**Spatial natural hedging: a general framework with application to the mortality of U.S. states**



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# Spatial Natural Hedging – A general framework with application to the mortality of U.S. States

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**Abstract**

It is well known that coupling life and death benefits within an insurance portfolio may be a beneficial longevity risk reduction technique, especially when policies are underwritten in the same geographical region. However, though desirable, the lack of available capacity of life insurance instruments in terms of underlying cohorts or duration of products underwritten within a given region can substantially constrain the use of natural hedging strategies for life insurance companies. That is why the primary objective of this paper is to investigate the implementation and effectiveness of natural hedging strategies when considering the geographical or spatial dimension. Starting from a well-known multi-population mortality model, we evaluate the relevance of natural hedging strategies and their susceptibility to basis risk resulting from age, period, and spatial effects. Our novel theoretical findings provide direct insights into specific and often complex positions necessary for optimal real-world hedging. In a practical numerical application predicated on U.S. mortality data, we demonstrate the situation of a U.S.-based insurance company capable of selling policies across different states. Though often unable to curtail product sales, an insurance company using our analytical tool can effectively, through marketing strategies, stimulate or destimulate sales to approach an optimal hedging position of an overall portfolio.

**Keywords**: spatial natural hedging; longevity risk; U.S. mortality rates; mortality uncertainty.

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# 1 Introduction

The spatial variability of mortality rates in U.S. has been established and modeled across various academic works, most recently for example by Cupido et al. [2021, 2020b], Yang et al. [2015], Chen et al. [2012] and Yang et al. [2011]. For life insurers in the United States that conduct business regionally, or across the entire country, the spatial variability of mortality rates should be seriously taken into account when pricing and hedging longevity risk.

In particular, when it comes to the hedging of longevity risk, the natural hedging provided by combining life and death benefits in a single portfolio can reduce longevity risk and increase profitability. This procedure ideally uses products underwritten on the same cohorts within the same population, which entail no basis risk. Unfortunately, an insurer usually runs into capacity constraints which prevent the implementation of first-best optimal hedging strategies, i.e. strategies where basis risk is not present. Hence, second-best (constrained) optimal combinations of life and death benefits are required, which use products underwritten on different cohorts and populations. Multiple sources of basis risk can be considered to affect optimal hedging strategies. Populations' mortalities can be heterogeneous due to socio-economic characteristics, environmental or geographical factors, etc. In the most comprehensive analysis published so far, Ludkovski and Padilla [2024] analyze state mortality trends over time in the USA. Their findings confirm documented features (see the references therein) of U.S. mortality in the late 2010s, in terms of its deterioration on pre-retirement ages and the wide and growing mortality differentials between states, based on economic and health determinants. Indeed, their analysis shows the existence of multiple clusters of states in terms of their mortality trends, largely connected with median income.

In this paper, we focus on the mortality differences due to the geographical location of the policyholders and their effects on static natural hedging strategies. Luckily, given the development of cross-border business, the integration of the insurance industry across states in the United States allows for the utilization of hedging capacity that go beyond the local dimension of an insurance company.[1] That is why the implementation of cross-border or *spatial natural hedging* strategies in the context of spatial variability of mortality rates is the main object of this paper. To our knowledge, practical methodologies for conducting natural hedging fully accounting for the spatial dimension have not been developed in the academic literature. This paper aims to provide a theoretical basis for practical spatial natural hedging.

## 1.1 Motivation

As defined by the National Association of Insurance Commissioners (NAIC)[2], a domestic insurer in the United States is an insurer formed under the laws of the domiciliary state. In contrast, a foreign insurer is formed under the laws of another state, district, territory, or commonwealth of the U.S. Therefore, differently stated, in the U.S., a foreign insurer is an insurance company that is located in one state but which may underwrite policies for consumers in other states. According to NAIC's *2021 Insurance Department Resources Report*[3] (see page

[1] A similar reasoning applies to states within the European Union, for instance.

[2] See: `https://content.naic.org/industry-ucaa-definitions`

[3] See: `https://content.naic.org/sites/default/files/publication-sta-bb-volume-one.pdf`

37-38), there were 883 domestic insurers in the U.S. selling life/annuity products. Most importantly, when it comes to licensed foreign insurers selling life/annuity products, on average there were 376 per state, with a minimum of 12 in American Samoa and a maximum of 460 in Mississippi. This suggests that there is a significant cross-border activity of life/annuity insurers. [4]

## 1.2 Literature Review

From an empirical perspective, the seminal paper on natural hedging by Cox and Lin [2007] shows that insurers with better matched life and annuity portfolios may gain a competitive advantage over competitors. Following this work, the work of Tsai et al. [2010] uses a risk minimization approach based on the Conditional-Value-at-Risk (CVaR) measure to find an optimal product mix and hedge against systemic mortality risk. Inspired by the immunization theory in finance, albeit in a different setting, the works of Wang et al. [2010], Tsai and Chung [2013], Lin and Tsai [2013], Lin and Tsai [2014], further explore the optimal insurance-annuity product mixes to hedge against longevity risk. In the context of a whole life insurance company, when accounting for interactions between assets and liabilities, the works of Gatzert and Wesker [2012] and Luciano et al. [2017] find that the impact of natural hedging on an insurer's situation is intricate, as longevity risk and interest-rate risk are not to be taken in isolation. In particular, Luciano et al. [2017] introduce single-and cross-generation natural hedging of longevity and financial risks, thus drawing further attention on basis risk when natural hedging strategies involve term insurance products written on different generations. A similar immunization-based approach is developed by Lin and Tsai [2020b], accounting for both mortality and interest rates. Similarly, the work of Li and Haberman [2015] assesses hedge effectiveness modeling different dynamics of mortality rates for annuitants/pensioners and life insurance policyholders. To find the optimal hedge, Yang et al. [2019b] minimize the variance of the change of the insurers profit function, while Lin and Tsai [2020a] develop a multiyear hedging approach to account for uncertainty in mortality rates and the number of insureds. The work of Chen et al. [2022] introduces a framework to account for both insurance product demand and optimal product mix considering CVaR. In the context of the mixed fractional Brownian motion mortality model, the work by Zhou et al. [2022] offer a natural hedging approach based on matching mortality sensitivity. Most recently, Nyegaard [2023] consider an explicit Markov chain governing the state of the insured and account for a disability state when investigating natural hedging strategies.

## 1.3 Contribution

In contrast to all the above literature, we consider alongside age (or cohort)-based basis risk, the basis risk due to the different geographical locations of the annuity and life insurance policyholders. The only paper who, up to our knowledge, has explored the geographical dimension of basis risk is De Rosa et al. [2021] which, however, limits the analysis to a 2-population

[4] When it comes to the European Economic Area (EEA), the European Insurance and Occupational Pensions Authority (EIOPA) reports similar cross-border activity of life insurers in Europe. See: `https://www.eiopa.europa.eu/tools-and-data/insurance-statistics_en#cross-border-premiums`

case. Similarly to Yang et al. [2019a] who use minimum variance approach, we consider natural hedging when multiple populations are involved. However, unlike them, in this paper, we focus specifically on the geographical segmentation of different populations in an insurer's portfolio. This choice allows us to tackle the most important issue in natural hedging strategies, which are capacity constraints. To illustrate the problem, let us consider an insurer which is only operating in one state in the U.S.. The size of the life insurance and/or annuity market may not be large enough for the insurer to sell the number of policies needed to achieve an optimal static natural hedging strategy. Moreover, the profitability of the products in that state may limit the insurer's ability to execute a natural hedging strategy. All in all, an insurer operating in several states may be better equipped to overcome these capacity constraints. However, this trades-off with basis risk considerations, due to the differences in the mortality dynamics of individuals from different geographical areas, which may be relevant, depending on the similarity of the populations involved in the strategy.

**Goal.** From a theoretical perspective, our primary goal is to develop a natural hedging strategy when an insurer can optimally allocate its sales to different states and within those states underwrite products with different underlying cohorts. That is, unfortunately, rarely possible. Indeed, an insurer can consider utilizing or expanding its life insurance and/or annuity business to another state to increase capacity. However, insurers often cannot stop product sales or precisely sell underwritten products (short of having access to certain bulk reinsurance arrangements). Thus, they always have some product capacity and operational constraints over space. Therefore the practical questions are twofold. First, how cross-border sales can be stimulated or destimulated via geographically attentive marketing strategies to reach an optimal hedging position of an overall portfolio? Practically, what should the directionality of those efforts be across space, and how to quickly ascertain that? Second, accounting for existing capacity and operational constraints across space, what are the optimal hedging positions available, and how different are they from the theoretically achievable ones?

