Bargaining over a Divisible Good in the Market for Lemons

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Bargaining over a Divisible Good in the Market for Lemons*

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Abstract

We study bargaining with divisibility and interdependent values. A buyer and a seller trade a divisible good. The seller is privately informed about its quality, which can be high or low. Gains from trade are positive and decreasing in quantity. The buyer makes offers over time. Divisibility introduces a new channel of competition between the buyer's present and future selves. The buyer's temptation to split the purchases of the high-quality good is detrimental to him. As bargaining frictions vanish and the good becomes arbitrarily divisible, the high-quality good is traded smoothly over time and the buyer's payoff shrinks to zero.

KEYWORDS: bargaining, gradual sale, Coase conjecture, divisible objects, interdependent valuations, market for lemons.

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1. Introduction

In many economic environments, agents bargain over goods that are divisible. Negotiations in financial markets typically involve both the amount of an asset and its price. Banks and institutional investors (e.g. pension funds) routinely bargain over how much of a securitized asset (pool of mortgages, credit-card debts, automotive loans) to trade and at what price. Similarly, after restructuring a company, an equity firm negotiates what fraction to sell and at what price. These negotiations are generally dynamic and decentralized. One party, typically the seller, is better informed about the quality of the asset.[1]

We study bargaining over a divisible good with asymmetric information, interdependent values and positive gains from trade. Gains from trade of a divisible good may depend not only on the quality of the asset, but also on how much of it has already been traded. We focus on the case of decreasing gains from trade, which leads to new insights into bargaining. Consider a bank negotiating the sale of a pool of mortgages to a pension fund. The quality of the asset can be either low or high, depending on its future cash flows from homeowners. As the pension fund is more interested in owning these promises of future cash flows, there are gains from trade. These gains are decreasing in the amount of the asset already traded between the parties, as they reflect the pension fund's desire to diversify its portfolio. The bank is directly involved in the process of securitization and hence has better information about the quality of these assets.

The main message of this paper is that divisibility introduces a new channel of competition between the buyer's present and future selves, and that this new channel has stark implications for the pattern of trade and for parties' payoffs. When assets are arbitrarily divisible and bargaining frictions vanish, high-quality assets are traded gradually. Divisibility is detrimental to the buyer; the competition between his present and future selves drives his payoffs to zero. This is in contrast to the outcome when the asset is indivisible. In that case, only the low-quality asset is traded in the beginning of the relationship.

[1] Consider the classic example of the synthetic CDO Hudson Mezzanine. As explained in McLean and Nocera [2011], Goldman Sachs selected all the securities in that CDO, strived to sell it as fast as possible and simultaneously bet against that security by taking a short position. See also Ashcraft and Schuermann [2008], Downing, Jaffee, and Wallace [2008] and Gorton and Metrick [2013]

A market freeze then follows, and only afterwards the high-quality asset is traded. The buyer of an indivisible asset obtains a positive payoff.

We extend the canonical bargaining model with incomplete information (Fudenberg, Levine, and Tirole [1985], and Gul, Sonnenschein, and Wilson [1986]) to account for interdependencies in values (Deneckere and Liang [2006] — DL henceforth) and divisibility. A buyer purchases a durable good from a seller who is privately informed about its quality. A high-type seller provides a high-quality good, while a low-type seller provides a low-quality good. The good is divided into finitely many units. There are positive gains from trade, which are decreasing in the number of units already traded by the parties. In every period, the buyer makes a take-it-or-leave-it offer that specifies a price and a number of units to be traded. The bargaining process continues until the parties have traded all available units. We assume that all learning is strategic. The buyer learns about the good's quality only through the seller's behavior; owning a fraction of the good does not provide the buyer with additional information about its quality.[2]

In equilibrium, the buyer employs only two types of offers: *screening* and *universal*. Screening offers are for all remaining units at a price lower than the high-type seller's cost. Universal offers are for some (or all) of the remaining units, at a price equal to the high-type seller's cost. The buyer alternates between screening the seller and purchasing some units through universal offers. The low-type seller randomizes between accepting and rejecting screening offers, while the high-type seller always rejects them. The rejection of screening offers makes the buyer more optimistic that the good is of high quality. Eventually, he is optimistic enough to purchase some (or all) of the remaining units through a universal offer. Both seller types accept this offer. After the purchase, the units that remain (if any) are less valuable, so the buyer returns to screening the seller.

Our main result characterizes the limit equilibrium outcome when bargaining frictions vanish and the good becomes arbitrarily divisible. We first let the length of each period converge to zero and we then let the number of units grow to infinity. In the limit, the buyer continuously makes both screening offers and universal offers for infinitesimal fractions of the good. At each point in time, he breaks even with either type of offer, so

[2] We elaborate on the assumption of strategic learning on page 10.

he obtains a payoff equal to zero. The high-type seller only accepts universal offers and thus sells the good smoothly over time. The low-type seller is indifferent between the two offers (screening and universal). He sells the good smoothly (pooling with the high-type seller) until a certain random time, and then concedes by selling the remaining fraction of the good at once.

Our work provides novel and testable predictions for markets for lemons. Our model highlights a rationale for markets of divisible goods (like markets for securities) to be more efficient than markets for indivisible goods (like real estate markets). We also show that these markets differ sharply on how parties split the gains from trade. Divisibility is detrimental to the buyer (uninformed party with bargaining power) and beneficial to the seller of lemons.

In order to understand the driving forces behind our main result, we first describe the pattern of trade when parties bargain over an indivisible good, as in DL. When bargaining frictions vanish, if the buyer can obtain a positive payoff, the usual Coasean forces imply that trade occurs without delay. In one of their main contributions, DL show that if the buyer must screen the seller, he does it through an impasse. During an impasse the market freezes: trade occurs with probability zero. After the impasse, the buyer is optimistic enough to pay the cost of the high-quality good. The impasse introduces delay, which is necessary to lower the price of screening offers before the impasse. In their path-breaking *double delay* result, DL show that the delay is twice the time necessary to make the low-type seller indifferent between the price after the impasse (which is the low-type seller's continuation payoff then) and the buyer's valuation of the low-quality good. This result has two important implications. First, before the impasse, the price of screening offers is strictly lower than the buyer's valuation of the low-quality good, so the buyer obtains a strictly positive payoff. Second, the larger the price after the impasse, the lower the price of screening offers before the impasse.

The driving force behind the gradual sale of the high-quality good when the good is divisible is that the buyer benefits from splitting his purchases. To see this, consider a simple example with ten remaining units. Suppose that the buyer is optimistic enough so that by making a universal offer, he obtains a positive payoff from the first five units (which are more valuable), a negative payoff from the last five units (which are less valuable), and overall, obtains a positive payoff from purchasing all ten units. If the buyer could only make offers for ten units, then he would purchase all of them through a universal offer. When the good is divisible, the buyer can instead purchase the more valuable units through a universal offer and by doing so essentially commit to pay a low price for the less valuable ones. Intuitively, when only the less valuable units remain, the buyer obtains a negative payoff from a universal offer, and so he must screen the seller. As in DL, screening occurs through impasses when the good is divisible. We extend their double delay result and show that the buyer obtains a strictly positive payoff before impasses. The buyer thus prefers to split the purchases of the high-quality good, instead of purchasing all remaining units through one transaction.

The temptation to split the purchases of the high-quality good generates a new channel of competition between the buyer's present and future selves. This new channel of competition is the driving force behind the buyer's zero payoff from trading an arbitrarily divisible good. To see this, consider again the simple example from the previous paragraph. Suppose now that the buyer is so pessimistic that he suffers a loss from a universal offer even for the most valuable of the ten remaining units. He must then screen the seller through an impasse. After this impasse, the buyer splits the purchases of the high-quality good, and so the low-type seller's payoff is lower than the one he would obtain if the buyer could only make offers for ten units. As the low-type seller's payoff after the impasse is lower, then the delay is shorter, which means that the price of the screening offers for ten units before the impasse must be larger. To sum up, since the buyer splits the purchases of the high-quality good *after* the impasse is resolved, then he must pay a higher price for screening offers *before* the impasse.

We show that the competition between the buyer's present and future selves is fierce when the good becomes arbitrarily divisible. Formally, as the good becomes arbitrarily divisible, the number of impasses goes to infinity but each of them becomes short: the price of screening offers before and after each impasse are close to each other, and thus screening does not take long. Between two consecutive impasses, the buyer purchases a vanishing fraction of the good through a universal offer. The driving forces described in the previous paragraphs lead to stark results: the high-quality good is traded smoothly over time and the buyer's payoff is zero.

Our analysis highlights the importance of the shape of gains from trade. If gains from trade are constant in the number of units already traded, the buyer cannot benefit from splitting the purchases of the high-quality good. Intuitively, the buyer cannot commit to pay a lower price for the last units by purchasing the first ones through a universal offer. All units are equally valuable, so if the buyer is willing to pay the cost of the high-quality good for the first ones, he must also be willing to pay that price for the last ones. Thus, with constant gains from trade all units are traded at the same time.[3]

1.1 Related literature

There is a large literature that studies bilateral bargaining with interdependent values (Samuelson [1984], Evans [1989], Vincent [1989], DL, Fuchs and Skrzypacz [2013], Gerardi, Hörner, and Maestri [2014], Hwang [2018a], Hwang [2018b] and Daley and Green [2020]). DL solves the one-unit version of the model in our paper. We take DL's construction as a stepping stone and extend the analysis to multiple units when there are two types of sellers.[4] Our paper uncovers a new role for divisibility in bargaining. The buyer gradually learns the seller's type as he makes two kinds of offers. On the one hand, he gradually makes universally accepted offers for small pieces of the good at large per-unit prices. On the other, he makes offers for all remaining units at large discounts.

Most of the literature focuses on bargaining between long-run players. Gerardi et al. [2014] study the role of commitment in negotiation environments under adverse selection.[5] Hwang [2018b] shows that bargaining deadlock arises when the seller receives random outside options over time. Hwang [2018a] focuses on a buyer who randomly becomes informed about the seller's type, while Daley and Green [2020] present a model with correlated but imperfect news that arise over time. Our findings of a continuous

[3]The pattern of trade with increasing gains from trade is equal to that with constant gains from trade.

[4]We restrict attention to the case of interdependent values because if instead values are private, divisibility plays no role; the Coase conjecture holds.

[5]In our paper, as in most of the literature, the uninformed party has all the bargaining power. This modelling choice reduces the informed party's ability to signal his type and yields strong predictions. See Gerardi et al. [2014] for the case of an informed party with all the bargaining power.

pattern of trade and of a zero payoff for the uninformed party are reminiscent of Fuchs and Skrzypacz [2013] and Ortner [2017]. Fuchs and Skrzypacz [2013] bridge the gap from DL by letting the gains from trade from the weakest type shrink to zero. Ortner [2017] studies a durable goods monopolist whose cost of production evolves stochastically over time.

Our paper is also related to the burgeoning body of literature that studies the effects of adverse selection in dynamic markets. Following the pioneering work of Inderst [2005], an important stream of this literature focuses on markets where a long-run player faces a sequence of short-run players. Hörner and Vieille [2009] analyze the role of private of offers in the market for lemons. Philippon and Skreta [2012] focus on optimal government interventions in such markets. Daley and Green [2012] study how noisy information about the value of a good is revealed to the market. Kim [2017] analyzes the role of time-on-the-market information in the market for lemons. Fuchs and Skrzypacz [2019] characterize optimal market design policies. Beyond the issue of divisibility, our paper differs from the above studies by analyzing the strategic effects that arise when two long-run players bargain under adverse selection.

The rest of the paper is organized as follows. We describe the model and the equilibrium concept in Section 2. In Section 3 we present our main result and discuss the economic implications of divisibility. In Section 4 we discuss equilibrium existence and present the intermediate results leading to our main result. In Section 5 we present extensions to our framework. Section 6 concludes. Proofs are relegated to the appendix.

2. The model

A buyer and a seller bargain over a good of size one. The seller is of one of two types $i \in \{L, H\}$. A seller of high type ($i = H$) provides a high-quality good, while a low-type seller ($i = L$) provides a low-quality good. The seller knows his own type, but the buyer does not. The seller is of high type with prior probability $\hat{\beta}$ that satisfies $0 < \hat{\beta} < 1$.

The buyer and the seller can trade fractions of the good. Let $z \in [0, 1]$ denote an infinitesimal unit of the good. We index units in *reverse order*. The buyer's first purchase consists of units $z \in [\overline{z}, 1]$, for some $0 \leq \overline{z} \leq 1$. A buyer who has already acquired units $z \in [\overline{z}, 1]$ can then buy subsequent units $z \in [\underline{z}, \overline{z}]$ from the seller, with $0 \leq \underline{z} \leq \overline{z}$.[6]

2.1 Parties' valuations

The buyer's valuation for the units $z \in [\underline{z}, \overline{z}]$ when the seller is of type i is equal to $\int_{\underline{z}}^{\overline{z}} \lambda(z) v_i \, dz$, where $\lambda(z)$ is a smooth function and $\lambda(z) > 0$ for all $z \in [0, 1]$. This valuation is higher if the seller is of high type: $0 < v_L < v_H$. The cost of the units $z \in [\underline{z}, \overline{z}]$ to the seller of type i is equal to $(\overline{z} - \underline{z})c_i$. The constant marginal cost of providing the good is higher for the high-type seller: $0 = c_L < c_H = c$.

We focus on the case with decreasing gains from trade. Since we index units in reverse order, this corresponds to a strictly increasing function $\lambda(z)$.[7] Without loss of generality we assume that $\min_{z \in [0,1]} \lambda(z) = \lambda(0) = 1$. We also assume that $0 < v_L < c < v_H$, so there are always gains from trade. Furthermore, we assume that

$$[\hat{\beta} v_H + (1 - \hat{\beta}) v_L] \lambda(1) < c. \tag{1}$$

The buyer's expected valuation from the first infinitesimal unit is lower than the high-type seller's cost. This assumption allows us to focus on the most interesting case: the buyer must screen the seller even to purchase the most valuable unit.[8]

We study the equilibrium behavior of the buyer and the seller as the good becomes arbitrarily divisible. We divide the good into m equally sized units and study the equilibrium behavior as m grows large. As with $z \in [0, 1]$, we also index units in reverse order, by $s \in \{1, \ldots, m\}$: $s = 1$ indicates the last unit, while $s = m$ indicates the first unit. The cost of each unit to the seller of type i is simply c_i/m. The buyer's valuation for the s'th

[6] We solve the game (with the good divided into finitely many units) using backward induction on the number of units *left* for trade, which is one of the state variables (see Section 4). The reverse order thus allows us to use the same index both for the state variable and for the unit number.

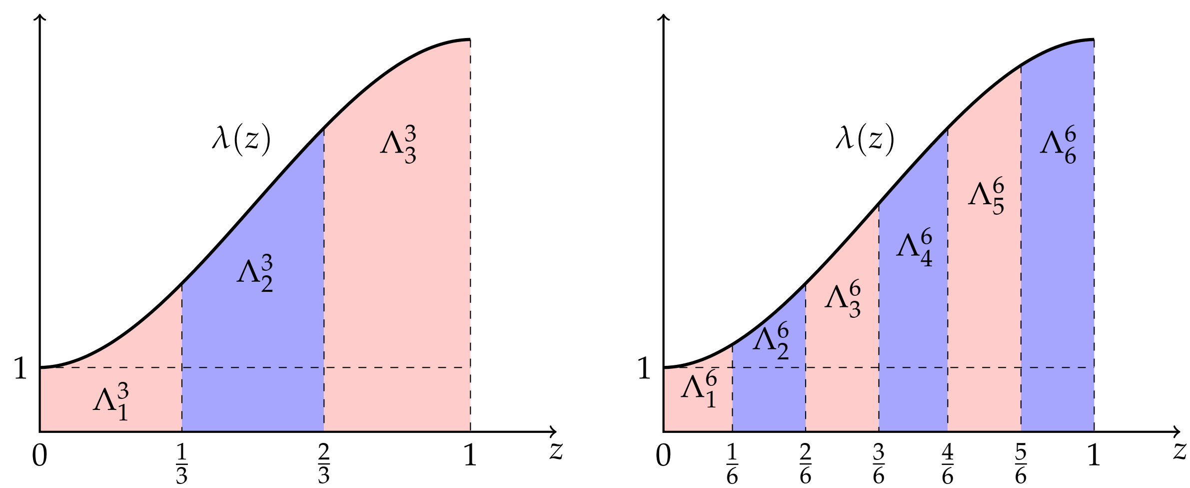
[7] There are two natural alternative environments: $\lambda(z)$ constant and $\lambda(z)$ strictly decreasing. In Section 5 we describe how divisibility plays no role in those cases.

[8] In Section 5 we extend our analysis to cases where equation (1) does not hold.

unit when the seller is of type i is $\Lambda_s^m v_i$ with

$$\Lambda_s^m \equiv \int_{(s-1)/m}^{s/m} \lambda(z)dz.$$

Figure 1 illustrates the buyer's valuation coefficients Λ_s^m of successive units of the good. In Figure 1(a) the good is divided into 3 units. Assume that the seller is of type i. The buyer's valuation for the first unit is $\Lambda_3^3 v_i$. The second unit gives the buyer intermediate valuation $\Lambda_2^3 v_i$. The last unit is the one with the lowest valuation to the buyer: $\Lambda_1^3 v_i$. Figure 1(b) illustrates the valuation coefficients of successive units of the good when it is divided into 6 units.



(a) Good divided into $m = 3$ units

(b) Good divided into $m = 6$ units

Figure 1: Valuation coefficients of successive units of a divided good

2.2 Timing and payoffs

The buyer and the seller trade sequentially over time. Time is discrete and periods are indexed by $t = 0, 1, \ldots$. In each period the buyer makes an offer $\varphi_t = (k, p)$, where $k \in \mathbb{Z}_+$ is the number of units requested and $p \in \mathbb{R}_+$ is the total payment offered. Without loss of generality, we assume that the number of units requested cannot exceed the number of remaining units. The seller can either accept ($a_t = A$) or reject ($a_t = R$) the offer. If the seller accepts, k units are traded and the buyer pays p to the seller. The game ends when all m units are traded.

The buyer and the seller share a discount factor $\delta = e^{-r\Delta}$ where $\Delta > 0$ represents the length of each period and $r > 0$ represents the discount rate. Suppose that the buyer and the seller of type i agree on trading a total of D times, indexed by $d \in \{1, \ldots, D\}$. In the first trade ($d = 1$), which takes place at time t_1, the buyer pays the seller p_1, in exchange for k_1 units, so the set of traded units is $S_1 = \{m, \ldots, m - k_1 + 1\}$. A generic trade $d > 1$ takes place at time t_d and involves a total payment p_d in exchange for k_d units. The set of traded units is $S_d = \{m - k_1 - \ldots - k_{d-1}, \ldots, m - k_1 - \ldots - k_d + 1\}$. Then, the total payoff to the buyer is:

$$\sum_{d=1}^{D} \delta^{t_d} \left[\sum_{s \in S_d} \Lambda_s^m v_i - p_d \right]$$

The seller, in turn, obtains

$$\sum_{d=1}^{D} \delta^{t_d} \left[p_d - \frac{c_i}{m} k_d \right].$$

The buyer does not learn about the quality of the good upon purchasing units of it. Therefore, all learning is strategic; the buyer only updates his belief based on the seller's behavior.

In our model, the buyer enjoys the benefits from a unit from the period in which he purchases it. However, realized payoffs do not provide additional information about the quality of the good to the buyer. By shutting down the possibility of learning from experiencing the good, this modelling choice allows us to focus on the effects of strategic learning.

Strategic learning is key in many environments, including markets for securities. To see this, consider a bank negotiating the sale of a pool of mortgages to a pension fund in a context of macroeconomic uncertainty. The bank has private information about the quality of its mortgages. In every period, an i.i.d. macroeconomic shock may occur. While no macroeconomic shock materializes, the housing market booms and borrowers associated to both high and low-quality mortgages are able to honor their debts. Thus, both types of mortgages provide the same cash flow to the pension fund. As soon as a shock occurs, both types of borrowers may become delinquent, with low type borrowers being more likely to default. Furthermore, after the shock, rating agencies downgrade mortgage securities, which prevents the pension fund from further negotiating them. Our model describes this environment with a slight reinterpretation of the parameters, where in particular δ incorporates the exogenous probability that a macroeconomic shock materializes in each period.

2.3 Strategies

The public history h^t, with $t \geq 1$, lists all offers made, together with all responses by the seller, from period 0 through period $t-1$: $h^t = ((\varphi_0, a_0), \ldots, (\varphi_{t-1}, a_{t-1}))$. We let $h^0 = \varnothing$ denote the initial public history and we let H^t denote the set of all possible histories h^t at the beginning of period t. Intermediate histories (h^t, φ_t) include the offer made after history h^t, but not the subsequent action chosen by the seller.

A buyer's (behavior) strategy $\sigma_B = \left(\sigma_B^t\right)_{t=0}^{\infty}$ assigns a random offer to every public history h^t, with $\sigma_B^t(h^t) \in \Delta\Phi(h^t)$, where $\Phi(h^t)$ is the set of available offers at h^t. A seller's (behavior) strategy $(\sigma_L, \sigma_H) = \left(\sigma_L^t, \sigma_H^t\right)_{t=0}^{\infty}$ assigns a random decision (A or R) to each intermediate history (h^t, φ_t), so $\sigma_i^t\left(h^t, \varphi_t\right) \in \Delta\{A, R\}$ for every $i \in \{L, H\}$. The system of beliefs $\beta(\cdot)$ is as follows. We let $\beta(h^t)$ and $\beta(h^t, \varphi_t)$ denote the buyer's belief that the seller is of high type after an arbitrary public history h^t, and an arbitrary intermediate history (h^t, φ_t), respectively.

2.4 Equilibrium concept and preliminary results

We work with *Stationary Perfect Bayesian Equilibria*. In this model, at any public history h^t there are two state variables: the number of remaining units $K(h^t)$ and the buyer's belief $\beta(h^t)$. A strict notion of stationarity would require strategies and value functions to depend only on the two state variables $K(h^t)$ and $\beta(h^t)$. As is standard in bargaining, there is no equilibrium that satisfies this strict notion. We then use a notion that places restrictions only on the seller's strategy. We require the seller's strategy to be a supply function and to depend only on state variables. In what follows we describe our definition in detail.

We first present some preliminary results which facilitate the exposition of our notion of stationarity. In any Perfect Bayesian Equilibrium (PBE), the buyer's system of beliefs $\beta(\cdot)$ must satisfy the following properties. Beliefs $\beta\left(h^t, \varphi_t, a_t\right)$ are derived from $\beta(h^t)$ according to Bayes' rule whenever action a_t occurs with positive probability after intermediate history (h^t, φ_t). Moreover, beliefs after intermediate histories are not affected by the buyer's offer: $\beta(h^t, \varphi_t) = \beta(h^t)$.

Lemma 1 provides a partial characterization of equilibria whenever the seller's strategy depends only on state variables. Let $V_H(h^t)$, $V_L(h^t)$ and $V_B(h^t)$ denote the continuation payoffs for, respectively, a seller of high type, a seller of low type and the buyer.

LEMMA 1. PARTIAL CHARACTERIZATION. *Let $\left(\sigma_B, (\sigma_L, \sigma_H), \beta\right)$ be an arbitrary PBE. Assume that whenever histories h^t and $\tilde{h}^{t'}$ have the same state variables: $\beta(h^t) = \beta(\tilde{h}^{t'})$ and $K(h^t) = K(\tilde{h}^{t'})$, then $\sigma_i\left(h^t, \varphi\right) = \sigma_i(\tilde{h}^{t'}, \varphi)$ for all $\varphi \in \Phi(h^t) = \Phi(\tilde{h}^{t'})$ and for both $i \in \{L, H\}$. Then,*

(a) Whenever $\beta(h^t) = 0$, the low-type seller gets zero payoffs: $V_L(h^t) = 0$.

(b) The buyer's continuation payoff $V_B(h^t)$ depends only on $\beta(h^t)$ and on $K(h^t)$.

(c) The high-type seller gets zero payoffs: $V_H(h^t) = 0$ for all h^t.

(d) The low-type seller's payoffs are bounded: $V_L(h^t) \leq \frac{c}{m}K(h^t)$ for all h^t.

See Appendix A.1 for the proof.

Lemma 1(*a*) states that the low-type seller cannot obtain positive payoffs after his type has been revealed. This result holds true in any PBE, so does not rely on stationarity.[9] Lemma 1(*b*) and (*c*) are direct results of the seller's strategy depending only on state variables. Lemma 1(*b*) states that the buyer's continuation payoff must depend only on beliefs and on the number of remaining units. Lemma 1(*c*) states that the high type seller always obtains zero profits. Hence, any offer of a payment larger than $\frac{c}{m}K(h^t)$ would be accepted with probability one by the high-type seller, and so also by the low type. This implies Lemma 1(*d*): the low-type seller continuation payoff is bounded above by $\frac{c}{m}K(h^t)$.

Our definition of stationary PBE incorporates the results from Lemma 1. The behavior of both types of sellers must be consistent with the payoffs that they obtain in a stationary

[9] This standard result is analogous to that in a model with common knowledge of types and a buyer who always makes the offer.

environment. Following Lemma 1(c), a high-type seller accepts any offer that leads to non-negative payoffs. Similarly, following Lemma 1(d), a low-type seller accepts any offer that the high-type seller also accepts.[10] But, does the low-type seller ever accept offers that the high-type seller rejects? If he does so, he immediately reveals his own type to the buyer. Moreover, if the low-type seller mixes, then a rejection increases the belief that the seller is of high type. Then, the behavior of the low-type seller is more subtle than that of the high-type seller. We impose that the acceptance decision of the low-type seller be governed by a function $\mathcal{V}_L(K,\beta)$ that depends on the number of remaining units K and on the beliefs β induced by a rejection.

DEFINITION. STATIONARY PERFECT BAYESIAN EQUILIBRIUM. *A PBE is stationary if there exists a (left-continuous) function $\mathcal{V}_L(K,\beta) : \{1,\dots,m\} \times [\hat{\beta},1] \to \mathbb{R}$ such that*

1. *The high-type seller accepts with probability one any payment greater or equal than $\frac{c}{m}k$ in exchange for any number of remaining units $k \le K(h^t)$. The high-type seller rejects any other offer with probability one.*

2. *The behavior of the low-type seller is as follows. Take any history h^t where the remaining number of units is $K(h^t)$ and the belief is $\beta(h^t) \ge \hat{\beta}$. Assume that the buyer offers a total payment p in exchange for $k \le K(h^t)$ remaining units. Then,*

   a. *If $p \ge \frac{c}{m}k$, then the low-type seller accepts the offer with probability one.*

   b. *If $p < \frac{c}{m}k$ and $p < \delta\mathcal{V}_L\left(K(h^t),\beta\right)$ for all $\beta \ge \beta(h^t)$, then the low-type seller rejects the offer with probability one.*

   c. *If $p < \frac{c}{m}k$ and there exists $\beta \ge \beta(h^t)$ with $p \ge \delta\mathcal{V}_L\left(K(h^t),\beta\right)$, then the low-type seller randomizes so that $\beta' = \max\left\{\beta : \delta\mathcal{V}_L(K(h^t),\beta) \le p\right\}$ is the next-period posterior after rejection.*

The function $\delta\mathcal{V}_L(K,\cdot)$ acts as a stationary supply when there are K units left. First, it acts as a supply function because when the buyer offers a higher price p, he induces a (weakly) higher posterior β' after rejection. Therefore, the probability of acceptance of the low-type seller is (weakly) increasing in the price offered by the buyer. Second, the

[10] This in turn implies that beliefs never decrease over time, and so they are bounded below by $\hat{\beta}$.

function $\delta\mathcal{V}_L(K,\cdot)$ acts as a *stationary* supply because the price that the buyer needs to pay to induce a posterior belief $\beta' \geq \beta(h^t)$ is independent of the current belief $\beta(h^t)$.[11]

The concept of stationary PBE (equilibrium henceforth), together with Lemma 1, allow for a characterization of the offers that can occur with positive probability in equilibrium. In particular, consider the family of *partial offers*. The buyer makes a partial offer when he requests less than the total number of remaining units and offers a payment that does not cover the costs of the high-type seller. These offers cannot be made and accepted with positive probability in equilibrium. The intuitive reason behind this is simple. A high-type seller never accepts a partial offer, since, by definition, a partial offer does not cover his costs. Then, only the low-type seller may accept partial offers with positive probability. The acceptance of a partial offers reveals that the seller is of low type, so remaining units are traded immediately, and the low-type seller gets no payoff from that trade. Instead of making a partial offer, the buyer could offer to buy *all remaining units* at the same (total) price. The low-type seller would get the same payoff from this alternative offer, so he would accept it, and with the same probability.[12] Trade would then speed up, with the buyer obtaining the additional surplus. Thus, the buyer could obtain a strictly higher payoff by making this alternative offer, i.e. asking for all remaining units, and offering the same payment. Lemma 2 formalizes this.

LEMMA 2. NO PARTIAL OFFERS. *Fix an equilibrium. Take any history h^t with $K(h^t) > 1$. Trades (k,p) with $k < K(h^t)$ and $p < \frac{c}{m}k$ occur with zero probability.*

See Appendix A.2 for the proof.

Consider the two remaining families of offers:

DEFINITION. UNIVERSAL AND SCREENING OFFERS. *The buyer makes a* universal offer *for $k \leq K(h^t)$ units when he offers a payment $p = \frac{c}{m}k$. Universal offers are then of the form $\left(k, \frac{c}{m}k\right)$ and both types accept them. The buyer makes a* screening offer *for all remaining units*

[11] Our definition of stationary PBE extends the notion of stationary equilibrium (see Gul and Sonnenschein [1988], Ausubel and Deneckere [1992], DL and Fuchs and Skrzypacz [2010]) to our setup. We conjecture generic uniqueness of PBE outcomes. However, we have not been able to show this.

[12] Our definition of stationary PBE implies that the randomization probability of the low-type seller depends on the number of units remaining, but not on the number of units requested by the buyer. This assumption is without loss of generality. In an earlier version of this paper we allow $\mathcal{V}_L$ to depend also on the number of units requested by the buyer. For generic values of the parameters, we obtain the same equilibrium outcome as with our definition of stationary PBE.

$K(h^t)$ when he offers a payment $p < \frac{c}{m}K(h^t)$. Screening offers are then of the form $\left(K(h^t), p\right)$ and the high-type seller never accepts them.

