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Spectral sequences and generalizations: applications and computational aspects

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Abstract

Spectral sequences, introduced by Jean Leray in 1946, play a central role in algebraic topology and homological algebra. In a typical setting, spectral sequences are employed to determine the homology of a filtered chain complex, trying to exploit the information encoded in the filtration. Being a very versatile construction, which arises quite naturally in a variety of situations, they often allow to clarify complex and meaningful relations: the Serre spectral sequence, for example, introduced by Jean-Pierre Serre in his celebrated doctoral dissertation, describes how the homology of the total space of a fibration is related to the homology of the base and fiber.

Spectral sequences have been recently generalized in the work of Benjamin Matschke to a wider scenario, by allowing the chain complexes to be filtered in a less restrictive way. By contrast with the classical case, which considers only linear filtrations (indexed over the set of integers \mathbb{Z}), the generalized construction applies to filtrations of chain complexes indexed over any partially ordered set (poset). The increased level of generality forces to reconsider several key aspects of the construction.

We present an application of generalized spectral sequences to the increasingly active field of topological data analysis. In particular, we consider persistent homology, a technique allowing to extract topological information from a filtration in the form of topological invariants. We describe the relationship between generalized spectral sequences and suitable generalizations of persistent homology, showing that under certain hypotheses on the poset of indices the invariants they produce can be considered equivalent. As a particularly relevant example we study the case of multidimensional persistence.

Another fundamental issue about generalized spectral sequences concerns the development of methods and programs to compute them. We describe how the technique of effective homology, introduced by Francis Sergeraert, can be applied to compute generalized spectral sequences even in difficult situations, for example for some infinitely generated chain complexes. Our work, including a series of new theoretical results on generalized spectral sequences, led to the development of an additional module, devoted to the computation of generalized spectral sequences, for the computer algebra system Kenzo. We improved the efficiency of our algorithms using discrete vector fields, a key component of discrete Morse theory. Besides considering the general scenario of chain complexes filtered over a poset, we focus on the generalized Serre spectral sequence for towers of fibrations, a subject requiring the development of ad hoc methods.

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Introduction

Homology theory plays a fundamental role in algebraic topology, the branch of mathematics that studies and classifies topological spaces and their properties via algebraic techniques. Ever since the last decade of the 19th century, mathematicians like Poincaré, Brouwer, Alexander and Lefschetz have been describing topological (or combinatorial) objects using numerical invariants related to them, such as Betti numbers. It was around 1927 that the key role of algebra as a tool at the service of topology became clear [Hil88], as the Betti numbers were recognized to be the ranks of some abelian groups associated with the topological objects: the homology groups. That realization prompted the development of a great deal of algebraic methods, inspired by the purposes of topology but completely stand-alone, which took the name of homological algebra [CE56] and started making use of the newly conceived language of category theory [EM45]. The purely algebraic notion of chain complex, freed by any connections with the topological spaces to which it can be associated, became a core concept in homological algebra.

Spectral sequences were introduced in the 1940s in the works of Jean Leray [Ler46, Ler50] and Jean-Louis Koszul [Kos47], and proved themselves to be an exceptional tool to describe and infer relations in algebraic topology and homological algebra. In a purely algebraic scenario, they arise for example from a filtered chain complex, that is a sequence of chain subcomplexes (indexed over the integer numbers)

$$\dots \subseteq F_{p-1}C_* \subseteq F_p C_* \subseteq F_{p+1}C_* \subseteq \dots$$

of a chain complex C_* . The spectral sequence associated with a filtered chain complex C_* produces a sequence of groups which, in a suitable sense, “converges” to the homology groups of C_* .

Spectral sequences have been recently generalized by Benjamin Matschke [Mat13] to a much broader perspective. The most innovative aspect of his remarkable construction is that it applies to filtrations indexed over any partially ordered set (poset), rather than being limited to filtrations indexed over the set \mathbb{Z} of integer numbers. Furthermore, the collection of groups produced by his generalized construction, which he calls a spectral system, is larger than in the classical scenario, as more parameters are taken into account.

Even though they were conceived as a technical tool to construct or understand complex relations in theoretical algebraic topology, spectral sequences present interesting connections with computational and applied topology, a field which has gone through an exciting development during the last few decades. In the present thesis we investigate some critical computational aspects of spectral sequences, starting from the recently studied results for the classical case and detailing how and to what extent they carry over to the generalized construction of spectral systems.

In particular, we identified two major areas in which classical spectral sequences are approached preeminently from a computational point of view. The first area is related to persistent homology,

the most important method of the emergent field of topological data analysis, and the role of spectral sequences in this context can be summarized with the question: What information do they reveal about the filtration from which they arise? While in a theoretical setting the focus is generally on the “limit” groups to which the spectral sequence converges, turning it on the whole collection of groups one can produce a numerical invariant that carries as much information about the filtration as the invariants of persistent homology. The second field in which spectral sequences have been extensively studied from a computational perspective is constructive homological algebra, which deals with classical constructions in algebraic topology and with the methods to implement them, in order to make them computable by a software. In this context, the question one may consider about spectral sequences is: What is the most general setting and which are the methods that allow one to compute them?

As mentioned, our contributions consist mainly in generalizing the results of both fields to the case of spectral systems. The increased level of generality opens a series of problems and requires the development of new methods and tools, as well as the capability to identify the particular cases which, because of their importance, deserve to be treated separately.

Persistent homology is a technique designed to extract information from a filtration (typically of simplicial complexes, with a canonically associated filtration of chain complexes) in the form of topological invariants. Intuitively, it consists in using homology to detect not only the topological features of every step of the filtration, but also how they evolve across the filtration seen as a whole. As it has been successfully used for a large number of applications in the last fifteen years, persistent homology is now recognized as an important tool in data analysis; its success in this field is due to the fact that if one manages to associate a filtration of simplicial complexes with some data, then the topological invariant produced represents a description of the data which is informative, easily computable and robust.

As a result of the intense development of the field, an algebraic theory of persistent homology has rapidly taken shape. At the same time, generalizations of the original technique have been proposed, which deal with filtrations indexed over sets different from \mathbb{Z} . The main example is multidimensional persistence, which deals with filtrations indexed over \mathbb{Z}^n and presents several additional issues.

Even though they were conceived for entirely different purposes, both spectral sequences and persistent homology are constructions in algebraic topology which originate from a filtration of chain complexes. The two concepts are in fact related, as it has been elegantly shown employing the notion of exact and derived couples introduced by William Massey in 1952.

In this thesis, we extend the relationship to the case of spectral systems and generalized persistent homology, considering filtrations indexed over any poset. This yields a relationship between the invariants of generalized persistent homology and spectral systems. Furthermore, we show that, for a particular class of posets, the notion of derived couples can be generalized, giving rise to an elegant construction connecting persistent homology and spectral sequences in a general setting. We show how this relationship can lead to interesting applications in both fields.

From a computational perspective, computing the homology of a finitely generated chain complex

amounts to determining the Smith normal form of a (finite) matrix, a task easily performed by a computer via efficient algorithms. Nonetheless, many theoretical constructions in algebraic topology give rise to infinitely generated chain complexes, whose homology is clearly more complicated to compute, because one cannot save the list of all generators and use it to perform computations. In this case, the technique of effective homology introduced by Francis Sergeraert around 1980 allows to tackle the issue: if the infinitely generated chain complex can be mapped to a finitely generated one via a particular type of chain equivalence, one can use the latter to perform the computations.

The effective homology technique can be successfully employed to compute spectral sequences associated with filtrations of (infinitely generated) chain complexes. A module developed by Ana Romero allows to handle spectral sequences with the software *Kenzo*, designed by Sergeraert and his collaborators. In addition to computing spectral sequences in the most general setting from filtrations of chain complexes, Romero's algorithms exploit homological perturbation theory to compute some particular spectral sequences, for example the Serre spectral sequence of a fibration.

In this thesis we present a series of theoretical results describing how the methods of effective homology can be applied in the generalized case of spectral systems. For instance, we show that not all the groups of a spectral system arising from an infinitely generated filtered chain complex are computable using effective homology, and we characterize the ones that are computable. We explain how the algorithms for classical spectral sequences must be modified to work in the generalized case, presenting the main features of an additional module for the software *Kenzo* we developed, together with Ana Romero. A section is devoted to presenting how effective homology can be applied to the generalized version of the Serre spectral sequence for towers of fibrations, a subject requiring the development of ad hoc methods.

The present thesis is therefore structured around two distinct yet strongly related themes, both characterized by the connections with the generalized theory of spectral sequences, which shows its utility as a theoretical tool as well as its interest from a computational perspective. In several respects, the two subjects complement each other to convey a more complete picture of computational aspects concerning spectral sequences and their generalizations. For instance, the structure of the thesis gives the opportunity to cover both the finite dimensional case, the domain of persistent homology theory, and the infinite dimensional, the typical context of application of the effective homology methods. Furthermore, it offers the chance of an interesting comparison between various topological and combinatorial objects associated with chain complexes and spectral sequences, and the computational adjustments they require: while in topological data analysis simplicial complexes are preferred, in effective homology simplicial sets are more versatile. A further occasion for a stimulating connection with different areas is offered by the theory of discrete vector fields, introduced by Robin Forman in 1998 and very successful in several fields, which we apply to simplify the computation of spectral systems.

In Chapter 1 we introduce some preliminary concepts and results in homological algebra we will use during this work. After presenting simplicial complexes and simplicial sets, two classes of objects which will play an important role in what follows, we illustrate the notion and the main properties of spectral sequences.

Chapter 2 contains an introduction to the theory of spectral systems. We explain how they arise from filtrations of chain complexes indexed over any poset, and show how they generalize the classical notion of spectral sequence, providing some concrete examples which will be employed in the rest of the thesis.

In Chapter 3 we study the connections between spectral systems and the most successful methods in topological data analysis. After a brief overview of the main notions related to persistent homology and its generalizations, such as multidimensional persistence, we illustrate their relation with spectral sequences and spectral systems.

In Chapter 4 we introduce the technique of effective homology, review how it can be applied to the computation of spectral sequences and study a generalization to the case of spectral systems. We present specific methods for the spectral system arising from a tower of fibrations and explain how, in some cases, discrete vector fields can be used to speed-up computations.

Chapter 1

Preliminaries

In this chapter we set the stage for the rest of the work. In Section 1.1 we introduce chain complexes and their homology, together with some fundamental properties and results of homological algebra. Then we introduce two classes of mathematical objects with which chain complexes can be associated: simplicial complexes (Section 1.2) and simplicial sets (Section 1.3). We will highlight the advantages of each class from a computational point of view: simplicial complexes, simpler and easier to define, can be employed in data analysis, and will be used in Chapter 3; simplicial sets, more general and versatile, are more suitable for theory-oriented computations in algebraic topology, as we will see in Chapter 4. In Section 1.4 we turn back to homological algebra to present one of the core concepts of this work: spectral sequences.

1.1 Basic notions of homological algebra

This section is intended as an overview of some fundamental concepts and results in homological algebra which will be employed in the present work. Our main references for homological algebra are [Wei94] and [Mac63], while for algebraic topology we refer to [Hat02].

In what follows, we call simply *ring* a commutative unital ring R . After introducing the basic notions of homological algebra in the relatively general framework of R -modules, we will focus principally on \mathbb{Z} -modules (abelian groups) and, in Chapter 3, on vector spaces. By virtue of the commutativity of R , we do not need to distinguish between left and right R -modules. For an introduction on R -modules we address the reader to [AM69, Ch. 2].

Definition 1.1. Given a ring R , a *chain complex* of R -modules is a sequence $C_* = (C_n, d_n)_{n \in \mathbb{Z}}$, where each C_n is an R -module, called the module of n -chains, and the d_n are morphisms $d_n : C_n \rightarrow C_{n-1}$ of R -modules, called *differentials* (or *boundary maps*), such that $d_{n-1}d_n = 0$, for all $n \in \mathbb{Z}$.

A chain complex $C_* = (C_n, d_n)_{n \in \mathbb{Z}}$ of R -modules is called *free* if each C_n is a free R -module. The chain complexes considered in this work are generally associated with topological or combinatorial objects, such as simplicial complexes or simplicial sets; in this case, as we will see, they are free by definition.

The kernel $Z_n := \ker d_n \subseteq C_n$ is called the submodule of n -cycles; the image $B_n := \operatorname{Im} d_{n+1} \subseteq C_n$ is called the submodule of n -boundaries. As the relations $d_{n-1}d_n = 0$ for all $n \in \mathbb{Z}$ are equivalent to the inclusions $B_n \subseteq Z_n$ for all $n \in \mathbb{Z}$, the quotients of these submodules are well-defined:

Definition 1.2. Let $C_* = (C_n, d_n)_{n \in \mathbb{Z}}$ be a chain complex of R -modules. For each $n \in \mathbb{Z}$, the n -homology module of C_* is the quotient module

$$H_n(C_*) := Z_n/B_n.$$

The elements of $H_n(C_*)$ are called n -homology classes. It is worth noting that the terminology *homology group* is very common, even if each $H_n(C_*)$ is in fact an R -module. Similarly, the modules C_n , Z_n and B_n are often addressed to as the *groups* of chains, cycles and boundaries. We use the notation $H_n(C_*; R)$ if we want to highlight the dependence on the ring R .

Since homology groups are defined as quotient modules, we recall here two simple results in module theory, which are proven for example in [AM69, Ch. 2.2].

Proposition 1.3. Let R be a ring and let $f : M \rightarrow M'$ a morphism of R -modules. If N is a submodule of M such that $N \subseteq \ker f$, then there exists a unique morphism of R -modules $\bar{f} : M/N \rightarrow M'$ such that $\bar{f}p = f$, where $p : M \rightarrow M/N$ is the canonical projection:

$$\begin{array}{ccc} M & & \\ p \downarrow & \searrow f & \\ M/N & \xrightarrow{\bar{f}} & M' \end{array}$$

Proposition 1.4. Let S be a submodule of an R -module M . Then there is an order-preserving bijection

$$\varphi : \{\text{submodules } N \text{ such that } S \subseteq N \subseteq M\} \rightarrow \{\text{submodules of } M/S\},$$

defined by $\varphi(N) := N/S$.

The following property, stated in [Mac63, p. 318], will also be useful to handle quotient modules:

Proposition 1.5. Let N, S, T be submodules of an R -module M . If $T \subseteq N$, then

$$N \cap (S + T) = N \cap S + T.$$

As a consequence, under the same hypotheses we have the canonical isomorphism of quotient modules

$$\frac{N}{N \cap S + T} = \frac{N}{N \cap (S + T)} \cong \frac{N + S}{T + S}.$$

Let us resume our introduction on chain complexes and homology. Since they both involve *graded* objects (indexed over the integers), the next definition may be helpful:

Definition 1.6. A *graded module* is a family of modules indexed over the integers: $M = (M_n)_{n \in \mathbb{Z}}$.

Let M and N be graded R -modules. A *graded map* (or *homomorphism of graded modules*) of *degree* $k \in \mathbb{Z}$ is a collection of morphisms of R -modules $f = (f_n : M_n \rightarrow N_{n+k})_{n \in \mathbb{Z}}$. We denote it shortly as $f : M \rightarrow N$ and write $\deg(f) = k$ to indicate that the degree of f is k .

A chain complex $C_* = (C_n, d_n)_{n \in \mathbb{Z}}$ can clearly be seen as a graded module $C = (C_n)_{n \in \mathbb{Z}}$ together with a graded map $d = (d_n : C_n \rightarrow C_{n-1})_{n \in \mathbb{Z}}$ of degree -1 satisfying $d_{n-1}d_n = 0$, for all $n \in \mathbb{Z}$.

Definition 1.7. Let $C_* = (C_n, d_n)_{n \in \mathbb{Z}}$ and $D_* = (D_n, d'_n)_{n \in \mathbb{Z}}$ be two chain complexes of R -modules. A *chain map* (or *morphism of chain complexes*) $f : C_* \rightarrow D_*$ is a sequence of morphisms of R -modules $f_n : C_n \rightarrow D_n$ which *commute* with the differentials of C_* and D_* , that is $d'_n f_n = f_{n-1} d_n$, for all $n \in \mathbb{Z}$.

An immediate consequence of the definition is that every chain map $f : C_* \rightarrow D_*$ induces well-defined morphisms of R -modules

$$f_* = H_n(f) : H_n(C_*) \rightarrow H_n(D_*),$$

for each $n \in \mathbb{Z}$.

Definition 1.8. Given two chain complexes $C_* = (C_n, d_n)_{n \in \mathbb{Z}}$ and $D_* = (D_n, d'_n)_{n \in \mathbb{Z}}$ and two chain maps $f, g : C_* \rightarrow D_*$, a *chain homotopy* h between f and g , denoted $h : f \simeq g$, is a sequence of morphisms of R -modules $h_n : C_n \rightarrow D_{n+1}$ such that $f_n - g_n = h_{n-1} d_n + d'_{n+1} h_n$, for all $n \in \mathbb{Z}$.

Definition 1.9. A chain map $f : C_* \rightarrow D_*$ is a *chain equivalence* if there exist a chain map $g : D_* \rightarrow C_*$ and homotopies $h_1 : \text{id}_{C_*} \simeq gf$ and $h_2 : \text{id}_{D_*} \simeq fg$.

Theorem 1.10. If there exists a chain homotopy $h : f \simeq g$ between two chain maps $f, g : C_* \rightarrow D_*$, then f and g induce the same morphisms on homology, that is

$$H_n(f) = H_n(g) : H_n(C_*) \rightarrow H_n(D_*),$$

for all $n \in \mathbb{Z}$.

Corollary 1.11. Let $f : C_* \rightarrow D_*$ be a chain equivalence. Then the induced morphism

$$H_n(f) : H_n(C_*) \rightarrow H_n(D_*)$$

is an isomorphism, for each $n \in \mathbb{Z}$.

A *chain subcomplex* $D_* = (D_n, d_n)_{n \in \mathbb{Z}}$ of a chain complex $C_* = (C_n, d_n)_{n \in \mathbb{Z}}$ is a sequence of submodules $D_n \subseteq C_n$ endowed with the differentials induced by the differentials of C_* (here denoted in the same way) and such that $d_n(D_n) \subseteq D_{n-1}$, for all $n \in \mathbb{Z}$. We use the notation $D_* \subseteq C_*$ to express the fact that D_* is a chain subcomplex of C_* .

Given $D_* \subseteq C_*$, there is a well-defined *quotient chain complex* C_*/D_* having C_n/D_n as group of n -chains, for each n , and differentials induced by those of C_* . In order to state an important result describing the relationship between the homology groups of D_* , C_* and C_*/D_* we need to introduce the notion of *exact sequence*:

Definition 1.12. A sequence of morphisms of R -modules

$$\cdots \rightarrow A_{n+1} \xrightarrow{f_{n+1}} A_n \xrightarrow{f_n} A_{n-1} \rightarrow \cdots$$

is *exact at* A_n if $\ker f_n = \text{Im } f_{n+1}$; it is *exact* if it is exact at A_n , for all n .

An exact sequence of the form $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is called a *short exact sequence*, while an infinite exact sequence is called a *long exact sequence*. Replacing the sequence of morphisms of modules of Definition 1.12 with a sequence of morphisms of chain complexes, one obtains the definition of exact sequence of chain complexes.

We state here a simplified version of a standard theorem in homological algebra:

Theorem 1.13. Given $D_* \subseteq C_*$, there exists a long exact sequence of homology groups

$$\cdots \rightarrow H_{n+1}(C_*/D_*) \rightarrow H_n(D_*) \rightarrow H_n(C_*) \rightarrow H_n(C_*/D_*) \rightarrow H_{n-1}(D_*) \rightarrow \cdots \quad (1.1)$$

We call (1.1) the *long exact sequence of (relative) homology*, because of its role in algebraic topology (see for example [Hat02, Theorem 2.13]). For the definition of all the maps involved and a more general statement of Theorem 1.13 we address the reader to [Wei94, Theorem 1.3.1] or [Mac63, Theorem 4.1]; here we just mention that the maps $\partial_n : H_n(C_*/D_*) \rightarrow H_{n-1}(D_*)$ are called *connecting homomorphisms* and defined from the differentials d_n .

We now introduce tensor products of modules and chain complexes following [Mac63, Ch. V], to which we refer the reader for further details. We then state the *universal coefficient theorem*, a fundamental result in homological algebra. Recall that, as we assume the ring R to be commutative, each left R -module M can be seen as a right R -module simply defining the right multiplication as $mr := rm$, for all $r \in R$ and $m \in M$ (and vice versa).

Definition 1.14. Let M, N be R -modules. The *tensor product* $M \otimes_R N$ is the abelian group generated by the set of symbols $\{m \otimes n \mid m \in M, n \in N\}$ subject to the following relations, for all $r \in R, m, m' \in M$ and $n, n' \in N$:

$$\begin{aligned} (m + m') \otimes n &= m \otimes n + m' \otimes n, \\ m \otimes (n + n') &= m \otimes n + m \otimes n', \\ mr \otimes n &= m \otimes rn. \end{aligned}$$

When $R = \mathbb{Z}$ we simply denote $M \otimes N$ the tensor product of M and N .

Proposition 1.15. If $f : M \rightarrow M'$ and $g : N \rightarrow N'$ are morphisms of R -modules, the map

$$f \otimes_R g : M \otimes_R N \rightarrow M' \otimes_R N'$$

defined on the generators by $(f \otimes_R g)(m \otimes n) := f(m) \otimes g(n)$ is a morphism of R -modules. Moreover, if $f' : M' \rightarrow M''$ and $g' : N' \rightarrow N''$ are morphisms of R -modules, we have

$$(f' \otimes_R g')(f \otimes_R g) = (f'f) \otimes_R (g'g).$$

Definition 1.16. Let $C_* = (C_n, d_n)_{n \in \mathbb{Z}}$ and $D_* = (D_n, d'_n)_{n \in \mathbb{Z}}$ be chain complexes of R -modules. The *tensor product* $C_* \otimes_R D_*$ is the chain complex of \mathbb{Z} -modules $C_* \otimes_R D_* = ((C_* \otimes_R D_*)_n, \partial_n)_{n \in \mathbb{Z}}$

with

$$(C_* \otimes_R D_*)_n := \bigoplus_{p+q=n} (C_p \otimes_R D_q)$$

and differentials defined on the generators $x \otimes y$ of each $C_p \otimes_R D_q$ as

$$\partial_n(x \otimes y) := d_p x \otimes y + (-1)^p x \otimes d'_q y.$$

Definition 1.17. Let $f : C_* \rightarrow C'_*$ and $g : D_* \rightarrow D'_*$ be morphisms of chain complexes (of R -modules). The *tensor product* $f \otimes_R g : C_* \otimes_R D_* \rightarrow C'_* \otimes_R D'_*$ is the morphism of chain complexes defined on the generators $x \otimes y$ of each $C_p \otimes_R D_q$ as $(f \otimes_R g)(x \otimes y) := f_p(x) \otimes g_q(y)$.

The tensor product $C_* \otimes_R M$ of a chain complex C_* of R -modules and an R -module M can be seen as a particular case of Definition 1.16 by considering M as a chain complex *concentrated* in degree 0 (that is: the only nonzero chain group is M at degree $n = 0$, and all the differentials are zero). If C_* is a chain complex of abelian groups and M is an abelian group, the homology groups $H_n(C_* \otimes M)$ are called *homology of C_* with coefficients in M* and are denoted also $H_n(C_*; M)$. Their relation with the *integer* homology groups $H_n(C_*; \mathbb{Z})$, denoted simply $H_n(C_*)$, is the following:

Theorem 1.18 (Universal coefficient theorem for homology). Let C_* be a chain complex of free abelian groups. Then for every $n \in \mathbb{Z}$ and every abelian group M there is a short exact sequence

$$0 \rightarrow H_n(C_*) \otimes M \rightarrow H_n(C_*; M) \rightarrow \mathrm{Tor}_1^{\mathbb{Z}}(H_{n-1}(C_*), M) \rightarrow 0$$

which splits (though not canonically):

$$H_n(C_*; M) \cong (H_n(C_*) \otimes M) \oplus \mathrm{Tor}_1^{\mathbb{Z}}(H_{n-1}(C_*), M).$$

For a proof of Theorem 1.18 see [Hat02, Ch. 3.A] or [Wei94, Ch. 3.6]. It is worth highlighting the deep meaning of the universal coefficient theorem: homology with coefficients in any abelian group can be described from integer homology, which can thus be seen as the most general (“universal”) case. We refer to [Wei94, Ch. 2–3] for more details on $\mathrm{Tor}(-, -)$. Here we only mention that $\mathrm{Tor}_1^{\mathbb{Z}}(M, N)$ is a \mathbb{Z} -module, for any \mathbb{Z} -modules M, N , and that $\mathrm{Tor}_1^{\mathbb{Z}}(M, N) = 0$ if either M or N is a projective \mathbb{Z} -module.

Since homology over \mathbb{Z} plays such an important role, we want to conclude this introductory section by recalling the decomposition theorem for (finitely generated) \mathbb{Z} -modules. We state it at first in its general form, for modules over a PID (principal ideal domain), see [DF04, Ch. 12.1].

Theorem 1.19 (Structure theorem for finitely generated modules over a PID). Let R be a PID and let M be a finitely generated R -module. Then M decomposes in a unique way as

$$M \cong R^\beta \oplus R/(a_1) \oplus R/(a_2) \oplus \cdots \oplus R/(a_k),$$

where β is an integer ≥ 0 and a_1, a_2, \dots, a_k are nonzero elements of R which are not units and satisfy the divisibility relations $a_1 \mid a_2 \mid \cdots \mid a_k$.

As a consequence, each \mathbb{Z} -module M decomposes as

$$M \cong \mathbb{Z}^\beta \oplus \mathbb{Z}_{a_1} \oplus \mathbb{Z}_{a_2} \oplus \cdots \oplus \mathbb{Z}_{a_k},$$

where the integer $\beta \geq 0$ is called the *rank* of M and the integers $a_1, a_2, \dots, a_k > 1$ and are called *torsion coefficients*. If X is an object for which a notion of homology can be defined (for example a topological space, a simplicial complex, a simplicial set; see Sections 1.2–1.3), then the coefficients $\beta_n, a_{n,1}, a_{n,2}, \dots, a_{n,k}$ appearing in the decomposition of each (integer) homology group

$$H_n(X) \cong \mathbb{Z}^{\beta_n} \oplus \mathbb{Z}_{a_{n,1}} \oplus \mathbb{Z}_{a_{n,2}} \oplus \cdots \oplus \mathbb{Z}_{a_{n,k}} \quad (1.2)$$

are *topological invariants* associated to X , meaning that (intuitively) they capture the topological properties of X . To make the notion of topological invariant more precise, we require that the following property is fulfilled: if $X \cong Y$ in an appropriate category, then their homology groups are isomorphic, and therefore they have the same “coefficients”. In particular, the integer β_n in (1.2) is called the *n-Betti number* of X . In Chapter 3 Betti numbers (and their generalizations) will assume a key role in devising topological techniques for data analysis.

1.2 Simplicial complexes

We introduce, following mainly [Mun84], simplicial complexes and their homology. Even if they can be thought as topological subspaces of \mathbb{R}^m , their topological properties are fully summarized by a combinatorial structure, which makes them suitable objects for modeling and implementation. They are in fact successfully employed in data analysis, as we will see in some examples which conclude the section.

Definition 1.20. An (*abstract*) *simplicial complex* is a collection K of finite nonempty sets such that if $\sigma \in K$ and τ is a nonempty subset of σ then $\tau \in K$. The elements of K are called *simplices*, and we say that $\sigma \in K$ is an *n-simplex* if its cardinality is $n + 1$. The vertex set V of K is the union $V := \cup_{\sigma \in K} \sigma$ of all the simplices of K . A *subcomplex* of K is subcollection $K' \subseteq K$ which is a simplicial complex by its own.

Abstract simplicial complexes are actually a purely combinatorial version of geometric simplicial complexes, which for decades played a central role in *combinatorial topology* [Ale56] before the development of the modern theory of algebraic topology. Geometric simplicial complexes are topological spaces which can be thought as made of simple “pieces”, the (geometric) simplices. If v_0, v_1, \dots, v_n are $n + 1$ points in \mathbb{R}^m such that the n vectors $v_1 - v_0, \dots, v_n - v_0$ are linearly independent, their convex hull

$$\{x = \sum_i t_i v_i \in \mathbb{R}^m \mid \sum_i t_i = 1 \text{ and } t_i \geq 0 \text{ for all } i\}$$

defines a (geometric) *n-simplex* σ ; the convex hull of any nonempty subset of $\{v_0, \dots, v_n\}$ is called a *face* of σ . The particular case where each v_i is the $(i + 1)$ -th vector $e_{i+1} = (0, \dots, 1, \dots, 0)$ of

the standard basis of \mathbb{R}^{n+1} defines the *standard n -simplex* Δ^n . Figure 1.1 represents simplices of dimension $n = 0, 1, 2, 3$.

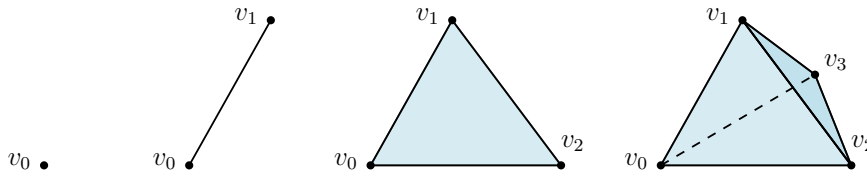


FIGURE 1.1: From the left: a 0-simplex (vertex), a 1-simplex (line segment), a 2-simplex (triangle) and a 3-simplex (tetrahedron).

Definition 1.21. A (*geometric*) *simplicial complex* is a collection X of simplices such that every face of a simplex of X belongs to X and the intersection of two simplices of X is either empty or a face of both.

An abstract simplicial complex K can be easily obtained from a geometric simplicial complex X by considering the set of vertices of each geometric simplex: in this case, we say that K is a *vertex scheme* of X and that X is a *geometric realization* of K . It is not difficult to prove that every abstract simplicial admits a geometric realization [Mun84, Theorem 3.1].

By virtue of their combinatorial nature, abstract simplicial complexes are more suitable for computational purposes; in the following we will therefore focus on them, calling them simply simplicial complexes. Nevertheless, keeping in mind their geometric counterpart is fundamental to understand the meaning of some notions, as we will see now for homology.

Consider a simplicial complex K . Supposing that its vertex set V is totally ordered, we can write each n -simplex σ as a list $\sigma = [x_0, x_1, \dots, x_n]$ with $x_0 < x_1 < \dots < x_n$ in V . A chain complex $C_*(K) = (C_n(K), d_n)_{n \in \mathbb{Z}}$, called the *chain complex associated with K* , can then be defined as follows:

- the group $C_n(K)$ of n -chains is the free \mathbb{Z} -module generated by the set of n -simplices of K . An n -chain $c \in C_n(K)$ is therefore a finite linear combination $c = \sum \lambda_i \sigma_i$, with $\lambda_i \in \mathbb{Z}$ and σ_i n -simplices of K ;
- the differential $d_n : C_n(K) \rightarrow C_{n-1}(K)$ is defined on each n -simplex $\sigma = [x_0, x_1, \dots, x_n]$ as

$$d_n(\sigma) := \sum_{i=0}^n (-1)^i [x_0, \dots, \hat{x}_i, \dots, x_n],$$

where \hat{x}_i means that x_i is removed; the definition of d_n is extended by linearity to all elements $c \in C_n(K)$.

We call *n -homology group of K* the n -homology group of the chain complex $C_n(K)$,

$$H_n(K) := H_n(C_*(K)) = \ker d_n / \text{Im } d_{n+1}.$$

As we said the homology groups can be constructed with coefficients in a ring R rather than in \mathbb{Z} , in which case we denote them by $H_n(K; R)$.

Geometrical intuition is helpful to understand how the homology groups $H_n(K)$, and in particular the Betti numbers $\beta_n := \text{rank } H_n(K)$, describe the “topological” properties of K . Intuitively, we can say that β_n counts n -dimensional holes of K : β_0 is the number of connected components, β_1 the number of “tunnels”, β_2 the number of “voids”, and so on (see Figure 1.2).

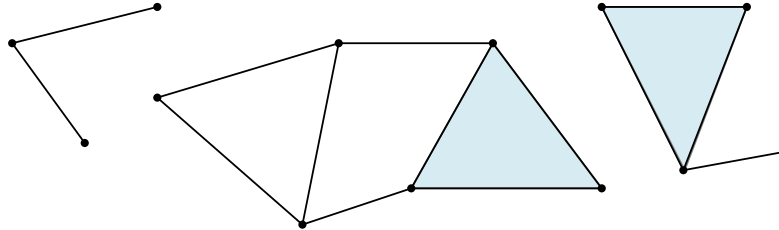


FIGURE 1.2: The Betti numbers of the represented simplicial complex are: $\beta_0 = 3$, $\beta_1 = 2$ and $\beta_n = 0$ for $n \geq 2$.

The simple combinatorial structure of simplicial complexes, together with their versatility, makes them a useful tool for data analysis, as witnesses their use in a large variety of contexts. As we will see in detail in Chapter 3, associating simplicial complexes with data represents the first step of the most important methods in the recently developed field of *topological data analysis*. In what follows we want to give an idea of how simplicial complexes can be associated with data, and what kind of information is revealed by their homology. In this brief overview we consider two types of data: networks and point clouds. A more complete treatment can be found in [EH10, Ch. III] and [Ghr14, Ch. 2].

Example 1.22 (Clique complex of a network). Networks are graphs (here assumed to be *undirected*) used to model and represent relations in a variety of discrete physical systems. The objects of the system are represented as vertices (or *nodes*), and an interaction between two nodes is represented as an edge (or *link*). By their very nature, networks are therefore particularly suitable for capturing pairwise relations, although sometimes they miss “higher dimensional” information. For example, the interaction between three objects is represented in a network as three pairwise interactions, that is three edges. To overcome this limitation, we can represent with a triangle (2-simplex) interactions between triples of objects, and in general with n -simplices interactions between $(n + 1)$ -tuples of objects; in this way we can construct a simplicial complex containing the information of the network plus information on relations between any number of objects. The easiest and most used method to associate a simplicial complex with a network is to consider its *clique complex*: each clique (complete subgraph) of $n + 1$ vertices of the network is seen as an n -simplex.

As simplicial complexes are suitable to be studied also with topological methods, in addition to standard graph-theoretical ones, they can reveal information which is hidden if we only consider the original network. To give an example, considering higher dimensional information in networks representing neural data led to remarkable insights [GGB16]. In particular, computing the homology of the associated simplicial complex can provide interesting information on the network and the physical system it describes. For instance, the homology of simplicial complexes associated with sensor networks [DSG06] can reveal poorly covered zones.

Example 1.23 (Čech and Vietoris-Rips complexes of a point cloud). A *point cloud* is a dataset that can be represented as a discrete collection of points in \mathbb{R}^m . A typical example is the output of 3D scanners, which detect the position of a sample of points on the surface of the object one wants to scan. Clearly one can be interested in reconstructing a global object from the points, and in particular a simplicial complex approximating the original surface; furthermore, it seems reasonable to require that the reconstructed object reflects the topological features of the original object (see for example [DSW11] for the case of homological features). A first kind of simplicial complex which can be constructed from a point cloud is the *Čech complex*. For each point v of the given point cloud $V \subseteq \mathbb{R}^m$ consider the open ball $B_\epsilon(v) := \{x \in \mathbb{R}^m \mid \|x - v\| < \epsilon\}$ of center v and radius ϵ , where $\epsilon > 0$ is a fixed parameter. The Čech complex \check{C}_ϵ has V as vertex set, and as n -simplices the collections of $n + 1$ points $\{v_i\} \subseteq V$ whose balls have nonempty intersection:

$$\sigma = \{v_i\} \in \check{C}_\epsilon \iff \bigcap_i B_\epsilon(v_i) \neq \emptyset. \quad (1.3)$$

The *nerve lemma* [Hat02, Corollary 4G.3] guarantees that the Čech complex \check{C}_ϵ is always homotopy equivalent to the union of the balls $B_\epsilon(v)$ over all $v \in V$. This is clearly a desirable property, as it means that \check{C}_ϵ captures well the homological features of the cover $\{B_\epsilon(v)\}_{v \in V}$. As it is easy to observe, the whole construction and its accuracy strongly depend on the choice of the radius ϵ ; even if in general finding an optimal value for ϵ can be difficult, in Chapter 3 we will see some methods to address this issue.

In order to define the n -simplices, instead of condition (1.3) we may only require *pairwise* intersections of balls to be nonempty, thus obtaining the definition of the *Vietoris-Rips complex* VR_ϵ :

$$\sigma = \{v_i\} \in \text{VR}_\epsilon \iff B_\epsilon(v_i) \cap B_\epsilon(v_j) \neq \emptyset, \text{ for all } v_i, v_j \in \sigma.$$

The Vietoris-Rips complex VR_ϵ is very used in practice because it is the clique complex of a graph,

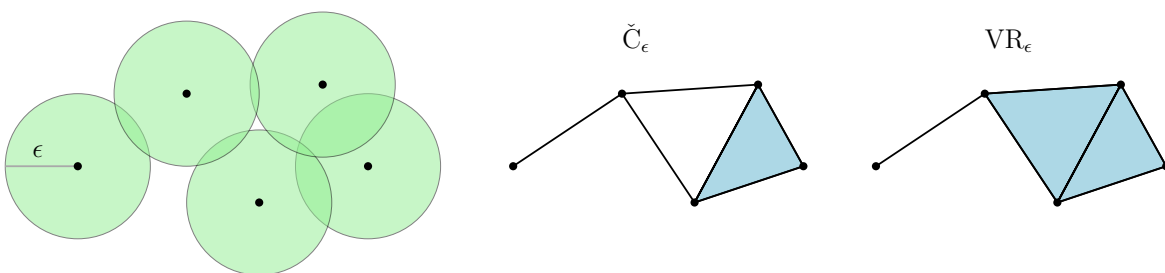


FIGURE 1.3: From the left: a point cloud with balls of radius ϵ centered at each point, the associated Čech complex \check{C}_ϵ and the associated Vietoris-Rips complex VR_ϵ .

namely the graph with V as set of vertices and an edge (v_i, v_j) if and only if $\|v_j - v_i\| < 2\epsilon$. This represents a remarkable advantage in implementations, as one does not need to save the list of all simplices: to check if a collection of points $\{v_i\} \subseteq V$ is a simplex of VR_ϵ it is sufficient to compute the pairwise distances between its points. Furthermore, it can be proven that the Vietoris-Rips complex is a good approximation of the Čech complex, as we have [EH10, Ch. III.2]

$$\check{C}_\epsilon \subseteq \text{VR}_\epsilon \subseteq \check{C}_{\sqrt{2}\epsilon},$$

for any fixed radius $\epsilon > 0$.

1.3 Simplicial sets

Simplicial sets, introduced in [EZ50] by Eilenberg and Zilber, have become a central concept in modern simplicial homotopy theory [GJ99]. Being more general objects than simplicial complexes, they are suitable for theory-oriented computations in algebraic topology. In this section we introduce them following [May67].

Definition 1.24. A *simplicial set* K is a collection of sets K_n , for each integer $n \geq 0$, together with maps $\partial_i : K_n \rightarrow K_{n-1}$ (called *face maps*) and $\eta_i : K_n \rightarrow K_{n+1}$ (called *degeneracy maps*), defined for each $0 \leq i \leq n$, which satisfy the *simplicial identities*:

$$\begin{aligned} \partial_i \partial_j &= \partial_{j-1} \partial_i & \text{if } i < j \\ \eta_i \eta_j &= \eta_{j+1} \eta_i & \text{if } i \leq j \\ \partial_i \eta_j &= \begin{cases} \eta_{j-1} \partial_i & \text{if } i < j \\ \text{id} & \text{if } i = j, j+1 \\ \eta_j \partial_{i-1} & \text{if } i > j+1 \end{cases} \end{aligned}$$

The elements of K_n are referred to as the n -*simplices* of K .

Given two simplicial sets K and L , a *simplicial map* $f : K \rightarrow L$ is a collection of maps $f_n : K_n \rightarrow L_n$ which commute with the face and degeneracy operators: $f_{n-1} \partial_i = \partial_i f_n$ and $f_{n+1} \eta_i = \eta_i f_n$, for all $0 \leq i \leq n$.

An n -simplex x of K is called *degenerate* if $x = \eta_i y$ for some simplex $y \in K_{n-1}$ and some degeneracy map η_i ; otherwise x is said to be *non-degenerate*. Any simplex can be expressed in a canonical form as follows:

Proposition 1.25. In a simplicial set K , any degenerate n -simplex $x \in K_n$ can be written in a unique way as iterated degeneracy of a non-degenerate simplex $y \in K_r$, $0 \leq r < n$, in the following form:

$$x = \eta_{i_s} \cdots \eta_{i_1} y,$$

where $s = n - r$ and $0 \leq i_1 < \cdots < i_s < n$.

We now introduce homology for simplicial sets and show that it can be defined disregarding degenerate simplices. Starting from a simplicial set K , a chain complex $C_*(K) = (C_n(K), d_n)_{n \in \mathbb{N}}$, called the *chain complex associated with K* , can be produced as follows:

- the group $C_n(K)$ of n -chains is the free \mathbb{Z} -module generated by K_n . An n -chain $c \in C_n(K)$ is therefore a finite linear combination $c = \sum \lambda_i x_i$, with $\lambda_i \in \mathbb{Z}$ and $x_i \in K_n$;
- the differential $d_n : C_n(K) \rightarrow C_{n-1}(K)$ is defined on n -simplices $x \in K_n$ as the alternating sum of faces, $d_n(x) := \sum_{i=0}^n (-1)^i \partial_i(x)$, and it is extended by linearity to all elements $c \in C_n(K)$.

We call n -homology group of K the n -homology group of the chain complex $C_n(K)$,

$$H_n(K) := H_n(C_*(K)).$$

Of course this construction can be adapted with coefficients in a ring R rather than in \mathbb{Z} , in which case we denote the homology modules by $H_n(K; R)$. In what follows for simplicity we consider integer homology.

A chain subcomplex of $C_*(K)$ can be defined by considering only the degenerate simplices of K . Denote $D_n(K)$ the set of degenerate elements of K_n , and let $C_n^D(K)$ be the free \mathbb{Z} -module generated by $D_n(K)$, for each n . The simplicial identities of Definition 1.24 (more precisely, the third equation) guarantee that the differential of a degenerate simplex is a combination of degenerate simplices, that is $d_n(C_n^D(K)) \subseteq C_{n-1}^D(K)$, and hence, as claimed, $C_*^D(K)$ is a chain subcomplex of $C_*(K)$. The quotient

$$C_*^N(K) := C_*(K)/C_*^D(K)$$

is called the *normalized chain complex associated with K* and, as shown in the following result, it has the same homology as the chain complex $C_*(K)$.

Theorem 1.26 (Normalization Theorem, [Mac63]). For each simplicial set K the canonical projection $C_*(K) \rightarrow C_*^N(K) := C_*(K)/C_*^D(K)$ is a chain equivalence. As a consequence, it induces isomorphisms in homology:

$$H_n(K) = H_n(C_*(K)) \cong H_n(C_*^N(K)), \quad \text{for each } n.$$

In order to show how general and versatile are simplicial sets, we briefly illustrate how the homology of simplicial complexes and topological spaces can be defined as the homology of particular simplicial sets.

