Cournot oligopoly when the competitors operate under capital constraints

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* Several Formulae are cut in the two-column Version

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(i) The corrections made in this section will be reviewed and approved by a journal production editor.

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Abstract

In this paper we extend and generalize the interesting duopoly model proposed by Tönu Puu (1991). An n-firm oligopoly is considered without product differentiation, when the price function is hyperbolic and the production cost function is linear. It is assumed that in addition the firms have limited budgets to cover production costs, so if they exceed their budgets, then they have to borrow extra capital with given unit costs. This additional cost makes the payoff functions of the firms only piece-wise differentiable and the best response functions are not unimodal making the equilibrium analysis more complex than in the classical case. In this paper the mathematical model is first introduced and then the best response functions of the firms are determined. A graphical approach is shown to prove the existence of positive equilibrium. The stability of the equilibrium is examined in the simulation study.

JEL classification: C72; C73

Abbreviations

No keyword abbreviations are available

Keywords:

Oligopolies, Repeated games, Complex dynamics

~~1.~~1 Introduction

Q4 Oligopoly models have a long history, this research area originates from the work of Cournot [2]. Since then many researchers devoted their efforts for the different versions and extensions of the classical Cournot model including oligopolies with and without product differentiation, multi-product models, labor-managed oligopolies, rent seeking and market sharing games among others.

Okuguchi [3] provided a comprehensive summary of the most important results up to the 70s, and their multi-product generalizations were discussed in Okuguchi and Szidarovszky [4]. In the early stages the main focus was on the existence and uniqueness of the equilibrium, and in later stages dynamic extensions became more and more important. Linear dynamics were first considered, when local asymptotic stability implies global stability and the stability theory of linear systems was well established. See for example, Szidarovszky and Bahill [5]. The nonlinear and nondifferentiable models became the main research focus. Puu [6] offered a survey of different aspects and methods to examine nonlinear phenomena in economics. Several model variants with different methods for stability analysis of dynamic oligopolies were discussed in Puu and Sushko [7]. A comprehensive summary of the main results on nonlinear oligopolies is given in Bischi et al. [8]. There are many ways to introduce nonlinearities into oligopoly models. If nonlinear price and/or cost functions are introduced, then in the differentiable cases local linearization is the usual approach to examine the local asymptotical stability of the equilibrium. In the discrete case the critical curve method is a useful approach for investigating the global asymptotic behavior of the equilibrium [8]. In discrete and continuous cases bifurcation diagrams can serve to show global dynamics. Production capacity limits, production adjustment costs and bounds introduce nonlinearities into the models. Reynolds [9,10], Szidarovszky and Yen [11] introduced additional capacity increasing costs. A differential game with production adjustment cost was introduced and examined by Driskill and McCafferty [12]. Howroyd and Rickard [13] and MacLeod [14] were interested in the effect of production adjustment cost on the long-term behavior of the equilibrium. Zhao and Szidarovszky [15] presented a complete best response and equilibrium analysis. Burr et al. [16] examined the best responses of the firms, the existence and global stability analysis of the equilibrium was performed with lower and upper bounds on the production adjustments. Cartelizing groups with antitrust thresholds were added to oligopolies by Matsumoto et al. [17–19]. Contingent workforce and in addition to production adjustment costs additional investment costs were introduced by Merlone and Szidarovszky [20]. Unemployment insurance systems were added to an extended oligopoly model by Matsumoto et al. [21]. In Szidarovszky and Matsumoto [22] a discontinuous oligopoly model was introduced in which the existence of at least one positive equilibrium was proved, and examples illustrated the existence of multiple equilibria.

Functions

In this paper special oligopolies are introduced when the firms have limited budgets, and if production costs exceed their budgets, then they have to borrow extra capital with given unit costs. The payoff ~~function~~s of the firms become piece-wise differentiable and the best response functions are not unimodal as it is the case for classical Cournot models with hyperbolic price function. The paper develops as follows. After the mathematical model is introduced, the best response ~~function~~s of the firms ~~are are is~~ derived. Then a graphical approach is shown to prove the existence of positive equilibria in the case of duopolies which is followed by a simulation study of the stability of the equilibria. The last section concludes the paper with further research directions.

Functions are

~~2.~~2 The ~~M~~mathematical ~~M~~model and ~~B~~best ~~R~~responses

$\sum_{k=1}^{n}$

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We consider a single-product oligopoly of n firms without product differentiation and isoelastic price function, $p(s) = \frac{A}{s}$ where s is the output of the industry. Let x_k denote the output of firm k, then $s = \sum_{k=1}^{n} x_k$. We assume that the firms have linear cost functions, $C_k(x_k) = c_k + d_k x_k$ with $c_k, d_k > 0$. In addition we assume that firms have a fixed amount of capital E_k to invest in production and can borrow extra capital at unit cost α_k. This additional cost can be modeled as

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$$\bar{C}_k(x_k) = \begin{cases} 0 & \text{if } c_k + d_k x_k \leq E_k \\ \alpha_k (c_k + d_k x_k - E_k) & \text{if } c_k + d_k x_k > E_k \end{cases} \tag{1}$$

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as the additional cost in time period t.

If $E_k \leq c_k$ for firm k, then the first case never occurs and the second case always holds resulting in a linear overall cost function. This case is already discussed in the literature, so we assume that $E_k > c_k$ for all firms.

Therefore the profit of firm k can be given as follows,

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$$\Pi_k(x_k) = \begin{cases} \varphi_1(x_k) = \frac{x_k A}{x_k + s_k} - (c_k + d_k x_k) & \text{if} 0 \le x_k \le \frac{E_k - c_k}{d_k} \\ \varphi_2(x_k) = \frac{x_k A}{x_k + s_k} - (c_k + d_k x_k) - \alpha_k (c_k + d_k x_k - E_k) & \text{if} x_k > \frac{E_k - c_k}{d_k} \end{cases} \quad (2)$$

X

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X X X

$\sum_{l \neq k}^{n}$

where $s_k = \sum_{l \neq k}{}^{n} x_l$ is the output of the rest of the industry.

Notice that for $0 \le x_k \le \frac{E_k - c_k}{d_k}$,

$$\frac{\partial \Pi_k}{\partial x_k} = \frac{A s_k}{(x_k + s_k)^2} - d_k = \varphi_1'(x_k) \quad (3)$$

and as $x_k > \frac{E_k - c_k}{d_k}$,

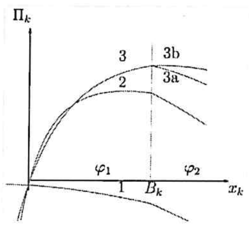
$$\frac{\partial \Pi_k}{\partial x_k} = \frac{A s_k}{(x_k + s_k)^2} - d_k (1 + \alpha_k) = \varphi_2'(x_k). \quad (4)$$

Clearly, both derivatives decrease in x_k, and $\varphi_2'(x_k) < \varphi_1'(x_k)$. Let $B_k = \frac{E_k - c_k}{d_k} > 0$, then Fig. 1 shows the possible shapes of $\Pi_k(x_k)$.

Images are optimised for fast web viewing. Click on the image to view the original version.

alt-text: Fig. 1

~~Figure 1:~~Fig. 1



Possible shapes of $\Pi_k(x_k)$.

We have now three major cases:

- Case 1. If $\varphi_1'(0) \le 0$, then $R_k(s_k) = 0$.

  This is the case, of $\frac{A}{s_k} - d_k \le 0$, or $s_k \ge \frac{A}{d_k}$ in which case, if $c_k > 0$, profit is negative at zero.

- Case 2. $\varphi_1'(0) > 0$ and $\varphi_1'(B_k) \le 0$.