It is without loss of generality to restrict attention only to universal and screening offers. To see this, suppose that in equilibrium, at history h^t, the buyer makes a partial offer (k, p), which the seller rejects (by Lemma 2). Replace this offer with the screening offer $(K(h^t), p)$. Stationarity implies that this offer is also rejected by the seller. By replacing all partial offers this way, we obtain an outcome equivalent equilibrium in which no partial offer is ever made. In this sense, there is no equilibrium with partial offers.[13]

Before presenting our main result, we perform a convenient change of variables. We work with the transformed beliefs $q(\beta) : [\hat{\beta}, 1] \to [0, 1-\hat{\beta}]$ given by the continuous and strictly increasing mapping

$$q\left(\beta\right) = 1 - \frac{\hat{\beta}}{\beta}.$$

For convenience we write $\hat{q} = 1 - \hat{\beta}$ and, with a slight abuse of notation, we let $q\left(h^t\right) = q\left(\beta\left(h^t\right)\right)$. A transformed belief equal to $\hat{q}$ means that the buyer assigns probability one to the seller being of high type. This transformation allows for a simple expression for the probability that the low-type seller accepts screening offers. Assume that after the rejection of a screening offer, the buyer updates his transformed belief from q to q'. This means that the low-type seller accepts such offer with probability $(q' - q)/(\hat{q} - q)$. Moreover, as we show in Appendix A.3, the buyer's value function is linear in transformed beliefs $q\left(h^t\right)$.[14]

3. Main results

We characterize the *limit equilibrium outcome*, that is, the pattern of trade when bargaining frictions vanish and the good becomes arbitrarily divisible. We first let the time between

[13]In fact, a stronger result holds: for generic values of the parameters, partial offers are never made in equilibrium.

[14]This change of variables is also explored in several papers in bargaining with incomplete information. Some readers may find useful the following interpretation for the variable q. Assume that the sellers's type q is uniformly distributed in the unit interval. Whenever $q \in [0, \hat{q})$ then the seller is of low type. If instead $q \in [\hat{q}, 1]$, the seller is of high type. Under this interpretation for q, the function $P(K, \cdot)$ that we introduce in Appendix A.3 represents the reservation price $P(K, q)$ for type $q \in [0, \hat{q})$.

offers shrink to zero and we then let the number of units grow to infinity. With this order of limits we can use an inductive argument on the number of remaining units and develop an algorithm that characterizes the limit equilibrium outcome as bargaining frictions vanish. In addition, our order of limits captures the following features of most real-world environments. On the one side, parties can trade essentially at any point in time. On the other side, there is typically a lower bound on the size of each unit of the good. This is true even for goods that can be divided into a large number of units, like financial assets.[15]

3.1 The limit equilibrium outcome

In our main result (Theorem 1) we show that in the limit equilibrium outcome the high-quality good is traded smoothly over time, the buyer's payoff converges to zero, and the low-type seller's payoff converges to $\left(\int_0^1 \lambda(z)dz\right)v_L$. In order to convey our main message swiftly, we introduce in this section only the minimum necessary notions to state Theorem 1. In this section we also take advantage of some intermediate results (such as the characterization of the equilibrium for fixed Δ and fixed m) that we present formally and discuss at length later, in Section 4.

We next provide a formal definition of the notion of limit equilibrium outcome. Consider the environment with a fixed time between offers $\Delta > 0$ and a fixed number of units m. We show in Section 4.1 that the buyer's equilibrium behavior is deterministic; at each period t he either makes a universal or a screening offer. Both types of seller always accept universal offers. While a seller of high type never accepts a screening offer, a low-type seller randomizes between accepting and rejecting them. We can then describe the equilibrium outcome in a simple way. Consider the history where all screening offers are rejected. For any period $t = 0, 1, \ldots$, we let $\widetilde{K}_m^\Delta(t)$ and $\tilde{q}_m^\Delta(t)$ denote respectively the number of remaining units and the buyer's transformed belief along that history. Whenever the buyer makes a universal offer, $\widetilde{K}_m^\Delta(\cdot)$ decreases between two consecutive periods while $\tilde{q}_m^\Delta(\cdot)$ remains unchanged. In contrast, when the buyer makes a screening offer,

[15]We discuss the implications of inverting the order of limits in the conclusion.

$\widetilde{K}_m^\Delta(\cdot)$ remains unchanged while $\tilde{q}_m^\Delta(\cdot)$ increases.[16] The functions $\widetilde{K}_m^\Delta(\cdot)$ and $\tilde{q}_m^\Delta(\cdot)$ thus summarize the equilibrium outcome.

We first let the time between offers Δ converge to zero, so that the discount factor $\delta = e^{-r\Delta}$ converges to one. To make meaningful comparisons between games with different period lengths Δ, we express the number of remaining units and the transformed belief as functions of time elapsed $\tau \in \mathbb{R}_+$. We show that these functions converge pointwise as Δ shrinks to zero and we let the functions $K_m : \mathbb{R}_+ \to [0, m]$ and $q_m : \mathbb{R}_+ \to [0, 1]$ denote their limits. We define the fraction of the good left for trade $z_m(\tau) : \mathbb{R}_+ \to [0, 1]$ by setting $z_m(\tau) = K_m(\tau)/m$. Finally, we define the *limit equilibrium outcome* as the limit of the functions $z_m(\cdot)$ and $q_m(\cdot)$ as m grows large.

We next describe two simple functions that, as we show in Theorem 1, characterize the limit equilibrium outcome. The function $z^* : \mathbb{R}_+ \to [0, 1]$ describes the fraction of the good left for trade and the function $q^* : \mathbb{R}_+ \to [0, \hat{q}]$ describes the evolution of beliefs. In order to describe these functions, we let $\bar{q}(z)$ denote the belief that makes the buyer break even when he makes a universal offer for the infinitesimal unit z:

$$[\hat{q} - \bar{q}(z)]\,[\lambda(z)v_L - c] + (1 - \hat{q})\,[\lambda(z)v_H - c] = 0$$

The function $\bar{q} : [0, 1] \to [0, \hat{q}]$ is strictly decreasing. We let $\psi : [\bar{q}(1), \bar{q}(0)] \to [0, 1]$ denote its inverse.

The construction of the functions $q^*(\cdot)$ and $z^*(\cdot)$ is simple, and can be better understood through the following artificial pattern of trade. At time elapsed $\tau = 0$, the buyer makes a screening offer and breaks even. The low-type seller accepts this offer with probability $\bar{q}(1)/\hat{q}$, so the belief at time $\tau = 0$ satisfies $q^*(0) = \bar{q}(1)$. From that point on, the buyer continuously makes both screening offers and universal offers for infinitesimal units. At any point in time $\tau \in \mathbb{R}_+$, the buyer breaks even with either type of offer. Finally, the low-type seller is indifferent between accepting and rejecting any screening offer. The functions $q^*(\cdot)$ and $z^*(\cdot)$ are the results of this artificial pattern of trade.

In the artificial pattern of trade, the buyer breaks even every time he makes a universal

[16] The rejection of a screening offer makes the buyer more optimistic about the quality of the good since only the low-type seller may accept a screening offer.

offer for the infinitesimal unit z. Thus, at any point in time $\tau \in \mathbb{R}_+$, the belief $q^*(\tau)$ and the fraction of remaining units $z^*(\tau)$ must satisfy $q^*(\tau) = \bar{q}\left(z^*(\tau)\right)$. Furthermore, since the buyer also breaks even whenever he makes a screening offer, at any point in time $\tau \in \mathbb{R}_+$ he offers to purchase the fraction $z^*(\tau)$ at the price $v_L \int_0^{z^*(\tau)} \lambda(z)dz$. Finally, the low-type seller is indifferent between accepting a screening offer at time τ or mimicking the high-type seller's behavior from τ to $\tau + \Delta\tau$ and then accepting a screening offer at time $\tau + \Delta\tau$:

$$v_L \int_0^{z^*(\tau)} \lambda(z)dz = \int_\tau^{\tau+\Delta\tau} e^{-r(s-\tau)} c(-z^{*\prime}(s))ds + e^{-r\Delta\tau} v_L \int_0^{z^*(\tau+\Delta\tau)} \lambda(z)dz \qquad (2)$$

We next let $\Delta\tau \to 0$ and, through a first order approximation of the right hand side of equation (2), obtain that $z^{*\prime}(\tau)\left[v_L \lambda\left(z^*(\tau)\right) - c\right] = r v_L \int_0^{z^*(\tau)} \lambda(z)dz$. Together with the fact that $q^*(\tau) = \bar{q}\left(z^*(\tau)\right)$, this implies that

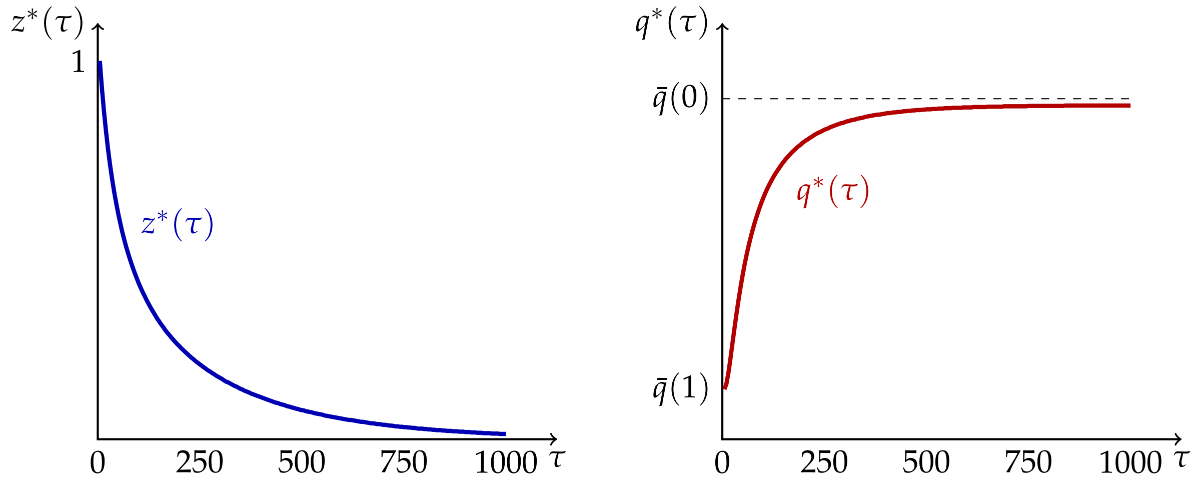
$$q^{*\prime}(\tau) = \frac{r v_L \int_0^{\psi(q^*(\tau))} \lambda(z)dz}{\psi'\left(q^*(\tau)\right)\left[v_L \lambda\left(\psi\left(q^*(\tau)\right)\right) - c\right]} \quad \text{and} \qquad (3a)$$

$$z^{*\prime}(\tau) = \psi'\left(q^*(\tau)\right) q^{*\prime}(\tau) = \frac{r v_L \int_0^{z^*(\tau)} \lambda(z)dz}{v_L \lambda\left(z^*(\tau)\right) - c}. \qquad (3b)$$

This, together with the initial conditions $q^*(0) = \bar{q}(1)$ and $z^*(0) = 1$ pins down the functions $q^*(\cdot)$ and $z^*(\cdot)$. Conditions (3a) and (3b) guarantee that the functions $q^*(\cdot)$ and $z^*(\cdot)$ are smooth and that $q^*(\cdot)$ converges to $\bar{q}(0)$ and $z^*(\cdot)$ shrinks to zero as $\tau \to \infty$.

THEOREM 1. LIMIT EQUILIBRIUM OUTCOME. *The sequence $\{(z_m(\cdot), q_m(\cdot))\}_{m=1}^\infty$ converges pointwise to $(z^*(\cdot), q^*(\cdot))$. Thus, in the limit equilibrium outcome, the high-quality good is traded smoothly over time, the low-type seller's payoff is $\left(\int_0^1 \lambda(z)dz\right)v_L$ and the buyer's payoff is zero.*

Figure 2 illustrates the limit equilibrium outcome $(z^*(\cdot), q^*(\cdot))$. The buyer's belief evolves smoothly and the high-quality good is sold gradually over time. At any point in time, a positive fraction of the good is left for trade and bargaining continues forever. On page 10 we provide an alternative interpretation of the model that includes an exogenous probability of a macroeconomic shock. In this case, Figure 2 illustrates the pattern of trade conditional on no shock ever taking place.



(a) Fraction of the good left $z^*(\cdot)$

(b) Transformed beliefs $q^*(\cdot)$

Note: These figures depict the limit equilibrium outcome for the following primitives: $v_H = 35$, $v_L = 1$, $c = 30$, $r = 0.1$ and $\hat{q} = 0.9$. Finally, $\lambda(z) = 1 + 0.1z + 15z^2 - 10z^3$ (this is the function shown in Figure 1).

Figure 2: Limit equilibrium outcome $(z^*(\cdot), q^*(\cdot))$: Pattern of trade as bargaining frictions vanish and the good becomes arbitrarily divisible

The proof of Theorem 1 consists of two parts. In the first one (Proposition 3) we fix the number of units m and characterize the equilibrium outcome as bargaining frictions vanish ($\Delta \to 0$). We show that the equilibrium outcome takes a simple form: phases of fast trade alternate with impasses. In the phases of fast trade, parties trade without delay. The buyer purchases chunks of the good from both seller types (through universal offers) and, with positive probability, he also purchases all remaining units from the low-type seller (through screening offers). Instead, the market freezes during an impasse. The buyer screens the seller with delay, as in DL. At each impasse the buyer's continuation payoff is zero, as he would have an incentive to speed up trade otherwise. In Proposition 3 we construct an algorithm that pins down the entire sequence of phases of fast trade and impasses.

In the second part of the proof of Theorem 1 (Proposition 4) we let the number of units m grow to infinity. We study the limit of the equilibrium outcome uncovered by the algorithm in Proposition 3. We show that the number of impasses grows to infinity and the length of each impasse goes to zero. The fraction of the good traded in each phase of fast trade through universal offers also goes to zero. Proposition 4 leads directly to the pattern of trade described in Theorem 1. Divisibility, together with decreasing gains from trade, introduces a new source of temptation for the buyer, which is reminiscent of the Coase conjecture. As we highlight in Section 4.3, the buyer is tempted to purchase the most valuable (small) fractions of the good. This is the driving force behind the gradual sale of the high quality good.

3.2 Implications of divisibility

Before diving into the explanation of Propositions 3 and 4, we take advantage of the characterization in Theorem 1 to shed light on the pattern of trade of an arbitrarily divisible good.[17]

How does the pattern of trade of securities look like? Theorem 1 shows that high-quality securities are traded in dribs and drabs. In contrast, when securities are of low quality, the parties either trade small portions or make a final transaction for all remaining securities.

We provide novel and testable predictions on market efficiency. Our model highlights a rationale for markets of divisible goods (like markets for securities) to be more efficient than markets for indivisible goods (like real estate markets). We also show that these markets differ sharply on the split of the gains from trade. Divisibility is detrimental to buyers (who are uninformed and have bargaining power) and beneficial to sellers of lemons. Finally, markets of securities and real estate markets share some features. We show that when adverse selection worsens, the speed of trade of high quality mortgages becomes slower.

We now describe in detail the implications of divisibility presented in the previous paragraphs. We first discuss the efficiency of the limit equilibrium outcome. We compare the (expected) gains from trade in the limit equilibrium outcome to those 1) under the buyer's optimal mechanism with commitment, 2) under the most efficient mechanism

[17] The reader more interested in the explanation of the driving force behind our main result may safely skip the next subsection and proceed directly to Section 4.

with commitment and 3) in the limit, as bargaining frictions vanish, of the model *without* divisibility (DL).[18]

The gains from trade are equal to $\hat{q}\big(\int_0^1 \lambda(z)dz\big)v_L$ in the limit equilibrium outcome. This follows from Theorem 1, as the low-type seller's payoff is equal to $\big(\int_0^1 \lambda(z)dz\big)v_L$, both the high-type seller and the buyer obtain a payoff of zero and the seller is of low type with probability $\hat{q}$.

The limit equilibrium outcome is as efficient as the buyer's optimal mechanism with commitment. Under this mechanism, the buyer purchases the whole good immediately from the low-type seller, and pays him a price of zero. The buyer and the high-type seller do not trade.[19] Thus, the buyer extracts all the surplus. So although the gains from trade are equal in these two environments, the split of the surplus between the parties is starkly different.

Samuelson [1984] shows that the buyer's optimal mechanism does not achieve the second best: the gains from trade under the buyer's optimal mechanism are lower than those under the most efficient mechanism with commitment.[20] Thus the limit equilibrium outcome is bounded away from the second best.

The gains from trade in our limit equilibrium outcome are higher than those in the model without divisibility (DL).[21] Without divisibility, both the low-type seller and the buyer obtain a positive payoff (the high-type seller obtains a zero payoff). Thus, although divisibility improves overall efficiency, it is detrimental for the buyer. Furthermore, while divisibility also increases the gains from trade conditional on the good being of high quality, this is not necessarily the case if the good is instead of low quality. Finally, we compare the speed of trade of the high-quality good with and without divisibility. We show that when the good is divisible, the high-quality good is traded faster.[22]

[18]The gains from trade under both mechanisms with commitment are independent of the number of units m and the period length Δ.

[19]To see why the buyer cannot improve upon this mechanism, note that for the buyer to purchase a marginal unit from the high-type seller, he must pay the marginal cost c to both types. Equation (1) implies that it is not profitable for the buyer to do so.

[20]Both the high-type seller and the buyer obtain a payoff equal to zero under the most efficient mechanism with commitment.

[21]The outcome without divisibility (DL) is similar to the one when only one unit remains in our model. We describe this in detail in footnote 30, on page 30.

[22]Formally, $-\int_0^\infty e^{-r\tau} z^{*\prime}(\tau)d\tau > e^{-rT_{DL}}$, where T_{DL} represents the time at which the high-quality good

We next show how the primitives of the model affect the speed of trade for both the high-quality and the low-quality good. We start with a configuration of the primitives $(\hat{q}, \lambda(\cdot), c, v_L, v_H, r)$, modify one of them (resulting in a new configuration that also satisfies the assumptions of our model) and compare the resulting limit equilibrium outcomes.

PROPOSITION 1. SPEED OF TRADE OF THE HIGH-QUALITY GOOD. *Let $(z^*(\cdot), q^*(\cdot))$ denote the limit equilibrium outcome associated to the primitives $(\hat{q}, \lambda(\cdot), c, v_L, v_H, r)$. Consider next an alternative configuration of primitives with associated limit equilibrium outcome $(\tilde{z}^*(\cdot), \tilde{q}^*(\cdot))$. For any of the following alternative configurations of primitives, the high-quality good is traded faster, i.e. $\tilde{z}^*(\tau) < z^*(\tau)$ for every $\tau > 0$:*

(a) $\left(\hat{q}, \tilde{\lambda}(\cdot), c, v_L, v_H, r\right)$ *with* $\tilde{\lambda}(z) > \lambda(z)$ *for all* $z \in (0, 1]$.

(b) $(\hat{q}, \lambda(\cdot), \tilde{c}, v_L, v_H, r)$ *with* $\tilde{c} < c$.

(c) $(\hat{q}, \lambda(\cdot), c, \tilde{v}_L, v_H, r)$ *with* $\tilde{v}_L > v_L$.

(d) $(\hat{q}, \lambda(\cdot), c, v_L, v_H, \tilde{r})$ *with* $\tilde{r} > r$.

Finally, the parameters v_H and $\hat{q}$ do not affect the speed of trade of the high-quality good.

See Appendix A.7 for the proof.

The intuition behind Proposition 1 is simple. The speed of trade of the high-quality good is such that the low-type seller is always indifferent between accepting the current screening offer or rejecting all screening offers and obtaining the discounted value of future universal offers. An increase in either v_L or in the function $\lambda(\cdot)$ makes each screening offer more attractive. Similarly, a decrease in c or an increase in r lower the value of future universal offers. In all these four cases the high-quality good must be traded faster to keep the low-type seller indifferent.

Unlike the high-quality good, the low-quality good is not always traded smoothly. Trade occurs smoothly while the low-type seller mimics the high-type seller's behavior. However, the buyer purchases the whole remaining fraction of the good as soon as the low-type seller accepts a screening offer. Therefore, the fraction of the low-quality good

is traded in DL as bargaining frictions vanish. This inequality follows from the expressions of the low-type seller's payoffs and from the fact that the low-type seller is indifferent in equilibrium (both with and without divisibility).

remaining at time elapsed τ is a random variable that takes a value of zero with probability $\frac{q^*(\tau)}{\hat{q}}$ and a value of $z^*(\tau)$ with the remaining probability. Then $g^*(\tau) = \frac{\hat{q}-q^*(\tau)}{\hat{q}} z^*(\tau)$ is the expected remaining fraction of the low-quality good at time elapsed τ and reflects the speed of trade of the low-quality good.

The following corollary, which follows directly from Proposition 1 and from the fact that $q^*(\tau) = \bar{q}(z^*(\tau))$, describes how changes in the parameters r, v_H and $\hat{q}$ affect the speed of trade of the low-quality good.

COROLLARY 1. SPEED OF TRADE OF THE LOW-QUALITY GOOD. *Whenever either r increases, or v_H decreases, or $\hat{q}$ increases, then the low-quality good is traded faster, i.e. $g^*(\tau)$ decreases for every $\tau > 0$.*

The remaining primitives ($\lambda(\cdot)$, v_L and c) have ambiguous effects on the speed of trade of the low-quality good. It is easy to construct examples where changes in these primitives can either increase or decrease $g^*(\tau)$ for some τ.

4. Mechanism behind the limit equilibrium outcome

We now turn back to the explanation of our main result, Theorem 1. We first study the environment with a fixed time between offers $\Delta > 0$ and a fixed number of units m. We discuss equilibrium existence (Proposition 2) and describe in detail the pattern of trade. We then describe the pattern of trade as bargaining frictions vanish (Proposition 3). We finally let the number of units grow to infinity and present Proposition 4, which directly leads to Theorem 1.

4.1 Equilibrium existence

In this subsection we study the bargaining game when the good is divided into a fixed number of units equal to m and the period length is fixed and equal to Δ.

PROPOSITION 2. EXISTENCE. *An equilibrium exists.*

See Appendix A.3 for the proof.

We show equilibrium existence by construction. Within our construction, we introduce the function $P_m^{\Delta}(K,q) : \{1,\ldots,m\} \times [0,\hat{q}] \to \mathbb{R}$, which plays a key role in the description and the analysis of the equilibrium. We derive this function from $\mathcal{V}^L(\cdot,\cdot)$ (see Section 2.4), and show that $P_m^{\Delta}(K,\cdot)$ is an increasing and left-continuous step function for every $K \in \{1,\ldots,m\}$. The function $P_m^{\Delta}(\cdot,\cdot)$ describes the relevant screening offers available to the buyer in equilibrium. Its interpretation is as follows. Suppose that there are K units left and that the current belief is $q \in [0,\hat{q}]$. Consider any discontinuity point q' of the function $P_m^{\Delta}(K,\cdot)$ with $q' \geq q$. Then, if the buyer makes a screening offer $\left(K, P_m^{\Delta}(K,q')\right)$ and it is rejected, his posterior belief is q'.

We solve the buyer's dynamic optimization problem. For any state (K,q), we let $W_m^{\Delta}(K,q) : \{1,\ldots,m\} \times [0,\hat{q}] \to \mathbb{R}$ denote the (normalized) buyer's continuation payoff.[23] When it is optimal for the buyer to make a screening offer $\left(K, P_m^{\Delta}(K,q')\right)$ for some discontinuity point q', the low-type seller accepts it with probability $(q'-q)/(\hat{q}-q)$. The buyer's continuation payoff satisfies

$$W_m^{\Delta}(K,q) = \left(q'-q\right)\left(\sum_{s=1}^{K} \Lambda_s^m v_L - P_m^{\Delta}(K,q')\right) + \delta W_m^{\Delta}\left(K,q'\right).$$

If instead it is optimal for the buyer to make a universal offer $\left(k, \frac{c}{m}k\right)$, the buyer's continuation payoff satisfies

$$W_m^{\Delta}(K,q) = \left(\sum_{s=K-k+1}^{K} \Lambda_s^m\right)\left[(\hat{q}-q)v_L + (1-\hat{q})v_H\right] - (1-q)\frac{c}{m}k + \delta W_m^{\Delta}(K-k,q).$$

We show that the low-type seller is indifferent between accepting and rejecting all screening offers that he receives in equilibrium (see Appendix A.3). Assume that in equilibrium the buyer makes a screening offer $\left(K, P_m^{\Delta}(K,q)\right)$. If the low-type seller accepts it, he obtains a continuation payoff of $P_m^{\Delta}(K,q)$. If he instead rejects it, the number of units left stays at K and the buyer's posterior is q. The buyer's subsequent offer can be either screening or universal. If the buyer makes a screening offer $\left(K, P_m^{\Delta}(K,q')\right)$, then the low-type seller's indifference requires that the prices of these consecutive screening offers be

[23] $W_m^{\Delta}(K,q)$ is normalized in the sense that we multiply the buyer's continuation payoff by $1-q$.

linked: $P_m^\Delta(K,q) = \delta P_m^\Delta(K,q')$. Assume instead that the buyer makes a universal offer $(k, \frac{c}{m}k)$ after the rejection of the screening offer $\left(K, P_m^\Delta(K,q)\right)$. This universal offer must be followed by a screening offer $\left(K-k, P_m^\Delta(K-k,q'')\right)$.[24] The low-type seller's indifference then requires that $P_m^\Delta(K,q) = \delta \frac{c}{m}k + \delta^2 P_m^\Delta(K-k,q'')$.

The focus on the equilibrium that we construct in Proposition 2 is without loss of generality as, for generic values of the parameters, the equilibrium outcome is unique.[25]

We show that the game ends after finitely many periods in the proof of Proposition 2. Let $h_m^{*\Delta}$ denote the on-path history along which the seller rejects all screening offers. The history $h_m^{*\Delta}$ is the longest on-path history. We let $T_m^{*\Delta}$ denote its length. As desxcribed in Section 3, we let $\tilde{q}_m^\Delta(t)$ denote the transformed beliefs at the beginning of period t along the history $h_m^{*\Delta}$, for any $t \leq T_m^{*\Delta}$. Similarly, $\widetilde{K}_m^\Delta(t)$ denotes the number of units left at the beginning of period t along the history $h_m^{*\Delta}$, for any $t \leq T_m^{*\Delta}$. Together with $P_m^\Delta(\cdot,\cdot)$, the functions $\widetilde{K}_m^\Delta(\cdot)$ and $\widetilde{q}_m^\Delta(\cdot)$ completely characterize the equilibrium pattern of trade.

4.2 Limit equilibrium outcome as bargaining frictions vanish

In this subsection we fix the number of units m and let the time between offers Δ converge to zero. To do so, we first characterize the equilibrium outcome as a function of time elapsed $\tau \in \mathbb{R}_+$. In a game with period-length Δ, the time elapsed τ after t periods is $\tau = t\Delta$. We express the number of remaining units and the transformed beliefs as functions of time elapsed τ:

$$K_m^\Delta(\tau) = \widetilde{K}_m^\Delta\left(\min\left\{\lfloor \tau/\Delta \rfloor, T_m^{*\Delta}\right\}\right)$$
$$q_m^\Delta(\tau) = \tilde{q}_m^\Delta\left(\min\left\{\lfloor \tau/\Delta \rfloor, T_m^{*\Delta}\right\}\right)$$

To examine the limit equilibrium outcome as bargaining frictions vanish, we take a sequence $\{\Delta_n\}_{n=1}^\infty \to 0$ and study the limit of its associated sequence $\left\{\left(K_m^{\Delta_n}(\cdot), q_m^{\Delta_n}(\cdot)\right)\right\}_{n=1}^\infty$. In Lemma 3 (Appendix A.4) we show that for any $\{\Delta_n\}_{n=1}^\infty \to 0$, the associated sequence

[24] We show this result in the proof of Proposition 2. The intuition behind it is simple. In equilibrium, the buyer's continuation payoff is positive at every state. Thus, he has an incentive to combine any two consecutive universal offers.

[25] An earlier version of this paper contains the result of generic uniqueness of the equilibrium outcome.

$\left\{\left(K_m^{\Delta_n}(\cdot), q_m^{\Delta_n}(\cdot)\right)\right\}_{n=1}^{\infty}$ converges pointwise to the same limit functions $(K_m(\cdot), q_m(\cdot))$. Similarly, for any $\{\Delta_n\}_{n=1}^{\infty} \to 0$, the associated sequence $\left\{\left(P_m^{\Delta_n}(K,\cdot), W_m^{\Delta_n}(K,\cdot)\right)\right\}_{n=1}^{\infty}$, with $K \in \{1, \ldots, m\}$, converges pointwise to the same limit functions $(P_m(K,\cdot), W_m(K,\cdot))$. The functions $(K_m(\cdot), q_m(\cdot))$ describe the limit equilibrium outcome as bargaining frictions vanish.

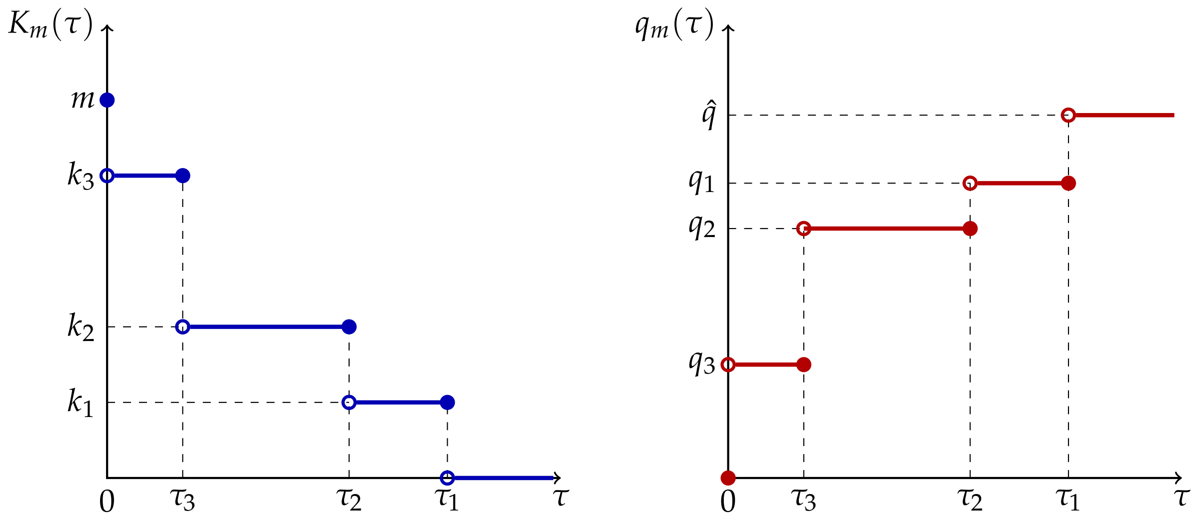
The pattern of trade that emerges as bargaining frictions vanish is simple: there is a sequence of phases of fast trade, mediated by impasses. We show this in Proposition 3 but first we provide a formal definition of this pattern of trade.

Definition. Phases of Fast Trade and Impasses. *We say that the limit equilibrium outcome as bargaining frictions vanish consists of a sequence of phases of fast trade, mediated by impasses whenever $K_m(\cdot)$ and $q_m(\cdot)$ are (left-continuous) step functions that are discontinuous at the same points in time. Moreover, we say that the collection of quantities and beliefs $\{(k_j, q_j)\}_{j=1}^{J}$ characterizes this limit equilibrium outcome as bargaining frictions vanish whenever there exist times $\tau_1 > \ldots > \tau_{J+1} = 0$ such that*

$$(K_m(\tau), q_m(\tau)) = \begin{cases} (m, 0) & \text{if } \tau = 0 \\ (k_j, q_j) & \text{if } \tau \in (\tau_{j+1}, \tau_j] \quad \text{for } j \in \{1, \ldots, J\} \\ (0, \hat{q}) & \text{if } \tau > \tau_1. \end{cases}$$

The phases of fast trade correspond to jumps in $K_m(\cdot)$ and $q_m(\cdot)$, while $K_m(\cdot)$ and $q_m(\cdot)$ are constant during each impasse. Each pair (k_j, q_j) describes quantities and beliefs during an impasse. The total number of impasses is $J \leq m$. We index impasses in reverse order, so $j = 1$ corresponds to the last impasse (k_1, q_1), while $j = J$ corresponds to the impasse (k_J, q_J) that occurs first. Therefore, $k_{j+1} > k_j$ and $q_{j+1} < q_j$ for all j.