Example 1.27 (Simplicial complexes). Let K be a simplicial complex and let V be its vertex set, as in Definition 1.20. A simplicial set \tilde{K} can be associated to K in a canonical way. Define the n -simplices of \tilde{K} as the sequences $(v_{j_0}, \dots, v_{j_n})$ of elements of V (possibly with repeated elements) such that the set $\{v_{j_0}, \dots, v_{j_n}\}$ (where repeated elements are removed) is an m -simplex of K , for some $m \leq n$. The face and degeneracy maps of \tilde{K} are defined by

$$\begin{aligned} \partial_i(v_{j_0}, \dots, v_{j_n}) &= (v_{j_0}, \dots, v_{j_{i-1}}, v_{j_{i+1}}, \dots, v_{j_n}), \\ \eta_i(v_{j_0}, \dots, v_{j_n}) &= (v_{j_0}, \dots, v_{j_i}, v_{j_i}, v_{j_{i+1}}, \dots, v_{j_n}). \end{aligned}$$

Suppose that V is totally ordered and restrict \tilde{K} to the set of sequences $(v_{j_0}, \dots, v_{j_n})$ such that $v_{j_0} \leq v_{j_1} \leq \dots \leq v_{j_n}$. Then, for each n -simplex of the simplicial complex K , there is exactly one non-degenerate n -simplex of the simplicial set \tilde{K} . This correspondence yields an isomorphism between the chain complexes $C_*(K)$ and $C_*^N(\tilde{K})$. In particular, we can conclude that $H_n(K) \cong H_n(\tilde{K})$ for each n , as a consequence of Theorem 1.26.

Example 1.28 (Simplicial homology). If X is a topological space, the family of its singular simplices carries the structure of a simplicial set. Recall that the standard geometric n -simplex is the subset of \mathbb{R}^{n+1}

$$\Delta^n := \{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid \sum_i t_i = 1 \text{ and } t_i \geq 0 \text{ for all } i\}$$

and that the set of singular n -simplices of X is defined as

$$S_n(X) := \{f : \Delta^n \rightarrow X \mid f \text{ is continuous}\}.$$

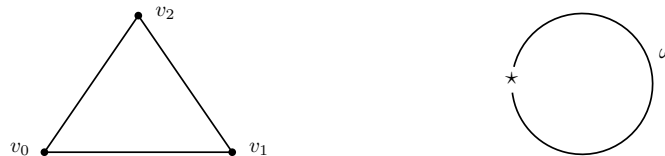
Then, $S(X)$ becomes a simplicial set with face and degeneracy maps $\partial_i : S_n(X) \rightarrow S_{n-1}(X)$ and $\eta_i : S_n(X) \rightarrow S_{n+1}(X)$ defined by

$$\begin{aligned} (\partial_i f)(t_0, \dots, t_{n-1}) &= f(t_0, \dots, t_{i-1}, 0, t_i, \dots, t_{n-1}), \\ (\eta_i f)(t_0, \dots, t_{n+1}) &= f(t_0, \dots, t_{i-1}, t_i + t_{i+1}, t_{i+2}, \dots, t_{n+1}). \end{aligned}$$

The simplicial homology of X is thus, by definition, the homology of the simplicial set $S(X)$.

Simplicial sets can be geometrically realized in a similar way as simplicial complexes (see for example [May67, Ch. III]).

It can be useful to see a more concrete example to clarify the difference between simplicial complexes and simplicial sets. Consider the circle S^1 . The simplest description of S^1 as a simplicial complex is $\{\{v_0\}, \{v_1\}, \{v_2\}, \{v_0, v_1\}, \{v_0, v_2\}, \{v_1, v_2\}\}$, which consists of three 0-simplices (vertices) and three 1-simplices (edges), and can be pictured as the frontier of a triangle. On the other hand,



for the simplest description of S^1 as a simplicial set one only needs two non-degenerate simplices: a 0-simplex (the *basepoint* \star) and a 1-simplex ω , which is sent to the basepoint by the face maps: $\partial_0(\omega) = \star = \partial_1(\omega)$. Clearly, the complete description of S^1 as a simplicial set K has to take account also of degenerate simplices, and is therefore of the form

$$K_0 = \{\star\}, \quad K_1 = \{\eta_0\star, \omega\}, \quad K_2 = \{\eta_0\eta_0\star, \eta_0\omega, \eta_1\omega\}, \quad \dots$$

There is a more modern and concise definition of simplicial sets, which exploits category theory and leads to immediate generalizations of this concept. Consider the *simplicial category* Δ , whose objects are sequences of integers $[n] := \{0, 1, \dots, n\}$, for $n \geq 0$, and whose morphisms are monotonic maps between these sets, i.e. maps $\phi : [n] \rightarrow [m]$ such that $\phi(i) \leq \phi(j)$ if $i < j$. There are two special classes of maps in Δ ,

$$\begin{aligned} \delta_i : [n] &\rightarrow [n+1] & 0 \leq i \leq n \\ \sigma_j : [n+1] &\rightarrow [n] & 0 \leq j \leq n+1, \end{aligned}$$

defined by

$$\delta_i(k) = \begin{cases} k & \text{if } k < i \\ k + 1 & \text{if } k \geq i \end{cases} \quad \sigma_j(k) = \begin{cases} k & \text{if } k \leq j \\ k - 1 & \text{if } k > j. \end{cases}$$

It is not difficult to see that every morphism $\phi : [n] \rightarrow [m]$ in Δ can be decomposed in a unique way as $\phi = \delta_{i_1} \cdots \delta_{i_r} \sigma_{j_1} \cdots \sigma_{j_s}$, with $i_1 \leq \cdots \leq i_r$ and $j_1 \leq \cdots \leq j_s$, and $m = n - s + r$. This implies that the class of morphisms in Δ is generated by the maps $\{\delta_i\}$ and $\{\sigma_j\}$; as it can be verified explicitly, these generators are subject to a set of relations which are “dual” to the *simplicial identities* of Definition 1.24.

A *simplicial object* in a category \mathcal{C} is defined as a contravariant functor $X : \Delta \rightarrow \mathcal{C}$, and a *simplicial morphism* is a natural transformation between such functors. In particular, when \mathcal{C} is the category of sets, we have a simplicial set, and Definition 1.24 is nothing but the explicit list of its “properties”. Similarly, we call for example a *simplicial (abelian) group* a simplicial object in the category of (abelian) groups.

1.4 Spectral sequences

In this section we introduce spectral sequences, one of the key notions of the present work. There are a number of references which treat the subject, among which we cite [McC01] as one of the most complete. For this treatment we follow mainly [Wei94, Ch. 5], [Mac63, Ch. 11] and [Rot09, Ch. 10].

We begin with some fundamental definitions and properties, presenting then in details the case of spectral sequences associated with filtrations of chain complexes. All the modules of this section are over a fixed ring R .

1.4.1 Definitions and properties

Spectral sequences are usually considered to be an arduous subject, and the cause of this reputation partly resides in the number of indices involved in their definition. For a major clarity, we first define “ungraded” spectral sequences, introducing then the usual *bigrading* characterizing the “standard” notion of spectral sequence, as presented for example in [Wei94] and [Mac63].

Definition 1.29. A *spectral sequence* is a sequence $(E^r, d^r)_{r \geq r_0}$ where the E^r are modules, $d^r : E^r \rightarrow E^r$ are homomorphisms such that $d^r d^r = 0$ and there are specified isomorphisms $E^{r+1} \cong \ker d^r / \text{Im } d^r$.

Let us explain better some points of Definition 1.29 and introduce some terminology. For each integer $r \geq r_0$, we call the module E^r the *r-page* of the spectral sequence $(E^r, d^r)_{r \geq r_0}$. The non-negative integer r_0 indicates that the spectral sequence *starts with the r_0 -page* (or *with E^{r_0}*), and its choice is part of the definition of a spectral sequence. In practice, the most common choices are $r_0 = 0, 1, 2$, depending on the context in which a spectral sequence arises and on a matter of grading which we will clarify in what follows; in order to simplify the notations, we will use $r_0 = 1$ as a default choice, pointing out when a different value is employed. The maps d^r are called *differentials*. The isomorphisms between E^{r+1} and $\ker d^r / \text{Im } d^r$ represent the fact that the

$(r+1)$ -page can be obtained as the “homology” of the r -page (in a sense we will make precise). They are often written as equalities, even though they are part of the definition of a spectral sequence; we will follow this convention, which will not lead to any ambiguity as in all the cases considered in the present work the isomorphisms are canonical and clearly understandable from the context.

Let $(E^r, d^r)_{r \geq 1}$ and $(E'^r, d'^r)_{r \geq 1}$ be two spectral sequences. A *morphism of spectral sequences* $f : E \rightarrow E'$ is a collection of morphisms $(f^r : E^r \rightarrow E'^r)_{r \geq 1}$ which are compatible with the differentials (that is $d^r f^r = f^r d'^r$) and such that each f^{r+1} is the map induced by f^r between $E^{r+1} \cong \ker d^r / \text{Im } d^r$ and $E'^{r+1} \cong \ker d'^r / \text{Im } d'^r$.

As we said before, Definition 1.29 is usually applied in the case where the modules E^r are *bigraded*.

Definition 1.30. A *bigraded module* is a collection of modules indexed by pairs of integers:

$$M = (M_{p,q})_{p,q \in \mathbb{Z}}.$$

Let M and N be bigraded modules. A *bigraded map* (or *morphism of bigraded modules*) of *bidegree* $(a, b) \in \mathbb{Z}^2$ is a collection of homomorphisms $f = (f_{p,q} : M_{p,q} \rightarrow N_{p+a, q+b})_{p,q \in \mathbb{Z}}$. We denote it shortly as $f : M \rightarrow N$ and write $\text{deg}(f) = (a, b)$ to indicate that the bidegree of f is (a, b) .

In order to introduce the standard definition of a spectral sequence, we have to require that each page E^r is a bigraded module $E^r = (E_{p,q}^r)_{p,q \in \mathbb{Z}}$. Moreover, we have to consider a particular class of bigraded maps: differentials $d^r : E^r \rightarrow E^r$ of bidegree $(-r, r-1)$, which are collections of morphisms $d^r = (d_{p,q}^r : E_{p,q}^r \rightarrow E_{p-r, q+r-1}^r)_{p,q \in \mathbb{Z}}$ such that $d_{p-r, q+r-1}^r d_{p,q}^r = 0$, for all $p, q \in \mathbb{Z}$. The relations $d_{p-r, q+r-1}^r d_{p,q}^r = 0$ suggest to define the *homology of E^r* as the bigraded module $H(E^r, d^r) := (H(E^r, d^r)_{p,q})_{p,q \in \mathbb{Z}}$, where

$$H(E^r, d^r)_{p,q} := \frac{\ker d_{p,q}^r}{\text{Im } d_{p+r, q-r+1}^r}.$$

We can now state the standard definition of a spectral sequence as a particular case of Definition 1.29. As said before, for simplicity we assume $r_0 = 1$.

Definition 1.31. A *spectral sequence* is a sequence $(E^r, d^r)_{r \geq 1}$ of bigraded modules $E^r = (E_{p,q}^r)_{p,q \in \mathbb{Z}}$, each with a bigraded differential

$$d^r = (d_{p,q}^r : E_{p,q}^r \rightarrow E_{p-r, q+r-1}^r)_{p,q \in \mathbb{Z}}$$

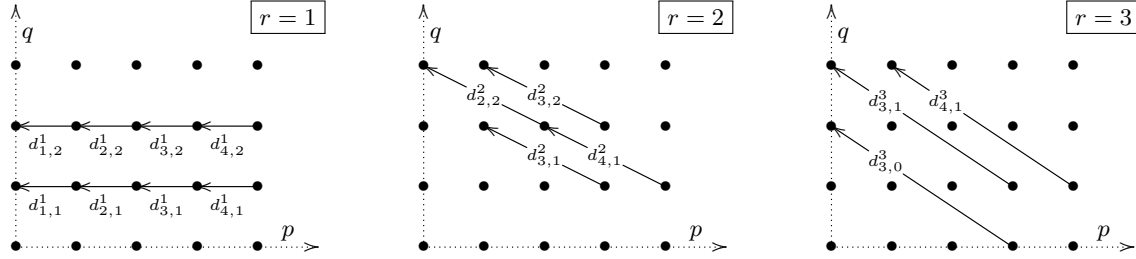
of bidegree $(-r, r-1)$, and such that there are isomorphisms $E^{r+1} \cong H(E^r, d^r)$ of bigraded modules.

Notice that an isomorphism $E^{r+1} \cong H(E^r, d^r)$ of bigraded modules consists of a collection of isomorphisms

$$E_{p,q}^{r+1} \cong \frac{\ker d_{p,q}^r}{\text{Im } d_{p+r, q-r+1}^r}.$$

There is a convenient way to represent graphically a spectral sequence, making clear the role of the bigraded differentials. Each page $E^r = (E_{p,q}^r)_{p,q \in \mathbb{Z}}$ is seen as a lattice $\mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z}$ with each

point (p, q) representing the corresponding *term* $E_{p,q}^r$. Each point (p, q) is the origin of an arrow pointing to $(p - r, q + r - 1)$, representing the differential $d_{p,q}^r$, and the target of an arrow coming from $(p + r, q - r + 1)$, representing the differential $d_{p+r,q-r+1}^r$. In the following scheme there are some terms (in the first quadrant $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$) of the first three pages of a spectral sequence, with some differentials explicitly denoted.



From this point on, when we treat spectral sequences we implicitly refer to Definition 1.31, assuming thus the presence of a bigraded structure. The merit of Definition 1.29 is to show how the bigraded structure given by the indices (p, q) is unnecessary to describe the essence of a spectral sequence, and it will be helpful in the following to understand how the notion of spectral sequence can be generalized; however, we will see in Section 1.4.2 that the bigrading arises in a very natural way in the most significant examples of spectral sequences.

Let us make explicit the notion of morphism of spectral sequences accounting for the bigraded structure of Definition 1.31. Let $(E^r, d^r)_{r \geq 1}$ and $(E'^r, d'^r)_{r \geq 1}$ be two spectral sequences. A *morphism of spectral sequences* $f : E \rightarrow E'$ is a collection of bigraded maps $(f^r : E^r \rightarrow E'^r)_{r \geq 1}$ of bidegree $(0, 0)$ which are compatible with the differentials (that is $d^r f^r = f^r d'^r$) and such that each f^{r+1} is the map induced by f^r between $E^{r+1} \cong H(E^r, d^r)$ and $E'^{r+1} \cong H(E'^r, d'^r)$.

A spectral sequence $(E^r, d^r)_{r \geq 1}$ determines a chain of submodules

$$0 = B^1 \subseteq B^2 \subseteq \dots \subseteq B^r \subseteq B^{r+1} \subseteq \dots \subseteq Z^{r+1} \subseteq Z^r \subseteq \dots \subseteq Z^2 \subseteq Z^1 = E^1$$

such that $E^r = Z^r/B^r$, for each $r \geq 1$, allowing to see each E^r as a *subquotient* (that is, a quotient of a submodule) of E^1 . In order to understand how the submodules B^r and Z^r are defined, one might consider the following property: let $S \subseteq T$ be submodules of a module M , and let $S' \subseteq T'$ be submodules of T/S ; then T'/S' is canonically isomorphic to T''/S'' , where $S'' \subseteq T''$ are the unique submodules of M containing S such that $S''/S = S'$ and $T''/S = T'$, as explained in Proposition 1.4. (Note that, here and in what follows, we often denote canonical isomorphisms as equalities). Thus, $B^2 \subseteq Z^2$ are the unique submodules of E^1 such that $Z^2/B^2 = E^2$; then, $B^3 \subseteq Z^3$ are the unique submodules of E^1 containing B^2 such that $Z^3/B^3 = E^3$, and so on.

Now, introducing the submodules

$$B^\infty := \bigcup_{r=1}^{+\infty} B^r \quad \text{and} \quad Z^\infty := \bigcap_{r=1}^{+\infty} Z^r$$

of E^1 , we can define the ∞ -page or *limit page* of the spectral sequence $(E^r, d^r)_{r \geq 1}$ as the bigraded module $E^\infty = Z^\infty/B^\infty$. Making the bigrading explicit, the ∞ -page is the bigraded module $E^\infty = (E_{p,q}^\infty)_{p,q \in \mathbb{Z}}$ where $E_{p,q}^\infty = Z_{p,q}^\infty/B_{p,q}^\infty$.

Theorem 1.32 ([Wei94]). Let $(E^r, d^r)_{r \geq 1}$ and $(E'^r, d'^r)_{r \geq 1}$ be two spectral sequences, and let $f : E \rightarrow E'$ be a morphism of spectral sequences. If $f^s : E^s \rightarrow E'^s$ is an isomorphism (of bigraded modules) then for all $r \geq s$ also $f^r : E^r \rightarrow E'^r$ is an isomorphism. Furthermore, $f^\infty : E^\infty \rightarrow E'^\infty$ is an isomorphism.

1.4.2 The spectral sequence of a filtered chain complex

Definition 1.33. A *filtration* $F = (F_p)_{p \in \mathbb{Z}}$ of a chain complex $C_* = (C_n, d_n)_{n \in \mathbb{Z}}$ is a collection of chain subcomplexes $F_p C_*$ of C_* indexed over the integers and such that

$$\cdots \subseteq F_{p-1} C_* \subseteq F_p C_* \subseteq F_{p+1} C_* \subseteq \cdots \subseteq C_*. \quad (1.4)$$

We call *filtered chain complex* a chain complex C_* endowed with a filtration F , and we denote it (C_*, F) . Observe that (1.4) is a compact notation for the following commutative diagram, where the columns are chain complexes and the horizontal arrows represent inclusions of modules:

$$\begin{array}{ccccccc} & \cdots & & \cdots & & \cdots & & \cdots \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \hookrightarrow & F_{p-1} C_{n+1} & \hookrightarrow & F_p C_{n+1} & \hookrightarrow & F_{p+1} C_{n+1} & \hookrightarrow \cdots & \hookrightarrow & C_{n+1} \\ & & \downarrow d_{n+1} & & \downarrow d_{n+1} & & \downarrow d_{n+1} & & & \downarrow d_{n+1} \\ \cdots & \hookrightarrow & F_{p-1} C_n & \hookrightarrow & F_p C_n & \hookrightarrow & F_{p+1} C_n & \hookrightarrow \cdots & \hookrightarrow & C_n \\ & & \downarrow d_n & & \downarrow d_n & & \downarrow d_n & & & \downarrow d_n \\ \cdots & \hookrightarrow & F_{p-1} C_{n-1} & \hookrightarrow & F_p C_{n-1} & \hookrightarrow & F_{p+1} C_{n-1} & \hookrightarrow \cdots & \hookrightarrow & C_{n-1} \\ & & \downarrow & & \downarrow & & \downarrow & & & \downarrow \\ & \cdots & & \cdots & & \cdots & & \cdots & & \cdots \end{array}$$

Given a filtered chain complex (C_*, F) , for each $p \in \mathbb{Z}$ we can consider the quotient chain complex $F_p C_*/F_{p-1} C_*$ of two subsequent subcomplexes $F_{p-1} C_* \subseteq F_p C_*$ of the filtration. In this section we show that a filtered chain complex (C_*, F) determines in a natural way a spectral sequence with

$$E_{p,q}^0 = \frac{F_p C_{p+q}}{F_{p-1} C_{p+q}} \quad \text{and} \quad E_{p,q}^1 = H_{p+q} \left(\frac{F_p C_*}{F_{p-1} C_*} \right)$$

which, under suitable assumptions, *converges* to the homology $H_*(C_*)$, in a sense that we will make precise.

As we said, our standard assumption is that spectral sequences start with the 1-page E^1 , a choice which will find its motivation in Chapter 2. Nevertheless, the construction of the spectral sequence associated with a filtered chain complex results easier starting with the 0-page which, as we stated, is defined as $E_{p,q}^0 = F_p C_{p+q}/F_{p-1} C_{p+q}$. We illustrate here the construction starting with

E^0 , advising the reader that in the next chapters we usually “forget” about the 0-page and consider instead E^1 as the starting page.

Consider the following chain of submodules of $E_{p,q}^0 = F_p C_{p+q} / F_{p-1} C_{p+q}$,

$$0 = B_{p,q}^0 \subseteq B_{p,q}^1 \subseteq \dots \subseteq B_{p,q}^r \subseteq \dots \subseteq B_{p,q}^\infty \subseteq Z_{p,q}^\infty \subseteq \dots \subseteq Z_{p,q}^r \subseteq \dots \subseteq Z_{p,q}^1 \subseteq Z_{p,q}^0 = E_{p,q}^0,$$

constructed as follows: let

$$A_{p,q}^r := \{c \in F_p C_n \mid d(c) \in F_{p-r} C_{n-1}\}, \quad n := p + q \quad (1.5)$$

and define

$$Z_{p,q}^r := \frac{A_{p,q}^r + F_{p-1} C_n}{F_{p-1} C_n}, \quad B_{p,q}^r := \frac{d(A_{p+r-1,q-r+2}^{r-1}) + F_{p-1} C_n}{F_{p-1} C_n}.$$

The terms of the spectral sequence are then defined by the canonically isomorphic quotients

$$E_{p,q}^r = \frac{Z_{p,q}^r}{B_{p,q}^r} = \frac{A_{p,q}^r + F_{p-1} C_n}{d(A_{p+r-1,q-r+2}^{r-1}) + F_{p-1} C_n} = \frac{A_{p,q}^r}{d(A_{p+r-1,q-r+2}^{r-1}) + A_{p-1,q+1}^{r-1}}, \quad (1.6)$$

where the last isomorphism can be seen as an application of Proposition 1.5, with the observation that $A_{p,q}^r \cap F_{p-1} C_n = A_{p-1,q+1}^{r-1}$. The differentials $d_{p,q}^r : E_{p,q}^r \rightarrow E_{p-r,q+r-1}^r$ of the spectral sequence are the morphisms induced by the differential d of C_* on the subquotients. There is a standard terminology for the indices p, q, n appearing in the previous definitions: p is the *filtration degree* (or *filtration index*), q is called *complementary degree* and n is called *total degree*. The motivation of this terms is evident for example from (1.5). Let us stress the fact that the indices p, q, n represent a standard notation, and when they appear together in the context of spectral sequences they are always related by $n = p + q$.

It can be shown (see [Wei94, § 5.4]) that there is a canonical isomorphism

$$E_{p,q}^{r+1} = \frac{\ker(d_{p,q}^r : E_{p,q}^r \rightarrow E_{p-r,q+r-1}^r)}{\operatorname{Im}(d_{p+r,q-r+1}^r : E_{p+r,q-r+1}^r \rightarrow E_{p,q}^r)} \quad (1.7)$$

for all $p, q \in \mathbb{Z}$ and for all $r \geq 0$, thus proving that (1.6) actually defines a spectral sequence. Notice that, since the 0-page consists of the terms $E_{p,q}^0 = F_p C_n / F_{p-1} C_n$, the isomorphisms (1.7) with $r = 0$ and for each $p, q \in \mathbb{Z}$ indicate that the 1-page consists of the homology groups of the chain complexes $F_p C_* / F_{p-1} C_*$, for each $p \in \mathbb{Z}$; more precisely, as we mentioned before, $E_{p,q}^1 = H_n(F_p C_* / F_{p-1} C_*)$.

Now that we have presented the construction of the spectral sequence associated with a filtered chain complex (C_*, F) , we can move on to study its *convergence*, the property which motivates the introduction of such a sophisticated machinery. We saw that the pages E^0 and E^1 have “meaningful” definitions, and said that, knowing the differential d^r , one can obtain E^{r+1} from E^r by computing homology. Under suitable assumptions, this procedure eventually produces the ∞ -page E^∞ , which allows to obtain the homology groups $H_n(C_*)$ “in pieces”. Hence, the spectral sequence of a filtered chain complex (C_*, F) can be thought as a recipe to obtain the homology groups $H_n(C_*)$ from the homology groups $H_n(F_p C_* / F_{p-1} C_*)$. In the typical scenario, the homology of C_* is the goal of investigation and it is difficult to determine directly; the spectral sequence provides a tool to study

it starting from the groups $H_n(F_p C_* / F_{p-1} C_*)$, which may be easier to determine if the filtration is defined properly. We will explain in what sense we obtain the homology of C_* in pieces, introducing a filtration of $H_*(C_*)$ naturally induced by F , pointing out how some extension problems have to be solved in order to actually reach a full knowledge of the groups $H_n(C_*)$.

A filtration $F = (F_p)_{p \in \mathbb{Z}}$ of a chain complex C_* induces a filtration $\mathcal{F} = (\mathcal{F}_p)_{p \in \mathbb{Z}}$ of the graded module $H_*(C_*) = (H_n(C_*))_{n \in \mathbb{Z}}$ defined, for all n , as

$$\mathcal{F}_p H_n(C_*) := \text{Im}(j_p : H_n(F_p C_*) \rightarrow H_n(C_*)),$$

where j_p is the map induced in homology by the inclusion $F_p C_* \hookrightarrow C_*$. An equivalent expression for the modules $\mathcal{F}_p H_n(C_*)$ is clearly

$$\mathcal{F}_p H_n(C_*) = \frac{\ker d_n \cap F_p C_n + \text{Im } d_{n+1}}{\text{Im } d_{n+1}}.$$

We treat convergence under suitable hypotheses for our purposes, following [Rot09]. The theme of convergence of spectral sequences is treated in a more general way in [Boa99] and [McC01] under very technical assumptions.

Definition 1.34. A filtration $F = (F_p)_{p \in \mathbb{Z}}$ of a graded module $M = (M_n)_{n \in \mathbb{Z}}$ is called *bounded* if, for each n , there exist two integers $s = s(n)$ and $t = t(n)$ such that $F_s M_n = 0$ and $F_t M_n = M_n$.

A first advantage in considering a filtered chain complex (C_*, F) where the filtration F is bounded is that, as it can be easily shown, for each p and q we have $E_{p,q}^\infty = E_{p,q}^r$, for r large enough. As we have seen, $E_{p,q}^r$ is a subquotient of $F_p C_n / F_{p-1} C_n$, for each r and for each p, q . But $F_p C_n / F_{p-1} C_n = 0$ if $p \leq s(n)$ or $p > t(n)$, so in these cases also its subquotients are zero. Now consider the differentials of the r -page

$$\cdots \leftarrow E_{p-r, q+r-1}^r \xleftarrow{d_{p,q}^r} E_{p,q}^r \xleftarrow{d_{p+r, q-r+1}^r} E_{p+r, q-r+1}^r \leftarrow \cdots$$

For r large enough the terms $E_{p-r, q+r-1}^r$ and $E_{p+r, q-r+1}^r$ are zero for what we said, then both $d_{p,q}^r$ and $d_{p+r, q-r+1}^r$ are the zero map, therefore $E_{p,q}^r = E_{p,q}^{r+1}$. Iterating, we get $E_{p,q}^r = E_{p,q}^{r+1} = E_{p,q}^{r+2} = \dots$ which clearly implies $E_{p,q}^r = E_{p,q}^\infty$.

Definition 1.35. A spectral sequence (E^r, d^r) is said to *converge* to a graded module $H = (H_n)_{n \in \mathbb{Z}}$ if there exists a bounded filtration $\mathcal{F} = (\mathcal{F}_p)_{p \in \mathbb{Z}}$ of H such that

$$E_{p,q}^\infty \cong \frac{\mathcal{F}_p H_n}{\mathcal{F}_{p-1} H_n}$$

for all $p, q \in \mathbb{Z}$ (with $n := p + q$).

Theorem 1.36 (Convergence). If the filtration $F = (F_p)_{p \in \mathbb{Z}}$ of a chain complex C_* is bounded, then also the induced filtration $\mathcal{F} = (\mathcal{F}_p)_{p \in \mathbb{Z}}$ of the graded module $H_*(C_*)$ is bounded, and the

spectral sequence (E^r, d^r) associated with (C_*, F) converges to $H_*(C_*)$:

$$E_{p,q}^\infty \cong \frac{\mathcal{F}_p H_n(C_*)}{\mathcal{F}_{p-1} H_n(C_*)}, \quad (1.8)$$

for all $p, q \in \mathbb{Z}$.

We have now introduced all the necessary results to understand the purpose for which spectral sequences were originally conceived: determining the homology of a chain complex C_* using the information provided by the filtration $F = (F_p)_{p \in \mathbb{Z}}$. In particular, for a bounded filtration F we have seen that it is possible to determine, starting from the 1-page, the terms of the ∞ -page in a finite number of steps (provided that one knows also the differentials d^r). Knowing all the terms $E_{p,q}^\infty$ provides a large amount of information about $H_*(C_*)$, but not its complete knowledge, because of the so-called *extension problems*. We explain extension problems in an intuitive way here, starting with the case where there is actually *no* problem. Suppose that the ring R of coefficients is a field \mathbb{F} , so that all the modules are \mathbb{F} -vector spaces. For vector spaces we know that $V/U = W$ is equivalent to $V \cong W \oplus U$. In other words, for vector spaces the extension problem is trivial: every time we have a short exact sequence

$$0 \rightarrow U \rightarrow V \rightarrow W \rightarrow 0$$

we know that there is an isomorphism of vector spaces $V \cong W \oplus U$. For vector spaces, it *is* indeed possible to determine $H_*(C_*)$ (up to isomorphism) from the terms $E_{p,q}^\infty$. More precisely, $H_n(C_*)$ can be determined from the terms $E_{p,q}^\infty$ such that $p + q = n$ in the simple way

$$H_n(C_*) \cong \bigoplus_{p+q=n} E_{p,q}^\infty, \quad (1.9)$$

using (1.8) and applying iteratively $V/U = W \iff V \cong W \oplus U$. However, in general (for R -modules) the central term of a short exact sequence cannot always be determined knowing the other two terms. An easy example for abelian group is that, having a short exact sequence

$$0 \rightarrow \mathbb{Z}_2 \rightarrow M \rightarrow \mathbb{Z}_2 \rightarrow 0$$

we have two non-isomorphic choices for M , namely $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ or \mathbb{Z}_4 . This in brief is the extension problem related to spectral sequences: in general (1.8) does not imply (1.9). Sometimes some additional information on the involved modules may help, but in other cases one might not be able to decide.

We conclude the section presenting a classical result which represents a version of Theorem 1.32 accounting for the convergence properties of spectral sequences associated with filtrations of chain complexes. This gives us the occasion to point out the functoriality of the construction of spectral sequences from filtered chain complexes. We refer to [Mac63, § 11.3].

Definition 1.37. Let (C_*, F) and (D_*, F') be two filtered chain complexes. A *morphism of filtered chain complexes* $f : (C_*, F) \rightarrow (D_*, F')$ is a chain map $f : C_* \rightarrow D_*$ compatible with the filtrations,

that is satisfying

$$f(F_p C_*) \subseteq F'_p D_*,$$

for all $p \in \mathbb{Z}$. We use the expression *chain map compatible with the filtrations* as a synonym of morphism of filtered chain complexes.

Let us denote $E(C_*, F)$ and $E(D_*, F')$ the spectral sequences associated respectively with the filtered chain complexes (C_*, F) and (D_*, F') . A morphism of filtered chain complexes $f : (C_*, F) \rightarrow (D_*, F')$ induces a morphism of spectral sequences $f : E(C_*, F) \rightarrow E(D_*, F')$. In other words, it induces for each r a morphism of bigraded modules

$$f^r : E(C_*, F)^r = (E(C_*, F)_{p,q}^r) \longrightarrow E(D_*, F')^r = (E(D_*, F')_{p,q}^r)$$

between the r -pages of the two spectral sequences. The construction of the spectral sequence associated with a filtered chain complex is *functorial*, in the following sense: the identity $\text{id}_{C_*} : C_* \rightarrow C_*$, which is clearly compatible with every filtration F of C_* , induces the identity for each page of the spectral sequence $E(C_*, F)$, that is $(\text{id}_{C_*})^r = \text{id}_{E(C_*, F)^r}$, for all r ; for each pair of morphisms f, g of filtered chain complexes such that the composition gf is defined, we have $(gf)^r = g^r f^r$, for each r .

The following theorem describes what can be concluded if a morphism of filtered chain complexes $f : (C_*, F) \rightarrow (D_*, F')$ induces an isomorphism on some page of the associated spectral sequences.

Theorem 1.38. Let $f : (C_*, F) \rightarrow (D_*, F')$ be a morphism of filtered chain complexes, and suppose that the filtrations F and F' are bounded. If, for some s , the induced morphism

$$f^s : E(C_*, F)^s \rightarrow E(D_*, F')^s$$

is an isomorphism (of bigraded modules), then also $f^r : E(C_*, F)^r \rightarrow E(D_*, F')^r$ for all $r \geq s$ and $f^\infty : E(C_*, F)^\infty \rightarrow E(D_*, F')^\infty$ are isomorphisms. In addition, $H_*(f) : H_*(C_*) \rightarrow H_*(D_*)$ is an isomorphism (of graded modules).

Chapter 2

Generalized spectral sequences

As we have seen, the construction of a spectral sequence arises in a quite natural way from a filtration $F = (F_p)_{p \in \mathbb{Z}}$ of a chain complex. Benjamin Matschke, in his impressive work [Mat13], proposed a generalization of the notion of spectral sequence, formulating a theory which allows to construct a generalized spectral sequence, or *spectral system*, from filtrations indexed over any partially ordered set (poset). One of the main motivations of Matschke was to describe a mathematical object general enough to unify several spectral sequences which usually one would apply in succession; as an instance, we will see in Chapter 4 the successive Serre spectral sequences associated with a tower of fibrations. The original motivation causes some posets to assume a particular relevance as sets of indices of a filtration, like \mathbb{Z}^m or, to achieve more generality, the poset of downsets of \mathbb{Z}^m , which we will introduce. Nonetheless, the theory of spectral systems is undoubtedly of interest in its most general form.

In this chapter we introduce the notion of spectral system, illustrating its properties and explaining how it generalizes the notion of spectral sequence. We did an effort to present Matschke's very technical work in a relatively intuitive way, respecting at the same time the original notations. We refer to [Mat13], the lecture notes [Mat14b] and the preprint [Mat14a]. Most of the figures in this chapter are inspired by those in [Mat13] and [Mat14b].

Throughout this chapter we assume \mathbb{Z} as the ring of coefficients.

2.1 Introduction to spectral systems

A partially ordered set or *poset* (I, \leq) is a set I endowed with a partial order \leq .

Definition 2.1. Let (I, \leq) be a poset. An *I-filtration* $F = (F_i)_{i \in I}$ of a chain complex C_* is a collection of chain subcomplexes $F_i C_*$ such that $F_i C_* \subseteq F_j C_*$ whenever $i \leq j$.

We call *I-filtered chain complex* (C_*, F) a chain complex C_* endowed with an *I-filtration* $F = (F_i)_{i \in I}$. In Section 1.4.2 we have seen that a \mathbb{Z} -filtered chain complex C_* gives rise to a spectral sequence converging, under suitable hypotheses, to the homology of C_* . Classical spectral sequences associated with \mathbb{Z} -filtrations are the running example of this chapter, being the natural term of comparison for understanding how the notion of spectral system and its properties represent a generalization of the usual setting. Having acquired familiarity with examples of spectral systems over the poset \mathbb{Z} , we will introduce examples over different posets, like \mathbb{Z}^m or the poset $D(\mathbb{Z}^m)$ of the *downsets* of \mathbb{Z}^m .

Before defining spectral systems, let us state a few conventions of notation that we observe, in order to guarantee legibility despite the large number of indices involved. We usually drop the grading of the chain complex C_* , as well as the gradings it induces. Then, for example, we use the notation $H(C_*)$ instead of $H_n(C_*)$ when n can be deduced from the context, and we simply denote d the differential, without further indices. Also the analogue of the total degree of spectral sequences is usually not denoted, leading for instance to notations like E_p^r instead of $E_{p,q}^r$ (the complementary degree q gives the same information of the total degree n , as $n = p + q$). In addition, we shorten the notation for $F_i C_*$ or $F_i C_n$ by writing simply F_i . For clarity, we will point out to the reader a few of the first uses of the conventions just stated, sure that they will not cause any confusion and will ensure an easier understanding. We state here another useful convention of notation we will observe:

Notational convention 2.2. In the context of spectral systems, if A and B are subgroups of an abelian group we briefly denote A/B the quotient $A/(A \cap B) \cong (A + B)/B$.

Definition 2.3. Let (C_*, F) be an I -filtered chain complex. Given a 4-tuple of indices $z \leq s \leq p \leq b$ in I we define

$$S[z, s, p, b] := \frac{F_p \cap d^{-1}(F_z)}{d(F_b) + F_s}. \quad (2.1)$$

The nomenclature for the indices z, s, p, b is the following: z is the *cycle degree*, s is the *quotient degree*, p is the *filtration degree* and b is the *boundary degree*. We call the collection $(S[z, s, p, b])_{z \leq s \leq p \leq b}$ the *spectral system* associated with the I -filtration $F = (F_i)_{i \in I}$ of C_* , and we call each abelian group $S[z, s, p, b]$ a *term* of the spectral system.

Let us clarify the conventions of notation we have used in (2.1). First, as said before, we have dropped the *total degree*, that is the grading induced by that of C_* . We use the notation $S_n[z, s, p, b]$ if we want to explicitly reintroduce the total degree; in this case equation (2.1) becomes

$$S_n[z, s, p, b] := \frac{F_p C_n \cap d_n^{-1}(F_z C_{n-1})}{d_{n+1}(F_b C_{n+1}) + F_s C_n}. \quad (2.2)$$

The total degree can be easily made explicit considering that the differential d is a graded map of degree -1 , motivating the choice of dropping it in general, except in the cases in which we want to give it relevance.

We want to stress the fact that the Convention 2.2 is employed in (2.1) and (2.2) as, for example, neither $d(F_b)$ nor F_s are necessarily subgroups of $F_p \cap d^{-1}(F_z)$. For the sake of clarity, let us state here the formula (2.1) also in a more explicit (although less immediate) way, without employing the Convention 2.2:

$$S[z, s, p, b] = \frac{F_p \cap d^{-1}(F_z)}{F_p \cap d(F_b) + F_s \cap d^{-1}(F_z)} = \frac{F_p \cap d^{-1}(F_z) + F_s}{F_p \cap d(F_b) + F_s}, \quad (2.3)$$

where the equal signs as usual mean canonical isomorphisms, and in the quotients the denominator is a subgroup of the numerator (recall Proposition 1.5). By comparison, we can notice how (2.1) conveys in a clearer way the role of the indices z, s, p, b .

From (2.1) we see that the inclusions $F_z \subseteq F_p$, $F_s \subseteq F_b$ and $F_s \subseteq F_p$, corresponding to the inequalities $z \leq p$, $s \leq b$ and $s \leq p$ between the indices, are the “reasonable” ones, which render

all the subgroups figuring in (2.1) relevant. The stronger assumption $z \leq s \leq p \leq b$ is motivated by a construction that we will introduce in the next chapter, called exact couple system. The particular cases $z = s \leq p = b$, for all $s \leq p$ in I , give rise to the terms $S[s, s, p, p] = H(F_p/F_s)$, whose collection is called the 1-page of the spectral system.

Example 2.4 (Classical spectral sequences). Consider a \mathbb{Z} -filtration $F = (F_p)_{p \in \mathbb{Z}}$ of a chain complex, and the spectral sequence $(E_{p,q}^r)$ it originates. Referring to the formula (1.6) of Section 1.4.2 we show that $E_{p,q}^r = S_n[p - r, p - 1, p, p + r - 1]$, where $n = p + q$ and, as said before, both n and q are dropped in notation. We observe that the definition of $A_{p,q}^r$ given by (1.5) can be expressed as

$$A_p^r = F_p \cap d^{-1}(F_{p-r});$$

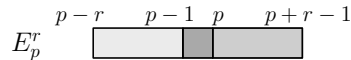
then (1.6) becomes

$$E_p^r = \frac{F_p \cap d^{-1}(F_{p-r}) + F_{p-1}}{F_p \cap d(F_{p+r-1}) + F_{p-1}} = S[p - r, p - 1, p, p + r - 1].$$

The spectral sequence (E_p^r) is then a subset of the collection $(S[z, s, p, b])$ composed of terms with a particular “symmetry” between the indices z, s, p, b . Following [Mat14b] we introduce an intuitive graphic representation for a term $S[z, s, p, b]$ with $z \leq s \leq p \leq b$ in \mathbb{Z}



with three stripes colored in light gray \blacksquare , medium gray \blacksquare and dark gray \blacksquare going respectively from z to s , from p to b and from s to p . The terms E_p^r of the spectral sequence are characterized by a more symmetric shape:



We point out that the 0-page of the spectral sequence is constituted of the terms $E_p^0 = F_p/F_{p-1} = S[p, p - 1, p, p - 1]$, which actually violate our restriction $z \leq s \leq p \leq b$ for the indices. Nonetheless, it is better to keep the assumption on the indices and add “artificially” the 0-page to the spectral system when needed.

2.2 Differentials and extensions

Given two 4-tuples of indices $z_1 \leq s_1 \leq p_1 \leq b_1$ and $z_2 \leq s_2 \leq p_2 \leq b_2$ in I , it is easy to check that d induces a well-defined differential

$$d : S[z_2, s_2, p_2, b_2] \rightarrow S[z_1, s_1, p_1, b_1]$$

whenever the additional inequalities $z_2 \leq p_1$ and $s_2 \leq b_1$ are satisfied. With a small abuse of notation, we denote with d also the induced differentials. In the case of $z_2 = p_1$ and $s_2 = b_1$, a direct computation shows that the kernel and cokernel of $d : S[z_2, s_2, p_2, b_2] \rightarrow S[z_1, s_1, p_1, b_1]$ have

the following nice expressions as terms of the spectral system:

$$\begin{aligned}\ker d &= S[s_1, s_2, p_2, b_2], \\ \operatorname{coker} d &= S[z_1, s_1, p_1, p_2].\end{aligned}$$

The following result describes how the homology of a sequence of such differentials can be expressed.

Proposition 2.5. Consider three 4-tuples of indices in I satisfying the relations

$$\begin{array}{ccccccc} & & & & z_3 & \leq & s_3 & \leq & p_3 & \leq & b_3 \\ & & & & \parallel & & \parallel & & & & \\ & & & z_2 & \leq & s_2 & \leq & p_2 & \leq & b_2 \\ & & & \parallel & & \parallel & & & & & \\ z_1 & \leq & s_1 & \leq & p_1 & \leq & b_1\end{array}$$

and the sequence of differentials between the corresponding terms:

$$S[z_3, s_3, p_3, b_3] \xrightarrow{d'} S[z_2, s_2, p_2, b_2] \xrightarrow{d} S[z_1, s_1, p_1, b_1]. \quad (2.4)$$

Then the homology at the middle term is given by

$$\frac{\ker d}{\operatorname{Im} d'} = S[s_1, s_2, p_2, p_3].$$

Note that, as the differential d of C_* is a graded map of degree -1 , so are the induced differentials between terms of the spectral system (with respect to the total degree). Then, for example, making explicit the total degree in (2.4) one obtains $S_{n+1}[z_3, s_3, p_3, b_3] \xrightarrow{d'} S_n[z_2, s_2, p_2, b_2] \xrightarrow{d} S_{n-1}[z_1, s_1, p_1, b_1]$, whose homology at the middle term is $\ker d / \operatorname{Im} d' = S_n[s_1, s_2, p_2, p_3]$.

Example 2.6 (Classical spectral sequences, continued). For spectral sequences associated with \mathbb{Z} -filtered chain complexes, Proposition 2.5 corresponds to $E_p^{r+1} = \ker d_p^r / \operatorname{Im} d_{p+r}^r$, a property which can be proved directly from the definition of the terms E_p^r (see (1.7), Section 1.4.2). Recall that $d_p^r : E_p^r \rightarrow E_{p-r}^r$ and $d_{p+r}^r : E_{p+r}^r \rightarrow E_p^r$ and notice that the indices of $E_{p-r}^r = S[p-2r, p-r-1, p-r, p-1]$, $E_p^r = S[p-r, p-1, p, p+r-1]$ and $E_{p+r}^r = S[p, p+r-1, p+r, p+2r-1]$ satisfy the hypothesis of Proposition 2.5, which for spectral sequences can be represented graphically as in Figure 2.1. Proposition 2.5 of course is valid for all the terms of the spectral system over \mathbb{Z} ; the general situation is depicted in Figure 2.2.

