the proactive role of the [expert participant, i.e. here the teacher, is] in establishing an appropriate classroom culture, in choosing and introducing instructional tasks, organising group work, framing topics for discussion, and orchestrating discussion". And Stein and collegues (2008, p. 320) also emphasise the importance of "using student-developed work as the launching point of whole-class discussions in which the teacher actively shapes the ideas that students produce to lead them to more powerful, efficient, and accurate mathematical thinking." In the Discursive Approach to mathematics education by Sierpinska (2005), the teachers' role in classroom conversations is characterised by an obligation to lead the discussion in the direction of relevant mathematical ideas, themes, and issues. Yet Sfard adds a major condition: the opportunity for meta-level learning arises when learners encounter a discourse that is *incommensurable* with their own. That is, on a semantic level, when in the same discourse, the same word is used in different ways. Sfard (2007) echoes

this concept from Rorty (1979, after Kuhn, 1962), who by the adjective "commensurable" means the ability "to be brought under a set of rules that [tell how a rational agreement can be reached on] would settle the issue on every point where statements seem to conflict" (*ibid*., p. 316).
For the sake of a better understanding, the word "conflict" deserves to be explored.
Conflicts of different kinds, from epistemological to cognitive, mark possible evolutions in the way mathematical understanding develops in students, and sometimes they also appear, at another scale, in mathematics teachers' professional development. This is an old repeated story in mathematics: from the discovery of irrational numbers in the Greek scientific world to recent findings about deterministic chaos, bafflements are an usual way, according to which old paradigms are broken and mathematics and its knowledge(s) go on in their development. In this way new, mathematical knowledge is often generated through conceptual and cognitive discontinuities: they challenge mathematical sense-making for students and for those knowledgeable of the discipline, generally creating what many authors call *conflicts* (Sfard, 2008; Tall, 1977; Tall & Schwarzenberger, 1978), others *contradiction* – which refers to the "accumulating structural tensions within and between activity systems" (Engeström, 2001, p. 137) –, and others *obstacles* (Brousseau, 1997), often accompanying the substantive with adjectives, like *cognitive*, *epistemological*, etc., according to their main focus, on students' or teachers' processes or on the discipline content. For example, in Brousseau the notion of epistemological obstacle is introduced as follows:

> Obstacles of really epistemological origin are those from which one neither can nor should escape, because of their formative rôle in the knowledge being sought. They can be found in the history of the concepts themselves (Brousseau, 1997, p. 87).

Hence, an obstacle is "a piece of knowledge or a conception, not a difficulty or a lack of knowledge" (*ibid*., p. 99). As such, an obstacle can be revealed by learners' errors, but it must not be confused with errors, which are the effect of the obstacle, namely "of a previous piece of knowledge which was interesting and successful, but which now is revealed as false or simply unadapted" (*ibid*., p. 82). However, an obstacle has also a cognitive nature, insofar it entails the necessity of a fresh way of thinking that the new knowledge would require but apparently is not coherent with the previous one and encounters difficulties to be activated. Difficulties in an obstacle may be particularly subtle since, at a first glance, the relationship between the old and the new knowledge seems a contradiction between the two, but generally it is not so: the new frame simply enlarges the old one putting forward a new standpoint, which allows to embrace the previous one in a new setting, which does contradict the older one in case the older framework is still used. Sfard thus defines what she names *commognitive conflict* as follows:

> a situation in which communication is hindered by the fact that different discursants are acting according to different meta-rules (and thus possibly using the same words in differing ways). Usually, the differences in meta-rules that are the source of the conflict find their explicit, most salient expression in the fact that different participants endorse contradicting narratives. (2007, p. 374)

Because the same words are used in different discourses, incommensurability may be invisible to discourse users. Instead, they may perceive an apparent *incompatibility* of narratives. But these narratives are not talking about the same thing. For example, the Euclidean Geometry is incompatible with the Hyperbolic one, but it is only incommensurable with the geometry necessary to describe "strange" objects like the Sierpiński triangle (Apkarian, Tabach, Dreyfus, & Rasmussen, 2019).

Thus, with the term *conflict*, is indicate a piece of new knowledge added to the old one, when this reveals inadequate to solve a fresh problem (epistemological side); the new knowledge is incommensurable but not contradictory with the old one and consequently asks for new ways of reasoning (cognitive side).

Sfard (2007, p. 574), however, specifies that "the notion of commognitive conflict should not be confused with the acquisitionist idea of cognitive conflict, central to the well-known, well-developed theory of conceptual change (Schnotz, Vosniadou, & Carretero, 1999; Vosniadou, 1994)". She lists three reasons for this distinction:

- the first within the locus of the conflict, that is, by contrasting the truth-falsity of a concept – of which the world is the arbiter –, with the idea of incommensurability between discourses;
- the second is in their significance for learning, that is, form "an optional pedagogical move, particularly useful when students display 'misconceptions'" (*ibid*., p. 575), to an indispensable source of metalevel mathematical learning;
- the third in the way the conflict is to be resolved, that is, moving from the principles of incompatibility and noncontradiction – two supposed contradictory narratives are also mutually exclusive, with common criterion to reject or endorse and label one as true –, to a conflict resolution as making sense of other people's thinking-talking about the world with "a gradual acceptance, 'customization', and rationalization – figuring out the inner logic – of other people's discourses" (*ibid*., p. 576).

Considering again the excerpt quoted at the beginning, since this my research work focuses on teachers' professional development, the framework of the participatory perspective needs to be adapted: networked with Transformative Learning, it is possible to make more explicit the kind of learning that took place in this moment, which can now interpret as a conflict. Let us recall the excerpts of Nicoletta, already introduced above (p. 3):

> **Nicoletta**: [...] I was thinking about this stuff here [she points to the Bartolini Bussi and Ramploud (2018) book] that we are facing: it is a great opportunity that is beyond the specific characteristics of our school. I mean, **the difficulty I see**, which I think is everywhere, but I see it here because I work here, **is the fact of having the willingness to dedicate time**. Then, "**we don't have time!** We don't have time to plan everything". But yesterday morning I was talking to some colleagues who at 8 a.m. – I arrive at 7.30 a.m. to do things because it's convenient for me to arrive at 7.30 a.m. instead of taking my notebooks home, and then I never manage to do it because **one colleague comes and says: "ah but you did that thing with the straws?** Ah well, it's nice to do it like that, but **tell my colleague to do it too**. Why do you do

> it this way, or that way?...” **No.** We cannot talk to each other about work like that. [...] but **we have to think about it, we have to talk about it**. **It’s not** that **in 30 seconds**... then I go down to the canteen with the children and **while I’m giving the parmesan cheese**: “So tell them how you did it...”. I mean, I thought about it for three or four hours, I discussed it with him [Ezio] because it was a mathematical thing [...]. This also undermines our professionalism [...].

As analysed above, the learning process of Nicoletta’s hypothetical colleague is part of the transformation of points of view. The colleague is interested in learning about new methodologies but lacks critical self-reflection. But this observation is not sufficient to describe what happens in the extract. With her words, Nicoletta implicitly highlights a conflict arising from the encounter of the teaching practices she knows with Lesson Study. Time is required for shared critical reflection, not just for being told something and trying something new received. The conflict described here is undoubtedly commognitive: the professional development discourse of the Lesson Study is at a different level than that described in the reality of Nicoletta and her colleagues.

Yet, this analysis does not seem sufficient to describe the meaning behind this extract. There are details that seem to get lost. Why is it on a different level? Where and how did this conflict arise? Answering these questions could help to resolve the conflict.

The theories presented in my research work make it clear that to answer these questions the extract might have to be immersed in a wider context. It is not only its institutional and ecological context, but as the meaning-making of Nicoletta’s dialogue is investigated, the semiotic and cultural aspects involved must be taken into account. The extract becomes text and, as such, embedded in the semiosphere.

It is from these reflections that the first outcome of this my research work emerges: the introduction of the concept of *cultural conflict*. Although it is not a novel terminology, its application to the field of research in Mathematics Education is highlighted.

THE CULTURAL CONFLICT

Stella Patrick Essien (2020), echoing Alexandre Grewe (2005, in Essien, 2020), defines cultural conflict as what “occurs when people’s expectations of a certain behavior coming from their cultural backgrounds are not met, as others have different cultural backgrounds and different expectations” (*ibid*., p. 140). Following on from the definition of culture given above, situating the research work in a Semiotics of Culture, it is not a matter of hierarchical dynamic in which there’s the “main” culture and then the “other” culture, and conflict results almost always in the logic of difference being determined by the main culture. The “other” is not the lesser, the less cultured, the less civilised. By encouraging people (students and teachers, but also researchers and scholars) to move into a different cultural paradigm, people can gain awareness of what Jullien (2005) defines as *unthoughts*, i.e. those aspects that escape people’s consciousness when they are immersed in their own culture. It is not so much a matter of understanding foreign cultures, thought of as homogeneous spheres with marked boundaries, but of an “interaction with foreignness” (Welsch, 1999, in Barton, 2008). Meeting foreign cultures leads to looking at one’s own practices.

The concept of cultural conflict therefore extends the idea of conflict presented in the previous section, but bearing in mind the possible change of context: here it is about teachers’ professional

development and not about students' learning. Ramploud, Funghi and Mellone (2021) in this context call "experience of" or "moment of *crisis*" – the main assumption of the teachers' professional development process through Cultural Transposition – what others call *obstacle*, *contradiction,* or *conflict*.
The concept of cultural conflict does not modify the framework but embeds it within the theoretical framework of the semiosphere. Table 1. 2 describes the differences between the concepts of cognitive, commognitive and cultural conflict.

Table 1. 2 Comparison of Concepts (adapted from Sfard, 2007, p. 756).

Concept	*Cognitive Conflict*	*Commognitive Conflict*	*Cultural Conflict*
Ontology: The conflict is between	the interlocutor and the world	incommensurable discourses	asymmetrical contexts
Role in learning	is an optional way for removing misconceptions	is practically indispensable for metalevel learning	is a transformative process through a redefinition of the problem and a critical self-reflection of the assumptions
How is it resolved?	by student's rational effort	by student's acceptance and rationalization (individualization) of the discursive ways of an expert interlocutor	by making explicit learner's *unthought* [but potentially activated by the encounter with foreignness]

THE CULTURAL TRANSPOSITION construct and all its splendour
Two initial premises to this section: the first is that "the purpose of this condition [i.e. the Cultural Transposition construct] is to reflect on and to rethink mathematics educational practices" on its original context (Mellone, Ramploud, Di Paola, & Martignone, 2019, p. 201); the second premise is that a mere description of the state of art related to this theoretical construct is not made here, but the purpose is to present the Cultural Transposition as re-read under the lens of Lotman's Semiosphere. A revolution of the construct itself is neither expected nor sought, but this re-reading was fundamental for me in achieving a greater awareness of the construct. Explicating this awareness by writing it down and sharing it with the reader is the purpose of this paragraph.

As defined by the researchers who devised this construct, Cultural Transposition is "a process activated by researchers, educators, and teachers who deconstruct those educational practices adopted in other cultural contexts in order to reconsider the issues of educational intentionality, which is the background of any educational practice" (Mellone *et al*., 2019, pp. 201-202).
However, in order to fully understand these words, it is appropriate to delve into the historical references, the state of art and the theoretical framework on which it is grounded.
Since the 1980s, a great deal of attention to cultural sensitivity has developed around the world. Several agencies dedicated to the scientific and academic study of the social, political, economic, religious, and cultural values of people around the world have sprung up. One example is the

World Values Survey (WVS), one of the most authoritative cross-national surveys actually used, "covering almost 120 countries/societies (nearly 95% of the world's population)" (Haerpfer, 2021). WVS is a global research project that explores the values and beliefs of people in different countries around the world, to assess what impact its stability or change over time has on the social, political and economic development of countries and societies. The WVS research work is useful to get a picture of the world situation, now and over time, of massive cultural change and the persistence of distinctive cultural traditions. Ronald Inglehart and Christian Welzel's map, for example, draws the world situation on two main dimensions of cross-cultural variation in the world: Traditional versus Secular-Rational values, and Survival versus Self-Expression values (Inglehart & Welzel, 2005). Figure 1. 16 helps to raise awareness of the world situation at the time of my research, even though it is a pre-pandemic analysis, and its unstable and provisional nature (demonstrated by the rapid evolution of the map over time – - see website for overview). Moreover, the map helps to define an acknowledged distinction between what might be different *cultural contexts*.

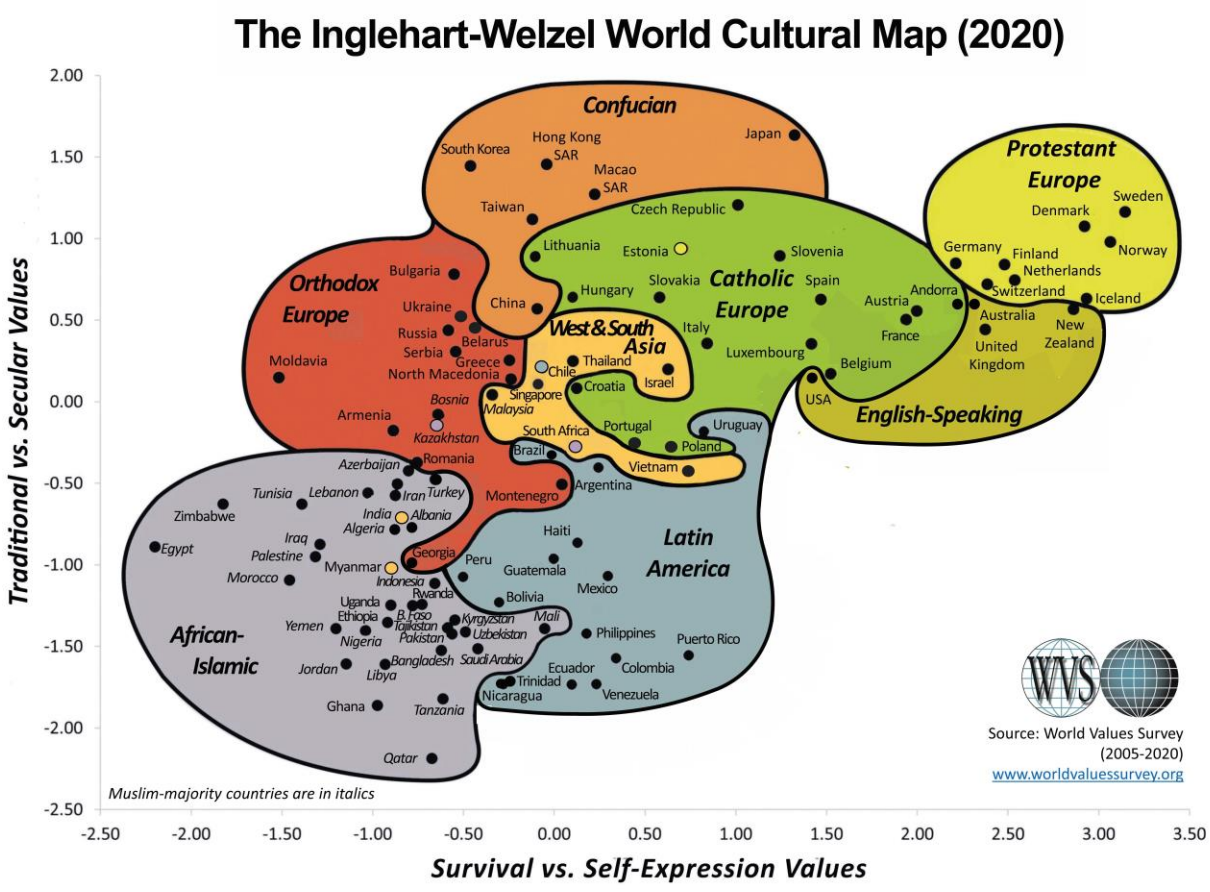


Figure 1. 16 The Inglehart-Welzel World Cultural Map – World Values Survey 7 (2020) [Provisional version]. Source: http://www.worldvaluessurvey.org/

Conscious of these differences, Bishop's (1988) trailblazing studies highlight the revolutionary importance of recognising mathematical practices as social phenomena embedded in the cultures/societies that generated them. Bishop (1991) has brought to the attention of the Mathematics Education scientific community the recognition of similarities and analogies in the basic mathematical (particularly numerical and geometric) skills in the different cultural contexts. Furthermore, the ethnomathematics trend of studies (Ascher, 1991; D'Ambrosio 2006) has proved that, when inquiring into mathematical practices, care and sensitivity about cultural and social issues contribute to the understanding of mathematics itself and, above all, shed light on a political level: the recognition of the existence of different mathematics and the scientific knowledge of the different mathematics also contribute to the scientific understanding of different cultures (Might mathematical practices analysis even add information to Inglehart-Welzel map?!).
Nowadays, as already seen above, there are several research approaches (Barton 2008; Sfard, 2012; Saxe, 2014; Albanese, Adamuz-Povedano, & Bracho-López, 2017; Nemirovsky, 2020) working from the crucial assumption that culture permeates all aspects of both mathematical practices and mathematics education practices. Nevertheless, what is rather new is the interest in having a theoretical framework that allows for the reading and for the explicit account of the aspects of cultural diversity in mathematics and also the ways in which culture affects mathematics teacher education. It is in this line of research that Mellone and colleagues propose:

> the paradigm of Cultural Transposition with the aim of using the differences among mathematical education practices adopted in different cultures and societies to design professional development that aims to develop teachers' awareness and, eventually, change their mathematical education practices (Mellone *et al*., 2019, p. 200).