**Contribution.** To achieve this goal, our contribution to the literature is three-fold. First, in an unconstrained optimization setting, we propose a general natural hedging strategy, which stems from the well-established risk-minimization approach of natural hedging and extends it to account for the spatial dimension. Second, in a constrained optimization setting, we show how second-best hedging strategies can be implemented by considering hedging through different populations, when the capacity of the life insurance market is limited, as it happens in reality. Third, we propose a real-world based analysis of competing hedging strategies when an insurer needs to hedge an annuity position. For this purpose, we develop an application based on U.S. CDC data. We fit the Li and Lee [2005] mortality model using the data of the 48 contiguous U.S. states and then produce forecasts to be able to price life annuities and insurance products. We then illustrate the effectiveness of different natural hedging strategies, before implementing optimal and second-best optimal strategies when capacity constraints are introduced. We measure the quality of hedging using two measures: variance and value-at-risk, as a measure of tail risk, as in Wong et al. [2017]. Finally, we provide a measure of cost/efficiency, which is the increment in the hedged liability value per percentage point of effectiveness improvement.

The paper unfolds as follows. Section 2 describes the mortality model we use throughout the paper as well as introduces basic life insurance products and relevant notation. Section 3 illustrates the framework and the hedging strategies. Section 4 provides an application, fitting the model to the data, deriving optimal natural hedging strategies and discussing their effectiveness. Section 5 concludes the paper.

# 2 Model Background

In this section, we introduce the mortality model and the relevant insurance products we use as instruments in our spatial natural hedging framework.

## 2.1 The Li-Lee Mortality Model

The modelling approach introduced by Li and Lee [2005] extends the popular discrete-time Lee-Carter model [Lee and Carter, 1992] to a multi-population framework, by introducing an assumption to ensure coherence in the mortality projections between groups in the long run. Let $m_{x,t,s}$ be the year-$t$ central death rate for an age $x$, belonging to population/state $s$. The Li-Lee model is given by

$$\ln m_{x,t,s} = \alpha_{x,s} + B_x K_t + \beta_{x,s}\kappa_{t,s} + \epsilon_{x,t,s}, \tag{1}$$

where

- $\alpha_{x,s}$ is an age-state-specific parameter,
- $\beta_{x,s}$ is an age-state-specific parameter which interacts with the state-specific time-index $k_{t,s}$,
- $B_x$ is an age-specific parameter which interacts with the common time-index $K_t$, and
- $\epsilon_{x,t,s}$ is the error term.

The parameters of the Li-Lee model are estimated in steps. First, the parameter $\alpha_{x,s}$ is estimated by calculating the average death rate for each state and age across the time period, $T$, as

$$\hat{\alpha}_{x,s} = \frac{\sum_{t=0}^{T} \ln(m_{x,s,t})}{T+1}.$$

The common parameters $B_x$ and $K_t$ are then estimated by a singular value decomposition from the combined data for all the states, with normalization conditions $\sum_x B_x = 1$ and $\sum_t K_t = 0$. Finally, parameters $\beta_{x,s}$ and $k_{t,s}$ are estimated using a singular value decomposition to the population-specific residuals left unexplained by the common factors, with normalization conditions: $\sum_s \beta_{x,s} = 1$ and $\sum_t \kappa_{t,s} = 0$.

As the Li-Lee model deconstructs age and time patterns as two separate components, projections of mortality rates simply involve extrapolating the time-dependent variables of the model. As outlined by Li and Lee [2005], time series models such as the random walk without drift or a first-order autoregressive model are used to forecast $K_t$ and $\kappa_{t,s}$. To ensure coherence of the forecasted log-mortality rates, the common parameters, $K_t$, are assumed to be non-stationary with a linear trend, and the population-specific processes, $\kappa_{i,t}$, must be stationary.

## 2.2 Insurance Products

We consider a typical life insurance company, which can sell both life insurance policies and annuities. These products can be sold to different age/cohort groups, possibly living in different states. We assume that the age, period and state affect the mortality of the individuals. The goal of natural hedging is to find an optimal portfolio mix between the number of life insurance products sold and the number of annuities sold across states and ages.

### 2.2.1 Annuity

We consider a single premium $\tau$-year deferred whole life annuity, sold to an individual from state $s$, who has survived to age $x$ at the beginning of year $t$. After the deferral period, an annuity payment of $C$ is payable to the annuitant at the end of each year until death. Assuming a constant interest rate[5] $r$, the present value of the liabilities associated to the annuity is

$$\mathcal{V}^a(x, s, \tau) = C \cdot \sum_{T=1}^{\infty} e^{-r(\tau+T)} p_{x,t,s}(\tau + T), \tag{2}$$

where we denote with $p_{x,t,s}(\tau + T)$ the probability that an individual from state $s$ aged $x$ at the beginning of year $t$ would survive another $\tau + T$ years.

While we model mortality rates as a stochastic process, we assume that the force of mortality is constant during a year. Contingent on a specific mortality path, we can express the survival probability $p_{x,t,s}(\tau + T)$ as

$$p_{x,t,s}(\tau + T) = \exp\left( - \sum_{u=0}^{\tau+T-1} m_{x+u,t+u,s} \right), \tag{3}$$

where $m_{x+u,t+u,s}$, $u = 0, \ldots, \tau + T - 1$, is the central death rate at age $x + u$ in year $t + u$ for state $s$. The present value of the liabilities $V^a$ associated to the annuity has a distribution, because its value depends upon the realization of mortality rates. It is the variability of this value that the insurer needs to hedge.

### 2.2.2 Life Insurance

Consider a single premium $\tau$-year term life insurance policy with $D$ death benefit payable at the end of the year of death to an individual from state $s$ aged $x$ at the beginning of year $t$. The present value of the life insurance liabilities is

$$\mathcal{V}^l(x, s, \tau) = D \cdot \sum_{T=0}^{\tau-1} e^{-r(T+1)} p_{x,t,s}(T) q_{x+T,t+T,s}, \tag{4}$$

where $q_{x,t,s}$ is the probability that an individual from state $s$, who has survived to age $x$ at the beginning of year $t$, dies in year $t$ (between time $t - 1$ and $t$). Under the constant force of mortality assumption, and contingent on a mortality path, we have

$$q_{x,t,s} = 1 - \exp\left(-m_{x,t,s}\right),$$

where $m_{x,t,s}$ is described by the assumed (spatial) mortality model.

[5]Notice that throughout the paper, we abstract from the presence of interest-rate risk in liability valuation. The interested reader may find this issue tackled in Wong et al. [2017] and Luciano et al. [2017], for instance.

# 3 Spatial Natural Hedging

In this section, we develop a spatial natural hedging framework. First, we introduce a notion of hedging strategies and then discuss optimal hedging strategies under increasing levels of generality.

## 3.1 Hedging Strategies

We assume that the insurer holds a portfolio of $M$ annuity (indexed $a$) policies with yearly installments $C = 1$ and a deferral period of $\tau_j^a$ years, issued to individuals from state $s_j^a$ aged $x_j^a$, $j = 1, \ldots, M$. The present value of the insurer's (unhedged) annuity liabilities is then

$$\mathcal{U} = \sum_{j=1}^{M} \mathcal{V}^a(x_j^a, s_j^a, \tau_j^a). \tag{5}$$

Being a life insurance company, the undertaking can sell both life insurance products and annuities. Hence, the insurer can evaluate the benefit from the natural hedging diversification introduced when coupling life insurance products and the annuity portfolio. We consider the position of an insurer that is able to sell $N$ different $\tau_i^l$-year term life (indexed $l$) insurance policies with a death benefit of $D = 1000$ to individuals aged $x_i^l$ from state $s_i^l$, $i = 1, \ldots, N$. We can then represent the present value of the insurer's total liabilities as

$$\mathcal{P} = \sum_{j=1}^{M} \mathcal{V}^a(x_j^a, s_j^a, \tau_j^a) + \sum_{i=1}^{N} h_i \mathcal{V}^l(x_i^l, s_i^l, \tau_i^l), \tag{6}$$

where $h_i$ is the number of the $i$-th life insurance policies sold.

Equation (6) provides a general representation of a multi-population insurance portfolio and it is the basis for our spatial natural hedging framework. Indeed, the insurer may choose to aim at creating a portfolio with different numbers of life insurance policies $N$, different volumes $h_i$ involving different states $s_i$ and ages $x_i$. These choices constitute a spatial natural hedging strategy. Here, it is worth discussing briefly the implementation of the strategy. Surely, the insurer has some control over $N$, because it can control the geographical dimension of the portfolio, deciding to be active in certain states only. Also, it can place upper limits on the number $h_i$ of policies to sell to different cohorts $x_i$, but does not have full control on the quantities.

Before moving to the description of optimal strategies, let us now provide some comments on examples of simple naturally hedged portfolios, accounting for the spatial dimension. Suppose for the sake of simplicity that the insurer has sold only one 15-year deferred annuity to an individual aged 50 from California (i.e., $M = 1$, $x_1^a = 50$, $\tau_1^a = 15$ and $s_1^a =$ California). Then, the insurer may employ one of the following natural hedging strategies to mitigate its longevity risk exposure:

1. One 15-year term life insurance policy sold to an individual aged 50 from California (i.e., $N = 1$, $x_1^l = 50$, $\tau_1^l = 15$ and $s_1^l =$ California).