Consider two terms $S[z_1, s_1, p_1, b_1]$ and $S[z_2, s_2, p_2, b_2]$ of the spectral system associated with an I -filtered chain complex. If $z_1 \leq z_2$, $s_1 \leq s_2$, $p_1 \leq p_2$ and $b_1 \leq b_2$ the inclusion of chain complexes induces a well-defined map

$$\ell : S[z_1, s_1, p_1, b_1] \rightarrow S[z_2, s_2, p_2, b_2].$$

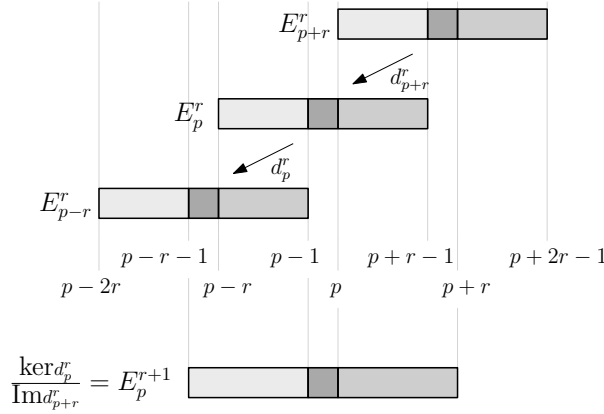


FIGURE 2.1: Proposition 2.5 for standard spectral sequences.

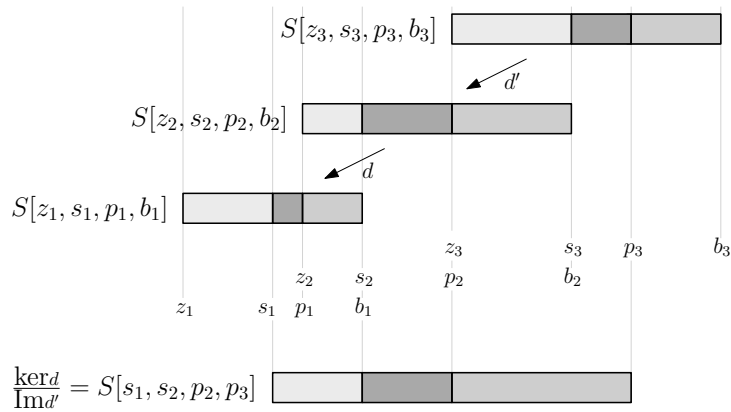


FIGURE 2.2: Proposition 2.5 for a spectral system over \mathbb{Z} .

As it can be easily verified, if $z \leq p \leq p' \leq p'' \leq b$ are indices in I , there exists a short exact sequence

$$0 \rightarrow S[z, p, p', b] \xrightarrow{\ell} S[z, p, p'', b] \xrightarrow{\ell} S[z, p', p'', b] \rightarrow 0. \tag{2.5}$$

If we want to denote explicitly the total degree, note that the maps ℓ are of degree zero with respect to it, being induced by inclusion.

Example 2.7 (Classical spectral sequences, continued). In Section 1.4.2 we introduced the filtration $\mathcal{F} = (\mathcal{F}_p)_{p \in \mathbb{Z}}$ of $H(C_*)$ induced by the filtration $F = (F_p)_{p \in \mathbb{Z}}$ of a chain complex C_* . We saw in Theorem 1.36 that $E_p^\infty = \mathcal{F}_p H(C_*) / \mathcal{F}_{p-1} H(C_*)$ when F is bounded, and said that extensions are what still misses to determine $H(C_*)$ from the ∞ -page (E_p^∞). It is interesting to see how all these abelian groups can be expressed as terms of the spectral system. Consider the poset $\overline{\mathbb{Z}} := \mathbb{Z} \cup \{-\infty, \infty\}$ where we define $F_{-\infty} := 0$ and $F_\infty := C_*$. Clearly we have

$$H(C_*) = S[-\infty, -\infty, \infty, \infty],$$

which is filtered by

$$\begin{aligned}\mathcal{F}_p &:= \text{Im}(H(F_p) \xrightarrow{\text{incl}_*} H(C_*)) \\ &= \text{Im}(S[-\infty, -\infty, p, p] \xrightarrow{\ell} S[-\infty, -\infty, \infty, \infty]) \\ &= S[-\infty, -\infty, p, \infty].\end{aligned}$$

The quotients $\mathcal{F}_p/\mathcal{F}_s$ for $s \leq p$ can be expressed as

$$\frac{\mathcal{F}_p}{\mathcal{F}_s} = \frac{\frac{F_p \cap d^{-1}(0)}{d(C_*)}}{\frac{F_s \cap d^{-1}(0)}{d(C_*)}} = \frac{F_p \cap d^{-1}(0)}{F_s} = S[-\infty, s, p, \infty]$$

and in particular

$$E_p^\infty = \mathcal{F}_p/\mathcal{F}_{p-1} = S[-\infty, p-1, p, \infty].$$

Supposing to have determined the ∞ -page (E_p^∞), an extension problem amounts to determining, for example, $\mathcal{F}_p/\mathcal{F}_{p-2}$ from $E_p^\infty = \mathcal{F}_p/\mathcal{F}_{p-1}$ and $E_{p-1}^\infty = \mathcal{F}_{p-1}/\mathcal{F}_{p-2}$. This situation is depicted here, as an instance of the short exact sequence (2.5):

$$\begin{array}{ccccccc} S[-\infty, p-2, p-1, \infty] & \cdots & \begin{array}{c} p-2 \quad p-1 \\ \hline \text{shaded box} \end{array} & \cdots & & & \\ & & \downarrow \ell & & & & \\ S[-\infty, p-2, p, \infty] & \cdots & \text{shaded box} & \cdots & & & \\ & & \downarrow \ell & & & & \\ S[-\infty, p-1, p, \infty] & \cdots & \begin{array}{c} p-1 \quad p \\ \hline \text{shaded box} \end{array} & \cdots & & & \end{array}$$

Let us state here a useful result related to the previous example and valid in general for I -filtrations:

Proposition 2.8. Consider the spectral system $(S[z, s, p, b])_{z \leq s \leq p \leq b}$ of an I -filtered chain complex.

(i) If $p \leq t$ in I , then

$$S[-\infty, -\infty, p, t] = \text{Im}(\ell : H(F_p) \rightarrow H(F_t)),$$

where ℓ is the map induced in homology by the inclusion $F_p \hookrightarrow F_t$, and where $-\infty$ is an element of I (which, if necessary, can be added to the poset) such that $F_{-\infty} = 0$.

(ii) If $S[-\infty, -\infty, p, b]$ and $S[-\infty, -\infty, u, t]$ are terms such that $p \leq u$ and $b \leq t$ (which imply $p \leq t$), then

$$S[-\infty, -\infty, p, t] = \text{Im}(\ell : S[-\infty, -\infty, p, b] \rightarrow S[-\infty, -\infty, u, t]).$$

2.3 Morphisms of spectral systems and functoriality

Morphisms of I -filtered chain complexes are defined generalizing in the natural way the notion of morphisms of \mathbb{Z} -filtered chain complexes (Definition 1.37):

Definition 2.9. Let (C_*, F) and (D_*, F') be two I -filtered chain complexes. A *morphism of I -filtered chain complexes* $f : (C_*, F) \rightarrow (D_*, F')$ is a chain map $f : C_* \rightarrow D_*$ compatible with the filtrations, that is satisfying

$$f(F_i C_*) \subseteq F'_i D_*,$$

for all $i \in I$.

Denote by $(S[z, s, p, b])$ and $(S'[z, s, p, b])$ the spectral systems associated respectively with the I -filtered chain complexes (C_*, F) and (D_*, F') . A morphism of I -filtered chain complexes $f : (C_*, F) \rightarrow (D_*, F')$ induces morphisms

$$f^{z,s,p,b} : S[z, s, p, b] \rightarrow S'[z, s, p, b]$$

for any 4-tuple of indices $z \leq s \leq p \leq b$ in I . The construction of the spectral system associated with an I -filtered chain complex is *functorial*, meaning that for each 4-tuple of indices $z \leq s \leq p \leq b$ it holds $\text{id}^{z,s,p,b} = \text{id}_{S[z,s,p,b]}$ and, for each pair of morphisms f, g of I -filtered chain complexes such that the composition gf is defined, it holds $(gf)^{z,s,p,b} = g^{z,s,p,b} f^{z,s,p,b}$. The last equality appears evident from the following commutative diagram (recalling Proposition 1.3):

$$\begin{array}{ccccc} F_p \cap d^{-1}(F_z) & \xrightarrow{f} & F'_p \cap d^{-1}(F'_z) & \xrightarrow{g} & F''_p \cap d^{-1}(F''_z) \\ \downarrow \pi & & \downarrow \pi' & & \downarrow \pi'' \\ S[z, s, p, b] & \xrightarrow{f^{z,s,p,b}} & S'[z, s, p, b] & \xrightarrow{g^{z,s,p,b}} & S''[z, s, p, b]. \end{array}$$

2.4 Spectral systems over $D(\mathbb{Z}^m)$

In this section we present some ideas from [Mat13, §3], focusing on filtrations over the posets \mathbb{Z}^m and the poset of its downsets $D(\mathbb{Z}^m)$. First, we illustrate a typical situation in which a filtration over \mathbb{Z}^m or $D(\mathbb{Z}^m)$ may appear.

Let C_* be a chain complex and suppose that m different \mathbb{Z} -filtrations $F^{(i)}$ of C_* are defined, $1 \leq i \leq m$. Each filtration $F^{(i)} = (F_k^{(i)})_{k \in \mathbb{Z}}$ consists of chain subcomplexes of C_* indexed over \mathbb{Z} ,

$$\dots \subseteq F_{k-1}^{(i)} \subseteq F_k^{(i)} \subseteq F_{k+1}^{(i)} \subseteq \dots,$$

and is by convention extended to $\overline{\mathbb{Z}} = \mathbb{Z} \cup \{-\infty, \infty\}$ by defining $F_{-\infty}^{(i)} := \bigcap_{k \in \mathbb{Z}} F_k^{(i)}$ and $F_{\infty}^{(i)} := C_*$. Consider now \mathbb{Z}^m , seen as the poset (\mathbb{Z}^m, \leq) with the order relation \leq defined coordinate-wise: $P = (p_1, \dots, p_m) \leq Q = (q_1, \dots, q_m)$ if and only if $p_i \leq q_i$, for all $1 \leq i \leq m$. The poset $\overline{\mathbb{Z}}^m$ is defined similarly. For our purposes, the difference between the use of \mathbb{Z}^m and $\overline{\mathbb{Z}}^m$ will not be of particular relevance; we will therefore use always the notation \mathbb{Z}^m , remembering that when needed it can be extended to $\overline{\mathbb{Z}}^m$ with the described conventions.

Definition 2.10. A *downset* of \mathbb{Z}^m is a subset $p \subseteq \mathbb{Z}^m$ such that if $P \in p$ and $Q \leq P$ in \mathbb{Z}^m then $Q \in p$. We denote $D(\mathbb{Z}^m)$ the collection of all downsets of \mathbb{Z}^m , which is a poset with respect to the inclusion \subseteq .

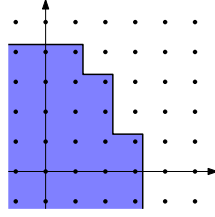


FIGURE 2.3: The set of points of \mathbb{Z}^2 in the colored region represent an example of a downset of \mathbb{Z}^2 .

For any $P = (p_1, \dots, p_m) \in \mathbb{Z}^m$ define

$$F_P := F_{p_1}^{(1)} \cap \dots \cap F_{p_m}^{(m)}.$$

It is evident that $(F_P)_{P \in \mathbb{Z}^m}$ is a \mathbb{Z}^m -filtration of C_* . We now define a $D(\mathbb{Z}^m)$ -filtration $(F_p)_{p \in D(\mathbb{Z}^m)}$ of C_* by setting, for each $p \in D(\mathbb{Z}^m)$,

$$F_p := \sum_{P \in p} F_P.$$

Filtrations over the poset $D(\mathbb{Z}^m)$ and the spectral systems associated with them are the subject of the present section. Although we have illustrated how they can be defined from \mathbb{Z}^m -filtrations

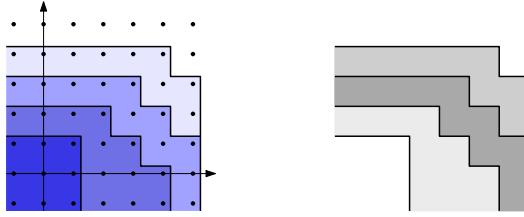


FIGURE 2.4: Graphical representations of a term $S[z, s, p, b]$ of a spectral system over $D(\mathbb{Z}^2)$. On the left, the four downsets $z \leq s \leq p \leq b$ in $D(\mathbb{Z}^2)$ defining the term $S[z, s, p, b]$ are colored in decreasing shades. On the right, the same term is represented following the convention introduced for \mathbb{Z} -spectral systems: $s \setminus z$ is colored in light gray \blacksquare , $p \setminus s$ in dark gray \blacksquare and $b \setminus p$ in medium gray \blacksquare .

(which, in turn, may originate from m different \mathbb{Z} -filtrations) as a notable case, the results we state here apply to any $D(\mathbb{Z}^m)$ -filtration $(F_p)_{p \in D(\mathbb{Z}^m)}$, with the additional hypothesis that it is *distributive*: $F_{p \cap q} = F_p \cap F_q$ and $F_{p \cup q} = F_p + F_q$, for all $p, q \in D(\mathbb{Z}^m)$. A simple example of a distributive filtration is the case in which C_* admits a decomposition $\bigoplus_{P \in \mathbb{Z}^m} C_P$ as graded abelian group and $F_p := \bigoplus_{P \in p} C_P$ are chain subcomplexes.

Isomorphic terms. Consider the spectral system $(S[z, s, p, b])$ associated with a distributive $D(\mathbb{Z}^m)$ -filtration. Some terms of the spectral system are naturally isomorphic, depending on the sets $s \setminus z$, $p \setminus s$ and $b \setminus p$. In order to make this precise, we have to introduce a few concepts.

A subset $A \subseteq \mathbb{Z}^m$ is *connected* if the graph having the elements of A as vertices and edges $\{(P, Q) \in A \times A \mid P \leq Q \text{ or } P \geq Q \text{ in } \mathbb{Z}^m\}$ is connected. Given four downsets $z \leq s \leq p \leq b$ in $D(\mathbb{Z}^m)$, define

$$Z := Z(z, s, p, b) \subseteq \mathbb{Z}^m$$

as the union of the connected components of $p \setminus z$ intersecting $p \setminus s$, and

$$B := B(z, s, p, b) \subseteq \mathbb{Z}^m$$

as the union of the connected components of $b \setminus s$ intersecting $p \setminus s$. The subsets Z and B determine the term $S[z, s, p, b]$ up to isomorphism:

Lemma 2.11 ([Mat13], Lemma 3.8). Let (F_p) be a distributive $D(\mathbb{Z}^m)$ -filtration of C_* . All the 4-tuples of downsets $z \leq s \leq p \leq b$ in $D(\mathbb{Z}^m)$ with the same Z and B determine naturally isomorphic terms $S[z, s, p, b]$.

The situation of Lemma 2.11 is represented in Figure 2.5.

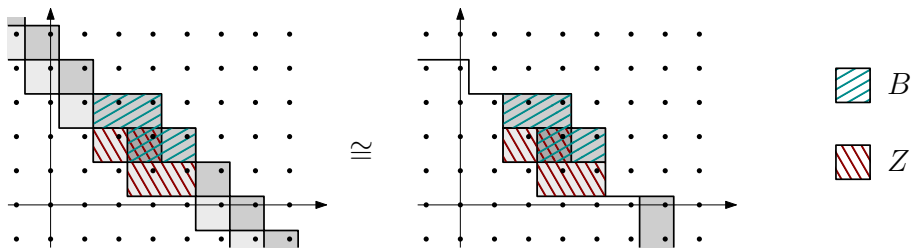


FIGURE 2.5: Two terms $S[z, s, p, b]$ and $S[z', s', p', b']$ of a spectral system associated with a distributive $D(\mathbb{Z}^m)$ -filtration, which are naturally isomorphic by virtue of Lemma 2.11. The three shades of gray have the usual meaning, while the ruled areas represent B and Z , as indicated in the legend.

Lexicographic connections. Given a distributive $D(\mathbb{Z}^m)$ -filtration of C_* , we call *connection* any procedure which, starting with the 1-page of the spectral system $(S[z, s, p, b])$ over $D(\mathbb{Z}^m)$, determines $H(C_*)$ via a succession of homology computations, natural isomorphisms and extensions. In this context, we define the 1-page of the spectral system $(S[z, s, p, b])$ as the collection of the terms $S[s, s, p, p] = H(F_p/F_s)$ such that p covers s , that is $s < p$ and there exists no $x \in D(\mathbb{Z}^m)$ with $s < x < p$. Notice that this agrees with the notion of 1-page $(E_p^1)_{p \in \mathbb{Z}} = (S[p-1, p-1, p, p])_{p \in \mathbb{Z}}$ for classical spectral sequences seen as part of a spectral system over \mathbb{Z} . In order to make the concept of connection clear, let us recall the connection for the spectral sequence of a \mathbb{Z} -filtered chain complex C_* in a series of steps, which will constitute a useful term of comparison for connections in spectral systems over $D(\mathbb{Z}^m)$. The situation is summarized in Figure 2.6:

- (i) Start from the 1-page $(E_p^1) = (H(F_p/F_{p-1}))$.
- (ii) For each page there are differentials $E_{p+r}^r \xrightarrow{d'} E_p^r \xrightarrow{d} E_{p-r}^r$ and taking the homology $\ker d / \text{Im } d'$ one obtains (E_p^{r+1}) .
- (iii) If the filtration is bounded, $E^\infty = E^r$ for r large enough.
- (iv) Via successive extensions, one can connect for example $S[-\infty, p-2, p, \infty]$ to E_p^r and E_{p-1}^r , and so on.
- (v) Obtain $S[-\infty, -\infty, \infty, \infty] = H(C_*)$.

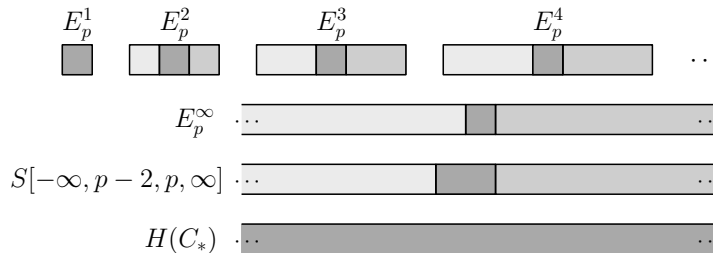


FIGURE 2.6: Connection from the 1-page to $H(C_*)$ for classical spectral sequences.

The study of connections for spectral systems over $D(\mathbb{Z}^m)$ is a very technical subject, mainly due to the large number of possible choices. Our purpose is to convey the main ideas and state the results we use in the present work; for a complete treatment we address the reader to [Mat13, § 3.2] and [Mat14a].

The idea on which the study of connections for $D(\mathbb{Z}^m)$ -spectral systems is based is actually simple: having selected a suitable collection of downsets to employ as indices of terms of the spectral system, we use Proposition 2.5 to describe how computing homology affects the “shape” of the downsets; employing Lemma 2.11 to identify 4-tuples of downsets with different shape which determine the same terms, we can then iterate the procedure. The use of Proposition 2.5 limits our choice to 4-tuples of downsets satisfying its hypotheses; for this reason, very often the downsets we consider are simply obtained as translations in \mathbb{Z}^m of a single downset. Here we introduce in a simplified form the *lexicographic connections* presented in [Mat13], which make use of the lexicographic order on \mathbb{Z}^m for defining the collection of downsets used in the connection. We will then describe the *secondary connections*, the most useful type of connection for the generalized Serre spectral sequence we treat in Chapter 4.

Recall that the *lexicographic order* \leq_{lex} on \mathbb{Z}^m is defined as follows: $X = (x_1, \dots, x_m) <_{\text{lex}} Y = (y_1, \dots, y_m)$ if and only if there exists $1 \leq k < m$ such that $x_i = y_i$ for all $i < k$ and $x_k < y_k$. Notice that $X \leq Y$ imply $X \leq_{\text{lex}} Y$. We use \leq_{lex} only to define some particular downsets of \mathbb{Z}^m , while we keep considering the order \leq on \mathbb{Z}^m . For each $1 \leq k \leq m$ and each $p_1, \dots, p_k \in \mathbb{Z}$ define the downset

$$A(p_1, \dots, p_k) := \{X \in \mathbb{Z}^m \mid (x_1, \dots, x_k) \leq_{\text{lex}} (p_1, \dots, p_k)\}.$$

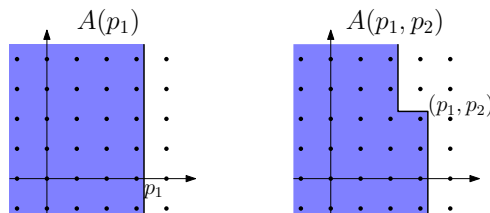


FIGURE 2.7: Example of downsets $A(p_1)$ and $A(p_1, p_2)$ in \mathbb{Z}^2 , with $(p_1, p_2) = (3, 2)$.

The downsets $A(p_1, \dots, p_k)$ are used to define, for each $r \geq 1$, the following 4-tuples of downsets

$$\begin{aligned} p(p_1, \dots, p_k; r) &:= A(p_1, \dots, p_{k-1}, p_k) \\ s(p_1, \dots, p_k; r) &:= A(p_1, \dots, p_{k-1}, p_k - 1) \\ z(p_1, \dots, p_k; r) &:= A(p_1, \dots, p_{k-1}, p_k - r) \\ b(p_1, \dots, p_k; r) &:= A(p_1, \dots, p_{k-1}, p_k + r - 1), \end{aligned}$$

whose associated terms are denoted by

$$S(p_1, \dots, p_k; r) := S[z(p_1, \dots, p_k; r), s(p_1, \dots, p_k; r), p(p_1, \dots, p_k; r), b(p_1, \dots, p_k; r)].$$

By convention, for $k = 0$ and $r = 1$ we set $p(-; 1) = b(-; 1) := C_*$ and $s(-; 1) = z(-; 1) := 0$, where $-$ is the empty list. We state in a series of steps the results of [Mat13, Lemma 3.11] describing one of the simplest ways to connect the 1-page to the homology $H(C_*)$:

- (i) Start from the 1-page, formed by the terms $S(p_1, \dots, p_n; 1) = H(F_{A(p_1, \dots, p_n)} / F_{A(p_1, \dots, p_n) \setminus \{P\}})$, for all $P = (p_1, \dots, p_n) \in \mathbb{Z}^m$.
- (ii) For each k and r there are differentials $d : S(p_1, \dots, p_k; r) \rightarrow S(p_1, \dots, p_k - r; r)$. Taking homology at $S(p_1, \dots, p_k; r)$ yields $S(p_1, \dots, p_k; r + 1)$.
- (iii) If the filtration is bounded, $S(p_1, \dots, p_k; \infty) = S(p_1, \dots, p_k; r)$ for r large enough.
- (iv) For each $1 \leq k \leq m$ there is a canonical \mathbb{Z} -filtration of $S(p_1, \dots, p_{k-1}; 1)$,

$$0 \subseteq \dots \subseteq \mathcal{F}_{i-1} \subseteq \mathcal{F}_i \subseteq \dots \subseteq S(p_1, \dots, p_{k-1}; 1),$$

such that $S(p_1, \dots, p_k; \infty) \cong \mathcal{F}_{p_k} / \mathcal{F}_{p_k-1}$. Therefore, via successive extensions, one can connect the terms $S(p_1, \dots, p_k; \infty)$ to the terms $S(p_1, \dots, p_{k-1}; 1)$, decreasing k by 1 and resetting r to 1.

- (v) Obtain $S(-; 1) = H(C_*)$.

Figure 2.8 represents the connection we described in the case $m = 2$. The method, as presented in

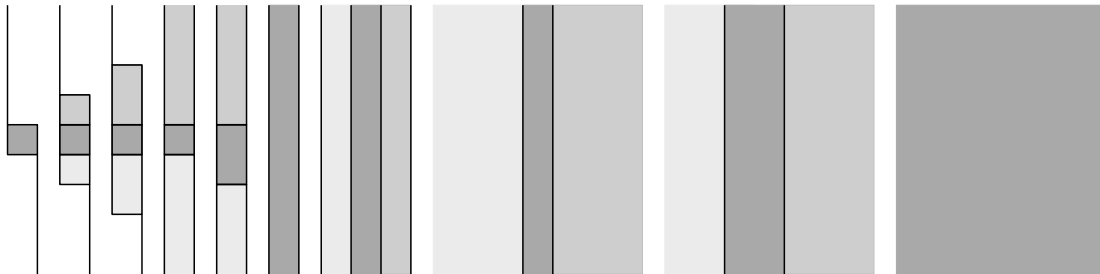


FIGURE 2.8: The first four terms correspond to $k = 2$ and $r = 1, 2, 3, \infty$. The fifth term represents an extension process. Then the terms for $k = 1$ and $r = 1, 2, \infty$ are represented, followed by another extension process and the last term for $H(C_*)$.

[Mat13, § 3.2], takes into account some additional parameters to modify the shape of the downsets,

in order to achieve more flexibility and to cover in a unified framework a larger amount of possible connections. As we aim to present in an intuitive way the notion of connection for spectral systems, we refer the interested reader to the treatment by Matschke.

Secondary connections. We have defined the 1-page of a spectral system over any poset I as the collection of terms $S[s, s, p, p] = H(F_p/F_s)$, specifying that for spectral systems over $D(\mathbb{Z}^m)$ we make the additional assumption that p covers s . We now give a definition of the 2-page of a spectral system over $D(\mathbb{Z}^m)$, which agrees with the usual one for classical spectral sequences and will appear even more “natural” in light of the results we present in Chapter 4. Following [Mat13, § 3.2], we introduce a way to connect the 1-page to the 2-page computing homology m times, and explain how the 2-page can be then connected to the homology $H(C_*)$.

For $1 \leq k \leq m$ define the automorphism $\varphi_k : \mathbb{Z}^m \rightarrow \mathbb{Z}^m$ as the map sending $X = (x_1, \dots, x_m)$ to

$$\varphi_k(X) := \left(x_{k+1}, x_{k+2}, \dots, x_m, \sum_{i=1}^k x_i, \sum_{i=2}^k x_i, \dots, x_k \right).$$

For $P \in \mathbb{Z}^m$ and $1 \leq k \leq m$ define the downset

$$T_P^k := \{X \in \mathbb{Z}^m \mid \varphi_k(X) \leq_{\text{lex}} \varphi_k(P)\}.$$

Let $e_i = (0, \dots, 1, \dots, 0)$ be the element of \mathbb{Z}^m whose only non-null entry is a 1 at position i , for

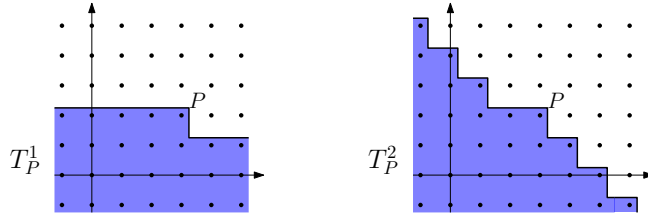


FIGURE 2.9: Example of the downsets T_P^1 and T_P^2 in \mathbb{Z}^2 , with $P = (3, 2)$.

$1 \leq i \leq m$. Given $P = (p_1, \dots, p_m) \in \mathbb{Z}^m$ and $1 \leq k \leq m$ define the following downsets, where by convention $e_0 := 0$ and $e_{-1} := -e_m$:

$$\begin{aligned} p(P; k) &:= T_P^k \\ s(P; k) &:= p(P; k) \setminus \{P\} = T_{P+e_{k-1}-e_k}^k \\ z(P; k) &:= p(P; k) - e_k = T_{P-e_k}^k \\ b(P; k) &:= s(P; k) + e_k = T_{P+e_{k-1}}^k \\ z^*(P; k) &:= z(P; k) \setminus \{P - e_k\} = T_{P+e_{k-1}-2e_k}^k \\ b^*(P; k) &:= b(P; k) \cup \{P + e_k\} = T_{P+e_k}^k. \end{aligned} \tag{2.6}$$

The 4-tuples of downsets $z \leq s \leq p \leq b$ and $z^* \leq s \leq p \leq b^*$ define respectively the terms

$$\begin{aligned} S(P; k) &:= S[z(P; k), s(P; k), p(P; k), b(P; k)], \\ S^*(P; k) &:= S[z^*(P; k), s(P; k), p(P; k), b^*(P; k)]. \end{aligned}$$

It is clear that the 1-page of the spectral system can be expressed as the collection of the terms $S(P; 1) = H(F_{p(P;1)}/F_{p(P;1)\setminus\{P\}})$, for all $P \in \mathbb{Z}^m$.

Definition 2.12. We call 2-page of a spectral system over $D(\mathbb{Z}^m)$ the collection of terms $S^*(P; m)$, for all $P \in \mathbb{Z}^m$.

The following two lemmas describe a way to connect the terms $S(P; 1)$ of the 1-page to the terms $S^*(P; m)$ of the 2-page.

Lemma 2.13 ([Mat13], Lemma 3.14). There exist differentials *in direction* $-e_k$,

$$d : S(P; k) \rightarrow S(P - e_k; k),$$

such that the homology at the middle term of

$$S(P + e_k; k) \xrightarrow{d'} S(P; k) \xrightarrow{d} S(P - e_k; k)$$

is $S^*(P; k)$.

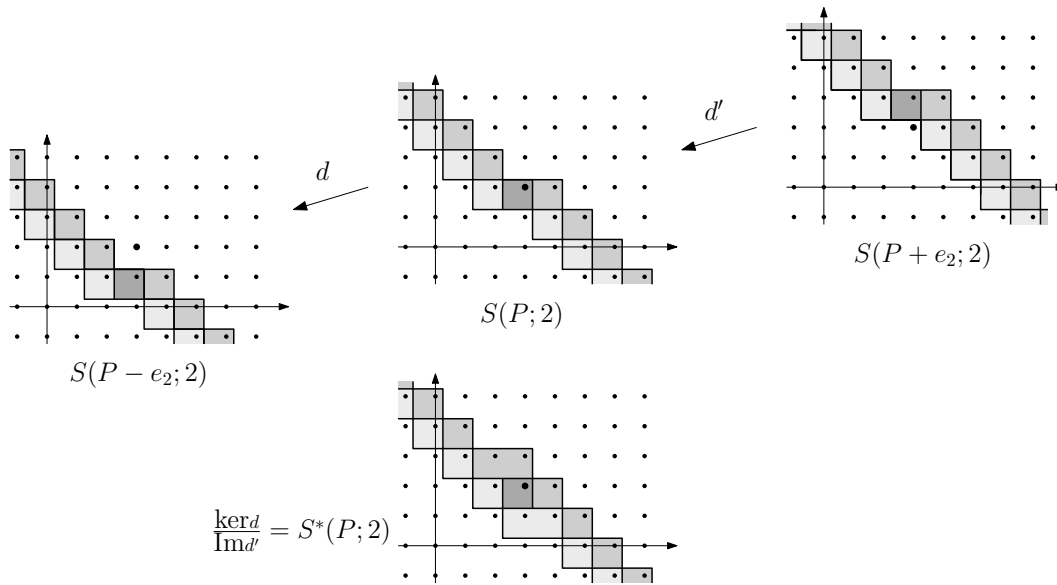


FIGURE 2.10: Representation of Lemma 2.13 for $m = k = 2$. The differentials d and d' are in direction $-e_2 = -(0, 1)$ and the homology $\ker d / \text{Im } d'$ is a term of the *second page*. The point $P = (3, 2)$ is highlighted in the picture.

Lemma 2.13 is an application of Proposition 2.5 (see Figure 2.10) to our current situation: a spectral system over $D(\mathbb{Z}^m)$ where some distinguished downset are defined by (2.6). In this framework we can also rephrase Lemma 2.11 on naturally isomorphic terms:

Lemma 2.14 ([Mat13], Lemma 3.15). For any $1 \leq k \leq m - 1$ there is a natural isomorphism

$$S^*(P; k) \cong S(P; k + 1).$$

The core idea of secondary connections consists in using Lemma 2.13 and Lemma 2.14 alternately: starting from the 1-page $S(P; 1)$ and taking homology in direction $-e_1$ one determines $S^*(P; 1) \cong$

$S(P; 2)$, then taking homology in direction $-e_2$ one determines $S^*(P; 2) \cong S(P; 3)$; continuing alternating homology and natural isomorphisms one eventually determines the 2-page $S^*(P; m)$. The 2-page can then be connected to the homology $H(C_*)$ in different ways, for example using lexicographic connections (see Figure 2.11).

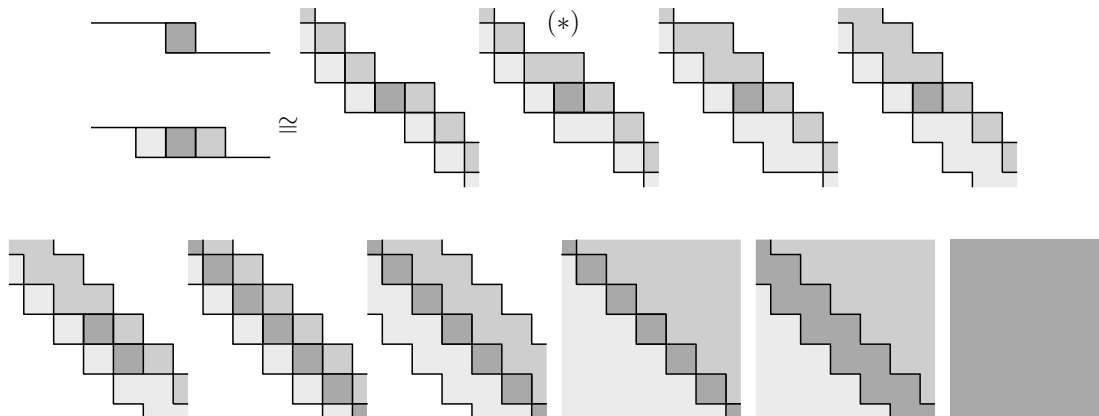


FIGURE 2.11: Secondary connection for $m = 2$. The second and third terms are respectively $S^*(P; 1)$ and $S(P; 2)$, which are isomorphic via Lemma 2.14. The term marked with an asterisk (*) belongs to the second page. The following terms represent a possible way to connect the second page to the homology $H(C_*)$ based on the lexicographic connection. See [Mat13, § 3.2] for further details.

Generalizations. We have seen that secondary connections describe a precise way to connect the 1-page to the 2-page, while one can then decide how to connect the 2-page to $H(C_*)$ via lexicographic connections (Figure 2.11). In [Mat14a] a more flexible framework is introduced, which allows to choose at each step the *direction* of the differential we want to consider, in a sense similar to Lemma 2.13. For example, starting from the 1-page, one can choose among m possible differentials, which intuitively correspond to the possible permutations of the m coordinates. The choices operated on the directions of the differentials are “summarized” in a word ω to which a new letter is added every time a homology computation takes place. The words ω are over the alphabet

$$\{1, \dots, m, 1^\infty, \dots, m^\infty, \mathbf{x}\},$$

where each letter $j \in \{1, \dots, m\}$ means considering the j th differential for taking homology, j^∞ means doing it infinite times and \mathbf{x} represents an extension process. A notion of ω -page $S(P; \omega)$ can thus be introduced, and Theorem 3.6 of [Mat14a] describes the relations between different ω -pages. For example, it can be shown that there are differentials

$$S(P + r_\omega^j; \omega) \xrightarrow{d'} S(P; \omega) \xrightarrow{d} S(P - r_\omega^j; \omega)$$

(where r_ω^j is an element of \mathbb{Z}^m which depends on the directions chosen for the differentials) such that the homology at the middle term is $S(P; \omega * j)$. This represents one of the fundamental results that allow to connect the 1-page to $H(C_*)$ in a large range of different ways.

We preferred to present secondary connection in the simpler and more explicit form of [Mat13], especially since we are specifically interested in the connection between the 1-page and the 2-page. All the results on secondary connections we present in this work (Chapter 4) hold in the general scenario of [Mat14a], considering that the 2-page corresponds to the collection of terms $S(P; \omega)$ with $\omega = 123 \cdots m$.

Chapter 3

Spectral systems and persistent homology

Persistent homology is a technique in applied algebraic topology conceived to summarize the information of a filtration (usually of simplicial complexes) in the form of topological invariants. Homology is used not only to study the topological features at each fixed point of the filtration, but also to track their evolution across the whole filtration. Since simplicial complexes, as we have mentioned, are in many situations convenient objects to be associated with data, persistent homology represents a very versatile method for the analysis of data; having proved itself effective in a large number of areas, it significantly contributed to the development of the field of *topological data analysis*.

Even if introduced with completely different aims, both persistent homology and spectral sequences stem from filtrations. The two notions are in fact related, as explained in [BP17] using exact couples, a classical construction in algebraic topology. Furthermore, they carry the same topological information about the filtration, in a sense that we will make precise. A different approach to link the two notions is presented in [RHR14].

In their original setting, both spectral sequences and persistent homology are defined from filtrations with indices in \mathbb{Z} . Nevertheless, generalizations of both concepts have been proposed which originate from a different choice of the set of indices of the filtration. In Chapter 2 we have introduced the notion of spectral system over a poset. On the other hand, multidimensional persistence [Car09] is a generalized version of persistent homology for filtrations with indices in \mathbb{Z}^n .

The main goal of this chapter is to illustrate the relation between persistent homology and spectral sequences and to show how it still stands if we consider the generalized constructions.

Firstly, we introduce persistent homology (Section 3.1) and its multidimensional generalization (Section 3.2). In Section 3.3 we introduce exact couples and use them to explain the connections between spectral sequences and persistent homology. Furthermore, we discuss why the topological invariants provided by persistent homology and spectral sequences can be considered equivalent. In Section 3.4 we study the connection between the generalizations of persistent homology and spectral sequences. We show that, for a particular class of posets, the construction of exact couples can be generalized to this framework, and we detail the case of multidimensional persistence as a particularly significant example.

The results of this chapter are the subject of the paper [GV18], in preparation.

3.1 Persistent homology

In order to work in the typical framework of persistent homology we make two assumptions, valid for the whole chapter: first, we always consider *finite* simplicial complexes, meaning that we require their vertex sets to be finite; second, we assume the coefficients for defining chain complexes and homology to be in a fixed field \mathbb{F} . The second assumption means that every module in this chapter (chain groups, homology groups, terms of a spectral sequence or system, and so on) is in fact an \mathbb{F} -vector space; for legibility we do not make explicit the dependence on \mathbb{F} , using for example the notation $H_n(K)$ instead of $H_n(K, \mathbb{F})$. The choice of field coefficients for persistent homology is motivated both by theoretical and computational reasons, as we will explain. Let us remark that Betti numbers, which will play an important role in what follows, are affected by the choice of coefficients, as an immediate consequence of the universal coefficient theorem (Theorem 1.18); we refer to [BM14] for a thorough treatment in the framework of persistent homology from the algorithmic point of view.

Let us illustrate the fundamental concepts of persistent homology theory. For more details and examples of applications we refer the reader to the surveys [Ker16, EM12]. A *finite filtration* of a simplicial complex K is a sequence of subcomplexes

$$\emptyset = \dots = K_{-1} = K_0 \subseteq K_1 \subseteq \dots \subseteq K_p \subseteq K_{p+1} \subseteq \dots \subseteq K_N = K_{N+1} = \dots = K. \quad (3.1)$$

To understand the motivation behind persistent homology, assume that the filtration (3.1) is associated with some dataset. For instance, recalling Example 1.23 we can think of the Vietoris-Rips complex VR_ϵ (or the Čech complex \check{C}_ϵ) associated with a given point cloud; increasing the value of ϵ we obtain a filtration of simplicial complexes depending on a real parameter, which can be discretized to assume the form (3.1). Considering Example 1.22 and supposing that the given network is weighted, we can similarly produce a filtration of the associated simplicial complex by adding the edges in decreasing (or increasing) order of weight.

The general idea of persistent homology is to detect, using homology, the topological features which “persist” across the filtration. In order to do this, for every pair of indices $s \leq t$ in the filtration consider the map $f_n^{s,t} : H_n(K_s) \rightarrow H_n(K_t)$ induced in homology by the inclusion of simplicial complexes $K_s \hookrightarrow K_t$.

Definition 3.1. For every pair of indices $s \leq t$ we define a *persistent n -homology group* $H_n^{s,t}(K)$ as the subspace of $H_n(K_t)$ given by the image of the map $f_n^{s,t}$:

$$H_n^{s,t}(K) := \text{Im}(f_n^{s,t} : H_n(K_s) \rightarrow H_n(K_t)).$$

We denote its dimension (as \mathbb{F} -vector space) $\beta_n^{s,t} := \dim_{\mathbb{F}} H_n^{s,t}(K)$, called a *persistent Betti number*.

It is clear that $H_n^{s,s}(K) = H_n(K_s)$, so $\beta_n^{s,s}$ equals the n th Betti number of K_s . Obviously, the definition of persistent homology groups can be applied also to a filtration of chain complexes, as it involves only the homology groups. When the filtration we are considering is evident from the context, we simply denote the persistent homology groups as $H_n^{s,t}$.

The collection of all persistent Betti numbers $\{\beta_n^{s,t}\}$, for all $n, s, t \in \mathbb{Z}$ with $s \leq t$, represents a sort of “topological signature” of the filtered simplicial complex. In persistent homology theory an intuitive terminology has been developed, which describes in an immediate way the meaning of persistent Betti numbers. Imagine the set of indices $p \in \mathbb{Z}$ of (3.1) as increasing instants of time, so that the filtration can be seen as describing the “growth process” of a simplicial complex. We say that a homology class is *born* at time $i \in \mathbb{Z}$ if it is an element of $H_n(K_i)$ not belonging to the image $\text{Im } f_n^{i-1,i}$. A homology class in $H_n(K_{j-1})$ is then said to *die* at time $j \in \mathbb{Z}$ if its image under $f_n^{j-1,j}$ is zero, otherwise it is said to *persist*; the homology classes which persist until the last step $N \in \mathbb{Z}$ of the filtration are said to *live forever* or *die at ∞* . Then, for example, one can observe that the difference $\beta_n^{i,j} - \beta_n^{i-1,j}$ corresponds to the number of distinct n -homology classes that are born at time i and persist until time j .

Using this terminology, it is easy to see that for all $i < j$ the non-negative integer

$$\mu_n^{i,j} = (\beta_n^{i,j-1} - \beta_n^{i,j}) - (\beta_n^{i-1,j-1} - \beta_n^{i-1,j}) \quad (3.2)$$

is the number of distinct n -homology classes that are born at time i and die at time j . Each homology class born at time i and dying at time j can be graphically represented by a bar from i to j (see Figure 3.1), hence the term *barcode* for the collection of all bars. It is easy to see that the collection of all *multiplicities* $\mu_n^{i,j}$, for all $n \in \mathbb{N}$ and $i < j$, completely characterizes the barcode. Equation (3.2) defines the barcode in terms of the persistent Betti numbers, but the relation can be

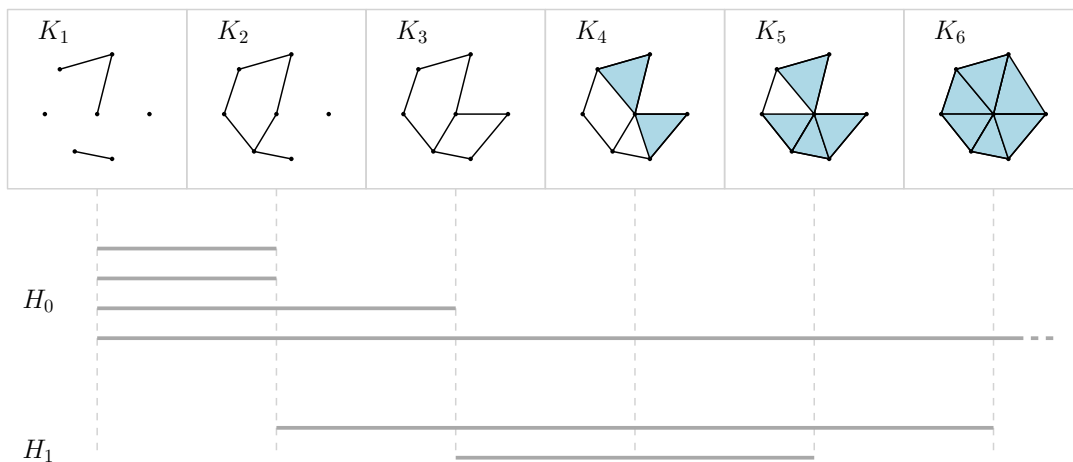


FIGURE 3.1: Barcode for H_0 and H_1 of a filtered simplicial complex.

inverted (as proven for instance in [EH10, §VII.1], “Fundamental Lemma of persistent homology”), giving

$$\beta_n^{s,t} = \sum_{i \leq s} \sum_{j > t} \mu_n^{i,j}. \quad (3.3)$$

Clearly, the combination of (3.2) and (3.3) expresses the fact that the persistent Betti numbers and the barcode encode the same topological information, as it is possible to compute the collection $\{\mu_n^{i,j}\}$ from the collection $\{\beta_n^{s,t}\}$ and vice versa.