  In this case the stationary point in interval $(0, B_k)$ is the best response:

$$R_k(s_k) = \sqrt{\frac{A s_k}{d_k}} - s_k \quad (5)$$

This is the case, when $s_k < \frac{A_k}{d_k}$ and

$$\frac{As_k}{(B_k+s_k)^2} - d_k \le 0. \tag{6}$$

which can be rewritten as

$$s_k^2 d_k + (2B_k d_k - A)\, s_k + B_k^2 d_k \ge 0 \tag{7}$$

The left hand side has no root if its discriminant

$$D = 4B_k^2 d_k^2 - 4AB_k d_k + A^2 - 4B_k^2 d_k^2 = A\,(-4B_k d_k + A) < 0$$

or $A < 4B_k d_k$. In this case Eq. (7) holds for all s_k. If $A = 4B_k d_k$, then Eq. (7) also holds, since there is only one real root. However if $A > 4B_k d_k$, then the roots are positive:

$$s_k^{(1),(2)} = \frac{A - 2B_k d_k \pm \sqrt{A\,(-4B_k d_k + A)}}{2d_k} \qquad \left(0<s_k^{(1)}<s_k^{(2)}\right) \tag{8}$$

It is easy to show that roots $s_k^{(1)}$ and $s_k^{(2)}$ are smaller than $\frac{A}{d_k}$, so this case occurs if either $0 \le s_k \le s_k^{(1)}$ or $s_k^{(2)} \le s_k < \frac{A}{d_k}$.

- Case 3. $\varphi_1'(B_k) > 0$, and Eq. (7) is violated with strict inequality. Notice that if $A \le 4B_k d_k$, then this case cannot occur, otherwise this is the case if $s_k^{(1)} < s_k < s_k^{(2)}$. We have now two subcases.
  - Case 3a. $\varphi_2'(B_k) \le 0$, when $R_k(s_k) = B_k$. This is the case, when $s_k < \frac{A_k}{d_k}$ and

$$\frac{As_k}{(B_k+s_k)^2} - d_k\,(1+\alpha_k) \le 0. \tag{9}$$

This is again a quadratic inequality

$$s_k^2 d_k\,(1+\alpha_k) + (2B_k d_k\,(1+\alpha_k) - A)\, s_k + B_k^2 d_k\,(1+\alpha_k) \ge 0 \tag{10}$$

Similarly to inequality (6) we consider two cases. If the discriminant is non-positive, that is, when $A \le 4B_k d_k(1+\alpha_k)$, then Eq. (10) always holds. Otherwise there are two positive roots

$$s_k^{(1)*,(2)*} = \frac{A - 2B_k d_k\,(1+\alpha_k) \pm \sqrt{A\,(-4B_k d_k\,(1+\alpha_k) + A)}}{2d_k\,(1+\alpha_k)}. \tag{11}$$

Notice that at $s_k^{(1)*,\,(2)*}$, Eq. (9) is satisfied with equality, and Eq. (6) violated with strict inequality implying that

$$s_k^{(1)} < s_k^{(1)*} < s_k^{(2)*} < s_k^{(2)}, \tag{12}$$

Do not break line

and Eq. (10) holds if either $s_k^{(1)} < s_k < s_k^{(1)*}$ or $s_k^{(2)*} < s_k < s_k^{(2)}$.

- Case 3b. $\varphi_2'(B_k) > 0$, which is the case if $s_k^{(1)*} < s_k < s_k^{(2)*}$, and in this case the best response is the stationary point behind B_k, which exists since $\lim_{x_k \to \infty} \Pi_k(x_k) = -\infty$ with all values of s_k, and Π_k is strictly concave in the interval $[B_k, \infty)$. So

$$R_k(s_k) = \sqrt{\frac{As_k}{d_k(1+\alpha_k)}} - s_k \qquad (13)$$

Do not break line

Fig. 2 shows the most general shape of $R_k(s_k)$, where some middle segments might be missing. If $A \leq 4B_k d_k$, then the entire segment $(s_k^{(1)}, s_k^{(2)})$ is missing. If $4B_k d_k < A \leq 4B_k d_k(1+\alpha_k)$, then only segment $(s_k^{(1)*}, s_k^{(2)*})$ is missing. If $A > 4B_k d_k(1+\alpha_k)$, then all segments are present.

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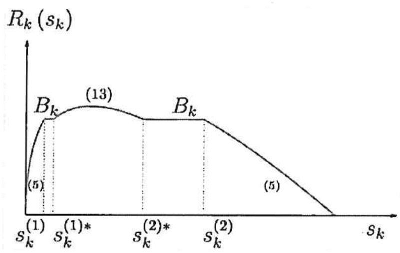
alt-text: Fig. 2

~~Figure 2:~~Fig. 2



Graph of the best response function $R_k(s_k)$.

It can be also proved that the maximum of Eq. (13) is between $s_k^{(1)*}$ and $s_k^{(2)*}$, and the maximum of Eq. (5) is between $s_k^{(1)}$ and $s_k^{(2)}$.

So Eq. (5) increases in $(0, s_k^{(1)})$ and decreases in $\left(s_k^{(2)}, \frac{A}{d_k}\right)$.

~~3.~~3 Existence of ~~D~~duopoly ~~E~~equilibrium

From the graphical representation of Fig. 2 the existence of an equilibrium is obvious although the trivial equilibrium – in which both firms have zero production level – is not feasible as the denominator of the price function vanishes. Notice that the derivative of Eq. (5) is the following:

$$\frac{\frac{A}{d_k}}{2\sqrt{\frac{As_k}{d_k}}} - 1 = \frac{1}{2}\sqrt{\frac{A}{d_k s_k}} - 1 \qquad (14)$$

which tends to infinity as $s_k \to 0$, so Eq. (5) has a vertical tangent line at zero. If we interchange the coordinate lines, then the best response of the other firm has horizontal tangent line, so there is at least one positive equilibrium.

Derivative of best response is

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$$R_k'(s_k) = \begin{cases} \frac{1}{2}\sqrt{\frac{A}{d_k s_k}} - 1 & \text{in } \left[0, s_k^{(1)}\right) \text{ and } \left(s_k^{(2)}, \frac{A}{d_k}\right) \\ \frac{1}{2}\sqrt{\frac{A}{d_k(1+\alpha_k)s_k}} - 1 & \text{in } \left(s_k^{(1)*}, s_k^{(2)*}\right) \\ 0 & \text{in } \left(s_k^{(1)}, s_k^{(1)*}\right), \left(s_k^{(2)*}, s_k^{(2)}\right) \text{ and } \left(\frac{A}{d_k}, \infty\right) \end{cases} \quad (15)$$

Determining equilibrium depends on which segments cross at the equilibrium. We have $5 \times 5 = 25$ regions:

Make size of "{" and "}" consistent

1. $\mathcal{R}_{1,1} = \{(x_1, x_2) : 0 \leq x_1 < s_1^{(1)} \wedge 0 \leq x_2 < s_2^{(1)}\}$
2. $\mathcal{R}_{1,2} = \{(x_1, x_2) : 0 \leq x_1 < s_1^{(1)} \wedge s_2^{(1)} \leq x_2 \leq s_2^{(1)*}\}$
3. $\mathcal{R}_{1,3} = \{(x_1, x_2) : 0 \leq x_1 < s_1^{(1)} \wedge s_2^{(1)*} < x_2 < s_2^{(2)*}\}$
4. $\mathcal{R}_{1,4} = \{(x_1, x_2) : 0 \leq x_1 < s_1^{(1)} \wedge s_2^{(2)*} \leq x_2 \leq s_2^{(2)}\}$
5. $\mathcal{R}_{1,5} = \left\{(x_1, x_2) : 0 \leq x_1 < s_1^{(1)} \wedge s_2^{(2)} < x_2 < \frac{A}{d_2}\right\}$
6. $\mathcal{R}_{2,1} = \{(x_1, x_2) : s_1^{(1)} \leq x_1 \leq s_1^{(1)*} \wedge 0 \leq x_2 < s_2^{(1)}\}$
7. $\mathcal{R}_{2,2} = \{(x_1, x_2) : s_1^{(1)} \leq x_1 \leq s_1^{(1)*} \wedge s_2^{(1)} \leq x_2 \leq s_2^{(1)*}\}$
8. $\mathcal{R}_{2,3} = \{(x_1, x_2) : s_1^{(1)} \leq x_1 \leq s_1^{(1)*} \wedge s_2^{(1)*} < x_2 < s_2^{(2)*}\}$
9. $\mathcal{R}_{2,4} = \{(x_1, x_2) : s_1^{(1)} \leq x_1 \leq s_1^{(1)*} \wedge s_2^{(2)*} \leq x_2 \leq s_2^{(2)}\}$
10. $\mathcal{R}_{2,5} = \left\{(x_1, x_2) : s_1^{(1)} \leq x_1 \leq s_1^{(1)*} \wedge s_2^{(2)} < x_2 < \frac{A}{d_2}\right\}$
11. $\mathcal{R}_{3,1} = \{(x_1, x_2) : s_1^{(1)*} \leq x_1 \leq s_1^{(2)*} \wedge 0 \leq x_2 < s_2^{(1)}\}$
12. $\mathcal{R}_{3,2} = \{(x_1, x_2) : s_1^{(1)*} \leq x_1 \leq s_1^{(2)*} \wedge s_2^{(1)} \leq x_2 \leq s_2^{(1)*}\}$
13. $\mathcal{R}_{3,3} = \{(x_1, x_2) : s_1^{(1)*} \leq x_1 \leq s_1^{(2)*} \wedge s_2^{(1)*} < x_2 < s_2^{(2)*}\}$
14. $\mathcal{R}_{3,4} = \{(x_1, x_2) : s_1^{(1)*} \leq x_1 \leq s_1^{(2)*} \wedge s_2^{(2)*} \leq x_2 \leq s_2^{(1)}\}$
15. $\mathcal{R}_{3,5} = \left\{(x_1, x_2) : s_1^{(1)*} \leq x_1 \leq s_1^{(2)*} \wedge s_2^{(2)} < x_2 < \frac{A}{d_2}\right\}$
16. $\mathcal{R}_{4,1} = \{(x_1, x_2) : s_1^{(2)*} \leq x_1 \leq s_1^{(2)} \wedge 0 \leq x_2 < s_2^{(1)}\}$
17. $\mathcal{R}_{4,2} = \{(x_1, x_2) : s_1^{(2)*} \leq x_1 \leq s_1^{(2)} \wedge s_2^{(1)} \leq x_2 \leq s_2^{(1)*}\}$
18. $\mathcal{R}_{4,3} = \{(x_1, x_2) : s_1^{(2)*} \leq x_1 \leq s_1^{(2)} \wedge s_2^{(1)*} < x_2 < s_2^{(2)*}\}$
19. $\mathcal{R}_{4,4} = \{(x_1, x_2) : s_1^{(2)*} \leq x_1 \leq s_1^{(2)} \wedge s_2^{(2)*} \leq x_2 \leq s_2^{(2)}\}$
20. $\mathcal{R}_{4,5} = \left\{(x_1, x_2) : s_1^{(2)*} \leq x_1 \leq s_1^{(2)} \wedge s_2^{(2)} < x_2 < \frac{A}{d_2}\right\}$
21. $\mathcal{R}_{5,1} = \left\{(x_1, x_2) : s_1^{(2)} < x_1 < \frac{A}{d_1} \wedge 0 \leq x_2 < s_2^{(1)}\right\}$
22. $\mathcal{R}_{5,2} = \left\{(x_1, x_2) : s_1^{(2)} < x_1 < \frac{A}{d_1} \wedge s_2^{(1)} \leq x_2 \leq s_2^{(1)*}\right\}$
23. $\mathcal{R}_{5,3} = \left\{(x_1, x_2) : s_1^{(2)} < x_1 < \frac{A}{d_1} \wedge s_2^{(1)*} < x_2 < s_2^{(2)*}\right\}$
24. $\mathcal{R}_{5,4} = \left\{(x_1, x_2) : s_1^{(2)} < x_1 < \frac{A}{d_1} \wedge s_2^{(2)*} \leq x_2 \leq s_2^{(2)}\right\}$
25. $\mathcal{R}_{5,5} = \left\{(x_1, x_2) : s_1^{(2)} < x_1 < \frac{A}{d_1} \wedge s_2^{(2)} < x_2 < \frac{A}{d_2}\right\}$.

~~4~~.4 Equilibria in duopoly

When considering a duopoly a non trivial equilibrium can occur in each of the positive segments of the best reply function, the two firms' reaction functions intersect. Although each best reply function is defined on five regions, when considering the functional form we have only nine cases:

1.

$$\begin{cases} x_1 = \sqrt{\frac{Ax_2}{d_1}} - x_2 \\ x_2 = \sqrt{\frac{Ax_1}{d_2}} - x_1 \end{cases} \tag{16}$$

either in $\mathscr{R}_{1,1}$, $\mathscr{R}_{1,5}$, $\mathscr{R}_{5,1}$ or $\mathscr{R}_{5,5}$;

2.

$$\begin{cases} x_1 = \sqrt{\frac{Ax_2}{d_1}} - x_2 \\ x_2 = B_2 \end{cases} \tag{17}$$

either in $\mathscr{R}_{1,2}$, $\mathscr{R}_{1,4}$, $\mathscr{R}_{5,2}$ or $\mathscr{R}_{5,4}$;

3.

$$\begin{cases} x_1 = \sqrt{\frac{Ax_2}{d_1}} - x_2 \\ x_2 = \sqrt{\frac{Ax_1}{d_2(1+\alpha_2)}} - x_1 \end{cases} \tag{18}$$

either in $\mathscr{R}_{1,3}$ or $\mathscr{R}_{5,3}$;

4.

$$\begin{cases} x_1 = B_1 \\ x_2 = \sqrt{\frac{Ax_1}{d_2}} - x_1 \end{cases} \tag{19}$$

either in $\mathscr{R}_{2,1}$, $\mathscr{R}_{2,5}$, $\mathscr{R}_{4,1}$ or $\mathscr{R}_{4,5}$;

5.

$$\begin{cases} x_1 = B_1 \\ x_2 = B_2 \end{cases} \tag{20}$$

either in $\mathscr{R}_{2,2}$, $\mathscr{R}_{2,4}$, $\mathscr{R}_{4,2}$ or $\mathscr{R}_{4,4}$;

6.

$$\begin{cases} x_1 = B_1 \\ x_2 = \sqrt{\frac{Ax_1}{d_2(1+\alpha_2)}} - x_1 \end{cases} \tag{21}$$

either in $\mathscr{R}_{2,3}$ or $\mathscr{R}_{4,3}$;

7.

$$\begin{cases} x_1 = \sqrt{\frac{Ax_2}{d_1(1+\alpha_1)}} - x_2 \\ x_2 = \sqrt{\frac{Ax_1}{d_2}} - x_1 \end{cases} \tag{22}$$

either in $\mathscr{R}_{3,1}$ or $\mathscr{R}_{3,5}$;

8.

$$\begin{cases} x_1 = \sqrt{\frac{Ax_2}{d_1(1+\alpha_1)}} - x_2 \\ x_2 = B_2 \end{cases} \tag{23}$$

either in $\mathscr{R}_{3,2}$ or $\mathscr{R}_{3,4}$;

9.

$$\begin{cases} x_1 = \sqrt{\frac{Ax_2}{d_1(1+\alpha_1)}} - x_2 \\ x_2 = \sqrt{\frac{Ax_1}{d_2(1+\alpha_2)}} - x_1 \end{cases} \tag{24}$$

in $\mathscr{R}_{3,3}$.

The equilibrium outputs of the firms can be determined by considering the two best response equations as a system of two equations for two unknowns and solve them, which is easy in all cases by the special equation forms. In particular when equilibrium is in regions $\mathscr{R}_{1,2}$, $\mathscr{R}_{1,4}$, $\mathscr{R}_{5,2}$, $\mathscr{R}_{5,4}$, $\mathscr{R}_{2,1}$, $\mathscr{R}_{2,5}$, $\mathscr{R}_{4,1}$, $\mathscr{R}_{4,5}$, $\mathscr{R}_{2,2}$, $\mathscr{R}_{2,4}$, $\mathscr{R}_{4,2}$, $\mathscr{R}_{4,4}$, $\mathscr{R}_{2,3}$, $\mathscr{R}_{4,3}$, $\mathscr{R}_{3,2}$ and $\mathscr{R}_{3,4}$ its derivation is a matter of trivial substitution. When the equilibrium is in regions $\mathscr{R}_{1,1}$, $\mathscr{R}_{1,5}$, $\mathscr{R}_{5,1}$, $\mathscr{R}_{5,5}$, $\mathscr{R}_{1,3}$, $\mathscr{R}_{5,3}$, $\mathscr{R}_{3,1}$, $\mathscr{R}_{3,5}$ and $\mathscr{R}_{3,3}$ we need to solve a system having the general form

$$\begin{cases} x_1 = \sqrt{\frac{x_2}{\gamma_1}} - x_2 \\ x_2 = \sqrt{\frac{x_1}{\gamma_1}} - x_1 \end{cases} \tag{25}$$