Figure 3 depicts an example of the limit equilibrium outcome as bargaining frictions vanish. At the beginning of the game, there is a phase of fast trade. The transformed belief $q_m(\cdot)$ jumps to q_3 at time elapsed $\tau = 0$, which reflects that the buyer makes (a sequence of) screening offers. The low-type seller accepts with total probability $q_3/\hat{q}$. The number of units left $K_m(\cdot)$ jumps to k_3 at time elapsed $\tau = 0$, which reflects that the buyer makes a universal offer for $m - k_3$ units. Although for any given $\Delta > 0$ these offers occur in different periods, as $\Delta \to 0$ the total time it takes to jump to k_3 and q_3 converges to zero.



(a) Units left $K_m(\tau)$

(b) Transformed beliefs $q_m(\tau)$

Figure 3: Pattern of trade $(K_m(\tau), q_m(\tau))$ as bargaining frictions vanish

After the first phase of fast trade, an impasse follows. Intuitively, an impasse is an interval of time elapsed in which no trade occurs. The first impasse depicted in Figure 3 takes place in the interval $(0, \tau_3]$. Within this interval, $K_m(\cdot)$ remains constant at k_3 and $q_m(\cdot)$ remains constant at q_3. First, the fact that the number of units left is constant reflects that, in the limit, the buyer makes a sequence of screening offers after the first universal offer. As $\Delta \to 0$ the total number of such screening offers goes to infinity. Crucially, it does so sufficiently fast so that the total time elapsed while making these offers converges to $\tau_3 > 0$. Second, the fact that the belief $q_m(\cdot)$ is constant reflects that, in the limit, the low-type seller accepts these screening offers with total probability zero. This is possible because as $\Delta \to 0$, the probability of acceptance of each screening offer goes to zero fast enough to overcome that the total number of screening offers goes to infinity. Finally, Figure 3 illustrates that after the first impasse, there are three phases of fast trade, mediated by impasses.

We construct an algorithm that pins down the phases of fast trade and the impasses that take place as bargaining frictions vanish. This algorithm also identifies some key properties of the limit functions $P_m(\cdot,\cdot)$ and $W_m(\cdot,\cdot)$. In order to state these properties, we define the belief $\bar{q}_m(K)$ for any $K \in \{1,\ldots,m\}$ as follows. Assume that the buyer makes an offer $\varphi = (1, \frac{c}{m})$ when there are K units left; that is, he offers to pay the high-type's cost in exchange of one unit. Then, $\bar{q}_m(K) \in (0,\hat{q})$ is the transformed belief that makes the buyer break even:[26]

$$[\hat{q} - \bar{q}_m(K)] \left(\Lambda_K^m v_L - \frac{c}{m}\right) + (1-\hat{q}) \left(\Lambda_K^m v_H - \frac{c}{m}\right) = 0$$

Note that $\bar{q}_m(m) < \ldots < \bar{q}_m(1)$, as gains from trade are decreasing. Finally, for any (K,q) let $P_m^-(K,q) = \lim_{q'\uparrow q} P_m(K,q')$ and $P_m^+(K,q) = \lim_{q'\downarrow q} P_m(K,q')$.

PROPOSITION 3. EQUILIBRIUM OUTCOME AS BARGAINING FRICTIONS VANISH. *Fix m. The limit equilibrium outcome as bargaining frictions vanish consists of a sequence of phases of fast trade and impasses characterized by a collection of quantities and beliefs $\{(k_j,q_j)\}_{j=1}^J$ with $1 \le J \le m$, $(k_1,q_1) = (1,\bar{q}_m(1))$ and $\bar{q}_m(k_j+1) < q_j < \bar{q}_m(k_j)$ for all $j > 1$.*

Moreover, $W_m(k_j,q_j) = 0$ for every $j \in \{1,\ldots J\}$. Finally,

$$P_m^+(k_1,q_1) = P_m^+(1,\bar{q}_m(1)) = \frac{c}{m}, \tag{4a}$$

$$P_m^-(k_j,q_j) = \left(\frac{v_L \sum_{s=1}^{k_j} \Lambda_s^m}{P_m^+(k_j,q_j)}\right)^2 P_m^+(k_j,q_j) \qquad \forall\, j \in \{1,\ldots,J\} \quad \text{and} \tag{4b}$$

$$P_m^+(k_{j+1},q_{j+1}) = (k_{j+1}-k_j)\frac{c}{m} + P_m^-(k_j,q_j) \qquad \forall\, j \in \{1,\ldots,J-1\}. \tag{4c}$$

See Appendix A.5 for the proof.

Proposition 3 shows that there is at least one impasse. The last impasse always occurs at $(1,\bar{q}_m(1))$, that is, when only one unit remains and the belief is $\bar{q}_m(1)$. The buyer's continuation payoff is zero at all impasses. Finally, Proposition 3 describes the limit functions $P_m(\cdot,\cdot)$ around each impasse (k_j,q_j).

Equation (4c) links the limit functions $P_m(\cdot,\cdot)$ between two consecutive impasses. After the impasse (k_{j+1},q_{j+1}) is resolved, the state shifts without delay to (k_j,q_j). To fix

[26] We define $\bar{q}_m(K)$ in an analogous way to $\bar{q}(z)$ from Section 3. While $\bar{q}(z)$ depends on the infinitesimal unit z, $\bar{q}_m(K)$ depends on the number of remaining units K. For convenience, we set $\bar{q}_m(m+1) = 0$.

ideas, suppose that the shift consists of one universal offer for $k_{j+1} - k_j$ units followed by a screening offer $\left(k_j, P_m^-(k_j, q_j)\right)$.[27] The low-type seller obtains a continuation payoff $\left(k_{j+1} - k_j\right) \frac{c}{m} + P_m^-(k_j, q_j)$, which must be equal to the price $P_m^+(k_{j+1}, q_{j+1})$ that the buyer has to pay in the limit to induce a belief $q > q_{j+1}$ close to q_{j+1}. Equation (4a) follows the same logic as equation (4c): after the last impasse is resolved the buyer purchases without delay the last unit at the price $\frac{c}{m}$.

Equation (4b) shows that the limit function $P_m(k_j, \cdot)$ is discontinuous at q_j. The jump between $P_m^-(k_j, q_j)$ and $P_m^+(k_j, q_j)$ pins down the length of the impasse (k_j, q_j). Let $\tilde{\tau}$ be the necessary time elapsed for the buyer's valuation for the low-quality good $v_L \sum_{s=1}^{k_j} \Lambda_s^m$ to be equal to the discounted value of $P_m^+(k_j, q_j)$: $v_L \sum_{s=1}^{k_j} \Lambda_s^m = e^{-r\tilde{\tau}} P_m^+(k_j, q_j)$. Equation (4b) shows that the delay is of length $2\tilde{\tau}$:

$$P_m^-(k_j, q_j) = e^{-2r\tilde{\tau}} P_m^+(k_j, q_j) = \left(\frac{v_L \sum_{s=1}^{k_j} \Lambda_s^m}{P_m^+(k_j, q_j)} \right)^2 P_m^+(k_j, q_j)$$

This finding is in line with DL's double delay result, which characterizes the length of each impasse.[28] We extend this result to the case of a divisible good.

4.2.1 Description of the algorithm

In what follows we describe the construction of the algorithm that characterizes the limit equilibrium outcome as bargaining frictions vanish. The algorithm follows an inductive approach. In the base step, we identify the last impasse. We show that it occurs when only one unit remains, and pin down both the length of the impasse, and the belief at which it occurs. In the inductive step we take an impasse and construct the previous one. Throughout this explanation, we focus directly on the "limit game", in the sense that the low-type seller's behavior is given by the limit function $P_m(\cdot, \cdot)$.[29]

When only one unit remains and the belief is higher than $\bar{q}_m(1)$, the buyer can guaran-

[27] Equation (4c) holds regardless of the particular sequence of offers that characterizes the shift from (k_{j+1}, q_{j+1}) to (k_j, q_j).

[28] The key feature behind DL's double delay result is the symmetry of the steps of the function $P_m^\Delta(1, \cdot)$ around the buyer's valuation (see page 1323 in DL).

[29] By continuity, all results of the "limit game" hold for Δ sufficiently close to zero.

tee a positive continuation payoff by making a universal offer for the last unit. Since his continuation payoff is strictly positive, the usual Coasean forces imply that the buyer has an incentive to speed up trade. Thus, the buyer purchases the remaining unit without delay. In equilibrium, the low-type seller is indifferent between accepting and rejecting the buyer's screening offers. As the price of a screening offer represents the low-type seller's continuation payoff, then $P_m(1,q) = c/m$ for all states $(1,q)$ with $q > \bar{q}_m(1)$.

The limit price $P_m(1,\cdot)$ must be discontinuous at $\bar{q}_m(1)$, with $P_m(1,q) \leq \Lambda_1^m v_L$ for beliefs $q < \bar{q}_m(1)$. If this were not the case, the buyer's continuation payoff at $q < \bar{q}_m(1)$ would be negative, as $W_m(1,\bar{q}_m(1)) = 0$. The discrete jump in $P_m(1,\cdot)$ at $\bar{q}_m(1)$ implies that there must be delay. DL show that the length of the impasse is twice the necessary time elapsed for the buyer's valuation for the low-quality good $v_L\Lambda_1^m$ to be equal to the discounted value of the price of a screening offer after the impasse $(1,\bar{q}(1))$ is resolved. Thus,

$$P_m^-(1,\bar{q}_m(1)) = \left(\frac{v_L\Lambda_1^m}{c/m}\right)^2 c/m = \left(\frac{v_L\Lambda_1^m}{c/m}\right) v_L\Lambda_1^m < v_L\Lambda_1^m.$$

The inequality above implies that the buyer's continuation payoff is strictly positive at any state $(1,q)$ with $q < \bar{q}_m(1)$. Intuitively, the buyer can make a screening offer with a price $P_m^-(1,\bar{q}_m(1))$ that the low-type seller accepts with strictly positive probability. Since the buyer has a positive continuation payoff, the usual Coasean forces kick in, and the state $(1,\bar{q}_m(1))$ is reached without delay. Therefore $P_m(1,q) = P_m^-(1,\bar{q}_m(1))$ for any $q < \bar{q}_m(1)$.[30]

In our model, there must be at least one impasse. If there were none, then the buyer would buy all units without delay, pay c/m for each of them, and obtain a negative payoff. Moreover, the last impasse must occur at state $(1,\bar{q}_m(1))$. Assume instead that the last impasse occurs at state (K,q), with $K > 1$. First, notice that q must be strictly smaller than $\bar{q}_m(1)$. This is because for any $q \geq \bar{q}_m(1)$ the buyer can obtain a strictly positive continuation payoff by making a universal offer for all remaining units, and so there cannot

[30]The arguments presented in these paragraphs also apply (with minor modifications) to the model with an indivisible good (DL). In DL's environment the buyer trades immediately with the low-type seller at a price $\left(\frac{v_L\int_0^1\lambda(z)dz}{c}\right)^2 c$ with positive probability. If there is no immediate trade, an impasse follows. After the impasse, the buyer purchases the whole good from both types at a price c and breaks even.

be delay. Second, after the impasse (K, q) is resolved, the buyer purchases all remaining units without delay and therefore pays c/m for each of them. However, because of divisibility, there exists an alternative course of action that gives the buyer a higher continuation payoff. The buyer can instead first make a universal offer for $K-1$ units. Then, he can make a screening offer $(1, P_m^-(1, \bar{q}_m(1)))$, which is accepted by the low-type seller with probability $(\bar{q}_m(1)-q)/(\hat{q}-q) > 0$. If instead the offer is rejected, the buyer pays c/m for the remaining unit. Thus, divisibility allows the buyer to take advantage of the positive profits from the screening offer before the impasse $(1, \bar{q}_m(1))$, and so he has a profitable deviation.

We now describe the inductive step of our algorithm. We explain how it identifies the penultimate impasse. At the end of this section, we discuss how the argument generalizes to arbitrary impasses. To identify the penultimate impasse, we consider a simple course of action that allows the buyer to take advantage of the positive profits from screening offers before the last impasse. This course of action brings the buyer from any state (K, q) with $K > 1$ and $q < \bar{q}_m(1)$ to the last impasse $(1, \bar{q}_m(1))$, where the buyer's continuation payoff is zero. The buyer first makes the universal offer $\left(K-1, \frac{c}{m}(K-1)\right)$ and then the screening offer $(1, P_m^-(1, \bar{q}_m(1)))$. We let $\check{q}(K)$ be the threshold belief that makes the buyer break even when he follows this course of action:[31]

$$\begin{aligned}\left[(\hat{q}-\check{q}(K))\, v_L + (1-\hat{q})\, v_H\right] \sum_{s=2}^{K} \Lambda_s^m - (1-\check{q}(K))\,(K-1)\frac{c}{m} + \\ (\bar{q}_m(1)-\check{q}(K))\left[\Lambda_1^m v_L - P_m^-(1, \bar{q}_m(1))\right] = 0\end{aligned}$$

A buyer who follows this simple course of action from state (K, q) obtains a negative payoff if $q < \check{q}(K)$ and a positive payoff if $q > \check{q}(K)$.

The penultimate impasse (k_2, q_2) occurs at the quantity $k_2 = \arg\max_{K \geq 2} \{\check{q}(K)\}$ and belief $q_2 = \check{q}(k_2)$. This result relies on the shape of $\check{q}(K)$: it first increases, then reaches a maximum at $\check{q}(k_2)$ and after that it decreases. We show this and the following string of

[31] To ease the exposition, we drop the dependence of $\check{q}(K)$ on m. We also set $\check{q}(K) = 0$ whenever the simple course of action leads to a positive payoff for every belief $q \in [0, \bar{q}_m(1))$. If $\check{q}(K) = 0$ for all $K \geq 2$, then there is no penultimate impasse. In what follows, we assume that $\check{q}(K) > 0$ for some $K \geq 2$.

inequalities, in Appendix A.5:[32]

$$\bar{q}_m(k_2+1) < q_2 < \bar{q}_m(k_2) \tag{5}$$

We explain why the penultimate impasse occurs at (k_2, q_2) through an example with $m = 6$ and $k_2 = 4$. Starting at any state (K, q) with $K \in \{2,3,4\}$ and $\check{q}(K) < q \leq \bar{q}_m(1)$, the buyer must reach $(1, \bar{q}_m(1))$ without delay. This is because the simple course of action yields a positive payoff to the buyer and $\check{q}(2) < \check{q}(3) < \check{q}(4)$.[33] As the price of a screening offer represents the low-type seller's continuation payoff, the limit equilibrium price is

$$P_m(K,q) = (K-1)\,\frac{c}{m} + P_m^-(1, \bar{q}_m(1)). \tag{6}$$

When instead $q < \check{q}(K)$, the limit equilibrium price is (we show this in Appendix A.5):

$$P_m(K,q) = \left(\frac{v_L \sum_{s=1}^{K} \Lambda_s^m}{(K-1)\frac{c}{m} + P_m^-\left(1, \bar{q}_m(1)\right)}\right)^2 \left((K-1)\,\frac{c}{m} + P_m^-\left(1, \bar{q}_m(1)\right)\right) \tag{7}$$

Equations (6) and (7) characterize the limit equilibrium price for $K \in \{2,3,4\}$. The discontinuity point at $(K, \check{q}(K))$ reflects that there is a (potentially off-path) impasse at $(K, \check{q}(K))$. The fact that $P_m(K,q)$ is constant for $q < \check{q}(K)$ reflects that starting at any state (K,q) with $q < \check{q}(K)$, the buyer's optimal course of action is to reach the impasse $(K, \check{q}(K))$ without delay.

Why is there a (potentially off-path) impasse at $(K, \check{q}(K))$ for $K \in \{2,3,4\}$? Consider a buyer at a (potentially off-path) state (K,q) with $K \in \{2,3,4\}$ and $q = \check{q}(K) - \varepsilon$. The buyer cannot reach the state $(1, \bar{q}_m(1))$ immediately, as this would yield a negative payoff. Moreover, since $\check{q}(2) < \check{q}(3) < \check{q}(4)$, the buyer would also obtain a negative payoff from any universal offer. Consequently, the delay must arise when K units remain. This

[32] To see the link between $\check{q}(K+1)$ and $\check{q}(K)$, notice that a simple course of action starting at state $(K+1, q)$ can be decomposed into a universal offer for one unit and a simple course of action from state (K, q). While the threshold belief for the simple course of action from (K,q) is $\check{q}(K)$, the belief that makes the buyer break even with a universal offer is $\bar{q}(K+1)$. The inequalities in (5) are strict because we focus on generic values of the parameters (for details, see Remark 1 on page 20 of the Appendix).

[33] For example, starting at $(3, q)$ with $q > \check{q}(3)$, the buyer can never reach a (potentially off-path) impasse with two units, since $q > \check{q}(3) > \check{q}(2)$.

argument holds for any small ε, so the (potentially off-path) impasse must take place at $(K, \check{q}(K))$. Equations (6) and (7) highlight that DL's double delay result extends to arbitrary impasses: the delay is twice the time necessary to make the low-type seller indifferent between the price after the impasse (equation (6)) and the buyer's valuation for the remaining units of the low-quality good ($v_L \sum_{s=1}^{K} \Lambda_s^m$).

There is double delay at all (potentially off-path) impasses $(K, \check{q}(K))$ for $K \in \{2, 3, 4\}$. Thus, as the buyer's belief approaches $\check{q}(K)$ from the left, the limit equilibrium price is lower than the buyer's valuation for the remaining units of the low-quality good. Therefore, the buyer has a course of action that guarantees a strictly positive payoff whenever $q < \check{q}(K)$. This in turn implies that some impasse must be reached without delay. Using equations (6) and (7) and some straightforward calculations, we show that the best course of action for a buyer in state (K, q) with $K \in \{2, 3, 4\}$ and $q < \check{q}(K)$ is to reach the impasse $(K, \check{q}(K))$ immediately. This is why $P_m(K, q)$ is constant for $q < \check{q}(K)$.

So finally, why is it that, on-path, the penultimate impasse must be at $(4, \check{q}(4))$? Consider a buyer at a state (K, q) with $K > 4$ and $q < \check{q}(4)$ who is contemplating making a universal offer for at least $K - 4$ units. Any such offer can be decomposed into two; first, an offer for exactly $K - 4$ units, and second, an offer for some extra units. The offer for $K - 4$ units takes the buyer to a state $(4, q)$ with $q < \check{q}(4)$. As mentioned above, from that state, it is optimal for the buyer to reach the impasse $(4, \check{q}(4))$ immediately. Therefore, of all universal offers for at least $K - 4$ units, the optimal one is for *exactly* $K - 4$ units. Then, starting at a state (K, q) with $K \geq 4$ and $q < \check{q}(4)$, the buyer never reaches a state (K', q') with $K' \in \{1, 2, 3\}$ and $q' < \check{q}(4)$. This shows two things. First, the penultimate impasse cannot be at $(2, \check{q}(2))$ or $(3, \check{q}(3))$, since $\check{q}(2) < \check{q}(3) < \check{q}(4)$. Second, there must be a penultimate impasse and this impasse cannot arise when five or six units remain. Otherwise, there would be a state (K, q) with $K \in \{5, 6\}$ and $q < \check{q}(4)$ from which the buyer immediately takes advantage of the low price associated with the last impasse, instead of first taking advantage of the low price associated with the impasse $(4, \check{q}(4))$.

Our algorithm proceeds by induction by taking the impasse (k_j, q_j) and identifying the previous impasse (k_{j+1}, q_{j+1}). To do this, we construct a simple course of action, analogous to the one before, where the buyer takes advantage of the positive profits from screening offers before the impasse (k_j, q_j). This course of action brings the buyer from any state (K, q) with $K > k_j$ and $q < q_j$ to the impasse (k_j, q_j). As before, we define for each $K > k_j$ the threshold belief such that the buyer breaks even following this simple course of action. The previous impasse occurs at (k_{j+1}, q_{j+1}), where k_{j+1} is the number of units that maximizes the threshold belief, and q_{j+1} is the threshold belief when k_{j+1} units remain. The algorithm ends in finitely many steps and there are at most m impasses.

4.3 Arbitrarily divisible good

In this subsection we describe the limit of the equilibrium outcome identified in Proposition 3 as the number of units m grows to infinity. In order to keep track of the number of units, we add the index m to the collection of quantities and beliefs $\left\{\left(k_j^m, q_j^m\right)\right\}_{j=1}^{J_m}$ that characterize impasses and let J_m denote the number of impasses when the good is divided into m units. We also let $z_j^m = k_j^m/m$ represent the fraction of the good left for trade at impasse j. Thus, we denote impasses by $\left\{\left(z_j^m, q_j^m\right)\right\}_{j=1}^{J_m}$ in this subsection.

Proposition 4. Impasses for an Arbitrarily Divisible Good. *The limit equilibrium outcome satisfies*

$$\lim_{m\to\infty}\left(\max\left\{z_j^m - z_{j-1}^m\right\}_{j=2}^{J_m}\right) = 0 \tag{8a}$$

$$\lim_{m\to\infty}\left(\max\left\{q_{j-1}^m - q_j^m\right\}_{j=2}^{J_m}\right) = 0 \tag{8b}$$

$$\lim_{m\to\infty} z_{J_m}^m = 1 \tag{8c}$$

$$\lim_{m\to\infty}\left(\max\left\{\left|q_j^m - \bar{q}\left(z_j^m\right)\right|\right\}_{j=1}^{J_m}\right) = 0 \tag{8d}$$

$$\lim_{m\to\infty}\left(\max\left\{\left|P_m^-\left(mz_j^m, q_j^m\right) - v_L\int_0^{z_j^m}\lambda(z)dz\right|\right\}_{j=1}^{J_m}\right) = 0 \tag{8e}$$

$$\lim_{m\to\infty}\left(\max\left\{\left|P_m^+\left(mz_j^m, q_j^m\right) - v_L\int_0^{z_j^m}\lambda(z)dz\right|\right\}_{j=1}^{J_m}\right) = 0 \tag{8f}$$

See Appendix A.6 for the proof.

Proposition 4 directly leads to Theorem 1. As the good becomes arbitrarily divisible, the number of impasses goes to infinity. The fraction that the buyer purchases through each universal offer shrinks to zero (equation (8a)). The change in the buyer's belief between two consecutive impasses also shrinks to zero (equation (8b)). Thus, in the limit, the high-quality good is traded smoothly over time, and beliefs also evolve continuously. The first impasse takes place with the whole good left for trade (equation (8c)). Let $z(\tau) = \lim_{m\to\infty} z_m(\tau)$ and $q(\tau) = \lim_{m\to\infty} q_m(\tau)$ denote respectively the limit fraction of the good left and the limit belief at time elapsed τ. Equation (8d) shows that these functions are linked: $q(\tau) = \bar{q}\left(z(\tau)\right)$. Furthermore, at any screening offer for a fraction $z(\tau)$ of the good, the buyer offers a price $v_L \int_0^{z(\tau)} \lambda(z)dz$, and so he breaks even (equations (8e) and (8f)).

The limit equilibrium outcome then coincides with the artificial pattern of trade described in Section 3. At time zero the buyer makes a screening offer for the whole good at price $v_L \int_0^1 \lambda(z)dz$. The low-type seller accepts this offer with probability $\bar{q}_m(1)/\hat{q}$. Then, the buyer continuously makes both universal and screening offers. The low-type seller's indifference between accepting different screening offers implies that the fraction $z(\tau)$ must satisfy equation (2). This pins down the pattern of trade in the limit: $z(\tau) = z^*(\tau)$ and $q(\tau) = q^*(\tau)$, as stated in Theorem 1.

The proof of Proposition 4 relies on the key properties identified in Proposition 3. Equation (8d) directly results from the condition $\bar{q}_m(k_j^m + 1) < q_j^m < \bar{q}_m(k_j^m)$ in Proposition 3. As m goes to infinity, $\frac{k_j^m+1}{m} \to \frac{k_j^m}{m} = z_j^m$. We next explain, through a unified argument, why equations (8a), (8b), (8e) and (8f) hold true.

We say that an impasse (z_j^m, q_j^m) is short whenever $P_m^-\left(mz_j^m, q_j^m\right)$ and $P_m^+\left(mz_j^m, q_j^m\right)$ are close (and so both are close to the valuation $v_L \int_0^{z_j^m} \lambda(z)dz$).[34] The buyer makes a profit with a screening offer before each impasse. Whenever an impasse is short, this profit is low. The driving force behind Proposition 4 is that whenever m is large, if an impasse (z_j^m, q_j^m) is short, then the previous impasse (z_{j+1}^m, q_{j+1}^m) must also be short. Moreover, the fraction $z_{j+1}^m - z_j^m$ that the buyer purchases between these two impasses

[34] Proposition 3 guarantees that $P_m^-\left(mz_j^m, q_j^m\right) < v_L \int_0^{z_j^m} \lambda(z)dz < P_m^+\left(mz_j^m, q_j^m\right)$ for every impasse (z_j^m, q_j^m). Whenever $P_m^-\left(mz_j^m, q_j^m\right)$ and $P_m^+\left(mz_j^m, q_j^m\right)$ are close and different from zero, their ratio is close to one. The low-type seller must be indifferent between accepting and rejecting screening offers, so it takes a short time for the price to go from $P_m^-\left(mz_j^m, q_j^m\right)$ to $P_m^+\left(mz_j^m, q_j^m\right)$. In this sense the impasse is short.

must be small. To show this, we link two consecutive impasses (z^m_{j+1}, q^m_{j+1}) and (z^m_j, q^m_j). The buyer obtains a zero continuation payoff at every impasse. Thus, the difference $W_m(mz^m_{j+1}, q^m_{j+1}) - W_m(mz^m_j, q^m_j)$, which we express in equation (9), is also zero:

$$\overbrace{\left(\hat{q} - q^m_{j+1}\right)\left[\int_{z^m_j}^{z^m_{j+1}} [\lambda(z)v_L - c]\, dz\right] + (1-\hat{q})\left[\int_{z^m_j}^{z^m_{j+1}} [\lambda(z)v_H - c]\, dz\right]}^{(*)} \\ + \underbrace{\left(q^m_j - q^m_{j+1}\right)\left[\int_0^{z^m_j} \lambda(z)v_L dz - P^-_m\left(mz^m_j, q^m_j\right)\right]}_{(**)} = 0 \tag{9}$$

From one impasse to the next one, the buyer makes a loss with a universal offer $(*)$, and a profit with a screening offer $(**)$. This profit is close to zero since the price $P^-_m\left(mz^m_j, q^m_j\right)$ of the screening offer is close to the buyer's valuation. Therefore, the loss associated to the universal offer must also be close to zero, which can only happen if z^m_j is close to z^m_{j+1}.[35]

We next show that the previous impasse (z^m_{j+1}, q^m_{j+1}) must also be short. Equations (4b) and (4c) in Proposition 3 imply that:

$$P^-_m\left(mz^m_{j+1}, q^m_{j+1}\right) = \left(\frac{v_L \int_0^{z^m_{j+1}} \lambda(z)dz}{\left(z^m_{j+1} - z^m_j\right)c + P^-_m\left(mz^m_j, q^m_j\right)}\right)^2 P^+_m\left(mz^m_{j+1}, q^m_{j+1}\right)$$

Since z^m_j and z^m_{j+1} are close and the price $P^-_m\left(mz^m_j, q^m_j\right)$ is close to the buyer's valuation, then the first term on the right hand side is close to one.

We argue next that the last impasse must be short as m grows large. The last impasse occurs when only one unit remains. Equations (4a) and (4b) in Proposition 3 imply that $P^-_m(1, \bar{q}_m(1))$ and $P^+_m(1, \bar{q}_m(1))$ both converge to zero as m grows large.

To complete the argument it remains to be shown that there are no cumulative effects in the sense that if one impasse is short, all previous ones must also be short. This is a technical part that we present in Appendix A.6.

[35]Equation (8d) implies that for m large, the buyer is close to breaking even if he makes a universal offer for an arbitrarily small unit at state $\left(mz^m_{j+1}, q^m_{j+1}\right)$. Since gains from trade are decreasing, any non-negligible universal offer would lead to a loss bounded away from zero.

Finally, the intuition behind equation (8c) is simple. If it does not hold, then for large m the buyer reaches the first impasse after purchasing a strictly positive fraction of the good though a universal offer. This offer yields a loss to the buyer. At each impasse the price of the screening offer is close to the buyer's valuation, so the buyer's profit from this offer is negligible. Therefore, if equation (8c) is violated, the buyer obtains a negative continuation payoff at the beginning, which can never happen.

5. Extensions

In our first extension, we study the limit equilibrium outcome when equation (1) does not hold. Equation (1) reflects an extreme form of adverse selection: under the prior belief, the buyer's expected valuation from any fraction of the good exceeds the high-type seller's cost. Therefore, the buyer needs to screen the seller even to purchase the most valuable fraction of the good.

We first assume that $\left[\hat{\beta}v_H + (1-\hat{\beta})v_L\right]\lambda(\bar{z}) = c$ for some $\bar{z} \in (0,1]$, so the buyer obtains a positive payoff if he buys any infinitesimal unit $z \in [\bar{z},1]$ through a universal offer. Our analysis directly extends to this case.[36] In the limit equilibrium outcome, the buyer purchases the first fraction $1-\bar{z}$ from both types without delay, paying $c(1-\bar{z})$. The environment after the units $z \in [\bar{z},1]$ are traded resembles that from our baseline model. Theorem 1 pins down the pattern of trade for the remaining fraction $\bar{z}$. Similarly to the case when equation (1) holds, divisibility is detrimental to the buyer. Although he obtains a profit from the units $z \in [\bar{z},1]$, he must pay the high-type seller's cost for them. He then obtains a zero profit from the remaining units. Furthermore, like in the benchmark model, the high-quality good is traded smoothly, but only for the units $z \in [0,\bar{z}]$.

We next assume that $\left[\hat{\beta}v_H + (1-\hat{\beta})v_L\right]\lambda(0) \geq 0$. In this case, the buyer obtains a positive payoff if he buys any fraction through a universal offer, so the standard Coasean forces apply. For any m, as bargaining frictions vanish, the buyer purchases the whole

[36]The proof of the characterization of the limit equilibrium outcome in this case is analogous to the proof of Theorem 1 so we omit it.

good from both types without delay and pays c.[37]

In our second extension we assume that $\lambda(\cdot)$ is either constant or strictly decreasing, which correspond, respectively, to constant gains from trade or increasing gains from trade. In either of these cases divisibility plays no role: for any m the buyer only makes offers (m, p) with $p \leq c$ in equilibrium. The equilibrium outcome is identical to the one when the good is indivisible. Proposition 5 formalizes this.