The research activity directed towards the just described purpose has its origins in the preliminary works of the Italian research group (Spagnolo & Di Paola, 2010; Ramploud & Di Paola, 2013; Bartolini Bussi, Sun, & Ramploud, 2014; Mellone & Ramploud, 2015; Di Paola, 2016) where, through qualitative and/or quantitative approaches, is point out how Mathematics Education researchers could elaborate new interpretative keys of the educational practice of their own cultural context and use the results to design teachers' professional development programmes. An example is the effective work done to encourage discussion and reflection on the didactics of Whole Numbers Arithmetic (Cai & Knuth, 2011). In the light of the Chinese "problems with variation" and the Davydov's Russian "visionary curriculum", the researchers have worked with teachers to foster new epistemological awareness of addition and subtraction in unified approaches (Bartolini Bussi & Sun, 2018). It is worth mentioning that the background of these reflections, as my research work, as of this current research work, is the Italian cultural context (Bartolini Bussi & Martignone 2013), since it has just been said that research in Mathematics Education is not "invariant under translations" of context.
The Italian cultural context is presented in the next sub-section, but in order to clarify the perspective of cultural transposition I here follow the traces of Mellone and colleagues (2019) who analyse the meaning of the two terms of the expression "cultural transposition" (C: cultural, and

T: transposition) and then indicate its main features by situating it theoretically.

(C) The term *Cultural*, which carries with it the semiotic perspective in which it is embedded and the philosophical concept of *deconstruction*:

> Strategically starting from the adjective *cultural*, in the Oxford Dictionary edited by A. S. Hornby, this entry has as a definition: "having to do with culture"; hence, the adjective cultural harkens back to the word culture. […] stimulated by the thought of Jurij Michajlovič Lotman, who defines culture as a complex semantic system made by different interlaced linguistic signs (Lotman and Uspenkij, 1975). Therefore, transposition in a cultural perspective is strictly related to the signs and those linguistic systems in which it develops.
>
> Focusing especially on the semiotic dimension, we want also to take into account those habits, those imaginations and that philosophy which feed and shape every culture (D'Ambrosio 2006). By means of the *deconstruction* movement introduced by Jaques Derrida (1967), we consider the semiotic perspective in which every culture is fed; this movement entails an analysis of the different levels on which a culture becomes stratified.[16] During an interview for an Italian newspaper, Derrida stated the following:
>
> > The word [deconstruction] comes from a Heidegger expression, 'Destruktion', meaning 'de-destruct' and not as 'destruction'. I use it in the sense of an analysis of the different layers in which it stratifies culture.
>
> Therefore, Derrida intends deconstruction (destructuration) as a methodology, or rather a critical exercise on cultural stratifications. Throughout his philosophical activity, Derrida developed the idea of deconstruction as a process that arises as an attitude that serves to continually deconstruct a culture, that is, to put in place a radical critique. In this sense, we would like to accomplish a didactic deconstructionism in our research, through a reflection that handles the differences among the didactics of mathematics in different cultures. According to this view, which differs from Bishop's approach engaged in the search of equivalence among cultures, we are more concerned with the investigation of differences among mathematics education practices (Mellone *et al*., 2019, pp. 200-201).

In these lines, the semiotic approach to culture, although left in the background, is given as required. The levels of culture should not be understood from a Kroeberian stratigraphic perspective, but rather from the synthetic and semiotic perspective of Geertz. Furthermore, deconstruction as a *methodology* engages researchers, professional educators, and teacher-researchers in Mathematics Education to re-interpret components of other cultures' teaching practices through their own culture.

> It is done with a dual intent, i.e., to make the new teaching practice compatible with the beliefs, customs, and values of one's own cultural context, and to maintain features that are not usually associated with one's own cultural context. […] In this perspective, the brutal import-export

[16] Deconstruction is defined as "an analysis of the different levels in which a culture is stratified" (Derrida, 2002, p. 1)

> practice of educational methods is seen not only as failing but also dangerous. Exposure to mathematical practices that come from different cultures, if suitably managed, can create spaces of refection and awareness development for researchers, educators, and teachers (Mellone, Pacelli & Liljedahl, 2021, p. 786).

In this way, the embedding of some of the new features is intentional and educational as it serves to allow participants to come into contact with otherness, i.e. to give rise to semiospherical movements.

(T) The term *Transposition*, for which the words of the authors are left to speak for themselves:

> the etymology of the word is a starting point; the word transposition comes from the Latin *transponere*, in which both the prefix *trans*- and the verb *ponere* are easily recognizable. The prefix indicates a passage, a transition, a change from one condition to another, while the verb means to place, to put. Therefore, the noun transposition is composed of two elements, the first identifying a passage, a transition, a change, while the latter provides a more static image suggested by the verb to place.
>
> Accepting these premises, we use the term transposition exactly to describe something placed after a transition from some initial conditions. In the field of education, we define the paradigm of Cultural Transposition as a process to decentralize the educational practice of one's own cultural context through the contact with educational practices of other cultural contexts (Mellone *et al*., 2019, p. 201).

The object of transposition then is not the educational practice of other contexts (in our case the Chinese Lesson Study), but the educational practice of one's own cultural context (the Italian educational practice). Mellone and colleagues (2019, p. 199) propose the cultural transposition construct "as *a condition* for decentralizing the didactic practice of a specific cultural context through contact with the didactic practices of different cultural contexts" (my emphasis in the text). The term "condition" is not accidentally chosen. It indicates "the state that something or someone is in" (from the Cambridge Dictionary). This is to indicate that it is the prerequisite to frame and design the encounter of teachers' and researchers' professionality with Mathematics Education tools and methodologies coming from different cultural contexts (not only understood as the cultural zones of the Inglehart-Welzel map – e.g. Italian context and Chinese context, but also as socio-historical situations – e.g. pre- and post-pandemic educational situation (Ramploud, Funghi & Mellone, 2021)).

In this transposed condition-position, encountering the meanings embedded in otherness, it is possible to "re-think" the meanings embedded in one's own practices. From this perspective, the process of cultural transposition encourages a political vision of Mathematics Education, that is:

- a *critical* vision of Mathematics Education as understood by Ernest (2016)

> No culture could claim dominance or precedence for the creation of mathematics conceived as a pan-cultural activity characterized by playing, designing, locating, explaining, counting and measuring (Bishop, 1988). In the same way, no mathematics

education practice should claim dominance in terms of effectiveness or success and no international assessments should be read accordingly (Mellone *et al*., 2019, p. 201);

- an *emancipatory* vision of Mathematics and Mathematics Education as understood by Habermas (1971, 1984) and Skovsmose (1994),

> […] mathematics is seen as an invisible structure that plays an important role in related societies. Mathematics, in particular, can represent a powerful means of emancipation for learners. In our perspective, the contact with different mathematics school practices can represent an experience of emancipation for teachers (Mellone *et al.*, 2019, p. 199);
>
> […] learners are not seen as passive recipients for institutionalized knowledge, rather they are seen as actively part of an educational process in which they are those who question, challenge and even shape the nature of their own learning experience. Similarly, teachers should shift from being passive receptacles of institutionalized knowledge into determining the nature of the mathematics teaching experience which they offer (Mellone *et al*., 2019, p. 201).

Teachers are asked to critical reflect on their usual pedagogical and didactical premises and beliefs. So, as also the authors point out (see Bartolini Bussi, Funghi & Ramploud, 2020), the Cultural Transposition within mathematics teacher education is not like a comparative study, rather it is more similar to that process described by François Jullien in his *chantier* [his philosophical construction yard], where he explores the *gap (écart)* between Chinese and European thought:

> This is not about comparative philosophy, about paralleling different conceptions, but about a philosophical dialogue in which every thought, when coming towards the other, questions itself about its own *unthought* (Jullien, 2006, p. 8)[17].

The pivotal idea of "unthought" conceived by Jullien is referred to all the implicit assumptions in which a cultural paradigm is rooted. As implicit, assumptions are "invisible". People have unconsciously incorporated them, albeit always with a certain degree of re-elaboration and personalisation. They are therefore taken for granted and, in a sense, considered obligatory within their own culture. In this situation the asymmetry of the semiosphere is not taken into account. But this cannot be a long-lasting status. Hence activities in the semiosphere are conceived as a dialogue between different praxeologies during which "each teaching choice, in contact with a different one, can become more aware" (Jullien, 2005). Cultural Transposition, in accordance with existing lines of research on mathematics teacher education (Wood, 2008; Potari & Chapman, 2020; Goos & Beswick, 2021), is thus proposed as a perspective in which the role of researchers is crucial, since it is they who introduce original interpretative keys for "the same" educational practice related to their own cultural context, through the deconstruction of the several levels in which an educational practice is stratified – in the sense of Derrida (2002) and Sfard (2007), i.e. to stimulate

[17] My translation. Original text in French: "Il ne s'agit pas là de philosophie comparée, par mise en parallèle des conceptions; mais d'un dialogue philosophique, où chaque pensée, à la rencontre de l'autre, s'interrogue sur son *impensé*."

dialogue. The Semiosphere allows to keep identifying the constituent elements of a reality even from the identification of elements external to it. In fact, precisely because of its asymmetric and non-homogeneous character, based on dialogue, the Semiosphere creates not only its own internal organization, but also its own type of external disorganization. It defines what is not itself. For example, the Lesson Study teachers' meetings of my research, can be pictured as a multidimensional dialogue in the Semiosphere during which each choice of teaching/learning, in contact with another, can become "more aware". Here the critical dialogue and reflection of the teachers, if read from the point of view of the Semiosphere, do not lose contact with the reality in which they are born.

The Italian educational context

The Italian education system is organised on the basis of the principles of subsidiarity and of autonomy of educational institutions[18]. That is, the State has exclusive legislative competence for the general rules on education and for determining the essential levels of services that must be guaranteed throughout the national territory. The State also defines the fundamental principles that Regions must respect in exercising their specific powers. The Regions have concurrent legislative powers in the field of education and exclusive legislative powers in the field of teachers' education and professional development. Public educational institutions, on the other hand, have (a) teaching, (b) organisational and (c) research, experimentation and development autonomy.

Recent reforms (2010 and 2012 respectively for secondary and primary school) stressed the importance of inclusiveness (law 133 and 169/2008). Italian school is structured around the concept of equity, and special schools do not exist: all students are given the same opportunities to reach the same goal, plus aids if needed. The Ministry of Education provides the *Indicazioni Nazionali* (national guidelines – par. 3, art. 10, law 89/2010; MIUR, 2012), which contain contents and aims for each subject and its number of hours in a year. National guidelines do not express specific teaching obligations, rather outline broad goals to be achieved by specific years. The contents are not prescriptive, but at the end of the 8th and 13th grades there are two national exams. Each teacher has the responsibility of the didactical plan for her/his classes, also according to two documents: the *Piano Triennale dell'Offerta Formativa* (Three-year Educational Plan – describing the cultural-pedagogical inspiration and the curricular, extracurricular, didactic and organisational design of the proposed activities), and the *Rapporto di Autovalutazione* (Self-Evaluation Report – providing, on a national online platform, an image of the school through an analysis of its performance and constituting the basis for identifying the development priorities for an improvement plan). The contents of these documents are specific of each educational institution and decided by the collegiality of teachers and school staff.

The teacher has a fundamental role. Freedom of teaching is understood as professional autonomy in carrying out teaching activities and free cultural expression of the teacher. It is guaranteed, since

[18] An Istituzione Scolastica [educational institution] is an institutionalised group of schools, operating under the same school manager.

1948, as a constitutional right due to the attempt – in times of post-Fascism – to defend teaching from authoritarian deviations of democracy. Article 33 of the Constitution enshrines: "Art and science are free and free is their teaching". Institutionally, the duration of the lesson is 60 minutes. The teacher can have up to three consecutive lessons in the same class, without interruptions. During the lesson, the teacher is usually the only adult figure in the class.

The general Italian cultural context and, in particular, school culture concerning education in Mathematics and STEM subjects, are still affected by the historical events that took place at the beginning of the last century and by the school reform dating back to those years. The protagonists of this school reform are, on the one hand, the great mathematician Federigo Enriques – a strong supporter of the centrality of the exact sciences for the technological and more widely cultural development of the country – and, on the other hand, Benedetto Croce and Giovanni Gentile – who tended, instead, to limit the weight of mathematics in the cultural landscape of the country. The school reform was signed, in the end, by Gentile, and not by Enriques, with the consequent downgrading of the exact sciences in the cultural development of the country. These historical events supported a humanistic perspective on knowledge, at the expense of a good development of the relationship with mathematical-scientifc-technological knowledge. In addition to this historical element, one of the major current challenges is the infrastructural development of the educational environment (for example, the first infrastructural projects aimed at making the connection cables reach the schools date back to around twenty years ago) – due also to a geographic conformation of the Italian territory, which constituted an obstacle to the physical building of Internet connection networks. Associated with this are attitudes of "technological rejection" amongst many teachers (Ramploud, Funghi, & Mellone, 2021). The gap between society and the educational system has grown, especially in these years of the COVID-19 pandemic. In addition, these issues discourage people with disabilities and specific learning disabilities from entering science education and subsequently preclude them from entering STEM professions. Technology in general, and assistive technology and compensatory tools in particular, are indeed essential for the education of these learners (Ahmetovic *et al*., 2019). Added to this is the fact that "there is no tradition of specific care in the design of mathematics lessons, based mainly on the teacher's theoretical explanations and the execution of stereotyped exercises by students" (Mellone, Pacelli & Liljedahl, 2021, p. 788), with the exception of small virtuous realities such as the NdRD[1] or autonomous groups of teachers.