$\frac{d_1}{A}$ $\frac{d_1(1+\alpha_1)}{A}$

with $\gamma_1 \in \left\{\cancel{\frac{A}{d_1}}, \cancel{\frac{A}{d_1(1+\alpha_1)}}\right\}$ and $\gamma_2 \in \left\{\cancel{\frac{A}{d_2}}, \cancel{\frac{A}{d_2(1+\alpha_2)}}\right\}$. In these regions the solution is the Cournot point as in ([23], p. 49)

$\frac{d_2}{A}$ $\frac{d_2(1+\alpha_2)}{A}$

$$\begin{cases} x_1^* = \frac{\gamma_2}{(\gamma_1+\gamma_2)^2} \\ x_2^* = \frac{\gamma_1}{(\gamma_1+\gamma_2)^2} \end{cases} . \tag{26}$$

~~5.~~5 Best ~~R~~responses as ~~F~~functions of ~~I~~industry ~~O~~output

Now we determine the best response functions of the firms as the functions of the industry output.

In Case 1, $\overline{R}_k(s) = x_k = 0$ and since $s = s_k$, this is the case when

$$s \geq \frac{A}{d_k} \tag{27}$$

In Case 2, $s = s_k + x_k$, so from Eq. (5)

$$s = x_k + s_k = \sqrt{\frac{As_k}{d_k}},$$

so

$$s - x_k = s_k = \frac{s^2 d_k}{A}, \tag{28}$$

implying that

$$\overline{R}_k(s) = x_k = s - \frac{s^2 d_k}{A} \tag{29}$$

This is the case when $s_k < \frac{A}{d_k}$ or $\frac{s^2 d_k}{A} < \frac{A}{d_k}$ showing that $s < \frac{A}{d_k}$, and from Eqs. (6), (8),

$$0 \leq \frac{s^2 d_k}{A} \leq s_k^{(1)} \quad \text{or} \quad s_k^{(2)} \leq \frac{s^2 d_k}{A} < \frac{A}{d_k} \tag{30}$$

that is

$$0 \leq s \leq \sqrt{\frac{As_k^{(1)}}{d_k}} \text{ or } \sqrt{\frac{As_k^{(2)}}{d_k}} \leq s < \frac{A}{d_k} \tag{31}$$

Consider now Case 3a, when $x_k = B_k$, so

$$s = s_k + B_k$$

and Eq. (10) holds if either

$$s_k^{(1)} + B_k < s < s_k^{(1)*} + B_k \ or \ s_k^{(2)*} + B_k < s < s_k^{(2)} + B_k \tag{32}$$

with best response

$$\overline{R}_k(s) = B_k \tag{33}$$

Notice that $\sqrt{\frac{As_k^{(1)}}{d_k}} = s_k^{(1)} + B_k$ and $\sqrt{\frac{As_k^{(2)}}{d_k}} = s_k^{(2)} + B_k$, since both can be rewritten as Eq~~uation~~. (6). It is also easy to see that $s_k^{(2)} + B_k < \frac{A}{d_k}$ based on Eq~~uation~~. (8).

Consider finally Case 3b. Since

$$s = s_k + x_k = \sqrt{\frac{As_k}{d_k(1+\alpha_k)}}$$

we have

$$s - x_k = s_k = \frac{s^2 d_k (1+\alpha_k)}{A} \tag{34}$$

implying that

$$\overline{R}_k(s) = x_k = s - \frac{s^2 d_k(1+\alpha_k)}{A} \tag{35}$$

This case occurs, when $s_k^{(1)*} < \frac{s^2 d_k(1+\alpha_k)}{A} < s_k^{(2)*}$ which can be rewritten as

$$\sqrt{\frac{As_k^{(1)*}}{d_k(1+\alpha_k)}} < s < \sqrt{\frac{As_k^{(2)*}}{d_k(1+\alpha_k)}}. \tag{36}$$

It is easy to see based on Eq~~uation~~. (11) that

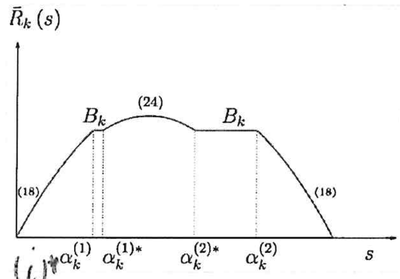
$$\sqrt{\frac{As_k^{(1)*}}{d_k(1+\alpha_k)}} = s_k^{(1)*} + B_k \text{ and } \sqrt{\frac{As_k^{(2)*}}{d_k(1+\alpha_k)}} = s_k^{(2)*} + B_k$$

The graph of $\overline{R}_k(s)$ is shown in Fig. 3.

Images are optimised for fast web viewing. Click on the image to view the original version.

alt-text: Fig. 3

~~Figure 3:~~Fig. 3



(i) α_K

(i) α_K

XX

Graph of $\overline{R}_k(s)$, where $\alpha_i = s_k^{(i)} + B_k$ and $\alpha_i^* = s_k^{(i)*} + B_k$.

The equilibrium is the solution of equation

$$\sum_{k=1}^{n} \overline{R}_k(s) - s = 0 \tag{37}$$

Notice that $\overline{R}_k'(0) = 1$, so if $n \geq 2$, then the derivative of the left hand side at $s = 0$ is greater than unity. The left hand side is zero at $s = 0$ and also at $\max\left\{\frac{A}{d_k}\right\}$ implying that there is at least one positive root of Eq~~uation~~. (37), therefore there is at least one equilibrium besides to the meaningless zero equilibrium.

~~6.~~6 Dynamic ~~E~~extension and ~~S~~simulation ~~S~~study

Assume discrete time scales and that the firms adjust their output quantities partially towards best responses. Let K_k denote the speed of adjustment of firm k, then this adjustment process can be described by the first order difference equations:

$$x_k(t) = x_k(t-1) + K_k\left[R_k\left(s_k(t-1)\right) - x_k(t-1)\right] \quad (38)$$

K n

for ~~k~~ $= 1, 2, \ldots,$ ~~N~~ n, where we assume that $0 < K_k \leq 1$. The selection of $K_k = 0$ would lead to constant trajectories, the case of $K_k = 1$ corresponds to best response dynamics.

Remove 6 but not the reference which is cited elsewhere

We start our analysis considering two firms with different amounts of capital to invest; this way we obtain a model which is a clear generalization of the duopoly model of [1,~~6~~] (Chapter 7). His special duopoly model can be obtained with $A = 1$ and by selecting either $\alpha_1 = \alpha_2 = 0$ or sufficiently large values of E_1 and E_2.

In this particular case, the only fixed point, except the origin, is, of course, the Cournot equilibrium point:

$$\begin{cases} x_1^* = \dfrac{d_2}{(d_1+d_2)^2} \\ x_2^* = \dfrac{d_1}{(d_1+d_2)^2} \end{cases}$$

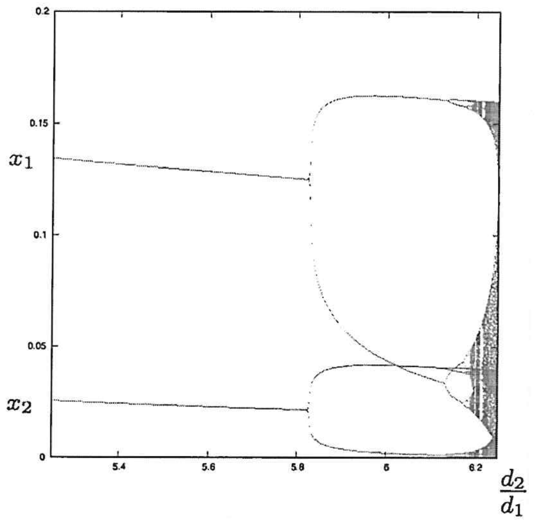
In Puu [6] it is proved that whenever one of the ratios of the marginal costs of the duopolists falls outside the interval $\left[3-\sqrt{2}, 3+\sqrt{2}\right]$, the Cournot point is not stable.