Proposition 5. Constant or increasing gains from trade. *When gains from trade are constant or increasing, the buyer only makes screening offers in equilibrium.*

See Appendix A.8 for the proof.

The intuition behind Proposition 5 is simple. Whenever the buyer is happy to pay the high-type seller's cost for some units, then he is also happy to pay that cost for subsequent units. As gains from trade are constant or increasing, those subsequent units are at least as valuable as the previous ones.

The limit equilibrium outcome with constant (or increasing) gains from trade differs starkly from that when gains from trade are decreasing. This difference highlights that there is a discontinuity in the shape of the gains from trade. Consider a family of strictly increasing functions $\lambda_n(\cdot)$ converging (pointwise) to a constant function $\lambda(\cdot)$. For every n, the high-quality good is traded smoothly, since gains from trade are decreasing (Theorem 1). Instead, when $\lambda(\cdot)$ is constant, in the limit equilibrium outcome the high-quality good is sold all at once (this follows directly from Proposition 5). Because of this discontinuity, one should be wary in applying Theorem 1 in environments where gains from trade may be close to constant. Instead, Theorem 1 provides clear predictions when there are strong economic forces that lead to decreasing gains from trade (e.g. the buyer benefits from diversifying his portfolio).

[37]In our model, the buyer's valuation of the last infinitesimal unit of the low quality good $\lambda(0)v_L$ exceeds the seller's cost c_L, which we normalize to zero. An alternative extension of our model would be to set $\lambda(0)v_L = c_L > 0$. In such environment, Proposition 3 holds for any number of units m. We do not know whether Proposition 4 holds in this environment (some steps in the current proof do not extend to it). However, we can show that most of the qualitative features of the limit equilibrium outcome continue to hold: as the good becomes arbitrarily divisible, the number of impasses goes to infinity and the fraction of the high quality good left for trade converges asymptotically to zero.

6. Conclusion

We study bargaining over a divisible good. We characterize the limit equilibrium outcome as bargaining frictions vanish and the good becomes arbitrarily divisible. Our model generates novel and testable predictions for dynamic markets with adverse selection. When gains from trade are constant or increasing, the pattern of trade is identical to that of parties negotiating over an indivisible good. Time on the market is the main signaling device and the buyer keeps some of his bargaining power. On the other hand, when there are decreasing gains from trade, the high-quality good is traded smoothly over time and the buyer loses all the bargaining power in the limit.

In this paper we first let the time between offers shrink to zero and we then let the number of units grow to infinity. This order of limits both reflects many real-world environments and allows for tractability. The tools developed in this paper do not allow for a complete characterization of the pattern of trade if we instead invert the order of limits. However, one of our main findings extends to that environment. If we invert the order of limits, the number of transactions of the high-quality good (and the number of impasses) must also grow without bound.[38]

Our model relies on some simplifying assumptions that make the analysis tractable. First, we assume that the quality of the good can take only two values. Our results extend to a model with finitely many types provided that the buyer's valuation for a good of any intermediate quality is sufficiently close to his valuation for the good of the highest quality. Future research can shed further light on bargaining with divisibility and many types.

Second, we focus on the benchmark case in which all learning is strategic: the buyer learns about the quality of the good only through the seller's behavior. Although this assumption is reasonable in a number of important applications, there are many markets

[38]The intuition behind this is simple. Assume instead that in the limit there is a finite number of transactions, and take the last transaction for a positive fraction of the good. Consider the last impasse before this transaction (such an impasse must exist; otherwise the buyer would pay the high-type seller's cost for the whole good and obtain a negative payoff). At this impasse, the buyer's payoff is zero and his belief is such that he breaks even if he makes a universal offer for the remaining fraction of the good. But then the buyer has a profitable deviation; because of decreasing gains from trade, he obtains a positive payoff by making a universal offer for less than the remaining fraction of the good.

where buyers obtain information as they purchase parts of the good.[39] This new channel of endogenous arrival of information opens many relevant paths for future research: information could be public or private, perfect or imperfect. A natural extension of our model is to allow for the buyer to receive public and imperfect information as he purchases parts of the good. We have constructed examples with a good divided into a finite number of units that suggest that some of our findings extend to such environment. In equilibrium the buyer alternates between screening and universal offers. Furthermore, while we do not have a general algorithm that characterizes the limit equilibrium outcome as bargaining frictions vanish, we can show that the pattern of trade is characterized by impasses mediated by phases of fast trade.

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A. Appendix for Online Publication for "Bargaining over a Divisible Good in the Market for Lemons"

A.1 Proof of Lemma 1

Proof. We show that (a) holds for the weaker solution concept of PBE. For any $K \in \{1, \dots, m\}$ and for any PBE $(\sigma_B, (\sigma_L, \sigma_H), \beta)$, let $H_K(\sigma_B, (\sigma_L, \sigma_H), \beta)$ denote the set of histories h^t with $K(h^t) = K$ and $\beta(h^t) = 0$.

We show first that (a) holds when only one unit remains. Let $\bar{u}_L$ denote the supremum, over all PBE $(\sigma_B, (\sigma_L, \sigma_H), \beta)$, of the low-type seller's continuation payoff at histories $h^t \in H_1(\sigma_B, (\sigma_L, \sigma_H), \beta)$. Assume towards a contradiction that $\bar{u}_L > 0$ and take $\varepsilon = \left(\frac{1-\delta}{2}\right)\bar{u}_L$. There must exist a PBE $(\sigma_B, (\sigma_L, \sigma_H), \beta)$ and a history $\bar{h}^t \in H_1(\sigma_B, (\sigma_L, \sigma_H), \beta)$ at which the buyer offers $\varphi_t = (1, p)$ for some $p \in [\bar{u}_L - \varepsilon, \bar{u}_L]$. The low-type seller must accept this offer with probability one. To see why, notice that if the low-type seller rejects this offer with positive probability, then $(\bar{h}^t, (\varphi_t, R)) \in H_1(\sigma_B, (\sigma_L, \sigma_H), \beta)$ and therefore the low-type seller's continuation payoff is at most $\bar{u}_L$. But then, since $\bar{u}_L - \varepsilon > \delta \bar{u}_L$, it is not optimal for the low-type seller to reject φ_t. For the same reason, the low-type seller must accept the offer $\varphi_t' = \left(1, \bar{u}_L - \frac{3}{2}\varepsilon\right)$ with probability one. Thus, the buyer has a profitable deviation at $\bar{h}^t$ since he strictly prefers the offer φ_t' to φ_t.

We show next that (a) holds for any number of remaining units K. We proceed by induction. Fix $K \in \{2, \dots, m\}$ and assume that for any PBE $(\sigma_B, (\sigma_L, \sigma_H), \beta)$ and for any $h^t \in H_1(\sigma_B, (\sigma_L, \sigma_H), \beta) \cup \dots \cup H_{K-1}(\sigma_B, (\sigma_L, \sigma_H), \beta)$, the low-type seller"s continuation payoff is zero. Again, let $\bar{u}_L$ denote the supremum, over all PBE $(\sigma_B, (\sigma_L, \sigma_H), \beta)$, of the low-type seller's continuation payoff at histories $h^t \in H_K(\sigma_B, (\sigma_L, \sigma_H), \beta)$. Towards a contradiction, assume that $\bar{u}_L > 0$ and take $\varepsilon = \left(\frac{1-\delta}{2}\right)\bar{u}_L$. There must exist a PBE $(\sigma_B, (\sigma_L, \sigma_H), \beta)$ and a history $\bar{h}^t \in H_K(\sigma_B, (\sigma_L, \sigma_H), \beta)$ at which the buyer offers $\varphi_t = (k, p)$ for some $p \in [\bar{u}_L - \varepsilon, \bar{u}_L]$ and some $k \leq K$. Using the induction hypothesis and an argument similar to the one presented in the previous paragraph, we conclude that the low-type seller must accept this offer with probability one. However, the same is true for the offer $\varphi_t' = \left(k, \bar{u}_L - \frac{3}{2}\varepsilon\right)$ which is, therefore, strictly preferred to φ_t. Again, this shows that the buyer has a profitable deviation at $\bar{h}^t$ and concludes the proof of part (a) of Lemma 1.

We show (b) by contradiction. Assume that there exist two histories h^t and $\tilde{h}^{t'}$ with the same state variables but with $V_B(h^t) < V_B\left(\tilde{h}^{t'}\right)$. The buyer then has a profitable deviation after history h^t. He can choose the same actions as he chooses after history $\tilde{h}^{t'}$. Since the seller's strategy depends only on state variables, then he reacts as he does after history $\tilde{h}^{t'}$, and so the buyer's continuation payoff increases.

We show (c) by contradiction. Assume instead that there is a history h^t where the high-type seller obtains a positive continuation payoff: $V_H(h^t) > 0$. Over all histories with positive continuation payoffs, pick those with the smallest number of remaining units $\underline{K} = \min\left\{K(h^t) : V_H(h^t) > 0\right\}$. Let $\alpha = \sup\left\{V_H(h^t) : K(h^t) = \underline{K}\right\}$ denote an upper bound for the high-type seller's continuation payoff when only $\underline{K}$ units remain. Finally, let $\varepsilon \equiv (1-\delta)\alpha/3$.

There must exist a history h^t with $K(h^t) = \underline{K}$ at which the buyer makes an offer (k, p) that the high-type seller accepts, and the offer satisfies $1 \leq k \leq \underline{K}$ and $p > \frac{c}{m}k + \alpha - \varepsilon$. This in turn implies that the low-type seller also accepts this offer (otherwise, by Lemma 1(a), he gets a total payoff of zero). Consider instead the following deviation by the buyer; he offers $\left(k, \frac{c}{m}k + \alpha - \varepsilon\right)$. If the high-type seller rejects this offer, he obtains a continuation payoff of at most $\delta\alpha < \alpha - \varepsilon$, so he accepts it. For the same reason as above, the low-type seller also accepts this offer. Both the original offer and the deviation lead to the same state variables, and therefore to the same continuation payoff to the buyer, as shown in Lemma 1(b). This implies that the deviation is profitable. This shows part (c) of Lemma 1.

Consider next part (d) of Lemma 1. Whenever $\beta(h^t) = 0$, the result follows immediately from Lemma 1(a). Otherwise, the zero bound on the continuation payoff for the high type seller directly implies a $\frac{c}{m}K(h^t)$ upper bound for the continuation payoff for the low-type seller. ■

A.2 Proof of Lemma 2

Proof. In the case $\beta(h^t) = 0$ all units are traded in the first period (this follows immediately from Lemma 1(*a*)). Assume instead that $\beta(h^t) > 0$ and consider an offer $\varphi_t = (k, p)$ with $k < K(h^t)$ and $p < \frac{c}{m}k$. We show that such an offer is not accepted with positive probability. By contradiction, assume that this offer is accepted with positive probability. A high-type seller would never accept such an offer, so it must be the low-type seller who accepts this offer with probability $\sigma_L^t(h^t, \varphi_t) > 0$.

A rejection then leads to a posterior $\beta' \in \big(\beta(h^t), 1\big)$. Whenever the low-type seller accepts, the buyer immediately learns that the seller is of low type. Then, in the following period all remaining units are traded, at zero cost. The buyer obtains the following payoff from this offer:

$$[1-\beta(h^t)]\sigma_L^t(h^t,\varphi_t)\left[\sum_{s=K(h^t)-k+1}^{K(h^t)} \Lambda_s^m v_L - p + \delta \sum_{s=1}^{K(h^t)-k} \Lambda_s^m v_L\right] + \left[1-\beta(h^t)\left(1-\sigma_L^t(h^t,\varphi_t)\right)\right] V_L(\beta', K)$$

Consider instead an offer to pay p in exchange for *all* remaining units. If the low-type seller accepts, he obtains the same payoff as from accepting the previous offer. Moreover, because of stationarity, a rejection leads to the same belief β' as before. Then, the low-type seller accepts this offer with the same probability as the previous offer. The buyer, however, obtains the following *higher* payoff from this offer:

$$[1-\beta(h^t)]\sigma_L^t(h^t,\varphi_t)\left[\sum_{s=K(h^t)-k+1}^{K(h^t)} \Lambda_s^m v_L - p + \sum_{s=1}^{K(h^t)-k} \Lambda_s^m v_L\right] + \left[1-\beta(h^t)\left(1-\sigma_L^t(h^t,\varphi_t)\right)\right] V_L(\beta', K)$$

Then, if an offer for $k < K(h^t)$ remaining units was accepted with positive probability, the buyer would rather make an offer for all remaining units, so there would be a profitable deviation. ■

A.3 Proof of Proposition 2

In this proof we consider a good divided into a fixed number of units equal to m a fixed period length equal to Δ. We thus suppress the dependence of all variables on m and Δ.

The proof is divided into two parts. In Part A we define the notion of a consistent quadruplet $(\mathcal{V}_L, P, W, y)$ of intertwined functions. We show that whenever a consistent quadruplet $(\mathcal{V}_L, P, W, y)$ exists, then a stationary PBE must exist. Our proof is constructive: we derive equilibrium strategies and beliefs from the consistent quadruplet. In Part B we construct a consistent quadruplet $(\mathcal{V}_L, P, W, y)$.

Part A. The consistent quadruplet $(\mathcal{V}_L, P, W, y)$

We first describe the components of the quadruplet $(\mathcal{V}_L, P, W, y)$. The function $\mathcal{V}_L(K, q)$: $\{1, \ldots, m\} \times [0, \hat{q}] \to \mathbb{R}$ determines the strategy of the low-type seller, as described in the definition of stationary PBE. The function $P(K, q) : \{1, \ldots, m\} \times [0, \hat{q}] \to \mathbb{R}$ pins down the screening offer $(K, P(K, q))$ that induces (transformed) posterior belief q if rejected. The function $W(K, q) : \{1, \ldots, m\} \times [0, \hat{q}] \to \mathbb{R}$ represents the buyer's (normalized) continuation payoff. Finally, the function $y(K, q) : \{1, \ldots, m\} \times [0, \hat{q}] \to \{1, \ldots, m\} \cup [0, \hat{q}]$ specifies the offers that the buyer makes on the equilibrium path.

Part A contains four steps. The first three define the notion of a consistent quadruplet $(\mathcal{V}_L, P, W, y)$. In step 1 we derive the function P from the function $\mathcal{V}_L$. In step 2 we turn to the buyer's optimization problem. We take as given the behavior of the low-type seller, which is summarized by P. We define the buyer's value function W and his best response correspondence. From this best response correspondence, in step 3 we select the offer $y(K, q)$ that the buyer makes in state (K, q). We construct a candidate value function $\mathcal{V}'_L$ for the low-type seller from the functions y and P. Finally, we say that the quadruplet $(\mathcal{V}_L, P, W, y)$ is *consistent* if $\mathcal{V}'_L = \mathcal{V}_L$.

In step 4 we construct strategies from the consistent quadruplet $(\mathcal{V}_L, P, W, y)$ and show that these strategies (together with appropriate beliefs) form a stationary PBE.

Step 1. From $\mathcal{V}_L$ to P. Consider a (left-continuous) candidate function $\mathcal{V}_L$ with $0 \leq \mathcal{V}_L(K, q) \leq \frac{c}{m}K$ for all (K, q). This function determines the low-type seller's behavior, following the definition of stationary PBE.[40] This same definition also pins down the high-type seller behavior: he accepts any offer for k units if and only if he receives in exchange a payment greater or equal than $\frac{c}{m}k$.

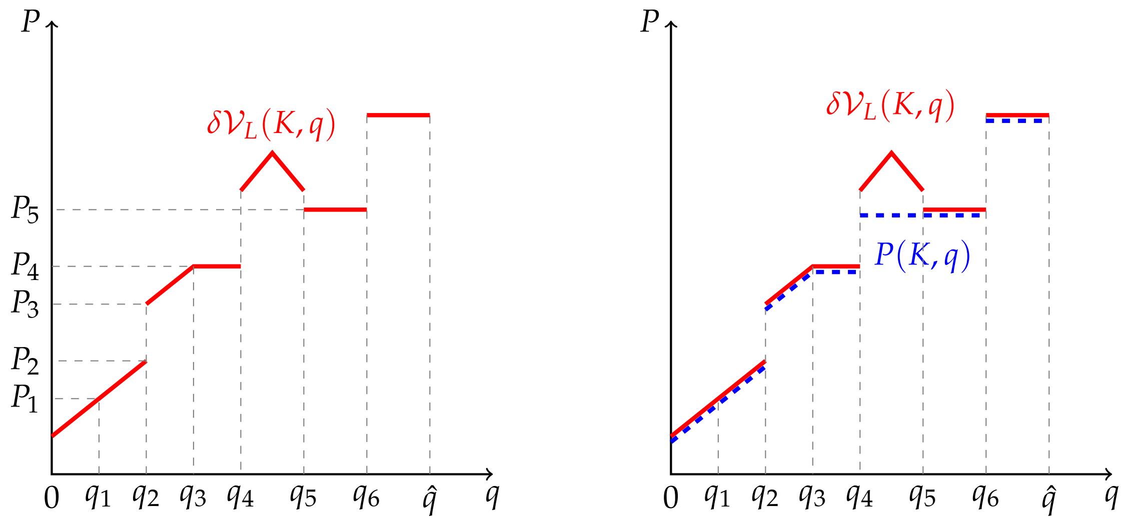
We study the buyer's best response to the seller's behavior implied by $\mathcal{V}_L(K,q)$. We can restrict attention to two types of offers: universal and screening. Universal offers are simple: the buyer offers a payment $\frac{c}{m}k$ for some (or all) remaining units $k \leq K$, both sellers accept and beliefs do not change.

Screening offers involve both a price and a transformed posterior belief. A price induces a probability of acceptance, which in turn leads to a transformed posterior belief after the offer is rejected. As we show below, different prices may induce the same posterior. Moreover, there may be some posteriors that no price can induce. We define a modified problem where the buyer who starts a period with a (transformed) belief $q \in [0, \hat{q}]$ can induce any (transformed) posterior belief $q' \in [q, \hat{q}]$ by choosing a unique price $P(K, q')$. We show in step 4 that solutions to the modified problem coincide with those of the original one.

We first illustrate how we derive $P(K,q)$ from $\mathcal{V}_L(K,q)$ and then provide the formal definition of $P(K,q)$. Consider the function $\delta\mathcal{V}_L(K,q)$ shown in Figure 4(a). It is simple to see that the price $P_1 = \delta\mathcal{V}_L(K,q_1)$ induces posterior belief q_1. This is because the function $\delta\mathcal{V}_L(K,q)$ lies above P_1 for posteriors greater than q_1. In fact, obtaining $P(K,q)$ would be straightforward if $\mathcal{V}_L(K,q)$ was continuous and strictly increasing. However, consider for example posterior belief q_2, which is induced by all prices in the range $[P_2, P_3]$. The buyer's preferred price in that range is the lowest: P_2; and thus we set $P(K,q_2) = P_2$.

The set of induced beliefs may be non-convex. The price P_4 induces posterior belief q_4, but no price induces posterior beliefs on the range $[q_3, q_4)$. To restore convexity, in the modified problem we allow the buyer to induce any belief $q \in [q_3, q_4)$ by paying the price $P(K,q) = P_4$. Similarly, the buyer cannot induce posterior beliefs in the range (q_4, q_6). We allow the buyer to induce any belief $q \in (q_4, q_6)$ by paying the price $P(K,q) = P_5$. Differently than before, $P(K,q) < \delta\mathcal{V}_L(K,q)$ for the interval $q \in (q_4, q_5]$.

[40] The function $\mathcal{V}_L(K,q)$ maps one-to-one to a function $\mathcal{V}_L(K,\beta) : m \times [\hat{\beta}, 1] \to \mathbb{R}$. The definition of stationary PBE pins down the behavior of the low-type seller through the function $\mathcal{V}_L(K,\beta)$.



(a) Multiplicity and non-convexity

(b) The function $P(K,q)$

Figure 4: Derivation of $P(K,q)$ from $\mathcal{V}(K,q)$

Formally, we let $P(K,q)$ be the largest weakly increasing function below $\delta\mathcal{V}_L(K,q)$. As an example, the dashed line in Figure 4(b) depicts the function $P(K,q)$ derived from $\delta\mathcal{V}_L(K,q)$ in Figure 4(a). Whenever the buyer can induce a posterior q but cannot induce posteriors in some range $(q-\eta, q)$, our definition implies that $P(k,q') = \delta\mathcal{V}_L(K,q)$ for all $q' \in (q-\eta, q)$. By doing so, the function $P(K,q)$ becomes flat in some region. Claim 1 in step 4 shows that the buyer never chooses interior points in flat regions, which guarantees that the solutions to the modified problem coincide with those of the original one.

Step 2. From P to W. The buyer's modified problem. We now formalize the buyer's (modified) dynamic optimization problem. With a slight abuse of notation, let $V_B(K,q)$ denote the buyer's continuation payoff when the state is (K,q). For convenience, we work directly with the buyer's *normalized* continuation payoff

$$W(K,q) \equiv (1-q)V_B(K,q).$$

We set $W(0,q) = 0$ and

$$W(K,\hat{q}) = (1-\hat{q})\left[\left(\sum_{s=1}^{K} \Lambda_s^m\right) v_H - \frac{c}{m}K\right].$$

For all other cases, we define $W(K,q)$ recursively by:

$$W(K,q) = \max\Bigg\{ \max_{q'\in[q,\hat{q}]} \overbrace{(q'-q)\left[\left(\sum_{s=1}^{K}\Lambda_s^m\right)v_L - P(K,q')\right] + \delta W(K,q')}^{(*)\text{ Screening. Offer } P(K,q') \text{ for } K \text{ units. If rejected, induced belief is } q'} \;,$$
$$\max_{0\le k\le K-1} \underbrace{\left\{\left(\sum_{s=k+1}^{K}\Lambda_s^m\right)\left[(\hat{q}-q)\,v_L + (1-\hat{q})\,v_H\right] - (1-q)\frac{c}{m}(K-k) + \delta W(k,q)\right\}}_{(**)\text{ Universal offer. Request next } K-k \text{ units in exchange for payment } \frac{c}{m}(K-k)} \Bigg\} \tag{10}$$

The first component $(*)$ of equation (10) provides the continuation payoff when the buyer induces belief q' through a screening offer. The second component $(**)$ of equation (10) provides the continuation payoff when the buyer makes a universal offer for $K-k$ units. The buyer compares the value of the best screening offer (optimal q') with the value of the best universal offer (optimal k) to choose which kind of offer to make.[41]

Equation (10) defines the buyer's *modified* problem. When the state is (K,q) with $q \in [0,\hat{q})$ we allow the buyer to induce any state (K,q') with $q' \geq q$ by making the screening offer $(K, P(K,q'))$. This includes states that cannot be reached in the original game, like (K,q_5) in Figure 4.

Let $Y(K,q)$ denote the set of solutions to the problem in equation (10). A screening offer that induces posterior q' is of the form $(K, P(K,q'))$. When such offer is optimal, we let $q' \in Y(K,q)$. A universal offer for $K-k$ units is of the form $\left(K-k, \frac{c}{m}(K-k)\right)$. When such offer is optimal, we let $k \in Y(K,q)$.

Step 3. From P and W to y and $\mathcal{V}_L'$. The notion of consistent quadruplet. We combine the low-type seller's behavior, implicit in P, with the buyer's optimal behavior to construct a *candidate* value function $\mathcal{V}_L'(K,q)$ for the low-type seller. Let $\mathcal{V}_L'(K,q)$ be defined recursively by:

$$\mathcal{V}_L'(K,q) = \min\left\{ \min_{q'\in Y(K,q)} P\left(K,q'\right), \min_{k\in Y(K,q)} \frac{c}{m}(K-k) + \delta\mathcal{V}_L'(k,q) \right\} \tag{11}$$

As equation (11) shows, we construct $\mathcal{V}_L'$ by always selecting the offer that minimizes

[41] The buyer's continuation payoff is always positive, so his individual rationality constraint is satisfied. To see this, note that the buyer can always choose $q' = q$ in equation (10).

the low-type seller's continuation payoff from all of the buyer's optimal choices $Y(K,q)$. Let $y(K,q) \in Y(K,q)$ denote the buyer's choice that solves (11). There may be many solutions to (11), but if so, one of them is universal.[42] In such case, we let $y(K,q)$ be the universal offer associated to the lowest k.

Finally, we say that a quadruplet $(\mathcal{V}_L, P, W, y)$ is *consistent* if its components are linked as described in steps 1 to 3 and if the derived $\mathcal{V}_L'$ satisfies $\mathcal{V}_L' = \mathcal{V}_L$.

Step 4. From the consistent quadruplet $(\mathcal{V}_L, P, W, y)$ to a stationary PBE.

a. Definition of strategies and beliefs. Fix a consistent quadruplet $(\mathcal{V}_L, P, W, y)$. Our definition of stationary PBE, together with $\mathcal{V}_L$, fully pins down the seller's strategy. Both types accept with probability one any offer (k,p) with $p \geq \frac{c}{m}k$. The high-type seller rejects offers (k,p) with $p < \frac{c}{m}k$ with probability one, while the low-type seller accepts them with probability pinned down by $\mathcal{V}_L$.

We next specify the buyer's strategy and beliefs. We first define for each t a set of histories $\widehat{H}^t$ that is not reached on the equilibrium path. We say that $h^t \in \widehat{H}^t$ whenever h^t contains either 1) a rejected offer (k,p) with $p \geq \frac{c}{m}k$, or 2) an accepted partial offer. Whenever $h^t \in \widehat{H}^t$, we let the buyer assign probability zero to the seller being of high type. Also, we let the buyer offer a payment of zero for all remaining units after any history $h^t \in \widehat{H}^t$.[43]

If instead $h^t \notin \widehat{H}^t$, the buyer's offer depends on the state $\left(K(h^t), q(h^t)\right)$ and on the actions (φ_{t-1}, a_{t-1}) in $t-1$. The buyer's strategy and beliefs are as follows:

1. If $(\varphi_{t-1}, a_{t-1}) = ((k,p), A)$ with $p \geq \frac{c}{m}k$, then the belief is unchanged: $q(h^t) = q(h^{t-1})$. The buyer makes the offer implied by $y(K(h^t), q(h^t))$.

2. If $(\varphi_{t-1}, a_{t-1}) = ((k,p), R)$ with $p < \frac{c}{m}k$, then

    a. If $p \leq P\left(K(h^{t-1}), q(h^{t-1})\right)$, then the belief is unchanged: $q(h^t) = q(h^{t-1})$. The buyer makes the offer implied by $y(K(h^t), q(h^t))$.

[42] To see why, assume that $P(K,q') = P(K,\tilde{q}')$ for $q' \in Y(K,q)$ and $\tilde{q}' \in Y(K,q)$. Since $P(K,q)$ is weakly increasing, then $P(K,q)$ is constant between q' and $\tilde{q}'$. But this cannot happen; Claim 1 shows that the buyer never chooses interior points in flat regions of $P(k,q)$.

[43] The set $\widehat{H}^t$ contains some but not all off-path histories. Below we specify the buyer's strategy and beliefs for all histories on path, and also for the remaining off-path histories.

b. If $p > P\left(K(h^{t-1}), q(h^{t-1})\right)$ and $p = P\left(K(h^{t-1}), q)\right)$ for some $q > q(h^{t-1})$, then the belief $q(h^t)$ is given by the probability of acceptance implied in the definition of stationary PBE. The buyer makes the offer implied by $y(K(h^t), q(h^t))$.

c. If $p > P\left(K(h^{t-1}), q(h^{t-1})\right)$ and $p \neq P\left(K(h^{t-1}), q)\right)$ for all $q > q(h^{t-1})$, then the belief $q(h^t)$ is given by the probability of acceptance implied in the definition of stationary PBE. The buyer randomizes among the elements of $Y(K(h^t), q(h^t))$ to rationalize the probability of acceptance of the low-type seller in $t-1$.[44]

b. Verification that strategies and beliefs form a stationary PBE. The strategy of the high-type seller is optimal. On-path, the buyer never pays more than $\frac{c}{m}k$ for any k. Then, it is optimal to accept any offer greater or equal than $\frac{c}{m}k$ for any k and to reject otherwise.

The optimality of the low-type seller's strategy follows from $\mathcal{V}_L = \mathcal{V}_L'$. Assume that the buyer and the seller follow the equilibrium strategies specified above. Then, in any on-path history h^t with state $(K, q) = \left(K(h^t), q(h^t)\right)$ the function $\mathcal{V}_L(K, q)$ satisfies:

$$\mathcal{V}_L(K, q) = \begin{cases} \frac{c}{m}(K-k) + \delta\mathcal{V}_L(k, q) & \text{if } \ y(K, q) = k \\ P\left(K, q'\right) = \delta\mathcal{V}_L\left(K, q'\right) & \text{if } \ y(K, q) = q' \end{cases} \tag{12}$$

Equation (12) follows from the definition of $\mathcal{V}_L'$ in equation (11), the equality $\mathcal{V}_L' = \mathcal{V}_L$, the definition of $P(K, q)$ and the fact that the buyer never chooses an induced posterior in a flat region of $P(K, q)$. Therefore, $\mathcal{V}_L(K, q)$ is the on-path continuation payoff of the low-type seller.

The low-type seller obtains a continuation payoff of zero if he rejects a universal offer. The first line of equation (12) shows that he obtains a strictly positive payoff if he instead accepts it. Then, it is optimal for the low-type seller to accept a universal offer.[45] The sec-

[44]Supoose that $p > P\left(K(h^{t-1}), q(h^{t-1})\right)$, $p \neq P\left(K(h^{t-1}), q)\right)$ for all $q > q(h^{t-1})$ and that the new belief is $q(h^t)$. Then, $\delta\mathcal{V}_L\left(K(h^t), q(h^t)\right) < p < \delta \lim_{q \downarrow q(h^t)} \mathcal{V}_L\left(K(h^t), q\right)$. One element of $Y\left(K(h^t), q(h^t)\right)$ yields a continuation payoff of $\mathcal{V}_L\left(K(h^t), q(h^t)\right)$ to the low-type seller, while another one yields a continuation payoff of $\lim_{q \downarrow q(h^t)} \mathcal{V}_L\left(K(h^t), q\right)$ to the low-type seller. In period t the buyer randomizes between these two elements of $Y\left(K(h^t), q(h^t)\right)$ so that the low-type seller's continuation payoff in period $t-1$ (if he rejects the screening offer) is exactly p. Note that this implies that off-the-equilibrium path the low-type seller's continuation payoff may depend not only on the state but also on the offer in the previous period.

[45]For this same reason it is optimal for the low-type seller to accept any offer (k, p) with $p > \frac{c}{m}k$.

ond line of equation (12) shows that the low-type seller is indifferent between accepting and rejecting the screening offers that the buyer makes on path. Consider instead a buyer who deviates and makes a partial offer $(k, P(K, q'))$ with $k < K$. If the low-type seller accepts, he obtains $P(K, q')$ in the current period and zero from then on. If he instead rejects, his continuation payoff is $\delta \mathcal{V}_L(K, q')$. Thus, the low-type seller is also willing to randomize in this case.[46]

We construct the strategy of the buyer by choosing for every history h^t elements from the set $Y\left(K(h^t), q(h^t)\right)$ of best responses in the modified problem. The difference between the original and modified problem lies in the set of posteriors that screening offers can induce. While in the modified problem the buyer can induce the whole set of posteriors $[q, \hat{q}]$ at any state (K, q), the set of posteriors that he can induce in the original game may be limited. Claim 1 shows that the best response correspondence $Y(K, q)$ in the modified problem only induces posteriors that are feasible in the original game.