The Italian teacher works at school from one to six hours a day, dedicated to classroom lessons. The planning of individual lessons is not part of the working hours, nor there are places in the school dedicated to this activity: the teacher's paradidactic activity takes place in personal and private time and space. There are no compulsory contents or practices for teachers' professional development, they are chosen by teachers according to their own needs. In-service teachers' professional development is defined by the *Buona Scuola* law as "compulsory, permanent and structural" (law 107/2015, art.1, paragraph 124), but there is no minimum number of hours per year and must be carried out outside working hours. Teachers' career advancement is based

exclusively on seniority, although some economic incentives are given to those that take relevant roles in the school organization (Blandino, 2008; Capperucci, 2008). The National Plan for the Professional Development of School Staff, states: "In-service professional development is not a formal or contractual obligation, it is a professional choice that allows broad cultural, planning, teaching and research autonomy, within the framework of teaching freedom and scientific innovation" (par. 124, art. 1, law 107/2015).

On paper, teachers have numerous occasions for improving their professionalism. Official documents from the Ministry[19] attest more than 500 agencies offering teachers' professional development opportunities. Universities, academic associations, teachers' associations, and educational companies which fulfil quality standards defined by the Ministry, are registered in a national database and can publish their teachers' professional development proposals on a digital platform (S.O.F.I.A.). The in-service professional development "system" is conceived as a "lifelong learning environment" for teachers and is intended as a "network of opportunities for professional growth and development for teachers" (law 107/2015). At national level, proposals come from the national education centre (INDIRE), academic associations (UMI-CIIM, AIRDM), teachers associations (Mathesis), educational companies. At regional level, regional school offices (Ufficio Scolastico Regionale – USR) intervene by supporting, managing, and publicising the proposals. At local level, experienced individual teachers also offer specific courses in their school, sometime opens to teachers in the surrounding area. No official account is given on how many teachers participate in teachers' professional development each year. Yet, the general impression is that this vastity of opportunities does not correspond to a generally-high-quality offer: official documents from the Ministry state that the quality of teachers' professional development programmes is compromised because of the general "low quality of models and methodologies" (law 107/2015) suggesting that teachers, although free to choose among the variety of opportunities for advancing their professionalism, might be easily lost and caught in low quality programmes. At this moment, though, the Ministry does not provide guidance to orientate in the labyrinth of such offers. The 2018 *National Guidelines and New Scenarios* (MIUR, 2018) emphasises the extent to which peer learning experiences and the shared development of practices and cultures can produce positive results and lasting change; calling for opportunities for collaborative work among peers in educational institutions to be increasingly encouraged.
On the Italian context,

> la communication entre chercheurs et enseignants est complexe à cause de la diversité des exigences: la description et l'analyse des comportements des élèves pour les chercheurs, la construction de situations didactiques qui permettent de rejoindre les objectifs d'apprentissage pour les enseignants. Mais la diversité peut s'orienter vers la complémentarité, parce que le rapport entre les deux exigences peut être dialectique: l'enseignant peut voir l'analyse des comportements des enfants comme nécessaire à la réalisation des objectifs d'apprentissage, en tenant compte des obstacles à différents niveaux (Scali, 2007, p. 2).

[19] https://www.miur.gov.it/accreditamento-enti-e-qualificazione-associazioni

In Italy, this important collaboration between teachers and researchers was formalised in 1975, when the Comitato per la Matematica del Centro Nazionale delle Ricerche (CNR) and the Unione Matematica Italiana (UMI) promoted the establishment of NdRD[1], active in several universities, at the Departments of Mathematics. These were mixed groups, involving university teachers and primary and secondary school teachers with the common aim of promoting educational research projects and carrying out innovative educational experiments, consistent with the international research of the moment and verified in periodic meetings. In our country, the research carried out in NdRD should be seen as the result of a progressive integration between two different traditions: the attention given to content by university mathematicians – which is of fundamental importance in highlighting the basic epistemological nuclei of the discipline; and the attention paid to improving mathematics teaching practices by teachers – through field research carried out directly in their own classrooms. CNR's grants, which were sustained for some twenty years, also made it possible to intensify relations with the international community, encouraging foreign researchers to come to Italy or allowing Italian researchers to participate in international conferences (Ciarrapico & Berni, 2017). Thanks to the international conferences, Italian researchers deepened different research traditions: they found educational contexts in which the classroom became a sort of laboratory, where problem-solving sessions were planned and observed, and the processes triggered in the students during problem-solving were highlighted, also using methods borrowed from other disciplines such as psychology, sociology, pedagogy. In the course of time, the interaction between the internal traditions of NdRD (reflection on the contents; field research) and the external traditions (observations in the laboratory classroom; construction of a theory of didactic phenomena) led to the clarification of the aims and methods of the "original paradigm that has *research for innovation* at its core" (Arzarello & Bartolini Bussi, 1998, p. 244).

The teacher-researchers, a new role for the Italian reality and conceived in the context of the Nuclei, played an essential role by shifting the attention from the observation of short-term processes – the focus of research conducted in other countries within the class-laboratory –, to the observation of long-term processes – important in Italy because of the long permanence of a teacher in the same class (at least at the Primary School level). The role of the teacher was clarified, in that he or she was not an object of observation, but the subject of decisions in the research. The impact on the national education system has not been negligible: the new knowledge on teaching-learning processes, acquired through the research of the Nuclei, is today held in high regard. Nevertheless, an objective problem remains in Italy: the number of scholars revolving around the initiatives of AIRDM (Italian Association for Research in Didactics of Mathematics) or similar associations, which includes researchers, university professors, teachers of all levels and some students in training is probably still very low compared to the overall number of Italian mathematics teachers. Thinking about how to give institutional weight to the scholars of the Departments of Mathematics or the Departments of Primary Education has become a necessity. They have to make sure that the mathematical community can continue to exert a cultural influence on the educational system and that the traditional national and international presence of the Italian research in Mathematics Education is not diminished.

LESSON STUDY

Concerning active professional development,
teachers are a strange category,
if the proposed project is transmissive they complain,
if it is active they get flustered.
We take note of this, but we persevere,
especially on mathematics, which is really an emergency,
which nobody seems to realise.
(Headmaster of a secondary school in northern Italy,
designing a Lesson Study professional development course for mathematics teachers)

Lesson study, in particular with its Chinese roots, is a paradigmatic example for my research work. Lesson Study is a teacher professional development methodology that can trigger a *perspective transformation*, as understood by Mezirow in transforming of habit of mind, i.e. a *critical reflection* (as action-oriented and historically embedded, social and political process) through the spiral of *self-reflection* (Kemmis, 1958).
With its features of:
- collaboration between participants stimulation;
- active and transformative critical learning;
- belonging to a foreign culture;
if studied with a cultural-semiotic lens capable of grasping the meaning-making of context-dependent processes, Lesson study is indeed a methodology capable of responding to the educational needs of the present-day society (Vermunt, Vrikki, van Halem, Warwick & Mercer, 2019; Skott & Møller, 2020).

Why the Chinese heritage?

Due to the excellent performance of Shanghai students on PISA (OECD, 2013), researchers around the world have been interested in understanding mathematics teachers' professional development in China (Tang, 2014). In particular, it has seemed necessary to study how and whether these skills develop from the early grades of primary school and whether there are paradigmatic elements to be considered a priori. Beside this element there is an important context fact. In Italy there is a large Chinese community, which causes the need to constantly create new occasions for dialogue among schools, teachers, universities and social context (Spagnolo 2002; Spagnolo & Di Paola, 2010; Bartolini Bussi, 2010). This collaboration between school and university is here interpreted in the attitude expressed by G. Prodi:

> It is important to maintain contact with school: the "University-School mixed groups" can play a very important role both in the spread of didactic innovation and in the didactic research. These groups require the adoption of a really joint relationship: pre-university school teachers invited to participate in a didactic research have to do it fully, and not only as auxiliaries responsible to collect data and protocols (Prodi, 1991).

It is in this context that Bartolini Bussi, Ramploud and other colleagues have begun to address some aspects of early arithmetic learning in the early grades of primary school (Bartolini Bussi, Sun, & Ramploud, 2014).

Of the "remote" Chinese cultural context, the first element that Westerners come across with are obviously the ideograms: a "living fossil". Indeed, they retain a very ancient, stylised representation of what they are meant to signify, and more, through them today's Chinese can understand messages written thousands of years ago. But above all, ideograms are "an instrument of civilization" (Granet, 1934), codified over time by the emperor.

> The merits of the Chinese script are of a different order: practical, not intellectual. This script can be used by people speaking different dialects - or even idioms, with the reader reading in his or her own way what the writer has written, thinking of words that have the same meaning, but which he or she could pronounce quite differently. Unaffected by changes in pronunciation over time, this script is an admirable organ of traditional culture. Independent of local pronunciations, which it tolerates, its main advantage is that it is what could be called a civilisation script (Granet, 1934, p. 29)[20].

There is no such instrument in the West. The ancient Greek alphabet is one of the major "tears" that occurred in the West. Indeed, it is:
- an instrument of democracy: it is used by the Sophists to construct dialogues to debate. Socrates died for the dialogues: the game is played practically only on the dialogues;
- an instrument of transcendentalisation (Jullien, 2009). Once the vowels have been invented, after an initial proximity, the Greek alphabet becomes completely detached from reality. Even more, any letter taken into consideration has no bearing on what is done, it is already transcendent with what is said.

In Western language, between signified and signifier, there is no more transit.
The Korean alphabet, on the other hand, for example, still has a connotation linked to immanence, even if different from the Chinese one, that is, it traces the shape of the position of the mouth. In the Chinese ideograms, even the simplified ones, there are references to very evocative elements: the "horse" (mǎ – trad. 馬, sempl. 马) has a mane, and legs (see Figure 1. 17). Either the "teeth" (yá chǐ – trad. 牙齒, sempl. 牙齿) are plainly recognizable to the trained eye (see Figure 1. 18). And "Chinese people in general are good at imagery thinking, a picturesque way to see the world" (Chen, 2017, p. 285).

[20] My translation. Original text in French: "Les mérites de l'écriture chinoise sont d'un ordre tout autre: pratique et non pas intellectuel. Cette écriture peut être utilisée par des populations parlant des dialectes — ou même des idiomes différents, le lecteur lisant à sa manière ce que l'éc rivain a écrit en pensant à des mots de même sens, mais qu'il pouvait prononcer de façon toute différente. Indé pendante des changements de la prononciation au cours des temps, cette écriture est un admirable organe de culture traditionnelle. Indépendante des prononciations locales qu'elle tolère, elle a pour principal avantage d'être ce qu'on pour rait appeler une écriture de civilisation."

Figure 1. 17 Evolution of the character "ma", horse.

Figure 1. 18 Evolution of the characters "yá" and "chǐ", teeth.

However, reading again Granet:

> The combinations of features which are properly called radicals are by no means characters symbolising fundamental notions. It will suffice to point out that one of these so-called radicals claims to represent canine teeth and another incisor, but that there are none that correspond to the «general» idea of teeth. In fact, these radicals correspond to *lists*[rubriques] intended to facilitate not a classification with a claim to objectivity, but a practical search in the lexicons and, no doubt, an easier learning of writing (Granet, 1934, p. 30)[21].

That is, the Chinese language is immanent and based on "rubrication". "Classification is meaningless, as haves and have-nots are constantly inter-changing into each other (有无相生), like Yin and Yang" (Chen, 2017, p. 286). And these are the two big differences from Western languages. Because the Western language, as previously mentioned, is instead transcendent and Westerners generalize and categorize, classify. This boost to transcendentalisation is a fundamental boost of Greek culture, even beyond the alphabet (Jullien, 2009). This boost of a cultural nature persists, characterises, and is reflected everywhere (i.e., Plato's ideas) and it is probably that same boost that allowed that conceptual "leap" whereby the ancient Greeks, at a certain point, imagined they could separate signified and signifier.

Another interesting logical-mathematical-philosophical contextual aspect that distinguishes Eastern from Western thought is the idea of contradiction. In the West, the *principium tertii exclusi* (or the law *tertium non datur*, "no third [possibility] is given") is undisputed, i.e. if I say A and you say not A, or I say true and you say false or vice versa, we cannot both be right. In China the problem of establishing what is "the truth" about things is not there; there is rather a problem of

[21] My translation. Original text in French: "Les combinaisons de traits que l'on nomme proprement des radicaux ne sont nullement des caractères symbolisant des notions fondamentales. Il suffira d'indiquer qu'un de ces soi-disant radicaux prétend représenter les dents canines et un autre les incisives, mais qu'il n'y en a aucun répondant à l'idée «générale» de dents. A vrai dire, ces radicaux correspondent à des rubriques destinées à faciliter non pas un classement à prétention d'objectivité, mais une recherche pratique dans les lexiques et, sans doute, un apprentissage plus aisé de l'écriture."

civil coexistence among many people. All Confucian thought is based substantially on the effort to minimize friction between people. It is taught from childhood with a legend from a collection of Chinese thoughts from the 3rd century (Chieng, 2020). The story tells of a vendor, who boasted of his indestructible spear, capable of piercing any shield even the best ones, and impervious shield, repelling every attack even the most powerful spears. A man then asked what would happen if he tried to stick his spear into the shield. The vendor was taken aback and struck in his pride. They clashed and the result of the collision was that both tools broke. This persists in the Chinese language. The word *contradiction*, in Chinese (máo dùn – 矛盾), is translated by the ideograms of *spear* (矛) and *shield* (盾). But the spear and the shield coexist. That is, contradiction is something that exists, that coexists, like shield and spear.

The Chinese heritage is thus a context rich in pivotal junctures to activate new critical reflections on our heritage.

My personal research on the Chinese heritage and cultural context was conducted not only through the literature, but also on the basis of five valuable contacts: (1) with Huixin Zhang, a Chinese girl who moved to Italy for work when she turned 18; (2) with Lorenzo Tabasso, an Italian boy perfectly fluent Chinese-speaker who worked and lived in China for 10 years after graduation; (3) with Yu Chen, a university professor at the same "G. Peano" Department of Mathematics; (4) with Liu Baocun, professor of Comparative Education at Beijing Normal University and visiting professor at the Department of Primary Education Sciences in Turin in 2019; (5) Xiaoli Lu, a Chinese young researcher with a research position in the School of Mathematics of the East China Normal University who has experiences on Chinese lesson study.

Why Lesson Study?

A recent paper by Bakker and colleagues stresses the key role represented by professional development for teachers as a research topic in Mathematics Education (Bakker, Cai & Zenger, 2021). It has been a growing research trend since the early 2000s, with the presentation of the survey *Professional Development of Mathematics Teachers* at ICME10 (Adler, Ball, Krainer, Lin, & Novotna, 2005), on which Sfard (2005) noted:

> I am pleased to find out that the last few years have been the era of the teacher as the almost uncontested focus of researchers' attention (Sfard, 2005, p. 409).

Furthermore, in-service teachers' professional development has acquired a central position in the international debate on educational policies. It should offer tools and opportunities to investigate situations while involving teachers in activities structured to encourage them to break out of routinized practices and move towards re-elaborating and re-planning their teaching (Brophy, 2006). Teachers agree that in-service professional development is the driving force of innovation, and the teaching-learning processes cannot remain *static* (Arzarello, Robutti, Sabena, Cusi, Garuti, Malara & Martignone, 2014; Weber, Gold, Prilop & Kleinknecht, 2018; Capone & Spagnolo, 2019).