As first noted in [ZC05] by Zomorodian and Carlsson, the barcode is a *complete* topological invariant, intuitively meaning that it captures all the topological information of a filtration. Let us

briefly illustrate their argument. Applying n -homology to the filtration (3.1) one obtains a sequence of \mathbb{F} -vector spaces and \mathbb{F} -linear maps

$$H_n(K_1) \xrightarrow{f_n^{1,2}} H_n(K_2) \xrightarrow{f_n^{2,3}} \dots \xrightarrow{f_n^{N-1,N}} H_n(K_N) \quad (3.4)$$

which can be extended on the right with the spaces $H_n(K_{N+1}), H_n(K_{N+2}), \dots$ and identity maps. Then

$$V := \bigoplus_{i=1}^{\infty} H_n(K_i)$$

can be given the structure of a (graded) $\mathbb{F}[x]$ -module if we define the action of x by

$$x \cdot (v_1, v_2, v_3, \dots) := (0, f_n^{1,2}(v_1), f_n^{2,3}(v_2), f_n^{3,4}(v_3), \dots).$$

Since it can be shown that V is a *finitely generated* positively graded $\mathbb{F}[x]$ -module, and since $\mathbb{F}[x]$ is a PID, a graded version of Theorem 1.19 gives a decomposition

$$\left(\bigoplus_{i=1}^k \sum^{a_i} \mathbb{F}[x] \right) \oplus \left(\bigoplus_{j=1}^{\ell} \sum^{b_j} \mathbb{F}[x]/(x^{c_j}) \right),$$

where \sum^a denotes a grading shift by a . Let us observe that in the described situation the importance of the assumption of coefficients in a field \mathbb{F} is evident, as $R[x]$ is a PID if and only if R is a field (for example, no simple classification of $\mathbb{Z}[x]$ -modules is known). One can see that each free summand $\sum^{a_i} \mathbb{F}[x]$ corresponds to a homology class born at a_i and living forever in the barcode, while each torsion summand $\sum^{b_j} \mathbb{F}[x]/(x^{c_j})$ corresponds to a homology class born at b_j and dying at $b_j + c_j$ in the barcode. To summarize, the positive integers a_i, b_j, c_j characterize the decomposition of the $\mathbb{F}[x]$ -module V , which encodes all the topological information that homology can capture. Let us mention that the same conclusion can be reached using *quiver representation theory*, which we do not introduce here, referring the interested reader to [Oud15]: in extreme brevity, (3.4) can be seen as a representation of an A_N -type quiver, which can be uniquely expressed as sum of indecomposable representations corresponding to the intervals of the barcode.

In order to give a complete (yet very concise) account of the theory of persistent homology, we must mention two aspects which substantially contributed to its success: algorithmic efficiency and stability.

The topological invariants of persistent homology (barcode or persistent Betti numbers) can be computed with efficient matrix reduction algorithms which in general are variants of Gaussian elimination (see [ZC05] or [EH10, Ch. VII]), providing thus further justification to the choice of field coefficients. The usual choice for the field of coefficients is \mathbb{Z}_2 , which leads to the fastest algorithms. Several speed-ups and optimizations of the original algorithms led to the release of a number of software implementations, whose efficiency is studied and compared in [OPT⁺17].

A desirable property for every technique devised to analyze data is *stability*, which in our case consists in requiring that small perturbation of data produce small perturbation of the topological invariant. A notion of distance between barcodes can be defined, which allows rigorous proofs of

stability of the methods of persistent homology [CSEH07, CCSG⁺09].

Since in Section 3.3 we will explain the relation between persistent homology and spectral sequences, let us conclude this section introducing the subject with a brief overview of the uses of spectral sequences in persistence theory. The connection between spectral sequences and persistent homology was first mentioned in the introduction of [ZC05], even if it was not detailed there. In the book by Edelsbrunner and Harer [EH10, § VII.4] an algorithm to compute the barcode based on the block reduction of a matrix is presented, which is inspired by spectral sequences: first, one reduces the blocks corresponding to the 1-page E^1 , then the blocks corresponding to the 2-page, repeating the process until the whole matrix is reduced. Some clarifications on the relation between persistent homology and spectral sequences appear in [RHRS14]; eventually the paper [BP17] illustrates the connection in a very explicit form, which we will generalize in the following. It is worth mentioning the completely different approach introduced in [LSVJ11], where the authors show how the Mayer-Vietoris spectral sequence (see for example [Bro82, Ch. VII.4]) can be used for parallelized computation of persistent homology, an idea further developed in [LM15]. The Mayer-Vietoris spectral sequence is employed also in [GS17] to prove an approximate version of the nerve lemma in the context of persistent homology. Lastly, in [Knu08] a spectral sequence involving Tor modules (see for example [Wei94, Ch. 5]) is used to determine a collection of “higher order” invariants for multidimensional persistence.

3.2 Multidimensional persistence

Consider again the filtration of simplicial complexes (3.1) of the previous section, here denoted shortly by

$$K_1 \hookrightarrow K_2 \hookrightarrow \cdots \hookrightarrow K_N,$$

recalling that it can be interpreted as a family of simplicial complexes which “grow” with respect to a single parameter. In some applications a setting in which simplicial complexes vary according to two or more parameters may be more interesting, for example because the interplay of the parameters can reveal information on the data. For instance, consider again the filtration of Vietoris-Rips (or Čech) complexes which can be constructed from a point cloud by letting the radius parameter ϵ vary (Example 1.23). Since outliers in the point cloud can sometimes compromise the effectiveness of topological methods, we can introduce (as in [CZ09, § 1.1]) a second parameter ρ related to the density of the points, which allows to discard points located “far from the others”. Combining the

two parameters, we can build a filtration along two dimensions of the following form

$$\begin{array}{ccccccc}
 K_{1N'} & \hookrightarrow & K_{2N'} & \hookrightarrow & \cdots & \hookrightarrow & K_{NN'} \\
 \uparrow & & \uparrow & & & & \uparrow \\
 \cdots & & \cdots & & & & \cdots \\
 \uparrow & & \uparrow & & & & \uparrow \\
 K_{12} & \hookrightarrow & K_{22} & \hookrightarrow & \cdots & \hookrightarrow & K_{N2} \\
 \uparrow & & \uparrow & & & & \uparrow \\
 K_{11} & \hookrightarrow & K_{21} & \hookrightarrow & \cdots & \hookrightarrow & K_{N1}
 \end{array} \tag{3.5}$$

where each row is a filtration with respect to the first parameter for a fixed value of the second one, and each column is a filtration with respect to the second parameter for a fixed value of the first one. This bidimensional filtration potentially encodes more information than two linear filtrations, as it can show the interaction between the two parameters. Here the filtration (3.5) is assumed to be *finite*, in accordance with the previous section. Let us make this concept more rigorous in the general case of a filtration along m dimensions. Consider \mathbb{Z}^m , endowed with the usual coordinate-wise order \leq . A collection of simplicial complexes $(K_v)_{v \in \mathbb{Z}^m}$ such that $K_w \subseteq K_{w'}$ if $w \leq w'$ is called a \mathbb{Z}^m -filtration of simplicial complexes (indicated in [CZ09] with the term *multifiltration*). A \mathbb{Z}^m -filtration $(K_v)_{v \in \mathbb{Z}^m}$ of simplicial complexes is *finite* if there exists $w = (w_1, \dots, w_m) \in \mathbb{Z}^m$ such that, for each $i = 1, \dots, m$, the \mathbb{Z} -filtration obtained fixing $m - 1$ parameters except the i -th, here denoted $(\hat{K}_p^{(i)})_{p \in \mathbb{Z}}$, is finite, with

$$\emptyset = \dots = \hat{K}_{-1}^{(i)} = \hat{K}_0^{(i)} \subseteq \hat{K}_1^{(i)} \subseteq \dots \subseteq \hat{K}_{w_i}^{(i)} = \hat{K}_{w_i+1}^{(i)} = \dots$$

For example, the finite \mathbb{Z}^2 -filtration of (3.5) can be completely represented in a diagram involving only the indices $v \in \mathbb{Z}^2$ such that $(1, 1) \leq v \leq w = (N, N')$.

Let us remark that clearly every \mathbb{Z}^m -filtration of simplicial complexes determines a \mathbb{Z}^m -filtration of chain complexes, in the sense of Chapter 2. Although we have chosen to introduce persistence theory using filtrations of simplicial complexes in order to highlight its close connection with data analysis, in the next sections we will consider the more general framework of filtered chain complexes.

Topological invariants of \mathbb{Z}^m -filtrations. *Multidimensional persistence* [CZ09] is a generalization of persistent homology which deals with \mathbb{Z}^m -filtrations instead of usual \mathbb{Z} -filtrations. The goal is (again) to use homology to describe the evolution of topological features across a \mathbb{Z}^m -filtration of simplicial complexes. As we have seen, the ultimate purpose of persistent homology is to provide an *invariant*, an object associated with a filtration which summarizes its topological properties. In the case of \mathbb{Z} -filtrations, the collection of persistent Betti numbers (or, equivalently, the barcode) is the invariant that captures the behavior of homology across the filtration; furthermore, it is a *complete* invariant, allowing to distinguish non-isomorphic structures. Unlike the 1-dimensional case, there is no discrete complete invariant for multidimensional persistence. This should not be a surprise: the homology of a \mathbb{Z}^m -filtration can be given the structure of a $\mathbb{F}[x_1, \dots, x_m]$ -module (in a similar way as we constructed the $\mathbb{F}[x]$ -module V from the homology of a \mathbb{Z} -filtration, where now the action of each

x_i is defined by morphisms “in direction e_i ”), and the classification of $\mathbb{F}[x_1, \dots, x_m]$ -modules is known to be very hard. The impossibility to produce a complete invariant in the multidimensional case has been proven in [CZ09] through algebraic geometry arguments, but more recently also arguments from quiver representation theory have been proposed (see for instance [Oud15]). Nevertheless, invariants can be defined for multidimensional persistence which are informative and relatively easy to compute. The most relevant in applications is the *rank invariant*, an immediate generalization of persistent Betti numbers proposed in [CZ09].

Definition 3.2. Let $(K_v)_{v \in \mathbb{Z}^m}$ be a \mathbb{Z}^m -filtration of simplicial complexes and let $v \leq w$ in \mathbb{Z}^m . We denote $f_n^{v,w} : H_n(K_v) \rightarrow H_n(K_w)$ the map induced in homology by the inclusion $K_v \hookrightarrow K_w$ and define

$$\beta_n^{v,w} := \dim_{\mathbb{F}} \operatorname{Im}(f_n^{v,w} : H_n(K_v) \rightarrow H_n(K_w)).$$

The collection of all $\beta_n^{v,w}$, for every pair of indices $v \leq w$ and for every n , is called *rank invariant* of the \mathbb{Z}^m -filtration.

In the next sections we will focus on the rank invariant as topological summary associated with a \mathbb{Z}^m -filtration, and we will illustrate its relation with spectral systems. Even if it represents by far the most exploited invariant in applications, during the last years several ideas have been proposed to define different invariants or to compute or improve the existing ones, making multidimensional persistence one of the most active research areas in topological data analysis. Let us briefly review the main results.

In [CSZ10] the authors propose an efficient algorithm to compute invariants associated with resolutions of modules constructed from \mathbb{Z}^m -filtrations, although some restrictive assumptions are made on the type of filtrations; a more general framework is studied in [CSV17]. In [CDFF⁺13] the study of a \mathbb{Z}^m -filtration is reduced to a family of \mathbb{Z} -filtrations corresponding to linear sections of (3.5) with different slopes. This idea has been further developed in [LW15], together with the theoretical bases of the software RIVET for visualizing 2-dimensional persistence, available at <http://rivet.online/doc/obtaining-rivet/>. Generalized persistent homology and its relation with filtrations of weighted graphs is studied in [VPP15]. In [SCL⁺17] an algebraic definition of noise (negligible topological features) for multidimensional persistence is introduced and some related invariants are studied, opening new perspectives at least from a theoretical point of view. Finally, the recent work [HOST17] proposes new computable invariants based on notions in commutative algebra like the multigraded Hilbert series.

3.3 Spectral sequences and persistence

In this section we introduce exact and derived couples, an elegant construction related to spectral sequences which we will employ to explain (following [BP17]) their relationship with persistent homology. Using this relation, we will show in what sense persistent Betti numbers and spectral sequences carry the same amount of topological information on the filtration from which they arise.

For the sake of generality we will consider filtrations of a chain complex $C_* = (C_n, d_n)_{n \in \mathbb{Z}}$ as our starting point; moreover, we assume the filtrations to be *finite*, of the form

$$0 = \dots = F_{-1} = F_0 \subseteq F_1 \subseteq \dots \subseteq F_p \subseteq F_{p+1} \subseteq \dots \subseteq F_N = F_{N+1} = \dots = C_*. \quad (3.6)$$

As usual, F_p denotes the chain complex $F_p C_*$, while $F_p C_n$ denotes the space of n -chains of F_p .

3.3.1 Exact and derived couples

The machinery of exact and derived couples, introduced by William Massey in [Mas52], represents a very general way to produce a spectral sequence. We present it here following [Rot09]. Even if we keep assuming to work with vector spaces over a fixed field \mathbb{F} , the construction is exactly the same for modules over a ring R .

Definition 3.3. An *exact couple* is a 5-tuple $(D, E, \alpha, \beta, \gamma)$, where D, E are \mathbb{F} -vector spaces and α, β, γ are linear maps such that the triangular diagram

$$\begin{array}{ccc} D & \xrightarrow{\alpha} & D \\ & \swarrow \gamma & \searrow \beta \\ & E & \end{array} \quad (3.7)$$

is exact at each vertex: $\ker(\beta) = \text{Im}(\alpha)$, $\ker(\gamma) = \text{Im}(\beta)$ and $\ker(\alpha) = \text{Im}(\gamma)$.

In our situation, D and E will be bigraded \mathbb{F} -vector spaces (recall Definition 1.30), but for now we do not denote the bigrading in order to keep the notation as simple as possible. In the diagram (3.7) the map $d := \beta\gamma : E \rightarrow E$ is a differential, i.e. $dd = (\beta\gamma)(\beta\gamma) = \beta(\gamma\beta)\gamma = 0$, the last equality coming from exactness. Thus, (E, d) is a differential vector space, and we can consider $E' := H(E, d) = \ker d / \text{Im} d$. We also define $D' := \text{Im} \alpha \subseteq D$ and the linear maps $\alpha' : D' \rightarrow D'$, $\beta' : D' \rightarrow E'$ and $\gamma' : E' \rightarrow D'$ as

$$\begin{aligned} \alpha' &:= \alpha|_{D'}, \\ \beta'(\alpha x) &:= \beta(x) + \text{Im} d, \text{ for all } x \in D \\ \gamma'(z + \text{Im} d) &:= \gamma(z) \in D', \text{ for all } z \in \ker d. \end{aligned} \quad (3.8)$$

It is easy to show, by diagram-chasing, that these maps are well-defined and that the following result holds (see [Rot09, Proposition 10.9] for the details):

Proposition 3.4. The diagram

$$\begin{array}{ccc} D' & \xrightarrow{\alpha'} & D' \\ & \swarrow \gamma' & \searrow \beta' \\ & E' & \end{array} \quad (3.9)$$

(with $D', E', \alpha', \beta', \gamma'$ defined as above) is an exact couple.

Definition 3.5. We call (3.9) the *derived couple* of the exact couple (3.7).

Of course it is possible to iterate the process of constructing the derived couple of an exact couple. For this reason we slightly modify the notations used so far and write $(D^1, E^1, \alpha^1, \beta^1, \gamma^1)$ for (3.7), $(D^2, E^2, \alpha^2, \beta^2, \gamma^2)$ for (3.9), and in general $(D^r, E^r, \alpha^r, \beta^r, \gamma^r)$ for the derived couple of $(D^{r-1}, E^{r-1}, \alpha^{r-1}, \beta^{r-1}, \gamma^{r-1})$, which we call the $(r-1)$ -derived couple of (3.7):

$$\begin{array}{ccc} D^r & \xrightarrow{\alpha^r} & D^r \\ & \swarrow \gamma^r & \searrow \beta^r \\ & E^r & \end{array} \quad (3.10)$$

There is a standard way to produce an exact couple from a filtration like (3.6). For every inclusion $F_{p-1} \subseteq F_p$ of chain complexes, the short exact sequence of chain complexes

$$0 \longrightarrow F_{p-1} \xrightarrow{i_{p-1}} F_p \xrightarrow{\pi_p} F_p/F_{p-1} \longrightarrow 0$$

(where i_{p-1} is the inclusion map and π_p is the natural projection) induces a long exact sequence in homology:

$$\cdots \rightarrow H_n(F_{p-1}) \xrightarrow{f_n^{p-1,p}} H_n(F_p) \xrightarrow{g_n^p} H_n(F_p/F_{p-1}) \xrightarrow{(\partial_p)_n} H_{n-1}(F_{p-1}) \rightarrow \cdots$$

(where $f_n^{p-1,p}, g_n^p$ are the maps induced in homology respectively by i_{p-1}, π_p , and ∂_p is the *connecting homomorphism*; see Theorem 1.13). We now define, using the standard notational convention $n := p + q$ like in the previous chapters, the vector spaces

$$\begin{aligned} D_{p,q} &:= H_n(F_p), \\ E_{p,q} &:= H_n(F_p/F_{p-1}), \end{aligned}$$

which we see as the components of the bigraded vector spaces $D := (D_{p,q})$ and $E := (E_{p,q})$, and the linear maps

$$\begin{aligned} \alpha_{p,q} &:= f_n^{p,p+1}, \\ \beta_{p,q} &:= g_n^p, \\ \gamma_{p,q} &:= (\partial_p)_n, \end{aligned}$$

which we see as the components of the bigraded maps $\alpha := (\alpha_{p,q})$, $\beta := (\beta_{p,q})$ and $\gamma := (\gamma_{p,q})$. We have then obtained an exact couple $(D^1, E^1, \alpha^1, \beta^1, \gamma^1)$ associated with the filtration (3.6), where all the vector spaces and the linear maps are bigraded. It is easy to see, rewriting the long exact sequence above as

$$\cdots \rightarrow D_{p-1,q+1}^1 \xrightarrow{\alpha_{p-1,q+1}^1} D_{p,q}^1 \xrightarrow{\beta_{p,q}^1} E_{p,q}^1 \xrightarrow{\gamma_{p,q}^1} D_{p-1,q}^1 \rightarrow \cdots,$$

that the bidegrees of the maps involved are as follows:

$$\begin{aligned}\deg \alpha^1 &= (1, -1), \\ \deg \beta^1 &= (0, 0), \\ \deg \gamma^1 &= (-1, 0).\end{aligned}$$

Furthermore, the bidegree of $d^1 = \beta^1 \gamma^1 : E^1 \rightarrow E^1$ is

$$\deg d^1 = \deg \beta^1 + \deg \gamma^1 = (-1, 0).$$

It holds in general, as one can easily verify recalling the definitions (3.8), that if $\deg \alpha^1 = (a, a')$, $\deg \beta^1 = (b, b')$, $\deg \gamma^1 = (c, c')$ then in the derived couple $\deg \alpha^2 = (a, a')$, $\deg \beta^2 = (b - a, b' - a')$, $\deg \gamma^2 = (c, c')$. Thus in our case we have, iterating the process:

$$\begin{aligned}\deg \alpha^r &= (1, -1), \\ \deg \beta^r &= (1 - r, r - 1), \\ \deg \gamma^r &= (-1, 0),\end{aligned}\tag{3.11}$$

and for $d^r = \beta^r \gamma^r : E^r \rightarrow E^r$

$$\deg d^r = \deg \beta^r + \deg \gamma^r = (-r, r - 1).\tag{3.12}$$

It is now clear that the spaces $E_{p,q}^r$ and the maps $d_{p,q}^r$ in the exact and derived couples associated with the filtration (3.6) form a spectral sequence (starting with the 1-page E^1): in particular, the bidegree (3.12) agrees with the bidegrees of the differentials in Definition 1.31, and the discussion before Proposition 3.4 ensures that the isomorphisms required in Definition 1.31 hold. Furthermore, it can be proven (see [Wei94, Theorem 5.9.4]) that this spectral sequence produced via exact and derived couples is naturally isomorphic to the spectral sequence (associated with the same filtration) we constructed more explicitly in Section 1.4.2.

Using the bidegrees of the maps, we can write the proper bigrading if we want to “unroll” the $(r - 1)$ -th derived couple (3.10) into long exact sequences:

$$\cdots \rightarrow D_{p+r-2, q-r+2}^r \xrightarrow{\alpha_{p+r-2, q-r+2}^r} D_{p+r-1, q-r+1}^r \xrightarrow{\beta_{p+r-1, q-r+1}^r} E_{p,q}^r \xrightarrow{\gamma_{p,q}^r} D_{p-1, q}^r \xrightarrow{\alpha_{p-1, q}^r} D_{p, q-1}^r \rightarrow \cdots\tag{3.13}$$

Convergence. As the filtration we are considering is finite, hence bounded (Definition 1.34), we can conclude by Theorem 1.36 that we have convergence:

$$E_{p,q}^\infty \cong \mathcal{F}_p H_n(C_*) / \mathcal{F}_{p-1} H_n(C_*),$$

for all p, q . Recall that

$$\mathcal{F}_p H_n(C_*) := \text{Im}(j_p : H_n(F_p C_*) \rightarrow H_n(C_*)),$$

where j_p is the map induced in homology by the inclusion $F_p C_* \hookrightarrow C_*$. With little effort, using again the finiteness of the filtration, one can prove that $E_{p,q}^r = E_{p,q}^\infty$ if $r \geq \max(p, N - p + 1)$.

For the purpose of convergence the focus is on the limit terms $E_{p,q}^\infty$ rather than on “non-limit terms” $E_{p,q}^r$; however, we now show that the whole collection of the terms $E_{p,q}^r$ is necessary to describe the relationship between spectral sequences and persistent homology.

3.3.2 Spectral sequences and persistent homology

We now want to clarify the relation between the persistent Betti numbers of a filtration (3.6) and the integers $\dim_{\mathbb{F}} E_{p,q}^r$, i.e. the dimensions as \mathbb{F} -vector spaces of the terms of the spectral sequence associated with (3.6). We follow [BP17].

First of all, consider the spaces $D_{p,q}^r$ which appear in the derived couples of (3.6). Recalling that $D^r = \alpha^{r-1} D^{r-1}$ by construction, and considering the bigrading, we have (setting $r = 2$) $D_{p,q}^2 = \alpha_{p-1,q+1}^1 D_{p-1,q+1}^1 = f_n^{p-1,p} H_n(F_{p-1}) = \text{Im } f_n^{p-1,p}$, which by definition is the persistent homology group $H_n^{p-1,p}$. With a simple inductive argument we obtain the following result:

Proposition 3.6 ([BP17]). For all integers p, q, r , with $r \geq 1$ and $n := p + q$,

$$D_{p,q}^r = \text{Im } f_n^{p-r+1,p} = H_n^{p-r+1,p}.$$

Now, recalling that the $(r - 1)$ -th derived couple $(D^r, E^r, \alpha^r, \beta^r, \gamma^r)$ can be “unrolled” into long exact sequences like in (3.13), and using Proposition 3.6 to rewrite the spaces $D_{p,q}^r$ in (3.13) as persistent homology groups, we obtain the long exact sequence

$$\dots \rightarrow H_n^{p-1,p+r-2} \xrightarrow{\alpha_{p+r-2,q-r+2}^r} H_n^{p,p+r-1} \rightarrow E_{p,q}^r \rightarrow H_{n-1}^{p-r,p-1} \xrightarrow{\alpha_{p-1,q}^r} H_{n-1}^{p-r+1,p} \rightarrow \dots \quad (3.14)$$

It is easy to see, using the definition of the maps α^r , that

$$\text{Im}(\alpha_{p+r-2,q-r+2}^r : H_n^{p-1,p+r-2} \rightarrow H_n^{p,p+r-1}) = H_n^{p-1,p+r-1}, \quad (3.15)$$

for any p, q, r .

We are now ready to show the relation between the persistent Betti numbers $\beta_n^{s,t}$ and the integers $\dim_{\mathbb{F}} E_{p,q}^r$, using the following simple fact about exact sequences:

Remark 3.7. In an exact sequence of finite dimensional \mathbb{F} -vector spaces

$$\dots \xrightarrow{f} U \xrightarrow{g} V \xrightarrow{h} W \xrightarrow{l} \dots$$

we can extract the short exact sequence $0 \rightarrow \text{Im } g \rightarrow V \rightarrow \text{Im } h \rightarrow 0$. From it, as $\text{Im } g \cong U / \ker g$ and $\text{Im } h = \ker l$, passing to the dimensions of the involved vector spaces we obtain the relation

$$\dim_{\mathbb{F}} V = (\dim_{\mathbb{F}} U - \dim_{\mathbb{F}} \text{Im } f) + (\dim_{\mathbb{F}} W - \dim_{\mathbb{F}} \text{Im } l).$$

Applying Remark 3.7 and (3.15) to (3.14) we get to the sought relation:

Theorem 3.8 ([BP17]). For all integers p, q, r with $r \geq 1$ and $n := p + q$ we have

$$\dim_{\mathbb{F}} E_{p,q}^r = (\beta_n^{p,p+r-1} - \beta_n^{p-1,p+r-1}) + (\beta_{n-1}^{p-r,p-1} - \beta_{n-1}^{p-r,p}). \quad (3.16)$$

3.3.3 Inverting the relation

We want now to “invert” the relation (3.16) in order to express every persistent Betti number $\beta_n^{s,t}$ as a combination of the dimensions $\dim_{\mathbb{F}} E_{p,q}^r$. The existence of an inverse relation intuitively would mean that the collections of integers $\{\beta_n^{s,t}\}$ and $\{\dim_{\mathbb{F}} E_{p,q}^r\}$ carry the same amount of topological information about the filtration. This fact can be easily deduced from the results in [BP17], but we state here a more direct formula.

In order to proceed by induction, we have to observe that $\beta_n^{s,t} = 0$ whenever $s \leq 0$ (by finiteness of the filtration) or $n < 0$ (by non-negativity of the chain complexes). Then, for instance, setting $n = 0$ and $p = 1$ in (3.16) we get $\beta_0^{1,r} = \dim_{\mathbb{F}} E_{1,-1}^r$ (for all $r \geq 1$), and by induction on p we obtain

$$\beta_0^{p,p+r-1} = \sum_{i=1}^p \dim_{\mathbb{F}} E_{i,-i}^{p+r-i}, \quad \text{for all } r \geq 1. \quad (3.17)$$

Now we can use (3.17) as base case and proceed by induction on n ; as the computations are simple, but long and tedious, we just state the result:

Proposition 3.9. For all n, p and for all $r \geq 1$ the following formulas hold:

$$\begin{aligned} \text{(i)} \quad \beta_n^{p,p+r-1} &= \sum_{i=1}^p \dim_{\mathbb{F}} E_{i,n-i}^{p+r-i} + N_p^{(n-1)}(r), \quad \text{where} \\ N_p^{(n-1)}(r) &:= \sum_{j=0}^{\lfloor \frac{p}{2} \rfloor - 1} \left(\beta_{n-1}^{p-r-2j,p-j} - \beta_{n-1}^{p-r-2j,p-1-j} \right), \\ \text{(ii)} \quad N_p^{(n-1)}(r) &= \sum_{i=1}^{p-r} \left(\dim_{\mathbb{F}} E_{i,n-i-1}^{p+1-i} - \dim_{\mathbb{F}} E_{i,n-i-1}^{p-i-\lfloor \frac{p-r-i}{2} \rfloor} \right) + \\ &\quad + \sum_{j=0}^{\lfloor \frac{p}{2} \rfloor - 1} \left(N_{p-r-2j}^{(n-2)}(r+j+1) - N_{p-r-2j}^{(n-2)}(r+j) \right). \end{aligned}$$

As $N_p^{(-1)}(r) = 0$, it is clear from Proposition 3.9 that every persistent Betti number can be computed from the dimensions $\dim_{\mathbb{F}} E_{p,q}^r$, even if expressing the relation in a non-recursive formula could be difficult. For example, while for $n = 1$ we have

$$\beta_1^{p,p+r-1} = \sum_{i=1}^p \dim_{\mathbb{F}} E_{i,1-i}^{p+r-i} + \sum_{i=1}^{p-r} \left(\dim_{\mathbb{F}} E_{i,-i}^{p+1-i} - \dim_{\mathbb{F}} E_{i,-i}^{p-i-\lfloor \frac{p-r-i}{2} \rfloor} \right)$$

for all $p, r \geq 1$, for $n = 2$ a similar formula is already quite complicated.

3.4 Spectral systems and persistence

We explain now how the previous results and constructions can be generalized to the case of spectral systems. After introducing a suitable generalization of the notion of exact couple, we describe how it can be applied to a particular class of posets. We detail then the case of multidimensional persistence, which can be seen as related to both \mathbb{Z}^m -filtrations and $D(\mathbb{Z}^m)$ -filtrations.

3.4.1 Generalized exact couples

In [Mat13, § 4] a generalized notion of exact couple is introduced, called an *exact couple system*. An exact couple system is a collection of exact triangles (particular “rolled up” long exact sequences) which induce a spectral system starting from the 1-page, that is containing only terms $S[z, s, p, b]$ with $z \leq s \leq p \leq b$.

In this section we assume that the poset I has a maximum ∞ and a minimum $-\infty$; since these elements can be added “artificially” to the poset when needed, the hypothesis is not restrictive.

Definition 3.10 ([Mat13]). An *exact couple system* over a poset (I, \leq) is a collection of abelian groups $(K_s^p)_{s \leq p}$ connected by homomorphisms $\ell_{p',s'}^{p,s} : K_s^p \rightarrow K_{s'}^{p'}$ whenever $p \leq p'$ and $s \leq s'$, and homomorphisms $k_{p,s} : K_s^p \rightarrow K_{-\infty}^s$, such that the following hold:

- (i) $\ell_{p'',s''}^{p',s'} \ell_{p',s'}^{p,s} = \ell_{p'',s''}^{p,s}$,
- (ii) The following triangles are exact:

$$\begin{array}{ccc} K_{-\infty}^s & \xrightarrow{\ell_{p,-\infty}^{s,-\infty}} & K_{-\infty}^p \\ & \swarrow k_{p,s} & \searrow \ell_{p,s}^{p,-\infty} \\ & K_s^p & \end{array}$$

- (iii) The following diagrams commute:

$$\begin{array}{ccc} K_s^p & \xrightarrow{k_{p,s}} & K_{-\infty}^s \\ \ell_{p',s'}^{p,s} \downarrow & & \downarrow \ell_{s',-\infty}^{s,-\infty} \\ K_{s'}^{p'} & \xrightarrow{k_{p',s'}} & K_{-\infty}^{s'} \end{array}$$

In an exact couple system over I , differentials $d_{psz} : K_s^p \rightarrow K_z^s$ can be defined as $d_{psz} := \ell_{s,z}^{s,-\infty} k_{p,s}$ for any $z \leq s \leq p$. Then, a *spectral system* over I can be defined as the collection of terms

$$S[z, s, p, b] := \frac{\ker(d_{psz} : K_s^p \rightarrow K_z^s)}{\text{Im}(d_{bps} : K_p^b \rightarrow K_s^p)}, \quad (3.18)$$

for any $z \leq s \leq p \leq b$. Notice that this definition of spectral system generalizes the one we used so far: as shown in [Mat13], if we start from an I -filtration $(F_i)_{i \in I}$ of a chain complex C_* , we can obtain an exact couple system by considering the graded modules $K_s^p := (H_n(F_p/F_s))_{n \in \mathbb{Z}}$, with the

morphisms $\ell_{p',s'}^{p,s}$ induced by inclusion and the morphisms $k_{p,s}$ induced by the differential d ; then, the resulting spectral system having terms (3.18) coincides with our usual definition of spectral system associated with an I -filtered chain complex (Definition 2.3). The properties of differentials and extensions of Section 2.2 can be proven in an abstract context for spectral systems arising from exact couple systems, as shown in [Mat13, § 4].

3.4.2 Spectral systems and rank invariant

We want to show that, under certain hypotheses (namely that the poset I is a partially ordered abelian group), it is possible to describe in an easy way some exact sequences contained in the exact couple system and, by similarity with the case of classical spectral sequences, deduce a relation between the terms of the spectral sequence and a generalization of the rank invariant. After presenting the result in its generality, we show in particular how it can be applied to multidimensional persistence.

Firstly, let us state the natural generalization of the rank invariant (Definition 3.2) that we will use in what follows. Given an I -filtration $(F_i)_{i \in I}$ and $v \leq w$ in I , we define

$$\beta_n(v, w) := \dim_{\mathbb{F}} \text{Im}(\ell : H_n(F_v) \rightarrow H_n(F_w)),$$

where ℓ is the map induced by the inclusion $F_v \hookrightarrow F_w$; we call *rank invariant* the collection of all $\beta_n(v, w)$, for any n and any $v \leq w$.

A *partially ordered abelian group* $(I, +, \leq)$ is an abelian group $(I, +)$ endowed with a partial order \leq that is *translation invariant*: for all $p, t, t' \in I$, if $t \leq t'$ then $p + t \leq p + t'$. For filtrations indexed over partially ordered abelian groups it is possible to construct the exact couple system and to describe quite simply its “derived” version, which plays a similar role to derived couples of an exact couple. Let $(I, +, \leq)$ be a partially ordered abelian group and let $(F_i)_{i \in I}$ be an I -filtration. The exact couple system associated is then composed by the following long exact sequences, one for each pair $s \leq p$, that we can see as the “unrolled” exact triangles of Definition 3.10:

$$\begin{aligned} \cdots \rightarrow S_n[-\infty, -\infty, s, s] \xrightarrow{\ell} S_n[-\infty, -\infty, p, p] \xrightarrow{\ell} S_n[s, s, p, p] \xrightarrow{k} S_{n-1}[-\infty, -\infty, s, s] \xrightarrow{\ell} \\ \xrightarrow{\ell} S_{n-1}[-\infty, -\infty, p, p] \rightarrow \cdots \end{aligned} \quad (3.19)$$

We can now proceed as we did in Section 3.3 to construct the derived couple of an exact couple. Remember that by Proposition 2.8, if $p \leq u$ and $b \leq t$ in I then

$$S_n[-\infty, -\infty, p, t] = \text{Im}(\ell : S_n[-\infty, -\infty, p, b] \rightarrow S_n[-\infty, -\infty, u, t]), \quad (3.20)$$

and in particular $S_n[-\infty, -\infty, p, t] = \text{Im}(\ell : H_n(F_p) \rightarrow H_n(F_t))$ which yields by definition $\dim_{\mathbb{F}} S_n[-\infty, -\infty, p, t] = \beta_n(p, t)$.

Now, if we focus on a pair of indices $s \leq p$ and denote $v := p - s$, among all the long exact sequences of the form (3.19) defined above we are interested in those whose indices satisfy the

hypothesis of Proposition 2.5: we consider then three 4-tuples of indices in I of the type

$$\begin{array}{ccccccc}
 & & & & p & \leq & p & \leq & p+v & \leq & p+v \\
 & & & & & & \parallel & & \parallel & & \\
 & & & & p-v & \leq & p-v & \leq & p & \leq & p \\
 & & & & & & \parallel & & \parallel & & \\
 p-2v & \leq & p-2v & \leq & p-v & \leq & p-v & & & &
 \end{array}$$

Imitating the construction of the derived couple of Section 3.3 and using Proposition 2.5 to determine homology and (3.20) to determine the terms $\text{Im } \ell$, we obtain a long exact sequence

$$\begin{aligned}
 \cdots \rightarrow S_n[-\infty, -\infty, p-v, p] \xrightarrow{\ell} S_n[-\infty, -\infty, p, p+v] \xrightarrow{\ell} S_n[p-2v, p-v, p, p+v] \xrightarrow{k} \\
 \xrightarrow{k} S_{n-1}[-\infty, -\infty, p-2v, p-v] \xrightarrow{\ell} S_{n-1}[-\infty, -\infty, p-v, p] \rightarrow \cdots
 \end{aligned}$$

which can be seen as part of the “derived” exact couple system. This construction clearly can be iterated, as for usual derived couples.

We can slightly modify our starting point, the family of long exact sequences of the form (3.19), in such a way that the involved indices keep satisfying the condition of Proposition 2.5. We do it by introducing an element $w \geq 0$ of I and modifying (3.19) into

$$\begin{aligned}
 \cdots \rightarrow S_n[-\infty, -\infty, p-v, p-v+w] \xrightarrow{\ell} S_n[-\infty, -\infty, p, p+w] \xrightarrow{\ell} S_n[p-v-w, p-v, p, p+w] \xrightarrow{k} \\
 \xrightarrow{k} S_{n-1}[-\infty, -\infty, p-v-w, p-v] \xrightarrow{\ell} S_{n-1}[-\infty, -\infty, p-w, p] \rightarrow \cdots,
 \end{aligned}$$

from which the construction of the derived couple can be carried out exactly as above. We summarize what we said in the following result.

Theorem 3.11. Let $(I, +, \leq)$ be a partially ordered abelian group, and let $(F_i)_{i \in I}$ be an I -filtration of chain complexes. Then, for any $v, w \in I$ such that $v, w \geq 0$ there is an exact sequence

$$\begin{aligned}
 \cdots \rightarrow S_n[-\infty, -\infty, p-v, p-v+w] \xrightarrow{\ell} S_n[-\infty, -\infty, p, p+w] \xrightarrow{\ell} S_n[p-v-w, p-v, p, p+w] \xrightarrow{k} \\
 \xrightarrow{k} S_{n-1}[-\infty, -\infty, p-v-w, p-v] \xrightarrow{\ell} S_{n-1}[-\infty, -\infty, p-w, p] \rightarrow \cdots \quad (3.21)
 \end{aligned}$$

which can be “derived” as

$$\begin{aligned}
 \cdots \rightarrow S_n[-\infty, -\infty, p-v, p+w] \xrightarrow{\ell} S_n[-\infty, -\infty, p, p+v+w] \xrightarrow{\ell} S_n[p-2v-w, p-v, p, p+v+w] \xrightarrow{k} \\
 \xrightarrow{k} S_{n-1}[-\infty, -\infty, p-2v-w, p-v] \xrightarrow{\ell} S_{n-1}[-\infty, -\infty, p-v-w, p] \rightarrow \cdots.
 \end{aligned}$$

Under the hypotheses of Theorem 3.11, using Remark 3.7 we immediately obtain the following numerical relation between the set $\{\dim_{\mathbb{F}} S_n[z, s, p, b]\}$ and the rank invariant $\{\beta_n(v, w)\}_{v \leq w \in I}$.

Corollary 3.12. Each long exact sequence of the form (3.21) yields the relation

$$\begin{aligned} \dim_{\mathbb{F}} S_n[p - v - w, p - v, p, p + w] &= \beta_n(p, p + w) - \beta_n(p - v, p + w) \\ &\quad + \beta_{n-1}(p - v - w, p - v) - \beta_{n-1}(p - v - w, p). \end{aligned}$$

3.4.3 The example of multidimensional persistence

Let us show how this result can be used in the context of multidimensional persistence to describe the behavior of the spectral systems and their relationship with the rank invariant. We consider two different partially ordered abelian groups which can be associated with a multidimensional filtration.

Starting from a (finite) \mathbb{Z}^m -filtration of simplicial complexes $(K_P)_{P \in \mathbb{Z}^m}$ (see Section 3.2) one can naturally define a \mathbb{Z}^m -filtration of chain complexes by setting $F_P := C_*(K_P)$, for all $P \in \mathbb{Z}^m$.

Corollary 3.13. The results of Theorem 3.11 apply to the case of multidimensional persistence by considering the partially ordered abelian group $(\mathbb{Z}^m, +, \leq)$ and the \mathbb{Z}^m -filtration $(F_P)_{P \in \mathbb{Z}^m}$. This gives us a relation (expressed by Corollary 3.12) between the rank invariant and the dimension of the terms of the spectral system over \mathbb{Z}^m .

Since the \mathbb{Z}^m -filtration $(F_P)_{P \in \mathbb{Z}^m}$ is finite, we can apply the arguments of Section 3.3.3 in order to obtain the inverse relation between the rank invariant and the dimension of the terms of the spectral system. Notice that the hypothesis of finiteness of the filtration $(F_P)_{P \in \mathbb{Z}^m}$ allows to start the inductive argument as in Section 3.3.3, since we have $\beta_n^{v,w} = 0$ whenever $v := (v_1, \dots, v_m) \in \mathbb{Z}^m$ has at least one coordinate ≤ 0 . We can conclude that the spectral system over \mathbb{Z}^m carries the same amount of topological information on the filtration as the rank invariant of Definition 3.2, as the collections $\{\dim_{\mathbb{F}} S[z, s, p, b]\}$ and $\{\beta_n^{v,w}\}$ can be determined one from the other.

Consider now the poset $D(\mathbb{Z}^m)$ of downsets of \mathbb{Z}^m (Definition 2.10). Let us show how there is a natural way (sketched also in [Mat13]) to construct a $D(\mathbb{Z}^m)$ -filtration of chain complexes from a (finite) \mathbb{Z}^m -filtration of simplicial complexes $(K_P)_{P \in \mathbb{Z}^m}$. Let B_P be the set of simplices that *enter the filtration* at $P \in \mathbb{Z}^m$, that is the simplices $\sigma \in K_P$ such that $\sigma \notin K_Q$, for any $Q < P$ (see Figure 3.2 for an example). Since \mathbb{Z}^m , for $m > 1$, is not totally ordered, a simplex σ can enter the filtration at several points $P_1, \dots, P_r \in \mathbb{Z}^m$. Then we set $F_p := C_*(\cup_{P \in p} B_P)$, for every $p \in D(\mathbb{Z}^m)$, obtaining a $D(\mathbb{Z}^m)$ -filtration $(F_p)_{p \in D(\mathbb{Z}^m)}$ associated with the filtration of simplicial complexes $(K_P)_{P \in \mathbb{Z}^m}$.

It is possible to endow $D(\mathbb{Z}^m)$ with an additive structure, defining the sum of $p, p' \in D(\mathbb{Z}^m)$ as $p + p' := \{P + P' \mid P \in p, P' \in p'\}$. Nevertheless, $(D(\mathbb{Z}^m), +, \subseteq)$ is not a partially ordered abelian group: as the zero element is the empty set, in general the inverse of an element does not exist.

The easiest way to construct a partially ordered abelian group starting from $D(\mathbb{Z}^m)$ is to consider the translation of a fixed downset. Define the *translation* of a downset $p \in D(\mathbb{Z}^m)$ by an element $v \in \mathbb{Z}^m$ as the standard notion of translation of a subset of \mathbb{Z}^m . If we fix a downset $p \in D(\mathbb{Z}^m)$ and denote T_p the family of all downsets of $D(\mathbb{Z}^m)$ obtained translating p by any $v \in \mathbb{Z}^m$, we see that $(T_p, \text{translation}, \subseteq)$ is a partially ordered abelian group.