CLAIM 1. THE BUYER NEVER CHOOSES A POSTERIOR IN A FLAT REGION OF $P(K, \cdot)$. *If $q' \in Y(K, q)$, then $P(K, q'') > P(K, q')$ for every $q'' > q'$.*

See Section T.1 of the Technical Addendum for the proof.

This proves that the strategy of the buyer is optimal.

Part B. Construction of the consistent quadruplet $(\mathcal{V}_L, P, W, y)$

We construct a consistent quadruplet $(\mathcal{V}_L, P, W, y)$ through two processes of induction (and a fixed point argument). In the base step of the first process of induction we construct the quadruplet $(\mathcal{V}_L(1, \cdot), P(1, \cdot), W(1, \cdot), y(1, \cdot))$, which deals with the case when only one unit remains. In the inductive step there are K units left, with $1 < K \leq m$. We assume that the quadruplet $(\mathcal{V}_L(k, \cdot), P(k, \cdot), W(k, \cdot), y(k, \cdot))$ has already been constructed for all $k \in \{1, \ldots, K-1\}$ and construct the quadruplet $(\mathcal{V}_L(K, \cdot), P(K, \cdot), W(K, \cdot), y(K, \cdot))$.

The second process of induction is nested within the first one. We explain this process in detail in steps 1 to 3 below. Let K be the number of remaining units and assume that the quadruplet $(\mathcal{V}_L(k, \cdot), P(k, \cdot), W(k, \cdot), y(k, \cdot))$ has already been constructed for all $k \in$

[46] The buyer could also deviate by making an offer (k, p) with $k \leq K$ and $p \neq P(K, q')$. The equilibrium strategies that we define also guarantee that the low-type seller behaves optimally. We omit the details.

$\{1, \ldots, K-1\}$. In the base step, we construct $(\mathcal{V}_L(K,\cdot), P(K,\cdot), W(K,\cdot), y(K,\cdot))$ for $q \in [\overline{q}, \hat{q}]$ for some $\bar{q} < \hat{q}$ (see step 1 below). In the inductive step (indexed by n), we assume that the quadruplet $(\mathcal{V}_L(K,\cdot), P(K,\cdot), W(K,\cdot), y(K,\cdot))$ has already been constructed for $q \in [q_n, \hat{q}]$. We extend $(\mathcal{V}_L(K,\cdot), P(K,\cdot), W(K,\cdot), y(K,\cdot))$ to $q \in [q_{n+1}, \hat{q}]$ with $q_{n+1} < q_n$ (we explain this in step 2a below). This inductive step involves a fixed point argument that we describe in detail in step 2b. Finally, we show that in a finite number ($\tilde{n}$) of steps $q_{\tilde{n}} = 0$ (step 3 below).

Step 1. Quadruplet in interval $q \in [\overline{q}, \hat{q}]$. Claim 2 describes the simple form that the quadruplet $(\mathcal{V}_L, P, W, y)$ takes when transformed beliefs are sufficiently close to $\hat{q}$. The intuition behind Claim 2 is simple. If the buyer is sufficiently convinced that the seller is of high type, he is better off trading right away. He offers to pay the high type's cost in exchange for all remaining units. Both types accept and the game ends. This leads directly to the quadruplet's form in Claim 2.

Claim 2. *There exists* $\overline{q} < \hat{q}$, *such that any consistent quadruplet* $(\mathcal{V}_L, P, W, y)$ *must satisfy*

$$\begin{aligned}
\mathcal{V}_L(K,q) &= \frac{c}{m}K, \\
P(K,q) &= \delta \frac{c}{m}K, \\
W(K,q) &= \sum_{s=1}^{K} \Lambda_s^m \left[(\hat{q}-q)\, v_L + (1-\hat{q})\, v_H\right] - (1-q)\frac{c}{m}K > 0 \quad \textit{and} \\
y(K,q) &= K
\end{aligned}$$

for every $q \in [\overline{q}, \hat{q}]$ *and for every* $K \in \{1, \ldots, m\}$.

Proof. Assume that there are K remaining units. A buyer who makes a screening offer obtains a (normalized) continuation payoff bounded above by

$$(\hat{q}-q)\sum_{s=1}^{K} \Lambda_s^m v_L + (1-\hat{q})\,\delta\left(\sum_{s=1}^{K} \Lambda_s^m v_H - \frac{c}{m}K\right).$$

Moreover, for a sufficiently high $q < \hat{q}$, the expression above is strictly smaller than

$$\sum_{s=1}^{K} \Lambda_s^m \left[\left(\hat{q} - q \right) v_L + \left(1 - \hat{q} \right) v_H \right] - \left(1 - q \right) \frac{c}{m} K$$

which represents the continuation payoff for the buyer when he makes a universal offer for all remaining units. This continuation payoff is strictly positive for sufficiently high $q < \hat{q}$. This, in turn, implies that there exists $\bar{q} < \hat{q}$ such that for any $q \in [\bar{q}, \hat{q}]$ and for any $K \in \{1, \ldots, m\}$, screening offers are strictly dominated by a universal offer for all remaining units, and this universal offer leads to strictly positive payoffs. Therefore, the best universal offer is to buy all units immediately, which leads to the expressions for W and y outlined above. These expressions, in turn, imply that $\mathcal{V}_L$ and P are as above. ■

Step 2. Extension of quadruplet from interval $[q_n, \hat{q}]$ **to interval** $q \in [q_{n+1}, \hat{q}]$. The extension of the quadruplet consists of two sub-steps. In the first one (a), we only allow the buyer to make screening offers. We find an interval $[q_{n+1}, q_n]$ where the optimal screening offer induces posterior belief above q_n. If universal offers were not allowed (i.e., if there were only one unit left, as in DL), this would conclude the extension to $[q_{n+1}, q_n]$. In the second sub-step (b), we give the buyer the possibility of making universal offers. This modifies the low-type seller's continuation payoff – and therefore the function $P(K, \cdot)$ – in the interval $[q_{n+1}, q_n]$. We allow the buyer to re-optimize, given the modified function $P(K, \cdot)$, which in turn changes the low-type seller's continuation payoff. We continue this process until we reach a fixed point.

a. Only screening offers. Fix the number of remaining units K. Assume that the quadruplet $(\mathcal{V}_L(k, \cdot), P(k, \cdot), W(k, \cdot), y(k, \cdot))$ is already defined for all $1 \leq k \leq K - 1$ and that the quadruplet $(\mathcal{V}_L(K, \cdot), P(K, \cdot), W(K, \cdot), y(K, \cdot))$ is defined for $q \in [q_n, \hat{q}]$.

We define two auxiliary value functions for the buyer that represent continuation payoffs from making screening offers. First, for $q \in [0, q_n]$ we let $W^I(K, q)$ represent the buyer's payoff from making a screening offer that leads to posterior $q' \geq q_n$:

$$W^I(K, q) = \max_{q' \geq q_n} \left(q' - q \right) \left(\sum_{s=1}^{K} \Lambda_s^m v_L - P\left(K, q'\right) \right) + \delta W\left(K, q'\right) \tag{13}$$

Let $X(K,q) \in [q_n, \hat{q}]$ denote the set of solutions to the above maximization problem, and let $\underline{x}(K,q)$ and $\overline{x}(K,q)$ denote respectively the smallest and largest elements of $X(K,q)$.

Second, let $P^I(K,q) = \delta P\left(K, \underline{x}(K,q)\right)$ denote an auxiliary pricing function for $q \in [0, q_n]$. The function $W^{II}(K,q)$ represents the buyer's payoff from making a screening offer $\left(K, P^I(K,q)\right)$ that leads to posterior $q' \in [q, q_n]$ (and to a continuation payoff W^I afterwards):

$$W^{II}(K,q) = \max_{q' \in [q,q_n]} (q' - q) \left(\sum_{s=1}^{K} \Lambda_s^m v_L - P^I(K,q') \right) + \delta W^I(K,q') \quad \text{for } q \in [0, q_n]$$

Let the endpoint q_{n+1} be defined by $q_{n+1} = \max\left\{q \in [0,q_n] : W^I(K,q) \le W^{II}(K,q)\right\}$ if the set is non-empty and $q_{n+1} = 0$ otherwise.

CLAIM 3. *Endpoints are strictly decreasing: $q_{n+1} < q_n$. Moreover, the continuation payoff $W^I(K,q)$ is continuous and satisfies $W^I(K,q) > 0$ for all $q \in [q_{n+1}, q_n]$.*

Proof. The continuation payoff $W^I(K,q_n)$ is strictly positive because it is bounded below by $\delta W(K,q_n) > 0$. By definition, $W^{II}(K,q_n) = \delta W^I(K,q_n)$, and so $W^{II}(K,q_n) < W^I(K,q_n)$. Finally, the theorem of the maximum guarantees that the functions $W^I(K,\cdot)$ and $W^{II}(K,\cdot)$ are continuous. Therefore, $q_{n+1} < q_n$. Next, by definition, for any $q \in (q_{n+1}, q_n]$, we have $W^I(K,q) > W^{II}(K,q) \ge \delta W^I(K,q)$. Thus, for any $q \in (q_{n+1}, q_n]$, we have $W^I(K,q) > 0$. It only remains to be shown that $W^I(K,q_{n+1}) > 0$, which we do in Section T.2 of the Technical Addendum. ■

b. Fixed Point. We define a sequence of quadruplets

$$\left\{ \left(\mathcal{V}_L^\ell(K,\cdot), P^\ell(K,\cdot), W^\ell(K,\cdot), y^\ell(K,\cdot) \right) \right\}_{\ell=1,2,\ldots} \quad \text{for the interval } [q_{n+1}, \hat{q}].$$

The first element of the sequence is as follows. For $q \in (q_n, \hat{q}]$, we set

$$\left(\mathcal{V}_L^1(K,q), P^1(K,q), W^1(K,q), y^1(K,q) \right) = \left(\mathcal{V}_L(K,q), P(K,q), W(K,q), y(K,q) \right).$$

For $q \in [q_{n+1}, q_n]$ we instead set

$$W^1(K,q) = \max\Bigg\{ W^I(K,q), \max_{0 \le k \le K-1} \left\{ \sum_{s=k+1}^{K} \Lambda_s^m \left[(\hat{q} - q)\, v_L + (1-\hat{q})\, v_H \right] - (1-q)\frac{c}{m}(K-k) + \delta W(k,q) \right\} \Bigg\}$$

and we let $y^1(K,q)$ be the solution that gives the lowest continuation payoff to the low-type seller.[47] The screening offer in W^I leads to a state (K, q') with $q' \ge q_n$. The continuation payoff $\mathcal{V}_L(K,q')$ is already defined for this state. Similarly, a universal offer leads to a state (k,q) with $k < K$, for which the continuation payoff $\mathcal{V}_L(k,q)$ is already defined. Thus, we extend $\mathcal{V}_L^1(K,\cdot)$ to the interval $[q_{n+1}, q_n]$ as follows:

$$\mathcal{V}_L^1(K,q) = \begin{cases} \delta \mathcal{V}_L\,(K,q') & \text{if } y^1(K,q) = q' \\ \frac{c}{m}(K-k) + \delta \mathcal{V}_L\,(k,q) & \text{if } y^1(K,q) = k \end{cases}$$

Finally, in the interval $[q_{n+1}, q_n]$, we define $P^1(K,\cdot)$ to be the largest weakly increasing function below $\delta \mathcal{V}_L^1\,(K,\cdot)$.

We define the remaining elements of the sequence of quadruplets recursively. For any $\ell \ge 1$, we define the $\ell+1$'th element of the sequence as follows. First, we set

$$W^{\ell+1}(K,q) = \max\Bigg\{ \max_{q' \in [q,\hat{q}]} (q'-q) \left[\left(\sum_{s=1}^{K} \Lambda_s^m \right) v_L - P^\ell(K,q') \right] + \delta W^\ell(K,q'), \max_{0 \le k \le K-1} \left\{ \left(\sum_{s=k+1}^{K} \Lambda_s^m \right) \left[(\hat{q}-q)\, v_L + (1-\hat{q})\, v_H \right] - (1-q)\frac{c}{m}(K-k) + \delta W(k,q) \right\} \Bigg\}.$$

Next, we let $y^{\ell+1}(K,q)$ be the solution to the above problem that gives the lowest continuation payoff to the low-type seller. Denote that continuation payoff by $\mathcal{V}_L^{\ell+1}(K,q)$. Finally, let $P^{\ell+1}(k,\cdot)$ be the largest weakly increasing function below $\delta V_L^{\ell+1}(K,\cdot)$.

[47] As in Step 3 of Part A, whenever there are many solutions with the same continuation payoff, then there must exist at least one that implies a universal offer $\left(K-k, \frac{c}{m}(K-k)\right)$. Of all such universal offers, we pick the one with the lowest k.

Claim 4. *There exists ℓ^* such that*

$$\left(\mathcal{V}_L^{\ell^*}(K,\cdot), P^{\ell^*}(K,\cdot), W^{\ell^*}(K,\cdot), y^{\ell^*}(K,\cdot)\right)$$
$$= \left(\mathcal{V}_L^{\ell^*+1}(K,\cdot), P^{\ell^*+1}(K,\cdot), W^{\ell^*+1}(K,\cdot), y^{\ell^*+1}(K,\cdot)\right).$$

Proof. For every $q \geq q_{n+1}$ and for every $\ell > 1$, $W^\ell(k,q) \geq W^1(k,q) > 0$. Then, there exists $\eta > 0$ such that for $q \in [q_{n+1}, q_n]$ and for every $\ell > 1$, $W^\ell(k,q) > \eta$.

If the claim fails, for any positive integer T there exist ℓ, $q \in [q_{n+1}, q_n)$, and a sequence $\{q^\tau\}_{\tau=0}^T$ with $q^0 = q$, $q^T < q + \frac{1}{T}$ and $y^\ell(K, q^{\tau-1}) = q^\tau$ for all $\tau \in \{1, \ldots, T\}$. The buyer's continuation payoff $W^\ell(K,q)$ is bounded above:

$$W^\ell(K,q) < \left(\frac{1}{T} + \delta^T\right) \sum_{s=1}^{K} \Lambda_s^m v_H$$

Finally, pick T so that

$$\left(\frac{1}{T} + \delta^T\right) \sum_{s=1}^{K} \Lambda_s^m v_H < \eta.$$

But $W^\ell(K,q) > \eta$, so we have reached a contradiction. ∎

At the end of the n'th inductive step, the quadruplet is already defined for $q \geq q_n$. We extend the quadruplet to $q \in [q_{n+1}, q_n)$ by setting it equal to the fixed point defined above:

$$(\mathcal{V}_L(K,q), P(K,q), W(K,q), y(K,q)) = \left(\mathcal{V}_L^{\ell^*}(K,q), P^{\ell^*}(K,q), W^{\ell^*}(K,q), y^{\ell^*}(K,q)\right).$$

Step 3. Extension to interval $[0, \hat{q}]$ takes finitely many steps. In the last step of the construction, we show that it takes finitely many steps to extend the quadruplet to the whole interval $[0, \hat{q}]$.

Claim 5. *There exists $\tilde{n}$ so that $q_{\tilde{n}} = 0$.*

See Section T.3 of the Technical Addendum for the proof.

Finally, note that $W(K,q) > 0$ for every (K,q). Thus it is never optimal for the buyer to make two consecutive universal offers. Formally, if $k \in Y(K,q)$ for some (K,q), then

$k' \notin Y(k,q)$. Assume towards a contradiction that $k \in Y(K,q)$ and $k' \in Y(k,q)$. Then,

$$\begin{aligned} W(K,q) &= \left(\sum_{s=k+1}^{K} \Lambda_s^m\right)\left[(\hat{q}-q)\,v_L + (1-\hat{q})\,v_H\right] - (1-q)\frac{c}{m}(K-k) + \delta W(k,q) \\ &< \left(\sum_{s=k+1}^{K} \Lambda_s^m\right)\left[(\hat{q}-q)\,v_L + (1-\hat{q})\,v_H\right] - (1-q)\frac{c}{m}(K-k) + W(k,q) \\ &= \left(\sum_{s=k'+1}^{K} \Lambda_s^m\right)\left[(\hat{q}-q)\,v_L + (1-\hat{q})\,v_H\right] - (1-q)\frac{c}{m}(K-k') + \delta W(k',q) \end{aligned}$$

This shows that, at state (K,q), the buyer strictly prefers to make a universal offer for $K-k'$ units, instead of making one for $K-k$ units. Thus, $k \notin Y(K,q)$. ■

A.4 Convergence as bargaining frictions vanish

Lemma 3. Convergence as bargaining frictions vanish. *Fix m.*

(a) *Consider an arbitrary sequence of vanishing frictions $\{\Delta_n\}_{n=1}^{\infty} \to 0$. The associated sequences $\left\{K_m^{\Delta_n}(\cdot)\right\}_{n=1}^{\infty}$, $\left\{q_m^{\Delta_n}(\cdot)\right\}_{n=1}^{\infty}$, $\left\{\left\{P_m^{\Delta_n}(K,\cdot)\right\}_{K=1}^{m}\right\}_{n=1}^{\infty}$ and $\left\{\left\{W_m^{\Delta_n}(K,\cdot)\right\}_{K=1}^{m}\right\}_{n=1}^{\infty}$ have subsequences that converge pointwise.*

(b) *There exist functions $K_m(\cdot)$, $q_m(\cdot)$, $\{P_m(K,\cdot)\}_{K=1}^{m}$ and $\{W_m(K,\cdot)\}_{K=1}^{m}$ such that for any sequence of vanishing frictions $\{\Delta_n\}_{n=1}^{\infty} \to 0$, the associated sequences $\left\{K_m^{\Delta_n}(\cdot)\right\}_{n=1}^{\infty}$, $\left\{q_m^{\Delta_n}(\cdot)\right\}_{n=1}^{\infty}$, $\left\{\left\{P_m^{\Delta_n}(K,\cdot)\right\}_{K=1}^{m}\right\}_{n=1}^{\infty}$ and $\left\{\left\{W_m^{\Delta_n}(K,\cdot)\right\}_{K=1}^{m}\right\}_{n=1}^{\infty}$ converge pointwise to $K_m(\cdot)$, $q_m(\cdot)$, $\{P_m(K,\cdot)\}_{K=1}^{m}$ and $\{W_m(K,\cdot)\}_{K=1}^{m}$, respectively, except for finitely many points.*[48]

Proof of part (a). For any $\Delta > 0$, the functions $K_m^{\Delta}(\cdot)$ and $q_m^{\Delta}(\cdot)$ are monotonic in time elapsed τ and the function $P_m^{\Delta}(K,\cdot)$ is monotonic in q for all $K \in \{1,\ldots,m\}$. Therefore, they all have bounded variation. Moreover, all these functions are bounded above and below by bounds that do not depend on Δ. By Helly's First Theorem (Theorem 6.1.18

[48]The finitely many points where pointwise convergence may not occur correspond to impasses. At any impasse at state (K,q), $P_m^{-}(K,q)$ and $P_m^{+}(K,q)$ are well defined. We set $P_m(K,q) = P_m^{+}(K,q)$. This is without loss of generality, as the limit equilibrium outcome as bargaining frictions vanish does not depend on this choice.

in Kannan and Krueger [1996]), $\left\{K_m^{\Delta_n}(\cdot)\right\}_{n=1}^{\infty}$, $\left\{q_m^{\Delta_n}(\cdot)\right\}_{n=1}^{\infty}$ and $\left\{\left\{P_m^{\Delta_n}(K,\cdot)\right\}_{K=1}^{m}\right\}_{n=1}^{\infty}$ all have subsequences that converge pointwise.

Fix $K \in \{1,\ldots,m\}$. The functions $\left\{W_m^{\Delta_n}(K,\cdot)\right\}_{n=1}^{\infty}$ are uniformly equicontinuous since they all have the same Lipschitz constant $v_H \sum_{s=1}^{K} \Lambda_s^m$. They are also uniformly bounded. Then, the Arzelà-Ascoli Theorem guarantees that $\left\{W_m^{\Delta_n}(K,\cdot)\right\}_{n=1}^{\infty}$ has a subsequence that converges uniformly. ■

Proof of part (b). In Proposition 3 we show that all convergent sequences $\left\{K_m^{\Delta_n}(\cdot)\right\}_{n=1}^{\infty}$, $\left\{q_m^{\Delta_n}(\cdot)\right\}_{n=1}^{\infty}$, $\left\{\left\{P_m^{\Delta_n}(K,\cdot)\right\}_{K=1}^{m}\right\}_{n=1}^{\infty}$ and $\left\{\left\{W_m^{\Delta_n}(K,\cdot)\right\}_{K=1}^{m}\right\}_{n=1}^{\infty}$ have the same limit. ■

A.5 Proof of Proposition 3

In this proof we introduce an algorithm that characterizes the limit equilibrium outcome as bargaining frictions vanish.[49] Proposition 3 follows immediately from this characterization.

We consider a sequence of vanishing bargaining frictions $\{\Delta_n\}_{n=1}^{\infty} \to 0$ with associated sequences $\left\{\left\{P_m^{\Delta_n}(K,\cdot)\right\}_{K=1}^{m}\right\}_{n=1}^{\infty}$, $\left\{\left\{W_m^{\Delta_n}(K,\cdot)\right\}_{K=1}^{m}\right\}_{n=1}^{\infty}$ and $\left\{\left(K_m^{\Delta_n}(\cdot), q_m^{\Delta_n}(\cdot)\right)\right\}_{n=1}^{\infty}$ that converge pointwise, by Lemma 3(a). We characterize the limits of these associated sequences, which we denote by $\{P_m(K,\cdot)\}_{K=1}^{m}$, $\{W_m(K,\cdot)\}_{K=1}^{m}$ and $(K_m(\cdot), q_m(\cdot))$.

We describe both on-path and off-path behavior: we specify how quantities and beliefs evolve starting from any state (K,q). We let $K_m(\tau;(K,q))$ and $q_m(\tau;(K,q))$ denote respectively the number of remaining units and the belief at time elapsed τ if the starting state at time elapsed zero is (K,q).[50] The on-path limit equilibrium outcome as bargaining frictions vanish $(K_m(\tau), q_m(\tau))$ then corresponds to $(K_m(\tau;(m,0)), q_m(\tau;(m,0)))$.

Our algorithm proceeds by induction. In each step we characterize the limit functions $\{P_m(K,\cdot)\}_{K=1}^{m}$, $\{W_m(K,\cdot)\}_{K=1}^{m}$ and $(K_m(\cdot), q_m(\cdot))$ for different subsets of the state space $\{1,\ldots,m\} \times [0,\hat{q}]$. In the base step ($j = 0$), we identify a candidate impasse $(k_1, q_1) = (1, \bar{q}_m(1))$. We characterize the limit functions for all states $(1,q)$ with $q < q_1$ (Claim 6)

[49] We do this for generic values of the parameters (for details, see Remark 1 on page 20 of this Appendix).

[50] As in the main body of the paper, these functions are left-continuous in τ. These functions are uniquely identified at all states, except at finitely many states, which correspond to (on- and off-path) impasses. For these states, the functions $K_m(\tau;(K,q))$ and $q_m(\tau;(K,q))$ reflect the evolution after the impasse is resolved.

and for all states (K,q) with $q \geq q_1$ (Claim 7). At each (non-final) step $j \geq 1$ of the inductive process we identify a candidate impasse (k_{j+1}, q_{j+1}) with $k_{j+1} > k_j$ and $q_{j+1} < q_j$. Claims 8, 9 and 10 characterize the limit functions for all states (K,q) with either 1) $K \in \{k_j+1, \ldots, k_{j+1}\}$ and $q \in [0, q_j)$, or 2) $K \in \{k_{j+1}+1, \ldots, m\}$ and $q \in [q_{j+1}, q_j)$. In particular, these claims show that the candidate impasse (k_j, q_j) is reached from the candidate impasse (k_{j+1}, q_{j+1}).

The algorithm ends after finitely many steps with a characterization of the limit functions for the whole state space $\{1, \ldots, m\} \times [0, \hat{q}]$ and with a collection $\{(k_j, q_j)\}_{j=1}^{J}$ of J candidate impasses. All candidate impasses are on-path: the limit equilibrium outcome as bargaining frictions vanish consists of a sequence of phases of fast trade and impasses summarized by $\{(k_j, q_j)\}_{j=1}^{J}$.

The base step (j = 0)

In the base step we obtain the first candidate impasse $(k_1, q_1) = (1, \bar{q}_m(1))$. Claim 6 shows that the candidate impasse $(1, \bar{q}_m(1))$ is reached without delay starting from any state $(1, q)$ with $q < \bar{q}_m(1)$.

CLAIM 6. *For all $q < \bar{q}_m(1)$, we have*

$$
\begin{aligned}
P_m(1,q) &= \frac{\left(\Lambda_1^m v_L\right)^2}{c/m}, \\
W_m(1,q) &= \left(\bar{q}_m(1) - q\right)\left(\Lambda_1^m v_L\right)\left(1 - \frac{\Lambda_1^m v_L}{c/m}\right) \quad \textit{and}
\end{aligned}
$$

$$
\left(K_m\left(\tau;(1,q)\right), q_m\left(\tau;(1,q)\right)\right) = \begin{cases} (1, \bar{q}_m(1)) & \textit{if } \tau \leq \tau_1 \\ (0, \hat{q}) & \textit{if } \tau > \tau_1 \end{cases} \qquad \textit{with } \tau_1 = \frac{2}{r}\ln\left(\frac{c/m}{\Lambda_1^m v_L}\right).
$$

The proof of Claim 6 is in DL, so we omit it.

Claim 7 shows that starting at any state (K,q) with $K \in \{1, \ldots, m\}$ and $q \in [\bar{q}_m(1), \hat{q}]$, the game ends without delay.

CLAIM 7. *For all (K,q) with $K \in \{1,\ldots,m\}$ and $q \in [\bar{q}_m(1), \hat{q}]$ we have*

$$P_m(K,q) = K\frac{c}{m},$$
$$W_m(K,q) = (\hat{q}-q)\left(\sum_{s=1}^{K}\Lambda_s^m v_L - K\frac{c}{m}\right) + (1-\hat{q})\left(\sum_{s=1}^{K}\Lambda_s^m v_H - K\frac{c}{m}\right) \quad \textit{and}$$
$$\left(K_m\left(\tau;(K,q)\right), q_m\left(\tau;(K,q)\right)\right) = (0,\hat{q}) \quad \forall \tau \geq 0.$$

Proof. At all states (K,q) with $K \in \{1,\ldots,m\}$ and $q \in [\bar{q}_m(1), \hat{q}]$, except for $(1, \bar{q}_m(1))$, the buyer can guarantee a strictly positive continuation payoff by making a universal offer for K units. Thus, the game ends without delay. The low-type seller can always mimic the high-type seller's behavior. Therefore, as bargaining frictions vanish, the price that the low-type seller is willing to accept for K units must converge to $K\frac{c}{m}$. Then, the function $P_m(1,\cdot)$ is discontinuous at $(1,\bar{q}_m(1))$. We assign $P_m(1,\bar{q}_m(1)) = P_m^+(1,\bar{q}_m(1))$. We do the same with the outcome $\left(K_m\left(\tau;(1,\bar{q}_m(1))\right), q_m\left(\tau;(1,\bar{q}_m(1))\right)\right)$, i.e. we take the limit from the right. In this way, these functions evaluated at $(1,\bar{q}_m(1))$ reflect what happens right after the impasse $(1,\bar{q}_m(1))$ is resolved. We follow this convention also for the next impasses. ■

The algorithm then continues to the first inductive step ($j = 1$).

The inductive step (j $\geq$ 1)

The previous step $j-1$ provides a (candidate) impasse (k_j, q_j) of length τ_j. The impasse (k_j, q_j) satisfies $\bar{q}_m(k_j+1) < q_j$ and $k_j < m$. All previous steps together provide a characterization of the limit functions for all states (K,q) with either $K \leq k_j$, or $q \geq q_j$, or both.

As we do in the main body of the paper, throughout this proof we focus on the "limit game" in the sense that the low-type seller's behavior is summarized by the limit function $P_m(\cdot,\cdot)$. We consider a simple course of action that brings the buyer from any state (K,q) with $K \in \{k_j+1,\ldots,m\}$ and $q \in [0,q_j]$ to the impasse (k_j,q_j). The buyer first makes the universal offer $\left(K-k_j, \frac{c}{m}(K-k_j)\right)$ and then the screening offer $\left(K, P_m^-(k_j,q_j)\right)$. The function $\mathcal{W}(K,q) : \{k_j+1,\dots,m\} \times [0,q_j] \to \mathbb{R}$, defined in equation (14), denotes the buyer's (normalized) payoff from following this simple course of action.

$$\mathcal{W}(K,q) \equiv (\hat{q}-q)\left[\sum_{s=k_j+1}^{K} \Lambda_s^m v_L - (K-k_j)\frac{c}{m}\right] + (1-\hat{q})\left[\sum_{s=k_j+1}^{K} \Lambda_s^m v_H - (K-k_j)\frac{c}{m}\right] + \left(q_j - q\right)\left[\sum_{s=1}^{k_j} \Lambda_s^m v_L - P_m^-\left(k_j, q_j\right)\right] \tag{14}$$

Remark 1. *The following two conditions hold for generic values of the parameters:*

$$\mathcal{W}(K,0) \neq 0 \quad \textit{for all } K \in \{k_j+1,\dots,m\} \tag{15a}$$

$$\mathcal{W}(K,\bar{q}_m(K)) \neq 0 \quad \textit{for all } K \in \{k_j+1,\dots,m\} \tag{15b}$$

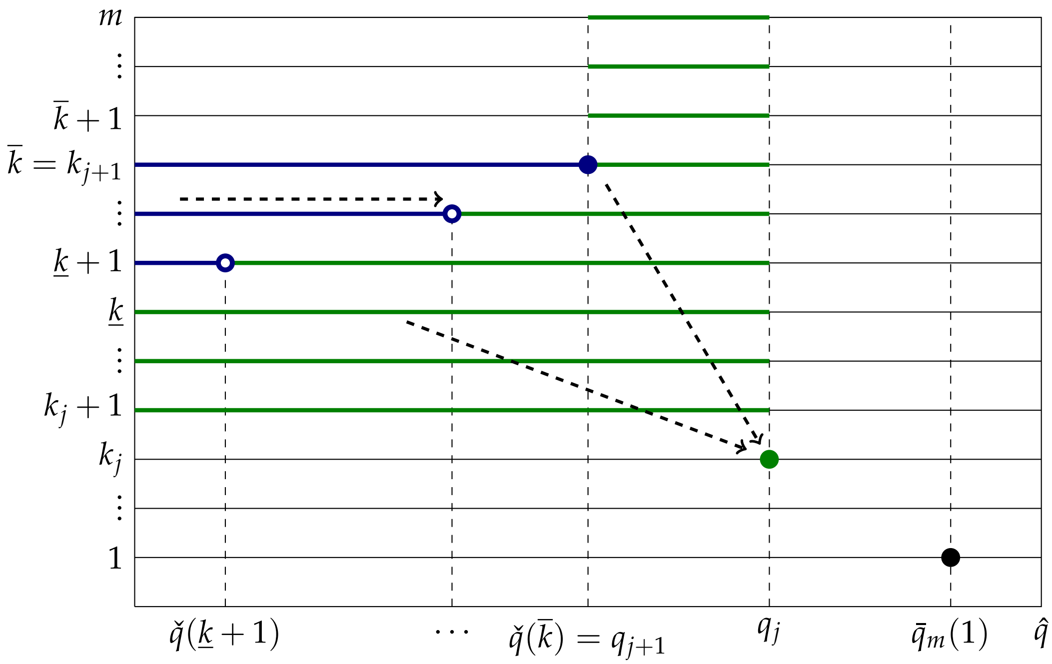
Throughout this proof we restrict attention to parameters that satisfy these two conditions.