Yet, while there is strong evidence in favor of student *active* learning, including group work and peer instruction (e.g., Gillies & Haynes, 2011; Freeman *et al*., 2014), the adoption of active learning in teachers' professional development – such as peer coaching or studying teaching cases (Shulman, 1987; Anderson & Pellicer, 2001) – still needs a push to further increase. This kind of professional development is clamoured for by education ministries around the world and by teachers themselves but often confusing activeness with collaboration (e.g., I have a personal experience of this issue working in the Italian and French contexts and knowing the ministerial requirements of the two countries, to which reference is made). Collaborating does not always mean active working. Current studies and courses result in practices where collaboration between teachers is enhanced, but the focus does not always remain on active learning. Collaboration between teachers is likely to result in exchanges of advice, opinions and good practice and even run out of steam during the professional development course itself.
Besides, Stigler and Hiebert remark,

> Disseminating models of effective teaching through static documents might work if teaching were a noncultural activity. If teachers learned to teach by studying books and memorizing techniques, written recommendations might have their intended effect. But everything we have learned indicates that teaching is a cultural activity, and consequently the writing and dissemination of reform documents is an unrealistic way to improve education. [...] If you want to improve teaching, the most effective place to do so is in the context of a classroom lesson. If you start with lessons, the problem of how to apply research findings in the classroom disappears. The improvements are devised within the classroom in the first place (Stigler & Hiebert, 1999, p. 109-111).

Thus, there is the need for theoretically based research about the way cultural aspects influence students' learning in mathematics (Bartolini Bussi, Canalini, & Ramploud, 2013) as well as mathematics teachers' critical reflection during their professional development experiences (Skott & Møller, 2020). Yet, teachers' professional development studies and courses that take cultural factors into account are sparse (e.g., Mellone, Ramploud, & Carotenuto, 2021), although their importance is recognized (Guala & Boero, 2017; Stylianides & Delaney, 2011; Jaworski, 2004).
In other words, while good progress has been made in improving teaching-learning processes, their widespread use and triggering of a capacity for their critical and intentional use remains a challenge (Scheiner *et al*., 2019), not to mention studying to improve culturally sensitive understanding of educational processes. A number of suggestions have been proposed for enhancing the professional development of teachers beyond the frontal lecture, the few-hours workshop, or the ubiquitous sharing-chatting on good classroom practices (Robutti *et al*., 2016). Starting with the United States, where the reception has been maximum, since *The Teaching Gap* (Stigler & Hiebert, 1999; Stigler, Gonzalez, Kawanaka, Knoll & Serrano, 1999) showcased the videotaped lesson of the Third International Mathematics and Science Study (TIMSS) (from the United States, Germany and Japan), Japanese Lesson Study has been highlighted as a milestone in contributing to a supportive way to improve in-service teachers' classroom instruction (Fernandez, Cannon, & Chokshi, 2003; Elliott, 2012).

Lesson Study is a collaborative teachers' professional development methodology, focused on the co-responsibility in the lesson-planning process of the involved teachers and knowledgeable others (Huang *et al*., 2019). It originated in Japan around 1870 to answer professional development needs for qualified teachers (Isoda, Stephens, Ohara & Miyakawa, 2007), and it has been in place for more than a century in both China and Japan nationwide (Chen & Yang, 2013; Fernandez & Yoshida, 2004; Li, 2019). The Lesson Study was introduced in China to foster, through peer-to-peer interaction, the in-service professional development of hundreds of thousands of teachers needed for Mao Zedong's literacy project in the 1950s (see the history of the *Great Proletarian Cultural Revolution* (无产阶级文化大革命) and the reference to it in the interview with Professor Baocun on p. 77).

Only recently Lesson Study has been gaining momentum on a national scale (and no longer through the intervention of individual researchers) in Western nations as well.

An interesting example is the French education system, which just in 2020 approved, for now for primary schools only and within French language and mathematics subjects, a 6-year ministerial mandate for an innovative professional development model based on group work and peer instruction:

> Professional development is grounded in the local teaching context (classrooms become the sites of professional development), and it articulates the exploitation of theoretical resources and variations in the classroom, allowing teachers to rely on research findings to find appropriate responses to their needs and those of their students (from French National Professional Development Plan[22]).

The plan radically changes the French in-service teachers' professional development. The official guide to the plan announces: "a new approach to in-service professional development, incorporating an accompanied reflective analysis: working within a small group [the *constellation*] of 6-8 teachers, led by a local trainer and set up as close to the classroom as possible"[22]. The constellation is declaredly inspired by the model of "lesson studies". The guide says: "The sequence developed collectively with external help is then implemented in a classroom by one or more teachers. The other teachers in the group observe this implementation, in particular to assess the effectiveness of student learning. The next step is to discuss the observations in order to identify what has been learned and what can be improved". The schedule for each constellation is 30 hours per year, distributed as illustrated in Figure 1. 19. Nevertheless, for the moment, the ministerial proposal is not fully accepted by French teachers. The web page of a French school labour union reports: "The lesson study model has been proposed a lot by the OECD since Pisa 2015. In Japan, volunteer teachers form groups that jointly design a lesson. One of the group members implements it in front of his colleagues. After observing and analysing the session, the teachers meet again:

[22] https://www.sgen-cfdt.fr/actu/communique-formations-constellations/ http://50.sgenbn.fr/formations-en-constellation-plan-francais-maths/ http://www.cafepedagogique.net/LEXPRESSO/Pages/2020/06/29062020Article637290123216351278.aspx

the one who led the session speaks first, then his colleagues each present their analysis in turn. This model is proving to be effective, but in a society and educational system totally different from ours.[22]"

Figure 1. 19 The "constellation" work: distribution of the compulsory 30 hours of French teachers' professional development.

Since 1999, indeed, a great deal of research focused on the dissemination of Lesson Study around the world (i.e., Huang & Shimizu, 2016; Lewis & Lee, 2017; Quaresma *et al*., 2018). Thanks to Catherine Lewis' research (2000) and Makoto Yoshida's doctoral thesis (Yoshida, 1999) the Japanese 授業研究[23] (*jugyoukenkyuu*), known in the Western world as Japanese Lesson Study, has been introduced in the United States, creating an Anglo-American tradition. Only then it has been adopted and adapted globally, gaining the educators' and researchers' interest from worldwide to promote collaboration and cooperation between teachers (Verhoef, Tall, Coenders, & Van Smaalen, 2014; Bartolini Bussi, Bertolini, Ramploud, & Sun, 2017; Adler & Alshwaikh, 2019; Huang, Takahishi, & da Ponte, 2019).

> With the high reputation in practical features such as job-embedded and teacher-oriented but student learning-focused (Lewis, 2016), lesson study has been well recognized as one of the most effective [professional development] (Huang, Lai, Huang, 2021, p. 202).

[23] *Nomen omen*. Concretely verifiable through any web translator, *kenkyuu jugyou* means *lesson study*, "and refers to the lessons that teachers jointly plan, observe and discuss. *Jugyou kenkyuu* – using the same two words in the reverse order – means *lesson research*, and refers to the process of instructional improvement of which the research lesson is the core piece" (Lewis, 2000, p. 4).

A glimpse of the global impact can be gained by peeking at the World Association of Lesson Studies (WALS) activity, which organizes yearly conferences with teachers and researchers *deeply committed to refining our collective understanding and execution of lesson study* (quoted from the web page), and at the journal IJLLS, that is dedicated to research on Lesson Study.

Structurally similar to the Japanese Lesson Study – built on collaborative lesson planning, cycles of teaching with classroom observation, post-lesson reflection and lesson revision –, Chinese scholars have shown the world the existence of a Chinese local form of Lesson Study called Teaching Research Group activity (Yang & Ricks, 2012). Teaching research groups are school-based realities that, since 1950, are in place as local professional learning communities (DuFour, 2004) within a well-established teaching research system (Chen, 2020).

> **Professor Liu Baocun:** Lesson Study, it doesn't mean the Lesson Study in the international context, because in the Chinese terms, we have a Chinese term 课 (kè, i.e. lesson). Here we have different activities around kè, it means "planning kè", or "planning lesson", or "talking lessons", or "teaching lessons", or "observing lesson", and also "evaluating lesson", so is a different term. But sometimes, you know, is very difficult to translate in English or other languages, so the meanings are similar, but in Chinese they are different from Lesson Study. It is a Chinese way. […] in China, in 1950, in Chinese universities we started our ***school-based teaching research***. You must do teaching research! It is a kind of measure related to the model of **soviet union**. For each subject we teach in school, we establish a **teaching research group** (English, Mathematics, etc.). The teachers are expected to do teaching research, within this unit. We develop our microplanning together and we develop our lesson teaching together, and the **teachers collaborate**. They observe the lessons, give some comments, give some suggestions on how to improve individual teaching, especially for the junior teachers. So that's the beginning of the Chinese LS.
>
> personal interview, with Simona Giordanengo
> Torino, Palazzo Nuovo, 3rd December 2019

> **Xiaoli:** May I ask **what's your definition of Lesson Study?**
> Guanmo lesson in Chinese, just as its name implies, is **observing (guan) and emulating (mo)**. That is, the **lesson is for people to observe and emulate**. Therefore, most of the guanmo lessons in China are the outcome of teaching competitions. Teachers are selected in school level, city-level, and then larger distict level to attend teaching competitions, and then the selected lessons would become guanmo lesson. **Another type** of guanmo lessons are delivered **by expert teachers**, for peer teachers (in particular young teachers) to learn. In my opinion, guanmo lessons are one type of public lessons in China but different from the Japanese definition of lesson study.

> Lessons provided by after-class tutorials mean that these lessons are not provided by the schools. They are sort of shadow education.
> […] Of course, the lesson study does **not only lead to teaching competition, but also help improve teachers professional development, individually and collectively**.
> Let us take the guanmo lesson comes from competition as an example. Firstly, every young teacher is encouraged to attend some teaching competition. Then he/she would receive supports from the lesson preparation group he/she situated in. He/she will **select one topic** and **teach in a real class** for **peer teachers to observe**, **evaluate and discuss** which aspects could be improved. **Such a period is called "mo"**, a different Chinese character different from the "mo" in "guanmo", **refers to polishing (the lesson), to make the lesson improved**.
>
> personal interview by e-mail exchange
> 18th July 2019, 7:36 a.m.

From these two interviews it is possible to go deeper into an understanding of what the Chinese Lesson Study can be. It is now a point of reference that it is not understandable if not taken in its cultural context (Chen, 2017). Introducing one of the terms used to refer to the Chinese Lesson Study (观摩课 – guān mó kè), Ramploud and Munarini (2015) report:

> An educator of Chinese origin participated in this work and said: «Guan means watching, Mo rubbing, smoothing and it also means clash-friction, Ke is lesson. Guan and mo together mean watching and learning. To observe and learn from mistakes. The Chinese term for Lesson Study already tells you that you can learn by watching others and comparing yourself with others; you are not born learned, but you can always improve» (Ramploud & Munarini, 2015, p. 61).

Here, too, *nomen omen*. Opening the classroom door and having others, having public audience, in the classroom, making of public lessons (公开课), and even being videotaped "seems to be great comfort" (Stigler, Thompson & Ji, 2012, p. 277) and a norm for Chinese teachers (Li, 2019). So, 观摩课 – guān mó kè, such a name refers explicitly to another an inspiring metaphor in Chinese tradition:

> "Those stones from other hills can be used to polish the jade"
> 他山之石的那些可用于抛光玉石
>
> Practices in other countries can serve as food for the improvement of one's own practice. There is no need to label such practices as "good" ones (let alone "best ones"), since whether it is good or not depends on how one uses it, and also on cultural backgrounds. By reflecting on the practices of these regions, one reflects on **one's own culture**, understands oneself more, and forms a basis of moving forward in one's own way (Fan, Wong, Cai, & Li, 2004).

Thus, Chinese Lesson Study unique features are (1) reiterative research lesson teaching to different groups of students, in order to refine its execution; and 23) experts' facilitation throughout the Lesson Study (Chen, 2017; Huang, Lai, & Huang, 2021). Yet these features need a deep

understanding:

> The traditional Chinese idiom "Proficiency comes from familiarity" (熟能生巧) can provide a proof for this kind of deliberate practice. Teaching is a craft, whose perfection takes practice over time. Only by practicing it again and again, those movements of the teaching artistry, which are difficult to grasp, will take hold in their body (Chen, 2017, p. 287).

This instruction comes from the Buddhist tradition, from which the tea ceremony also comes: one of Japan's most fascinating traditional rituals. Sixteenth-century Buddhist monks codified every step, even the simplest gestures, because "There are things that you can try as hard as you want but you don't understand until the right moment comes. But when you do understand them one day, you can't pretend that you don't" (Morishita, 2020).

Furthermore, to follow a reiterate cycle, repeated teaching provides an opportunity for teachers to make mistakes. Planning does not have to be perfect, and if something happens differently in the classroom, it is an opportunity to reflect on expectations. Only "practice makes perfect" (Li, 2006).

> The above theorization of the three features of Chinese LS can be viewed as efforts in filling the gap between theory and practice. Public lessons create a real learning site that bridges the mind and body divide. Practical reasoning in repeated teaching resorts to useful standards in actually improving the quality of a lesson, more than "correct" standards codified in theoretical books or official documents. "Good" examples embodied in "good" teachers' actions epitomize the hidden criteria for "good" teaching, which are crucial for teacher learning and identity formation. What overrides the above efforts is the "supremacy of practice" (Arendt, 2009)[24]. Since Plato, the divide between theory and practice has been hindering the western

[24] The supremacy of practice is also reflected in mathematical ontology and epistemology and thus in mathematics education. And, like a circle, it comes back to the whole "Chinese mentality" (Kirkpatrick & Xu, 2012). Indeed, Professor Chen has an interesting, and for him unexpected, reflection on this issue:

> **Prof. Yu Chen**: [...] in, let's say, "ancient Chinese mathematics" there was no such thing as demonstration, which is the proof. Even school doesn't prove anything to you. And I didn't know, and in fact **I didn't notice this point**. I went to look at the history of Chinese mathematics and I see that the Chinese mathematics really didn't know what "proof" meant. **It only knew how to make examples**. That is to say, by means of examples he came up with rules: from examples to rules. And this "more or less right" **leap** is the Chinese mentality: **the "more or less".** Although I mean this: most of the teaching of modern mathematics is Western.
>
> personal interview by webex meet
> 28th October 2020 (01:04:45 to 01:05:36)

Indeed, in their study on the articulation of Chinese thought, from a linguistic point of view, concerning the argumentation, Kirkpatrick & Xu (2012), show how Chinese argumentation is structured "by addition of examples". That is, the authors show how argumentation performed by Chinese-speakers is not linked, as we are used to thinking in Western thought (and this is a

> mainstream academia from seeing the value of practice. The life of theoretical contemplation has always been considered superior to that of action. In Chinese LS (and those of other countries), teachers have subverted this false hierarchy, by giving more weight to practice, or theory that is derived from practice. Enactment of understanding, practical reasoning, and emulation of "good" examples all take place in LS practices. As practice has a moral purpose and creative power, teachers can create their own knowledge and identity through embodied action. Emphasizing the supremacy of practice does not mean that theory is unimportant. In the Chinese epistemology, theory comes from practice or they are two sides of the same coin. The underlying belief in conducting LS is that theory and practice are one, embedded in teachers' activities. (Chen, 2017, p. 290).