   *Comment.* In this strategy, both the cohort and state of the annuitant and life insurance policyholder coincide. In other words, the insurer has sold both policies to individuals with homogeneous mortality dynamics. This strategy is thus not exposed to the basis risk resulting from differences in the mortality experiences of the annuitant and the life insurance policyholder, at least for the deferral period. Indeed, some residual risk remains, because the insurer remains exposed to the period effect of mortality affecting the rates after age 66, because the two insurance contracts' payment durations differ.

2. One 15-year term life insurance policy sold to an individual aged 35 from California (i.e., $N = 1$, $x_1^l = 35$, $\tau_1^l = 15$ and $s_1^l =$ California).

   *Comment.* This strategy may be in reality the most typical one followed by insurers, because the life insurance market for older individuals, aged 50 f.i., is limited in capacity. Usually, the insurer is able to sell life insurance policies in sufficient amount to younger generations only (f.i. to 35-year old). This strategy, differently from the previous one, exposes the insurer to risks arising from differences in the mortality experience between the two generations (aged 35 and aged 50). This is the most commonly used setup in previous studies on natural hedging (see for instance Jevtić and Regis [2015]).

3. One 15-year term life insurance policy sold to an individual age 35 from a different state, for instance, Arizona (i.e., $N = 1$, $x_1^l = 35$, $\tau_1^l = 15$ and $s_1^l =$ Arizona).

   *Comment.* This strategy assumes that the life insurance market in California is not available to the insurer, but that the insurer can sell life insurance policies in a neighbour state. The strategy, relative to the previous ones, is thus exposed to the risk related to spatial mortality differences. It is still a viable strategy, as commonalities exist in the mortality dynamics of the different states, as modeled in (1), not only because of the common trend in mortality in the U.S., but also because of geographical proximity.

4. One 15-year term life insurance policy sold to an individual aged 35 from New York (i.e., $N = 1$, $x_1^l = 35$, $\tau_1^l = 15$ and $s_1^l =$ New York).

   *Comment.* This strategy further assumes that the life insurance market is not available to the insurer not only in California but also in all its nearby states. The insurer has to consider expanding its business to a more distant state, for instance in the East coast. We can think of the insurer in this case as a nation-wide insurance company that has business all over the country.

5. Two life insurance policies: one 15-year term life sold to an individual age 35 from California and one 15-year term life sold to an individual age 35 from New York (i.e., $N = 2$, $x_1^l = 35$, $\tau_1^l = 15$, $s_1^l =$ California, $x_2^l = 55$, $\tau_2^l = 15$ and $s_2^l =$ New York).

   *Comment.* This strategy assumes that the insurer can complement the "imperfect" matching of the cohort within its annuity liability's state (California), selling a policy to that cohort in another state. This strategy might be useful when the cohort effect is strong or when the capacity of the life insurance market in the states in which the company has sold the annuity is limited.

## 3.2 Optimal Natural Hedging Strategies

The portfolios described in the previous section are simple sub-optimal hedges, because the life insurance positions are not optimally selected according to any criterion, but simply one life insurance policy is sold for each annuity in the portfolio. From now on, we assume that the insurer can control, up to some capacity limitations, the number of insurance policies it sells, to achieve the best (or second-best) longevity risk diversification. Indeed, we are going to analyze the optimal trade-off between longevity risk diversification through the portfolio mix of annuities and life insurance policies and the basis risk introduced by hedging positions written on different cohort and states, accounting for the geographical dimension.

Let us now more formally describe the problem of implementing an optimal natural hedging strategy in our framework. Here, the term optimal refers to the problem of finding, given an annuity portfolio and a set $\mathcal{N}$ of possible life insurance products, the positions $h_i, i = 1, ..., N$, such that a chosen risk measure on the hedged portfolio of liablities $\mathcal{P}$ is minimized. Following Yang et al. [2019a] and other studies on natural hedging, we consider variance minimization.

Hence, we look for the optimal mix of $h_i^*$, $i = 1, \dots, N$ that solves

$$\min_{h_1,\dots,h_N \in \mathbb{R}} \text{Var}\left( \sum_{j=1}^{M} \mathcal{V}^a(x_j^a, s_j^a, \tau_j^a) + \sum_{i=1}^{N} h_i \mathcal{V}^l(x_i^l, s_i^l, \tau_i^l) \right). \tag{7}$$

We follow Cairns et al. [2014] and Yang et al. [2019a] and define the hedge effectiveness as the risk measure percentage reduction achieved by implementing the hedge. When we consider the variance reduction, we have:

$$R^2(h) = 1 - \frac{Var(\mathcal{P})}{Var(\mathcal{U})} = 1 - \frac{Var\left(\mathcal{V}^a(x^a, s^a, \tau^a) + h\mathcal{V}^l(x^l, s^l, \tau^l)\right)}{Var\left(\mathcal{V}^a(x^a, s^a, \tau^a)\right)} \tag{8}$$

When evaluating the hedge effectiveness, on top of looking at the variance reduction, we consider also a measure of the reduction in tail risk, namely the value-at-risk (VaR) percentage reduction. When we can consider the percentage reduction in Value-at-Risk at a significance level of $\eta$, we call this measure of effectiveness $T(h; \eta)$:

$$T(h; \eta) = 1 - \frac{\text{VaR}_\eta\left(\frac{\mathcal{P}(h) - \mathbb{E}[\mathcal{P}]}{\mathbb{E}[\mathcal{P}]}\right)}{\text{VaR}_\eta\left(\frac{\mathcal{U} - \mathbb{E}[\mathcal{U}]}{\mathbb{E}[\mathcal{U}]}\right)}. \tag{9}$$

Notice that, since the unhedged and hedged portfolios have different values, we compute effectiveness by calculating the change in the ratio between the unexpected increase in the liability value (at a certain $\eta$ confidence level) and the expected value of liabilities.

Let us now examine some situations in which the optimal hedge can be computed explicitly.

### 3.2.1 Case 1: $M = 1$ and $N = 1$

In the simple case where $M = 1$ and $N = 1$, i.e. the insurer has an annuity portfolio of policies issued to one cohort in one state and it is hedging it with life insurance policies written on one cohort in a certain state, we have:

$$\min_{h \in \mathbb{R}} \text{Var}\left(\mathcal{V}^a(x^a, s^a, \tau^a) + h\mathcal{V}^l(x^l, s^l, \tau^l)\right), \tag{10}$$

where, as in Wang et al. [2010], $h$ is the so-called hedge ratio, i.e. the number of life insurance policies needed to hedge the longevity risk of one annuity policy.

The optimal mix $h^*$ can be found, following Cairns et al. [2014], as:

$$h^* = -\frac{\text{Cov}\left(\mathcal{V}^a(x^a, s^a, \tau^a), \mathcal{V}^l(x^l, s^l, \tau^l)\right)}{\text{Var}\left(\mathcal{V}^l(x^l, s^l, \tau^l)\right)}$$
$$= -\text{Corr}\left(\mathcal{V}^a(x^a, s^a, \tau^a), \mathcal{V}^l(x^l, s^l, \tau^l)\right)\sqrt{\frac{\text{Var}\left(\mathcal{V}^a(x^a, s^a, \tau^a)\right)}{\text{Var}\left(\mathcal{V}^l(x^l, s^l, \tau^l)\right)}}.$$

It follows that the value of $h^*$ depends on the correlation between $\mathcal{V}^a(x^a, s^a, \tau^a)$ and $\mathcal{V}^l(x^l, s^l, \tau^l)$ and their variance. The covariance between $V^a$ and $V^l$ is expected to be negative because longevity improvements (decrements) simultaneously increase (decrease) the value of annuity liabilities and decrease (increase) the value of life insurance ones. Thus, $h^*$ is expected to be positive. Its magnitude depends crucially on the assumed underlying mortality model, which is the only risk source that makes $V^a$ and $V^l$ stochastic in our framework. Indeed, hedge effectiveness is intimately related to the ability of the assumed model to capture the correlation and variances accurately. Thus, if a natural hedge is established using insurance products from multiple populations/states, and the geographical effects are important, then the use of a mortality model that can capture the spatial dependence among multiple populations/states is crucial.