The function $\mathcal{W}(\cdot,\cdot)$ satisfies $\mathcal{W}(K,q_j) > 0$ because $\bar{q}_m(K) \leq \bar{q}_m(k_j+1) < q_j$. Moreover, $\mathcal{W}(\cdot,0)$ is strictly decreasing in K. Given the genericity condition (15a), we next let

$$\underline{k} = \begin{cases} \max\left\{K \in \{k_j+1,\dots,m\} : \mathcal{W}(K,0) > 0\right\} & \text{if } \mathcal{W}(k_j+1,0) > 0 \\ k_j & \text{if } \mathcal{W}(k_j+1,0) < 0 \end{cases}$$

We split the remainder of the inductive step into two parts, *a* and *b*. If $\underline{k} = m$, the algorithm proceeds with part *a* and then ends. If $k_j < \underline{k} < m$, the algorithm proceeds first with part *a* and then with part *b*. If $\underline{k} = k_j$, the algorithm skips part *a* and moves directly to part *b*. Throughout the description of these two parts, we refer to Figure 5 to facilitate their exposition.

Part a. In this part we characterize the equilibrium outcome for all states (K,q) with $K \in \left\{k_j+1,\dots,\underline{k}\right\}$ and $q \in [0,q_j)$. At any such state, the buyer can guarantee a positive continuation payoff by following the simple course of action described above. We represent this area of the state space with thick green lines in Figure 5. We show in Claim 8



Notes: The green circle at state (k_j, q_j) denotes the candidate impasse from the previous step $j-1$. Thick green lines represent states (K,q) with $\mathcal{W}(K,q) > 0$, while thick blue lines represent states (K,q) with $\mathcal{W}(K,q) < 0$. Dashed black arrows illustrate transitions without delay. Filled circles represent on-path impasses, while empty circles represent off-path impasses.

Figure 5: The inductive step ($j \geq 1$) of the algorithm

how starting an any state (K,q) with $K \in \{k_j+1,\ldots,\underline{k}\}$ and $q \in [0,q_j)$, the state (k_j,q_j) is reached without delay. The state remains there for time elapsed τ_j, i.e. there is an impasse of length τ_j at state (k_j,q_j). After the impasse is resolved, the evolution of the number of remaining units and of beliefs is as specified in the previous step of the induction process.

CLAIM 8. *For all $K \in \{k_j+1,\ldots,\underline{k}\}$ and for all $q \in [0,q_j)$ we have:*

$$P_m(K,q) = (K-k_j)\frac{c}{m} + P_m^-\left(k_j,q_j\right)$$

$$W_m(K,q) = \mathcal{W}(K,q)$$

$$\left(K_m\left(\tau;(K,q)\right), q_m\left(\tau;(K,q)\right)\right) = \begin{cases} \left(k_j,q_j\right) & \text{if } \tau \leq \tau_j \\ \left(K_m\left(\tau-\tau_j;(k_j,q_j)\right), q_m\left(\tau-\tau_j;(k_j,q_j)\right)\right) & \text{if } \tau > \tau_j \end{cases}$$

See Section T.4 of the Technical Addendum for the proof.

If $\underline{k} = m$, then (k_j, q_j) is the first impasse and the algorithm ends. Otherwise, the algorithm proceeds to part *b*.

Part b. We first let

$$\bar{k} = \max\left\{K \in \{\underline{k}+1, \ldots, m\} : \mathcal{W}\left(K, \bar{q}_m(K)\right) > 0\right\}.$$

Furthermore, for all $K \geq \underline{k}+1$ we let $\check{q}(K) \in (0, q_j)$ be defined by $\mathcal{W}(K, \check{q}(K)) = 0$. In this part we derive the functions of interest for all states (K, q) with either 1) $K \in \{\underline{k}+1, \ldots, \bar{k}\}$ and $q < q_j$ or 2) $K > \bar{k}$ and $q \in [\check{q}(\bar{k}), q_j)$. To do so, we first prove the following fact.

FACT 1. *The following inequalities hold:*

$$\frac{\partial \mathcal{W}(K,q)}{\partial q} = (K - k_j)\frac{c}{m} + P_m^-\left(k_j, q_j\right) - \sum_{s=1}^{K} \Lambda_s^m v_L > 0 \quad \forall\, K > \underline{k} \tag{16a}$$

$$\bar{q}_m(\bar{k}+1) < \check{q}(\bar{k}) < \bar{q}_m(\bar{k}) \tag{16b}$$

$$\check{q}(\underline{k}+1) < \check{q}(\underline{k}+2) < \cdots < \check{q}(\bar{k}-1) < \check{q}(\bar{k}) \tag{16c}$$

where if $\bar{k} = m$, replace (16b) by $\check{q}(\bar{k}) < \bar{q}_m(\bar{k})$.

Proof. First, for (16a), note that $\mathcal{W}(K, 0) < 0$ and $\mathcal{W}(K, q_j) > 0$ for all $K > \underline{k}$. Moreover, $\mathcal{W}(K, q)$ is linear in q. Thus, $\mathcal{W}(K, q)$ is strictly increasing in q for all $K > \underline{k}$.[51] Second, for (16b), note that by the definition of $\bar{k}$, $\mathcal{W}\left(\bar{k}, \bar{q}_m(\bar{k})\right) > 0$. Since $\mathcal{W}(K, q)$ is strictly increasing, then $\check{q}(\bar{k}) < \bar{q}_m(\bar{k})$. If $\bar{k} = m$, this finishes the proof of (16b). Otherwise, note that the definition of $\bar{k}$ (and the genericity condition (15b)) imply that $\mathcal{W}\left(\bar{k}+1, \bar{q}_m(\bar{k}+1)\right) < 0$. Since $\mathcal{W}\left(\bar{k}+1, \bar{q}_m(\bar{k}+1)\right) = \mathcal{W}\left(\bar{k}, \bar{q}_m(\bar{k}+1)\right)$, then $\bar{q}_m(\bar{k}+1) < \check{q}(\bar{k})$. Finally, regarding equation (16c), note that:

$$\mathcal{W}(K, q) = \mathcal{W}(K-1, q) + (\hat{q} - q)\left[\Lambda_K^m v_L - \frac{c}{m}\right] + (1 - \hat{q})\left[\Lambda_K^m v_H - \frac{c}{m}\right]$$

Then, $\mathcal{W}(K, q) \geq \mathcal{W}(K-1, q) \Leftrightarrow q \geq \bar{q}_m(K)$. Suppose that $\check{q}(K) < \bar{q}_m(K)$. Then, $0 = \mathcal{W}(K, \check{q}(K)) < \mathcal{W}(K-1, \check{q}(K))$ and so $\check{q}(K-1) < \check{q}(K)$. Since, $\check{q}(\bar{k}) < \bar{q}_m(\bar{k})$, an inductive

[51] The strict monotonicity of $\mathcal{W}(K,q)$ together with the equality $\mathcal{W}\left(K, \bar{q}_m(K)\right) = \mathcal{W}\left(K-1, \bar{q}_m(K)\right)$ implies that $\mathcal{W}\left(K, \bar{q}_m(K)\right) > 0$ for all $K \in \{\underline{k}+1, \ldots, \bar{k}\}$. Furthermore, $\check{q}(K) < \bar{q}_m(K+1)$ for all $K \in \{\underline{k}+1, \ldots, \bar{k}-1\}$.

argument shows equation (16c).[52] ■

The buyer can guarantee a positive continuation payoff at any state (K, q) with $K \in \{\underline{k}+1, \ldots, \bar{k}\}$ and $q \in (\check{q}(K), q_j)$. This follows directly from the definition of $\check{q}(\cdot)$. The buyer can also guarantee a positive continuation payoff at any state (K, q) with $K \in \{\bar{k}+1, \ldots, m\}$ and $q \in [\check{q}(\bar{k}), q_j)$. This follows from the first inequality in equation (16b) and the fact that $\bar{q}_m(\cdot)$ is strictly decreasing in K. We represent these areas of the state space with thick green lines in Figure 5. As in Claim 8, starting from any state (K, q) with $\mathcal{W}(K, q) > 0$, the state (k_j, q_j) is reached without delay and an impasse of length τ_j occurs. Claim 9 summarizes these findings.[53] We omit the proof of Claim 9 since it is analogous to that of Claim 8.

CLAIM 9. *For all (K, q) with either 1) $K \in \left\{\underline{k}+1, \ldots, \bar{k}\right\}$ and $q \in \left[\check{q}(K), q_j\right)$ or 2) $K \in \left\{\bar{k}+1, \ldots, m\right\}$ and $q \in \left[\check{q}(\bar{k}), q_j\right)$ we have*

$$P_m(K, q) = (K - k_j)\frac{c}{m} + P_m^-\left(k_j, q_j\right),$$

$$W_m(K, q) = \mathcal{W}(K, q) \quad \textit{and}$$

$$(K_m(\tau; (K, q)), q_m(\tau; (K, q))) = \begin{cases} (k_j, q_j) & \textit{if } \tau \leq \tau_j \\ \left(K_m\left(\tau - \tau_j; (k_j, q_j)\right), q_m\left(\tau - \tau_j; (k_j, q_j)\right)\right) & \textit{if } \tau > \tau_j. \end{cases}$$

Claim 10 completes the description of the limit functions in the inductive step. States (K, q) with $K \in \{\underline{k}+1, \ldots, \bar{k}\}$ and $q < \check{q}(K)$ have $\mathcal{W}(K, q) < 0$. We represent these states with thick blue lines in Figure 5. Claim 10 shows that starting from any such (K, q), the state shifts without delay to $(K, \check{q}(K))$, where an impasse of length $\rho(K)$ occurs. The reason behind this impasse is that the function $P_m(K, \cdot)$ must be discontinuous at $\check{q}(K)$ for any $K \in \{\underline{k}+1, \ldots, \bar{k}\}$. If it were continuous, the buyer's continuation payoff would be negative at states (K, q) with q close (and to the left) of $\check{q}(K)$. This impasse makes the

[52]It is easy to show in a similar way that $\check{q}(\bar{k}) > \check{q}(\bar{k}+1) > \ldots > \check{q}(m)$. Thus $\bar{k} = \arg\max_{K \in \left\{k_j+1, \ldots, m\right\}} \{\check{q}(K)\}$, which is consistent with the definition of k_2 in section 4.2.1.

[53]In Claim 10 we show that there is a (potentially off-path) impasse at every state $(K, \check{q}(K))$ with $K \in \left\{\underline{k}+1, \ldots, \bar{k}\right\}$. Following the convention established in Claim 7, the limit functions evaluated at $(K, \check{q}(K))$ reflect the outcome after the impasse is resolved.

price $P^-_m\left(K,\check{q}(K)\right)$ low enough so that the buyer finds it optimal to move to state $(K,\check{q}(K))$ without delay.

Claim 10. *For all (K,q) with $K\in\{\underline{k}+1,\ldots,\overline{k}\}$ and $q\in[0,\check{q}(K))$ we have:*

$$P_m(K,q)=\frac{\left(\sum_{s=1}^{K}\Lambda_s^m v_L\right)^2}{(K-k_j)\frac{c}{m}+P^-_m\left(k_j,q_j\right)},$$
$$W_m(K,q)=\left(\check{q}(K)-q\right)\left(\sum_{s=1}^{K}\Lambda_s^m v_L\right)\left(1-\frac{\sum_{s=1}^{K}\Lambda_s^m v_L}{(K-k_j)\frac{c}{m}+P^-_m\left(k_j,q_j\right)}\right)$$

and

$$\left(K_m\left(\tau;(K,q)\right),q_m\left(\tau;(K,q)\right)\right)=\begin{cases}(K,\check{q}(K)) & \text{if } \tau\le\rho(K)\\ \Big(K_m\left(\tau-\rho(K);(K,\check{q}(K))\right), \\ \qquad q_m\left(\tau-\rho(K);(K,\check{q}(K))\right)\Big) & \text{if } \tau>\rho(K)\end{cases}$$
$$\text{with } \rho(K)=\frac{2}{r}\log\left(\frac{(K-k_j)\frac{c}{m}+P^-_m(k_j,q_j)}{\left(\sum_{s=1}^{K}\Lambda_s^m\right)v_L}\right).$$

See Section T.4 of the Technical Addendum for the proof.

We finally describe how the inductive step concludes. We let $(k_{j+1},q_{j+1})=\left(\overline{k},\check{q}\left(\overline{k}\right)\right)$ and $\tau_{j+1}=\rho(\overline{k})$. If $\overline{k}<m$, then the algorithm proceeds to the next inductive step. If $\overline{k}=m$, then the algorithm ends. Since m is finite, the algorithm ends in finitely many steps.

When the algorithm ends, it provides a collection $\left\{(k_j,q_j)\right\}_{j=1}^{J}$ of candidate impasses and a complete characterization of the limit functions. The last inductive step shows that starting at the initial state $(m,0)$, the state (k_J,q_J) is reached without delay and an impasse of length τ_J ensues. Each inductive step shows how after the impasse in state (k_j,q_j) is resolved, the state shifts without delay to (k_{j-1},q_{j-1}), where an additional impasse of length τ_{j-1} occurs. The base step shows that the game ends after the last impasse $(1,\bar{q}_m(1))$ is reached.

To sum up, all impasses in $\left\{(k_j,q_j)\right\}_{j=1}^{J}$ occur on-path.[54] Thus, the limit equilibrium

[54] All other impasses identified in Claim 10 in each inductive step are off-path.

outcome as bargaining frictions vanish consists of a sequence of phases of fast trade an impasses characterized by $\{(k_j, q_j)\}_{j=1}^{J}$. ■

A.6 Proof of Proposition 4

We first show equation (8d). We then proceed with the proof of equation (8b), which is the most involved part of the proof of Proposition 4 and includes several steps. We finally show how the remaining equations in Proposition 4 follow from equations (8b) and (8d).

Proof of equation (8d). Any impasse (k_j^m, q_j^m) must satisfy $\bar{q}_m(k_j^m + 1) < q_j^m < \bar{q}_m(k_j^m)$ (see Proposition 3). Together with the definitions of $\bar{q}(\cdot)$ and $\bar{q}_m(\cdot)$, and replacing $z_j^m = k_j^m/m$ when needed, this implies

$$\bar{q}\left(z_j^m + \frac{1}{m}\right) = \bar{q}\left(\frac{k_j^m + 1}{m}\right) < \bar{q}_m(k_j^m + 1) < q_j^m < \bar{q}_m(k_j^m) < \bar{q}\left(\frac{k_j^m - 1}{m}\right) = \bar{q}\left(z_j^m - \frac{1}{m}\right)$$

Notice that $\left|\frac{d\bar{q}(z)}{dz}\right|$ is bounded by some constant $\check{\rho} < \infty$ (because $\frac{d\lambda(\cdot)}{dz}$ is continuous). Thus,

$$\left|q_j^m - \bar{q}(z_j^m)\right| < \max\left\{\left|\bar{q}\left(z_j^m - 1/m\right) - \bar{q}(z_j^m)\right|; \left|\bar{q}\left(z_j^m + 1/m\right) - \bar{q}(z_j^m)\right|\right\} < \check{\rho}/m.$$

The bound $\check{\rho}$ is independent of j, so $\max\left\{\left|q_j^m - \bar{q}(z_j^m)\right|\right\}_{j=1}^{J_m} < \check{\rho}/m$, which leads to equation (8d):

$$\lim_{m\to\infty} \max\left\{\left|q_j^m - \bar{q}(z_j^m)\right|\right\}_{j=1}^{J_m} = 0$$

Proof of equation (8b). We split this proof in two parts. In the first one we construct a sequence of limits of consecutive impasses and show how to link these limits. In the second one we use this construction to show that limits of consecutive impasses must be arbitrarily close.

Construction of the sequence of limits of consecutive impasses. Assume towards a contradiction that

$$\limsup_{m\to\infty}\left(\max\left\{q_{j-1}^m - q_j^m\right\}_{j=2}^{J_m}\right) > 0.$$

Then, by taking a subsequence if necessary, we may assume that a sequence of consecutive impasses $\left\{\left(z^m_{j_m}, q^m_{j_m}\right), \left(z^m_{j_m-1}, q^m_{j_m-1}\right)\right\}_{m=1}^{\infty}$ that converges to $((z_0, q_0), (z_{-1}, q_{-1}))$ with $q_0 > q_{-1}$ exists. Equation (8d) guarantees that $q_0 = \bar{q}(z_0)$ and $q_{-1} = \bar{q}(z_{-1})$

The buyer obtains a zero continuation payoff at every impasse. Thus, the difference $W_m(mz^m_{j_m}, q^m_{j_m}) - W_m(mz^m_{j_m-1}, q^m_{j_m-1})$, which we express in equation (17), is also zero:[55]

$$\begin{aligned} \left(q^m_{j_m-1} - q^m_{j_m}\right)\left[\int_0^{z^m_{j_m}} \lambda(z) v_L dz - P^+_m\left(mz^m_{j_m}, q^m_{j_m}\right)\right] \\ + \left(\hat{q} - q^m_{j_m-1}\right)\int_{z^m_{j_m-1}}^{z^m_{j_m}} \left[\lambda(z) v_L - c\right] dz + (1-\hat{q})\int_{z^m_{j_m-1}}^{z^m_{j_m}} \left[\lambda(z) v_H - c\right] dz = 0 \end{aligned} \tag{17}$$

The left hand side of equation (17) is continuous in $\left(z^m_{j_m}, q^m_{j_m}\right)$, $\left(z^m_{j_m-1}, q^m_{j_m-1}\right)$ and $P^+_m\left(mz^m_{j_m}, q^m_{j_m}\right)$. Moreover it strictly decreases in $P^+_m\left(mz^m_{j_m}, q^m_{j_m}\right)$, with derivative bounded away from zero. Hence, since $\left\{\left(z^m_{j_m}, q^m_{j_m}\right)\right\}_{m=1}^{\infty}$ and $\left\{\left(z^m_{j_m-1}, q^m_{j_m-1}\right)\right\}_{m=1}^{\infty}$ converge, then $\left\{P^+_m\left(mz^m_{j_m}, q^m_{j_m}\right)\right\}_{m=1}^{\infty}$ must also converge. We let P^+_0 denote its limit. Equation (18) expresses equation (17) in the limit:

$$\begin{aligned} (q_{-1} - q_0)\left[\int_0^{\psi(q_0)} \lambda(z) v_L dz - P^+_0\right] \\ + (\hat{q} - q_{-1})\int_{\psi(q_{-1})}^{\psi(q_0)} \left[\lambda(z) v_L - c\right] dz + (1-\hat{q})\int_{\psi(q_{-1})}^{\psi(q_0)} \left[\lambda(z) v_H - c\right] dz = 0 \end{aligned} \tag{18}$$

with a change of variables taking advantage of $z_\ell = \psi(q_\ell)$ for $\ell \in \{0, -1\}$, where $\psi(\cdot)$ is the inverse of $\bar{q}(\cdot)$. Equation (18) links the limits (z_0, q_0) and (z_{-1}, q_{-1}).

We show next that $q_{-1} < \bar{q}(0)$ (and so $z_{-1} > 0$). Assume towards a contradiction that $q_{-1} = \bar{q}(0)$ and $z_{-1} = 0$. This implies that $P^+_0 = z_0 c$.[56] Using this, we rewrite the left hand side of equation (18) as

$$(\hat{q} - q_0)\left[\int_0^{\psi(q_0)} \left[\lambda(z) v_L - c\right] dz\right] + (1-\hat{q})\int_0^{\psi(q_0)} \left[\lambda(z) v_H - c\right] dz < 0$$

where the inequality follows from the definition of $\psi(\cdot)$. This leads to a contradiction.

[55]We use equation (4c) to obtain equation (17).

[56]Equation (4b) implies that $P^-_m\left(mz^m_{j_m-1}, q^m_{j_m-1}\right) < v_L \int_0^{z^m_{j_m-1}} \lambda(z) dz$, which converges to zero as $m \to \infty$. This and equation (4c) imply that $P^+_m\left(mz^m_{j_m}, q^m_{j_m}\right)$ becomes arbitrarily close to $(z^m_{j_m} - z^m_{j_m-1})c$ as $m \to \infty$.

For every (large enough) m there exists an impasse $\left(z^m_{j_m-2}, q^m_{j_m-2}\right)$ that occurs after $\left(z^m_{j_m-1}, q^m_{j_m-1}\right)$ is resolved. This is because the last impasse occurs at $z = \frac{1}{m}$ and $z_{-1} > 0$. Assume, by taking a subsequence if necessary, that the sequence $\left\{\left(z^m_{j_m-2}, q^m_{j_m-2}\right)\right\}_{m=1}^{\infty}$ converges to (z_{-2}, q_{-2}). By an argument like the one for q_{-1}, then also $q_{-2} < \bar{q}(0)$.

We show next that $q_{-1} < q_{-2}$. Assume instead that $q_{-1} = q_{-2}$ (so $z_{-1} = z_{-2}$). Equation (4c) then implies $\lim_{m\to\infty} P^+_m\left(mz^m_{j_m-1}, q^m_{j_m-1}\right) - P^-_m\left(mz^m_{j_m-2}, q^m_{j_m-2}\right) = 0$. Proposition 3 guarantees that in general

$$P^-_m\left(mz^m_{j_m-2}, q^m_{j_m-2}\right) < v_L \int_0^{z^m_{j_m-2}} \lambda(z)dz < v_L \int_0^{z^m_{j_m-1}} \lambda(z)dz < P^+_m\left(mz^m_{j_m-1}, q^m_{j_m-1}\right).$$

Thus, $q_{-1} = q_{-2}$ implies $\lim_{m\to\infty} P^+_m\left(mz^m_{j_m-1}, q^m_{j_m-1}\right) = \lim_{m\to\infty} P^-_m\left(mz^m_{j_m-2}, q^m_{j_m-2}\right) = v_L \int_0^{z_{-1}} \lambda(z)dz$. Finally, we link $P^+_m\left(mz^m_{j_m}, q^m_{j_m}\right)$ and $P^+_m\left(mz^m_{j_m-1}, q^m_{j_m-1}\right)$ using equations (4b) and (4c) and take limits to obtain

$$P^+_0 = (z_0 - z_{-1})\, c + v_L \int_0^{z_{-1}} \lambda(z)dz.$$

We plug this expression for P^+_0 in the left hand side of equation (18) and obtain the following contradiction:

$$(\hat{q} - q_0) \int_{\psi(q_{-1})}^{\psi(q_0)} [\lambda(z)v_L - c]\, dz + (1 - \hat{q}) \int_{\psi(q_{-1})}^{\psi(q_0)} [\lambda(z)v_H - c]\, dz < 0$$

The same argument that shows that the sequence $\left\{P^+_m\left(mz^m_{j_m}, q^m_{j_m}\right)\right\}_{m=1}^{\infty}$ must converge to P^+_0 also guarantees that the sequence $\left\{P^+_m\left(mz^m_{j_m-1}, q^m_{j_m-1}\right)\right\}_{m=1}^{\infty}$ must converge, and its limit, which we denote by P^+_{-1} must satisfy an equation like (18):

$$\begin{aligned} &(q_{-2} - q_{-1})\left[\int_0^{\psi(q_{-1})} \lambda(z)v_L dz - P^+_{-1}\right] \\ &+ (\hat{q} - q_{-2}) \int_{\psi(q_{-2})}^{\psi(q_{-1})} [\lambda(z)v_L - c]\, dz + (1 - \hat{q}) \int_{\psi(q_{-2})}^{\psi(q_{-1})} [\lambda(z)v_H - c]\, dz = 0 \end{aligned}$$

The previous equation links the limits (z_{-1}, q_{-1}) and (z_{-2}, q_{-2}) of the sequences of consecutive impasses $\left\{\left(z^m_{j_m-1}, q^m_{j_m-1}\right)\right\}_{m=1}^{\infty}$ and $\left\{\left(z^m_{j_m-2}, q^m_{j_m-2}\right)\right\}_{m=1}^{\infty}$.

We next link the limit prices P_0^+ and P_{-1}^+ using equations (4b) and (4c). Equation (4c) links $P_m^+\left(mz_{j_m}^m, q_{j_m}^m\right)$ and $P_m^-\left(mz_{j_m-1}^m, q_{j_m-1}^m\right)$. Equation (4b) links $P_m^-\left(mz_{j_m-1}^m, q_{j_m-1}^m\right)$ and $P_m^+\left(mz_{j_m-1}^m, q_{j_m-1}^m\right)$. Using these equations together, and taking limits, we obtain

$$P_0^+ = [\psi(q_0) - \psi(q_{-1})]\, c + \frac{\left(v_L \int_0^{\psi(q_{-1})} \lambda z dz\right)^2}{P_{-1}^+}. \tag{19}$$

We proceed recursively and construct, taking subsequences if necessary, a collection of sequences of impasses $\left\{\left\{\left(z_{j_m-\ell}^m, q_{j_m-\ell}^m\right)\right\}_{m=1}^{\infty}\right\}_{\ell=0}^{\infty}$, where, for every ℓ, the sequence $\left\{\left(z_{j_m-\ell}^m, q_{j_m-\ell}^m\right)\right\}_{m=1}^{\infty}$ converges to $(z_{-\ell}, q_{-\ell})$ as m grows to infinity. Furthermore, for every ℓ, the sequence $\left\{P_m^+\left(mz_{j_m-\ell}^m, q_{j_m-\ell}^m\right)\right\}_{m=1}^{\infty}$ converges to $P_{-\ell}^+$.

For every $\ell = 0, 1, \ldots$ the limits of consecutive impasses must satisfy equations (20) and (21).

$$\begin{aligned}&\left(q_{-(\ell+1)} - q_{-\ell}\right)\left[\int_0^{\psi(q_{-\ell})} \lambda(z) v_L dz - P_{-\ell}^+\right] \\ &\quad + \left(\hat{q} - q_{-(\ell+1)}\right)\int_{\psi(q_{-(\ell+1)})}^{\psi(q_{-\ell})} [\lambda(z)v_L - c]\, dz + (1-\hat{q})\int_{\psi(q_{-(\ell+1)})}^{\psi(q_{-\ell})} [\lambda(z)v_H - c]\, dz = 0\end{aligned} \tag{20}$$

$$P_{-\ell}^+ = \left[\psi(q_{-\ell}) - \psi(q_{-(\ell+1)})\right] c + \frac{\left(v_L \int_0^{\psi(q_{-(\ell+1)})} \lambda(z) dz\right)^2}{P_{-(\ell+1)}^+} \tag{21}$$

These conditions mirror equations (18) and (19). Finally, limit beliefs satisfy

$$q_0 < q_{-1} < \ldots < q_{-\ell} < \ldots < \bar{q}(0). \tag{22}$$

Bounding the distance between limits of consecutive impasses. In the remainder of the proof we focus on the collection $\left\{\left(q_{-\ell}, P_{-\ell}^+\right)\right\}_{\ell=0}^{\infty}$ which satisfies equations (20), (21), and (22). We show that the limit beliefs $\{q_{-\ell}\}_{\ell=0}^{\infty}$ are arbitrarily close to each other. To do this, we obtain explicit bounds that link successive limit impasses by using equations (20) and (21). These bounds link differences between consecutive beliefs and also differences between prices and valuations. Facts 2 and 3 state the first bounds (see Section T.5 of the Technical Addendum for their proof).

FACT 2. *There exists $\eta^* > 0$ such that for every $\ell \geq 1$, if $q_{-(\ell+1)} - q_{-\ell} < \eta^*$, then $q_{-\ell} - q_{-(\ell-1)} < \frac{4}{3}\left(q_{-(\ell+1)} - q_{-\ell}\right)$.*

FACT 3. *There exists constants $b_1 > 0$ and $b_2 > 0$ such that for every $\ell = 0, 1, \ldots,$ we have:*

$$\frac{\left[P^+_{-\ell} - \int_0^{\psi(q_{-\ell})} \lambda(z) v_L dz\right] - \left[P^+_{-(\ell+1)} - \int_0^{\psi(q_{-(\ell+1)})} \lambda(z) v_L dz\right]}{q_{-(\ell+1)} - q_{-\ell}} \leq b_1 \left(q_{-(\ell+1)} - q_{-\ell}\right) \quad (23)$$

$$q_{-(\ell+1)} - q_{-\ell} \leq b_2 \left[P^+_{-\ell} - \int_0^{\psi(q_{-\ell})} \lambda(z) v_L dz\right] \quad (24)$$

Using Facts 2 and 3 we prove Claims 11 and 12, which provide further bounds. Claim 11 links successive differences between prices and valuations and Claim 12 links differences between successive beliefs.