Thus, teaching becomes an experimental enquiry. In the Lesson Study experiences described below, indeed, teachers with decades of teaching experience never stop questioning themselves and are even a little fear and excited about the implementation in the classroom. The classroom becomes a laboratory, that as such needs the experts' facilitation (Watanabe, 2002; Gu & Gu, 2016). And experts from Buddhist tradition are aware that it is wise to let errors occur before correcting them (Morishita, 2020).

Therefore, the unity of knowing and doing, the practical reasoning for the most appropriate action in specific contexts, and the emulation-observation of who is better to self-polish, together represent the teachers' reconstruction of the Chinese cultural heritage in contemporary Lesson Study activities. So, no *a priori* theorisation of Lesson Study (be it Japanese or Chinese) can be said to exist. However, it is a Western need. As embedded in Western thought, we need to theoretically frame a practice, to read it as praxeology:

> As usual in scientific research, the role of a theoretical framework is to make our assumptions, reasonings, research objects and research questions precise and explicit. It is particularly important when the research objects include such complex and culturally embedded phenomena as teacher knowledge, and "settings" for developing it (Miyakawa & Winsløw, 2019, pp. 283-284).

Winsløw (2011), while fitting a Western needly theorising-based context, describes Lesson Study as part of the Japanese *paradidactic infrastructure* – that concern "teaching-related" practices, but are not themselves teaching. In other words, Winsløw attempts a comprehension of Lesson Study that is more linked to the systemic and ecological dimensions in which this practice is embedded.

Yet, despite the rising awareness on the importance of studying endemic cultural contexts and identities to contextualize global trends in Mathematics Education (Bakker, Cai & Zenger, 2021), many discussions in the WALS group and research teams in Mathematics Education around the world (e.g. Buchard and Martin's (2017) detailed review of the existing literature), would have liked to arrive at an identification of the "generic essential features" (*ibid*., p. 13) of the Lesson

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> cultural fact), to the tension towards a hypothetical-deductive organisation. Rather, it is linked to the "accumulation" of examples in favour of a certain statement.

Study methodology (see Figure 1. 20). Although there is not yet complete academic agreement, there seems to be a lack of awareness of what Lesson Study really could be. The majority of reports on Lesson Study around the world seems to depict Lesson Study as an *isolated* practice in the Japanese panorama of teachers' professional development practices (Miyakawa & Winsløw, 2019) and seemingly ignores that the Japanese definition of Lesson Study (as well as the Chinese one) is not as clear cut as the American one (Miyakawa & Winsløw, 2013). This suggests that "to develop a deeper understanding of Lesson Study in a post-modern global world, there is a need to seek views beyond those presented from an American perspective" (White & Lim, 2008, p. 915).

Through this theoretical framework, the research not only considers the ecological and systemic dimensions of the Lesson Study, but sees it as embedded in a semiosphere. In this way, the Lesson Study is seen as a significant practice of mathematics teachers and, as such, to be read through a semiotic lens (Radford, 2008). The assumption is that without a cultural-semiotic framework it may be incomplete to answer the research problems highlighted in the previous chapters.

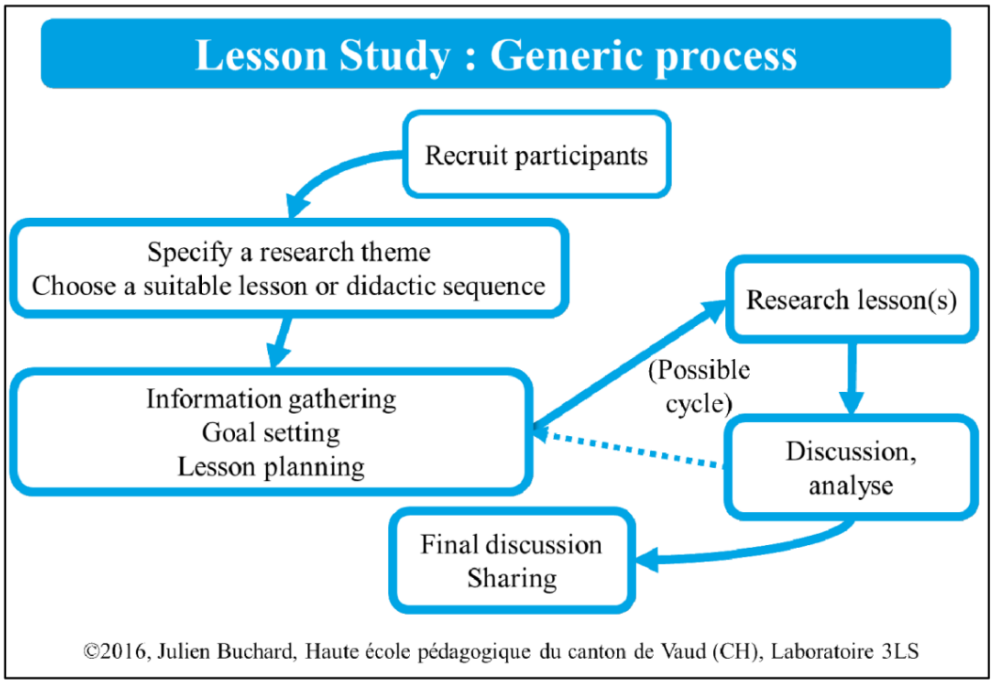

Figure 1. 20. The "essential Lesson Study features" (Buchard & Martin, 2017, p. 13).

THEORETICAL FRAMEWORK

Teachers' discourses within specific collective practices of professional development (i.e., Lesson Study) are the unit of analysis of my research. It is indeed within collective practices that people unconsciously reproduce and alter different cultural forms of representation and signification (Saxe, 2014). In particular, Lesson Study, embedded in a Cultural Transposition prespective, is here analised as the place where otherness of a different cultural paradigm is experienced to gain awareness of what Jullien (2005) defines *unthoughts*, i.e. those aspects that escape people's consciousness when they are immersed in their own culture. Meeting foreign cultures leads to looking at one's own practices. It is not so much a matter of understanding foreign cultures, thought of as homogeneous spheres with marked boundaries (Arzarello, 2020), but of an "interaction with foreignness" (Welsch, 1999, in Barton, 2008). Every culture (or semiosphere) indeed needs another culture to define its own essence and limits.

I intend to focus on mathematical education from a Bakhtinian sociocultural point of view, i.e., where communication is collaboration, in the sense of openly shared meaning, but conflicts are possible. Treading upon the consolidated Bakhtin's path on a dialogical approach to consciousness, I follow Lotman's understanding of the dynamics of cultural encounters.

Bakhtin and Lotman applied their dialogical point of view mainly on general cultural practices like literary practices or general philosophy of the humanities. A valuable application of these ideas in the Mathematics Education field, in particular in mathematics teachers' professional development, and in present time, requires some specifications. In the present research I take Bakhtin's thinking and Lotman conceptualisation of the Semiosphere as the fundamental background of the analysis of teachers transformative learning (Mezirow, 1981, in Kitchenham, 2008).

Mathematics teachers' discourses are the *texts*, the textualised culture, meant as the focus of analysis of the Semiotics of Culture theory.

Semiotics of Culture theory is the theoretical framework in which to study not only Radford's cultural semiotic system of activities, nor only the noosphere as outlined in the ATD, but rather the semiotic space of teachers participating in Lesson Study experiences, outside of which space "even the mere existence of semiosis is impossible" (Lotman, 1989, p. 44). This is in order to learn how to read and then manage the relationships, the webs of significance that exist within the whole semiosphere.

RESEARCH QUESTIONS

My research questions were initially 3:

RQ1. What methodology, what theoretical framework, could be used to analyse cultural conditions and constraints in mathematics teachers' professional development practices?

RQ2. How do cultural elements affect the professional development of mathematics teachers (as individuals and as a community)?

RQ3. How can the "culturally sensitive" understanding of teachers' critical reflection be improved?

However, as can be appreciated from the research trajectory traced above, it was as I progressed

with the study that I realised that these were not enough. Four further questions arose.
Two questions of an epistemic nature:

RQ4. What is meant by culture?

RQ5. How is culture characterised in Mathematics Education?

and two question of a theoretical-methodological nature:

RQ6. How to develop a semiotic-cultural analysis of the texts (discourses) of mathematics teachers?

RQ7. Which theoretical lens(es) allows such an analysis to be achieved?

The research, drawing on the studies of Bakhtin, Voloshinov and Radford, led me to consider Lotman's Semiosphere as a theoretical lens possibly suitable for my purpose. However, it was necessary to consider it within research in Mathematics Education. The construct of *Cultural Conflict* is thus the first outcome of my research. It is not a question of understanding the "non-semiotic" space, but of establishing an interaction with this foreignness in order to give significance to one's own space. This completely espouses the perspective of the Cultural Transposition (Mellone *et al*., 2019), the aim of which is at fostering teachers' observations concerning mathematics education with a consequent innovation of the consciousness on the related practices, that is, transforming mathematics teachers' habits of mind (Mezirow in Kitchenham, 2008).
Thus, the Lesson Study, as a methodology of teachers' professional development adhering to the construct of critical reflection (Kemmis, 1985), and in particular its Chinese vision (Chen, 2017), is proposed as a paradigmatic model for my research.
I have placed my research within the Italian cultural context.
All these examined and processed aspects made it possible to gradually refine my research questions. It was thus possible to achieve a second condensed and contextualised formulation of the questions, embedded in my theoretical framework and cultural research context.
I have combined research questions 1, 6 and 7 into a New first Research Question:

NRQ1. Can the Semiosphere be the theoretical lens that allows a unified reading of the transformative learning processes of Italian mathematics teachers?

Research questions 2 and 3 were then refined and reframed as follows:

NRQ2. How can the Semiosphere characterise the cultural transposition of Italian mathematics teachers' practices?

NRQ2 turned out to be a very complex question, which I therefore preferred to split into two:

NRQ2a. how can Lesson Study be studied as a significant practice for Italian mathematics teachers, in order to help them in becoming critical and active participants in their professionalism?

NRQ2b. what is the product of the cultural transposition of the Chinese Lesson Study into the Italian context?

The research questions RQ4 and RQ5 are still valid.
I summarise the developmental trajectory of my research questions just described in Table 1. 3.

Table 1. 3 Research questions' developmental trajectory.

FIRST FORMULATION **developed during the study**	**SECOND FORMULATION** **embedded within the theoretical framework and cultural research context**
RQ1 What methodology, what theoretical framework, could be used to analyse cultural conditions and constraints in mathematics teachers' professional development practices?	NRQ1 Can the Semiosphere be the theoretical lens that allows a unified reading of the transformative learning processes of Italian mathematics teachers?
RQ2 How do cultural elements affect the professional development of mathematics teachers (as individuals and as a community)?	NRQ2 How can the Semiosphere characterise the cultural transposition of Italian mathematics teachers' practices?
RQ3 How can the "culturally sensitive" understanding of teachers' reflection be improved?	NRQ2a How can Lesson Study be studied as a significant practice for Italian mathematics teachers, in order to help them in becoming critical and active participants in their professionalism? NRQ2b What is the product of the cultural transposition of the Chinese Lesson Study into the Italian context?
RQ4 What is meant by culture?	RQ4 What is meant by culture?
RQ5 How is culture characterised in Mathematics Education?	RQ5 How is culture characterised in Mathematics Education?
RQ6 How to develop a semiotic-cultural analysis of the texts (discourses) of mathematics teachers?	
RQ7 Which theoretical lens(es) allows such an analysis to be achieved?	

METHODOLOGY

The methodology of analysis follows the traces of Lotman's analytical work.

a) First and foremost, all the collected data - i.e. the audio and video recordings of the research lesson planning meetings, of the implemented research lessons, of the post-implementation discussions, of the large group returns of the work done in each working group - are considered as texts. As such, the texts allow the identification of the moments of movement of the semiosphere. This is a phase of interpretation and as such is never definitive and always dynamic. The interpreter who approaches these texts must therefore reciprocally situate himself and the considered text. The *comprehensiveness* of the semiosphere is identified and, where possible, an approximate perimeter is traced.

b) The moments of movement are categorised as outward push or inward reception (Figure 1. 8), depending on the locally interpreted interaction between semiosphere and "nonsemiotic" external space at that time of movement.

c) The Cultural Conflict generated by the movement is described, and what, if any, is the "detonator", i.e. the tool, the practice that according to the local interpretation triggered the movement. In the Lesson Study context, the detonators can be tools, such as the lesson plan, or the Lesson Study cycle phases. The description of the cultural conflict, meanwhile, is achieved through a narrative of the interpreter, who describes the encounter between the semiosphere and otherness and, where possible, makes the unthought of explicit.

Now the asymmetry is manifest. Asymmetry is classified according to the following 5 descriptors:

- mathematical content level;
- epistemological level;
- didactic-methodological level;
- pedagogical level;
- ecological level.

In order to determine which descriptors were appropriate and to choose these descriptors and not others, a groundwork analysis was conducted on 47 questionnaires completed by teachers enrolled in the ministerial professional development course (PDC) *il Lesson Study in Matematica* [the Lesson Study in Mathematics] carried out from the 3rd of September 2020 to the 20th of March 2021. The questionnaire was provided to teachers at the completion of the first of two Lesson Study cycles that were proposed to be carried out during the course.

The questionnaire, after a preliminary section on personal data, consists of 5 questions with the addition of two final sections asking for free comments and for expressing interest in being interviewed in person. The 5 questions are:

1. Reflecting on the experience of this first cycle, from the point of view of the Lesson Study as a proposal for teacher professional development, which aspects do you find most interesting? And which ones would you like to leave out? Give reasons for your answers
2. Have you felt contrasts or/and affinities between the demands due to the Lesson Study methodology and your teaching professionalism? Which ones and why, in your opinion?

3. Has the experience changed something in your way of teaching? In which way? And in your way of reflecting on your teaching practices compared to those of your colleagues?
4. With respect to teaching MATHEMATICS in particular, what do you consider peculiar to Lesson Study?
5. The dialogue between mathematics teachers colleagues can occur at different levels: at the content level (mathematical, e.g. the division algorithm or the formula for solving second degree equations), at the pedagogical/didactical level (methodologies adopted in the classroom, e.g. how to teach division or second degree equations), and at the epistemological level (the way a teacher thinks about content in relation to the idea he/she has of mathematics, e.g. the role of algorithms or formulas in relation to problems). In your opinion, in which of these areas does the Lesson Study most stimulate collaboration and possible innovation? Why? Please give reasons for your answers.

The answers to these questions as a whole corroborated the hypothesis that the following levels – here seen as loci (Sfard, 2008) – can be used as descriptors of asymmetries:

- mathematical content level;
- epistemological level;
- didactic-methodological level;
- pedagogical level;
- ecological level.

In fact, all and only these levels were used as labels to attach to the answers. No teacher presented answers that were not situated in one of these levels of comparison. These levels were then found in the literature, e.g., in the onto-semiotic framework of the Suitability Criteria (Hummes, Font, & Breda, 2019), or even within the same scale of level of didactic codeterminacy.

Data collection

The collected data originates from meetings, projects and experiments carried out during the time of my PhD program, from September 2017 to March 2021, see Table 1. 4.