Furthermore, the complexity of the dynamics is illustrated by a bifurcation diagram of firms' outputs versus the marginal cost ratio. When we set either $\alpha_1 = \alpha_2 = 0$ or sufficiently large E_1 and E_2, we obtain the bifurcation diagram reported in Fig. 4, which is identical to the one presented in [6] (p. 271).[1].

Images are optimised for fast web viewing. Click on the image to view the original version.

alt-text: Fig. 4

~~Figure 4:~~Fig. 4



Bifurcation diagram. Firms' outputs plotted vs. marginal cost ratio, with $A = 1$, $c_1 = c_2 = 0$, $d_1 = 1$ and initial condition $x_1(0) = x_2(0) = 0.1$; d_2 varies in the interval [5.25,6.25].

In the bifurcation diagram depicted in Fig. 4, the ratio of the marginal costs of the duopolists d_2/d_1 is smaller or equal to 25/4 in order to avoid the zero branches of the reply functions. When considering the model with capital constrains, this condition may need to be adapted depending on which segments cross at the equilibrium point.

In order to study the effect of capital constraints we compare the bifurcation diagrams to the original case as the cost of capital varies in three different scenarios:

1. the capital constraint affects only the firm with lower marginal cost;
2. the capital constraint affects only the firm with higher marginal cost;
3. the capital constraint affects both firms.

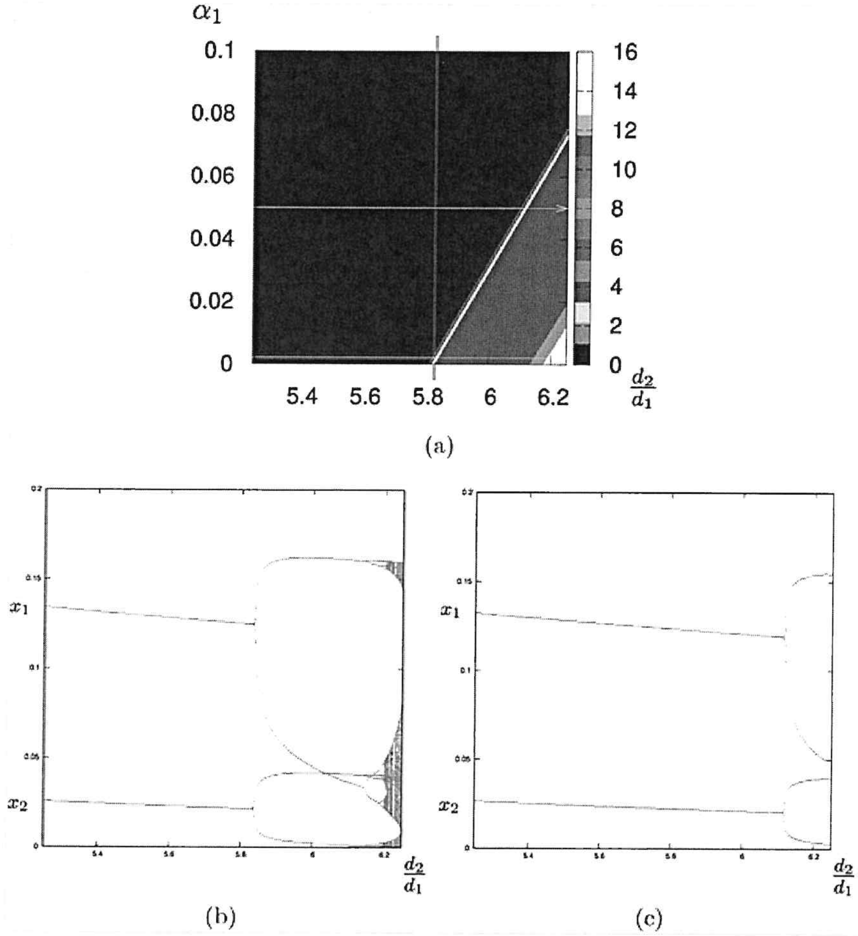
As $d_1 = 1$ and $d_2 \in [5.25,6.25]$, in the range of parameters we consider, the first firm has always the lower marginal cost.

In Fig. 5 the capital constraint is binding only for the firm with lower marginal cost as $E_1 = 0.05$ and α_1 varies, while $E_2 = 1$ and $\alpha_2 = 0$. From Fig. 5(a), we can see that, as the cost of capital for the lower marginal cost firm increases, the dynamics remains stable in a larger interval; this confirmed by the one ~~dimentional~~dimensional bifurcation diagrams in Fig. 5(b) and (c). In the two latter figures we can see that, when the equilibrium loses stability, we can see that a stable cycle of period 2 is created for the quantity produced by each firm. In Fig. 5(a) the associated color code corresponds to a period 4 cycle as for this figure periods are computed considering aggregate quantity and each firm alternates null production to positive production. When comparing these bifurcation diagrams to Fig. 4 we can see that the capital constraint makes the dynamics less complex, at least in the parameter range we consider.

(i)

alt-text: Fig. 5

~~Figure 5:~~Fig. 5



(a) Bifurcation diagram in the parameter plane d_2/d_1, α_1, with $\alpha_2 = 0$; the regions of different periodicity of the aggregate quantity are represented by different colors. The vertical red line represents $d_2/d_1 = 3 + 2\sqrt{2} \simeq 5.8284$, on which the Cournot point in Puu [6] loses stability. The horizontal yellow lines represent two cases, in (b), the bifurcation diagram is shown with $\alpha_1 = 0.2\%$, and in (c) the bifurcation diagram is shown with $\alpha_1 = 5\%$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

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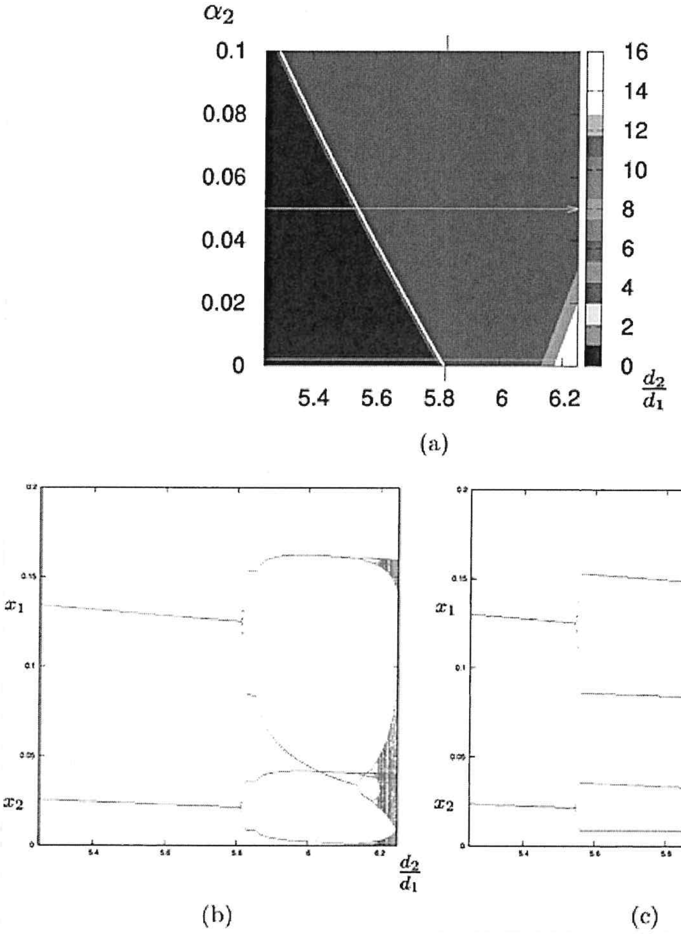
By contrast, in the second scenario, illustrated in Fig. 6, the capital constraint is binding only for the firm with larger marginal cost as $E_2 = 0.05$ and α_2 varies, while $E_1 = 1$ and $\alpha_1 = 0$, Here we can see that, as the cost α_2 of capital increases, the stability interval for d_2/d_1 becomes smaller. Furthermore, as the cost of capital for the higher marginal cost firm increases, the dynamics becomes less complex in contrary to the case with no capital constraints considered in Puu [6]. X

(i) Images are optimised for fast web viewing. Click on the image to view the original version.