CLAIM 11. *Consider ℓ' and ℓ'' with $0 \leq \ell' < \ell''$. Let $\varepsilon > 0$ and $\eta > 0$ be such that $q_{-(\ell+1)} - q_{-\ell} < \varepsilon$ for all $\ell \in \{\ell', \ldots, \ell'' - 1\}$ and $q_{-\ell''} - q_{-\ell'} < \eta$. Then, for every $\ell \in \{\ell', \ldots, \ell'' - 1\}$, we have:*

$$P^+_{-\ell} - \int_0^{\psi(q_{-\ell})} \lambda(z) v_L dz < P^+_{-\ell''} - \int_0^{\psi(q_{-\ell''})} \lambda(z) v_L dz + \varepsilon\eta b_1$$

Proof. For every $\ell \in \{\ell', \ldots, \ell'' - 1\}$ we have

$$\begin{aligned} P^+_{-\ell} - \int_0^{\psi(q_{-\ell})} \lambda(z) v_L dz &= P^+_{-(\ell+1)} - \int_0^{\psi(q_{-(\ell+1)})} \lambda(z) v_L dz \\ &\quad + \left(q_{-(\ell+1)} - q_{-\ell}\right) \frac{P^+_{-\ell} - \int_0^{\psi(q_{-\ell})} \lambda(z) v_L dz - \left(P^+_{-(\ell+1)} - \int_0^{\psi(q_{-(\ell+1)})} \lambda(z) v_L dz\right)}{q_{-(\ell+1)} - q_{-\ell}} \\ &< P^+_{-(\ell+1)} - \int_0^{\psi(q_{-(\ell+1)})} \lambda(z) v_L dz + \varepsilon b_1 \left(q_{-(\ell+1)} - q_{-\ell}\right) \end{aligned}$$

where the inequality follows from $q_{-(\ell+1)} - q_{-\ell} < \varepsilon$ and equation (23) in Fact 3. Applying the same argument recursively leads to

$$\begin{aligned} P^+_{-\ell} - \int_0^{\psi(q_{-\ell})} \lambda(z) v_L dz &< P^+_{-\ell''} - \int_0^{\psi(q_{-\ell''})} \lambda(z) v_L dz + \varepsilon b_1 \sum_{\tilde{\ell}=\ell}^{\ell''-1} \left(q_{-(\tilde{\ell}+1)} - q_{-\tilde{\ell}}\right) \\ &< P^+_{-\ell''} - \int_0^{\psi(q_{-\ell''})} \lambda(z) v_L dz + \varepsilon\eta b_1 \qquad \blacksquare \end{aligned}$$

Claim 12. *Consider ℓ' and ℓ'' with $1 \leq \ell' < \ell''$. Let $0 < \varepsilon < \eta^*$ and $0 < \eta < (3b_1b_2)^{-1}$ be such that $q_{-(\ell+1)} - q_{-\ell} < \varepsilon$ for all $\ell \in \{\ell', \ldots, \ell''-1\}$, $q_{-\ell''} - q_{-\ell'} < \eta$ and $P^+_{-\ell''} - \int_0^{\psi(q_{-\ell''})} \lambda(z) v_L dz < (3b_2)^{-1}\varepsilon$. Then, $q_{-\ell'} - q_{-(\ell'-1)} < \varepsilon$.*

Proof. We have

$$q_{-(\ell'+1)} - q_{-\ell'} \leq b_2 \left[P^+_{-\ell'} - \int_0^{\psi(q_{-\ell'})} \lambda(z) v_L dz\right] < b_2 \left(P^+_{-\ell''} - \int_0^{\psi(q_{-\ell''})} \lambda(z) v_L dz + \varepsilon\eta b_1\right)$$
$$< b_2 \left((3b_2)^{-1}\varepsilon + \varepsilon(3b_1b_2)^{-1}b_1\right) < \frac{2}{3}\varepsilon$$

where the first inequality follows from equation (24) in Fact 3 and the second one from Claim 11. This, together with Fact 2, implies that

$$q_{-\ell'} - q_{-(\ell'-1)} < \frac{4}{3}\left(q_{-(\ell'+1)} - q_{-\ell'}\right) < \left(\frac{4}{3}\right)\left(\frac{2}{3}\right)\varepsilon < \varepsilon \qquad \blacksquare$$

Claim 12 provides the last intermediate result to complete the proof of equation (8b). The sequence $\{q_{-\ell}\}_{\ell=0}^{\infty}$ is strictly increasing and bounded above by $\bar{q}(0)$. Then, it has a limit, which we denote by $q_{-\infty}$. With this, applying L'Hôpital's rule to equation (20) we obtain

$$\lim_{\ell\to\infty} P^+_{-\ell} - \int_0^{\psi(q_{-\ell})} \lambda(z) v_L dz = 0.$$

We focus on elements of the sequence $\{q_{-\ell}\}_{\ell=0}^{\infty}$ which are sufficiently close to $q_{-\infty}$. Let $\ell' = \min\left\{\ell : q_{-\ell} \geq q_{-\infty} - (6b_1b_2)^{-1}\right\}$. Fix $\varepsilon = \frac{1}{2}\min\{q_{-\ell'} - q_{-\ell'+1}; \eta^*\} > 0$ and pick ℓ'' such that:

$$\max\left\{q_{-(\ell''+1)} - q_{-\ell''}; P^+_{-\ell''} - \int_0^{\psi(q_{-\ell''})} \lambda(z) v_L dz\right\} < \min\left\{\varepsilon, (3b_2)^{-1}\varepsilon\right\}$$

Then, applying Claim 12 recursively, we obtain $q_{-\ell'} - q_{-\ell'+1} < \varepsilon$, which is a contradiction and completes the proof of equation (8b).

Proof of equations (8a)**,** (8c)**,** (8e) **and** (8f). Equations (8b) and (8d) together imply equation (8a). Equation (17) links *any* sequence of consecutive impasses. We take the limit of equation (17) as m grows large, use equations (8b) and (8d) and apply L'Hôpital's rule to obtain equation (8f). Equation (8e) follows from equation (8f) and equation (4b) in Proposition 3. Finally, we show equation (8c) by contradiction. Assume instead that, taking subsequences if necessary, $\lim_{m\to\infty} z^m_{J_m} = \bar{z} < 1$. This, together with equation (8e), implies that, in the limit, the buyer's continuation payoff at the beginning of the game is negative:

$$\lim_{m\to\infty} W_m(m,0) = \hat{q}\left[\int_{\bar{z}}^1 [\lambda(z)v_L - c]\, dz\right] + (1-\hat{q})\left[\int_{\bar{z}}^1 [\lambda(z)v_H - c]\, dz\right] < 0$$

This can never happen, so we have reached a contradiction. ■

A.7 Proof of Proposition 1

Proof. We present here the proof for (a). The cases (b), (c) and (d) follow the same argument. Assume towards a contradiction that the result does not hold for (a). Equation (3b) implies that

$$\tilde{z}^{*\prime}(0) = \frac{rv_L \int_0^1 \tilde{\lambda}(z)dz}{v_L\tilde{\lambda}(1) - c} < \frac{rv_L \int_0^1 \lambda(z)dz}{v_L\lambda(1) - c} = z^{*\prime}(0).$$

Let $\underline{\tau} = \min\{\tau > 0 : \tilde{z}^*(\tau) = z^*(\tau)\}$. It follows again from equation (3b) that

$$\tilde{z}^{*\prime}(\underline{\tau}) = \frac{rv_L \int_0^{\tilde{z}^*(\underline{\tau})} \tilde{\lambda}(z)dz}{v_L\tilde{\lambda}(\tilde{z}^*(\underline{\tau})) - c} < \frac{rv_L \int_0^{z^*(\underline{\tau})} \lambda(z)dz}{v_L\lambda(z^*(\underline{\tau})) - c} = z^{*\prime}(\underline{\tau}).$$

But then there exists $\tau' \in (0,\underline{\tau})$ with $\tilde{z}^{*\prime}(\tau') = z^{*\prime}(\tau')$, reaching a contradiction. Finally, notice that $z^*(0) = 1$ and that $z^{*\prime}(\cdot)$ does not depend on v_H or $\hat{\beta}$. ■

A.8 Proof of Proposition 5

Proof. Let $\lambda(z) = 1$ for every $z \in [0,1]$. Fix the number of units m and the period length Δ. Let $W(1,\cdot)$ and $P(1,\cdot)$ be respectively the buyer's normalized payoff and the price function when one unit remains. These functions are as in DL, so $W(1,q) > 0$ for every $q \in [0,\hat{q}]$. Suppose that for every $K \in \{1,\ldots,m\}$ and for every $q \in [0,\hat{q}]$, $W(K,q) = KW(1,q)$ and $P(K,q) = KP(1,q)$. Finally consider a belief $q' \in [0,\hat{q}]$ such that the buyer makes a screening offer at state $(1,q')$. The following argument shows that it is not optimal for the buyer to make a universal offer at any state (K, q') with $K \in \{2, \ldots, m\}$. Assume towards a contradiction that it is optimal to make a universal offer for $K-k$ units. Then,

$$W(K, q') = KW(1, q') \leq \frac{K-k}{m}\left[(\hat{q} - q')\, v_L + (1-\hat{q})\, v_H - (1-q')c\right] + \delta k W(1, q')$$
$$< \frac{K-k}{m}\left[(\hat{q} - q')\, v_L + (1-\hat{q})\, v_H - (1-q')c\right] + k W(1, q')$$

This in turn, implies that

$$W(1, q') < \frac{1}{m}\left[(\hat{q} - q')\, v_L + (1-\hat{q})\, v_H - (1-q')c\right]$$

which violates the assumption that a screening offer is optimal at state $(1, q')$. This argument directly implies Proposition 5 when gains from trade are constant.

An argument analogous to the one in the previous paragraphs extends the result to the case of increasing gains from trade. We omit the proof here.[57] ■

[57] An earlier version of our paper contains further details on the cases of constant and increasing returns.

NOT FOR PUBLICATION

Technical Addendum to "Bargaining over a Divisible Good in the Market for Lemons"

T.1 Proof of Claim 1

Proof. By contradiction. Assume that $q' \in Y(K,q)$ and that $P(K,q'') = P(K,q')$ for some $q'' > q'$ and consider the course of action started by choosing q'. We show that there exists an alternative course of action that leads to a strictly higher payoff than that from the course of action started by choosing q'. To simplify the algebra, in what follows we focus on a specific (optimal) course of action started by choosing q'. Assume that in the two periods following the screening offer $(P(K,q'),q')$, the buyer makes offers implied by $y(K,q') = k$ and $y(k,q') = q'''$ with $q''' > q''$. The alternative course of action involves inducing the belief q'' in the first period. In the following two periods, the buyer mimics the behavior from the first course of action. He makes a universal offer for $K-k$ units in the second period and induces belief q''' in the third period.

The difference in payoffs between the alternative course of action and the original one is given by:

$$(q''-q')\left[\overbrace{v_L\left[(1-\delta)\sum_{j=k}^{K}\Lambda_j + (1-\delta^2)\sum_{j=1}^{k}\Lambda_j\right]}^{>0} + \overbrace{\left[\delta\frac{c}{m}(K-k) + \delta^2 P\left(k,q'''\right) - P(K,q')\right]}^{\geq 0}\right] > 0$$

The weak inequality in the second term is a direct consequence of the definitions of P and $\mathcal{V}_L'$, together with the equality $\mathcal{V}_L = \mathcal{V}_L'$.[58] ■

[58] In general, consider an arbitrary optimal course of action started by choosing q' in period t. Let $t+T$ denote the first period in which the buyer makes a screening offer that leads to a posterior $q''' > q''$. The behavior in periods $t' \in \{t+1,\ldots,t+T-1\}$ encompasses screening and universal offers. Let T^1 be the subset of $\{t+1,\ldots,t+T-1\}$ at which the buyer makes screening offers and T^2 be those periods at which the buyer makes universal offers. We define an alternative course of action as follows. First, the buyer induces posterior q'' in period t. Second, the buyer makes no offers in periods $t' \in T^1$. Third, the buyer makes the same universal offers as in the optimal course of action in periods $t' \in T^2$. Finally, the buyer induces belief q''' in period $t+T$. The definitions of P and $\mathcal{V}_L'$, together with the equality $\mathcal{V}_L = \mathcal{V}_L'$ imply that the alternative course of action leads to a strictly higher payoff.

T.2 Details of the proof of Claim 3

Proof. In what follows we show, by contradiction, that $W^I(K, q_{n+1}) > 0$. Take $\varepsilon > 0$ with $q_{n+1} + \varepsilon < q_n$. Then, the buyer's continuation payoff $W^I(K, q_{n+1} + \varepsilon)$ is bounded below by the value of choosing the posterior $\overline{x}(K, q_{n+1})$:

$$\begin{aligned} W^I(K, q_{n+1} + \varepsilon) &\geq [\overline{x}(K, q_{n+1}) - q_{n+1} - \varepsilon] \left(\sum_{j=1}^{K} \Lambda_j^m v_L - P\left(K, \overline{x}(K, q_{n+1})\right) \right) \\ &\quad + \delta W\left(K, \overline{x}(K, q_{n+1})\right) \\ &= W^I(K, q_{n+1}) - \varepsilon \left(\sum_{j=1}^{K} \Lambda_j^m v_L - P\left(K, \overline{x}(K, q_{n+1})\right) \right) \end{aligned} \tag{T1}$$

Similarly, the buyer's continuation payoff $W^{II}(K, q_{n+1})$ is bounded below by the value of choosing the posterior $q_{n+1} + \varepsilon$:

$$W^{II}(K, q_{n+1}) \geq \varepsilon \left(\sum_{j=1}^{K} \Lambda_j^m v_L - P^I\left(K, q_{n+1} + \varepsilon\right) \right) + \delta W^I\left(K, q_{n+1} + \varepsilon\right) \tag{T2}$$

Assume towards a contradiction that $W^I(K, q_{n+1}) = W^{II}(K, q_{n+1}) = 0$. This, together with equations (T1) and (T2) leads to:

$$0 \geq \varepsilon \left[(1 - \delta) \sum_{j=1}^{K} \Lambda_j^m v_L - \delta \left[P\left(K, \underline{x}(K, q_{n+1} + \varepsilon)\right) - P\left(K, \overline{x}(K, q_{n+1})\right) \right] \right]$$

We show next that $\lim_{\varepsilon \downarrow 0} P\left(K, \underline{x}\left(K, q_{n+1} + \varepsilon\right)\right) = P\left(K, \overline{x}(K, q_{n+1})\right)$. This implies that the right hand side is strictly positive for $\varepsilon > 0$ low enough, which implies a contradiction. To show this, note that the objective function in (13) has strictly increasing differences in q at all maximizers. Thus, $X(K, \cdot)$ is a nondecreasing correspondence: if $q' > q$, then $\underline{x}\left(K, q'\right) \geq \overline{x}\left(K, q\right)$. Moreover, the theorem of the maximum guarantees that $X(K, \cdot)$ is upper hemicontinuous.

First, since $X(K, \cdot)$ is a non-decreasing upper hemicontinuous correspondence, then $\lim_{\varepsilon \downarrow 0} \underline{x}\left(K, q_{n+1} + \varepsilon\right) = \overline{x}(K, q_{n+1})$. If $P\left(K, \cdot\right)$ is continuous at $\overline{x}(K, q_{n+1})$, then this implies that $\lim_{\varepsilon \downarrow 0} P\left(K, \underline{x}\left(K, q_{n+1} + \varepsilon\right)\right) = P\left(K, \overline{x}(K, q_{n+1})\right)$. Second, if instead $P\left(K, \cdot\right)$ is discontinuous at $\overline{x}(K, q_{n+1})$, then $\underline{x}(K, q_{n+1}+\varepsilon) = \overline{x}(K, q_{n+1})$ for ε sufficiently small. This guarantees that $\lim_{\varepsilon\to 0} P\left(K, \underline{x}\left(K, q_{n+1}+\varepsilon\right)\right) - P\left(K, \overline{x}(K, q_{n+1})\right) = 0$, so $W^I(K, q_{n+1}) > 0$. ■

T.3 Proof of Claim 5

Proof. We show by contradiction that $q_{\tilde{n}} = 0$ for some $\tilde{n}$. Assume instead that $\lim_{n\to\infty} q_n = q^* > 0$. We split this proof in two exhaustive cases.

Case 1. Assume that there exists a sequence of transformed beliefs $\{q_j\}_{j=1}^{\infty}$ with $q_j > q^*$ for all j, with $\lim_{j\to\infty} q_j = q^*$, and such that at all those beliefs, the buyer makes screening offers: $y(K, q_j) = q'_j$. This implies that for any $\eta > 0$, there exists j with $q'_j - q_j < \eta$. Take a subsequence $\{q_{j_r}\}_{r=1}^{\infty}$ with $q_{j_r} < q'_{j_r} < q_{j_{r-1}}$. The function $P(K, \cdot)$ is non-decreasing and satisfies $P(K, q) \leq \delta \mathcal{V}_L(K, q)$ for all q. Moreover, whenever the buyer makes a screening offer $y(K, q) = q'$, it must be true that $\mathcal{V}_L(K, q) = P(K, q')$. Then $P(K, q_{j_r}) \leq \delta P(K, q'_{j_r}) \leq \delta P(K, q_{j_{r-1}})$. This implies that $\lim_{q\to q^*} P(K, q) = 0$, and so $\inf_{q\in(q^*,\hat{q}]} W(K, q) > 0$.

Fix $\varepsilon > 0$ so that

$$\left[\sum_{j=1}^{K} \Lambda_j^m v_H + \delta\right] \varepsilon < (1-\delta) \inf_{q\in(q^*,\hat{q}]} W(K, q). \tag{T3}$$

Uniform continuity of $W(K, \cdot)$ guarantees that there exists $\tilde{\eta} \in (0, \varepsilon)$ such that for every $(q, \tilde{q}) \in (q^*, \hat{q}] \times (q^*, \hat{q}]$, whenever $|q - \tilde{q}| < \tilde{\eta}$, then $|W(K, q) - W(K, \tilde{q})| < \varepsilon$. Pick $q_{\hat{j}} \in \{q_j\}_{j=1}^{\infty}$ such that $q'_{\hat{j}} - q_{\hat{j}} < \tilde{\eta}$. Then,

$$\min\left\{W(K, q_{\hat{j}}), W(K, q'_{\hat{j}})\right\} \leq W(K, q_{\hat{j}}) \leq \sum_{j=1}^{K} \Lambda_j^m v_H \varepsilon + \delta W(K, q'_{\hat{j}}) \leq$$

$$\sum_{j=1}^{K} \Lambda_j^m v_H \varepsilon + \delta\left(\min\left\{W(K, q_{\hat{j}}), W(K, q'_{\hat{j}})\right\} + \varepsilon\right) < \min\left\{W(K, q_{\hat{j}}), W(K, q'_{\hat{j}})\right\}$$

where the last inequality follows from equation (T3). We have reached a contradiction. If there is only one unit left ($K = 1$), case 1 covers all possibilities (as in DL). If there is more than one unit left ($K \geq 2$), the buyer may make no screening offers close to q^*. The following case covers this remaining possibility.

Case 2. Assume there exists an interval $(q^*, q^* + \eta')$ where the buyer only makes universal offers for some number $K - k$ of remaining units: $y(K, q) = k$ for all $q \in (q^*, q^* + \eta')$.

$W(k, \cdot)$ is bounded away from zero for all $k < K$. Thus, any universal offer for $K - k$ units must be followed by a screening offer. Furthermore, the low-type seller accepts the screening offer that the buyer makes with probability bounded away from zero. These two facts together imply that there exist n' and $\tilde{q} > q_{n'}$ such that for all $n \geq n'$ we have that $y(K, q_n) = k$ and $y(k, q_n) = \tilde{q}$. In what follows we show that $\lim_{n\to\infty} W(K, q_n) = 0$.

Consider a small $\varepsilon > 0$. Uniform continuity of $W^I(K, \cdot)$ guarantees that there exists $\eta \in (0, \varepsilon)$ such that for every $(q, \tilde{q}) \in (q^*, \hat{q}] \times (q^*, \hat{q}]$, whenever $|q - \tilde{q}| < \eta$, then $\left|W^I(K, q) - W^I(K, \tilde{q})\right| < \varepsilon$. Furthermore, there exists n'' such that $q_n - q_{n+1} < \eta$ for every $n \geq n''$. Therefore, for every $n \geq \overline{n} \equiv \max\{n', n''\}$ we have

$$
\begin{aligned}
W^I(K, q_{n+1}) &= W^{II}(K, q_{n+1}) \\
&= \max_{q' \in [q_{n+1}, q_n]} \left(q' - q_{n+1}\right) \left(\sum_{j=1}^{K} \Lambda_j v_L - P^I\left(K, q'\right) \right) + \delta W^I\left(K, q'\right) \\
&\leq \varepsilon \sum_{j=1}^{K} \Lambda_j v_L + \delta \max_{q' \in [q_{n+1}, q_n]} W^I\left(K, q'\right) \\
&\leq \varepsilon \sum_{j=1}^{K} \Lambda_j v_L + \delta \left(W^I\left(K, q_{n+1}\right) + \varepsilon \right)
\end{aligned}
$$

Then,

$$
W^I(K, q_{n+1}) \leq \frac{\varepsilon}{1 - \delta} \left(\sum_{j=1}^{K} \Lambda_j v_L + \delta \right)
$$

This implies that $\lim_{n\to\infty} W^I(K, q_n) = 0$. Moreover, for all $n \geq \overline{n}$ we have

$$
W_B^I(K, q_{n+1}) \geq -\varepsilon \frac{c}{m} K + \delta W(K, q_{n+1})
$$

which in turn implies that $\lim_{n\to\infty} W(K, q_n) = 0$.

We have

$$P(K, q_{\overline{n}}) \leq \delta \mathcal{V}_L(K, q_{\overline{n}}) = \delta \left[\frac{c}{m}(K-k) + \delta P(k, \tilde{q}) \right]. \tag{T4}$$

Suppose that the state is (K, q_n) and consider a screening offer $(K, P(K, q_{\overline{n}}))$. Then,

$$\begin{aligned} W(K, q_n) &\geq (q_{\overline{n}} - q_n) \left[\sum_{j=1}^{K} \Lambda_j^m v_L - P(K, q_{\overline{n}}) \right] \\ &\geq (q_{\overline{n}} - q_n) \left[\sum_{j=1}^{K} \Lambda_j^m v_L - \delta \left[\frac{c}{m}(K-k) + \delta P(k, \tilde{q}) \right] \right] \end{aligned}$$

Since $\lim_{n\to\infty} W(K, q_n) = 0$ and $q_{\overline{n}} - q_n$ is positive and bounded away from zero, then it must be true that

$$\delta \left[\frac{c}{m}(K-k) + \delta P(k, \tilde{q}) \right] \geq \sum_{j=1}^{K} \Lambda_j^m v_L.$$

In what follows we describe a course of action for the buyer that, when started in state (K, q) with $q < q_{\bar{n}}$ provides the buyer with continuation payoff of at least $R(q)$. We show next that $\lim_{n\to\infty} R(q_n)$ is positive and bounded away from zero. This contradicts our previous result that $\lim_{n\to\infty} W(K, q_n) = 0$. The course of action is as follows. In the first period, the buyer makes the screening offer $(K, P(K, q_{\bar{n}}))$. In the second period, the buyer makes the universal offer $\left(K-k, \frac{c}{m}(K-k)\right)$. In the third period the buyer is in state $(k, q_{\bar{n}})$. From that period on, he follows the optimal strategy. The continuation payoff from this alternative course of action at state (K, q) with $q < q_{\bar{n}}$ is bounded below by $R(q)$, given by:[59]

$$\begin{aligned} R(q) = {} & (q_{\bar{n}} - q) \left(\sum_{j=1}^{K} \Lambda_j^m v_L - \delta \left[\frac{c}{m}(K-k) + \delta P(k, \tilde{q}) \right] \right) \\ & + \delta \left[\left[(\hat{q} - q_{\bar{n}})\, v_L + (1-\hat{q})\, v_H \right] \sum_{j=k+1}^{K} \Lambda_j^m - (1 - q_{\bar{n}}) \frac{c}{m}(K-k) \right] \\ & + \delta^2 (\tilde{q} - q_{\bar{n}}) \left(\sum_{j=1}^{k} \Lambda_j^m v_L - P(k, \tilde{q}) \right) + \delta^3 W(k, \tilde{q}) \end{aligned}$$

[59]The bound is a direct consequence of equation (T4).

For all states (K,q) with $q \in (q^*, q_{\bar{n}}]$ the buyers's continuation payoff is given by:

$$
\begin{aligned}
W(K,q) = & (q_{\bar{n}} - q)\, v_L \sum_{j=k}^{K} \Lambda_j^m - (q_{\bar{n}} - q)\, \frac{c}{m}(K-k) \\
& + [(\hat{q} - q_{\bar{n}})\, v_L + (1-\hat{q})\, v_H] \sum_{j=k+1}^{K} \Lambda_j^m - (1-q_{\bar{n}})\, \frac{c}{m}(K-k) \\
& + \delta \left[(q_{\bar{n}} - q) \left(\sum_{j=1}^{k} \Lambda_j^m v_L - P\,(k,\tilde{q}) \right) + (\tilde{q} - q_{\bar{n}}) \left(\sum_{j=1}^{k} \Lambda_j^m v_L - P\,(k,\tilde{q}) \right) \right] \\
& + \delta^2 W(k,\tilde{q})
\end{aligned}
$$

Let $\bar{q} \leq q_{\bar{n}}$ be such that $R(\bar{q}) = W(K,\bar{q})$. Such $\bar{q}$ is well defined since it solves:

$$
\begin{aligned}
& (q_{\bar{n}} - q) \left(\sum_{j=1}^{k} \Lambda_j^m v_L + \left[\frac{c}{m}(K-k) + \delta P(k,\tilde{q}) \right] \right) \\
& \quad = [(\hat{q} - q_{\bar{n}})\, v_L + (1-\hat{q})\, v_H] \sum_{j=k+1}^{K} \Lambda_j^m - (1-q_{\bar{n}})\, \frac{c}{m}(K-k) \\
& \qquad + \delta \left[(\tilde{q} - q_{\bar{n}}) \left(\sum_{j=1}^{k} \Lambda_j^m v_L - P\,(k,\tilde{q}) \right) \right] + \delta^2 W(k,\tilde{q}) \qquad \text{(T5)}
\end{aligned}
$$

The right hand side of equation (T5) exceeds the left hand side for all $q \in (q^*, q_{\hat{n}}]$ because $W(K,q)$ is the value from following the optimal course of action. As $q \to -\infty$ the left hand side increases continuously without bound, while the right hand side is constant. Thus, there exists $\bar{q} \leq q^*$ with $R\,(\bar{q}) = W\,(K,\bar{q})$. Moreover, from the definition of $R(q)$ and equation (T5), we obtain

$$
R(\bar{q}) = V(\bar{q}) \geq (q_{\bar{n}} - \bar{q}) \sum_{j=1}^{k} \Lambda_j^m v_L > 0.
$$

Finally, note that $R(\cdot)$ is weakly increasing in q, so for all $n \geq \bar{n}$:

$$
W(K,q_n) \geq R(q_n) \geq R(\bar{q}) > 0
$$

which contradicts the fact that $\lim_{n\to\infty} W(K,q_n) = 0$. ■

T.4 Proof of Claims 8 and 10

We briefly discuss the links between the proofs of Claims 8 and 10 before presenting them. For the inductive step $j = 1$, the proof of Claim 8 only requires Claim 6 to hold. For inductive steps $j > 1$, the proof of Claim 8 uses the results (including Claims 9 and 10) from previous inductive steps. Similarly, for any inductive step j, the proofs of Claims 9 and 10 use results from previous inductive steps and Claim 8 from the current step j.

Throughout the following proofs we proceed as follows. We first provide an explicit characterization of the limit functions $(K_m(\tau;(K,q)), q_m(\tau;(K,q)))$. In equilibrium, the low-type seller is always indifferent between accepting or rejecting a screening offer. This, together with the limit functions $(K_m(\tau;(K,q)), q_m(\tau;(K,q)))$ pins down the function $P_m(K,q)$. We explicitly express $P_m^-(K,q)$ whenever there is an impasse at (K,q). For all other states, the expression of $P_m(K,q)$ is immediate. The buyer's continuation payoff $W_m(K,q)$ can be easily computed from the limit functions $(K_m(\tau;(K,q)), q_m(\tau;(K,q)))$ and $P_m(K,q)$ so we omit it.

Proof of Claim 8. For Δ sufficiently small, the buyer has a course of action with continuation payoff arbitrarily close to $\mathcal{W}(K,q)$. For all (K,q) with $K \in \{k_j+1,\ldots,\underline{k}\}$ and $q \in [0,q_j]$, $\mathcal{W}(K,q)$ is bounded away from zero. Then, for Δ sufficiently small the buyer can guarantee a strictly positive continuation payoff. This implies, as shown in section **??**, that there is no delay: $K_m(0;(K,q)) \leq k_j$. In what follows, we show that $(K_m(0;(K,q)), q_m(0;(K,q))) = (k_j, q_j)$ and that $\inf\{\tau : q_m(\tau;(K,q)) > q_j\} = \tau_j$.

First, assume by contradiction that $(K_m(0;(K,q)), q_m(0;(K,q))) = (k, q')$ with $k < k_j$. This leads to a continuation payoff (weakly) bounded above by

$$\begin{aligned}
&(\hat{q}-q)\left[\sum_{s=k_j+1}^{K}\Lambda_s^m v_L - (K-k_j)\frac{c}{m}\right] + (1-\hat{q})\left[\sum_{s=k_j+1}^{K}\Lambda_s^m v_H - (K-k_j)\frac{c}{m}\right] \\
&\quad + (\hat{q}-q)\left[\sum_{s=k+1}^{k_j}\Lambda_s^m v_L - (k_j-k)\frac{c}{m}\right] + (1-\hat{q})\left[\sum_{s=k+1}^{k_j}\Lambda_s^m v_H - (k_j-k)\frac{c}{m}\right] \\
&\quad + W_m(k,q)
\end{aligned}$$

$$< (\hat{q}-q)\left[\sum_{s=k_j+1}^{K}\Lambda_s^m v_L-(K-k_j)\frac{c}{m}\right]+(1-\hat{q})\left[\sum_{s=k_j+1}^{K}\Lambda_s^m v_H-(K-k_j)\frac{c}{m}\right]$$
$$+ W_m(k_j,q)$$

In the previous induction step we show that at state (k_j, q) there exists a unique course of action that yields $W_m(k_j, q)$. This leads to the strict inequality in the expression above. Thus, $K_m(0;(K,q)) = k_j$.

Second, assume towards a contradiction that $\inf\{\tau : q_m(\tau;(K,q)) > q_j\} = 0$. If so, the buyer's continuation payoff results from 1) making a universal offer for $K - k_j$ units and then 2) reaching the state (k_j, q'), with $q' > q_j$ without delay. Therefore, the buyer's continuation payoff is strictly bounded above by

$$(\hat{q}-q)\left[\sum_{s=k_j+1}^{K}\Lambda_s^m v_L-(K-k_j)\frac{c}{m}\right]+(1-\hat{q})\left[\sum_{s=k_j+1}^{K}\Lambda_s^m v_H-(K-k_j)\frac{c}{m}\right]+W_m(k_j,q).$$

Thus, $\inf\{\tau : q_m(\tau;(K,q)) > q_j\} > 0$. We know from the previous inductive step that there is no delay at any state (k_j, q) with $q < q_j$. Thus $q_m(0;(K,q)) = q_j$.

We finally show that $\inf\{\tau : q_m(\tau;(K,q)) > q_j\} = \tau_j$. The characterization of the limit functions from the previous inductive step implies that $\inf\{\tau : q_m(\tau;(K,q)) > q_j\} \leq \tau_j$. Assume by contradiction that $\inf\{\tau : q_m(\tau;(K,q)) > q_j\} \in (0, \tau_j)$. Then, in state (K, q) the low-type seller obtains a limit continuation payoff $\widetilde{V}_L(K,q)$ that satisfies:

$$\widetilde{V}_L(K,q) > (K-k_j)\frac{c}{m}+P_m^-(k_j,q_j) \tag{T6}$$

The buyer obtains a continuation payoff

$$(\hat{q}-q)\left[\sum_{s=k_j+1}^{K}\Lambda_s^m v_L-(K-k_j)\frac{c}{m}\right]+(1-\hat{q})\left[\sum_{s=k_j+1}^{K}\Lambda_s^m v_H-(K-k_j)\frac{c}{m}\right]$$
$$+ (q_j-q)\left[\sum_{s=1}^{k_j}\Lambda_s^m v_L-\left[\widetilde{V}_L(K,q)-(K-k_j)\frac{c}{m}\right]\right]+W_m(k_j,q_j)$$

$$
\begin{aligned}
&< (\hat{q} - q) \left[\sum_{s=k_j+1}^{K} \Lambda_s^m v_L - (K - k_j) \frac{c}{m} \right] + (1 - \hat{q}) \left[\sum_{s=k_j+1}^{K} \Lambda_s^m v_H - (K - k_j) \frac{c}{m} \right] \\
&+ (q_j - q) \left[\sum_{s=1}^{k_j} \Lambda_s^m v_L - P_m^-(k_j, q_j) \right] + W_m(k_j, q_j) \\
&= \mathcal{W}(K, q)
\end{aligned}
$$

where the strict inequality follows from equation (T6). Thus, we have reached a contradiction. ■

Proof of Claim 10. We first characterize the limit functions for all states (K, q) with $K \in \{\underline{k} + 1, \dots, \overline{k}\}$, $q < \check{q}(K)$ and $q \geq \check{q}(K-1)$ if $K \neq \underline{k} + 1$. In particular, we show that starting from any such state (K, q), the state $(K, \check{q}(K))$ is reached without delay. At state $(K, \check{q}(K))$ a (potentially off-path) impasse of length $\rho(K)$ occurs.