Table 1. 4 Numbers of participants in Lesson Study experiences.

	participants (teachers)					research team	LS cycles	period
	in-service			prospective				
	Primary school (grade1-5)	Middle school (grade 6-8)	Secondary school (grade 9-13)	Primary school	Middle/ Secondary school			
Pilot study n.1 – with prospective teachers					29	4 (one of whom is the me)	8	October 2018 - January 2019
Pilot study n.2 – with in-service teachers	5, one of whom a retired former teacher-researcher					1 (me)	4	November 2018 - April 2019
Val d'Aosta regional PDC	7	10		4		1 (me)	4	September 2018 – May 2020
Piemonte regional PDC	27, three of whom took part in Pilot study n. 2	6	14	6	4	1 (me)	23	September 2019 – March 2020
Total of involved teachers	36, since 3 took part in two experiences	16	14	10	33			
	Total in-service teachers: 66			Total pre-service teachers: 43		Total implemented LS cycles: 39		

My initial experiences with Lesson Study methodologies emerge from collaborations with the University of Modena-Reggio Emilia (Bartolini Bussi & Ramploud, 2018). Later on, two pilot studies were carried out, the first with prospective teachers at the University of Turin and the second with in-service teachers in Piossasco, a town near Torino.

The first pilot study was held from October 2018 to January 2019.

Pilot study n.1 – EMHS Lesson Study with prospective teachers

The research team consists of 1 professor and 2 PhD students (one of whom is me) from the University of Turin, and a researcher from the University of Salerno.

The participants to the experiment are 29 secondary school prospective teachers (from 22 to 24 years old, with no experience in real-classroom teaching – although some already engaged in two others Mathematics Education courses which represents for the students one of the first opportunity to deepen Mathematics Education topics), attending 'Elementary Mathematics from a Higher Standpoint' course (professors L. Giacardi and O. Robutti) in the first semester 2018-2019 at the University of Turin (Mathematics Department). Generally, students wishing to become secondary school teachers choose this course, and for this reason we considered them prospective teachers. Currently, in Italy, to be hired as a middle and secondary school teacher, it is necessary to attend the university degree course related to the specific teaching subject. It usually consists of 3 Under Graduate (UG[25]) years and 2 Post Graduate (PG[25]) years, plus 24 credits (ECTS) in psychological, anthropological, pedagogical and didactic subjects.
The course in question focuses on continued fractions, explored from epistemological, historical, and didactical standpoints. The course lasts 48 hours (6 ECTS): 30 hours are dedicated to continuous fractions in the history and epistemology of mathematics, 16 hours to didactical approaches to continuous fractions (Klein, 1908 [2016]), and 2 hours to the participation to a joint project with other courses for prospective teachers. In the 16 hours of didactical approaches, prospective teachers have to use the mathematical knowledge acquired in the course – working in groups in a Lesson Study setting – to design an activity (made of a task for students with institutional references to the National Curriculum) on continued fractions (ideally addressed to students in grades 6 to 10), to plan a lesson, to teach this lesson to their peers- pretend-students, and to observe it in front of their peers and of the researchers, and finally to discuss the efficacy of the lesson.

The goal for the researchers was mainly to challenge themselves with the Lesson Study as a professional development methodology. In fact, the research group of the Department of Mathematics "G. Peano" of the University of Turin has been working for years on the professional development of mathematics teachers in Italy and related research, at various levels, in collaboration with the Italian Association for Research in Didactics of Mathematics (AIRDM). Indeed, the key role represented by "professional development for teachers" as a research topic in Mathematics Education is now undisputed (Bakker, Cai & Zenger, 2021). In particular, the survey commissioned for ICME 13 "Teachers Working and Learning Through Collaboration" (Robutti *et al*., 2016) stresses the need to investigate teachers' collaboration for professional development, indicating a variety of emerging or well-established methodologies for mathematics teachers to work and learn through collaboration, as well as a variety of theoretical perspective to frame such methodologies. This survey led to ICMI Study 25 "Teachers of Mathematics Working and Learning in Collaborative Groups" – to which the young researchers in the team contributed (Capone, Manolino, & Minisola, 2020). The ICMI Study 25 Discussion Document again recognised:

[25] The labels UG and PG are from the European mobility program *Erasmus+*. PG years end with what can be called a Master's Degree (Mellone *et al*., 2021).

> Across education systems, and at all educational levels, mathematics teachers work and learn through various forms of collaboration. Such collaborative work of teachers has a long tradition in mathematics education as it is critical as a way to bring educational innovation into the everyday practice of teaching. (ICMI Study 25 Discussion Document, p. 2)

In this first pilot study, there was no shortage of difficulties, and the researchers particularly questioned themselves as professional educators. From this work the researchers experienced first-hand the effort and the need for a careful cultural transposition, thus paving the way for conscious and shared research (see Minisola, 2021).

The second pilot study (see Manolino *et al*., 2020; Manolino, 2020; 2021) was held from November 2018 to April 2019.

Pilot study n.2 – Piossasco Lesson Study with in-service teachers

The working group is made up of six people: the researcher (me, a PhD student), a retired former teacher-researcher (Ezio) and four teachers who teach in different primary school classes of the same institute. Three are 1st-grade teachers: Michela is a support teacher for low achievers, Nicoletta teaches Italian in her class, Marcello teaches mathematics, science, history, geography, and English. Valentina, the fourth teacher, teaches mathematics and science in 3rd grade. The Italian school system is characterized by high flexibility in teaching in primary school. Teachers teach several subjects and even the support teacher, supporting the class in which there is the low achiever, can take charge of teaching subjects to the whole class, according to his skills, if the team deems it appropriate.

The first part of the experiment consists of three complete cycles in the three 1st-grade classes. The topic of the lesson is the introduction of the "plus" sign for the addition and its institutionalization. The specific goal for children is to understand the concept of addition as the sum of two quantities in its meaning of "putting together" and relate it to the signs of mathematical language. In the second part of the experience, consistently with the previous three cycle, a new lesson is carried out in the 3rd-grade. The designed activity is part of the educational path that includes the knowledge of weight measurements and the study of state transitions, via experiments. The aim is to accompany students in reinvesting their mathematical knowledge and argumentation skills with respect to the transversely of the disciplines. Each teacher implements the lesson in his or her class but in the total co-responsibility of the group, which is there in agreement with the school headmaster. During the lesson, the other participants play the role of active observers: in 1st-grade classes they interact with the students as "hand-lenders", i.e. they transcribe the thoughts of not yet writing-skilled children (technique documented in one of the work units of the MIUR-DIMA Project[26]). The "hand-lenders" assistance technique consists of individualised assistance given to the child for the construction of a text suitable for writing, which gives order and form to the child's thoughts, and is dictated to the teacher-writer with the twofold aim of relieving the child's fatigue and accustoming him to constructing well in his mind the speech to be written.

The experience, for a total duration of 24 hours of group work, covers all four cycles. All the

[26] www://didmat.dima.unige.it

design (4 hours of initial formation and 8 hours of design de facto, 2 per cycle) and discussion moments (8 hours, 2 per cycle), but also the classroom lessons (1hour in each class – cycle –, for a total of 4 hours), were video-recorded. Some excerpts from these recordings were then transcribed by the researcher. In addition, for each planned lesson, the group produced a Lesson Plan (Manolino *et al*., 2020; Bartolini Bussi & Ramploud, 2018): a written table collecting the entire lesson planning, the objectives the group chose for the lesson, the positioning of the lesson within the long-term planning of the class, and the educational intentionality behind each choice of the group.

The terminology used to indicate the subjects of the two pilot studies intentionally differed. In the first, the prospective teachers were divided into 8 groups of their choice, from two to four per group, and planned-implemented-discussed one lesson per group. In fact, we speak of *research team*, to denote who conducted and assisted the course, and the *participants* are those who carried out the Lesson Study cycles (1 cycle per group). Instead, for the second pilot study we speak of a *working group*, consisting of the 4 implementing teachers, Ezio and me, as we all were member of the 4 Lesson Study cycles.

The "Lesson Study" project in Valle d'Aosta

In May 2019, a three-years agreement between the Department of Mathematics "G. Peano" of the University of Turin, the Regional Council for Education, Universities, Research and Youth Policies, the Superintendency of Studies Department, and the Val d'Aosta School Autonomy Support Office is signed. A PDC for in-service primary and middle school teachers within the Lesson Study methodology begin. The course responds to ministerial requests. The Professional Development Project "Lesson Study", referred to the note of the Superintendency of Studies prot. n. 20243/SS of October 21st, 2019 of the Region of Valle d'Aosta, born in order to respond to an expressed demand of STEM subjects teachers professional development in the Val d'Aosta area. Two schools join the project: the Istituzione Scolastica "Émile Lexert", with the two primary school complexes, "Quartiere Cogne" and "Ettore Ramires", and the middle school "Émile Lexert"; and the Istituzione Scolastica Unité des Communes Valdôtaines "Mont Rose A", with three primary school school complexes, "Vert", "Donnas" and "Hône", and the middle school "Pont-Saint-Martin". From these experiences, 4 Master's Degree dissertations result – of which I am supervisor – by four female students (prospective teachers) from the Primary Education Course of the University of Turin. Thus, a new knowledge of Lesson Study cultural transposition is emerging, a deep research experience not only to collect data, but to "transform the reality" (in Mezirow's terms) in which we are acting in order to improve the educational practice (Trinchero, 2002). The first Master's Degree thesis is a study of the Italian context, using the *cultural dimensions* of the Dutch psychologist Geert Hofstede[27] (2001, in Arani, Fukaya & Lassegard, 2010) and on the footsteps of the two Australian researchers Ebaeguin and Stephens (2014); the

[27] https://geerthofstede.com/; https://hi.hofstede-insights.com/national-culture; https://www.hofstede-insights.com/country/italy/; https://www.hofstede-insights.com/country/china/; https://www.hofstede-insights.com/country/japan/

second one, in the light of Zan and Di Martino's research (Zan, 2007; 2016; Di Martino, 2017; Di Martino & Zan, 2019; 2020)[28], considers how the Lesson Study can influence the formulation by teachers of the mathematics problem and the possible ways of solving it by students; the third one is a research on the collaborative analysis of the semiotic potential of an artifact and of the teacher's role, according to the Theory of Semiotic Mediation (Bartolini Bussi & Mariotti, 2008; Bartolini Bussi, Corni, Mariani, & Falcade, 2012), for the development of argumentative competence in Mathematics; the fourth is a survey on Italian teaching design in Mathematics through the Lesson Plan as a tool for inclusion – core element for the Italian school (Antonietti & Veneziani, 2018; Baccaglini-Frank, A., & Di Martino, 2019; 2021; Goei, Norwich, & Dudley, 2021)

Table 1. 5 Lesson Study cycles in Valle d'Aosta.

Lesson Study cycle nick-name	Students' School	Students' grade	pivotal Topic of the lesson	Teachers' Schools	No. of participating teachers			Period
					Primary school		Middle school	
					In-service	Prospective		
1°-5° group	Quartiere Cogne	5	"Three photos on a page": problem solving and argumentation	Quartiere Cogne	3[29]	2[30]	-	November 5 - November 21, 2019
				Ettore Ramires	1[31]			
3°-4° group	Quartiere Cogne	4	Argumentation: mathematizing reality.	Quartiere Cogne	2[32]	2[33]	-	November 5 - December 11, 2019
				Ettore Ramires	1[34]			
Secondary Lexert	middle school "Émile Lexert"	8	multi-solution problems and argumentation	Quartiere Cogne	-	-	5	December 13, 2019 - interrupted due to COVID 19
Mont Rose A	Pont-Saint-Martin	7	Problem solving and problem posing	Pont-Saint-Martin	-	-	3	October 30, 2019 (postponed due to COVID 19) - May 20, 2021
	Donnas	5		Vert	1		-	
				Donnas	1			

[28] https://www.giuntiscuola.it/progetto-problemi-al-centro
[29] Clea, Ivana e **Barbara.**
[30] Simona e Luisa.
[31] Loredana.
[32] Donatella e **Anna**.
[33] Noemi e Chiara.
[34] Edith.

With the intervention of Professor Ferdinando Arzarello, on May 15, 2019, a 2-hour informative seminar open to all teachers in Valle d'Aosta is offered. The two schools (Istituzione Scolastica "Émile Lexert" and Istituzione Scolastica "Mont Rose A") then accepted the proposal for the PDC in mathematics, accepting the proposal for primary and middle schools only (no high school join). The resulting cycles are described in Table 1. 5.
In September 2019, the participating teachers attended a four-hour meeting with the PhD student Manolino, who explained to them the main features and the way to approach the Lesson Study methodology.
The institutional reasons that prompted the administrative offices to propose the PDC stemmed from the fact that a marked decline in performance on Italian National Institute for the Evaluation of Education System (INVALSI[35]) tests in Mathematics, and primarily in argumentation and problem solving, is being recorded.

On the indication and support of the school manager, the "Émile Lexert" teachers decided among the different regional formative proposals to take part in this PDC, since in mathematics. In fact, the school leadership was very concerned about the lack of students' mathematical-argumentative competence, also found by the systematic observation carried out during the teaching activities by the class teachers, so much so that teachers agreed to join the regional project, and the "Émile Lexert" is proposed as the pilot school[36].
We present below some statements about this, reported by the teachers during the planning meetings of the groups:

> **Barbara**: ...we realised, through classroom observation and the results of the Invalsi tests, that children have difficulty in argumentative competence and experience a steep drop in the use of the Italian language during argumentation. In the specific area of mathematics, we have started to activate activities to provide children with models of argumentation. We help them to argue, or perhaps some children who are more fluent and competent orally and who use temporal connectives effectively help the others. So from that point of view we are building models.
>
> Lesson Study 1°-5° group Planning meeting
> 5th November 2019

[35] Italian large-scale mathematics surveys (referred to as "INVALSI tests" since they are produced, implemented, elaborated, and analysed by the INVALSI staff). All INVALSI data can be requested on-line, at www.invalsi.it (see Cascella, Giberti, & Bolondi, 2020).

[36] That is, the school that serves as a reference in the agreement with the University and maintains institutional relationships, while giving faculty from other schools the opportunity to participate. As in the case of "Mont Rose A".

Anna: [...] I'll tell you two or three little things about maths, because we are always quite low in the Invalsi tests, and in general in everything related to Italian and maths. But especially mathematics. So this project was chosen precisely to improve the offerings.
Donatella: We have a particular audience. More or less 50-60% of the students in each class are of foreign origin, with a migrant background, so the main difficulties are those related to language, and in mathematics there is language, so it's a big problem. We can do a lot of work, but if we don't get this aspect of language right, it's clear that we're going to struggle.
Simona: So the problem of argumentation is also linked to a linguistic problem.
Donatella: They have problems in understanding. We gave them a problem: "a building has 4 facades, if in each facade there are 6 windows, how many windows are there in all?". Out of 21 pupils none knew the meaning of the word "facade".
Anna: Or "each".
Donatella: No no, the problem was just the word "facade". Anyway, we worked on it, we even made a model with paper.
Anna: Because everything is very relative. For example, we have a Chinese child, who really has a hard time speaking Italian, but is very good at maths. So it's true that the obstacle of the language is enormous, but when he has understood it, he can solve it in two minutes. But when he sees a problem and has to explain how he did it, it takes him a quarter of an hour and the others agonise because he tries to find all the words and can't even find the right ones, so it's a language problem. You have to go beyond the pure aspect of arguing, you also have to understand what they have understood, what they have in their heads. So, he actually sometimes does calculations in another way, because his mother explains them to him...