alt-text: Fig. 6

~~Figure 6:~~Fig. 6



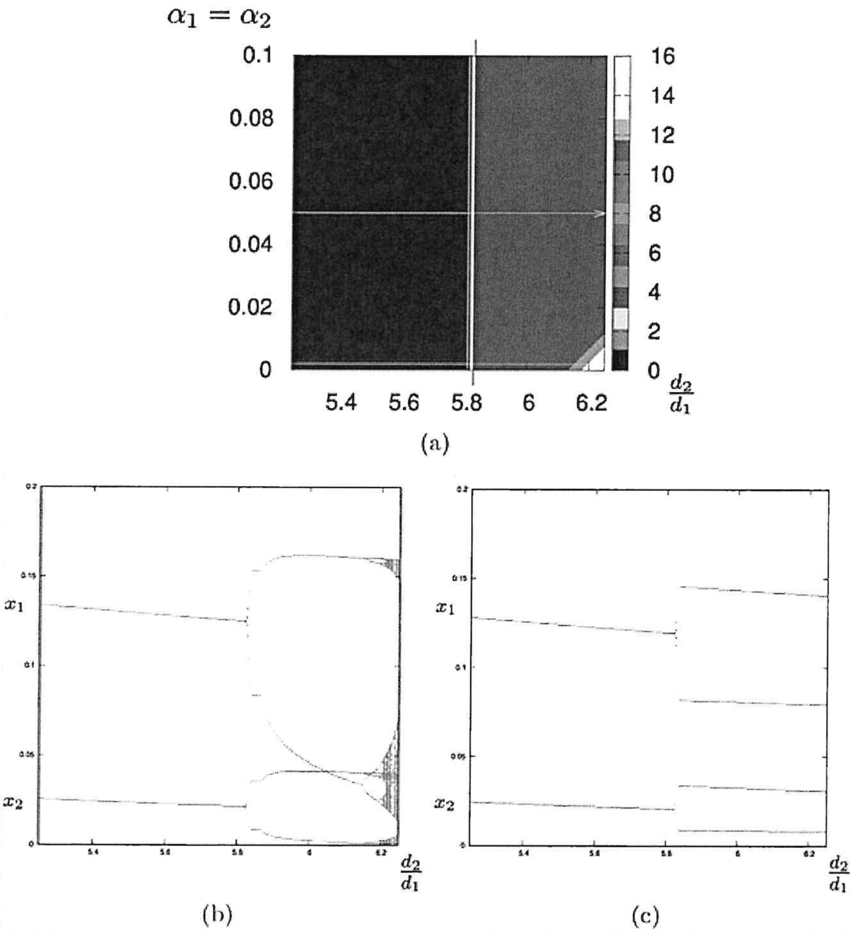
(a) Bifurcation diagram in the parameter plane d_2/d_1, α_1, with $\alpha_1 = 0$; the regions of different periodicity of the aggregate quantity are represented by different colors. The vertical red line represents the $d_2/d_1 = 3 + 2\sqrt{2} \simeq 5.8284$, on which the Cournot point in Puu [6] loses stability. The horizontal yellow lines represent two cases, in (b), the bifurcation diagram is shown with $\alpha_2 = 0.2\%$, and in (c) the bifurcation diagram is shown with $\alpha_2 = 5\%$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Finally, the case in which the capital constraint is binding for both firms is examined in Fig. 7(a). There we can see that the cost of capital seems to have no effect on the marginal cost ratio threshold for the stability of the Cournot point. Also, in this case, as the cost of capital increases, the dynamics becomes less complex in contrary to the case with no capital constraints analyzed in Puu [6].

Images are optimised for fast web viewing. Click on the image to view the original version.

alt-text: Fig. 7

~~Figure 7:~~Fig. 7



(b) (c)

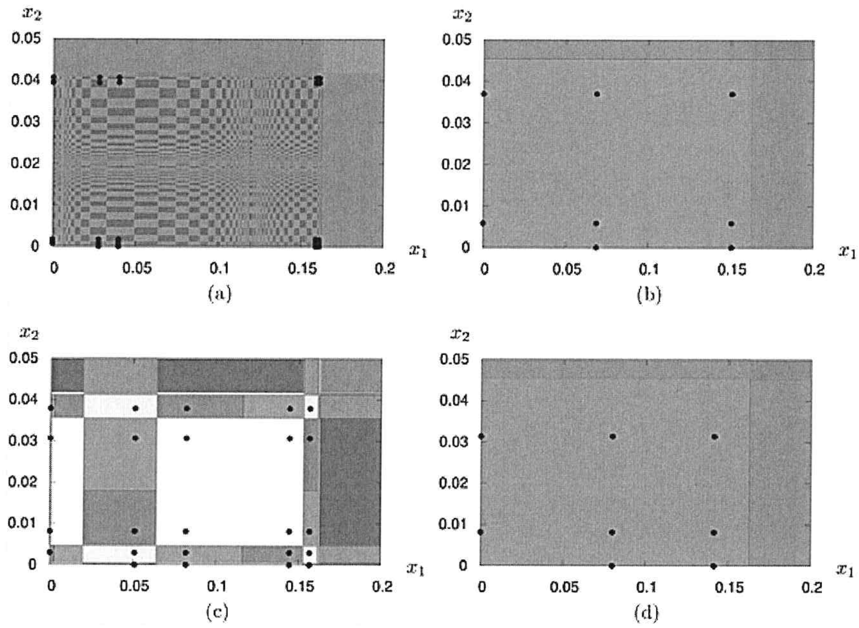
(a) Bifurcation diagram in the parameter plane d_2/d_1, $\alpha_1 = \alpha_2$ the regions of different periodicity of the aggregate quantity are represented by different colors. The vertical red line represents $d_2/d_1 = 3 + 2\sqrt{2} \simeq 5.8284$, on which the Cournot point in Puu [6] loses stability. The horizontal yellow lines represent two cases, in ~~(a)~~, the bifurcation diagram is shown with $\alpha_1 = \alpha_2 = 0.2\%$, and in ~~(b)~~ the bifurcation diagram is shown with $\alpha_1 = \alpha_2 = 5\%$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Another interesting aspect in Puu [6] is the coexistence. In the original model, Puu identifies two 4-period cycles in each variable alone which combine to produce two different coexisting 8-period cycles. They are represented in Fig. 8 (a) together with an additional 8-period cycle in which one of the variables is zero at every other iteration. As in the original paper, we can see that attraction basins consist of fragmented rectangular areas, getting smaller and smaller in the neighborhood of the (unstable) Cournot point; there fragmentation becomes denser. In the case of the best response dynamics, i.e., $K_k = 1$, the second iterate of the map is decoupled and this explains coexistence as discussed in Bischi et al. [24]. When the cost of capital affects the lower marginal cost firm we can find just a single 2-period ~~cycles~~ in each variable alone which produces a 4-period cycle; this cycle coexists with the 4-period cycle in which one of the variables is zero at every other iteration. By contrast, when the cost of capital affects only the firm with higher marginal cost, we have more coexisting basins. In fact, we can observe two 2-period cycles in each variable alone which produces two 4-period cycle together with relative 4-period cycles in which one of the variables is zero at every other iteration and at two 4-period cycles. Their basins are represented in Fig. 8(c), there we can see that all these cycles are generated by the possible combinations of two 2-period cycles in the two state variables. Finally, when both firms are affected by the cost of capital, the situation is quite similar to the case in which only the lower marginal cost firm is affected by cost of capital. As matter of fact, Fig. 8(b) and (d) are quite similar, although the numerical values are obviously different.

cycle

alt-text: Fig. 8

~~Figure 8:~~Fig. 8



Coexistent cycles and their basins of attraction for parameter value $d_2/d_1 = 6.15$. (a) Three coexistent 8-period cycles in Puu [6] model; (b) two coexistent 4-period cycles when $\alpha_1 = 5\%$, (c) six coexistent 4-period cycles when $\alpha_2 = 5\%$; (d) two coexistent 4-period cycles when $\alpha_1 = \alpha_2 = 5\%$.