Recalling Equation (8), we can further express the hedge effectiveness $R^2(h)$ in terms of the optimal mix $h^*$ and the correlation between $\mathcal{V}^a(x^a, s^a, \tau^a)$ and $\mathcal{V}^l(x^l, s^l, \tau^l)$:

$$R^2(h) = \left(\text{Corr}\left(\mathcal{V}^a(x^a, s^a, \tau^a), \mathcal{V}^l(x^l, s^l, \tau^l)\right)\right)^2 \left(1 - \left(\frac{h - h^*}{h^*}\right)^2\right).$$

It follows that the 'best' $R^2(h)$ is reached when $h = h^*$ (the optimal mix), and the resulting 'best' $R^2(h)$ equals

$$\left(\text{Corr}\left(\mathcal{V}^a(x^a, s^a, \tau^a), \mathcal{V}^l(x^l, s^l, \tau^l)\right)\right)^2.$$

In other words, the optimal natural hedge is achieved when the correlation (in absolute terms) between $\mathcal{V}^a(x^a, s^a, \tau^a)$ and $\mathcal{V}^l(x^l, s^l, \tau^l)$ is maximized. In terms of constructing a hedging strategy, then, it means that the smaller the "basis risk" in the hedge, the highest will be the degree of effectiveness achieved.

#### 3.2.2 Case 2: $M = 1$ and $N = 2$

When $N = 2$, a combination of two life insurances is used as the hedging set in a natural hedge. This case is important to represent the situation in which an insurer can sell policies in two different states, or to different cohorts. In this case, we have

$$\min_{h_1, h_2 \in \mathbb{R}} \text{Var}\left(\mathcal{V}^a(x^a, s^a, \tau^a) + h_1\mathcal{V}^l(x_1^l, s_1^l, \tau_1^l) + h_2\mathcal{V}^l(x_2^l, s_2^l, \tau_2^l)\right), \tag{11}$$

where $h_1$ and $h_2$ are the number of policies needed from each of two life insurances, respectively.

Again, the optimal mixes $h_1^*$ and $h_2^*$ can be solved analytically. Let

$$\begin{aligned}
\sigma_{l_1,l_1} &= \text{Var}\left(\mathcal{V}^l(x_1^l, s_1^l, \tau_1^l)\right), \\
\sigma_{l_2,l_2} &= \text{Var}\left(\mathcal{V}^l(x_2^l, s_2^l, \tau_2^l)\right), \\
\sigma_{a,l_1} &= \text{Cov}\left(\mathcal{V}^a(x^a, s^a, \tau^a), \mathcal{V}^l(x_1^l, s_1^l, \tau_1^l)\right), \\
\sigma_{a,l_2} &= \text{Cov}\left(\mathcal{V}^a(x^a, s^a, \tau^a), \mathcal{V}^l(x_2^l, s_2^l, \tau_2^l)\right), \\
\sigma_{l_1,l_2} &= \text{Cov}\left(\mathcal{V}^l(x_1^l, s_1^l, \tau_1^l), \mathcal{V}^l(x_2^l, s_2^l, \tau_2^l)\right).
\end{aligned}$$

The optimal positions $h_1^*$ and $h_2^*$ are

$$h_1^* = \frac{\sigma_{a,l_2}\sigma_{l_1,l_2} - \sigma_{a,l_1}\sigma_{l_2,l_2}}{\sigma_{l_1,l_1}\sigma_{l_2,l_2} - \sigma_{l_1,l_2}\sigma_{l_1,l_2}}, \tag{12}$$

$$h_2^* = \frac{\sigma_{a,l_1}\sigma_{l_1,l_2} - \sigma_{a,l_2}\sigma_{l_1,l_1}}{\sigma_{l_1,l_1}\sigma_{l_2,l_2} - \sigma_{l_1,l_2}\sigma_{l_1,l_2}}. \tag{13}$$

Similar to Case 1, the optimal mixes $h_1^*$ and $h_2^*$ depend on the variances and covariances of the annuity liability being hedged and the two hedging instruments (the two life insurance policies). If these insurance products are sold in different populations/states, then, in order to obtain the correct values of the optimal mixes $h_1^*$ and $h_2^*$, we need a mortality model that can accurately capture these variances and covariances.

### 3.2.3 Case 3: General $M$ and General $N$

For the general case of (6), the liabilities being hedged, denoted as $\mathcal{V}_a := \sum_{j=1}^{M} \mathcal{V}_a(x_j^a, s_j^a, \tau_j^a)$ can be viewed as a single portfolio. The optimal mixes, $h_1^*, \ldots, h_N^*$, are determined by solving

$$\min_{h_1,\ldots,h_N \in \mathbb{R}} \text{Var}\left(\mathcal{V}_a + \sum_{i=1}^{N} h_i \mathcal{V}_{l_i}(x_i^l, s_i^l, \tau_i^l)\right). \tag{14}$$

Have $\mathcal{V}_i = \mathcal{V}^l(x_i^l, s_i^l, \tau_i^l)$ and $h_a = 1$ then the optimization problem in equation (14) can be expressed as

$$\min_{h_1,\ldots,h_N \in \mathbb{R}} \sum_{j \in \{a,1,\ldots,N\}} \sum_{k \in \{a,1,\ldots,N\}} h_j h_k \text{Cov}(\mathcal{V}_j, \mathcal{V}_k).$$

Thus, by setting partial derivatives with respect to $h_i$ to zero, where $i \in \{1, \ldots, N\}$, we find the candidate solutions. Thus we proceed by having

$$\begin{aligned}
&\frac{\partial}{\partial h_i}\Big(\sum_{j \in \{a,1,\ldots,N\}} \sum_{k \in \{a,1,\ldots,N\}} h_j h_k \text{Cov}(\mathcal{V}_j, \mathcal{V}_k)\Big) \\
&= \frac{\partial}{\partial h_i}\Big(h_i h_i \text{Cov}(\mathcal{V}_i, \mathcal{V}_i) + \sum_{j \neq i} \sum_{k = i} h_j h_k \text{Cov}(\mathcal{V}_j, \mathcal{V}_k) \\
&+ \sum_{j = i} \sum_{k \neq i} h_j h_k \text{Cov}(\mathcal{V}_j, \mathcal{V}_k) + \sum_{j \neq i} \sum_{k \neq i} h_j h_k \text{Cov}(\mathcal{V}_j, \mathcal{V}_k)\Big)
\end{aligned}$$

leading to

$$2h_i^*\mathrm{Var}(\mathcal{V}_i) + \sum_{j\neq i} h_j^*\mathrm{Cov}(\mathcal{V}_j, \mathcal{V}_i) + \sum_{k\neq i} h_k^*\mathrm{Cov}(\mathcal{V}_j, \mathcal{V}_i) = 0$$

thus having a linear system

$$h_i^*\mathrm{Var}(\mathcal{V}_i) + \sum_{j\neq i,a} h_j^*\mathrm{Cov}(\mathcal{V}_j, \mathcal{V}_i) = -h_a\mathrm{Cov}(\mathcal{V}_a, \mathcal{V}_i).$$

In matrix form this system can be seen as $\Sigma h = \eta$, where $h := (h_1, \ldots, h_N)'$, $\Sigma := \{\mathrm{Cov}(V_j, V_j)\}_{j,k=1}^N$ and $\eta := (\mathrm{Cov}(V_1, V_a), \ldots, \mathrm{Cov}(V_N, V_a))'$. Thus, the solution for the optimal number of policies is $h_i^* = -(\Sigma^{-1}\eta)_i$. Differently from the case in the previous Section 3.2.1, the optimal policies $h_i^*$'s, which are unconstrained, can be negative. Indeed, we can interpret a negative position on a life insurance policy as the need to buy reinsurance on the mortality exposure to the specific cohort/state couple identified by the hedging instrument (see, for instance, Luciano et al. [2017]), or to adjust their marketing or sales efforts.

As a special case of the more general unconstrained case, or in fact, a more realistic case, we can consider a situation in which an insurance company operates across $N$ states and has access to a limited capacity of life insurance policies sold $h_1^{\max}, \ldots, h_N^{\max}$, and has no access to reinsurance markets. Then the question is how, under these constrained conditions, to choose optimally $\mathcal{V}_i = \mathcal{V}^l(x_i^l, s_i^l, \tau_i^l)$. Hence, the optimization problem becomes

$$\begin{aligned} &\min_{h_1,\ldots,h_N\in\mathbb{R}} \mathrm{Var}\left(\mathcal{V}_a + \sum_{i=1}^N h_i\mathcal{V}_{l_i}(x_i^l, s_i^l, \tau_i^l)\right) \\ &\text{s.t.} \quad 0 \le h_i \le h_i^{\max} \ \text{ where } \ i \in \{1, \ldots, N\}, \end{aligned} \tag{15}$$

or more generally, we can set an $h_i^{min}$ which will lead to

$$\text{s.t.} \quad h_i^{min} \le h_i \le h_i^{\max} \ \text{ where } \ i \in \{1, \ldots, N\}.$$

This optimization problem can be efficiently addressed using techniques of numerical optimization, for example available in Matlab *fmincon* procedure.