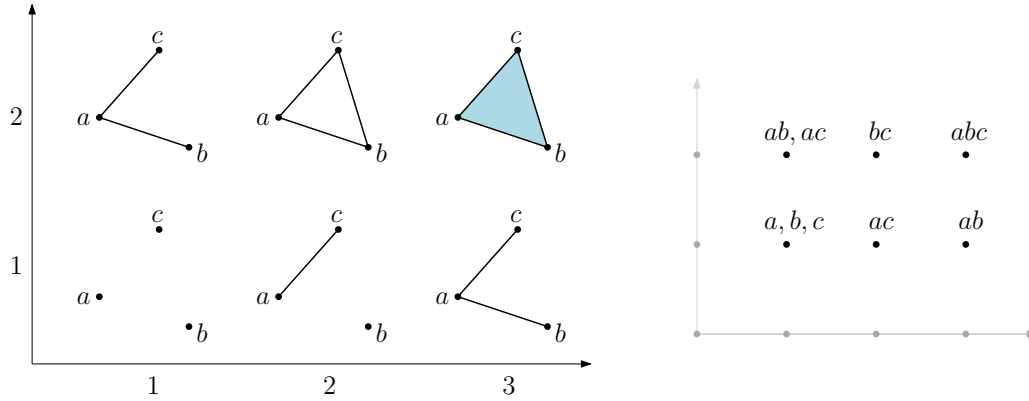


FIGURE 3.2: A \mathbb{Z}^2 -filtration of a triangle (left) and the simplices entering the filtration at each $P \in \mathbb{Z}^2$ (right). Notice, for example, that the edge ac enters the filtration at two different points: $(1, 2)$ and $(2, 1)$.

Corollary 3.14. The results of Theorem 3.11 apply to the case of multidimensional persistence by considering the partially ordered abelian group $(T_p, \text{translation}, \subseteq)$ and the $D(\mathbb{Z}^m)$ -filtration defined above, associated with the \mathbb{Z}^m -filtration of simplicial complexes $(K_v)_{v \in \mathbb{Z}^m}$.

We want now to show with a simple example that the spectral system over $D(\mathbb{Z}^m)$ associated with a filtration $(K_P)_{P \in \mathbb{Z}^m}$ carries more topological information than the rank invariant $\{\beta_n^{v,w}\}_{v \leq w \in \mathbb{Z}^m}$, as a consequence of the different underlying poset. Consider Figure 3.3, which represents a simplicial complex filtered over \mathbb{Z}^2 in two different ways. The rank invariant $\{\beta_n^{v,w}\}$ is unable to discriminate

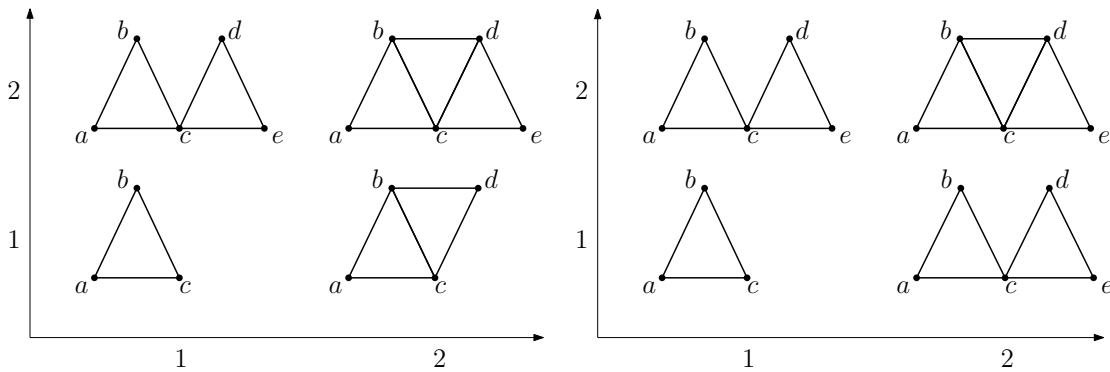
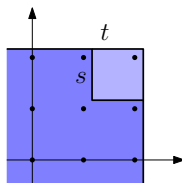


FIGURE 3.3: Two different filtrations over \mathbb{Z}^2 of a simplicial complex.

between the two filtrations (even if other invariants used in multidimensional persistence can distinguish the two cases). On the other hand, the use of the downset defined by the points $(1, 2)$ and $(2, 1)$ (that is, the smallest downset containing those points) allows to recognize the difference. Let us call s the downset defined by $(1, 2)$ and $(2, 1)$, and t the downset defined by the point $(2, 2)$. Moreover, let us denote $(F_p)_{p \in D(\mathbb{Z}^m)}$ the filtration of chain complexes associated with the filtration of



simplicial complexes on the left (denote $S[z, s, p, b]$ the terms of the associated spectral system), and $(F'_p)_{p \in D(\mathbb{Z}^m)}$ the filtration of chain complexes associated with the filtration of simplicial complexes on the right (denote $S'[z, s, p, b]$ the terms of the associated spectral system). We can easily see that, for example, $S_1[s, s, t, t] = H_1(F_t/F_s) = 0$ and $S'_1[s, s, t, t] = H_1(F'_t/F'_s) \cong \mathbb{F}$; in other words, $\dim_{\mathbb{F}} S_1[s, s, t, t] = 0$ and $\dim_{\mathbb{F}} S'_1[s, s, t, t] = 1$. The set $\{\dim_{\mathbb{F}} S_n[z, s, p, b]\}$ represents therefore a more accurate summary of the topological information of the filtration $(K_P)_{P \in \mathbb{Z}^m}$ than the set $\{\beta_n^{v,w}\}_{v \leq w \in \mathbb{Z}^m}$.

3.4.4 Conclusions and future work

We have illustrated the relation between the terms of a spectral system over I and a generalization of the rank invariant for I -filtrations. As we have seen, one of the advantages of the description via generalized exact couples is that, in addition to a purely numerical relation, it yields an algebraic relation in the form of long exact sequences involving terms of the spectral system, many of which are of the form $\text{Im } \ell$ for some map $\ell : H_n(F_s) \rightarrow H_n(F_t)$ induced by inclusion.

In the case of multidimensional persistence, from a filtration of simplicial complexes $(K_P)_{P \in \mathbb{Z}^m}$ one can construct in an obvious way a \mathbb{Z}^m -filtration $(F_P)_{P \in \mathbb{Z}^m}$; as we have shown, the associated spectral system carries the same topological information as the rank invariant. Associated with the filtration $(K_P)_{P \in \mathbb{Z}^m}$ there is also a natural $D(\mathbb{Z}^m)$ -filtration $(F_p)_{p \in D(\mathbb{Z}^m)}$; in this case, as we have seen, the associated spectral system carries more topological information than the rank invariant $\{\beta_n^{v,w}\}_{v \leq w \in \mathbb{Z}^m}$. For this reason, one can think to consider the dimensions of the terms of the spectral system over $D(\mathbb{Z}^m)$ as a topological invariant associated with the filtration $(K_P)_{P \in \mathbb{Z}^m}$. In Chapter 4 we describe an algorithm to compute the terms of a spectral system; furthermore, we introduce a method to increase computational efficiency, based on discrete vector fields. Even if this improvement usually allows to considerably speed up computations, in many cases the computation of the whole spectral system over $D(\mathbb{Z}^m)$ appears to be infeasible. A stimulating challenge (and a theme for future research) is to identify a subset of indices in $D(\mathbb{Z}^m)$ which allows to extract sufficient information at reasonable computational costs.

Chapter 4

Effective homology for generalized spectral sequences

Standard algorithms for computing the homology of a chain complex $C_* = (C_n, d_n)_{n \in \mathbb{Z}}$ are based on matrix reduction: each differential d_n is expressed as a matrix from which, through a diagonalization process, one can determine the homology groups of C_* . A necessary condition to express the differentials in matrix form is that each chain group C_n is finitely generated. Nonetheless, many fundamental constructions in algebraic topology give rise to chain complexes which may have some infinitely generated chain groups.

Effective homology [Ser94] is a method developed to compute homology in situations which cannot be tackled with standard algorithms. As we will explain, this technique has been used to design a software for computations in algebraic topology which has obtained remarkable results. As illustrated in [RRS06], the effective homology technique can also be applied to develop algorithms computing spectral sequences. In this respect, it is worth pointing out that a spectral sequence, in general, is not an *algorithm* allowing to compute iteratively the successive pages of a given one, since knowing $(E_{p,q}^r, d_{p,q}^r)$ one can determine $(E_{p,q}^{r+1})$ but not always $(d_{p,q}^{r+1})$. In other words, despite the differentials of each page are mathematically defined, the fact that their definition is not constructive prevents from seeing spectral sequences as general algorithms that can be implemented in a computer. In some significant situations, for instance whenever infinitely generated chain complexes are involved, the effective homology technique is the only known way to obtain actual algorithms for the computation of spectral sequences.

In this chapter we explain how the effective homology technique can be applied to spectral systems over a poset. Our study is complemented by the implementation of algorithms to compute spectral systems, presented in [GR18]. After introducing the basics of effective homology in Section 4.1, we review in Section 4.2 how the method can be used to compute spectral sequences. In this context, besides presenting the general case of spectral sequences associated with \mathbb{Z} -filtered chain complexes, we focus on the Serre spectral sequence of a fibration as a relevant example. In Section 4.3 we prove some theoretical results necessary to understand how the effective homology method carries over to the case of spectral systems. Section 4.4 is devoted to the Serre spectral system, a generalization of the Serre spectral sequence for towers of fibrations. In the last sections of the chapter we treat the computational aspects: in Section 4.5 we explain how discrete vector fields can be used to improve the efficiency of the algorithms computing spectral systems; in Section 4.6

we describe the algorithms we developed to compute spectral systems using effective homology, including examples and details on the implementation.

In this chapter we always consider homology with coefficients in \mathbb{Z} . Furthermore, all the chain complexes are assumed to be free.

4.1 Effective homology

Effective homology, introduced by Francis Sergeraert in [Ser87, Ser94] and explained in details in [RS97, RS02, RS06], is a technique designed to compute the homology of complicated spaces. In particular, it allows to develop algorithms computing the homology of some infinitely generated chain complexes, i.e. chain complexes such that some chain groups are not finitely generated. A key notion in this context is that of *effective chain complex*, which is (intuitively) a chain complex that can be handled with standard algorithms for determining bases, differentials and homology. If a chain complex can be associated (in a way we will make precise) with an effective one, the effective homology technique allows to use the latter in order to make computations on the former. Sergeraert and his coworkers developed a software, called *Kenzo*, to carry out computations with effective homology.

In this section we introduce the basic definitions and results necessary to understand the theory of effective homology. We then state the homological perturbation theorems (the so-called Trivial and Basic Perturbation Lemmas), important results allowing to use the effective homology technique in some “difficult” cases such as loop spaces [Rub91] or total spaces of fibrations (see Section 4.2). We conclude the section presenting some features of the system *Kenzo*.

4.1.1 Definitions and basic results

We introduce the fundamental concepts of the effective homology method, following mainly [RS06].

Definition 4.1. An *effective chain complex* is a free chain complex of \mathbb{Z} -modules $C_* = (C_n, d_n)_{n \in \mathbb{Z}}$ such that each group C_n is finitely generated, and there exists an algorithm which returns, for each degree n , a (distinguished) \mathbb{Z} -basis for C_n .

When a chain complex $C_* = (C_n, d_n)_{n \in \mathbb{Z}}$ is effective, each differential $d_n : C_n \rightarrow C_{n-1}$ can be expressed as a finite integer matrix. Several algorithms are available which compute, for instance, the subgroups $\ker d_n$ and $\text{Im } d_{n+1}$ of n -cycles and n -boundaries, and the homology groups $H_n(C_*)$. These are examples of *global* information about the chain complex C_* .

Unlike the case of an effective chain complex, if some group C_n has an infinite number of generators, global information about C_* cannot be easily computed in general. For example, the subgroups $\ker d_n$ and $\text{Im } d_{n+1}$ could be infinitely generated, and hence cannot be expressed by means of a finite list of generators. Nevertheless, in some cases *local* information can be computed, for example if there is an algorithm to compute the boundary of any given element; when this happens we call C_* a *locally effective chain complex*.

The theory of *effective homology* [Ser87, Ser94] consists in a set of methods to associate a locally effective chain complex with an effective one via chain equivalences, in order to allow the

use of standard algorithms to compute homology. We now introduce *reductions* and *strong chain equivalences*, the two fundamental tools of effective homology.

Definition 4.2. A *reduction* ρ between two chain complexes C_* and D_* , denoted by $\rho : C_* \rightrightarrows D_*$, is a triple of maps $\rho = (f, g, h)$

$$\begin{array}{ccc} & & \\ & \curvearrowright & \\ & h & \\ & \downarrow & \\ C_* & \begin{array}{c} \xrightarrow{f} \\ \xleftarrow{g} \end{array} & D_* \end{array}$$

where f and g are chain maps, h is a morphism of graded modules of degree $+1$, and the following identities hold:

- (i) $fg = \text{id}_{D_*}$,
- (ii) $d_C h + h d_C = \text{id}_{C_*} - gf$,
- (iii) $fh = hg = hh = 0$.

The relations (i) and (ii) tell that f (as well as g) is a particular case of chain equivalence between C_* and D_* , with $h : \text{id}_{C_*} \simeq gf$ and $\text{id}_{D_*} = fg$ (in other words, id_{D_*} and fg are homotopic via the null map). This would be enough to guarantee, by virtue of Corollary 1.11, the isomorphism of the homology groups $H_n(C_*)$ and $H_n(D_*)$, for each n . However, in the current situation an even stronger result holds, concerning the chain complexes themselves rather than the homology groups. The identities of Definition 4.2 express indeed the fact that C_* can be decomposed direct sum of D_* and an acyclic chain complex. More precisely, we have $C_* = \ker f \oplus \text{Im } g$, with $\text{Im } g \cong D_*$ and $H_*(\ker f) = 0$.

Let us point out that the relevance of the condition (iii) is marginal: it can be shown (see for instance [RS97]) that a triple $\rho = (f, g, h)$ satisfying (i) and (ii) can be slightly modified, in a canonical way, into a triple (f, g, h') that satisfies (iii) as well.

When we have a reduction $\rho : C_* \rightrightarrows D_*$ we call informally C_* the *big* chain complex and D_* the *small* chain complex. This nomenclature reflects the application we have in mind for reductions: they represent a bridge between a chain complex whose homology is hard to compute (because it is infinitely generated or it has a huge number of generators) and a chain complex whose homology can be determined by means of an algorithm. A deeper reason for this terminology lies of course in the decomposition $C_* = \ker f \oplus \text{Im } g \cong \ker f \oplus D_*$ mentioned above, which allows to see D_* as a chain subcomplex of C_* .

Given a chain complex C_* , the *trivial reduction* $\rho = (f, g, h) : C_* \rightrightarrows C_*$ is the one such that $f = g = \text{id}$ and $h = 0$. We now show that reductions have a “good behavior” with respect to composition and tensor product.

Proposition 4.3. Let $\rho = (f, g, h) : C_* \rightrightarrows D_*$ and $\rho' = (f', g', h') : D_* \rightrightarrows E_*$ be two reductions. Then a reduction $\rho'' = (f'', g'', h'') : C_* \rightrightarrows E_*$ is given by:

$$\begin{aligned} f'' &:= f' f \\ g'' &:= g g' \\ h'' &:= h + g h' f \end{aligned}$$

Proposition 4.4. Let $\rho = (f, g, h) : C_* \rightrightarrows D_*$ and $\rho' = (f', g', h') : C'_* \rightrightarrows D'_*$ be two reductions. Then a reduction $\rho'' = (f'', g'', h'') : C_* \otimes C'_* \rightrightarrows D_* \otimes D'_*$ is given by:

$$\begin{aligned} f'' &:= f \otimes f' \\ g'' &:= g \otimes g' \\ h'' &:= h \otimes \text{id}_{C'_*} + (gf) \otimes h' \end{aligned}$$

Reductions are employed to define another tool which connects two chain complexes and establishes isomorphisms between their homology groups: the *strong chain equivalence*.

Definition 4.5. A *strong chain equivalence* $\varepsilon : C_* \iff D_*$ between the chain complexes C_* and D_* is a triple $\varepsilon = (B_*, \rho_1, \rho_2)$, where B_* is a chain complex and ρ_1, ρ_2 are reductions from B_* respectively to C_* and D_* :

$$\begin{array}{ccc} & B_* & \\ \rho_1 \swarrow & & \searrow \rho_2 \\ C_* & & D_* \end{array}$$

Note that, like reductions, strong chain equivalences are particular cases of chain equivalences. In what follows, when it doesn't generate confusion we will refer to strong chain equivalences simply as *equivalences*.

Strong chain equivalences can be concatenated, as stated in the following result.

Proposition 4.6. Let $\varepsilon : A_* \iff C_*$ and $\varepsilon' : C_* \iff E_*$ be two equivalences, as in the following scheme:

$$\begin{array}{ccc} & B_* & \\ \rho_1 \swarrow & & \searrow \rho_2 \\ A_* & & C_* \end{array} \quad \begin{array}{ccc} & D_* & \\ \rho'_1 \swarrow & & \searrow \rho'_2 \\ C_* & & E_* \end{array}$$

Then an equivalence $\varepsilon'' : A_* \iff E_*$ can be constructed:

$$\begin{array}{ccc} & Z_* & \\ \rho''_1 \swarrow & & \searrow \rho''_2 \\ A_* & & E_* \end{array}$$

From the computational point of view, it is important to point out that the chain complex Z_* can be built in a canonical way from the chain complexes of ε and ε' , just like the reductions ρ''_1, ρ''_2 can be constructed from $\rho_1, \rho_2, \rho'_1, \rho'_2$. We refer the reader to [RS06, Ch. 8] for the details of these constructions.

We are now ready to describe one the main objects of our study in this chapter, namely chain complexes C_* such that there is an equivalence $\varepsilon : C_* \iff D_*$ between C_* and an effective chain complex D_* . We say that such chain complexes *have effective homology*, or that they are chain complexes *with effective homology*. By extension, if X is an object from which a chain complex $C_*(X)$ can be defined, and if $C_*(X)$ has effective homology, we say that X is an *object with effective homology*.

Using the software Kenzo (see Section 4.1.3) we will be able to handle simplicial sets with effective homology, i.e. simplicial sets K such that there is an equivalence $\varepsilon : C_*(K) \iff D_*$ where D_* is an effective chain complex.

4.1.2 Homological perturbation theorems

We devote this section to two important theorems, which describe how a reduction $\rho : C_* \rightrightarrows D_*$ is affected by a *perturbation*, i.e. a modification of the differential in one of the chain complexes. When the perturbation concerns the small chain complex D_* we have the so-called Trivial Perturbation Lemma, a simple but quite powerful result. If the perturbation is instead applied to the big chain complex C_* , the situation is much more complicated, and it is clarified in the Basic Perturbation Lemma, a fundamental theorem in homological algebra introduced by Weishu Shih [Shi62] and stated in the abstract modern form by Ronald Brown [Bro67].

Definition 4.7. Let $C_* = (C_n, d_n)_{n \in \mathbb{Z}}$ be a chain complex. A *perturbation* δ of the differential d is a family of morphisms $\delta = \{\delta_n : C_n \rightarrow C_{n-1}\}_{n \in \mathbb{Z}}$ such that the sum $d + \delta$ is again a differential, that is $(d + \delta)^2 = 0$ holds.

We call $C'_* = (C_n, d_n + \delta_n)_{n \in \mathbb{Z}}$ the *perturbed* chain complex obtained from C_* by introducing the perturbation δ .

Theorem 4.8 (Trivial Perturbation Lemma). Let $C_* = (C_n, d_{C_n})_{n \in \mathbb{Z}}$ and $D_* = (D_n, d_{D_n})_{n \in \mathbb{Z}}$ be two chain complexes, $\rho = (f, g, h) : C_* \rightrightarrows D_*$ a reduction, and δ_D a perturbation of the differential d_D . Then a reduction $\rho' = (f', g', h') : C'_* \rightrightarrows D'_*$ exists, where:

- (i) $C'_* = (C_*, d_C + g\delta_D f)$ is the perturbed chain complex obtained from C_* by introducing the perturbation $g\delta_D f$;
- (ii) $D'_* = (D_*, d_D + \delta_D)$ is the perturbed chain complex obtained from D_* by introducing the perturbation δ_D ;
- (iii) the maps of the new reduction $\rho' = (f', g', h')$ are given by $f' := f$, $g' := g$, $h' := h$.

As we see, the perturbation δ_D of the small chain complex D_* transfers to the big chain complex C_* in the simplest possible way, by using the morphisms f and g , and the maps of the new reduction ρ' are unchanged. By contrast, if a perturbation δ_C is introduced in the chain complex C_* , we need an additional hypothesis to ensure that a reduction still exists between the perturbed version of C_* and a perturbed version of D_* , and the involved maps are more complicated.

Theorem 4.9 (Basic Perturbation Lemma, [Bro67]). Let $C_* = (C_n, d_{C_n})_{n \in \mathbb{Z}}$ and $D_* = (D_n, d_{D_n})_{n \in \mathbb{Z}}$ be two chain complexes, $\rho = (f, g, h) : C_* \rightrightarrows D_*$ a reduction, and δ_C a perturbation of the differential d_C . Suppose that the composition $h\delta_C$ satisfies the following *nilpotency condition*: for every $x \in C_*$ there exists a non-negative integer $m = m(x) \in \mathbb{N}$ such that $(h\delta_C)^m(x) = 0$. Then a reduction $\rho' = (f', g', h') : C'_* \rightrightarrows D'_*$ exists, where:

- (i) $C'_* = (C_*, d_C + \delta_C)$ is the perturbed chain complex obtained from C_* by introducing the perturbation δ_C ;

- (ii) $D'_* = (D_*, d_D + \delta_D)$ is the perturbed chain complex obtained from D_* by introducing the perturbation $\delta_D := f\delta_C\varphi g = f\psi\delta_C g$;
- (iii) the maps of the new reduction $\rho' = (f', g', h')$ are given by

$$\begin{aligned} f' &:= f\psi = f(\text{id}_{C_*} - \delta_C\varphi h), \\ g' &:= \varphi g, \\ h' &:= \varphi h = h\psi, \end{aligned}$$

with the operators φ and ψ given by

$$\begin{aligned} \varphi &:= \sum_{i=0}^{\infty} (-1)^i (h\delta_C)^i, \\ \psi &:= \sum_{i=0}^{\infty} (-1)^i (\delta_C h)^i = \text{id}_{C_*} - \delta_C\varphi h, \end{aligned}$$

the convergence of these series being guaranteed by the nilpotency condition.

Let us briefly illustrate the main use of these perturbation theorems, and in particular of the Basic Perturbation Lemma, in the context of effective homology. Suppose that we have a reduction $\rho : C_* \Rightarrow D_*$ from a locally effective chain complex to an effective one, which allows (as we have said, and will see in details) the computation of the homology of C_* by handling the “smaller” chain complex D_* . The complex C_* usually corresponds to some object X (for example a simplicial set) in algebraic topology, and the reduction ρ is a tool we exploit for making the computation easier. If a different object X' can be seen as a “perturbed version” of X , meaning that $C'_* := C_*(X')$ is a perturbed chain complex obtained from $C_*(X)$, then the Basic Perturbation Lemma provides us with an adequate tool to compute the homology of C'_* , namely a reduction to an effective chain complex D'_* . This is in fact a quite common situation, and we will see an example in Section 4.2.

4.1.3 The software Kenzo

Kenzo is a symbolic computation software for Algebraic Topology and Homological Algebra implemented in about 16,000 lines of Common Lisp code [Gra96]. Written by Francis Sergeraert and collaborators, it is available at <https://www-fourier.ujf-grenoble.fr/~sergerar/Kenzo/>, together with its documentation [RSS99] and examples.

Kenzo can handle algebraic objects such as chain complexes and simplicial sets, and functions between them, such as chain maps, simplicial maps, reductions and strong chain equivalences. As we mentioned before, the computation of homology is based on the notion of object with effective homology. When a chain complex is effective, standard diagonalization algorithms applied to the matrices associated with the differentials can easily determine its homology groups (see [KMM04]). On the other hand, when a chain complex is only locally effective and has effective homology, Kenzo is able to exploit the constructions of Section 4.1.1 to compute its homology, making a strong use of

functional programming. Thanks to its capability to handle infinitely generated chain complexes, Kenzo managed to carry out computations of some homology groups which were still unknown.

Let us introduce some fundamental features of the software Kenzo with an example adapted from [RRS06]. Consider the Eilenberg-MacLane space $K(\mathbb{Z}, 1)$ (see [EM53] and [May67, Ch. V]), which can be seen as a simplicial set with $K(\mathbb{Z}, 1)_n = \mathbb{Z}^n$, for all n . The space $K(\mathbb{Z}, 1)$ is in fact a simplicial abelian group, whose simplices are represented in Kenzo using the *bar* notation: each n -simplex is seen as a list of integers $[a_1 | a_2 | \dots | a_n]$, which are all non-null if and only if the simplex is non-degenerate. The face maps are defined by

$$\begin{aligned}\partial_0[a_1 | a_2 | \dots | a_n] &= [a_2 | a_3 | \dots | a_n] \\ \partial_i[a_1 | a_2 | \dots | a_n] &= [a_1 | \dots | a_{i-1} | a_i + a_{i+1} | \dots | a_n] \quad \text{for } 1 < i < n \\ \partial_n[a_1 | a_2 | \dots | a_n] &= [a_1 | a_2 | \dots | a_{n-1}]\end{aligned}$$

and the degeneracy maps are defined by

$$\eta_i[a_1 | a_2 | \dots | a_n] = [a_1 | \dots | a_i | 0 | a_{i+1} | \dots | a_n]$$

for all $0 \leq i \leq n$; for more details see [RSS99, § 11.4]. Let us see how to construct and handle $K(\mathbb{Z}, 1)$ with Kenzo:

```
> (setf kz1 (k-z 1))
[K1 Abelian-Simplicial-Group]
```

The Kenzo function `k-z` allows to construct Eilenberg-MacLane spaces over \mathbb{Z} ; in our case, the object $K(\mathbb{Z}, 1)$ is assigned to the symbol `kz1`. It belongs to the Kenzo class `ABELIAN-SIMPLICIAL-GROUP`, which is a subclass of `CHAIN-COMPLEX`. If we consider for instance the 3-simplex $[7 | -2 | 2]$ of $K(\mathbb{Z}, 1)$, we can compute its faces by

```
> (dotimes (i 4)
  (print (face kz1 i 3 '(7 -2 2))))
<AbSm - (-2 2)>
<AbSm - (5 2)>
<AbSm 1 (7)>
<AbSm - (7 -2)>
nil
```

Notice that the face $\partial_2[7 | -2 | 2] = \eta_1[7]$ is degenerate. Determining the faces of a given simplex is an example of *local* information which can be easily computed. By contrast, for objects like $K(\mathbb{Z}, 1)$ we may be unable to compute *global* information such as a basis of $K(\mathbb{Z}, 1)_3 = \mathbb{Z}^3$:

```
> (basis kz1 3)
Error: The object [K1 Abelian-Simplicial-Group] is locally-effective.
```

Since the required basis is the set of all non-degenerate simplices in dimension 3, that is the (infinite) subset of \mathbb{Z}^3 formed by the points with non-null coordinates, it cannot be stored as an

explicit list of elements. This fact clearly prevents us from using the standard matrix reduction algorithms for computing the homology groups of $K(\mathbb{Z}, 1)$. Therefore we have to rely on effective homology:

```
> (efhm kz1)
[K22 Homotopy-Equivalence K1 <= K1 => K16]
```

The function `efhm` reveals that Kenzo, in order to compute the homology of $K(\mathbb{Z}, 1)$, constructs a reduction to an object `K16`, which is simply the circle S^1 :

```
> (orgn (k 16))
(circle)
```

Kenzo uses thus a theoretical result, namely the existence of a reduction $C_*(K(\mathbb{Z}, 1)) \Rightarrow C_*(S^1)$, to justify the use of $C_*(S^1)$ to compute the homology of $K(\mathbb{Z}, 1)$. The circle S^1 is represented in Kenzo with its minimal simplicial model, whose basis (in dimension 0, 1, 2) is

```
>(dotimes (i 3)
  (print (basis (k 16) i)))
(*)
(s1)
nil
nil
```

The homology of the finitely generated chain complex $C_*(S^1)$ can be easily determined with standard algorithms. For what we just said, it coincides with the result given by Kenzo if we ask for the homology of $K(\mathbb{Z}, 1)$:

```
> (homology kz1 0 3)
Homology in dimension 0 :
Component Z
----done----

Homology in dimension 1 :
Component Z
----done----

Homology in dimension 2 :
----done----
```

4.2 Computing spectral sequences

In this section we explain how the effective homology technique can be employed to compute spectral sequences. We start from the general case of a spectral sequence arising from a \mathbb{Z} -filtered chain complex: if the chain complex is infinitely generated, the effective homology method is necessary to determine the associated spectral sequence with an algorithm. As we will see, using effective homology one cannot always compute every page of a spectral sequence; sometimes one has to settle

for determining it from a certain page on, depending on the equivalence (or reduction) between the original chain complex and the effective one. We then introduce the Serre spectral sequence associated with a fibration. We explain, following [RRS06], how the Basic Perturbation Lemma plays an important role in computing the homology of the total space of a fibration, a fact which allows to compute the Serre spectral sequence by introducing a suitable \mathbb{Z} -filtration. In this context, homological perturbation theorems are also useful to assess the accuracy of the algorithms based on effective homology.

The considerations and results of this section led to the development by Ana Romero of a Kenzo module to compute spectral sequences, available at <http://www.unirioja.es/cu/anromero/research2.html>.

Our main references for this section are [RRS06, Rom07].

4.2.1 Effective homology and spectral sequences

Let us explain how a spectral sequence associated with a filtered chain complex can be computed via an algorithm. We consider at first an *effective* chain complex $C_* = (C_n, d_n)_{n \in \mathbb{Z}}$, which we suppose to be \mathbb{Z} -filtered by a filtration $F = (F_p)_{p \in \mathbb{Z}}$. Since we are assuming C_* to be a free chain complex, each chain group C_n has a finite basis $\{g_1^{(n)}, \dots, g_{b_n}^{(n)}\}$. Each differential $d_n : C_n \cong \mathbb{Z}^{b_n} \rightarrow C_{n-1} \cong \mathbb{Z}^{b_{n-1}}$ can therefore be represented as a $b_{n-1} \times b_n$ matrix D_n , with respect to the bases $\{g_1^{(n)}, \dots, g_{b_n}^{(n)}\}$ of C_n and $\{g_1^{(n-1)}, \dots, g_{b_{n-1}}^{(n-1)}\}$ of C_{n-1} . Recall now that the terms of the spectral sequence associated with (C_*, F) are given by

$$E_{p,q}^r = \frac{A_{p,q}^r + F_{p-1}C_n}{d_{n+1}(A_{p+r-1,q-r+2}^{r-1}) + F_{p-1}C_n}, \quad (4.1)$$

with $A_{p,q}^r := \{c \in F_p C_n \mid d_n(c) \in F_{p-r} C_{n-1}\}$. Each group in the expression of $E_{p,q}^r$ can be determined via elementary matrix operations on (submatrices of) D_n and D_{n+1} . For example, the subgroup $A_{p,q}^r$ can be determined as the kernel (which can be computed via the Smith Normal Form technique described in [KMM04]) of a submatrix of D_n , whose definition depends on the filtration F and on the indices p, q, r . As we will illustrate more in depth a similar algorithm in Section 4.6 for the case of spectral systems over a poset I , we address the reader to [RRS06, Rom07] for further details on this algorithm. We just mention that, knowing the generators of all the subgroups in (4.1), one can use again the Smith Normal Form technique to obtain a *basis-divisors* representation of $E_{p,q}^r$, that is a list of generators (c_1, \dots, c_t) and a list of integers $(a_1, \dots, a_k, 0, \dots, 0)$ (with $t = k + \beta$), where a_1, \dots, a_k are the torsion coefficients of $E_{p,q}^r$ and β is its rank (recall Theorem 1.19).

Let us consider now a \mathbb{Z} -filtered chain complex (C_*, F) such that C_* is only *locally effective*. If it has effective homology, we know that an equivalence $C_* \Leftarrow B_* \Rightarrow D_*$ between C_* and an effective chain complex D_* can be used to compute the homology of C_* . However, in order to use the equivalence to compute the spectral sequence associated with (C_*, F) we have to consider a further ingredient, namely filtrations. Clarifying the behavior of reductions between filtered chain complexes represents therefore a fundamental step to understand how the effective homology techniques can be employed to compute spectral sequences. We state a result from [RRS06].

Definition 4.10. Let (C_*, F) and (D_*, F') be \mathbb{Z} -filtered chain complexes. We say that a map $h : (C_*, F) \rightarrow (D_*, F')$ has *order* $\leq t$ if $h(F_p C_*) \subseteq F'_{p+t} D_*$, for every $p \in \mathbb{Z}$.

Theorem 4.11 ([RRS06]). Let $\rho = (f, g, h) : C_* \rightrightarrows D_*$ be a reduction and let F and F' be \mathbb{Z} -filtrations respectively of the chain complexes C_* and D_* . Suppose that the maps f and g are compatible with the filtrations and that the homotopy h has order $\leq t$. Then the induced morphism of spectral sequences $f : E(C_*, F) \rightarrow E(D_*, F')$ gives isomorphisms of bigraded modules $f^r : E(C_*, F)^r \cong E(D_*, F')^r$ between the r -pages of the two spectral sequences, for all $r > t$. In particular, $E(C_*, F)_{p,q}^r \cong E(D_*, F')_{p,q}^r$, for all $p, q \in \mathbb{Z}$ and $r > t$.

The theorem means that if we are able to compute the spectral sequence of the filtered chain complex (D_*, F') we also know the spectral sequence of (C_*, F) , but only from the $(t + 1)$ -page on.

When we deal with an equivalence we can simply apply Theorem 4.11 twice (once for each reduction) to obtain the following result:

Corollary 4.12. Let (C_*, F^C) be a \mathbb{Z} -filtered chain complex and suppose that there is an equivalence

$$\begin{array}{ccc} & B_* & \\ \rho_1 \swarrow & & \searrow \rho_2 \\ C_* & & D_* \end{array}$$

with $\rho_1 = (f_1, g_1, h_1)$ and $\rho_2 = (f_2, g_2, h_2)$. Suppose that the chain complexes B_* and D_* are also \mathbb{Z} -filtered, with filtrations F^B and F^D , that the maps f_1, g_1, f_2, g_2 are compatible with the filtrations and that the homotopies h_1, h_2 have both order $\leq t$. Then the spectral sequences associated with (C_*, F^C) and (D_*, F^D) are isomorphic from the $(t + 1)$ -page on; in particular, $E(C_*, F^C)_{p,q}^r \cong E(D_*, F^D)_{p,q}^r$, for all $p, q \in \mathbb{Z}$ and $r > t$.

When C_* is a chain complex with effective homology and $C_* \leftarrow B_* \rightrightarrows D_*$ is an equivalence between C_* and an effective chain complex D_* satisfying the hypotheses above, Corollary 4.12 ensures that we can compute $E(C_*, F^C)^r$, for any $r > t$, using the effective chain complex D_* for our computations (this is in fact the method implemented in [RRS06] as a new module for the Kenzo system). On the other hand, in general we are not able to exploit the equivalence $C_* \leftarrow B_* \rightrightarrows D_*$ to determine the “low” pages $E(C_*, F^C)^r$ with $r \leq t$.

4.2.2 The Serre spectral sequence

In this section we introduce the classical Serre spectral sequence [Ser51] associated with a fibration. We present it in a simplicial framework, which was first described by Shih in his memoir [Shi62]. We begin by recalling some definitions from [May67, Ch. IV].

Definition 4.13. The *Cartesian product* $X \times Y$ of two simplicial sets X and Y is the simplicial set whose set of n -simplices is $(X \times Y)_n := X_n \times Y_n$, with coordinate-wise defined face and degeneracy maps: if $(x, y) \in (X \times Y)_n$, then

$$\begin{aligned} \partial_i(x, y) &:= (\partial_i x, \partial_i y), & 0 \leq i \leq n; \\ \eta_i(x, y) &:= (\eta_i x, \eta_i y), & 0 \leq i \leq n. \end{aligned}$$

Definition 4.14. A *twisting operator* from a simplicial set B to a simplicial group G is a map $\tau : B \rightarrow G$ of degree -1 , that is a collection of maps $\tau = \{\tau_n : B_n \rightarrow G_{n-1}\}_{n \geq 1}$, satisfying the following identities, for any dimension n and for any $b \in B_n$:

$$\begin{aligned}\partial_i(\tau b) &= \tau(\partial_i b), & i < n - 1, \\ \partial_{n-1}(\tau b) &= \tau(\partial_n b)^{-1} \cdot \tau(\partial_{n-1} b), \\ \eta_i(\tau b) &= \tau(\eta_i b), & i \leq n - 1, \\ e_n &= \tau(\eta_n b),\end{aligned}$$

where e_n is the identity element of G_n .

We chose to define twisting operators in a slightly different (yet equivalent) way from [May67], in order to agree with the definition implemented in the Kenzo system.

Given a simplicial group G , a simplicial set B and a twisting operator $\tau : B \rightarrow G$, the *twisted (Cartesian) product* $E(\tau) := G \times_\tau B$ is the simplicial set whose set of n -simplices is $E(\tau)_n = (G \times_\tau B)_n := G_n \times B_n$ and whose face and degeneracy maps are defined in the following way: if $(g, b) \in (G \times_\tau B)_n$, then

$$\begin{aligned}\partial_i(g, b) &:= (\partial_i g, \partial_i b), & 0 \leq i < n, \\ \partial_n(g, b) &:= (\tau(b) \cdot \partial_n g, \partial_n b); \\ \eta_i(g, b) &:= (\eta_i g, \eta_i b), & 0 \leq i \leq n.\end{aligned}$$

It can be easily shown that the identities defining a twisting operator τ are equivalent to the simplicial identities of $G \times_\tau B$.

Definition 4.15. Given a simplicial group G , a simplicial set B and a twisting operator $\tau : B \rightarrow G$, we call (*principal*) *fibration* a sequence $G \hookrightarrow E \rightarrow B$, where $E := E(\tau) = G \times_\tau B$ and the maps are the inclusion as first factor $G \hookrightarrow G \times_\tau B$ and the projection on the second factor $G \times_\tau B \rightarrow B$. In a fibration, B is called the *base*, G is called the *fiber* and E is called the *total space*.

We can now state Serre's famous theorem, which makes clear the relation between the homology of the base, the fiber and the total space of a fibration.

Theorem 4.16 ([Ser51]). Let $G \hookrightarrow E \rightarrow B$ be a fibration and suppose the base B is a 1-reduced simplicial set, that is to say both B_0 and B_1 have as unique element the basepoint \star . Then there is a first quadrant spectral sequence converging to the homology $H_*(E)$ of the total space whose second page is given by

$$E_{p,q}^2 = H_p(B; H_q(G)).$$

The Serre spectral sequence can be constructed from a filtered chain complex, as we will explain in Section 4.2.4. Here we want to stress the fact that, in general, the Serre spectral sequence is not an *algorithm* to compute $H_*(E)$ from $H_*(B)$ and $H_*(G)$. A first reason is that the usual extension problems for spectral sequences may prevent to determine the homology groups $H_n(E)$ from the terms $E_{p,q}^\infty$ of the limit page. But there is a second (and sometimes bigger) problem, which may

make it impossible to even determine the groups $E_{p,q}^\infty$: knowing the terms of the r -page ($E_{p,q}^r$) does not necessarily imply the possibility to determine the terms of the $(r+1)$ -page ($E_{p,q}^{r+1}$), as the differentials $d_{p,q}^r$ must be provided too. In fact, knowing the terms of a certain page but not the differentials is a quite common situation: for instance, Serre's theorem gives a formula for the terms of the 2-page, but it does not provide the differentials $d_{p,q}^2$. Serre himself encountered this problem while using the spectral sequence to study the homotopy groups of spheres. Even if sometimes some extra information (for example, in the case of the homotopy of spheres, the multiplicative structure of cohomology) can allow to determine the differentials and continue with the computation, in other situations one might be unable to proceed further.

On the contrary, as we will see in what follows, the effective homology technique allows to actually compute $H_*(E)$, together with each page ($E_{p,q}^r$) of the Serre spectral sequence (for $r \geq 2$), when B and G have effective homology.

4.2.3 Effective homology of a twisted product

Let $G \hookrightarrow E \rightarrow B$ be a fibration, where the total space E is the twisted product $G \times_\tau B$, with $\tau : B \rightarrow G$ a twisting operator. Suppose further, like in the hypothesis of Theorem 4.16, that the base B is a 1-reduced simplicial set. We assume that the simplicial sets G and B have effective homology, i.e. there exist equivalences

$$\begin{array}{ccc} & \hat{C}G_* & \\ \swarrow & & \searrow \\ C_*(G) & & DG_* \end{array} \quad \begin{array}{ccc} & \hat{C}B_* & \\ \swarrow & & \searrow \\ C_*(B) & & DB_* \end{array} \quad (4.2)$$

where DG_* and DB_* are effective chain complexes, and we explain how an equivalence between $C_*(G \times_\tau B)$ and an effective chain complex can be constructed. The idea is to begin by following the proof of the Twisted Eilenberg-Zilber theorem, a result obtained by applying the Basic Perturbation Lemma (Theorem 4.9) to the classical Eilenberg-Zilber reduction. In this section we recall these results (presented in [RRS06]) and show, for the sake of our computational interests, how every object and map involved can be constructed.

In order to state the definitions and the result correctly, we suppose, for the rest of the section, that all the chain complexes associated with simplicial sets are normalized (see Section 1.3). In other words, in the present section the notation $C_*(K)$ for a chain complex has to be interpreted as representing the normalized chain complex $C_*^N(K)$.

As mentioned, we start from the Eilenberg-Zilber theorem, which describes the relation between the chain complex of a Cartesian product of simplicial sets $C_*(G \times B)$ and the tensor product $C_*(G) \otimes C_*(B)$.

Theorem 4.17 (Eilenberg-Zilber, [EZ53]). For any simplicial sets G and B there exists a reduction

$$\rho = (f, g, h) : C_*(G \times B) \rightrightarrows C_*(G) \otimes C_*(B).$$

The maps f, g, h , which are called respectively the *Alexander-Whitney*, *Eilenberg-MacLane* and *Shih maps*, are defined as follows:

$$\begin{aligned} f(x_n, y_n) &:= \sum_{i=0}^n \partial_{i+1} \cdots \partial_n x_n \otimes \partial_0 \cdots \partial_{i-1} y_n, \\ g(x_p \otimes y_q) &:= \sum_{(\alpha, \beta) \in \{(p, q)\text{-shuffles}\}} (-1)^{\text{sg}(\alpha, \beta)} (\eta_{\beta_q} \cdots \eta_{\beta_1} x_p, \eta_{\alpha_p} \cdots \eta_{\alpha_1} y_q), \\ h(x_n, y_n) &:= \sum_{\substack{0 \leq q \leq n-1, \\ 0 \leq p \leq n-q-1, \\ (\alpha, \beta) \in \{(p+1, q)\text{-shuffles}\}}} (-1)^{n-p-q+\text{sg}(\alpha, \beta)} (\eta_{\beta_q+n-p-q} \cdots \eta_{\beta_1+n-p-q} \eta_{n-p-q-1} \partial_{n-q+1} \cdots \partial_n x_n, \\ &\quad \eta_{\alpha_{p+1}+n-p-q} \cdots \eta_{\alpha_1+n-p-q} \partial_{n-p-q} \cdots \partial_{n-q-1} y_n), \end{aligned}$$

where a (p, q) -shuffle $(\alpha, \beta) = (\alpha_1, \dots, \alpha_p, \beta_1, \dots, \beta_q)$ is defined as a permutation of the set $\{0, 1, \dots, p+q-1\}$ such that $\alpha_i < \alpha_{i+1}$ and $\beta_j < \beta_{j+1}$, and $\text{sg}(\alpha, \beta) := \sum_{i=1}^p (\alpha_i - i - 1)$.