Lesson Study 3°-4° group Planning meeting
5th November 2019

Based on this, the PTOF of the "Émile Lexert" school is written, which mentions:

> [We confirm our membership as pilot school for the PDC titled:] "Il Lesson Study in Matematica: una metodologia di lavoro collaborativa per lo sviluppo professionale dei docenti. Attraverso una Riflessione Critica costruttiva". About twenty Primary and Secondary Mathematics teachers from É. Lexert schools will be the recipients of this [Lesson Study] methodology and it will last eleven hours in total for each working group - and therefore for each participant, at no cost. [...]
>
> The teachers education offer wants to be a response to a district context that does not offer cultural proposals also because of the gradual closure of public education centres.

The primary school teachers decided to work in two groups, which were respectively supported by two prospective teachers per group. The two groups were composed by teachers from classes of grade 3 and 4, and from grade 1 and 5 respectively.

“il Lesson Study in Matematica” project in Piemonte

During the 2020 spring, in-service mathematics teachers of all levels in Piemonte, struggling with the difficulties of distance learning and health emergencies, received a new challenge for their professional development from the regional school office (URS) of Piemonte. The proposal is a professional development course for teachers based on the Lesson Study methodology, as referred in the note of the USR, prot. n. 2683 of March 26th, 2020.

In the first meeting I (Carola) illustrate to the teachers the Lesson Study methodology embedded in a cultural transposition prospective. In the second meeting, which took place on 17 September 2020, the teachers met two teachers who already had Lesson Study experience. They then started to work in groups, at the end of which they were asked to report back to the large group on the experience.

Teachers are asked to get involved in the first person as part of a group of colleagues to carry out a Lesson Study cycle together and, at their discretion and not in a compulsory way, also a second cycle that can be thought of either as an iteration of the previous one, or as the planning of a different course, chosen by the group. The autonomy and freedom of choice that each group of teachers possesses is recognised from the outset.

The timeline for the first Lesson Study cycle of the project is shown in the diagram in Figure 1. 21, with a total workload of approximately 8 hours, to be completed by mid-January 2021, in time for the work report to the whole group.

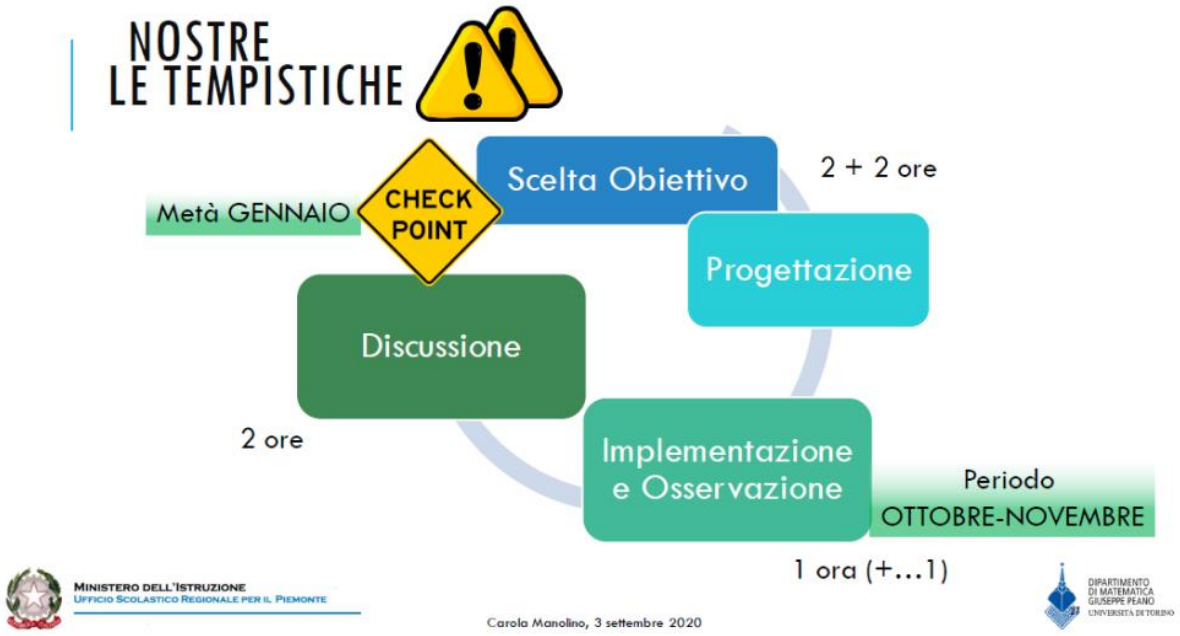

Figure 1. 21 Timeline of the first Lesson Study cycle of the project in Piemonte [September 3rd, 2020, PowerPoint slides].

The number of hours indicated *a priori* in Figure 1. 21 is not fixed. It is variable depending on the working groups, as is made clear at the very beginning of the course:

Carola: In our experience, we plan 2 hours for the planning phase with a possible addition of 2 hours, i.e. with a previous meeting to choose the purpose of the research lesson. Obviously, these 4 hours can be reduced to 3 or 2, if the group is well matched and if can immediately get to the heart of the matter, or rather can be extended.

Introductory meeting
3rd September 2020

The convenience of groups already cohesive and accustomed to collaboration was not, however, always possible: in-service teachers who took part in the project were 47, coming from the provinces of Torino, Alessandria, Asti and Cuneo, from different 17 institutes, and were divided into 12 working groups due to geographical proximity and school grade. 8 groups from Primary School and 4 from Secondary School (1 in Middle and 3 in Secondary School). 10 prospective teachers collaborated, 4 from the Master's Degree Course in Mathematics and 6 from the Primary Education Course of the University of Turin. All except one of them participated with the role of facilitator within a group of teachers.

At the end of the first cycle, each working group was asked to write a report on the first part of the project, answering questions about their experience in the phases of the cycle. In addition, each participant was provided with an individual questionnaire (see the questions on p. 85) in which they could give their thoughts.
From the group reports, among other considerations, it emerges that the amount of time necessary for the first two phases of the cycle (choice of objective and planning) was in most cases greater than that budgeted. Planning a lesson in the accuracy required by the Lesson Study is time-consuming, even though the lesson is supposed to last 60 minutes.
Out of the 9 groups that provided indications regarding the duration of the planning meetings,

- 3 groups (2 from Primary and 1 from Secondary) spent between 5 and 6 hours in the planning phase;
- 4 groups (3 from Primary and 1 from Secondary) spent between 7 and 8 hours in the planning phase;
- group from the Primary School, which had already had a Lesson Study experience in the past, had four planning meetings with a total duration of 9 hours and described this phase as "very intense".

This was despite the fact that the request was to plan only one hour of lessons:

Carola: The implementation will be of one hour, the time-module that the teacher has available in that class. We recommend that it should be one hour and not two, even though there may be two time-modules available, one after the other. Why? Because when you're planning in such detail, it doesn't make too much sense to ramble, it's better to keep concentrating on something short, concise and then actually think about it together.

Introductory meeting
3rd September 2020

In the second Lesson Study cycle, carried out in January-March 2021, out of 12 initial groups 7 participated.
At the end-of-cycle meetings where the groups were asked to report on their work, the Professor Bartolini Bussi (January 16th, 2021 – end of first cycle) and the researchers Ramploud and Funghi (March 27th, 2021– end of second cycle) were invited as experts.

The network of teachers that has been created and the collaboration with the USR and the Superintendency has led to other Lesson Study projects starting in the 2021-2022 school year in both Piedmont and Valle d'Aosta.

Of the four Lesson Study project experiences, all data (groups meeting and personal interviews) were audio recorded and transcribed. Part of this data is used for my research work.

DATA ANALYSIS

Using the above cultural-semiotic analysis tools (i.e., dynamic spatial modelling of Lotman's Semiosphere and the cultural conflict) it is possible to depict the relationships constituting the webs of signification existing in practice of mathematics teachers' education, that is, depict possible enabling of transformative learning (transforming habits of mind). Indeed, the purpose is, in the light of the construct of Cultural Transposition, to see teachers as active and critical participants in Mathematics Education.

Every moment of a practice could be analysed as part of the semiosphere in which it is immersed. The short episodes presented here are emblematic of many other episodes in the 4 described projects. I chose these four episodes mainly because: (a) representative of each project carried out with in-service teachers - there is at least one episode of each project; (b) representative of the geographical territory - from a rural background in the province of Cuneo (Revello) to the urban context of Mondovì and Piossasco, up to the frontier territory in Valle d'Aosta; (c) representative of different levels of co-determination but, as we shall see, all deeply permeated by cultural data; and (d) representative of both of the movements: outward push or inward reception.

The episodes are presented in the chronological order in which they occurred over the years.

The first episode I am presenting is part of the Lesson Study experience in Piossasco. Marcello is recounting to the group his reflections on the epistemology of the sum concept, before starting a group study on the use of the sum operation during a double purchase – i.e. when two or more objects are bought together and the final price is total – as introduction to the + sign in a grade 1 class.

> **Marcello**: I wanted to ask... […] **I would anticipate the = sign over the + sign**. I mean ... First of all, I have already worked in class on the > and < signs, which in my opinion can be linked to it. ...and I would like to go from that first and then go further. Also because, perhaps, for what you said before, to work **on + sign as an identity and therefore not as a result**, I thought it could be more... I mean, to work on >, <, = instead of... So, I don't know, to work on 4 = 4, and then from that move to complementaries, so to say that (4 and 6) = (7 and 3), and then instead… That is, then **do it first with the couples**…
>
> **Ezio**: …disjoint from addition.
>
> **Marcello**: Oh yes. In the sense that, to work a bit on **the identity discourse**, it seemed more useful to me to work on complementarity. I mean, pairs of numbers: that **a pair of numbers is equal to another pair of numbers**, rather than a pair of numbers is equal to a single number. This is my idea! I don't know if... but it seems easier to me, if we want to ensure that the children learn the concept of identity before of the concept of result. It seemed easier to me to convey it like this. Well, **this is my idea, I don't even know if it makes sense**.
>
> **Ezio**: I think it's a good idea because...
>
> Group study meeting on Double Purchase
> Piossasco, 15th February 2019

As required by dynamic spatial modelling, I describe the episode as a text of the semiosphere whose centre is Marcello. Here I identify a moment of movement of inward reception. Marcello, because stimulated by the group meetings, begins to reflect on his own way of conceiving the concept of sum. The detonator is the pre-planning group study/research time.

The conflict occurs between Marcello's habit of introducing the sign = as operational, the expression of a result and closely linked to the operation of addition, and the conceptualisation of the sign = as the expression of an identity. But we need to understand the significance of this reflection. It is the expression of a moment of conflict in Marcello as he is embedded in the Italian context, where the traditionally mathematical operations are four, very distinct from each other, and aimed at counting. By this I do not mean that Marcello was completely unaware of the possibility of introducing the sign = as the expression of a (mathematical) relation, but the interaction with the group leads Marcello to make explicit his epistemological view. and this is also a cultural fact: Italian teachers are not used to discussing their visions with other colleagues, there is no suitable space-time in school. There is no suitable place for professional discussion, especially on content – the mathematical content comes from outside (e.g., from textbooks or experts giving lectures), usually in primary school it is not part of a teacher's conscious cultural background –, and the scheduled meeting time is used to reason about methodologies and pedagogical or bureaucratic problems.
The asymmetry is therefore manifest. It can be labelled not only as epistemological and content-related, but also as ecological because of the reflections made on the spaces of discussion between teachers.

The second episode is part of the Lesson Study experience of the 3°-4° group of Aosta. Anna, the teacher who implemented the lesson in class, and her colleague Edith are expressing their impressions and reflections resulting from the classroom implementation of the planned lesson. In particular, they dialogue addressing to the two prospective teachers of the group.

> **Anna**: [...] But at that moment I realised that **I didn't like myself**, that I didn't do the things I would have liked to do, because I had to stick to a certain cliché, even in terms of things to say and things to do. And so, [the Lesson Plan] is great because it gives you a direction, you don't waste time, and **if you are going to ramble you don't do it**, and you stay on track...
> [...] So on the one hand I saw the Lesson Plan as a help, definitely. On the other hand, I felt it was a bit of an obstacle because it was **too bridling**.
> **Edith**: Well, because then you have to say that... you have to do that...
> **Anna**: **And if I don't, what happens? Nothing!** Because, for example the planned questions, you saw that **I skipped right over them**…
> […] The task explicitly planned in detail, the plot of the lesson, forces us to **an imagery of the act**, which requires self-knowledge as a teacher. What do I do when I give a task, how do I move, how do I speak? So, it questions us as teachers in the way of **acting professionally**... But also, the account of the classroom: how do the children react when I talk to them, what do they do? [...] the fact of planning in detail the task and the entire lesson means getting used to **being unable to improvise**.
>
> Lesson Study 3°-4° group
> post-implementation Discussion meeting
> 11th December 2019, 2.40 p.m.

The dynamic spatial modelling sees the episode as a text of the semiosphere whose centre is Anna, and the external is the school context in which Anna is embedded. Here I identify a

moment of movement of outward push. Anna did not like herself in carrying out what she, together with the group, has planned and she reacts by not intentionally fulfilling part of the lesson. The detonator is the implemented research lesson. The conflict occurs between the demand to plan a lesson in its details, the Italian teachers' habit of planning only starting and ending points and in between "react on the spot" following the argumentation and cognitive processes of the children, and the Anna's need to free herself from this duality: "And if I don't, what happens? Nothing!". The significance of this discourse lies in what Emma Castelnuovo called *the art of teaching*, that is, the "letting oneself be inspired each time by the needs of the class [...] the [teacher's] sensitivity to grasp…" (Castelnuovo, 1965). In the case of the 3°-4° group, it is not so much a matter of confusing "the art of the teacher" with arguing for the "futility of foresight". Anna does not theorize the "art of the teacher" as a postulate of total freedom from schemes and restrictions, and does not seek recipes. Anna sees the work of planning as a labor of trial and error. The implemented lesson and discussion with prospective teachers allowed her to speak about it.
The asymmetry is therefore manifest. It can be labelled as didactical as it is a professional matter for teachers.

The third episode is part of the Lesson Study experience of the Mondovì-Fossano group in the Piemonte project. Claudia, Margherita, and Donatella are teachers from different schools and do not usually work together. Margherita met Claudia and Donatella for the first time in this project. The three teachers are interacting online, because during the school closures due to the pandemic situation. In this their first meeting they reflect on the content of the lesson they would like to plan. Claudia, Margherita, and Donatella attended the Introductory meeting, in which the time of the lesson to plan was discussed (see related excerpt on p. 95).

Margherita: eh! So listen...
Claudia: ...we don't have anything else in mind!
Margherita: That's because **there's also a problem of timing**. Because, for us, now that we're planning, it's very difficult to predict where exactly we'll be when we do...
Claudia: uh uh!
Margherita: so I think we need to think of something a bit untied, which we can then pick up later, right?! But it has to be something a bit unrelated. So we don't have to struggle with. We mustn't hurry to get to that right then and there, because otherwise...
Claudia: all right!
Donatella: because that's a big theme... Do you understand? I think... that's what Margherita means. This is a big educational path: it's what I was telling you before!
Any activity we choose is within a path. Or we choose, I don't know... let's take Sudoku - I give an example - which is something you do on the spot... Even if that is again linked, again, to the path... to a lattice.. to a... I mean... No? Anything can be a big discourse. But what can be a very short 40-minute task...
Claudia: Eh!
Margherita: Eh! But, in my opinion...
Donatella: I don't know... Problem solving on a situation text comes to mind!