### 3.2.4 Case 4: Fully general $M_1 \times M_2 \times M_3$ and general $N_1 \times N_2 \times N_3$

For fully the general case, denoted as $\mathcal{V}_A$, the liabilities being hedged can be viewed as a single portfolio i.e. $\mathcal{V}_A := \sum_{a=1,1,1}^{M_1\times M_2\times M_3} \mathcal{V}(x_a^A, \tau_a^A, s_a^A)$. The optimal mixes of term contracts, $h_{1,1,1}^*, \ldots, h_{i,j,k}^*, \ldots, h_{N_1,N_2,N_3}^*$, are then determined by solving

$$\min_{h_{1,1,1},\ldots,h_{i,j,k},\ldots,h_{N_1,N_2,N_3}\in\mathbb{R}} \mathrm{Var}\left(\mathcal{V}_A + \sum_{i=1}^N h_{i,j,k}\mathcal{V}_{ijk}(x_i^L, \tau_j^L, s_k^L)\right). \tag{16}$$

Let $\mathcal{V}_{i,j,k} = \mathcal{V}_{ijk}(x_i^L, \tau_j^L, s_k^L)$, and $h_A = 1$ as well as $I = \{1, \ldots, N_1\} \times \{1, \ldots, N_2\} \times \{1, \ldots, N_3\}$. Then, the optimization problem in equation (16) can be expressed in a simplified form as

$$\min_{h_{1,1,1},\ldots,h_{i,j,k},\ldots,h_{N_1,N_2,N_3}\in\mathbb{R}} \sum_{\alpha\in\{A\}\cup I} \sum_{\beta\in\{A\}\cup I} h_\alpha h_\beta \mathrm{Cov}(\mathcal{V}_\alpha, \mathcal{V}_\beta).$$

Finally, by setting the partial derivatives with respect to $h_{i,j,k}$ to zero, we find the candidate solutions. Thus we proceed by having

$$\frac{\partial}{\partial h_{i,j,k}}\Big(\sum_{\alpha\in\{A\}\cup I}\sum_{\beta\in\{A\}\cup I} h_\alpha h_\beta \mathrm{Cov}(\mathcal{V}_\alpha,\mathcal{V}_\beta)\Big)$$
$$= \frac{\partial}{\partial h_{i,j,k}}\Big(h_{i,j,k}h_{i,j,k}\mathrm{Cov}(\mathcal{V}_{i,j,k},\mathcal{V}_{i,j,k}) + \sum_{\alpha\neq i,j,k}\sum_{\beta=i,j,k} h_\alpha h_\beta \mathrm{Cov}(\mathcal{V}_j,\mathcal{V}_k)$$
$$+ \sum_{\alpha=i,j,k}\sum_{\beta\neq i,j,k} h_\alpha h_\beta \mathrm{Cov}(\mathcal{V}_\alpha,\mathcal{V}_\beta) + \sum_{\alpha\neq i,j,k}\sum_{\beta\neq i,j,k} h_j h_k \mathrm{Cov}(\mathcal{V}_\alpha,\mathcal{V}_\beta)\Big)$$

leading to

$$2h^*_{i,j,k}\mathrm{Var}(\mathcal{V}_{i,j,k}) + \sum_{\alpha\neq i,j,k} h^*_\alpha \mathrm{Cov}(\mathcal{V}_\alpha,\mathcal{V}_{i,j,k}) + \sum_{\beta\neq i,j,k} h^*_\beta \mathrm{Cov}(\mathcal{V}_{i,j,k},\mathcal{V}_\beta) = 0$$

thus having a linear system

$$h^*_{i,j,k}\mathrm{Var}(\mathcal{V}_{i,j,k}) + \sum_{\alpha\neq\{i,jk\cup A\}} h^*_\alpha \mathrm{Cov}(\mathcal{V}_\alpha,\mathcal{V}_{i,j,k}) = -h_A \mathrm{Cov}(\mathcal{V}_A,\mathcal{V}_{i,j,k}).$$

In matrix form this system can be seen as $\Sigma h = \eta$, where $h := (h_{1,1,1},\ldots,h_{N_1,N_2,N_3})'$, $\Sigma := \{\mathrm{Cov}(V_\alpha,V_\beta)\}_{\alpha,\beta=1,1,1}^{N_1,N_2,N_3}$ and $\eta := (\mathrm{Cov}(V_A,V_{1,1,1}),\ldots,\mathrm{Cov}(V_A,V_{N_1,N_2,N_3}))'$. Thus, the solution for the optimal number of policies is $h^*_{i,j,k} = -(\Sigma^{-1}\eta)_{i,j,k}$.

# 4 Numerical Illustrations

We provide a number of numerical illustrations in this section to discuss the effectiveness of spatial natural hedges. In particular, we focus on the impact of spatial dependence on the effectiveness of a natural hedge.

## 4.1 The Data

We consider a mortality dataset composed of the death and exposure counts from different states of the United States. [6] Our data source is the Center for Disease Control and Prevention (CDC)'s WONDER database. We consider the male populations of 48 contiguous states in the United States[7] from 1968 to 2018 and estimate the mortality model described in Section 2.

[6]Due to data limitation regarding differential mortality rates of annuitants and term insurance policy holders, we consider the same mortality dynamics for annuitants and life insurance policyholders in the same state. Considering these two groups as sub-populations would introduce an additional source of basis risk between the hedging portfolio and the portfolio to be hedged. Empirical evidence however, points to little relevance of this basis risk source, see Kwon and Jones [2006]. Notice, however, that this basis risk source is additional to the basis risk arising from the different ages at which the two types of contracts payout cash-flows, for which we account for with our mortality model.

[7]We exclude the states of Alaska and Hawaii as they do not share a land-border with another state, as well as Washington, D.C..

For the purpose of this study, we need to estimate and forecast the mortality rates of single-year ages from 25-99 in order to calculate the actuarial present values of insurance products. This age range covers the typical ages of both life insurance policyholders and annuitants. Individual age mortality rates are not available however for the whole sample period. [8] For the period of 1999-2018, the 'Detailed Mortality' dataset from CDC provides the data needed. However, for the period of 1968-1998, the data are age-grouped. We thus first obtain single-year-of-age data from group ones using a data disaggregation method.

We apply the method proposed by Boot et al. [1967], as in Li et al. [2017].[9] While Boot et al. [1967]'s method was originally applied to infer quarterly figures from annual ones, we adapt it to obtain individual ages death counts and exposures, starting from age-groups. The method minimizes the squared differences (or the second order differences) between consecutive individual age death counts and exposures, under the constraint that the individual age quantities must sum up correctly as in the original data, once aggregated by age group. After applying this method, our dataset is composed of individual ages mortality rates for ages 25-95 for the 48 U.S. states between 1968 and 2018.

## 4.2 Calibration

We fit the Li-Lee mortality model to the dataset. Figures 9 and 10 in the Appendix report graphs of the estimated parameters of the model and central death rate estimates for each state, as well as goodness of fit statistics for the fitted model.

To be able to analyze the effectiveness of the hedging strategies, we need to simulate forecasted paths of the mortality rates for all the ages and states in our sample from our model. This allows us to evaluate the annuity and life insurance liabilities to be hedged and those that are employed in our hedging strategies over time. When forecasting the models, we assume that all the state-specific time indexes $k_{t,s}$'s and the common time-index $K_t$ follow AR(1) processes, whose parameters are estimated via maximum likelihood.

## 4.3 Hedging Strategies Effectiveness

In this section, we numerically test the effectiveness of the natural hedging strategies we derived in Section 3.2. Our baseline problem is that of hedging one 15-year deferred annuity issued to a 45-year old individual residing in one state (California) with one or more life insurance policies from the same or different states. As an illustrative example, we consider an annuity that pays \$100 yearly after the deferral period. The hedging portfolio is composed by

[8] The CDC provides two sets of mortality data: the 'Detailed Mortality' and the 'Compressed Mortality.' The 'Detailed Mortality' dataset is available for the period of 1999-2018 only, but contains single-year-of-age population and death counts from age 0 to 99 for all 50 states. The 'Compressed Mortality' dataset is separated into three periods: 1968-1978, 1979-1998, 1999-2016. The difference between them lies in the ICD (cause of death) code used, which is irrelevant to this paper. More importantly, the 'Compressed Mortality' dataset contains the population and death counts of all states for age groups only (5-year age groups for ages 1-24; 10-year age groups for ages 25-84; open interval for age 85+).