As we are interested in studying the twisted product $G \times_\tau B$ instead of the Cartesian product $G \times B$, we recall that the only difference between them lies in the face operators (see Definition 4.14). Consequently, the chain complexes $C_*(G \times B)$ and $C_*(G \times_\tau B)$ have the same underlying graded group, but different differentials. The differential of $C_*(G \times_\tau B)$ can be seen as a perturbed version of the differential of $C_*(G \times B)$, where the perturbation is given, for any $(g, b) \in G_n \times B_n$, by

$$\delta(g, b) := (-1)^n [(\tau(b) \cdot \partial_n g, \partial_n b) - (\partial_n g, \partial_n b)].$$

A very important result, known as the twisted Eilenberg-Zilber theorem [Bro59], is obtained applying the Basic Perturbation Lemma to the Eilenberg-Zilber reduction $\rho = (f, g, h) : C_*(G \times B) \Rightarrow C_*(G) \otimes C_*(B)$. This yields a new reduction $C_*(G \times_\tau B) \Rightarrow C_*(G) \otimes_t C_*(B)$, where $C_*(G) \otimes_t C_*(B)$ is a perturbed version of $C_*(G) \otimes C_*(B)$ called the *twisted tensor product* of $C_*(G)$ and $C_*(B)$.

Of course, in order to apply the Basic Perturbation Lemma, one has to verify that the composition $h\delta$ satisfies the nilpotency condition. In order to show this, we consider the *degeneracy degree* of the second factor in $G \times B$, defined in the following way: an element $b \in B_n$ has degeneracy degree $\leq p$ if $b = \eta_{i_{n-p}} \cdots \eta_{i_1} b_p$ for some $b_p \in B_p$. As a direct consequence of the definitions, the map h preserves the degeneracy degree of the factor B , while δ decreases it at least by 1. It is then clear that the nilpotency condition is satisfied.

Theorem 4.18 (Twisted Eilenberg-Zilber, [Bro59]). Let G, B be simplicial sets and let $\tau : B \rightarrow G$ be a twisting operator. Then there exists a reduction

$$\rho' = (f', g', h') : C_*(G \times_\tau B) \Rightarrow C_*(G) \otimes_t C_*(B),$$

where $C_*(G) \otimes_t C_*(B)$ is the perturbed chain complex obtained from $C_*(G) \otimes C_*(B)$ by introducing the perturbation ‘‘induced’’ (via the application of the Basic Perturbation Lemma) by δ .

Let us stress once again that, for computational purposes, knowing the explicit definition of the maps f, g, h of Theorem 4.17 is fundamental, as it allows to actually construct the maps f', g', h' of Theorem 4.18 using the operators φ, ψ defined in the statement of the Basic Perturbation Lemma.

Similarly, in what follows whenever we apply the Basic or Trivial Perturbation Lemma to obtain a reduction, we are able to construct every object and every map involved, making it possible to turn these theoretical results into an algorithm for actual computations.

Since our goal for the present section is to show how the homology of $C_*(G \times_\tau B)$ can be computed using effective homology, the reduction $\rho' = (f', g', h') : C_*(G \times_\tau B) \Rightarrow C_*(G) \otimes_t C_*(B)$ of the twisted Eilenberg-Zilber theorem is not enough. However, it is the first important step of the method we are presenting, whose second step consists in connecting $C_*(G) \otimes_t C_*(B)$ to an effective chain complex via an equivalence.

As we said, the chain complex $C_*(G) \otimes_t C_*(B)$ is obtained from $C_*(G) \otimes C_*(B)$ by introducing a perturbation $\bar{\delta}$ induced by δ which, recalling (ii) of the statement of Theorem 4.9, can be explicitly expressed as $\bar{\delta} := f\delta\varphi g = f\psi\delta g$. Now, we want to keep “propagating” the perturbation $\bar{\delta}$ through reductions of chain complexes. Specifically, we are interested in the couple of reductions ρ_1, ρ_2 of the equivalence

$$\begin{array}{ccc}
 & \hat{C}G_* \otimes \hat{C}B_* & \\
 \rho_1 \swarrow & & \searrow \rho_2 \\
 C_*(G) \otimes C_*(B) & & DG_* \otimes DB_*
 \end{array}$$

between $C_*(G) \otimes C_*(B)$ and the *effective* chain complex $DG_* \otimes DB_*$, obtained from the equivalences (4.2) by applying twice Proposition 4.4.

Consider the reduction ρ_1 . We perturb $C_*(G) \otimes C_*(B)$ by introducing the perturbation $\bar{\delta}$, which gives us the chain complex $C_*(G) \otimes_t C_*(B)$. Applying the Trivial Perturbation Lemma (Theorem 4.8), we obtain a perturbed version $\hat{C}G_* \otimes_t \hat{C}B_*$ of $\hat{C}G_* \otimes \hat{C}B_*$, with a perturbation $\hat{\delta}$ induced by $\bar{\delta}$.

Consider now the reduction ρ_2 . We perturb $\hat{C}G_* \otimes \hat{C}B_*$ by introducing the perturbation $\hat{\delta}$, which gives us the chain complex $\hat{C}G_* \otimes_t \hat{C}B_*$. Applying the Basic Perturbation Lemma (after checking that the nilpotency assumption is satisfied), we obtain a perturbed version $DG_* \otimes_t DB_*$ of the effective chain complex $DG_* \otimes DB_*$. As $DG_* \otimes_t DB_*$ is clearly still effective, we have finally achieved our goal to construct an effective chain complex which allows to compute the homology of $C_*(G \times_\tau B)$.

In order to fill the small gap in the exposed method, let us show that the nilpotency assumption is satisfied for the reduction $\rho_2 = (f_2, g_2, h_2)$. Define the *second factor degree* of an element $\sum_i c_i \otimes d_i$ of a chain complex $C_* \otimes D_*$ as the maximum p such that some d_i belongs to D_p . Under the assumption that B is a 1-reduced simplicial set, the perturbation $\hat{\delta}$ decreases the second factor degree at least by 2 (as a consequence of the results in [Bro59]; see [RS06, Prop. 132] for a direct and elementary proof), while the homotopy h_2 increases it at most by 1 (as a consequence of the definition of tensor product of reductions; we will detail this fact in Lemma 4.22 and Lemma 4.34). Thus, the nilpotency condition follows from the fact that $h_2\hat{\delta}$ strictly decreases the second factor degree.

The results presented in this section can be summarized in the following way: starting from the equivalences

$$\begin{array}{ccccc}
 & & C_*(G \times B) & & \hat{C}G_* \otimes \hat{C}B_* \\
 & \swarrow \text{id} & \searrow \rho & \swarrow \rho_1 & \searrow \rho_2 \\
 C_*(G \times B) & & C_*(G) \otimes C_*(B) & & DG_* \otimes DB_*
 \end{array}$$

the perturbation δ of the chain complex $C_*(G \times B)$ can be propagated through the reductions using the Basic/Trivial Perturbation lemmas, to obtain

$$\begin{array}{ccccc}
 & & C_*(G \times_\tau B) & & \hat{C}G_* \otimes_t \hat{C}B_* \\
 & \swarrow \text{id} & \searrow \rho' & \swarrow \rho'_1 & \searrow \rho'_2 \\
 C_*(G \times_\tau B) & & C_*(G) \otimes_t C_*(B) & & DG_* \otimes_t DB_*
 \end{array}$$

where $DG_* \otimes_t DB_*$ is effective. Notice that, by virtue of Proposition 4.6, an equivalence connecting directly $C_*(G \times_\tau B)$ and $DG_* \otimes_t DB_*$ can be constructed.

4.2.4 Computing the Serre spectral sequence

We can now use the results of the previous section to describe a method to compute the Serre spectral sequence of a fibration $G \hookrightarrow E \rightarrow B$, where $E = G \times_\tau B$ and we suppose that B is 1-reduced and that G, B have effective homology.

We have seen how the chain complex $C_*(G \times_\tau B)$ can be connected through a series of reductions to the effective chain complex $DG_* \otimes_t DB_*$, which can be used to compute its homology. Now, we want to introduce “naturally defined” filtrations for all the chain complexes involved, exploiting them to compute the sought Serre spectral sequence. The theoretical results of Section 4.2.1 will complete the picture, describing the accuracy of the method.

First of all we define two types of filtration: a filtration F^\times of a chain complex canonically associated with a Cartesian product of simplicial sets, and a filtration F^\otimes of a tensor product of two chain complexes.

Definition 4.19. Let X, Y be simplicial sets and consider the chain complex $C_*(X \times Y)$. We define a filtration F^\times of $C_*(X \times Y)$ looking at the degeneracy degree of the second component:

$$F_p^\times(C_n(X \times Y)) := \{c = \sum_i \lambda_i(x_i, y_i) \in C_n(X \times Y) \mid \text{the degeneracy degree of } y_i \text{ is } \leq p, \text{ for every } i\},$$

where the degeneracy degree of $y \in Y_n$ is equal to p if $y = \eta_{i_{n-p}} \cdots \eta_{i_1} y_p$ for some non-degenerate $y_p \in Y_p$.

Definition 4.20. Let C_*, D_* be chain complexes, and consider their tensor product $C_* \otimes D_*$. We define a filtration F^\otimes of $C_* \otimes D_*$ looking at the second factor degree:

$$F_p^\otimes(C_* \otimes D_*)_n := \bigoplus_{i \leq p} C_{n-i} \otimes D_i.$$

Of course the definitions of F^\times and F^\otimes apply verbatim to perturbed chain complexes, as long as they identify filtrations of chain complexes. This is the case of the chain complexes we are considering (that is: $C_*(G \times_\tau B)$, $C_*(G) \otimes_t C_*(B)$, $\hat{C}G_* \otimes_t \hat{C}B_*$ and $DG_* \otimes_t DB_*$), as the considerations of Section 4.2.3 imply that the respective perturbations (that is: $\delta, \bar{\delta}, \dots$) always decrease the filtration index. Let us also point out that the filtrations F^\times and F^\otimes are always bounded.

In Section 4.2.2 we stated Serre's famous Theorem 4.16, but we did not define explicitly the Serre spectral sequence of a fibration $G \hookrightarrow E \rightarrow B$ of simplicial sets. Now, the filtration F^\times just introduced makes the task easier.

Definition 4.21. The *Serre spectral sequence* of a fibration $G \hookrightarrow E \rightarrow B$, where $E = G \times_\tau B$ and B is 1-reduced, is defined as the spectral sequence associated with the filtration F^\times of $C_*(G \times_\tau B) = C_*(E)$.

Consider then the filtered chain complex $(C_*(G \times_\tau B), F^\times)$. As mentioned, we filter also the other chain complexes appearing in the equivalences

$$C_*(G \times_\tau B) \begin{array}{c} \xleftarrow{\text{id}} \\ \xrightarrow{\text{id}} \end{array} C_*(G \times_\tau B) \begin{array}{c} \xrightarrow{\rho'} \\ \xleftarrow{\rho'} \end{array} C_*(G) \otimes_t C_*(B) \begin{array}{c} \xleftarrow{\rho'_1} \\ \xrightarrow{\rho'_1} \end{array} \hat{C}G_* \otimes_t \hat{C}B_* \begin{array}{c} \xrightarrow{\rho'_2} \\ \xleftarrow{\rho'_2} \end{array} DG_* \otimes_t DB_*$$

using of course filtrations of the type F^\otimes , and study their behavior with respect to the reductions.

Let us state a useful result from [Rom07], which we will generalize and prove in what follows (see Lemma 4.34):

Lemma 4.22 ([Rom07]). Consider two reductions $\rho = (f, g, h) : C_* \rightrightarrows D_*$ and $\rho' = (f', g', h') : C'_* \rightrightarrows D'_*$ and their tensor product $\tilde{\rho} = (\tilde{f}, \tilde{g}, \tilde{h}) : C_* \otimes C'_* \rightrightarrows D_* \otimes D'_*$. Then the maps \tilde{f} and \tilde{g} are compatible with the filtrations of the type F^\otimes defined on $C_* \otimes C'_*$ and $D_* \otimes D'_*$, while the homotopy \tilde{h} has order ≤ 1 with respect to the same filtrations.

Looking at the scheme of successive reductions above, we can make the following observations:

- Consider the two equivalences separately, starting from the left one, which is essentially the reduction $\rho' = (f', g', h') : C_*(G \times_\tau B) \rightrightarrows C_*(G) \otimes_t C_*(B)$ of the twisted Eilenberg-Zilber theorem. As it is obtained by applying the Basic Perturbation Lemma to the Eilenberg-Zilber reduction $\rho = (f, g, h) : C_*(G \times B) \rightrightarrows C_*(G) \otimes C_*(B)$, we begin analyzing the behavior of the maps f, g, h with respect to the filtrations F^\times of $C_*(G \times B)$ and F^\otimes of $C_*(G) \otimes C_*(B)$. Using their explicit definition, it can be shown that f, g, h are compatible with the filtrations. The chain complex $C_*(G \times_\tau B)$ is obtained from $C_*(G \times B)$ by introducing the perturbation δ , which (rephrasing the observations of Section 4.2.3 on the degeneracy degree) decreases by one the index of the filtration F^\times . This fact implies that also the maps f', g', h' , defined as in Theorem 4.9, are compatible with the filtrations F^\times of $C_*(G \times_\tau B)$ and F^\otimes of $C_*(G) \otimes_t C_*(B)$. Then by Theorem 4.11 we can conclude that

$$E(C_*(G \times_\tau B), F^\times)_{p,q}^r \cong E(C_*(G) \otimes_t C_*(B), F^\otimes)_{p,q}^r, \quad \text{for all } r \geq 1. \quad (4.3)$$

- Consider now the right equivalence, composed by the reductions $\rho'_1 = (f'_1, g'_1, h'_1)$ and $\rho'_2 = (f'_2, g'_2, h'_2)$. They are the perturbed versions, obtained by means of the Trivial/Basic Perturbation Lemmas, of the reductions $\rho_1 = (f_1, g_1, h_1)$ and $\rho_2 = (f_2, g_2, h_2)$, to which we can apply Lemma 4.22 to conclude that f_1, g_1, f_2, g_2 are compatible with the filtrations of the type F^\otimes , while the homotopies h_1, h_2 have order ≤ 1 . Recalling how the maps involved in the Trivial/Basic Perturbation Lemmas are defined, it is immediate to see that introducing the perturbations the situation does not change: f'_1, g'_1, f'_2, g'_2 are compatible with the filtrations of the type F^\otimes , while the homotopies h'_1, h'_2 have order ≤ 1 . Then by Corollary 4.12

$$E(C_*(G) \otimes_t C_*(B), F^\otimes)_{p,q}^r \cong E(DG_* \otimes_t DB_*, F^\otimes)_{p,q}^r, \quad \text{for all } r \geq 2. \quad (4.4)$$

Putting together the isomorphisms (4.3) and (4.4) gives

$$E(C_*(G \times_\tau B), F^\times)_{p,q}^r \cong E(DG_* \otimes_t DB_*, F^\otimes)_{p,q}^r, \quad \text{for all } r \geq 2. \quad (4.5)$$

As the spectral sequence of the effective chain complex $DG_* \otimes_t DB_*$ is computable, (4.5) guarantees that it can be used to compute, from the second page on, the Serre spectral sequence of the fibration $G \hookrightarrow E \rightarrow B$.

4.3 Effective homology and spectral systems: theoretical results

In this section we generalize the theoretical results of Section 4.2.1 to the case of spectral systems over a poset. We prove a generalization of Theorem 4.11 for I -filtered chain complexes, illustrating then its consequences in the case of some particularly relevant posets. The results presented here allow to assess the accuracy of the algorithms based on effective homology we will present in Section 4.6. More in general, if an I -filtered chain complex is connected to an effective one via a sequence of reductions, they allow to establish which terms of the two spectral systems associated with them are isomorphic.

We begin by proving a useful generalization of Proposition 3.5 of [Mac63], which deals with \mathbb{Z} -filtrations, to the case of filtrations over any poset. Let (C_*, F) and (D_*, F') be I -filtered chain complexes, where I is a poset. In what follows we denote by $(S[z, s, p, b])$ and $(S'[z, s, p, b])$ the spectral systems associated respectively with (C_*, F) and (D_*, F') .

Proposition 4.23. Let $\alpha, \beta : (C_*, F) \rightarrow (D_*, F')$ be two chain maps compatible with the filtrations, and let $h : \alpha \simeq \beta$ be a chain homotopy. Given four indices $z \leq s \leq p \leq b$ in I , let $\alpha^{z,s,p,b}$ and $\beta^{z,s,p,b}$ be the morphisms from $S[z, s, p, b]$ to $S'[z, s, p, b]$ induced respectively by α and β (see Section 2.3). Then

$$\alpha^{z,s,p,b} = \beta^{z,s,p,b} : S[z, s, p, b] \rightarrow S'[z, s, p, b]$$

whenever the following conditions on the homotopy h hold:

$$\begin{aligned} h(F_z) &\subseteq F'_s \\ h(F_p) &\subseteq F'_b. \end{aligned} \quad (4.6)$$

Proof. Remember that

$$S[z, s, p, b] = \frac{F_p \cap d^{-1}(F_z) + F_s}{d(F_b) \cap F_p + F_s},$$

$$S'[z, s, p, b] = \frac{F'_p \cap d^{-1}(F'_z) + F'_s}{d(F'_b) \cap F'_p + F'_s},$$

where we have expressed the quotients (just on this occasion) without the usual convention of notation, in order to have denominators which are subgroups of the numerators. Now define $\gamma := \alpha - \beta$, which yields $h : \gamma \simeq 0$, and prove that $\gamma^{z,s,p,b} = 0$ (i.e. the map induced by γ between $S[z, s, p, b]$ and $S'[z, s, p, b]$ is the zero map). Let $c \in F_p \cap d^{-1}(F_z)$ and consider

$$\gamma c = d_D h c + h d_C c. \quad (4.7)$$

Focus on the two summands of the second member separately. We can immediately say that $h d_C c \in F'_s$, because $d_C c \in F_z$ and, by hypothesis, the first condition of (4.6) holds. On the other hand, knowing that $h c \in F'_b$ by the second condition of (4.6) and observing that $d_D h c = \gamma c - h d_C c \in F'_p$, we conclude that $h c \in F'_b \cap d^{-1}(F'_p)$. Thus $\gamma c \in d(F'_b) \cap F'_p + F'_s$, so it represents the zero class in $S'[z, s, p, b]$. \square

From the proof it is clear that (4.6) are not the “sharpest” sufficient conditions that imply the thesis, as we have actually used only the assumptions

$$h(d(F_p) \cap F_z) \subseteq F'_s$$

$$h(F_p \cap d^{-1}(F_z)) \subseteq F'_b.$$

Nevertheless, as the conditions (4.6) are enough for our purposes, we have decided to employ them to state Proposition 4.23 in a simpler way.

Now we use Proposition 4.23 to prove a result on isomorphic terms in spectral systems. We will see that, in the case of classical \mathbb{Z} -filtrations, it reduces to Theorem 4.11.

Theorem 4.24. Let $\rho = (f, g, h) : C_* \Rightarrow D_*$ be a reduction between the I -filtered chain complexes (C_*, F) and (D_*, F') , and suppose that f and g are compatible with the filtrations. Then, given four indices $z \leq s \leq p \leq b$ in I , the map f induces an isomorphism

$$f^{z,s,p,b} : S[z, s, p, b] \rightarrow S'[z, s, p, b]$$

whenever the homotopy $h : (C_*, F) \rightarrow (C_{*+1}, F)$ satisfies the conditions

$$h(F_z) \subseteq F_s$$

$$h(F_p) \subseteq F_b. \quad (4.8)$$

Proof. Remember the following properties of a reduction:

- $fg = \text{id}_D$,
- $gf + d_C h + h d_C = \text{id}_C$.

The first property implies that, for any 4-tuple of indices $z \leq s \leq p \leq b$ in I , we have the induced maps $(fg)^{z,s,p,b} = (\text{id}_D)^{z,s,p,b}$. Then, by functoriality,

$$f^{z,s,p,b} g^{z,s,p,b} = \text{id}_{S'[z,s,p,b]}.$$

The second property implies that $h : gf \simeq \text{id}_C$. Then, Proposition 4.23 yields $(gf)^{z,s,p,b} = (\text{id}_C)^{z,s,p,b}$ whenever h satisfies the conditions (4.8). Therefore, again by functoriality,

$$g^{z,s,p,b} f^{z,s,p,b} = \text{id}_{S[z,s,p,b]}, \quad \text{whenever } h \text{ satisfies (4.8).}$$

□

It is easy to observe that the conditions (4.8) are automatically satisfied for every choice of indices $z \leq s \leq p \leq b$ if also the homotopy h is compatible with the filtration. In this particular case, the spectral systems arising from (C_*, F) and (D_*, F') are therefore isomorphic:

Corollary 4.25. Let $\rho = (f, g, h) : C_* \Rightarrow D_*$ be a reduction between the I -filtered chain complexes (C_*, F) and (D_*, F') , and suppose that the maps f, g, h are compatible with the filtrations. Then the map f induces isomorphisms

$$f^{z,s,p,b} : S[z, s, p, b] \rightarrow S'[z, s, p, b]$$

for any 4-tuple of indices $z \leq s \leq p \leq b$ in I .

We now analyze how Theorem 4.24 applies in some relevant situations ($I = \mathbb{Z}, \mathbb{Z}^m, D(\mathbb{Z}^m)$). In some cases the conditions (4.8) can be rephrased in a more intuitive way, for example using the concept of order of the homotopy h .

Classical spectral sequences. Consider two \mathbb{Z} -filtered chain complexes (C_*, F) , (D_*, F') and the spectral sequences respectively associated with them: $(E(C_*, F)_{p,q}^r)$ and $(E(D_*, F')_{p,q}^r)$, with $p, q \in \mathbb{Z}$ and $r \geq 1$. Remember that, by definition, we have $E(C_*, F)_{p,q}^r = S_n[p-r, p-1, p, p+r-1]$ and $E(D_*, F')_{p,q}^r = S'_n[p-r, p-1, p, p+r-1]$ for all p, q, r , where $n := p+q$ is the total degree, which in this section we prefer to drop for readability. We want now to state the conditions (4.8), which in this case are

$$\begin{aligned} h(F_{p-r}) &\subseteq F_{p-1} \\ h(F_p) &\subseteq F_{p+r-1}, \end{aligned} \tag{4.9}$$

in terms of the *order* of the homotopy h (Definition 4.10). The conditions (4.9) hold for all $p \in \mathbb{Z}$ and $r \geq 1$ if and only if the homotopy h has order $\leq r-1$. It is then easy to see Theorem 4.11 as a corollary of Theorem 4.24.

Spectral systems over \mathbb{Z}^m . Suppose now that (C_*, F) and (D_*, F') are two \mathbb{Z}^m -filtered chain complexes, and consider the spectral systems $(S[z, s, p, b])$ and $(S'[z, s, p, b])$ respectively associated with them, with $z \leq s \leq p \leq b$ in \mathbb{Z}^m . Note that the case $m = 1$ corresponds to considering, in the

previous example, the whole spectral systems over \mathbb{Z} instead of the spectral sequences. The notion of order of the homotopy h can be generalized in a natural way:

Definition 4.26. Let (C_*, F) and (D_*, F') be \mathbb{Z}^m -filtered chain complexes. We say that a map $h : (C_*, F) \rightarrow (D_*, F')$ has *order* $\leq t \in \mathbb{Z}^m$ if $h(F_p) \subseteq F'_{p+t}$, for every $p \in \mathbb{Z}^m$.

Remember that \mathbb{Z}^m denotes here the poset (\mathbb{Z}^m, \leq) , where \leq is the coordinate-wise partial order. Considering the statement of Theorem 4.24 in the case of \mathbb{Z}^m -filtrations, if we suppose that the homotopy h has order $\leq t$, for an element $t \in \mathbb{Z}^m$, then a sufficient condition for (4.8) is clearly that $z + t \leq s$ and $p + t \leq b$ or, equivalently,

$$t \leq s - z \quad \text{and} \quad t \leq b - p.$$

In \mathbb{Z}^m we can express this by considering the coordinate-wise minimum of $s - z$ and $b - p$ and requiring that $t \leq (\min(s_i - z_i, b_i - p_i))_{i=1, \dots, m}$. We have therefore proved the following corollary of Theorem 4.24:

Corollary 4.27. Let $\rho = (f, g, h) : C_* \Rightarrow D_*$ be a reduction between the \mathbb{Z}^m -filtered chain complexes (C_*, F) and (D_*, F') . Suppose that f and g are compatible with the filtrations and that the homotopy h has order $\leq t \in \mathbb{Z}^m$. Given four indices $z \leq s \leq p \leq b$ in \mathbb{Z}^m , if $t \leq (\min(s_i - z_i, b_i - p_i))_{i=1, \dots, m}$ then the map f induces an isomorphism

$$f^{z, s, p, b} : S[z, s, p, b] \rightarrow S'[z, s, p, b].$$

Spectral systems over a partially ordered abelian group. Let us consider again Definition 4.26. It seems natural, for a generalized notion of *order* of a map between filtered chain complexes, to require that it satisfies the “desirable” property: if h has order $\leq t$ and $t \leq t'$, then h has order $\leq t'$. This clearly holds for filtrations over \mathbb{Z}^m , and it is a consequence of the *translation invariance of \leq* in the poset \mathbb{Z}^m : if $t \leq t'$, then $p + t \leq p + t'$, for all $p \in \mathbb{Z}^m$. We can then further generalize the notion of order to filtrations over posets which behave like \mathbb{Z}^m with respect to the desirable property stated above. Recall that a *partially ordered abelian group* is an abelian group $(I, +)$ endowed with a partial order \leq that is translation invariant: for all $p, t, t' \in I$, if $t \leq t'$ then $p + t \leq p + t'$. For partially ordered abelian groups we can immediately generalize Definition 4.26 and Corollary 4.27, following the same argument we presented for \mathbb{Z}^m -filtrations.

Definition 4.28. Let $(I, +, \leq)$ be a partially ordered abelian group, and let (C_*, F) and (D_*, F') be I -filtered chain complexes. We say that a map $h : (C_*, F) \rightarrow (D_*, F')$ has *order* $\leq t \in I$ if $h(F_p) \subseteq F'_{p+t}$, for every $p \in I$.

In the situation of Theorem 4.24 for filtrations over a partially ordered abelian group $(I, +, \leq)$, assuming that h has order $\leq t \in I$, a “natural” sufficient condition for (4.8) exploiting the order of h is that $t \leq s - z$ and $t \leq b - p$, as it is equivalent to $z + t \leq s$ and $p + t \leq b$ (note that we use the translation invariance of \leq), which in turn implies $h(F_z) \subseteq F_{z+t} \subseteq F_s$ and $h(F_p) \subseteq F_{p+t} \subseteq F_b$. To summarize:

Corollary 4.29. Let $(I, +, \leq)$ be a partially ordered abelian group and let $\rho = (f, g, h) : C_* \Rightarrow D_*$ be a reduction between the I -filtered chain complexes (C_*, F) and (D_*, F') . Suppose that f and g are compatible with the filtrations and that the homotopy h has order $\leq t \in I$. Given four indices $z \leq s \leq p \leq b$ in I , if $t \leq s - z$ and $t \leq b - p$ then the map f induces an isomorphism

$$f^{z,s,p,b} : S[z, s, p, b] \rightarrow S'[z, s, p, b].$$

Spectral systems over $D(\mathbb{Z}^m)$. We want to consider now the poset $D(\mathbb{Z}^m)$ of downsets of \mathbb{Z}^m and study if the notion of order of a map can be suitably generalized. In order to exploit what we know in the case of partially ordered abelian groups, we can proceed like in Section 3.4.3, fixing a downset $\bar{p} \in D(\mathbb{Z}^m)$ and considering all its translations by any $v \in \mathbb{Z}^m$. As we already pointed out, this situation occurs frequently with some of the connections of spectral systems we have presented in Chapter 2. In this setting, when the fixed downset \bar{p} is defined by a *finite* number of points $P_j \in \mathbb{Z}^m$ (that is to say, \bar{p} is the smallest downset containing the points P_j) we can see that

$$(\{\text{translations of } \bar{p} \text{ by all } v \in \mathbb{Z}^m\}, \text{translation}, \subseteq)$$

is isomorphic as partially ordered abelian group to $(\mathbb{Z}^m, +, \leq)$. Then we can apply Corollary 4.27, where the terms of the inequalities $t \leq s - z$ and $t \leq b - p$ (equivalent to the hypothesis $t \leq (\min(s_i - z_i, b_i - p_i))_{i=1, \dots, m}$) must be interpreted as elements of \mathbb{Z}^m describing the translations.

4.4 Serre spectral system for a tower of fibrations

One of the motivating examples of Matschke's work [Mat13] are *towers of fibrations*, i.e. sequences of fibrations such that the total space of each is the base of the next one. For example, the fibrations $M \rightarrow N \rightarrow B$ and $G \rightarrow E \rightarrow N$, which can be denoted also by

$$\begin{array}{ccc} G & \longrightarrow & E \\ & & \downarrow \\ M & \longrightarrow & N \\ & & \downarrow \\ & & B \end{array}$$

form a tower of two fibrations. In this situation, as the usual goal of computation is the homology $H(E)$ of the total space of the upper fibration, one typically applies twice the Serre spectral sequence, assuming that the homology of B, M, G is known. Leaving aside extension problems, one can think to determine $H(N)$ from $H(B)$ and $H(M)$ via a first Serre spectral sequence, using then a second Serre spectral sequence to try to determine $H(E)$ from $H(N)$ and $H(G)$. A suitably defined spectral system (over $D(\mathbb{Z}^2)$, in this case) represents a unified framework "containing" these spectral sequences and offering a larger number of connections to the limit $H(E)$. Moreover, the 2-page of the spectral system is defined by a formula which generalizes that of Serre's theorem (Theorem 4.16)

In this section we introduce the Serre spectral system associated with a tower of fibrations, and we explain how the effective homology technique can be used to compute its terms. Our idea was to retrace the arguments illustrated in Section 4.2 for the Serre spectral sequence, starting from a simple case in which the total space is the Cartesian product of the base and the fiber and introducing then twisting operators (Definition 4.14) to describe the general setting. However, this generalization turns out to be more complicated than in the case of one fibration, as the twisting operators, as we will explain, are not compatible with the filtration defining the Serre spectral system. Our results are therefore for now limited to the case of Cartesian products, which can be seen as towers of fibrations “without twist”.

In Section 4.4.1 we present our results with an overview in the case of two fibrations. We then prove all the results in the general case of m fibrations in Section 4.4.2 and 4.4.4, while in Section 4.4.3 we focus on a generalized formula for the terms of the 2-page. In Section 4.4.5 we explain how, by the very nature of the defined filtrations, there are problems to extend our approach to the twisted case.

4.4.1 Overview of the results

We introduce our results on towers of fibrations with an overview in the simple case of two fibrations (without twist). All the results, here stated in an intuitive way, will be proved in the following sections.

Consider the Cartesian product $G \times M \times B$ of the simplicial sets G , M and B , which we see as the total space of the tower of two fibrations (without twist)

$$\begin{array}{ccc} G & \longrightarrow & G \times (M \times B) \\ & & \downarrow \\ M & \longrightarrow & M \times B \\ & & \downarrow \\ & & B \end{array}$$

Suppose that the simplicial sets G , M and B have effective homology: there exist equivalences

$$C_*(G) \Leftarrow \hat{C}G_* \Rightarrow DG_*, \quad C_*(M) \Leftarrow \hat{C}M_* \Rightarrow DM_*, \quad C_*(B) \Leftarrow \hat{C}B_* \Rightarrow DB_*, \quad (4.10)$$

where DG_* , DM_* and DB_* are effective chain complexes. Using the tools and the results introduced in Section 4.2.3, we obtain a scheme of reductions connecting the chain complex $C_*(G \times M \times B)$ to the effective chain complex $DG_* \otimes DM_* \otimes DB_*$:

$$\begin{array}{ccc} C_*(G \times M \times B) & & \\ \text{EZ}_0 \Downarrow & & \\ C_*(G) \otimes C_*(M \times B) & \xrightarrow{\hat{C}G_* \otimes \hat{C}M_* \otimes \hat{C}B_*} & \\ \text{id} \otimes \text{EZ}_1 \Downarrow & \hat{\rho}_1 \swarrow & \searrow \hat{\rho}_2 \\ C_*(G) \otimes C_*(M) \otimes C_*(B) & \xrightarrow{\quad} & DG_* \otimes DM_* \otimes DB_* \end{array}$$

EZ_0 is the Eilenberg-Zilber reduction $C_*(G \times (M \times B)) \Rightarrow C_*(G) \otimes C_*(M \times B)$, which we introduced in the statement of Theorem 4.17, while $\text{id} \otimes EZ_1$ is the tensor product of the trivial reduction of $C_*(G)$ and the Eilenberg-Zilber reduction $C_*(M \times B) \Rightarrow C_*(M) \otimes C_*(B)$; the reductions $\hat{\rho}_1$ and $\hat{\rho}_2$ are given by the tensor product of the reductions of (4.10). Notations like $DG_* \otimes DM_* \otimes DB_*$ for 3-fold tensor products of chain complexes are not ambiguous, as a consequence of the associativity of the tensor product (of modules): $M_1 \otimes (M_2 \otimes M_3) \cong (M_1 \otimes M_2) \otimes M_3$; this remark generalizes to k -fold tensor products, for any k .

We define $D(\mathbb{Z}^2)$ -filtrations on each of the chain complexes appearing in the above scheme. As explained in Section 2.4, we only need to specify \mathbb{Z}^2 -filtrations of the type $(F_P)_{P \in \mathbb{Z}^2}$, as they identify canonical $D(\mathbb{Z}^2)$ -filtrations $(F_p)_{p \in D(\mathbb{Z}^2)}$ with $F_p := \sum_{P \in p} F_P$. The \mathbb{Z}^2 -filtrations we consider for the chain complexes of the scheme are as follows:

- $F_{(p_1, p_2)} C_n(G \times M \times B)$ is the free \mathbb{Z} -module generated by the elements $(g, m, b) \in (G \times M \times B)_n$ such that $\deg m \leq p_1$ and $\deg b \leq p_2$, where \deg denotes the degeneracy degree (see Section 4.2.3). This filtration represents a natural generalization of the filtration introduced in Definition 4.19.
- $F_{(p_1, p_2)}(C_*(G) \otimes C_*(M \times B))_n$ is the \mathbb{Z} -module $(C_*(G) \otimes C_*(\mathcal{F}_{(p_1, p_2)}(M \times B)))_n$, where $\mathcal{F}_{(p_1, p_2)}(M \times B)$ is the simplicial set consisting of all elements $(m, b) \in M \times B$ such that $\deg m \leq p_1$ and $\deg b \leq p_2$.
- $F_{(p_1, p_2)}(C_*(G) \otimes C_*(M) \otimes C_*(B))_n$ is defined as the \mathbb{Z} -module

$$\bigoplus_{\substack{i+j+k=n, \\ j \leq p_1 \text{ and } k \leq p_2}} C_i(G) \otimes C_j(M) \otimes C_k(B).$$

Filtrations are defined similarly on the chain complexes $\hat{C}G_* \otimes \hat{C}M_* \otimes \hat{C}B_*$ and $DG_* \otimes DM_* \otimes DB_*$. They represent a natural generalization of the filtration of Definition 4.20.

Informally, the idea is to look at M and B and filter the chain complexes by considering whichever index makes sense between the degeneracy degree and the dimension of the chain groups. We did not differentiate the notation of the various filtrations, as it will be clear by the context which one is employed.

Having defined filtrations of all the involved chain complexes, we can now study how they behave with respect to the reductions EZ_0 , $\text{id} \otimes EZ_1$, $\hat{\rho}_1$ and $\hat{\rho}_2$.

The reductions EZ_0 and $\text{id} \otimes EZ_1$ are compatible with the filtrations (meaning that all the three maps of each reduction are compatible with the filtrations), as it can be easily verified recalling the Eilenberg-Zilber theorem (Theorem 4.17) and the considerations we made in Section 4.2.4. The spectral systems arising from the filtrations of $C_*(G \times M \times B)$, $C_*(G) \otimes C_*(M \times B)$ and $C_*(G) \otimes C_*(M) \otimes C_*(B)$ are therefore isomorphic by virtue of Corollary 4.25.

The reductions $\hat{\rho}_1 = (\hat{f}_1, \hat{g}_1, \hat{h}_1)$ and $\hat{\rho}_2 = (\hat{f}_2, \hat{g}_2, \hat{h}_2)$ are such that the homotopies \hat{h}_1 and \hat{h}_2 satisfy

$$\hat{h}_i(F_{(p_1, p_2)}) \subseteq F_{(p_1+1, p_2)} + F_{(p_1, p_2+1)}, \quad i = 1, 2, \quad (4.11)$$

as we will prove in a general setting in Section 4.4.2. In this case, we cannot conclude that the spectral systems arising from the filtrations of $C_*(G) \otimes C_*(M) \otimes C_*(B)$, $\hat{C}G_* \otimes \hat{C}M_* \otimes \hat{C}B_*$ and $DG_* \otimes DM_* \otimes DB_*$ are isomorphic, as the conditions (4.8) of Theorem 4.24 must be taken into account. In order to apply Theorem 4.24, we can then study which terms $S[z, s, p, b]$ of the spectral system over $D(\mathbb{Z}^2)$ of $\hat{C}G_* \otimes \hat{C}M_* \otimes \hat{C}B_*$ satisfy

$$\begin{aligned} \hat{h}_i(F_z) &\subseteq F_s \\ \hat{h}_i(F_p) &\subseteq F_b \end{aligned} \tag{4.12}$$

for both homotopies ($i = 1, 2$). Recalling the definition of the downsets z, s, p, b for the terms $S^*(P; 2)$ of the 2-page (see Section 2.4, Secondary connections), it can be easily proved that a translation by $e_1 = (1, 0)$ or by $e_2 = (0, 1)$, motivated by (4.11), of the downset z (respectively p) produces a downset which is still contained in s (respectively b), meaning that the inclusions (4.12) hold for the terms of the 2-page. In Figure 4.1 we give an intuitive representation of this property, and in Section 4.4.4 we present a general result for spectral systems over $D(\mathbb{Z}^m)$. Note that for the pages “previous” to the 2-page in the secondary connection the inclusions (4.12) do not hold in general, as one can observe in Figure 4.2. Since the spectral systems associated with the $D(\mathbb{Z}^2)$ -filtrations of

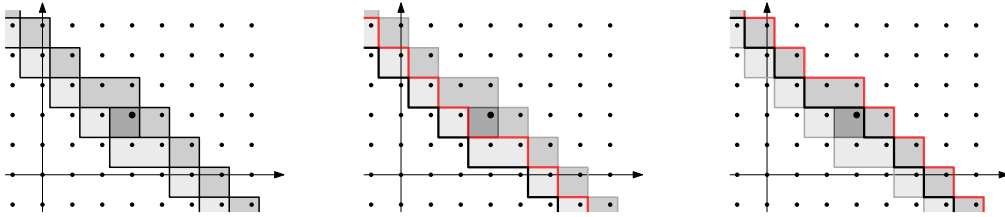


FIGURE 4.1: The three figures represent the term $S^*(P; 2)$ of the 2-page of a spectral system over $D(\mathbb{Z}^2)$, with $P = (3, 2)$. The first picture represents $S^*(P; 2)$ with our usual conventions; in the central picture the downsets z (black) and s (red) are highlighted; in the last picture the downsets p (black) and b (red) are highlighted.

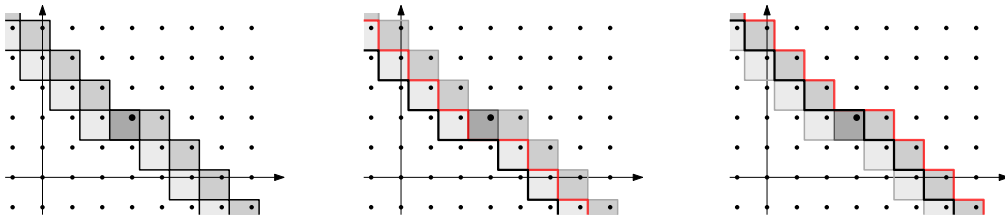


FIGURE 4.2: Representations of the term $S(P; 2)$ of a spectral system over $D(\mathbb{Z}^2)$, with $P = (3, 2)$. The first picture represents $S(P; 2)$ with our usual conventions; in the central picture we can notice that the downset z (black) shifted by $(0, 1)$ is not contained in the downset s (red); in the last picture the same situation holds for the downsets p (black) and b (red).

$C_*(G) \otimes C_*(M) \otimes C_*(B)$, $\hat{C}G_* \otimes \hat{C}M_* \otimes \hat{C}B_*$ and $DG_* \otimes DM_* \otimes DB_*$ are isomorphic from the 2-page, we can use the effective homology technique to compute the spectral system over $D(\mathbb{Z}^2)$ associated with $C_*(G \times M \times B)$ using the effective chain complex $DG_* \otimes DM_* \otimes DB_*$ and obtaining, from the 2-page on, isomorphic terms.

As stated in [Mat13], the terms $S^*(P; 2)$ of the 2-page can be expressed as

$$S_n^*(P; 2) = H_{p_2}(B; H_{p_1}(M; H_{n-p_1-p_2}(G))),$$

where $P = (p_1, p_2) \in \mathbb{Z}^2$. In Section 4.4.3 we provide a proof of this relation, which clearly generalizes Serre's formula (see Theorem 4.16), in our current setting, for general Cartesian products of simplicial sets.

To summarize, we have seen that, given a Cartesian product $G \times M \times B$ of simplicial sets, there is a spectral system over $D(\mathbb{Z}^2)$ with limit $S[-\infty, -\infty, \infty, \infty] = H(G \times M \times B)$ whose 2-page $S^*(P; 2)$ satisfies $S_n^*(P; 2) = H_{p_2}(B; H_{p_1}(M; H_{n-p_1-p_2}(G)))$ and is the first page during the secondary connection for which the effective homology technique provides provably correct computations. In what follows, we present and prove these results in the general case of an $(m + 1)$ -fold Cartesian product, corresponding to a tower of m fibrations (without twist).

4.4.2 Towers of fibrations: reductions and filtrations

Consider a tower of m fibrations (without twist)

$$\begin{array}{ccc} G_0 & \longrightarrow & E_0 \\ & & \downarrow \\ \dots & & \dots \\ & & \downarrow \\ G_{m-1} & \longrightarrow & E_{m-1} \\ & & \downarrow \\ & & B \end{array}$$

where we suppose that the simplicial sets G_0, \dots, G_{m-1}, B have effective homology. This means that there exist chain equivalences

$$C_*(G_0) \Leftarrow \hat{C}G_{0*} \Rightarrow DG_{0*}, \quad \dots, \quad C_*(G_{m-1}) \Leftarrow \hat{C}G_{(m-1)*} \Rightarrow DG_{(m-1)*}, \quad C_*(B) \Leftarrow \hat{C}B_* \Rightarrow DB_*, \quad (4.13)$$

where $DG_{0*}, \dots, DG_{(m-1)*}, DB_*$ are effective chain complexes.