First, assume towards a contradiction that $K_m(0; (K, q)) < K$. Then, the buyer's continuation payoff is bounded above by:

$$
(\hat{q} - q) \left[\Lambda_K^m v_L - \frac{c}{m} \right] + (1 - \hat{q}) \left[\Lambda_K^m v_H - \frac{c}{m} \right] + W_m(K - 1, q) = \mathcal{W}(K, q) < 0. \tag{T7}
$$

Since the continuation payoff cannot be strictly negative, we have reached a contradiction.

We next show that $P_m(K, \cdot)$ is discontinuous at $\check{q}(K)$. If it were not, then $P_m(K, q) > \sum_{s=1}^{K} \Lambda_s^m v_L$ for all $q \in [\check{q}(K) - \eta, \check{q}(K))$ for some $\eta > 0$.[60] This together with equation (T7), implies that the buyer's continuation payoff would be strictly negative at any state (K, q) with $q \in [\check{q}(K) - \eta, \check{q}(K))$, leading to a contradiction.

The discontinuity of $P_m(K, \cdot)$ at $\check{q}(K)$ implies that an impasse occurs at $(K, \check{q}(K))$. Because of an argument analogous to that in DL, the length of the impasse must be $\rho(K)$, as defined in Claim 10. The expression for $P_m^-(K, \check{q}(K))$ is a direct consequence of $P_m^+(K, \check{q}(K))$ and the length of the impasse:

$$
P_m^-(K, \check{q}(K)) = \frac{\left(\sum_{s=1}^{K} \Lambda_s^m v_L \right)^2}{(K - k_j) \frac{c}{m} + P_m^-(k_j, q_j)} < \sum_{s=1}^{K} \Lambda_s^m v_L
$$

[60] This follows from $P_m^+(K, \check{q}(K)) = (K - k_j) \frac{c}{m} + P_m^-(k_j, q_j) > \sum_{s=1}^{K} \Lambda_s^m v_L$, see equation (16a).

where the inequality follows from equation (16a). Since shifting to state $(K, \check{q}(K))$ gives the buyer a positive continuation payoff, there cannot be delay at state (K,q) with $K \in \{\underline{k}+1, \ldots, \overline{k}\}$, $q < \check{q}(K)$ and $q \geq \check{q}(K-1)$ if $k \neq \underline{k}+1$. Together with equation (T7), this implies that in fact the impasse $(K, \check{q}(K))$ is reached without delay. This concludes the characterization of the limit functions for all (K,q) with $K \in \{\underline{k}+1, \ldots, \overline{k}\}$, $q < \check{q}(K)$ and $q \geq \check{q}(K-1)$ if $K \neq \underline{k}+1$.

The remainder of the proof is by induction. The base step is that Claim 10 holds for $\underline{k}+1$, which follows from the first part of the proof of this claim. The inductive step is as follows. Assume that Claim 10 holds for all $k \in \{\underline{k}+1, \ldots, K-1\}$ with $\underline{k}+1 \leq K-1 < \overline{k}$. We show next that then it must also hold for K and $q < \check{q}(K-1)$.

Consider any state (K,q) with $q < \check{q}(K-1)$. The continuation payoff of the buyer is bounded away from zero:

$$W_m(K,q) \geq [\check{q}(K) - q] \left[\sum_{s=1}^{K} \Lambda_s^m v_L - P_m^- \left(K, \check{q}(K)\right) \right] > 0$$

This implies that there cannot be delay at state (K,q). To conclude this proof, we show that $K_m(0;(K,q)) = K$. Assume towards a contradiction that $K_m(0;(K,q)) < K$. Then the buyer's continuation payoff is bounded above by:[61]

$$\begin{aligned}
&(\hat{q} - q) \left[\Lambda_K^m v_L - \frac{c}{m}\right] + (1-\hat{q}) \left[\Lambda_K^m v_H - \frac{c}{m}\right] + W_m(K-1,q) \\
&= (\hat{q} - q) \left[\Lambda_K^m v_L - \frac{c}{m}\right] + (1-\hat{q}) \left[\Lambda_K^m v_H - \frac{c}{m}\right] \\
&\quad + (\check{q}(K-1) - q) \left[\sum_{s=1}^{K-1} \Lambda_s^m v_L - P_m^-(K-1, \check{q}(K-1))\right] \\
&\quad + (\hat{q} - \check{q}(K-1)) \left[\sum_{s=k_j+1}^{K-1} \Lambda_s^m v_L - (K-1-k_j)\frac{c}{m}\right] \\
&\quad + (1-\hat{q}) \left[\sum_{s=k_j+1}^{K-1} \Lambda_s^m v_H - (K-1-k_j)\frac{c}{m}\right]
\end{aligned}$$

[61] In the expression to the right of the equality sign, the third, fourth and fifth lines add up to zero. Nevertheless, we include them to make the comparison between payoffs easier. We proceed in a similar fashion in the expression for Ω_2 below.

$$
\begin{aligned}
& + \left(q_j - \check{q}(K-1)\right) \left[\sum_{s=1}^{k_j} \Lambda_s^m v_L - P_m^-\left(k_j, q_j\right)\right] \\
< & \ (\hat{q} - q) \left[\Lambda_K^m v_L - \frac{c}{m}\right] + (1-\hat{q}) \left[\Lambda_K^m v_H - \frac{c}{m}\right] \\
& + (\check{q}(K) - q) \left[\sum_{s=1}^{K-1} \Lambda_s^m v_L - P_m^-(K-1, \check{q}(K-1))\right] \\
& + (\hat{q} - \check{q}(K)) \left[\sum_{s=k_j+1}^{K-1} \Lambda_s^m v_L - (K-1-k_j)\frac{c}{m}\right] \\
& + (1-\hat{q}) \left[\sum_{s=k_j+1}^{K-1} \Lambda_s^m v_H - (K-1-k_j)\frac{c}{m}\right] \\
& + \left(q_j - \check{q}(K)\right) \left[\sum_{s=1}^{k_j} \Lambda_s^m v_L - P_m^-\left(k_j, q_j\right)\right] \\
\equiv & \ \Omega_1
\end{aligned}
$$

where the strict inequality follows from

$$
P_m^-(K-1, \check{q}(K-1)) < P_m^+(K-1, \check{q}(K-1)) = (K-1-k_j)\frac{c}{m} + P_m^-(k_j, q_j).
$$

Starting in state (K, q), the buyer could instead follow an alternative course of action and reach the state $(K, \check{q}(K))$ without delay. This would lead to a continuation payoff equal to

$$
\begin{aligned}
& (\check{q}(K) - q) \left[\sum_{s=1}^{K} \Lambda_s^m v_L - P_m^-(K, \check{q}(K))\right] \\
= & \ (\check{q}(K) - q) \left[\sum_{s=1}^{K} \Lambda_s^m v_L - P_m^-(K, \check{q}(K))\right] \\
& + (\hat{q} - \check{q}(K)) \left[\sum_{s=k_j+1}^{K} \Lambda_s^m v_L - (K-k_j)\frac{c}{m}\right] + (1-\hat{q}) \left[\sum_{s=k_j+1}^{K} \Lambda_s^m v_H - (K-k_j)\frac{c}{m}\right] \\
& + \left(q_j - \check{q}(K)\right) \left[\sum_{s=1}^{k_j} \Lambda_s^m v_L - P_m^-\left(k_j, q_j\right)\right] \\
\equiv & \ \Omega_2
\end{aligned}
$$

The difference $\Omega_2 - \Omega_1$ takes the following form

$$\Omega_2 - \Omega_1 = (\check{q}(K) - q)\left[\frac{c}{m} + P_m^-\left(K-1, \check{q}(K-1)\right) - P_m^-\left(K, \check{q}(K)\right)\right]$$
$$= (\check{q}(K) - q)\left[\frac{c}{m} + \frac{\left(\sum_{s=1}^{K-1}\Lambda_s^m v_L\right)^2}{(K-1-k_j)\frac{c}{m} + P_m^-\left(k_j, q_j\right)} - \frac{\left(\sum_{s=1}^{K-1}\Lambda_s^m v_L + \Lambda_K^m v_L\right)^2}{\frac{c}{m} + (K-1-k_j)\frac{c}{m} + P_m^-\left(k_j, q_j\right)}\right] > 0,$$

where the inequality holds because $\frac{c}{m} > \Lambda_K^m v_L$. Thus, we have reached a contradiction. ∎

T.5 Proof of Facts 2 and 3

Proof of Fact 2. We first plug the expression for $P_{-\ell}^+$ from equation (20) for ℓ into equation (21). We obtain an expression for $P_{-(\ell+1)}^+$ that we plug into equation (20) for $\ell - 1$. The resulting expression links the (limit) beliefs of three consecutive impasses $q_{-(\ell-1)}$, $q_{-\ell}$ and $q_{-(\ell+1)}$:

$$\begin{aligned}&\left(q_{-\ell} - q_{-(\ell-1)}\right)\left[\int_0^{\psi(q_{-(\ell-1)})}\lambda(z)v_L dz - \left[\psi(q_{-(\ell-1)}) - \psi(q_{-\ell})\right]c\right] \\ &- \left(q_{-\ell} - q_{-(\ell-1)}\right)\left[\frac{\left(v_L\int_0^{\psi(q_{-\ell})}\lambda(z)dz\right)^2}{\int_0^{\psi(q_{-\ell})}\lambda(z)v_L dz + \frac{\left(\hat{q}-q_{-(\ell+1)}\right)\int_{\psi(q_{-(\ell+1)})}^{\psi(q_{-\ell})}[\lambda(z)v_L - c]dz + (1-\hat{q})\int_{\psi(q_{-(\ell+1)})}^{\psi(q_{-\ell})}[\lambda(z)v_H - c]dz}{q_{-(\ell+1)} - q_{-\ell}}}\right] \\ &+ (\hat{q} - q_{-\ell})\int_{\psi(q_{-\ell})}^{\psi(q_{-(\ell-1)})}\left[\lambda(z)v_L - c\right]dz + (1-\hat{q})\int_{\psi(q_{-\ell})}^{\psi(q_{-(\ell-1)})}\left[\lambda(z)v_H - c\right]dz = 0\end{aligned} \tag{T8}$$

Rearranging terms, we obtain the following expression for the ratio of the difference of consecutive beliefs:

$$\frac{q_{-\ell} - q_{-(\ell-1)}}{q_{-(\ell+1)} - q_{-\ell}} = \frac{\int_0^{\psi(q_{-\ell})}\lambda(z)v_L dz}{\int_0^{\psi(q_{-\ell})}\lambda(z)v_L dz + \frac{\left(\hat{q}-q_{-(\ell+1)}\right)\int_{\psi(q_{-(\ell+1)})}^{\psi(q_{-\ell})}[\lambda(z)v_L - c]dz + (1-\hat{q})\int_{\psi(q_{-(\ell+1)})}^{\psi(q_{-\ell})}[\lambda(z)v_H - c]dz}{q_{-(\ell+1)} - q_{-\ell}}}$$
$$\times \frac{\frac{\left(\hat{q}-q_{-(\ell+1)}\right)\int_{\psi(q_{-(\ell+1)})}^{\psi(q_{-\ell})}[\lambda(z)v_L - c]dz + (1-\hat{q})\int_{\psi(q_{-(\ell+1)})}^{\psi(q_{-\ell})}[\lambda(z)v_H - c]dz}{\left(q_{-(\ell+1)} - q_{-\ell}\right)^2}}{\frac{\int_{\psi(q_{-\ell})}^{\psi(q_{-(\ell-1)})}[c - \lambda(z)v_L]dz}{q_{-\ell} - q_{-(\ell-1)}} - \frac{(\hat{q}-q_{-\ell})\int_{\psi(q_{-\ell})}^{\psi(q_{-(\ell-1)})}[\lambda(z)v_L - c]dz + (1-\hat{q})\int_{\psi(q_{-\ell})}^{\psi(q_{-(\ell-1)})}[\lambda(z)v_H - c]dz}{\left(q_{-\ell} - q_{-(\ell-1)}\right)^2}}$$

It follows from the definition of $\psi(\cdot)$ that the first term in the right hand side of previous equation is less than one. Therefore, the ratio $\frac{q_{-\ell}-q_{-(\ell-1)}}{q_{-(\ell+1)}-q_{-\ell}}$ is bounded above as follows:

$$\frac{q_{-\ell}-q_{-(\ell-1)}}{q_{-(\ell+1)}-q_{-\ell}} \leq \frac{\frac{\left(\hat{q}-q_{-(\ell+1)}\right)\int_{\psi(q_{-(\ell+1)})}^{\psi(q_{-\ell})}[\lambda(z)v_L-c]dz+(1-\hat{q})\int_{\psi(q_{-(\ell+1)})}^{\psi(q_{-\ell})}[\lambda(z)v_H-c]dz}{\left(q_{-(\ell+1)}-q_{-\ell}\right)^2}}{\frac{\int_{\psi(q_{-\ell})}^{\psi(q_{-(\ell-1)})}[c-\lambda(z)v_L]dz}{q_{-\ell}-q_{-(\ell-1)}}-\frac{(\hat{q}-q_{-\ell})\int_{\psi(q_{-\ell})}^{\psi(q_{-(\ell-1)})}[\lambda(z)v_L-c]dz+(1-\hat{q})\int_{\psi(q_{-\ell})}^{\psi(q_{-(\ell-1)})}[\lambda(z)v_H-c]dz}{\left(q_{-\ell}-q_{-(\ell-1)}\right)^2}} \tag{T9}$$

We next provide convenient expressions for the terms in the right hand side of the inequality above. To do so, we define the function $\hat{\lambda}(\cdot)$ by $\hat{\lambda}(q)=\lambda(\psi(q))$ and the function $v(\cdot)$ by

$$v(q)=\begin{cases} v_L & \text{if } q\in[0,\hat{q}] \\ v_H. & \text{if } q\in(\hat{q},1] \end{cases}$$

Using these definitions, we express

$$\begin{aligned}
&\left(\hat{q}-q_{-(\ell+1)}\right)\int_{\psi(q_{-(\ell+1)})}^{\psi(q_{-\ell})}[\lambda(z)v_L-c]\,dz+(1-\hat{q})\int_{\psi(q_{-(\ell+1)})}^{\psi(q_{-\ell})}[\lambda(z)v_H-c]\,dz \\
&=\int_{q_{-\ell}}^{q_{-(\ell+1)}}-\psi'(q)\left[\int_{q_{-(\ell+1)}}^{1}\left[\hat{\lambda}(q)v(s)-c\right]ds\right]dq \\
&=\left(\int_{q_{-(\ell+1)}}^{1}v(s)ds\right)\int_{q_{-\ell}}^{q_{-(\ell+1)}}\psi'(q)\left(\int_{q}^{q_{-(\ell+1)}}\hat{\lambda}'(u)du\right)dq \\
&=\left(\int_{q_{-(\ell+1)}}^{1}v(s)ds\right)\psi'\left(q'_{\ell,\ell+1}\right)\hat{\lambda}'(q''_{\ell,\ell+1})\frac{\left(q_{-(\ell+1)}-q_{-\ell}\right)^2}{2},
\end{aligned} \tag{T10}$$

for some $\left(q'_{\ell,\ell+1},q''_{\ell,\ell+1}\right)\in\left[q_{-\ell},q_{-(\ell+1)}\right]^2$. The first equality follows from a change of variables. For the second we use the fact that for all $q<q_{-(\ell+1)}$, then $\hat{\lambda}(q)=\hat{\lambda}(q_{-(\ell+1)})-\int_q^{q_{-(\ell+1)}}\hat{\lambda}'(s)ds$ and also that $\int_{q_{-(\ell+1)}}^{1}\left[\hat{\lambda}(q_{-(\ell+1)})v(s)-c\right]ds=0$. The third equality follows from the mean value theorem.

In a similar way we obtain

$$\begin{aligned}
&(\hat{q}-q_{-\ell})\int_{\psi(q_{-\ell})}^{\psi(q_{-(\ell-1)})}[\lambda(z)v_L-c]\,dz+(1-\hat{q})\int_{\psi(q_{-\ell})}^{\psi(q_{-(\ell-1)})}[\lambda(z)v_H-c]\,dz \\
&=\left(\int_{q_{-\ell}}^{1}v(s)ds\right)\psi'\left(q'_{\ell-1,\ell}\right)\hat{\lambda}'(q''_{\ell-1,\ell})\frac{\left(q_{-\ell}-q_{-(\ell-1)}\right)^2}{2}
\end{aligned} \tag{T11}$$

for some $\left(q'_{\ell-1,\ell}, q''_{\ell-1,\ell}\right) \in \left[q_{-(\ell-1)}, q_{-\ell}\right]^2$. Finally, again with a change of variables and using the mean value theorem, we obtain

$$\int_{\psi(q_{-\ell})}^{\psi(q_{-(\ell-1)})} \left[c - \lambda(z)v_L\right] dz = -\psi'(q'''_{\ell-1,\ell}) \left[c - \hat{\lambda}(q'''_{\ell-1,\ell})v_L\right] \left(q_{-\ell} - q_{-(\ell-1)}\right) \tag{T12}$$

for some $q'''_{\ell-1,\ell} \in \left[q_{-(\ell-1)}, q_{-\ell}\right]$.

We plug equations (T10), (T11) and (T12) into equation (T9) and obtain

$$\begin{aligned}\frac{q_{-\ell} - q_{-(\ell-1)}}{q_{-(\ell+1)} - q_{-\ell}} &\leq \frac{\frac{1}{2}\left(\int_{q_{-(\ell+1)}}^{1} v(s)ds\right) \psi'\left(q'_{\ell,\ell+1}\right) \hat{\lambda}'(q''_{\ell,\ell+1})}{-\psi'(q'''_{\ell-1,\ell}) \left[c - \hat{\lambda}(q'''_{\ell-1,\ell})v_L\right] - \frac{1}{2}\left(\int_{q_{-\ell}}^{1} v(s)ds\right) \psi'\left(q'_{\ell-1,\ell}\right) \hat{\lambda}'(q''_{\ell-1,\ell})} \\ &\equiv \Xi\left(q_{-(\ell-1)}, q_{-\ell}, q_{-(\ell+1)}\right)\end{aligned} \tag{T13}$$

where we do not express explicitly that $q'_{\ell,\ell+1}, q''_{\ell,\ell+1}, q'_{\ell-1,\ell}, q''_{\ell-1,\ell}, q'''_{\ell-1,\ell}$ also depend on $q_{-(\ell-1)}, q_{-\ell}$ and $q_{-(\ell+1)}$.

Fact 2 links $q_{-\ell} - q_{-(\ell-1)}$ and $q_{-(\ell+1)} - q_{-\ell}$ when $q_{-(\ell+1)} - q_{-\ell}$ is small. We study the function $\Xi(\cdot,\cdot,\cdot)$ when this difference is small. We fix $q_{-(\ell+1)}$, let $q_{-\ell} = q_{-(\ell+1)} - h$ and define $q_{-(\ell-1)}(q_{-(\ell+1)}, h)$ implicitly by equation (T8). Therefore, we directly study the function $\widetilde{\Xi}\left(h, q_{-(\ell+1)}\right) \equiv \Xi\left(q_{-(\ell-1)}(q_{-(\ell+1)}, h), q_{-(\ell+1)} - h, q_{-(\ell+1)}\right)$ in a neighborhood of $h = 0$.

First, we show that $\lim_{h\to 0} \widetilde{\Xi}\left(h, q_{-(\ell+1)}\right) = 1$ for every $q_{-\ell} < \bar{q}(0)$. It follows from equation (T8) that $\lim_{h\to 0} q_{-(\ell-1)}(q_{-(\ell+1)}, h) = q_{-(\ell+1)}$. Thus,

$$\begin{aligned}&\lim_{h\to 0} \widetilde{\Xi}\left(h, q_{-(\ell+1)}\right) \\ &\quad = \frac{\frac{1}{2}\left(\int_{q_{-(\ell+1)}}^{1} v(s)ds\right) \psi'\left(q_{-(\ell+1)}\right) \hat{\lambda}'(q_{-(\ell+1)})}{-\psi'(q_{-(\ell+1)}) \left[c - \hat{\lambda}(q_{-(\ell+1)})v_L\right] - \frac{1}{2}\left(\int_{q_{-(\ell+1)}}^{1} v(s)ds\right) \psi'\left(q_{-(\ell+1)}\right) \hat{\lambda}'(q_{-(\ell+1)})} \\ &\quad = \frac{\frac{1}{2}\left(\int_{q_{-(\ell+1)}}^{1} v(s)ds\right) \frac{\hat{\lambda}(q_{-(\ell+1)})v_L - c}{\int_{q_{-(\ell+1)}}^{1} v(s)ds}}{-\left[c - \hat{\lambda}(q_{-(\ell+1)})v_L\right] - \frac{1}{2}\left(\int_{q_{-(\ell+1)}}^{1} v(s)ds\right) \frac{\hat{\lambda}(q_{-(\ell+1)})v_L - c}{\int_{q_{-(\ell+1)}}^{1} v(s)ds}} = 1\end{aligned}$$

where the second equality follows from $\hat{\lambda}'(q) = \frac{\hat{\lambda}(q)v_L - c}{\int_q^1 v(s)ds}$.[62]

Second, it follows from the fact that $\lambda(\cdot)$ is smooth that there exists $\xi > 0$ and $\tilde{h} > 0$ such that $h' < \tilde{h}$ implies that $\left| \frac{\partial \widetilde{\Xi}\left(h, q_{-(\ell+1)}\right)}{\partial h} \Big|_{h=h'} \right| < \xi$ for every $q_{-(\ell+1)} < \bar{q}(0)$.

Putting together the last two results, it follows that for any $\varepsilon > 0$ there exists $\tilde{h}$ such that if $h < \tilde{h}$ then $\widetilde{\Xi}\left(h, q_{-(\ell+1)}\right) < 1 + \varepsilon$ for every $q_{-(\ell+1)} < \bar{q}(0)$. This directly leads to Fact 2. ■

Proof of Fact 3. It is straightforward to establish the first result in Fact 3 if $q_{-(\ell+1)} - q_{-\ell}$ is bounded away from zero. Therefore, we restrict attention to the case in which $q_{-(\ell+1)} - q_{-\ell}$ is small.

Consider the following three consecutive limit beliefs: $\left(q_{-\ell}, q_{-(\ell+1)}, q_{-(\ell+2)}\right)$. Equation (T13) guarantees

$$\frac{q_{-(\ell+2)} - q_{-(\ell+1)}}{q_{-(\ell+1)} - q_{-\ell}} \geq \frac{1}{\Xi\left(q_{-\ell}, q_{-(\ell+1)}, q_{-(\ell+2)}\right)}.$$

We fix $q_{-(\ell+1)}$ and let $h \equiv q_{-(\ell+1)} - q_{-\ell}$. We define $q_{-(\ell+2)}\left(h, q_{-(\ell+1)}\right)$ implicitly by equation (T8), but linking the consecutive limit beliefs: $\left(q_{-\ell}, q_{-(\ell+1)}, q_{-(\ell+2)}\right)$. We also define

$$\widehat{\Xi}\left(h, q_{-(\ell+1)}\right) \equiv \frac{1}{\Xi\left(q_{-(\ell)} - h, q_{-(\ell+1)}, q_{-(\ell+2)}\left(h, q_{-(\ell+1)}\right)\right)}.$$

The function $\widehat{\Xi}(\cdot,\cdot)$ satisfies $\lim_{h\to 0} \widehat{\Xi}\left(h, q_{-(\ell+1)}\right) = 1$. Moreover, for every $\tilde{h} > 0$ there exists ξ such that if $0 \leq h' < \tilde{h}$ then $\left| \frac{\partial \widehat{\Xi}\left(h, q_{-(\ell+1)}\right)}{\partial h} \Big|_{h=h'} \right| < \xi$ for every $q_{-(\ell+1)} < \bar{q}(0)$. Thus, through a Taylor approximation, there must exist $\tilde{h} > 0$ such that for all $h < \tilde{h}$:

$$\widehat{\Xi}\left(h, q_{-(\ell+1)}\right) > 1 - \xi h \quad \text{for every } q_{-(\ell+1)} < \bar{q}(0)$$

[62]This, in turn, follows from $\int_q^1 \left[\hat{\lambda}(q)v(s) - c\right] ds = 0$ for every q in the domain of $\hat{\lambda}(\cdot)$.

We restrict attention to $q_{-(\ell+1)} - q_{-\ell} < \tilde{h}$, which implies

$$\frac{q_{-(\ell+2)} - q_{-(\ell+1)}}{q_{-(\ell+1)} - q_{-\ell}} \geq 1 - \xi\left(q_{-(\ell+1)} - q_{-\ell}\right). \tag{T14}$$

We put together equation (20) and the first equality in (T10) to express the left hand side in (23). First, note that

$$P^{+}_{-\ell} - \int_0^{\psi(q_{-\ell})} \lambda(z) v_L dz = \frac{\int_{q_{-\ell}}^{q_{-(\ell+1)}} -\psi'(q)\left[\int_{q_{-(\ell+1)}}^{1}\left[\hat{\lambda}(q)v(s) - c\right] ds\right] dq}{\left(q_{-(\ell+1)} - q_{-\ell}\right)} \tag{T15}$$

and so

$$\begin{aligned}
&\frac{\left[P^{+}_{-\ell} - \int_0^{\psi(q_{-\ell})} \lambda(z) v_L dz\right] - \left[P^{+}_{-(\ell+1)} - \int_0^{\psi(q_{-(\ell+1)})} \lambda(z) v_L dz\right]}{q_{-(\ell+1)} - q_{-\ell}} \\
&= \frac{\int_{q_{-\ell}}^{q_{-(\ell+1)}} -\psi'(q)\left[\int_{q_{-(\ell+1)}}^{1}\left[\hat{\lambda}(q)v(s) - c\right] ds\right] dq}{\left(q_{-(\ell+1)} - q_{-\ell}\right)^2} \\
&\quad - \frac{\int_{q_{-(\ell+1)}}^{q_{-(\ell+2)}} -\psi'(q)\left[\int_{q_{-(\ell+2)}}^{1}\left[\hat{\lambda}(q)v(s) - c\right] ds\right] dq}{\left(q_{-(\ell+2)} - q_{-(\ell+1)}\right)^2}\left(\frac{q_{-(\ell+2)} - q_{-(\ell+1)}}{q_{-(\ell+1)} - q_{-\ell}}\right) \\
&= R\left(q_{-\ell}, q_{-(\ell+1)}\right) - R\left(q_{-(\ell+1)}, q_{-(\ell+2)}\right)\left(\frac{q_{-(\ell+2)} - q_{-(\ell+1)}}{q_{-(\ell+1)} - q_{-\ell}}\right)
\end{aligned} \tag{T16}$$

where for any $(q, q') \in [\bar{q}(1), \bar{q}(0)]^2$ with $q \leq q'$, we let:

$$R\left(q, q'\right) \equiv \begin{cases} \frac{\int_q^{q'} -\psi'(u)\left[\int_{q'}^{1}\left[\hat{\lambda}(u)v(s) - c\right] ds\right] du}{(q'-q)^2} & \text{if } q < q' \\ \frac{1}{2}\psi'(q)\hat{\lambda}'(q)\int_q^1 v(s) ds & \text{if } q = q' \end{cases}$$

The function $R(\cdot, \cdot)$ is continuous. We let $\underline{R} \equiv \min_{\bar{q}(1) \leq q \leq q' \leq \bar{q}(0)} R\left(q, q'\right) > 0$ and $\overline{R} \equiv \max_{\bar{q}(1) \leq q \leq q' \leq \bar{q}(0)} R\left(q, q'\right)$. If $\frac{q_{-(\ell+2)} - q_{-(\ell+1)}}{q_{-(\ell+1)} - q_{-\ell}} > \overline{R}/\underline{R}$ then the right hand side of equation (T16) is negative and the first inequality in Fact 3 holds trivially. Therefore, we restrict attention to:

$$\frac{q_{-(\ell+2)} - q_{-(\ell+1)}}{q_{-(\ell+1)} - q_{-\ell}} \leq \overline{R}/\underline{R} \tag{T17}$$

The function $R(\cdot,\cdot)$ has bounded partial derivatives. Then, there exist constants $\kappa_1 > 0$ and $\kappa_2 > 0$ such that:

$$\begin{aligned}
&R\left(q_{-\ell}, q_{-(\ell+1)}\right) - R\left(q_{-(\ell+1)}, q_{-(\ell+2)}\right)\left(\frac{q_{-(\ell+2)} - q_{-(\ell+1)}}{q_{-(\ell+1)} - q_{-\ell}}\right) \\
&\leq R\left(q_{-(\ell+1)}, q_{-(\ell+1)}\right) + \kappa_1\left(q_{-(\ell+1)} - q_{-\ell}\right) \\
&\quad - \left[R\left(q_{-(\ell+1)}, q_{-(\ell+1)}\right) - \kappa_2\left(q_{-(\ell+2)} - q_{-(\ell+1)}\right)\right]\left(\frac{q_{-(\ell+2)} - q_{-(\ell+1)}}{q_{-(\ell+1)} - q_{-\ell}}\right) \\
&\leq (\overline{R}\xi + \kappa_1 + \kappa_2\overline{R}/\underline{R})\left(q_{-(\ell+1)} - q_{-\ell}\right) - \kappa_2\xi\overline{R}/\underline{R}\left(q_{-(\ell+1)} - q_{-\ell}\right)^2
\end{aligned}$$

where the second inequality follows from the inequalities in (T14) and (T17), plus the definition of $\overline{R}$. This directly leads to (23) in Fact 3.

Next, we obtain the following simple bound for $q_{-(\ell+1)} - q_{-\ell}$ from equation (T15) and the definition of $R(\cdot,\cdot)$:

$$q_{-(\ell+1)} - q_{-\ell} = \frac{P^+_{-\ell} - \int_0^{\psi(q_{-\ell})} \lambda(z) v_L dz}{R\left(q_{-\ell}, q_{-(\ell+1)}\right)} \leq \frac{P^+_{-\ell} - \int_0^{\psi(q_{-\ell})} \lambda(z) v_L dz}{\underline{R}}.$$

This directly leads to (24) in Fact 3. ■