[9] Notice that we could have used several alternative methods. First, we could have splitted forecasted age-group rates instead of observed ones, as in Renshaw and Haberman [2003] and Villegas and Haberman [2014]. Second, we could have used other disaggregation methods, such as interpolation through splines as the Human Mortality Database does.

life insurance policies sold to 35, 40 or 45-year old policyholders, with terms 10,15 and 20 years as hedging instruments. We focus on term insurance policies rather than whole life, as the market capacity for the latter product is negligible.[10] We then analyze robustness of our results to different annuity specifications (immediate) and to different population mortalities (Illinois, Arizona) to be hedged. Table 1 reports the composition of the portfolios to be hedged in the different cases we consider throughout the section.

| Section/Appendix | Annuity Portfolio Composition |
|---|---|
| 4.3.1 | One 15-year deferred whole life annuity issued to a 45-year old from California |
| 4.3.2 | One 15-year deferred whole life annuity issued to a 45-year old from California |
| 4.3.3 | An array of annuities that are immediate and 15-year deferred, issued to 50 and 60-year old from Arizona, California and Illinois. |
| B | One 15-year deferred whole life annuity issued to a 45-year old from Illinois |

Table 1: The composition of the annuity portfolios to be hedged in our numerical illustrations.

This section proceeds as follows. First, in Section 4.3.1, we analyze standardized hedging strategies: a one-to-one (base) hedge and an optimized one-to-one hedge, where the annuity issued is hedged with one or an optimal number of life insurance policies sold in the same vs. a different state. This allows us to appreciate the effectiveness of "simple" natural hedging strategies. These strategies may reproduce what happens in reality when the set of hedging instruments is restricted because of capacity or penetration constraints on the side of the insurance company, and hedging opportunities are limited to particular age classes or states. There, basis risk arises because typically annuities and life insurance policies are sold to different age-cohorts.

In Section 4.3.2, then, we compare the hedging effectiveness obtained by focusing on a particular hedging instrument/a particular state, which may differ from the one of the portfolio to be hedged. In that case, basis risk involves both the age and the population dimension. However, we point out that hedging effectiveness can improve because the set of hedging instruments enlarges. To further build on this point, exploiting the general formulas in Section 3.2.4, we compute and test the effectiveness of strategies that may involve more instruments from several states, and show that almost all of the longevity risk can be hedged. This is the focus of Section 4.3.3, where we illustrate how the combination of different life insurance policies from different states can improve hedging effectiveness.

### 4.3.1 One-to-one-hedging

We provide first an illustration of the effectiveness of spatial natural hedging in the baseline case of a one-to-one (and thus sub-optimal) hedge. In Table 2, we report the unhedged position mean value, its standard deviation and coefficienct of variation. In Table 3 we evaluate the hedging effectiveness that can be achieved through the issuance of one specific life insurance policy (one-to-one hedging).

[10]The use of term insurance rather than whole-life insurance policies is indeed prudential in terms of the performance evaluation of the natural hedging strategy. Such performance would be improved if whole-life insurance was available and used.

| State | UH (Mean) | UH (Std) | UH (CV) |
|---|---|---|---|
| California | 806.78 | 25.14 | 3.12% |

Table 2: Unhedged annuity portfolio. r=3%

| Strategy | $R^2(h)$ | $T(h)$ | H(mean) | H (Std) | H (CV) | H(Cost) |
|---|---|---|---|---|---|---|
| 1 | 17.66% | 15.16% | 862.92 | 22.82 | 2.64% | 3.18 |
| 2 | 9.05% | 8.14% | 831.78 | 23.98 | 2.88% | 2.78 |
| 3 | 12.16% | 13.33% | 868.22 | 23.50 | 2.71% | 4.85 |
| 4 | 10.85% | 12.45% | 866.30 | 23.74 | 2.74% | 5.49 |

Table 3: One-to-one hedge effectiveness. r=3%

The table reports hedge effectiveness measures ($R^2$ and $T(h, 95\%)$) for hedging strategies that differ in the choice of the hedging instrument. Strategy 1 considers hedging with a life insurance policy sold to a 45-year old male from California. Strategy 2 considers hedging with a different cohort, more distant from the one to be hedged (35-year old). Strategies 3 and 4 consider hedging with a life insurance policy sold to a 45-year old from Arizona and New York, respectively. These cases allow us to highlight some important details. First, the issuance of one life insurance policy to an individual from the same state of the annuitant delivers the best hedge against longevity risk, as measured by the reduction in the variance of the hedged position vs. the unhedged one, only when the cohort to which the life insurance policy is sold is closer to the one that needs to be hedged (1). When instead we consider hedging with a policy issued to a younger (35-year old) policyholder (Strategy 2), the additional basis risk introduced reduces the hedging effectiveness. It goes down from 17.66% to 9.05%. Since in reality it is not always possible to hedge with policies sold to the same population, because of capacity constraints, we consider hedging with policies sold in other states. This is the case of an insurance company with branches in different states. We appreciate the difference between hedging with two different populations, Arizona and New York. The former mortality dynamics is more similar to that of California, as the hedging strategy is revealed to be more efficient, with an $R^2$ of 12.16% vs. 10.85%. Although the strategies are not selected to optimize the tail-risk measure $T(\cdot;\cdot)$, the Table shows that the same effectiveness ranking across strategies maintains when considering this alternative statistics. Importantly, we find that strategies that hedge across states using the same cohort may provide a higher reduction of longevity risk than those that hedge within state with a different cohort. However, they are less cost/effective than those hedging using life insurance policies sold in California. The most cost/effective is the Strategy 2 (2.78), which is even more cost/effective than the most effective Strategy 1. This is due to the fact that the life insurance cover for the younger generation used in this strategy is cheaper.

### 4.3.2 Single State Optimal Hedging

The strategies we have analyzed so far are sub-optimal, because the positions in the life insurance policies are fixed arbitrarily to 1 and are thus not solving (10). We then consider the same strategies described in the previous section, but optimize them in the number of life insurance policies sold. Table 4 reports the results and in particular the optimal policy $h^*$. Such policy is a real number, not necessarily integer. While this may seem impractical, as it represents a number of contracts to be sold, we remark that we are analysing the hedging of a single annuity and that, when hedging a portfolio of hundreds of contracts, the hedge ratio becomes – or may be safely rounded to – an integer number. Indeed, hedge effectiveness is greatly improved, and reaches almost 69% when the optimal number (7.26) of life insurance policies are sold to 45-year old from California. Indeed, for all other strategies the gain from optimally selecting the position is considerable relative to the one-to-one hedge. Still, the ranking among the strategies in terms of cost and cost/effectiveness maintains.

| Case | $h^*$ | $R^2(h)$ | $T(h)$ | H(mean) | H (Std) | H (CV) | H(Cost) |
|---|---|---|---|---|---|---|---|
| 1 | 7.26 | 68.82% | 61.98% | 1214.20 | 14.04 | 1.16% | 5.92 |
| 2 | 12.95 | 60.94% | 55.17% | 1132.10 | 15.71 | 1.39% | 5.34 |
| 3 | 5.84 | 40.45% | 45.67% | 1165.50 | 19.40 | 1.66% | 8.87 |
| 4 | 4.03 | 18.65% | 25.18% | 1121.86 | 22.68 | 2.02% | 9.03 |

Table 4: Simple optimal strategies hedge effectiveness. r=3%

Then, we enlarge the scope of our analysis, comparing the optimal hedging strategies which still use one single life insurance as a hedging instrument, but with different terms (10,15,20 years) and sold to policyholders of different cohorts (35-,40-,45-year old) from the 48 states we are analyzing. Figures 1 and 2 report the optimal positions and hedge effectiveness, respectively, of such strategies, for all the 48 states. Figure 3, instead, reports their cost/effectiveness.