The analysis of the basins of attraction is consistent with Figs. 5, 6 and 7: the cost of capital makes the dynamics less complex in contrary to the case with no capital constraints analyzed in Puu [6].

unity

When the speed of adjustment ~~are~~ is not ~~unitary~~, i.e. we are non considering best replies the analysis is much more complex. Following Puu [6] we can linearize the system by computing the Jacobian:

$$J(x_1, x_2) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and the characteristic polynomial is

$$\varphi(\lambda) = (a - \lambda)(\lambda - d) - bc = \lambda^2 - \lambda(a + d) + (ad - bc).$$

The eigenvalues are

$$\lambda_{1,2} = \frac{a + d \pm \sqrt{(a - d)^2 + 4bc}}{2}$$

Since Eq. (15) is defined by different segments, we can have 9 different cases to define the Jacobian and the eigenvalues

1. in regions $\mathcal{R}_{1,1}$, $\mathcal{R}_{1,5}$, $\mathcal{R}_{5,1}$ and $\mathcal{R}_{5,5}$:

$$J(x_1, x_2) = \begin{bmatrix} (1 - K_1) & K_1\left(\frac{1}{2}\sqrt{\frac{A}{d_1 x_2}} - 1\right) \\ K_2\left(\frac{1}{2}\sqrt{\frac{A}{d_2 x_1}} - 1\right) & (1 - K_2) \end{bmatrix}$$

and

one line

one line

* do not cut

$$
\begin{aligned}
a+d &= 2-K_1-K_2\\
ad-bc &= (1-K_1)(1-K_2)-K_1\left(\frac{1}{2}\sqrt{\frac{A}{d_2x_1}}-1\right)K_2\left(\frac{1}{2}\sqrt{\frac{A}{d_1x_2}}-1\right)=\\
&= 1-K_1-K_2-K_1K_2\left[\frac{1}{4}\sqrt{\frac{A^2}{d_1d_2x_1x_2}}-\frac{1}{2}\sqrt{\frac{A}{d_2x_1}}-\frac{1}{2}\sqrt{\frac{A}{d_1x_2}}\right]
\end{aligned}
\tag{39}
$$

2. in regions $\mathscr{R}_{1,2}$, $\mathscr{R}_{1,4}$, $\mathscr{R}_{5,2}$ and $\mathscr{R}_{5,4}$:

check formula is visible in the 2-column version

$$
J(x_1,x_2)=\begin{pmatrix}(1-K_1) & K_1\left(\frac{1}{2}\sqrt{\frac{A}{d_1x_2}}-1\right)\\ 0 & (1-K_2)\end{pmatrix}
$$

the matrix is triangular and we have

$$
\varphi(\lambda)=(1-K_1-\lambda)(1-K_2-\lambda) \tag{40}
$$

3. in regions $\mathscr{R}_{1,3}$ and $\mathscr{R}_{5,3}$:

$$
J(x_1,x_2)=\begin{pmatrix}(1-K_1) & K_1\left(\frac{1}{2}\sqrt{\frac{A}{d_1x_2}}-1\right)\\ K_2\left(\frac{1}{2}\sqrt{\frac{A}{d_2(1+\alpha_2)x_1}}-1\right) & (1-K_2)\end{pmatrix}
$$

and

one line

one line

* Do not cut

$$
\begin{aligned}
a+d &= 2-K_1-K_2\\
ad-bc &= (1-K_1)(1-K_2)-K_1\left(\frac{1}{2}\sqrt{\frac{A}{d_2(1+\alpha_2)x_1}}-1\right)K_2\left(\frac{1}{2}\sqrt{\frac{A}{d_1x_2}}-1\right)=\\
&= 1-K_1-K_2-K_1K_2\left[\frac{1}{4}\sqrt{\frac{A^2}{d_1d_2(1+\alpha_2)x_1x_2}}-\frac{1}{2}\sqrt{\frac{A}{d_2(1+\alpha_2)x_1}}-\frac{1}{2}\sqrt{\frac{A}{d_1x_2}}\right]
\end{aligned}
\tag{41}
$$

4. in regions $\mathscr{R}_{2,1}$, $\mathscr{R}_{2,5}$, $\mathscr{R}_{4,1}$ and $\mathscr{R}_{4,5}$:

$$
J(x_1,x_2)=\begin{pmatrix}(1-K_1) & 0\\ K_2\left(\frac{1}{2}\sqrt{\frac{A}{d_2x_1}}-1\right) & (1-K_2)\end{pmatrix}
$$

the matrix is triangular and we have

$$
\varphi(\lambda)=(1-K_1-\lambda)(1-K_2-\lambda) \tag{42}
$$

5. in regions $\mathscr{R}_{2,2}$, $\mathscr{R}_{2,4}$, $\mathscr{R}_{4,2}$ and $\mathscr{R}_{4,4}$:

$$J(x_1,x_2) = \begin{pmatrix} (1-K_1) & 0 \\ 0 & (1-K_2) \end{pmatrix}$$

diagonal

the matrix is ~~triangular and~~ we have

X

$$\varphi(\lambda) = (1-K_1-\lambda)(1-K_2-\lambda) \tag{43}$$

6. in regions $\mathscr{R}_{2,3}$ and $\mathscr{R}_{4,3}$:

$$J(x_1,x_2) = \begin{pmatrix} (1-K_1) & 0 \\ K_2\left(\frac{1}{2}\sqrt{\frac{A}{d_2(1+\alpha_2)x_1}}-1\right) & (1-K_2) \end{pmatrix}$$

the matrix is triangular and we have

$$\varphi(\lambda) = (1-K_1-\lambda)(1-K_2-\lambda) \tag{44}$$

7. in regions $\mathscr{R}_{3,1}$ and $\mathscr{R}_{3,5}$:

$$J(x_1,x_2) = \begin{pmatrix} (1-K_1) & K_1\left(\frac{1}{2}\sqrt{\frac{A}{d_1(1+\alpha_1)x_2}}-1\right) \\ K_2\left(\frac{1}{2}\sqrt{\frac{A}{d_2x_1}}-1\right) & (1-K_2) \end{pmatrix}$$

and

$$a+d = 2-K_1-K_2$$
$$ad-bc = (1-K_1)(1-K_2)\ldots\ldots$$

X Do not cut

$$\begin{aligned} a+d \\ = 2-K_1-K_2 \\ ad-bc \\ = (1-K_1)(1-K_2) - K_1\left(\tfrac{1}{2}\sqrt{\tfrac{A}{d_2x_1}}-1\right)K_2\left(\tfrac{1}{2}\sqrt{\tfrac{A}{d_1(1+\alpha_1)x_2}}-1\right) = \\ = 1-K_1-K_2-K_1K_2\left[\tfrac{1}{4}\sqrt{\tfrac{A^2}{d_1(1+\alpha_1)d_2x_1x_2}} - \tfrac{1}{2}\sqrt{\tfrac{A}{d_2x_1}} - \tfrac{1}{2}\sqrt{\tfrac{A}{d_1(1+\alpha_1)x_2}}\right] \end{aligned} \tag{45}$$

X X X

8. in regions $\mathscr{R}_{3,2}$ and $\mathscr{R}_{3,4}$:

$$J(x_1,x_2) = \begin{pmatrix} (1-K_1) & K_1\left(\frac{1}{2}\sqrt{\frac{A}{d_1(1+\alpha_1)x_2}}-1\right) \\ 0 & (1-K_2) \end{pmatrix}$$

the matrix is triangular and we have

$$\varphi(\lambda) = (1 - K_1 - \lambda)(1 - K_2 - \lambda) \tag{46}$$

9. in region $\mathscr{R}_{3,3}$:

$$J(x_1, x_2) = \begin{pmatrix} (1 - K_1) & K_1 \left(\frac{1}{2}\sqrt{\frac{A}{d_1(1+\alpha_1)x_2}} - 1\right) \\ K_2 \left(\frac{1}{2}\sqrt{\frac{A}{d_2(1+\alpha_2)x_1}} - 1\right) & (1 - K_2) \end{pmatrix}$$

and

single line

single line

$$\begin{aligned} a + d &= 2 - K_1 - K_2 \\ ad - bc &= (1 - K_1)(1 - K_2) - K_1\left(\frac{1}{2}\sqrt{\frac{A}{d_2(1+\alpha_2)x_1}} - 1\right) K_2 \left(\frac{1}{2}\sqrt{\frac{A}{d_1(1+\alpha_1)x_2}} - 1\right) = \\ &= 1 - K_1 - K_2 - K_1K_2\left[\frac{1}{4}\sqrt{\frac{A^2}{d_1(1+\alpha_1)d_2(1+\alpha_2)x_1x_2}} - \frac{1}{2}\sqrt{\frac{A}{d_2(1+\alpha_2)x_1}} - \frac{1}{2}\sqrt{\frac{A}{d_1(1+\alpha_1)x_2}}\right] \end{aligned} \tag{47}$$

X

X

*

Do not cut.