The simplicial sets E_i are of the form $E_i = G_i \times \dots \times G_{m-1} \times B$, as we are assuming to have no twist. We can connect via successive reductions the chain complex $C_*(E_0) = C_*(G_0 \times \dots \times G_{m-1} \times B)$

to the effective chain complex $DG_{0*} \otimes \dots \otimes DG_{(m-1)*} \otimes DB_*$ in the following way:

$$\begin{array}{ccc}
C_*(G_0 \times G_1 \times \dots \times G_{m-1} \times B) & & \\
\text{EZ}_0 \downarrow & & \\
C_*(G_0) \otimes C_*(G_1 \times \dots \times B) & & \\
\text{id} \otimes \text{EZ}_1 \downarrow & & \\
C_*(G_0) \otimes C_*(G_1) \otimes C_*(G_2 \times \dots \times B) & & \\
\text{id} \otimes \text{EZ}_2 \downarrow & & \\
\vdots & & \\
\text{id} \otimes \text{EZ}_{m-1} \downarrow & & \\
C_*(G_0) \otimes \dots \otimes C_*(G_{m-1}) \otimes C_*(B) & \xleftarrow{\hat{\rho}_1} \hat{C}G_{0*} \otimes \dots \otimes \hat{C}G_{(m-1)*} \otimes \hat{C}B_* & \xrightarrow{\hat{\rho}_2} DG_{0*} \otimes \dots \otimes DG_{(m-1)*} \otimes DB_*
\end{array}$$

The reductions of the type $\text{id} \otimes \text{EZ}_i$ are defined in the obvious way as tensor product of the trivial reduction of $C_*(G_0) \otimes \dots \otimes C_*(G_{i-1})$ and the Eilenberg-Zilber reduction

$$\text{EZ}_i = (f_i, g_i, h_i) : C_*(G_i \times (G_{i+1} \times \dots \times B)) \Rightarrow C_*(G_i) \otimes C_*(G_{i+1} \times \dots \times B).$$

The reductions $\hat{\rho}_1$ and $\hat{\rho}_2$ are the tensor product of the $m+1$ reductions forming the equivalences (4.13). In order to explicitly describe the maps of the reductions $\hat{\rho}_1$ and $\hat{\rho}_2$, we state the following generalization of Proposition 4.4:

Proposition 4.30. Given k reductions

$$\begin{aligned}
\rho_1 &= (f_1, g_1, h_1) : C_*^{(1)} \Rightarrow D_*^{(1)} \\
\rho_2 &= (f_2, g_2, h_2) : C_*^{(2)} \Rightarrow D_*^{(2)} \\
&\dots \\
\rho_k &= (f_k, g_k, h_k) : C_*^{(k)} \Rightarrow D_*^{(k)},
\end{aligned}$$

a reduction $\hat{\rho} = (\hat{f}, \hat{g}, \hat{h}) : C_*^{(1)} \otimes C_*^{(2)} \otimes \dots \otimes C_*^{(k)} \Rightarrow D_*^{(1)} \otimes D_*^{(2)} \otimes \dots \otimes D_*^{(k)}$ can be defined by

$$\begin{aligned}
\hat{f} &= f_1 \otimes f_2 \otimes \dots \otimes f_k \\
\hat{g} &= g_1 \otimes g_2 \otimes \dots \otimes g_k \\
\hat{h} &= (h_1 \otimes \text{id} \otimes \dots \otimes \text{id}) + (g_1 f_1 \otimes h_2 \otimes \dots \otimes \text{id}) + \dots + (g_1 f_1 \otimes g_2 f_2 \otimes \dots \otimes h_k).
\end{aligned}$$

The reduction $\hat{\rho}$ is called the tensor product of $\rho_1, \rho_2, \dots, \rho_k$ and is denoted also $\rho_1 \otimes \rho_2 \otimes \dots \otimes \rho_k$.

Proof. Proceed by induction on k and use Proposition 4.4. \square

Our purpose is to define a spectral system over $D(\mathbb{Z}^m)$ associated with the chain complex $C_*(G_0 \times \dots \times G_{m-1} \times B)$ and to exploit the successive reductions of the scheme above and the effective chain complex $DG_{0*} \otimes \dots \otimes DG_{(m-1)*} \otimes DB_*$ to compute its terms. In order to achieve it, we have to define $D(\mathbb{Z}^m)$ -filtrations of the involved chain complexes.

Definition 4.31. We define $D(\mathbb{Z}^m)$ -filtrations for all the chain complexes appearing in the scheme of successive reductions. As said before, it is actually enough to define \mathbb{Z}^m -filtrations of the form $(F_P)_{P \in \mathbb{Z}^m}$, as they identify canonical $D(\mathbb{Z}^m)$ -filtrations $(F_p)_{p \in D(\mathbb{Z}^m)}$ with $F_p := \sum_{P \in p} F_P$.

Given a Cartesian product of m simplicial sets $G_1 \times \cdots \times G_{m-1} \times B$, we define a \mathbb{Z}^m -filtration of simplicial sets as follows:

$$\mathcal{F}_P(G_1 \times \cdots \times G_{m-1} \times B) := \{(x_1, \dots, x_m) \in G_1 \times \cdots \times B \mid \deg x_i \leq p_i, \text{ for all } 1 \leq i \leq m\},$$

where $P = (p_1, \dots, p_m) \in \mathbb{Z}^m$ and $\deg x_i$ denotes the degeneracy degree of x_i . Note that

$$\mathcal{F}_P(G_1 \times \cdots \times G_{m-1} \times B) = \mathcal{F}_{p_1} G_1 \times \cdots \times \mathcal{F}_{p_{m-1}} G_{m-1} \times \mathcal{F}_{p_m} B,$$

where \mathcal{F}_{p_i} are the usual \mathbb{Z} -filtrations by degeneracy degree:

$$\mathcal{F}_{p_i} K := \{x \in K \mid \deg x \leq p_i\}.$$

Using the filtrations \mathcal{F}_P and \mathcal{F}_{p_i} , we define a \mathbb{Z}^m -filtration of $C_*(G_0 \times G_1 \times \cdots \times G_{m-1} \times B)$ by

$$F_P C_*(G_0 \times G_1 \times \cdots \times G_{m-1} \times B) := C_*(G_0 \times \mathcal{F}_P(G_1 \times \cdots \times G_{m-1} \times B)),$$

and a \mathbb{Z}^m -filtration of $C_*(G_0) \otimes C_*(G_1) \otimes \cdots \otimes C_*(G_{m-1}) \otimes C_*(B)$ by

$$F_P(C_*(G_0) \otimes \cdots \otimes C_*(B)) := C_*(G_0) \otimes C_*(\mathcal{F}_{p_1} G_1) \otimes \cdots \otimes C_*(\mathcal{F}_{p_{m-1}} G_{m-1}) \otimes C_*(\mathcal{F}_{p_m} B),$$

which can be expressed also as

$$F_P(C_*(G_0) \otimes \cdots \otimes C_*(B))_n = \bigoplus_{\substack{\sum_i k_i = n, \\ k_i \leq p_i, \forall 1 \leq i \leq m}} C_{k_0}(G_0) \otimes C_{k_1}(G_1) \otimes \cdots \otimes C_{k_{m-1}}(G_{m-1}) \otimes C_{k_m}(B)$$

by virtue of the fact that all the chain complexes are assumed to be *normalized*. Filtrations over \mathbb{Z}^m are defined similarly for the other $(m+1)$ -fold tensor product of chain complexes, namely $\hat{C}G_{0*} \otimes \cdots \otimes \hat{C}G_{(m-1)*} \otimes \hat{C}B_*$ and $DG_{0*} \otimes \cdots \otimes DG_{(m-1)*} \otimes DB_*$, by considering the dimension of the last m factors.

Lastly, we define \mathbb{Z}^m -filtrations on the remaining chain complexes, of the form $C_*(G_0) \otimes \cdots \otimes C_*(G_{i-1}) \otimes C_*(G_i \times \cdots \times B)$, in the following way:

$$\begin{aligned} F_P(C_*(G_0) \otimes C_*(G_1) \otimes \cdots \otimes C_*(G_{i-1}) \otimes C_*(G_i \times \cdots \times B)) \\ := C_*(G_0) \otimes C_*(\mathcal{F}_{p_1} G_1) \otimes \cdots \otimes C_*(\mathcal{F}_{p_{i-1}} G_{i-1}) \otimes C_*(\mathcal{F}_{(p_i, \dots, p_m)}(G_i \times \cdots \times B)). \end{aligned}$$

Remark 4.32. The filtrations of Definition 4.31 generalize both the filtrations F^\times and F^\otimes of Section 4.2.4 (corresponding to the case $m = 1$) and the filtrations introduced in Section 4.4.1 for $m = 2$. For readability, we simply denote $(F_P)_{P \in \mathbb{Z}^m}$ or $(F_p)_{p \in D(\mathbb{Z}^m)}$ all the filtrations of Definition 4.31.

The following results clarify the behavior of the reductions we are considering with respect to the filtrations of Definition 4.31.

Proposition 4.33. All the maps of the reduction $EZ_0 = (f_0, g_0, h_0)$ and all the maps of each reduction $\text{id} \otimes EZ_i = (\text{id} \otimes f_i, \text{id} \otimes g_i, \text{id} \otimes h_i)$ are compatible with the filtrations.

Proof. It is a direct consequence of the fact that the maps of an Eilenberg-Zilber reduction (see Theorem 4.17) never increase the filtration indices, as they are based on the degeneracy degrees. \square

Lemma 4.34. Given $m + 1$ reductions

$$\begin{aligned} \rho_0 &= (f_0, g_0, h_0) : C_*^{(0)} \Rightarrow D_*^{(0)} \\ \rho_1 &= (f_1, g_1, h_1) : C_*^{(1)} \Rightarrow D_*^{(1)} \\ &\dots \\ \rho_m &= (f_m, g_m, h_m) : C_*^{(m)} \Rightarrow D_*^{(m)}, \end{aligned}$$

let $\hat{\rho} = (\hat{f}, \hat{g}, \hat{h}) : C_*^{(0)} \otimes C_*^{(1)} \otimes \dots \otimes C_*^{(m)} \Rightarrow D_*^{(0)} \otimes D_*^{(1)} \otimes \dots \otimes D_*^{(m)}$ be the reduction $\rho_0 \otimes \rho_1 \otimes \dots \otimes \rho_m$. Denote $(F_P^C)_{P \in \mathbb{Z}^m}$ the \mathbb{Z}^m -filtration of $C_*^{(0)} \otimes C_*^{(1)} \otimes \dots \otimes C_*^{(m)}$ on the dimension of the last m factors:

$$F_P^C(C_*^{(0)} \otimes C_*^{(1)} \otimes \dots \otimes C_*^{(m)})_n = \bigoplus_{\substack{\sum_i k_i = n, \\ k_i \leq p_i, \forall 1 \leq i \leq m}} C_{k_0}^{(0)} \otimes C_{k_1}^{(1)} \otimes \dots \otimes C_{k_{m-1}}^{(m-1)} \otimes C_{k_m}^{(m)},$$

with $P = (p_1, \dots, p_m) \in \mathbb{Z}^m$; denote $(F_P^D)_{P \in \mathbb{Z}^m}$ the \mathbb{Z}^m -filtration of $D_*^{(0)} \otimes D_*^{(1)} \otimes \dots \otimes D_*^{(m)}$ defined similarly. Then \hat{f} and \hat{g} are compatible with the filtrations F_P^C and F_P^D , while

$$\hat{h}(F_P^C) \subseteq F_{P+(1,0,\dots,0)}^C + F_{P+(0,1,\dots,0)}^C + \dots + F_{P+(0,0,\dots,1)}^C, \quad (4.14)$$

for all $P \in \mathbb{Z}^m$.

Proof. The maps $\hat{f}, \hat{g}, \hat{h}$ of the reduction $\hat{\rho}$ are defined as in the statement of Proposition 4.30. Then, as $\hat{f} = f_0 \otimes f_1 \otimes \dots \otimes f_m$ and each f_i is a chain map, \hat{f} is compatible with the filtrations F_P^C and F_P^D ; the same conclusion holds for \hat{g} , by a similar argument. Since each h_i is a graded map of degree $+1$, the map

$$\hat{h} = (h_0 \otimes \text{id} \otimes \dots \otimes \text{id}) + (g_0 f_0 \otimes h_1 \otimes \text{id} \otimes \dots \otimes \text{id}) + \dots + (g_0 f_0 \otimes g_1 f_1 \otimes \dots \otimes g_{m-1} f_{m-1} \otimes h_m)$$

behaves differently, as each summand $g_0 f_0 \otimes \dots \otimes g_{i-1} f_{i-1} \otimes h_i \otimes \text{id} \otimes \dots \otimes \text{id}$ is a map sending $C_{k_0}^{(0)} \otimes \dots \otimes C_{k_i}^{(i)} \otimes \dots \otimes C_{k_m}^{(m)}$ to $C_{k_0}^{(0)} \otimes \dots \otimes C_{k_i+1}^{(i)} \otimes \dots \otimes C_{k_m}^{(m)}$, that is (for $1 \leq i \leq m$) sending F_P^C to $F_{P+e_i}^C$. Equation (4.14) then follows. \square

Since by Proposition 4.33 all the maps of the reductions $EZ_0, \text{id} \otimes EZ_1, \dots, \text{id} \otimes EZ_{m-1}$ are compatible with the filtrations, we can apply Corollary 4.25 to conclude that the spectral systems associated with the $D(\mathbb{Z}^m)$ -filtered chain complexes $C_*(G_0 \times \dots \times B)$, $C_*(G_0) \otimes C_*(G_1 \times \dots \times B)$, \dots , $C_*(G_0) \otimes \dots \otimes C_*(B)$ are isomorphic.

Nonetheless, Lemma 4.34 applied to the reductions $\hat{\rho}_1 : \hat{C}G_{0*} \otimes \cdots \otimes \hat{C}B_* \Rightarrow C_*(G_0) \otimes \cdots \otimes C_*(B)$ and $\hat{\rho}_2 : \hat{C}G_{0*} \otimes \cdots \otimes \hat{C}B_* \Rightarrow DG_{0*} \otimes \cdots \otimes DB_*$ reveals that we cannot reach the same conclusion for the spectral systems associated with $C_*(G_0) \otimes \cdots \otimes C_*(B)$, $\hat{C}G_{0*} \otimes \cdots \otimes \hat{C}B_*$ and $DG_{0*} \otimes \cdots \otimes DB_*$, as the homotopies \hat{h}_1 and \hat{h}_2 are not compatible with the filtrations. In Section 4.4.4 we prove that the spectral system of $C_*(G_0 \times \cdots \times B)$ is isomorphic from the 2-page to the spectral system of the effective chain complex $DG_{0*} \otimes \cdots \otimes DB_*$.

4.4.3 Serre's formula for the 2-page

We make a small detour to prove a generalization to the framework of spectral systems over $D(\mathbb{Z}^m)$ of Serre's formula for the 2-page of classical spectral sequences associated with fibrations. The argument we present here is based only on the existence of a reduction (the composition of $EZ_0, \text{id} \otimes EZ_1, \dots, \text{id} \otimes EZ_{m-1}$) from $C_*(G_0 \times \cdots \times B)$ to $C_*(G_0) \otimes \cdots \otimes C_*(B)$. It has therefore the advantage to be very intuitive, and it highlights with clarity the role played by secondary connections in a spectral system.

This section is a proof, in our current setting, of the following result of [Mat13], which is valid for general towers of fibrations (see [Mat13, § 5]):

Theorem 4.35. Consider a tower of m fibrations without twist or, equivalently, an $(m + 1)$ -fold Cartesian product of simplicial sets $G_0 \times \cdots \times G_{m-1} \times B$. There exists an associated spectral system over $D(\mathbb{Z}^m)$ with limit $S[-\infty, -\infty, \infty, \infty] = H(G_0 \times \cdots \times G_{m-1} \times B)$ and 2-page

$$S_n^*(P; m) = H_{p_m}(B; H_{p_{m-1}}(G_{m-1}; \dots H_{p_1}(G_1; H_{p_0}(G_0)))), \quad (4.15)$$

with $P = (p_1, \dots, p_m) \in \mathbb{Z}^m$ and $p_0 := n - p_1 - \cdots - p_m$.

The spectral system associated with $G_0 \times \cdots \times G_{m-1} \times B$ is the one arising from the $D(\mathbb{Z}^m)$ -filtration $F_p := \sum_{P \in p} F_P$ of $C_*(G_0 \times \cdots \times G_{m-1} \times B)$. The following result, stated in general for filtrations over a poset I , will be useful for proving Theorem 4.35:

Proposition 4.36. Let (C_*, F) and (D_*, F') be I -filtered chain complexes and let $\rho = (f, g, h) : C_* \Rightarrow D_*$ be a reduction. If the maps f, g and h are compatible with the filtrations, then for any $s \leq p$ in I there is an induced reduction

$$\tilde{\rho} = (\tilde{f}, \tilde{g}, \tilde{h}) : \frac{F_p C_*}{F_s C_*} \Rightarrow \frac{F'_p D_*}{F'_s D_*}.$$

Proof. The maps $\tilde{f}, \tilde{g}, \tilde{h}$ induced on the quotients by f, g, h are well defined as f, g, h are compatible with the filtrations, and satisfy the conditions of the definition of a reduction because f, g, h satisfy them. \square

One consequence of Proposition 4.36 is that, being \tilde{f} a chain equivalence, it induces isomorphisms in homology $H_n(F_p C_* / F_s C_*) \cong H_n(F'_p D_* / F'_s D_*)$. But we already know that from Corollary 4.25, which tells more: the whole spectral systems arising from (C_*, F) and (D_*, F') are isomorphic. This fact will play an important role in our proof, and Proposition 4.36 in a sense complements it describing what happens for the “0-page” F_p / F_s .

Cartesian product $G \times B$. The best way to understand the proof for filtrations over (downsets of) \mathbb{Z}^m is to see it as a generalization of the case over \mathbb{Z} , considering then $m = 1$ at first. We start from the statement of the Eilenberg-Zilber theorem: for any simplicial sets G and B , there exists a reduction $\text{EZ} = (f, g, h) : C_*(G \times B) \Rightarrow C_*(G) \otimes C_*(B)$. Consider the \mathbb{Z} -filtrations of $C_*(G \times B)$ and $C_*(G) \otimes C_*(B)$ defined in Section 4.2.4, that is $F^\times = (F_p^\times)_{p \in \mathbb{Z}}$ defined as

$$F_p^\times C_n(G \times B) := C_n(G \times \mathcal{F}_p B),$$

and $F^\otimes = (F_p^\otimes)_{p \in \mathbb{Z}}$ defined as

$$F_p^\otimes (C_*(G) \otimes C_*(B))_n := (C_*(G) \otimes C_*(\mathcal{F}_p B))_n = \bigoplus_{k \leq p} C_{n-k}(G) \otimes C_k(B).$$

As a consequence of Proposition 4.36, for each $p \in \mathbb{Z}$ the map

$$\tilde{f} : \frac{F_p^\times C_*(G \times B)}{F_{p-1}^\times C_*(G \times B)} \rightarrow \frac{F_p^\otimes C_*(G) \otimes C_*(B)}{F_{p-1}^\otimes C_*(G) \otimes C_*(B)}$$

is a chain equivalence, and in particular it induces an isomorphism in homology. Now, observing that for all n and p we have

$$\frac{F_p^\otimes (C_*(G) \otimes C_*(B))_n}{F_{p-1}^\otimes (C_*(G) \otimes C_*(B))_n} = \frac{\bigoplus_{k \leq p} C_{n-k}(G) \otimes C_k(B)}{\bigoplus_{k \leq p-1} C_{n-k}(G) \otimes C_k(B)} \cong C_{n-p}(G) \otimes C_p(B),$$

we conclude that the homology of $F_p^\times C_*(G \times B)/F_{p-1}^\times C_*(G \times B)$ is isomorphic to the homology of $C_{n-p}(G) \otimes C_p(B)$, endowed with the differential $d \otimes \text{id}$. Therefore we have

$$H_n \left(\frac{F_p^\times C_*(G \times B)}{F_{p-1}^\times C_*(G \times B)} \right) = H_{n-p}(G) \otimes C_p(B),$$

where the right member is obtained applying the universal coefficient theorem (Theorem 1.18), with $\text{Tor}_1^{\mathbb{Z}} = 0$ as $C_p(B)$ is a free \mathbb{Z} -module. We have thus shown that, associated with the filtration F^\times of $C_*(G \times B)$ there is a (first quadrant) spectral sequence (also, a spectral system over \mathbb{Z}) converging to $H_*(G \times B)$ and such that its 1-page is

$$E_{p,q}^1 = H_n \left(\frac{F_p^\times C_*(G \times B)}{F_{p-1}^\times C_*(G \times B)} \right) = H_q(G) \otimes C_p(B).$$

The 2-page of this spectral sequence is clearly given by the homology of $C_*(B)$ with coefficients in $H_q(G)$, that is:

$$E_{p,q}^2 = H_p(H_q(G) \otimes C_*(B)) = H_p(B; H_q(G)).$$

Notice that for $m = 1$ the classical notion of 2-page of the spectral sequence coincides with that of 2-page of the spectral system, as for $p \in \mathbb{Z}$ we have $S^*(p; 1) = S[p-2, p-1, p, p+1] = E_p^2$.

Tower of m fibrations, or $(m+1)$ -fold Cartesian product. Consider now a Cartesian product $G_0 \times \cdots \times G_{m-1} \times B$ of simplicial sets and the successive reductions $\text{EZ}_0, \text{id} \otimes \text{EZ}_1, \dots, \text{id} \otimes \text{EZ}_{m-1}$ from $C_*(G_0 \times \cdots \times G_{m-1} \times B)$ to $C_*(G_0) \otimes \cdots \otimes C_*(G_{m-1}) \otimes C_*(B)$ introduced in Section 4.4.2, and filter each chain complex using the $D(\mathbb{Z}^m)$ -filtrations of Definition 4.31. We want to show that the filtration $F = (F_p)_{p \in D(\mathbb{Z}^m)}$ of $C_*(G_0 \times \cdots \times G_{m-1} \times B)$ produces a spectral system over $D(\mathbb{Z}^m)$ whose 2-page (recall Definition 2.12) is given by the formula (4.15).

Since all the maps of each reduction $\text{EZ}_0, \text{id} \otimes \text{EZ}_1, \dots, \text{id} \otimes \text{EZ}_{m-1}$ are compatible with the filtrations, by Proposition 4.36 we can consider the induced reductions

$$\widetilde{\text{EZ}}_0, \text{id} \otimes \widetilde{\text{EZ}}_1, \dots, \text{id} \otimes \widetilde{\text{EZ}}_{m-1}$$

on subquotients of the form F_p/F_s , for any $s \leq p$ in $D(\mathbb{Z}^m)$. Their composition (recall Proposition 4.3) gives a reduction from the subquotient F_p/F_s of $C_*(G_0 \times \cdots \times G_{m-1} \times B)$ to the subquotient F_p/F_s of $C_*(G_0) \otimes \cdots \otimes C_*(G_{m-1}) \otimes C_*(B)$, which therefore have the same homology (it is furthermore important to remember that by Corollary 4.25 the whole spectral systems associated with the $D(\mathbb{Z}^m)$ -filtrations of $C_*(G_0 \times \cdots \times G_{m-1} \times B)$ and $C_*(G_0) \otimes \cdots \otimes C_*(G_{m-1}) \otimes C_*(B)$ are isomorphic). In particular consider the homology groups of the form $H_n(F_{p(P;1)}/F_{p(P;1) \setminus \{P\}})$, introduced in Section 2.4 as the terms $S_n(P;1)$ forming the 1-page of the spectral system, where $F_{p(P;1)}/F_{p(P;1) \setminus \{P\}}$ denotes a subquotient of the $D(\mathbb{Z}^m)$ -filtered chain complex $C_*(G_0 \times \cdots \times G_{m-1} \times B)$. As the difference between the two downsets $p(P;1)$ and $p(P;1) \setminus \{P\}$ contains only $P = (p_1, \dots, p_m) \in \mathbb{Z}^m$, the homology $H_n(F_{p(P;1)}/F_{p(P;1) \setminus \{P\}})$ is isomorphic to the n -homology of

$$C_{*-p_1-\dots-p_m}(G_0) \otimes C_{p_1}(G_1) \otimes \cdots \otimes C_{p_{m-1}}(G_{m-1}) \otimes C_{p_m}(B),$$

that is

$$H_{p_0}(G_0) \otimes C_{p_1}(G_1) \otimes \cdots \otimes C_{p_{m-1}}(G_{m-1}) \otimes C_{p_m}(B), \quad \text{with } p_0 := n - p_1 - \cdots - p_m. \quad (4.16)$$

This last formula appears evident if one considers the universal coefficient theorem, observing that $\text{Tor}_1^{\mathbb{Z}} = 0$ as the tensor product of projective modules is a projective module.

Consider now Lemma 2.13 and note that taking homology in direction $-e_k$ corresponds, in the isomorphic spectral system associated with $C_*(G_0) \otimes \cdots \otimes C_*(G_{m-1}) \otimes C_{p_m}(B)$, to taking homology with respect to the differential $d_{G_k} \otimes \text{id} \otimes \cdots \otimes \text{id}$. For example, the homology in direction $-e_1$ of $H_{p_0}(G_0) \otimes C_{p_1}(G_1) \otimes \cdots \otimes C_{p_{m-1}}(G_{m-1}) \otimes C_{p_m}(B)$ is the p_1 -homology of the chain complex

$$H_{p_0}(G_0) \otimes C_*(G_1) \otimes C_{p_2}(G_2) \otimes \cdots \otimes C_{p_{m-1}}(G_{m-1}) \otimes C_{p_m}(B),$$

which is

$$H_{p_1}(G_1; H_{p_0}(G_0)) \otimes C_{p_2}(G_2) \otimes \cdots \otimes C_{p_{m-1}}(G_{m-1}) \otimes C_{p_m}(B),$$

as one can verify using the universal coefficient theorem and observing that $C_{p_2}(G_2) \otimes \cdots \otimes C_{p_m}(B)$ is a projective \mathbb{Z} -module. It is easy to see that iterating this argument we obtain the formula (4.15) for the 2-page.

4.4.4 Effective homology for the 2-page

We now study the reductions $\hat{\rho}_1$ and $\hat{\rho}_2$ introduced in Section 4.4.2, in order to prove that the spectral system associated with $C_*(G_0 \times \cdots \times B)$ is isomorphic from the 2-page to the spectral system associated with the effective chain complex $DG_{0*} \otimes \cdots \otimes DB_*$. This result assures that if we use the effective homology technique to compute the spectral system associated with $C_*(G_0 \times \cdots \times B)$, from the 2-page on the results of the computations correspond, up to isomorphism, to the terms of the spectral system. Furthermore, we show that the 2-page is the first page of the secondary connection for which such an isomorphism holds in general.

Remember that we have already proved that the spectral system associated with $C_*(G_0 \times \cdots \times B)$ is isomorphic to the one associated with $C_*(G_0) \otimes \cdots \otimes C_*(B)$.

Theorem 4.37. Let $\hat{\rho}_1 = (\hat{f}_1, \hat{g}_1, \hat{h}_1)$ and $\hat{\rho}_2 = (\hat{f}_2, \hat{g}_2, \hat{h}_2)$ be the reductions

$$C_*(G_0) \otimes \cdots \otimes C_*(G_{m-1}) \otimes C_*(B) \begin{array}{c} \xleftarrow{\hat{\rho}_1} \\ \xleftarrow{\hat{\rho}_1} \end{array} \hat{C}G_{0*} \otimes \cdots \otimes \hat{C}G_{(m-1)*} \otimes \hat{C}B_* \begin{array}{c} \xrightarrow{\hat{\rho}_2} \\ \xrightarrow{\hat{\rho}_2} \end{array} DG_{0*} \otimes \cdots \otimes DG_{(m-1)*} \otimes DB_*$$

and consider the spectral systems over $D(\mathbb{Z}^m)$ associated with the chain complexes $C_*(G_0) \otimes \cdots \otimes C_*(B)$, $\hat{C}G_{0*} \otimes \cdots \otimes \hat{C}B_*$ and $DG_{0*} \otimes \cdots \otimes DB_*$ defined in Section 4.4.2. The maps \hat{f}_1 and \hat{f}_2 induce isomorphisms between the terms $S^*(P; m)$ of the 2-page of the spectral systems.

Proof. We give the proof for $\hat{\rho}_1 = (\hat{f}_1, \hat{g}_1, \hat{h}_1)$, as the case of $\hat{\rho}_2$ is analogous. In Section 2.4 we defined the downset $T_P^m := \{X \in \mathbb{Z}^m \mid \varphi_m(X) \leq_{\text{lex}} \varphi_m(P)\}$, where $\varphi_m(X) := (\sum_{i=1}^m x_i, \sum_{i=2}^m x_i, \dots, x_m)$, for all $X = (x_1, \dots, x_m) \in \mathbb{Z}^m$. Recall that each term $S^*(P; m)$ is defined as $S[z^*, s, p, b^*]$, where

$$\begin{aligned} z^* &:= T_{P+e_{m-1}-2e_m}^m, & s &:= T_{P+e_{m-1}-e_m}^m, \\ p &:= T_P^m, & b^* &:= T_{P+e_m}^m. \end{aligned} \tag{4.17}$$

By Theorem 4.24 we only need to check the conditions $\hat{h}_1(F_{z^*}) \subseteq F_s$ and $\hat{h}_1(F_p) \subseteq F_{b^*}$. Since the behavior of \hat{h}_1 with respect to the filtration we are considering is described by Lemma 4.34, we prove the following fact, which implies the thesis: $T_P^m + e_i \subseteq T_{P+e_m}^m$ for all $1 \leq i \leq m$ and for all $P \in \mathbb{Z}^m$. This can be easily shown directly. First, observe that $\varphi_m(e_i) \leq_{\text{lex}} \varphi_m(e_m)$, for all $1 \leq i \leq m$. If $X \in T_P^m$, then $X + e_i$ is such that $\varphi_m(X + e_i) = \varphi_m(X) + \varphi_m(e_i) \leq_{\text{lex}} \varphi_m(P) + \varphi_m(e_m) = \varphi_m(P + e_m)$, that is $X + e_i \in T_{P+e_m}^m$. \square

To see that the 2-page is the first page satisfying the isomorphism conditions, consider the terms of the form $S(P; m)$, belonging to the step immediately before the 2-page in the secondary connection. Each $S(P; m)$ is defined as $S[z, s, p, b]$, where s and p are as in (4.17), while z and b are

$$z := T_{P-e_m}^m, \quad b := T_{P+e_{m-1}}^m.$$

It is easy to see that $P + e_m \notin T_{P+e_{m-1}}^m$, for all $P \in \mathbb{Z}^m$, making false in general the sufficient conditions $\hat{h}_1(F_z) \subseteq F_s$ and $\hat{h}_1(F_p) \subseteq F_b$ for having an isomorphism. In fact, one can easily find non-isomorphic terms of the page ($S(P; m)$); we will show an example at the end of Section 4.6.2.

4.4.5 Twisting operators and towers of fibrations

As we mentioned at the beginning of this section, introducing twisted products to study towers of fibrations in the general scenario leads to complications regarding the associated spectral systems over $D(\mathbb{Z}^m)$. In particular, as we are going to explain, the issue concerns the definition of suitable filtrations on the involved chain complexes.

In order to better understand what goes wrong in the case of a tower of fibrations, let us briefly review the case of one fibration, where the problem does not manifest itself. As explained in Section 4.2, the total space of a fibration $G \rightarrow E \rightarrow B$ can be expressed as the twisted product $E = G \times_\tau B$, with $\tau : B \rightarrow G$ a twisting operator. For each n , the only difference between $(G \times B)_n$ and $(G \times_\tau B)_n$ regards the n -th face operator, which for the twisted product is defined by

$$\partial_n(g, b) := (\tau(b) \cdot \partial_n g, \partial_n b), \quad (4.18)$$

for all $(g, b) \in (G \times_\tau B)_n$. This implies that the differential of $C_*(G \times_\tau B)$ can be seen as a perturbed version of the differential of $C_*(G \times B)$, which allows to construct, from the Eilenberg-Zilber reduction $\text{EZ} = (f, g, h) : C_*(G \times B) \rightrightarrows C_*(G) \otimes C_*(B)$, the reduction $\text{EZ}' = (f', g', h') : C_*(G \times_\tau B) \rightrightarrows C_*(G) \otimes_t C_*(B)$ of Theorem 4.18. The key point now is that the filtrations F^\times and F^\otimes (Definitions 4.19 and 4.20), defined respectively on $C_*(G \times B)$ and $C_*(G) \otimes C_*(B)$, are filtrations also of the chain complexes $C_*(G \times_\tau B)$ and $C_*(G) \otimes_t C_*(B)$; in particular, notice that the perturbation given by (4.18) concerns only the first factor of $G \times_\tau B$ and therefore does not “affect” the filtration F^\times , defined using the degeneracy degree of the second factor. Then, as the maps f', g', h' are compatible with F^\times and F^\otimes , we concluded in Section 4.2.4 that the spectral sequences associated with the filtered chain complexes $(C_*(G \times_\tau B), F^\times)$ and $(C_*(G) \otimes_t C_*(B), F^\otimes)$ are isomorphic.

Let us now show that, for towers of fibrations, the filtrations we introduced for the case without twist (Definition 4.31) are (in general) no longer filtrations of chain complexes when we introduce the twisting operators.

For simplicity, let us consider a tower of two fibrations

$$\begin{array}{ccc} G_0 & \longrightarrow & E_0 \\ & & \downarrow \\ G_1 & \longrightarrow & E_1 \\ & & \downarrow \\ & & B \end{array}$$

where G_0, G_1 are simplicial groups and there are twisting operators $\tau_0 : E_1 \rightarrow G_0$ and $\tau_1 : B \rightarrow G_1$. This means that $E_1 = G_1 \times_{\tau_1} B$ and $E_0 = G_0 \times_{\tau_0} E_1 = G_0 \times_{\tau_0} (G_1 \times_{\tau_1} B)$. Consider at first

Cartesian products (without twist): proceeding as in Section 4.4.2, we obtain the reductions

$$\begin{array}{c} C_*(G_0 \times (G_1 \times B)) \\ \text{EZ}_0 \downarrow \\ C_*(G_0) \otimes C_*(G_1 \times B) \\ \text{id} \otimes \text{EZ}_1 \downarrow \\ C_*(G_0) \otimes C_*(G_1) \otimes C_*(B). \end{array}$$

Furthermore, we know that all the maps of the reductions EZ_0 and $\text{id} \otimes \text{EZ}_1$ are compatible with $D(\mathbb{Z}^m)$ -filtrations $(F_p)_{p \in D(\mathbb{Z}^m)}$ “naturally” defined (recall Definition 4.31) on the chain complexes $C_*(G_0 \times G_1 \times B)$, $C_*(G_0) \otimes C_*(G_1 \times B)$ and $C_*(G_0) \otimes C_*(G_1) \otimes C_*(B)$. Recall that the filtrations $(F_p)_{p \in D(\mathbb{Z}^m)}$ are of the type $F_p := \sum_{P \in p} F_P$, where for $P = (p_1, p_2) \in \mathbb{Z}^2$ we have

$$\begin{aligned} F_P C_*(G_0 \times G_1 \times B) &:= C_*(G_0 \times \mathcal{F}_P(G_1 \times B)), \\ F_P(C_*(G_0) \otimes C_*(G_1 \times B)) &:= C_*(G_0) \otimes C_*(\mathcal{F}_P(G_1 \times B)), \\ F_P(C_*(G_0) \otimes C_*(G_1) \otimes C_*(B)) &:= C_*(G_0) \otimes C_*(\mathcal{F}_{p_1} G_1) \otimes C_*(\mathcal{F}_{p_2} B). \end{aligned} \tag{4.19}$$

It is then easy to observe that, introducing the twist and considering therefore the chain complexes $C_*(G_0 \times_{\tau_0} (G_1 \times_{\tau_1} B))$, $C_*(G_0) \otimes_{t_0} C_*(G_1 \times_{\tau_1} B)$ and $C_*(G_0) \otimes_{t_0} (C_*(G_1) \otimes_{t_1} C_*(B))$ instead of $C_*(G_0 \times G_1 \times B)$, $C_*(G_0) \otimes C_*(G_1 \times B)$ and $C_*(G_0) \otimes C_*(G_1) \otimes C_*(B)$, the formulas (4.19) do not define in general filtrations of chain complexes. For example, if we focus on $C_*(G_0 \times_{\tau_0} (G_1 \times_{\tau_1} B))$ and consider a generic element $(g, g', b) \in (G_0 \times_{\tau_0} (G_1 \times_{\tau_1} B))_n$, we have

$$\begin{aligned} \partial_n(g, g', b) &= (\tau_0(g', b) \cdot \partial_n(g), \partial_n(g', b)) \\ &= (\tau_0(g', b) \cdot \partial_n(g), \tau_1(b) \cdot \partial_n(g'), \partial_n(b)). \end{aligned}$$

If we assume that $\deg g' \leq p_1$ and $\deg b \leq p_2$, we have no guarantee that the degeneracy degree of the second component of $\partial_n(g, g', b)$, namely $\tau_1(b) \cdot \partial_n(g')$, is again $\leq p_1$, because the twisting operator τ_1 can increase the degeneracy degree. Therefore, since $d_n := \sum_{i=0}^n (-1)^i \partial_i$, we may have that $d_n(F_P) \not\subseteq F_P$ showing that the first formula of (4.19) does not define a filtration of $C_*(G_0 \times_{\tau_0} (G_1 \times_{\tau_1} B))$. Similar problems arise with the other “twisted” chain complexes.

In conclusion, we have shown that, for our purposes, introducing the twisting operators for towers of fibrations is more difficult than for a single fibration, because of the need of non-trivial generalizations of the involved filtrations. As a theme for further work, we are interested in trying to define suitable filtrations, in order to extend our method for computing the Serre spectral system to the twisted case.

4.5 Discrete vector fields for algorithmic efficiency

The notion of *discrete vector field* was introduced by Robin Forman in his seminal work [For98], which gave birth to the combinatorial version of Morse theory called *discrete Morse theory*. Although Forman’s theory was conceived for CW-complexes and cellular homology, in what follows we refer

to the purely algebraic approach of [RS10] for the definitions and the main results. The choice of terminology, though, reveals a trace of the original framework as, for example, we will call *cells* the elements forming a basis of the groups of chains of a chain complex.

A discrete vector field can produce a reduction $\rho : C_* \Rightarrow D_*$; as usual, we are interested in the small (effective) chain complex D_* , which simplifies the homology computations for the chain complex C_* . Our aim is to introduce I -filtrations of the chain complexes C_* and D_* , and study how discrete vector fields can be used to produce a reduction which does not affect the associated spectral systems (Theorem 4.47). Extending some ideas presented in [RRS16], we determine easily checkable conditions which allow to speed up the computation of spectral systems associated with filtrations over \mathbb{Z}^m and $D(\mathbb{Z}^m)$.

Let us first introduce, following [RS10], some basic notions.

Definition 4.38. An *algebraic cellular complex* (C_*, β) is a free chain complex $C_* = (C_n, d_n)_{n \in \mathbb{Z}}$ such that every chain group C_n is endowed with a distinguished \mathbb{Z} -basis β_n . A basis chain $\sigma \in \beta_n$ is called an *n-cell*.

The bases β_n are not assumed to be finite, therefore the chain groups C_n are not necessarily of finite type. Clearly, there is an “obvious” algebraic cellular complex associated with each simplicial set K , with $\beta_n = K_n$. This is the object that the reader should have in mind for this section, as it is particularly suitable for examples; furthermore, when we consider filtrations of algebraic cell complexes in what follows, it is convenient to think them as induced by filtrations of simplicial sets.

Definition 4.39. Let (C_*, β) be an algebraic cellular complex. A *discrete vector field* V on C_* is a collection of pairs $V = \{(\sigma_j; \tau_j)\}_{j \in J}$ satisfying the following conditions:

- Every σ_j is an element of some β_n , in which case the corresponding τ_j is an element of β_{n+1} , the degree n depending on j .
- For each $j \in J$, σ_j is a *regular face* of the corresponding τ_j , meaning that the coefficient of σ_j in $d\tau_j$ is ± 1 .
- Each cell of C_* appears at most one time in V .

We will see that the cells of C_* not appearing in the vector field V are particularly important, since (under suitable hypotheses) they form a basis of the chain complex D_* figuring in a reduction $\rho : C_* \Rightarrow D_*$ which can be canonically defined from the discrete vector field V on C_* .

Definition 4.40. A pair $(\sigma_j; \tau_j)$ of V is called a *vector*; we use the notations $\tau_j = V(\sigma_j)$ or $\sigma_j = V^{-1}(\tau_j)$ to express the fact that σ_j and τ_j are the components of a vector of V . The cells σ_j and τ_j are called respectively a *source cell* and a *target cell*. A cell $\sigma \in \beta_n$ which does not appear in the discrete vector field V is called a *critical cell*.

Definition 4.41. Given a discrete vector field V , a *V-path* π of degree n and length m is a sequence $\pi = \{(\sigma_{j_k}; \tau_{j_k})\}_{0 \leq k < m}$ such that:

- Every pair $(\sigma_{j_k}; \tau_{j_k})$ is a vector of V and τ_{j_k} is an n -cell.

- For every $0 < k < m$, the component σ_{jk} is a *face* of τ_{jk-1} (meaning that the coefficient of σ_{jk} in $d\tau_{jk-1}$ is non-null), non necessarily regular but different from σ_{jk-1} .

Definition 4.42. A discrete vector field V is called *admissible* if, for every $n \in \mathbb{Z}$, a function $\lambda_n : \beta_n \rightarrow \mathbb{N}$ is provided such that the length of every V -path starting from $\sigma \in \beta_n$ is bounded by $\lambda_n(\sigma)$.

The following result, due to Forman [For98, § 8], has been generalized in [RS10] to the case of chain complexes not necessarily of finite type.

Theorem 4.43 ([RS10], Vector-Field Reduction Theorem). Let (C_*, β) be an algebraic cellular complex, with $C_* = (C_n, d_n)_{n \in \mathbb{Z}}$, and let $V = \{(\sigma_j; \tau_j)\}_{j \in J}$ be an admissible discrete vector field on C_* . Then V defines a canonical reduction $\rho = (f, g, h) : C_* \rightrightarrows C_*^c$, where $C_*^c = (C_n^c, d_n^c)_{n \in \mathbb{Z}}$ is the *critical* chain complex: each C_n^c is the free \mathbb{Z} -module generated by the critical n -cells, and d^c is a differential canonically defined from d and V .

Proof. We sketch the proof given in [RS10], as we will refer to it for proving Theorem 4.47. For each basis β_n , consider the partition $\beta_n^t \cup \beta_n^s \cup \beta_n^c$ into target, source and critical cells, which induces a decomposition (as \mathbb{Z} -modules) of the chain groups: $C_n = C_n^t \oplus C_n^s \oplus C_n^c$. By virtue of this decomposition, each differential d_n can be represented as a 3×3 matrix

$$d_n = \begin{bmatrix} d_{n,1,1} & d_{n,1,2} & d_{n,1,3} \\ d_{n,2,1} & d_{n,2,2} & d_{n,2,3} \\ d_{n,3,1} & d_{n,3,2} & d_{n,3,3} \end{bmatrix}.$$

It can be proven that $d_{n,2,1} : C_n^t \rightarrow C_{n-1}^s$ is an isomorphism, and that its inverse $d_{n,2,1}^{-1} : C_{n-1}^s \rightarrow C_n^t$ can be made explicit via the recursive formula

$$d_{n,2,1}^{-1}(\sigma) = \varepsilon(\sigma, V(\sigma)) \left(V(\sigma) - \sum_{\sigma' \in \beta_{n-1}^s \setminus \{\sigma\}} \varepsilon(\sigma', V(\sigma)) d_{n,2,1}^{-1}(\sigma') \right), \quad (4.20)$$

where $\varepsilon(\sigma, \tau)$ denotes the coefficient of σ in the differential $d\tau$. Then, the differential d^c and the maps f, g, h of the reduction can be explicitly defined as follows:

$$\begin{aligned} d_n^c &= d_{n,3,3} - d_{n,3,1} d_{n,2,1}^{-1} d_{n,2,3} & f_{n-1} &= \begin{bmatrix} 0 & -d_{n,3,1} d_{n,2,1}^{-1} & 1 \end{bmatrix} \\ g_n &= \begin{bmatrix} -d_{n,2,1}^{-1} d_{n,2,3} \\ 0 \\ 1 \end{bmatrix} & h_{n-1} &= \begin{bmatrix} 0 & d_{n,2,1}^{-1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned} \quad (4.21)$$

□

The reduction $\rho : C_* \rightrightarrows C_*^c$ of Theorem 4.43 provides in particular the isomorphisms $H_n(C_*) \cong H_n(C_*^c)$ and, as it was constructed explicitly, it allows to apply the effective homology technique whenever C_*^c is an effective chain complex. Discrete vector fields prove themselves extremely useful

also when the chain complex C_* , despite being effective, is too big (in terms of number of generators of the chain groups) to be treated with standard algorithms for homology in a reasonable amount of time; the reduction $\rho : C_* \Rightarrow C_*^c$ allows then to employ the smaller chain complex C_*^c to perform the computations.