| | AL | | | AZ | | | AR | | | CA | | | CO | | | CT | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Age/Term | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 |
| 35 | 7.75 | 4.9 | 3.46 | 24.51 | 14.27 | 8.86 | 4.56 | 3 | 2.1 | 16.97 | 12.95 | 9.5 | 17.79 | 9.88 | 5.44 | 8.01 | 13.93 | 10.51 |
| 40 | 5.6 | 3.68 | 2.51 | 16.14 | 9.2 | 5.68 | 3.42 | 2.28 | 1.55 | 17.04 | 10.66 | 6.94 | 10.96 | 5.5 | 3.57 | 19.82 | 10.47 | 6.34 |
| 45 | 4.3 | 2.67 | 1.98 | 10.31 | 5.84 | 3.94 | 2.63 | 1.66 | 1.24 | 12.59 | 7.26 | 4.58 | 5.93 | 3.65 | 3.64 | 10.7 | 6.25 | 4.44 |
| | DE | | | FL | | | GA | | | ID | | | IL | | | IN | | |
| Age/Term | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 |
| 35 | 2.74 | 3.03 | 4.63 | 3.83 | 5.5 | 6.4 | 5.09 | 4.11 | 3.7 | 6.16 | 5.45 | 3.08 | 5.4 | 5 | 4.37 | 4.73 | 3.35 | 2.97 |
| 40 | 4.86 | 6.77 | 4.73 | 10.49 | 8.48 | 5.46 | 5.32 | 4.29 | 3.56 | 7.38 | 3.32 | 1.97 | 6.99 | 5.28 | 3.93 | 4.12 | 3.45 | 3.05 |
| 45 | 9.66 | 4.15 | 1.92 | 9.91 | 5.37 | 3.76 | 5.54 | 4.01 | 3 | 3.44 | 1.94 | 2.14 | 6.59 | 4.34 | 2.86 | 4.54 | 3.58 | 3.11 |
| | IA | | | KS | | | KY | | | LA | | | ME | | | MD | | |
| Age/Term | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 |
| 35 | 4.76 | 2.59 | 2.25 | 4.48 | 2.5 | 1.91 | 2.64 | 1.97 | 1.89 | 10.09 | 6.72 | 4.92 | 2.24 | 2.23 | 2.87 | 5.85 | 9.48 | 7.53 |
| 40 | 2.8 | 2.49 | 2.66 | 2.81 | 2.11 | 1.71 | 2.49 | 2.25 | 2.15 | 8.11 | 5.42 | 3.59 | 3.15 | 4.02 | 4.55 | 15.91 | 6.54 | 3.48 |
| 45 | 3.32 | 3.35 | 3.68 | 2.7 | 2.01 | 1.89 | 3.07 | 2.63 | 2.36 | 6.43 | 3.83 | 2.61 | 6.66 | 5.85 | 3.53 | 5.29 | 3.03 | 2.52 |
| | MA | | | MI | | | MN | | | MS | | | MO | | | MT | | |
| Age/Term | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 |
| 35 | 13.1 | 15.69 | 10.11 | 23.32 | 14.99 | 7.64 | 15.49 | 10.47 | 9.62 | 10.63 | 5.58 | 3.38 | 4.33 | 3.23 | 2.96 | 31.06 | 17.21 | 9.98 |
| 40 | 20.06 | 9.42 | 4.96 | 17.14 | 7.71 | 4.06 | 12.38 | 10.75 | 7.49 | 5.9 | 3.43 | 2.16 | 4.05 | 3.5 | 3.04 | 19.03 | 10.09 | 6.06 |
| 45 | 9.05 | 4.55 | 2.59 | 7.89 | 3.9 | 2.87 | 13.36 | 7.51 | 3.34 | 3.75 | 2.21 | 1.56 | 4.68 | 3.59 | 3.04 | 10.91 | 6.09 | 3.31 |
| | NE | | | NV | | | NH | | | NJ | | | NM | | | NY | | |
| Age/Term | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 |
| 35 | 2.98 | 3.13 | 5.33 | 26.49 | 14.37 | 8.73 | 2.36 | 3.14 | 4.78 | 3.19 | 4.08 | 3.81 | 1.73 | 1.17 | 1.05 | 2.11 | 2.32 | 2.26 |
| 40 | 4.71 | 7.66 | 5.97 | 15.58 | 8.83 | 5.21 | 5.42 | 7.43 | 6.36 | 7.31 | 5.27 | 3.4 | 1.33 | 1.2 | 1.25 | 3.56 | 3.01 | 2.42 |
| 45 | 11.81 | 5.52 | 2.22 | 9.48 | 5.11 | 3.42 | 11.69 | 6.62 | 2.8 | 6.85 | 3.73 | 1.91 | 1.6 | 1.6 | 1.9 | 4.03 | 2.83 | 1.92 |
| | NC | | | ND | | | OH | | | OK | | | OR | | | PA | | |
| Age/Term | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 |
| 35 | 4.9 | 4.15 | 3.92 | 0.62 | 0.63 | 0.83 | 4.04 | 3.35 | 2.98 | 6.88 | 3.77 | 2.51 | 10.2 | 4.66 | 5.77 | 10.12 | 10.41 | 8.83 |
| 40 | 5.52 | 4.76 | 3.92 | 0.94 | 1.26 | 1.85 | 4.59 | 3.63 | 3.05 | 4.2 | 2.64 | 1.83 | 4.29 | 6.12 | 5.88 | 14.54 | 9.78 | 5.92 |
| 45 | 6.35 | 4.51 | 3.39 | 2.4 | 3.27 | 3.87 | 4.84 | 3.56 | 3.13 | 3.08 | 1.96 | 1.46 | 9.02 | 6.08 | 2.92 | 10.85 | 5.8 | 3.31 |
| | RI | | | SC | | | SD | | | TN | | | TX | | | UT | | |
| Age/Term | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 |
| 35 | 14.46 | 13.72 | 7.06 | 5.8 | 4.63 | 3.86 | 0.99 | 1.35 | 2.21 | 5.54 | 3.57 | 2.78 | 16.64 | 13.72 | 9.1 | 2.92 | 1.51 | 1.2 |
| 40 | 18.6 | 7.57 | 4.44 | 5.87 | 4.43 | 3.22 | 2.29 | 3.96 | 4.65 | 4.18 | 3.09 | 2.37 | 17.09 | 9.68 | 5.95 | 1.69 | 1.32 | 1.46 |
| 45 | 7.68 | 4.33 | 3.71 | 5.59 | 3.55 | 2.58 | 8.89 | 6.58 | 3.5 | 3.8 | 2.65 | 2.04 | 10.74 | 6.06 | 3.86 | 1.69 | 1.87 | 2.87 |
| | VT | | | VA | | | WA | | | WV | | | WI | | | WY | | |
| Age/Term | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 |
| 35 | 3.42 | 4.84 | 7.52 | 11.19 | 6.45 | 4.24 | 32.74 | 15.85 | 6.25 | 1.55 | 1.32 | 1.25 | 10.63 | 8.32 | 7.2 | 21.53 | 15.46 | 10.11 |
| 40 | 8.82 | 10.33 | 4.68 | 7.12 | 4.39 | 3.06 | 16.64 | 5.86 | 3.19 | 1.7 | 1.56 | 1.51 | 10.57 | 8.28 | 6.35 | 18.39 | 10.31 | 5.93 |
| 45 | 11.57 | 3.72 | 1.07 | 4.96 | 3.22 | 2.41 | 5.46 | 3.01 | 2.36 | 2.12 | 1.89 | 1.88 | 10.39 | 6.85 | 4.21 | 11.06 | 5.82 | 3.02 |

Figure 1: Positions of the optimal hedging strategy for an annuity issued in California using one single life insurance as a hedging instrument across different terms and cohorts from the 48 states.