When the equilibrium is in regions $\mathscr{R}_{1,1}$, $\mathscr{R}_{1,5}$, $\mathscr{R}_{5,1}$, $\mathscr{R}_{5,5}$, $\mathscr{R}_{1,3}$, $\mathscr{R}_{5,3}$, $\mathscr{R}_{3,1}$, $\mathscr{R}_{3,5}$, and $\mathscr{R}_{3,3}$ it is immediate to see that the eigenvalues are complex conjugates and we have a Neimark-Sacker bifurcation as observed in ~~(~~[23]~~;~~ (p. 57).

~~7.~~7 Conclusion and ~~F~~further ~~R~~research

In this paper the special classical oligopoly model discussed in Puu [6] is extended to the case when the firms have only limited budgets to cover production costs, and if they exceed the budgets, then the firms borrow extra capital with given unit costs. After model formulation, the best responses of each firm is determined as the function of the output of the rest of the industry, which is only piecewise differentiable in contrary to the classical model. In the duopoly case a simple graphical approach is shown for the existence of the nontrivial equilibrium. Then the best responses are reformulated as functions of the total industry output, which were the basis of proving the existence of at least one nontrivial equilibrium in the general n-firm case. In the simulation study of duopolies we first compared the asymptotic behavior of the extended model to that of Puu's classical case to illustrate the effect of capital constraints in three different scenarios: (a) for the firm with lower marginal cost; (b) for the firm with higher marginal costs; and (c) for both firms with equal marginal costs. We could see how the complexity of the dynamics depends on the unit cost of capital.

The research reported in this paper can be extended in different directions. First the sensitivity analysis can be extended to the analysis of the effects of the marginal costs and budgets on the dynamics. Second, more complex price and production cost functions can be introduced into the model and the study of this paper can be repeated in these cases. Third, production capacity limits can also be introduced with the additional constraints that if the output level change in a time period is smaller than a given threshold, then the interest of the firm is not to make the change (as in [16]).

CRediT authorship contribution statement

Ugo Merlone: Conceptualization, Methodology, Software, Formal analysis, Writing part of original draft X

Ferenc Szidarovszky: Methodology, Writing review, Formal analysis, Writing part of original draft X

Declaration of competing interest

Q5 The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

References

ⓘ The corrections made in this section will be reviewed and approved by a journal production editor. The newly added/removed references and its citations will be reordered and rearranged by the production team.

[1] Puu T. Chaos in duopoly pricing. Chaos, SolitonsFractals 1991;1:573–581.

[2] Cournot A. Recherches sur les Principes Mathematiques de la Theorie des Richesses. 1838. Paris: Hachette; 1960. English translation, Kelly, New York, NY.

[3] Okuguchi K. Expectations and stability in oligopoly models. Berlin/Heidelberg/New York: Springer-Verlag; 1976.

[4] Okuguchi K., Szidarovszky F. The theory of oligopoly with multi-product firms. Berlin/Heidelberg/New York: Springer-Verlag; 1999.

[5] Szidarovszky F., Bahill T. Linear systems theory. Boca Raton/London: CRC Press; 1997.

[6] Puu T. Attractors, bifurcations & chaos. Nonlinear phenomena in economics. second edition Berlin/Heidelberg/New York: Springer-Verlag; 2003.

[7] Puu T., Sushko I., editors Oligopoly dynamics. Models and tools. Berlin/Heidelberg/New York: Springer-Verlag; 2002.

[8] Bischi G.-I., Chiarella C., Kopel M., Szidarovszky F. Nonlinear oligopolies: stability and bifurcations. Berlin/New York: Springer-Verlag; 2010.

[9] Reynolds S.S. Capacity investment, preemption and commitment in an infinite horizon model. IntEconRev 1987;28:69–88.

[10] Reynolds S.S. Dynamic oligopoly with capacity adjustment costs. JEconDynControl 1991;15:491–514.

[11] Szidarovszky F., Yen J. Dynamic Cournot oligopoly with production adjustment costs. JMatheEcon 1995;24:95–101.

[12] Driskill R.A., McCafferty S. Dynamic oligopoly with adjustment costs: a differential game approach. JEconTheory 1989;49:324–338.

[13] Howroyd T.D., Rickard J.A. Cournot oligopoly and adjustment costs. EconLett 1981;7:113–117.

[14] MacLeod W.B. On adjustment costs and the stability of equilibria. Rev Econ Stud 1985;52:575–591.

[15]

Zhao J., Szidarovszky F. N-firm oligopolies with production adjustment costs: best responses and equilibrium. JEconBehavOrgan 2008;68:87–99.

[16] Burr C., Gardini L., Szidarovszky F. Discrete time dynamic oligopolies with adjustment constraints. JDynGames 2015;2(1):65–87.

[17] Matsumoto A., Merlone U., Szidarovszky F. Cartelizing groups in dynamic linear oligopoly with antitrust threshold. IntGame Theory Rev 2008;10(4):399–419.

[18] Matsumoto A., Merlone U., Szidarovszky F. Cartelising groups in dynamic hyperbolic oligopoly with antitrust threshold. AustEconPap 2010;49(4):289–300.

[19] Matsumoto A., Merlone U., Szidarovszky F. Dynamic oligopoly with partial cooperation and antitrust threshold. JEconBehavOrgan 2010;73:259–272.

[20] Merlone U., Szidarovszky F. Dynamic oligopolies with contingent workforce and investment costs. MathComputSimul 2015;108:144–154.

[21] Matsumoto A., Merlone U., Szidarovszky F. Oligopolies with contingent workforce and unemployment insurance systems. CommunNonlinear SciNumerSimul 2015;27:52–65.

[22] Szidarovszky F., Matsumoto A. On a discontinuous Cournot oligopoly. In: von Mouche Pierre, Quartieri Federico, editors. Equilibrium theory for Cournot oligopolies and related games: essays in honour of Koji Okuguchi. Berlin, New York: Springer-Verlag; 2016. p. 97–112.

[23] Puu T. Disequilibrium economics. Cham: Springer-Verlag; 2018.

[24] Bischi G.I., Mammana C., Gardini L. Multistability and cyclic attractors in duopoly games. Chaos, Solitons Fractals 2000;11(4):543–564. doi:10.1016/S0960-0779(98)00130-1. ISSN 0960-0779.

Footnotes

Text Footnotes

[1] The careful reader will notice that in [6] (p. 271) $d_2/d_1 \in [5.75, 6.25]$; here we have extended the interval to show in more detail the effects of capital constraints.

Highlights

- The paper extends and generalizes a classic duopoly model proposed by Puu in 1991.
- We introduce capital ~~contraints~~constraints and study how these constraints affect the dynamics.
- The map that we obtain is piecewise smooth.
- Stability ~~and~~and coexistence are studied by simulation.
- The results are compared to those in the original model.

Queries and Answers

Q1

Query: Your article is registered as belonging to the Special Issue/Collection entitled "Tribute to Tõnu Puu". If this is NOT correct and your article is a regular item or belongs to a different Special Issue please contact v.premarajan@elsevier.com immediately prior to returning your corrections.

Answer: OK

Q2

Query: Please confirm that given names and surnames have been identified correctly and are presented in the desired order, and please carefully verify the spelling of all authors' names.

Answer: OK

Q3

Query: The author names have been tagged as given names and surnames (surnames are highlighted in teal color). Please confirm if they have been identified correctly.

Answer: OK

Q4

Query: The citation "Cournot (1838)" has been changed to "Cournot (1960)" to match the author name/date in the reference list. Please check if the change is fine in this occurrence and modify the subsequent occurrences, if necessary.

Answer: OK

Q5

Query: Correctly acknowledging the primary funders and grant IDs of your research is important to ensure compliance with funder policies. We could not find any acknowledgement of funding sources in your text. Is this correct?

Answer: Yes