Applications to spectral systems. Let us now consider an I -filtration $F = (F_i)_{i \in I}$ of the chain complex C_* , where (I, \leq) is a poset. Clearly F induces a filtration on $C_*^c = (C_n^c, d_n^c)$, which a priori is only a filtration of graded abelian groups, since one should verify that $d'(F_i) \subseteq F_i$ for all i in order to prove that it is a filtration of chain complexes. As this will be proven in Theorem 4.47, we can safely assume that also the critical chain complex C_*^c is filtered, through the I -filtration induced by F . Now we actually want to consider I -filtrations of algebraic cellular complexes, which means assuming that an additional compatibility condition is satisfied:

Definition 4.44. An I -filtration $F = (F_i)_{i \in I}$ of an algebraic cellular complex (C_*, β) is an I -filtration F of C_* compatible with faces, that is: if σ is a face of a cell τ , then $\tau \in F_i$ implies $\sigma \in F_i$, for all $i \in I$.

We now generalize a result of [RRS16], proving that, under suitable hypotheses on the discrete vector field V , the maps of the reduction $\rho : C_* \Rightarrow C_*^c$ are compatible with the filtrations and therefore the spectral systems associated with C_* and C_*^c are isomorphic. We treat the subject at first from a general theoretical point of view, providing then more practical conditions useful for software implementation.

Definition 4.45. Given a discrete vector field $V = \{(\sigma_j; \tau_j)\}_{j \in J}$ on C_* and an I -filtration $F = (F_i)_{i \in I}$ of (C_*, β) , we say that σ_j and τ_j *appear together* in the filtration if $\sigma_j \in F_i \iff \tau_j \in F_i$, for all $i \in I$.

Remark 4.46. Let $F = (F_i)_{i \in I}$ be an I -filtration of (C_*, β) . By Definition 4.44, if σ is a face of a cell τ , then $\tau \in F_i \implies \sigma \in F_i$, for all $i \in I$. Then, if $V = \{(\sigma_j; \tau_j)\}_{j \in J}$ is a discrete vector field such that each paired σ_j and τ_j appear together in the filtration, for each V -path $\pi = \{(\sigma_{j_k}; \tau_{j_k})\}_{0 \leq k < m}$ we can conclude that

$$\sigma_{j_0} \in F_i \implies \tau_{j_m} \in F_i,$$

for each $i \in I$.

Theorem 4.47. Let $F = (F_i)_{i \in I}$ be an I -filtration of an algebraic cellular complex (C_*, β) , where $C_* = (C_n, d_n)$. Let $V = \{(\sigma_j; \tau_j)\}_{j \in J}$ be an admissible discrete vector field on C_* such that, for all $j \in J$, the cells σ_j and τ_j appear together in the filtration. Consider on C_*^c the I -filtration induced by F . Then the three maps of the canonical reduction $\rho = (f, g, h) : C_* \Rightarrow C_*^c$ described in Theorem 4.43 are compatible with the filtrations.

Proof. We refer to the proof of Theorem 4.43. Recall the decomposition $C_n = C_n^t \oplus C_n^s \oplus C_n^c$ of the chain groups; on the groups C_n^t , C_n^s and C_n^c consider the ‘‘obvious’’ I -filtrations (of abelian groups) induced by F . Clearly, each component $d_{n,k,\ell}$ (with $k, \ell = 1, 2, 3$) of the differential d is compatible with the filtrations. As the differential d' of C_*^c and the maps f, g, h of the reduction are given by

(4.21), we only need to prove that $d_{n,2,1}^{-1}$ is compatible with the filtrations in order to conclude that d', f, g, h are compatible with the filtrations. For each $\sigma \in C_{n-1}^s$, using the recursive formula (4.20) we can express $d_{n,2,1}^{-1}(\sigma)$ as a finite sum

$$d_{n,2,1}^{-1}(\sigma) = \sum \lambda_k \tau_{j_k},$$

where the λ_k are coefficients in \mathbb{Z} and each τ_{j_k} is at the end of a V -path starting from σ . Then from Remark 4.46 follows that $d_{n,2,1}^{-1}$ is compatible with the filtrations. \square

Corollary 4.48. Under the hypotheses of Theorem 4.47, the spectral systems associated with the I -filtrations of (C_n, d_n) and (C_n^c, d_n^c) are isomorphic.

Proof. Apply Corollary 4.25. \square

As said before, Theorem 4.47 is a generalized version of a result stated in [RRS16] in the case of \mathbb{Z} -filtrations. In the cited paper, the hypothesis that σ_j and τ_j of each vector of V appear together is expressed in the following equivalent form: the components σ_j and τ_j of each vector of V have the same filtration index, where we say that $\sigma \in \beta_n$ has *filtration index* $i \in \mathbb{Z}$ if $\sigma \in F_i C_n \setminus F_{i-1} C_n$. For filtrations over \mathbb{Z}^m , we can generalize the concept of filtration index in the following way:

$$\text{GenFlin}(\sigma) := \{P \in \mathbb{Z}^m \mid \sigma \in F_P \setminus \cup_{Q < P} F_Q\}.$$

For example, consider the \mathbb{Z}^2 -filtered chain complex associated with the filtration represented in Figure 4.3. If we consider, for instance, the cells of β_0 , we have: $\text{GenFlin}(a) = \text{GenFlin}(b) = \text{GenFlin}(c) = \{(1, 1)\}$, $\text{GenFlin}(d) = \{(1, 2), (2, 1)\}$ and $\text{GenFlin}(e) = \{(1, 2)\}$.

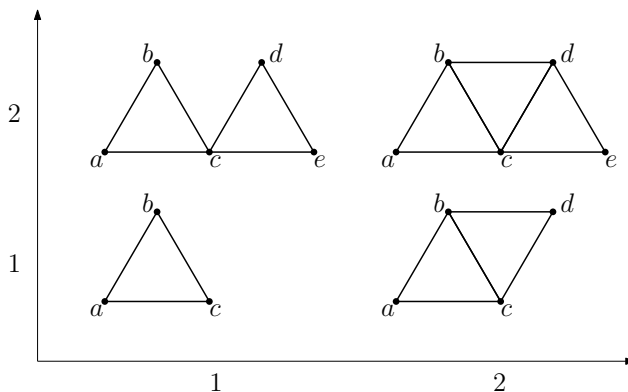


FIGURE 4.3: A simple filtration over \mathbb{Z}^2 of a simplicial set.

Consider a \mathbb{Z}^m -filtration $F = (F_P)_{P \in \mathbb{Z}^m}$ of a chain complex C_* and suppose that it is *coordinate-wise bounded below* in the following sense: for each $k = 1, \dots, m$ there exists an integer b_k such that $F_P = 0$ for all $P = (p_1, \dots, p_m)$ with $p_k < b_k$. Here we make the further assumption, motivated only by notational simplification, that $b_k = 0$ for all k , and we speak for brevity of a $\mathbb{Z}_{\geq 0}^m$ -filtration. In a $\mathbb{Z}_{\geq 0}^m$ -filtration, the notion of GenFlin of a cell σ provides a complete summary of the chain complexes F_P for which $\sigma \in F_P$:

Proposition 4.49. Let $F = (F_P)$ be a $\mathbb{Z}_{\geq 0}^m$ -filtration of C_* , and let $\sigma \in \beta_n$. Then $\sigma \in F_P$ if and only if $P \geq X$ for some $X \in \text{GenFlin}(\sigma)$.

Proof. The only non-trivial direction is the “only if” one. Suppose that $\sigma \in F_P$, for some $P = (p_1, p_2, \dots, p_m) \in \mathbb{Z}_{\geq 0}^m$. Consider the coordinates of P one at a time, starting from the first one: by the well-ordering principle there exists a point $P' = (x_1, p_2, \dots, p_m)$ with *minimal* first coordinate such that $\sigma \in F_{P'}$; then consider the second coordinate of P' , and so on. Repeating this simple argument for all the coordinates we obtain a point $X = (x_1, \dots, x_m) \in \mathbb{Z}_{\geq 0}^m$ such that $X \leq P$ and $X \in \text{GenFlin}(\sigma)$. \square

Proposition 4.49 has the following easy consequence: given a $\mathbb{Z}_{\geq 0}^m$ -filtration $F = (F_P)$ of C_* , we have $\text{GenFlin}(\sigma) = \text{GenFlin}(\tau)$ if and only if σ and τ appear together in the filtration F . This argument can be easily extended if we consider, instead of the $\mathbb{Z}_{\geq 0}^m$ -filtration (F_P) , the canonically associated $D(\mathbb{Z}_{\geq 0}^m)$ -filtration (F_p) , with $F_p = \sum_{P \in p} F_P$: two cells have the same GenFlin exactly when they appear together in the filtration (F_p) . In implementations, checking if two cells σ and τ have the same GenFlin is more convenient than using directly Definition 4.45. Thanks to these considerations, we generalized an algorithm of [RS10] to compute the spectral systems over $\mathbb{Z}_{\geq 0}^m$ or $D(\mathbb{Z}_{\geq 0}^m)$ using discrete vector fields and comparing the GenFlin of the components of each vector to ensure that the hypotheses of Theorem 4.47 are satisfied (see Algorithm 4.57 of Section 4.6.1).

Notice that to reach the conclusion of Proposition 4.49, assuming that the \mathbb{Z}^m -filtration is coordinate-wise bounded below is necessary, as the following counterexample shows:

Example 4.50. Let $F = (F_P)$ be a \mathbb{Z}^2 -filtration of a chain complex C_* such that, given two distinct cells σ and τ ,

- there exists $a \in \mathbb{Z}$ such that σ belongs exactly to the $F_{(p_1, p_2)}$ with $p_1 \geq a$;
- there exists $b \in \mathbb{Z}$ such that τ belongs exactly to the $F_{(p_1, p_2)}$ with $p_2 \geq b$.

Then, for example, $\sigma \in F_{(p_1, p_2)}$ and $\tau \notin F_{(p_1, p_2)}$ whenever $p_1 \geq a$ and $p_2 < b$, although by definition $\text{GenFlin}(\sigma) = \text{GenFlin}(\tau) = \emptyset$.

4.6 Algorithms and examples

In this section we present more in detail some algorithms and programs for computing spectral systems which we implemented as a new module for the system Kenzo, available at <http://www.unirioja.es/cu/anromero/research2.html>. We make use of the effective homology technique in order to handle infinitely generated chain complexes; moreover, we use discrete vector fields to improve the computational efficiency. Many of the theoretical results presented in the previous sections of this chapter find application in the programs we are going to introduce.

In Section 4.6.1 we describe the principal algorithms implemented in the new Kenzo module. In Section 4.6.2 we include some remarks on implementation and some examples of applications.

The subject of this section represents the core of the paper [GR18].

4.6.1 Effective algorithms for computing spectral systems

Let us consider at first an *effective* chain complex C_* with a filtration $F = (F_i)_{i \in I}$ over a poset (I, \leq) . Assume further to have an algorithm to compute each F_i , which is necessarily finitely generated. In this case, all the subgroups which appear in the definition of each term $S[z, s, p, b]$ of the associated spectral system, given by the formula

$$S[z, s, p, b] = \frac{F_p \cap d^{-1}(F_z) + F_s}{F_p \cap d(F_b) + F_s}, \quad (4.22)$$

can be determined and expressed as matrices, as we explained in Section 4.2.1 for spectral sequences. Then the corresponding quotient can be computed by means of diagonalization algorithms of matrices in a similar way to the algorithm used to compute homology groups by means of the Smith Normal Form technique [KMM04]. The result is a basis-divisors description of the group $S[z, s, p, b]$, that is to say, a list of combinations $(c_1, \dots, c_{k+\beta})$ which generate the group, together with the list of non-negative integers $(a_1, \dots, a_k, 0, \dots, 0)$, where a_1, \dots, a_k are the torsion coefficients of $S[z, s, p, b]$ and β is its rank (recall Theorem 1.19). The list of *divisors* can be seen as the list of the coefficients of the elements that appear in the denominator with regard to the list of combinations that generate the group. We obtain in this way the following algorithm:

Algorithm 4.51.

Input:

- an effective chain complex $C_* = (C_n, d_n)_{n \in \mathbb{Z}}$,
- an I -filtration $(F_i)_{i \in I}$ of C_* ,
- elements z, s, p, b in I such that $z \leq s \leq p \leq b$.

Output: a basis-divisors representation of the group $S[z, s, p, b]$.

For the computation of the differential maps $d : S[z_2, s_2, p_2, b_2] \rightarrow S[z_1, s_1, p_1, b_1]$, we compute first the basis-divisors representation of both $S_2 := S[z_2, s_2, p_2, b_2]$ and $S_1 := S[z_1, s_1, p_1, b_1]$ by means of Algorithm 4.51. Let us suppose that S_2 is generated by a list of combinations (c_1, \dots, c_t) with divisors $(a_1, \dots, a_t) = (a_1, \dots, a_k, 0, \dots, 0)$ (where $t = k + \beta$). Similarly, suppose that S_1 is generated by $(c'_1, \dots, c'_{t'})$, with divisors $(a'_1, \dots, a'_{t'}) = (a'_1, \dots, a'_{k'}, 0, \dots, 0)$.

Let a be a class of S_2 , given by means of its coefficients $(\lambda_1, \dots, \lambda_t)$ (with $\lambda_i \in \mathbb{Z}$) with respect to the set of generators (c_1, \dots, c_t) . Then, we consider the element $x = \sum_{i=1}^t \lambda_i c_i \in a$. To apply the differential map d to the class $a = [x]$, we compute an element $z \in a = [x]$ such that $z \in F_p \cap d^{-1}(F_z)$; in other words, we build the *projection* of $x \in F_p \cap d^{-1}(F_z) + F_s$ over the factor $F_p \cap d^{-1}(F_z)$. Then, we apply the differential map d to the element $z \in F_p \cap d^{-1}(F_z)$, we compute the coefficients of $d(z)$ with respect to the set of generators $(c'_1, \dots, c'_{t'})$, and we “simplify” them considering the corresponding divisors $(a'_1, \dots, a'_{t'})$. We obtain in this way the following algorithm:

Algorithm 4.52.

Input:

- an effective chain complex $C_* = (C_n, d_n)_{n \in \mathbb{Z}}$,
- an I -filtration $(F_i)_{i \in I}$ of C_* ,
- elements $z_1 \leq s_1 \leq p_1 \leq b_1$ and $z_2 \leq s_2 \leq p_2 \leq b_2$ in I such that $z_2 \leq p_1$ and $s_2 \leq b_1$ (see Section 2.2),
- a class $a \in S[z_2, s_2, p_2, b_2]$, given by means of the coefficients $(\lambda_1, \dots, \lambda_t)$ with respect to the basis (c_1, \dots, c_t) of the group $S[z_2, s_2, p_2, b_2]$ determined by Algorithm 4.51.

Output: the coefficients of the class $d(a) \in S[z_1, s_1, p_1, b_1]$ with respect to the basis (c'_1, \dots, c'_t) computed by means of Algorithm 4.51.

Let us consider now an I -filtered chain complex C_* which is not effective, but only *locally effective*. Since for some $n \in \mathbb{Z}$ the group of n -chains C_n can have an infinite number of generators, in general the subgroups appearing in the formula (4.22) cannot be determined and Algorithms 4.51 and 4.52 for computing the terms and differentials of the spectral system cannot be applied. Let us suppose that the chain complex C_* has effective homology: there exists an equivalence $C_* \xleftarrow{\rho_1} B_* \xrightarrow{\rho_2} D_*$ with $\rho_1 = (f_1, g_1, h_1)$ and $\rho_2 = (f_2, g_2, h_2)$, where D_* is an effective chain complex. In addition, let us suppose that the chain complexes B_* and D_* are also filtered over the poset I . Theorem 4.24 and Corollary 4.25 describe the conditions for which the spectral systems of the three filtered chain complexes are isomorphic; in this way it is possible to compute the spectral system of C_* by applying Algorithms 4.51 and 4.52 to the effective chain complex D_* .

Algorithm 4.53.

Input:

- a chain complex $C_* = (C_n, d_n)_{n \in \mathbb{Z}}$ with effective homology $C_* \xleftarrow{\rho_1} B_* \xrightarrow{\rho_2} D_*$,
- I -filtrations for C_* , B_* and D_* such that all the maps of the reductions ρ_1 and ρ_2 are compatible with the filtrations,
- elements $z \leq s \leq p \leq b$ in I .

Output: a basis-divisors representation of the group $S[z, s, p, b]$ of the spectral system associated with the filtered chain complex C_* .

Notice that we have stated the algorithm under the hypotheses of Corollary 4.25, namely that all the maps of the reductions ρ_1 and ρ_2 , including the homotopies h_1 and h_2 , are compatible with the filtrations. Algorithm 4.53 can be applied also if h_1 and h_2 are not compatible with the filtrations, but the correctness of the output is guaranteed only for the terms $S[z, s, p, b]$ satisfying the hypotheses of Theorem 4.24: $h_i(F_z) \subseteq F_s$ and $h_i(F_p) \subseteq F_b$ for both homotopies ($i = 1, 2$). The same remark applies to the following algorithm.

Algorithm 4.54.

Input:

- a chain complex $C_* = (C_n, d_n)_{n \in \mathbb{Z}}$ with effective homology $C_* \xleftarrow{\rho_1} B_* \xrightarrow{\rho_2} D_*$,

- I -filtrations for C_* , B_* and D_* such that all the maps of the reductions ρ_1 and ρ_2 are compatible with the filtrations,
- elements $z_1 \leq s_1 \leq p_1 \leq b_1$ and $z_2 \leq s_2 \leq p_2 \leq b_2$ in I such that $z_2 \leq p_1$ and $s_2 \leq b_1$,
- a class $a \in S[z_2, s_2, p_2, b_2]$, given by means of the coefficients $(\lambda_1, \dots, \lambda_t)$ with respect to the basis (c_1, \dots, c_t) of the group $S[z_2, s_2, p_2, b_2]$ determined by Algorithm 4.53.

Output: the coefficients of the class $d(a) \in S[z_1, s_1, p_1, b_1]$ with respect to the basis (c'_1, \dots, c'_t) computed by means of Algorithm 4.53.

Sometimes, even an effective (filtered) chain complex C_* can be so big that Algorithms 4.51 and 4.52 have efficiency problems and the computation of the associated spectral system cannot be done in a reasonable time. For this reason, as explained in Section 4.5, we are interested in using discrete vector fields to produce a smaller (filtered) chain complex, which is also effective, in such a way that the associated spectral systems are isomorphic and therefore we can compute the spectral system of the (big) initial chain complex by means of the one of the small chain complex. In Theorem 4.47 we saw that an admissible discrete vector field $V = \{(\sigma_j; \tau_j)\}_{j \in J}$ on C_* determines a canonical reduction $\rho = (f, g, h) : C_* \Rightarrow C_*^c$ whose maps are all compatible with the filtrations, provided that V satisfies the following condition: for all $j \in J$, the cells σ_j and τ_j appear together in the filtration (Definition 4.45). In what follows, we refer to this last condition saying that V is *compatible with the filtration* of C_* .

Theorem 4.47 allows therefore to determine the spectral system associated with a big chain complex by defining a discrete vector field producing a smaller chain complex (with isomorphic spectral system). Clearly now the problem is: how can we determine an admissible discrete vector field for a given chain complex C_* which is compatible with the given I -filtration of C_* ?

In [RS10], an algorithm is presented for computing an admissible discrete vector field (for a digital image), which furthermore is *maximal* in the following sense: it is impossible to add a new vector such that the new vector field is admissible. The idea of the algorithm is to consider the differential map for some degree n (which is given by an integer matrix M); the entries $M_{i,j}$ which are equal to $+1$ or -1 correspond to pairs (σ, τ) of generators $\sigma \in \beta_{n-1}$ and $\tau \in \beta_n$ which are possible vectors. The difficult part of the algorithm consists in choosing the correct $M_{i,j}$'s such that one obtains a discrete vector field which is admissible (see [RS10, Ch. 5] for details).

Algorithm 4.55. [RS10]

Input: an integer matrix M .

Output: a maximal admissible discrete vector field V for M .

Applying this algorithm to the differential map of C_* of some degree n , one obtains a discrete vector field $V = \{(\sigma_j; \tau_j)\}_{j \in J}$ for C_* , with all $\sigma_j \in \beta_{n-1}, \tau_j \in \beta_n$. Then, a canonical reduction (Theorem 4.43) $C_* \Rightarrow C_*^c$ is obtained, where C_*^c is frequently significantly smaller than C_* . The process can be iterated for C_*^c and a different degree producing again a reduction to a smaller chain complex.

Now, we have modified Algorithm 4.55 to obtain a discrete vector field which is compatible with the given I -filtration of C_* . This can be done by applying separately Algorithm 4.55 to some *submatrices* corresponding to the rows and columns of the elements which appear together in the filtration. In Section 4.5 we introduced the concept of *generalized filtration index* (GenFlin) and explained that, for some posets such as $\mathbb{Z}_{\geq 0}^m$ and $D(\mathbb{Z}_{\geq 0}^m)$, checking if two cells have the same GenFlin is equivalent to check if they appear together in the filtration, and it is more convenient from a computational point of view. We present the following algorithms for filtrations over the posets $\mathbb{Z}_{\geq 0}^m$ and $D(\mathbb{Z}_{\geq 0}^m)$ as we use the GenFlin to verify if two cells appear together; notice that these algorithms work for any poset I for which the equality of the GenFlin is equivalent to two cells appearing together, for each pair of cells.

Given an effective chain complex C_* with distinguished bases $\beta = (\beta_n)$ of the chain groups, a degree n and a filtration over $\mathbb{Z}_{\geq 0}^m$ or $D(\mathbb{Z}_{\geq 0}^m)$, we consider the (finite) list of generators β_n and we compute the set of all the generalized filtration indices $L_n = \{\text{GenFlin}(\sigma) \mid \sigma \in \beta_n\}$, which is finite and satisfies $\text{card}(L_n) \leq \text{card}(\beta_n)$. Then, for each $P \in L$ we determine the “submatrix” of the differential matrix with rows and columns corresponding to elements with generalized filtration index equal to P and we apply Algorithm 4.55. We obtain so a list of vectors (σ_j, τ_j) , with $\sigma_j \in \beta_{n-1}, \tau_j \in \beta_n$ and such that all σ_j and τ_j appear together in the filtration. Repeating the process for all the generalized filtration indices in L and concatenating the results we obtain an admissible discrete vector field which is compatible with the filtration and is *maximal* with respect to this property, in the following sense: it is impossible to add a new vector such that the new vector field is admissible and compatible with the filtration. We obtain in this way the following algorithm, which is a generalization of the one presented in [RRS16] (which produces an admissible discrete vector field that is compatible with a finite \mathbb{Z} -filtration).

Algorithm 4.56.

Input:

- an effective chain complex $C_* = (C_n, d_n)_{n \in \mathbb{Z}}$,
- a filtration of C_* over $\mathbb{Z}_{\geq 0}^m$ or $D(\mathbb{Z}_{\geq 0}^m)$,
- a degree n .

Output: a maximal admissible discrete vector field $V = \{(\sigma_j; \tau_j)\}_{j \in J}$ for C_* , with $\sigma_j \in \beta_{n-1}, \tau_j \in \beta_n$, which is compatible with the filtration.

Applying now Theorem 4.47, a reduction $\rho : C_* \Rightarrow C_*^c$ is obtained which is compatible with the filtrations. Then the process can be repeated for C_*^c and a different degree n' . Finally we obtain the following algorithm.

Algorithm 4.57.

Input:

- an effective chain complex $C_* = (C_n, d_n)_{n \in \mathbb{Z}}$,
- a filtration F of C_* over $\mathbb{Z}_{\geq 0}^m$ or $D(\mathbb{Z}_{\geq 0}^m)$.

Output:

- an effective chain complex D_* (usually) smaller than C_* ,
- a filtration F' of D_* over $\mathbb{Z}_{\geq 0}^m$ or $D(\mathbb{Z}_{\geq 0}^m)$ (induced by F),
- a reduction $\rho : C_* \Rightarrow D_*$ compatible with the filtrations.

4.6.2 Implementation and examples

Let us give an idea of how the algorithms presented in Section 4.6.1 have been implemented as a new module for the Kenzo system. The module consists of 2500 lines of Common Lisp code containing new definitions and functions which make it possible to determine the terms $S[z, s, p, b]$ and the corresponding differential maps.

The new programs work in a similar way to the module for computing spectral sequences explained in [RRS06]: if an I -filtered chain complex is effective, then the terms and differential maps can be directly computed by means of Algorithms 4.51 and 4.52; otherwise, the effective homology technique is used to determine the spectral system by means of the associated effective chain complex using Algorithms 4.53 and 4.54. Moreover, in some common cases Algorithm 4.57 can be used to reduce the size of effective chain complexes before computing the spectral system.

First of all, we have defined two new classes, `PARTIALLY-ORDERED-SET` and `GENERALIZED-FILTERED-CHAIN-COMPLEX`. In both cases, functional programming plays an important role and makes it possible to work with infinite objects. The definition of the class `PARTIALLY-ORDERED-SET` is the following (where some Lisp technical components without any interest have been omitted):

```
(DEFCLASS PARTIALLY-ORDERED-SET ()
  ((pocmpr :type (function (element element)
    '(member :less :equal :greater :undefined))))
```

The relevant slot is `pocmpr`, a function which inputs two elements of the poset and says if the first one is less, equal or greater than the second one or if they are not comparable. In this way we can easily implement posets with an infinite number of elements, such as for example \mathbb{Z}^2 :

```
> (setf z2 (z2))
[K23 Partially-Ordered-Set]
> (pocmpr z2 '(1 2) '(2 2))
:LESS
> (pocmpr z2 '(1 2) '(2 1))
:UNDEFINED
```

The class `GENERALIZED-FILTERED-CHAIN-COMPLEX` inherits from the Kenzo class `CHAIN-COMPLEX` and includes two new slots:

```
(DEFCLASS GENERALIZED-FILTERED-CHAIN-COMPLEX (chain-complex)
  ((pos :type partially-ordered-set )
   (gen-flin :type (function (generator) list-of-filtration-indexes))))
```

The first slot `pos` is the poset over which the generalized filtration is defined. The second one, `gen-flin`, is a function which inputs a generator of the chain complex and returns a list of elements of `pos`, the generalized filtration index (`GenFlin`).

The function for computing the associated spectral system is `gen-spsq-group`, which works as follows. We consider as a didactic example the chain complex associated with the \mathbb{Z}^2 -filtration of Figure 4.3 and compute the terms $S_1[(1, 1), (1, 2), (2, 2), (2, 2)]$ and $S_1[(1, 1), (1, 1), (2, 2), (2, 2)]$:

```
> (gen-spsq-group K '(1 1) '(1 2) '(2 2) '(2 2) 1)
Generalized spectral sequence S[(1 1),(1 2),(2 2),(2 2)]_{1}
Component Z
> (gen-spsq-group K '(1 1) '(1 1) '(2 2) '(2 2) 1)
Generalized spectral sequence S[(1 1),(1 1),(2 2),(2 2)]_{1}
Component Z
Component Z
```

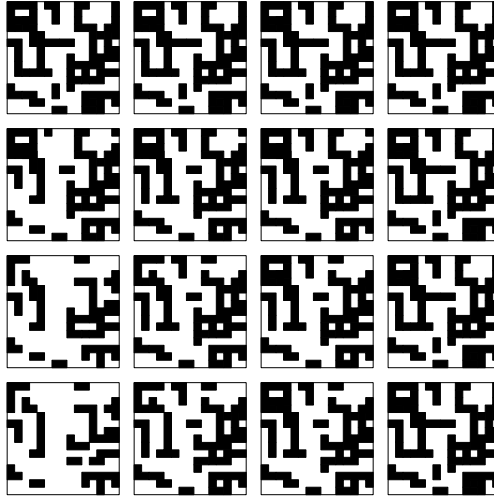
It is also possible to determine the generators of these groups by means of the function `gen-spsq-gnrts`. For example, the group $S_1[(1, 1), (1, 1), (2, 2), (2, 2)]$ is generated by the combinations $-1 * bd + 1 * cd$ and $1 * cd - 1 * cd + 1 * de$.

```
> (gen-spsq-gnrts K '(1 1) '(1 1) '(2 2) '(2 2) 1)
(
-----{CMBN 1}
<-1 * BD>
<1 * CD>
-----
-----{CMBN 1}
<1 * CD>
<-1 * CE>
<1 * DE>
-----
)
```

Finally, as explained in Section 4.6.1, we can also compute the differential maps $d : S_2 := S[z_2, s_2, p_2, b_2] \rightarrow S_1 := S[z_1, s_1, p_1, b_1]$. To this aim we must introduce a class a of S_2 , given by a list of coefficients $(\lambda_1, \dots, \lambda_t)$ in \mathbb{Z} with respect to the set of generators of S_2 . For instance, the differential map $d : S[(1, 1), (1, 2), (2, 2), (2, 2)] \cong \mathbb{Z} \rightarrow S[(0, 1), (0, 1), (2, 2), (2, 2)] \cong \mathbb{Z}$ is the null morphism:

```
> (gen-spsq-dffr K '(1 1) '(1 2) '(2 2) '(2 2)
      '(0 1) '(0 1) '(2 2) '(2 2) 1 '(1))
(0)
```

In the previous example, since the chain complex is small (we have only 5 generators of degree 0 and 7 of degree 1), the computation of the terms and differential maps of the spectral system has been done directly by means of diagonalization algorithms on matrices as explained for Algorithms 4.51 and 4.52. Now, let us consider a bigger chain complex such as the one corresponding to the digital image shown in Figure 4.4, filtered again over \mathbb{Z}^2 .

FIGURE 4.4: Digital image filtered over \mathbb{Z}^2 .

In this case the associated chain complex (constructed as described in [RRS16]) has 203 vertices, 408 edges and 208 triangles. Even if the chain complex is not very big, it is convenient to use discrete vector fields to reduce it and construct a smaller one by means of Algorithm 4.57. This is done directly by Kenzo, which stores the reduction as part of the slot `efhm` containing the effective homology of the initial chain complex. The effective chain complex determined by Algorithm 4.57 has in this case 21 vertices, 23 edges and 5 triangles.

```
> (efhm K2)
[K176 Homotopy-Equivalence K145 <= K145 => K159]
> (setf efK2 (rbcc (efhm K2)))
[K159 Generalized-Filtered-Chain-Complex]
> (length (basis efK2 0))
21
> (length (basis efK2 1))
23
> (length (basis efK2 2))
5
```

The computation of the terms and differential maps of the spectral system is done now in a more efficient way by means of Algorithms 4.53 and 4.54 (even if the initial chain complex is already effective).

```
> (gen-spsq-group K2 '(1 1) '(1 1) '(4 4) '(4 4) 1)
Generalized spectral sequence S[(1 1),(1 1),(4 4),(4 4)]_{1}
Component Z
Component Z
Component Z
Component Z
Component Z
Component Z
Component Z
Component Z
```

Let us conclude this section with an example that better illustrates the role of effective homology and of the theoretical results of Section 4.4.4 on equivalences and isomorphic terms. We want to show how to compute with Kenzo the Serre spectral system associated with the 3-fold Cartesian product $K(\mathbb{Z}_2, 2) \times K(\mathbb{Z}_2, 3) \times K(\mathbb{Z}_2, 4)$.

The Eilenberg-MacLane spaces $K(\mathbb{Z}_2, 2), K(\mathbb{Z}_2, 3), K(\mathbb{Z}_2, 4)$ can be constructed using the Kenzo function `k-z2`.

```
> (setf kz22 (k-z2 2) kz23 (k-z2 3) kz24 (k-z2 4))
[K213 Abelian-Simplicial-Group]
```

Since the chain complexes associated with the spaces $K(\mathbb{Z}_2, 2), K(\mathbb{Z}_2, 3), K(\mathbb{Z}_2, 4)$ are effective, Kenzo can handle them directly. In alternative, Kenzo can construct equivalences to smaller (effective) chain complexes, and use the latter for computations. For example, an equivalence between $C_*(K(\mathbb{Z}_2, 2))$ and an effective chain complex K230 is constructed automatically by Kenzo.

```
> (efhm kz22)
[K322 Homotopy-Equivalence K189 <= K312 => K230]
```

It may be interesting to compare the two chain complexes, in order to establish if the size has reduced significantly. In this case we can verify for example that, in dimension 4, a basis of $C_*(K(\mathbb{Z}_2, 2))$ has 41 elements, while a basis of the smaller chain complex K230 has only 2 elements.

```
> (length (basis kz22 4))
41
> (length (basis (k 230) 4))
2
```

Now, we can build (iteratively) the Cartesian product $K(\mathbb{Z}_2, 2) \times K(\mathbb{Z}_2, 3) \times K(\mathbb{Z}_2, 4)$

```
> (setf y (crts-prdc (crts-prdc kz22 kz23) kz24))
[K328 Simplicial-Set]
```

and turn it into a filtered chain complex $C_*(K(\mathbb{Z}_2, 2) \times K(\mathbb{Z}_2, 3) \times K(\mathbb{Z}_2, 4))$ (where the filtration is like in Definition 4.31) using the function `crpr3-gflin`.

```
> (setf yf (change-chcm-to-gflcc y (dz2) crpr3-gflin 'crpr3-gflin))
[K333 Generalized-Filtered-Chain-Complex]
```

Starting from the chain complex $C_*(K(\mathbb{Z}_2, 2) \times K(\mathbb{Z}_2, 3) \times K(\mathbb{Z}_2, 4))$, Kenzo automatically constructs a reduction to the tensor product $C_*(K(\mathbb{Z}_2, 2)) \otimes C_*(K(\mathbb{Z}_2, 3)) \otimes C_*(K(\mathbb{Z}_2, 4))$ (constructed iteratively) by applying iteratively Eilenberg-Zilber reductions, as explained in Section 4.4.2. Then, it uses also the equivalences

$$C_*(K(\mathbb{Z}_2, 2)) \Leftarrow \hat{C}_* \Rightarrow D_*, \quad C_*(K(\mathbb{Z}_2, 3)) \Leftarrow \hat{C}'_* \Rightarrow D'_*, \quad C_*(K(\mathbb{Z}_2, 4)) \Leftarrow \hat{C}''_* \Rightarrow D''_*$$

to construct the equivalence

$$C_*(K(\mathbb{Z}_2, 2)) \otimes C_*(K(\mathbb{Z}_2, 3)) \otimes C_*(K(\mathbb{Z}_2, 4)) \begin{array}{c} \xleftarrow{\hat{\rho}_1} \\ \xrightarrow{\hat{\rho}_2} \end{array} \hat{C}_* \otimes \hat{C}'_* \otimes \hat{C}''_* \begin{array}{c} \xrightarrow{\hat{\rho}_2} \\ \xleftarrow{\hat{\rho}_1} \end{array} D_* \otimes D'_* \otimes D''_*$$

we described in Section 4.4.2, which allows to use the smaller chain complex $D_* \otimes D'_* \otimes D''_*$ for computations. The use of the effective homology technique is then made possible by the equivalence between $C_*(K(\mathbb{Z}_2, 2) \times K(\mathbb{Z}_2, 3) \times K(\mathbb{Z}_2, 4))$ and $D_* \otimes D'_* \otimes D''_*$ resulting from the composition of the equivalences we introduced.

The filtration of the effective chain complex $D_* \otimes D'_* \otimes D''_*$ we introduced in Definition 4.31 can then be constructed using the function `tnpr3-gflin`.

```
> (setf ey (rbcc (efhm y)))
[K650 Chain-Complex]
> (setf eyf (change-chcm-to-gflcc ey (dz2) tnpr3-gflin 'tnpr3-gflin))
[K671 Generalized-Filtered-Chain-Complex]
```

As we said, Kenzo can compute the Serre spectral system associated with $C_*(K(\mathbb{Z}_2, 2) \times K(\mathbb{Z}_2, 3) \times K(\mathbb{Z}_2, 4))$ in two different ways: either directly, using the chain complex $C_*(K(\mathbb{Z}_2, 2) \times K(\mathbb{Z}_2, 3) \times K(\mathbb{Z}_2, 4))$, or with effective homology, using the chain complex $D_* \otimes D'_* \otimes D''_*$.

In according with Theorem 4.37, we have verified that (up to total degree $n = 3$) the terms of the 2-page ($S_n^*(P; 2)$) computed with the two methods coincide. The only non-null terms we obtained (with both methods) are the following:

```
(0 0) Generalized spectral sequence S[((1 -2)),((1 -1)),((0 0)),((0 1) (1 0))]._{0}
Component Z

(2 0) Generalized spectral sequence
      S[((0 0) (1 -1) (3 -2)),((0 1) (1 0) (3 -1)),((0 1) (2 0)),((0 2) (2 1) (3 0))]._{2}
Component Z/2Z

(0 3) Generalized spectral sequence S[((1 1) (2 0)),
      ((1 2) (2 1) (3 0)),((0 3) (1 2) (2 1) (3 0)),((0 4) (1 3) (2 2) (3 1) (4 0))]._{3}
Component Z/2Z
```

where the pair of integers at the beginning of a line represents the point $P = (p_1, p_2) \in \mathbb{Z}^2$ for which we are computing the term $S_n^*(P; 2)$. Notice further that the downsets z, s, p, b of each term are stored as lists of points: each list of points in \mathbb{Z}^2 represents the smallest downset of $D(\mathbb{Z}^2)$ containing those points. Using effective homology we were able to efficiently compute the remaining non-null terms up to total degree $n = 5$:

```
(0 0) Generalized spectral sequence S[((1 -2)),((1 -1)),((0 0)),((0 1) (1 0))]._{4}
Component Z/2Z

(4 0) Generalized spectral sequence S[((0 2) (1 1) (2 0) (3 -1) (5 -2)),
      ((0 3) (1 2) (2 1) (3 0) (5 -1)),((0 3) (1 2) (2 1) (4 0)),((0 4) (1 3) (2 2) (4 1) (5 0))]._{4}
```

Component $Z/4Z$

(0 5) Generalized spectral sequence $S[((1\ 3)\ (2\ 2)\ (3\ 1)\ (4\ 0)),((1\ 4)\ (2\ 3)\ (3\ 2)\ (4\ 1)\ (5\ 0)),$
 $((0\ 5)\ (1\ 4)\ (2\ 3)\ (3\ 2)\ (4\ 1)\ (5\ 0)),((0\ 6)\ (1\ 5)\ (2\ 4)\ (3\ 3)\ (4\ 2)\ (5\ 1)\ (6\ 0))]]_{\{5\}}$

Component $Z/2Z$

(2 3) Generalized spectral sequence $S[((0\ 3)\ (1\ 2)\ (3\ 1)\ (4\ 0)),((0\ 4)\ (1\ 3)\ (3\ 2)\ (4\ 1)\ (5\ 0)),$
 $((0\ 4)\ (2\ 3)\ (3\ 2)\ (4\ 1)\ (5\ 0)),((0\ 5)\ (2\ 4)\ (3\ 3)\ (4\ 2)\ (5\ 1)\ (6\ 0))]]_{\{5\}}$

Component $Z/2Z$

(5 0) Generalized spectral sequence $S[((0\ 3)\ (1\ 2)\ (2\ 1)\ (3\ 0)\ (4\ -1)\ (6\ -2)),$
 $((0\ 4)\ (1\ 3)\ (2\ 2)\ (3\ 1)\ (4\ 0)\ (6\ -1)),((0\ 4)\ (1\ 3)\ (2\ 2)\ (3\ 1)\ (5\ 0)),$
 $((0\ 5)\ (1\ 4)\ (2\ 3)\ (3\ 2)\ (5\ 1)\ (6\ 0))]]_{\{5\}}$

Component $Z/2Z$

We can now show with an example that, for the terms of the pages previous to the 2-page, one does not necessarily obtain isomorphic results using the two methods (see the discussion after Theorem 4.37). For example, the following term of the 1-page is different if computed directly

(3 0) Generalized spectral sequence $S[((0\ 2)\ (1\ 1)\ (2\ 0)\ (4\ -1)),$
 $((0\ 2)\ (1\ 1)\ (2\ 0)\ (4\ -1)),((0\ 2)\ (1\ 1)\ (3\ 0)),((0\ 2)\ (1\ 1)\ (3\ 0))]]_{\{3\}}$

Component Z

Component Z

Component Z

Component Z

or using effective homology

(3 0) Generalized spectral sequence $S[((0\ 2)\ (1\ 1)\ (2\ 0)\ (4\ -1)),$
 $((0\ 2)\ (1\ 1)\ (2\ 0)\ (4\ -1)),((0\ 2)\ (1\ 1)\ (3\ 0)),((0\ 2)\ (1\ 1)\ (3\ 0))]]_{\{3\}}$

Component Z

Conclusions

Since their introduction in the mid 1940s, spectral sequences have played a fundamental role in algebraic topology and homological algebra, as they can be seen as a method to determine homology groups via successive approximations. In a classical setting, spectral sequences often arise from filtrations indexed over \mathbb{Z} of a chain complex. The recent generalization of this notion to the case of filtrations indexed over any poset led to the development of the rich and interesting theory of spectral systems.

This thesis shows how spectral systems, besides opening new possibilities from a theoretical point of view, provide interesting applications outside the domain of theoretical algebraic topology and set new challenges from a computational perspective. In this respect, our contribution is twofold: on the one hand, we study the connection between spectral systems and generalizations of persistent homology, the most important method of the rapidly developing field of topological data analysis; on the other hand, we provide theoretical results which allow to understand how the effective homology technique can be applied to compute spectral systems, to implement actual algorithms and to assess their correctness.

Our new results on the relation between spectral systems and generalized persistent homology enhance our understanding of generalized persistence theory and open new perspectives for the application of spectral systems in the context of topological data analysis. However, since efficiency is a key factor when dealing with real data, spectral systems still cannot stand comparison with the most used topological invariants computed from data.

Our contributions on the use of the effective homology technique for computing spectral systems found direct application in a module we implemented for the computer algebra system Kenzo. Our algorithms can be used to compute spectral systems associated with filtered chain complexes which cannot be handled with other methods, for example because of their infinite number of generators. The increased efficiency we obtained through the use of discrete vector fields makes our programs suitable for a larger range of applications, from theoretical algebraic topology to the analysis of digital images.

Some questions remain open, and constitute interesting directions for future studies. In order to practically use the connection between spectral systems and persistent homology, identifying a subset of terms of the spectral system which carry enough topological information and are easily computable would represent a significant advancement. With regard to the use of effective homology to compute spectral systems, we are interested in further studying if our methods can be used in some relevant particular cases of spectral systems. We explained that a full understanding of how homological perturbation theory can be applied in the case of generalized filtrations is necessary to compute a generalized version of the Serre spectral sequence. In addition, we believe that some other

particular cases of spectral systems represent a stimulating theme of research, such as generalized versions of the Eilenberg-Moore and Adams-Novikov spectral sequences.

In conclusion, the generalization of spectral sequences obtained from chain complexes filtered over any poset, in addition to being useful from a theoretical perspective, raises new questions from an applied and computational point of view. The results presented in this work suggest new possibilities for their application and tackle in a systematic way the issues and questions related to their computation.

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