Margherita: Eh, but they just talked about a mathematical concept. For example, they, who worked on the concept of addition, on the symbol of addition ... So, in my opinion, we don't have to think about an activity, a content. We have to think of an ob... a mathematical concept...
Claudia: I am thinking of **a didactic unit** linked to...
Margherita: eh! ... **that you can practically consider finished in 40 minutes**. Then of course we will come back to it again in class. But considered to be taught in 40 minutes. That's my guess! And I personally, for grade 5, was thinking about something on the **circle**. Not because... Because it's **a bit of self-contained thing**. So **you can do it a little earlier, a little later**... I mean, of course it's related to everything else... But I mean... The work, for example, on pi, or... I mean, it seems to me... or...
Donatella: the circle is no longer there... The circle no longer exists in the National Guidelines!
Margherita: uh ... yes, but ... On a level ... the circle and its features! I mean, in my opinion, yes ... it's important to give it some attention! That is, for example, **"the circle and its features" seems to me something that can fit in 40 minutes**. By thinking up something nice and for which they might also be experiencing... um...
Donatella: Well, then the circumference rectification!
Margherita: Yes, for example! Sure!
Donatella: So I propose robots!
Margherita: Oh yes, but I have never worked with robots. Instead I know that ...
Claudia: 40 minutes, Dona! It's true, we ...
Donatella: The circumference rectification can be done even...
Claudia: Can we complete it in 40 minutes?
Donatella: Oh yes, because you let them test it with... Think about the little robot...

Mondovì-Fossano group
First planning meeting
22nd September 2020, 3.45 p.m

As required by dynamic spatial modelling, I describe the episode as a text of the semiosphere whose centre are Claudia, Margherita, and Donatella. Here I identify a moment of movement of inward reception.

The teachers describe the circle as:

- "a bit of self-contained thing";
- Something that "you can do it a little earlier, a little later", that is, "related to everything else";
- something that "no longer exists in the National Guidelines";
- "important" enough to require "to give it some attention";
- "something that can fit in 40 minutes".

The conflict occurs between what in the Italian context is considered the time frame necessary to address a mathematical concept with the class, and the requirement to plan one-hour lesson. Or rather, the conflict occurs within the very idea of planning a mathematics teaching moment. The detonator is the planning phase, in particular the moment of choosing the research objective and the teaching content.

According to Claudia, Margherita and Donatella, the question of time concerns two different aspects: (1) of scheduling, i.e. when to include the experience in the long-term educational

planning of the class - due to the pandemic, they might not be able to foresee well in advance a moment in which to plan the implementation of the lesson – and this also concerns the relationship that is considered to exist between the different mathematical contents, i.e there is content that can be worked out in class "a bit detached from the others"; (2) of time-content correlation – there is content that can be worked out in class in a short time, others that just can not, or content that at least requires "distended" time.

To better understand what we mean, here is presented an excerpt of a Lesson Study experience by Ezio, an expert teacher, during the introductory meeting of the new Lesson Study course that has just started in September 2021. This excerpt explains the reasons for the habit of Italian mathematics teachers of giving their class a relaxed time. In fact, teachers are aware of the low importance of mathematics in the cultural landscape of my country and of the urgency of recovering negative attitudes towards mathematics.

> **Ezio**: So. There is another aspect that we have been thinking about. We are told that **in order to recover negative attitudes**, in particular towards mathematics, "it is extremely important to have a methodology that enhances the role of time in mathematical activity, helping to **unhinge the idea that success in mathematics consists in giving quickly the correct answer**". This sentence, taken from a book of Zan and Baccaglini-Frank (2017), very clearly reflects the attention that the teacher must pay to the time given to students. So, it is clear that in recent years there has been a growth in teachers' awareness of the need for distended time. On the other hand, we have seen the Lesson Study as a development of intentionality in the educational act. That is to say, the fact that we realise that we need to have reasons for planning that kind of educational action. So, these two aspects, which are both important, can also be combined in the Lesson Study experience, where **distended times are not always extended times**, they are **adequate times to be able to achieve a well-defined objective**. **Defined by the intentionality** that the teacher has put into that type of activity.
>
> Lesson Study 2021-2021
> Introductory meeting
> 7th October 2021, 5.45 p.m.

The cultural conflict over time and planning led teachers to make explicit their own unthoughts related to the mathematical content of "the circle and its features". The asymmetry is therefore manifest. It can be labelled not only as didactic-methodological, but also as epistemological and content-related.

The fourth and last episode is part of the Lesson Study experience of the Revello group in the Piemonte project. The educational institution of Revello is a *Scuola Senza Zaino school* (literally, school without book bag)[37], i.e. part of a network of schools offering a different teaching method with respect to the traditional method used in the Italian schools: for example, there are no longer individual desks but "large islands" of 5 students to work together, or even, the teaching content is not always dealt with by the whole class at the same time, but through a rotation system there is

[37] https://www.senzazaino.it/

the possibility of offering differentiation of teaching, i.e. proposing and using different activities and approaches at the same time, etc.

> **Anna Maria**: the fact of **not having used that lesson ~~study~~ [plan] model** you gave us **has yet another reason in our case**. In parallel with this, we are doing another course, on evaluation, with our school manager... in fact the model, the outline, was created by her... Together we thought of creating something, an outline, that would allow us to document meaningful activities. Such as a real-life task (Tessaro, 2014) or whatever, and on our online platform we made a repository space, divided by topics, where you can find the proposals. Instead of being called lesson plans, they're called **IPUs, [Istruzioni Per l'Uso] instructions for use** of... for example, a mathematics activity in grade 1, a cooperative activity in grade 2... We put everything there **because it's not always easy to talk to each other**, but having a common repository where you can go and **take someone else's material and modify it to make it useful to you** seemed really significant. This virtual part of the repository is under construction, and **with this [the Lesson Study project] experience we put also our contribution**. In addition to the virtual, **there is also a small practical deposit**, because there are also tools, objects and manufactured objects... for example, the decimal wheel is a piece of wood with a plastic-coated wheel, and together with it there are instructions on how to use it. Both instructions for teachers and for children to use it. These **physical tools** are in a cupboard. So it's there, if somebody needs it, they take the tool, **the IPU sheet**, the children, **and it's done: already fill a rotation.**

The dynamic spatial modelling sees the episode as a text of the semiosphere whose centre is Revello school's practices. Here I identify a moment of movement of outward push. The Revello group chose not to use the Lesson Plan model proposed in the introductory meeting. The teachers have already worked hard to develop what is a foundational tool for their school: the IPU sheets (the instructions for use are a habit of the school without book bag). They know its didactic intentionality, the potential and the constraints. This allows teachers, within the Lesson Study, to use a tool external to the methodology in a consistent way. The detonator is the Lesson Plan. The cultural conflict occurs between the production of a document that simply describes how to carry out a teaching activity, and a document aimed at keeping fixed the points of didactic intentionality that guided the planning choices: "it's not always easy to talk to each other, but having a common repository where you can go and take someone else's material and modify it to make it useful to you seemed really significant".
The asymmetry is manifest. It can be labelled as didactic-methodological.

The asymmetrical contexts identified here are the locus of cultural conflict. To have made explicit the dimensions and forms of these asymmetries is, for researchers in teacher education, the starting point for designing and implementing culturally sensitive professional development practices.

CONCLUSIONS

The main points of the chapters of the dissertation are briefly recalled here to show how they constitute a reasoned answer to the research questions, as formulated in the last elaboration at p.83.

The international research community in Mathematics Education has recognised didactics in general and teachers' professional development in particular as a "natural process" and, as such, "dealing with culture" (Eraut, 1977; Gallimore, 1996). The chosen frame of reference to define and to study the professional development of mathematics teachers is made of the correlation between Mezirow's Transformative Learning and Kemmis' Critical Refection. Indeed, in this frame, teachers' professional development is defined as political research practice and a dynamic collaborative process in action, embedded in a cultural dimension.

One of the possible learning processes, the so-called *transforming habits of mind*, is conceptualized by Mezirow as the "learning through meaning transformation". That is, "the learner encounters a problem or anomaly that cannot be resolved through present meaning schemes or through learning new meaning schemes; the resolution comes through a redefinition of the problem. Transformation occurs by critical self-reflection of the assumptions that supported the meaning scheme or perspective in use" (Mezirow in Kitchenham, 2008, p. 112). Furthermore, according to Kemmis, reflection as an emancipatory, research process that "must to be studied and analysed in action", in particular through the "*spiral of self-reflection*", consisting of cycles of: planning action (on the basis of reflection); implementing plans in action (praxis); observing or monitoring processes, conditions and consequences of action; and evaluating actions in the light of the collected evidence (returning to reflection) as a basis for replanning and further action.

Since I place my research in a perspective that considers learning "a social, cultural, and historical activity", or even better, that defines learning as the development of a situated "critical discourse with others" (Mezirow in Kitchenham, 2008; Sfard, 2008), I propose Semiotics, and especially Semiotic of Culture as a comprehensive method of analysis of teachers' discourses. Indeed, texts are the object of study of semiotics. Over time, the concept of text has gradually been redefined, to the point of taking into consideration "any carrier of integral ('textual') meaning" (Uspenskij *et al*., 1998, p. 38). The discourses of mathematics teachers are the texts analysed in this my research work. The proposal of semiotics as a framework of analysis is not innovative, but well established in research in Mathematics Education. However, despite the definitions given so far of socio-cultural context (Vygotsky), of dialogue and sign community (Bakhtin), of territory of signs (Voloshinov), and of cultural semiotic system (Radford), the development of a semiotic-cultural analysis of the networks shaping the text, the dialogue, was still an obscure task. By answering my research questions, however, I think I have made a contribution in this respect to the international research community in Mathematics Education.

As anticipated at the beginning of this chapter, the answers to the final elaboration of my research questions (NRQ1, NRQ2, RQ4, RQ5: see p. 83) are contained in the discussion developed in the different chapters of the dissertation, as I will now recall.

For Research Question 4 (RQ4: What is meant by culture?) I proposed to the reader the metaphorical journey that starts at page 30 – *Definition(s?) of Culture*

: its issues constitute an articulate answer to this question. Then, to answer Research Question 5, (RQ5: How is culture characterised in Mathematics Education?), given by its *essentially semiotic* nature and its extreme charge of significance, I described Geertz's definition of culture (p. 34) as the most suitable choice for a frame of analysis of mathematics teachers' professional development discourses. Culture is, then, the "webs of significance" span by the individual and within which she/he her/himself is suspended. Indeed "we do not inhabit a mere concrete, material world, but a world full of meaning, and that meaning belongs to the order of signs" (Voloshinov, in Radford, 1998, p. 7). At the same time, however, signs do not dwell in a world made up only of ideas and abstraction either. Therefore, there is a (non-physical, but relational) meeting place, which is the sign, that synthesizes the biological organism and the external world. It goes beyond physiological aspects, beyond symbolic aspects, and so its products cannot be analysed as things, but understood and interpreted as signs. So Lotman's Semiosphere emerges as the theoretical lens suitable for my purpose.

However, it is necessary to consider it within research in Mathematics Education. Proposing to the Mathematics Education community the new construct of Cultural Conflict, I rethink Lotman's dynamic spatial modelling of Semiosphere, schematised in Figure 1. 8, as methodological analysis that allows a unified reading of the transformative learning processes of mathematics teachers. Thus, the answer to NRQ1 (Can the Semiosphere be the theoretical lens that allows a unified reading of the transformative learning processes of Italian mathematics teachers?) is yes: the Semiosphere can be the theoretical lens suitable for a unified reading of the transformative learning processes of Italian mathematics teachers. And the construct of Cultural Conflict is the first (theorical) outcome of my research.

Withal I needed a real context in which to test it. Cultural Conflict is not only roped into teacher professional development, but emerged as a response to the need to study such practices. Thus, the Lesson Study, and in particular its Chinese vision (Chen, 2017), as a methodology of teachers' professional development adhering to the construct of critical reflection (Kemmis, 1985) is proposed as a paradigmatic model for my research. I place my research in the Italian cultural context – this is also to propose myself as an active researcher-educator in my area, capable of answering the strong demand for mathematics teacher education. I proceeded in turn following a spiral of self-reflection: planning, implementing, and testing 4 cycles of proposals for mathematics teacher professional development courses. My aim was at offering proposal to potentially transforming mathematics teachers' habits of mind, that is fostering the explosion of a cultural conflict.

Using the lens of the Semiosphere, I was able to face the research question NRQ2 (How can the Semiosphere characterise the cultural transposition of Italian mathematics teachers' practices?) in its more structured form of the two main subquestions NRQ2a and NRQ2b:

NRQ2a. how can Lesson Study be studied as a sign-ificant practice for Italian mathematics teachers, in order to help them in becoming critical and active participants in their professionalism?

NRQ2b. what is the product of the cultural transposition of the Chinese Lesson Study into the Italian context?

This was possible through the analysis of the discourses of the 109 teachers I met during the 39 Lesson Study cycles carried out, of which I gave some emblematic samples in the chapter *Data Analysis* (p. 97 and ff.).

First of all, it was not a question of comparing the Chinese practice with the Italian one. But it has been a matter of establishing an interaction with a foreignness and then of reading the relations, the space of webs of significance existing in Italian professional development practice. So the answer to the first question is: Lesson Study can be studied as a sign-ificant practice for Italian mathematics teachers, in order to help them in becoming critical and active participants in their professionalism, through the analysis of the lens of the Semiosphere itself.

In fact, for example, we have seen how teachers, only because placed in an active practice of professional development capable of triggering the encounter of the self with a foreignness, have come to re-consider the space of professional discussion among colleagues and to question their own epistemological assumptions (in Marcello's episode) or didactical-methodological (in the Revello's group episode); how teachers have come to overcome the apparent duality between the "art of teaching" and the planning of teaching action – i.e., by making their didactic intentionalities explicit (in Anna's episode); or rather, how teachers found themselves expressing and justifying their ontological and epistemological visions of mathematical content and its relation to time (in the Mondovì-Fossano group episode). The manifestation of asymmetries is in fact a moment of revelation of the significance of practices. And, secondly, precisely these manifest asymmetries are the product of the cultural transposition of the Chinese Lesson Study into the Italian context.

Therefore, asymmetries can be useful both for teachers themselves, within their professional life, and for researchers, to re-think a culturally sensitive teacher education.

In conclusion, these results are both a response to a theoretical need for research in the field of Mathematics Education and a practical proposal for teacher education. Indeed, since semiosis is unlimited – as in the Peircean view, but here extended to the semiosphere and embedded in a mathematics education context – the process of interpreting the significance of practices can never end. The asymmetry pushes teachers to elaborate their own transposition processes differently from school to school and sometimes also within the same school but in successive different steps. What is generated is an unending process of semiosis. This should not demoralise us, but rather keep high our attention because everything can always be a source of new consciousness.

Of course, my research is just beginning.

There are many possible future developments that come to mind: for instance, our already begun study of the "unending process" of semiosis of the Lesson Study practice in a network of "hybridizations" (Arzarello, 2016) within the Semiosphere; or a theoretical study of the relationship between the concept of didactic suitability of the ontosemiotic approach and the

asymmetry.
Moreover, it could also be explored whether the Semiosphere could be a useful theoretical lens to describe disability and inclusion aspects. Inclusion is indeed a characteristic aspect of our curriculum, and the Italian inclusive school is a *unicum* in the European context, but fraught with problems. Defining what is meant by disability is one of the greatest cultural challenges of our time.

Yet, as I have already mentioned, the network of teachers that has been created in our area is considerable, and the collaboration with the USR and the Superintendency has led to the establishment of new Lesson Study projects that are starting in the 2021-2022 school year in both Piemonte and Valle d'Aosta. This gives me hope that the challenge of a culturally sensitive teacher education has only just begun, and yet with great determination to tread into the future

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