| | AL | | | AZ | | | AR | | | CA | | | CO | | | CT | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Age/Term | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 |
| 35 | 0.1 | 0.17 | 0.24 | 0.27 | 0.4 | 0.49 | 0.06 | 0.1 | 0.15 | 0.38 | 0.61 | 0.81 | 0.16 | 0.23 | 0.26 | 0.06 | 0.3 | 0.46 |
| 40 | 0.12 | 0.2 | 0.27 | 0.28 | 0.41 | 0.49 | 0.07 | 0.13 | 0.17 | 0.46 | 0.68 | 0.84 | 0.16 | 0.21 | 0.27 | 0.27 | 0.38 | 0.46 |
| 45 | 0.15 | 0.23 | 0.32 | 0.29 | 0.4 | 0.51 | 0.09 | 0.14 | 0.2 | 0.49 | 0.69 | 0.85 | 0.14 | 0.22 | 0.42 | 0.25 | 0.37 | 0.52 |
| | DE | | | FL | | | GA | | | ID | | | IL | | | IN | | |
| Age/Term | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 |
| 35 | 0.03 | 0.09 | 0.27 | 0.04 | 0.16 | 0.38 | 0.06 | 0.13 | 0.23 | 0.05 | 0.12 | 0.14 | 0.05 | 0.13 | 0.23 | 0.05 | 0.09 | 0.17 |
| 40 | 0.09 | 0.32 | 0.43 | 0.2 | 0.4 | 0.5 | 0.11 | 0.22 | 0.36 | 0.1 | 0.12 | 0.14 | 0.11 | 0.23 | 0.34 | 0.07 | 0.16 | 0.28 |
| 45 | 0.29 | 0.3 | 0.26 | 0.3 | 0.39 | 0.5 | 0.18 | 0.33 | 0.47 | 0.08 | 0.11 | 0.25 | 0.18 | 0.3 | 0.38 | 0.13 | 0.27 | 0.44 |
| | IA | | | KS | | | KY | | | LA | | | ME | | | MD | | |
| Age/Term | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 |
| 35 | 0.04 | 0.06 | 0.1 | 0.04 | 0.07 | 0.11 | 0.03 | 0.07 | 0.13 | 0.14 | 0.24 | 0.35 | 0.02 | 0.05 | 0.15 | 0.07 | 0.27 | 0.42 |
| 40 | 0.04 | 0.09 | 0.2 | 0.05 | 0.09 | 0.15 | 0.05 | 0.12 | 0.23 | 0.19 | 0.31 | 0.4 | 0.05 | 0.16 | 0.37 | 0.28 | 0.28 | 0.3 |
| 45 | 0.08 | 0.2 | 0.44 | 0.07 | 0.14 | 0.25 | 0.11 | 0.23 | 0.38 | 0.24 | 0.35 | 0.43 | 0.17 | 0.39 | 0.45 | 0.14 | 0.21 | 0.34 |
| | MA | | | MI | | | MN | | | MS | | | MO | | | MT | | |
| Age/Term | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 |
| 35 | 0.11 | 0.36 | 0.47 | 0.25 | 0.41 | 0.44 | 0.12 | 0.21 | 0.4 | 0.16 | 0.22 | 0.27 | 0.05 | 0.09 | 0.18 | 0.31 | 0.43 | 0.51 |
| 40 | 0.29 | 0.36 | 0.38 | 0.3 | 0.36 | 0.38 | 0.16 | 0.36 | 0.52 | 0.14 | 0.21 | 0.26 | 0.07 | 0.17 | 0.29 | 0.3 | 0.41 | 0.49 |
| 45 | 0.22 | 0.28 | 0.31 | 0.23 | 0.29 | 0.41 | 0.29 | 0.42 | 0.37 | 0.14 | 0.21 | 0.28 | 0.14 | 0.27 | 0.43 | 0.28 | 0.4 | 0.42 |
| | NE | | | NV | | | NH | | | NJ | | | NM | | | NY | | |
| Age/Term | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 |
| 35 | 0.03 | 0.07 | 0.24 | 0.3 | 0.43 | 0.53 | 0.02 | 0.07 | 0.22 | 0.03 | 0.1 | 0.19 | 0.02 | 0.04 | 0.06 | 0.02 | 0.06 | 0.11 |
| 40 | 0.07 | 0.29 | 0.46 | 0.3 | 0.43 | 0.5 | 0.07 | 0.27 | 0.48 | 0.11 | 0.22 | 0.28 | 0.02 | 0.05 | 0.11 | 0.06 | 0.12 | 0.2 |
| 45 | 0.28 | 0.34 | 0.27 | 0.3 | 0.4 | 0.5 | 0.28 | 0.41 | 0.34 | 0.18 | 0.25 | 0.24 | 0.04 | 0.11 | 0.24 | 0.11 | 0.19 | 0.24 |
| | NC | | | ND | | | OH | | | OK | | | OR | | | PA | | |
| Age/Term | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 |
| 35 | 0.06 | 0.13 | 0.25 | 0.01 | 0.02 | 0.04 | 0.05 | 0.1 | 0.18 | 0.09 | 0.13 | 0.18 | 0.09 | 0.11 | 0.29 | 0.1 | 0.27 | 0.48 |
| 40 | 0.11 | 0.24 | 0.39 | 0.01 | 0.05 | 0.14 | 0.08 | 0.17 | 0.29 | 0.09 | 0.15 | 0.21 | 0.07 | 0.24 | 0.47 | 0.24 | 0.43 | 0.52 |
| 45 | 0.21 | 0.36 | 0.5 | 0.06 | 0.2 | 0.44 | 0.15 | 0.27 | 0.45 | 0.11 | 0.17 | 0.24 | 0.23 | 0.39 | 0.37 | 0.3 | 0.41 | 0.45 |
| | RI | | | SC | | | SD | | | TN | | | TX | | | UT | | |
| Age/Term | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 |
| 35 | 0.13 | 0.33 | 0.35 | 0.08 | 0.16 | 0.27 | 0.01 | 0.03 | 0.11 | 0.07 | 0.12 | 0.19 | 0.17 | 0.37 | 0.5 | 0.03 | 0.03 | 0.05 |
| 40 | 0.28 | 0.3 | 0.36 | 0.13 | 0.25 | 0.35 | 0.03 | 0.16 | 0.36 | 0.09 | 0.17 | 0.25 | 0.3 | 0.43 | 0.53 | 0.02 | 0.05 | 0.11 |
| 45 | 0.2 | 0.29 | 0.47 | 0.2 | 0.31 | 0.42 | 0.22 | 0.41 | 0.41 | 0.13 | 0.22 | 0.32 | 0.31 | 0.44 | 0.53 | 0.04 | 0.11 | 0.32 |
| | VT | | | VA | | | WA | | | WV | | | WI | | | WY | | |
| Age/Term | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 |
| 35 | 0.03 | 0.11 | 0.34 | 0.11 | 0.17 | 0.23 | 0.27 | 0.35 | 0.28 | 0.02 | 0.05 | 0.09 | 0.09 | 0.19 | 0.34 | 0.22 | 0.4 | 0.52 |
| 40 | 0.12 | 0.38 | 0.34 | 0.12 | 0.19 | 0.27 | 0.23 | 0.22 | 0.24 | 0.04 | 0.09 | 0.17 | 0.15 | 0.31 | 0.48 | 0.3 | 0.42 | 0.48 |
| 45 | 0.27 | 0.22 | 0.12 | 0.14 | 0.23 | 0.33 | 0.13 | 0.18 | 0.28 | 0.07 | 0.16 | 0.3 | 0.25 | 0.42 | 0.49 | 0.29 | 0.38 | 0.38 |

Figure 2: Effectiveness of the optimal hedging strategy for an annuity issued in California using one single life insurance as a hedging instrument across different terms and cohorts from the 48 states.

| | AL | | | AZ | | | AR | | | CA | | | CO | | | CT | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Age/Term | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 |
| 35 | 17.72 | 11.04 | 7.99 | 16.76 | 10.64 | 7.74 | 17.01 | 10.38 | 7.53 | 6.75 | 5.34 | 4.38 | 16.44 | 10.3 | 7.45 | 17.74 | 10.85 | 7.73 |
| 40 | 15.67 | 9.96 | 7.38 | 14.81 | 9.56 | 7.1 | 15.08 | 9.4 | 6.98 | 8.29 | 5.86 | 4.56 | 14.49 | 9.13 | 6.77 | 14.9 | 9.46 | 6.94 |
| 45 | 14.21 | 9.25 | 7.08 | 13.5 | 8.87 | 6.78 | 13.73 | 8.77 | 6.74 | 8.73 | 5.92 | 4.38 | 13.04 | 8.43 | 6.51 | 13.22 | 8.64 | 6.53 |
| | DE | | | FL | | | GA | | | ID | | | IL | | | IN | | |
| Age/Term | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 |
| 35 | 15.38 | 10.08 | 7.55 | 16.63 | 10.55 | 7.72 | 16.98 | 10.75 | 7.75 | 18.22 | 11.1 | 7.9 | 17.25 | 10.89 | 7.84 | 16.7 | 10.43 | 7.56 |
| 40 | 14.17 | 9.45 | 7.15 | 14.81 | 9.61 | 7.18 | 14.94 | 9.63 | 7.12 | 15.49 | 9.79 | 7.16 | 15.04 | 9.67 | 7.14 | 14.76 | 9.4 | 6.96 |
| 45 | 13.42 | 8.99 | 6.94 | 13.63 | 9.02 | 6.92 | 13.62 | 8.94 | 6.84 | 14.13 | 9.06 | 6.76 | 13.61 | 8.93 | 6.84 | 13.4 | 8.73 | 6.68 |
| | IA | | | KS | | | KY | | | LA | | | ME | | | MD | | |
| Age/Term | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 |
| 35 | 16.53 | 10.34 | 7.49 | 15.58 | 9.65 | 7.05 | 16.19 | 10.27 | 7.47 | 16.23 | 10.49 | 7.72 | 16.3 | 10.07 | 7.34 | 14.6 | 10.11 | 7.63 |
| 40 | 14.78 | 9.31 | 6.86 | 13.88 | 8.83 | 6.59 | 14.49 | 9.39 | 7 | 14.51 | 9.5 | 7.13 | 14.42 | 9.19 | 6.83 | 14.7 | 9.72 | 7.2 |
| 45 | 13.42 | 8.63 | 6.52 | 12.79 | 8.32 | 6.44 | 13.36 | 8.83 | 6.8 | 13.35 | 8.88 | 6.87 | 13.21 | 8.64 | 6.61 | 14.45 | 9.22 | 6.89 |
| | MA | | | MI | | | MN | | | MS | | | MO | | | MT | | |
| Age/Term | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 |
| 35 | 17.12 | 10.56 | 7.57 | 16.7 | 10.54 | 7.54 | 15.58 | 9.97 | 7.27 | 16.75 | 10.56 | 7.67 | 16.95 | 10.66 | 7.71 | 17.03 | 10.76 | 7.75 |
| 40 | 14.82 | 9.4 | 6.9 | 14.76 | 9.45 | 6.95 | 13.91 | 9.06 | 6.71 | 14.91 | 9.56 | 7.11 | 14.92 | 9.55 | 7.07 | 14.95 | 9.55 | 7.04 |
| 45 | 13.22 | 8.63 | 6.54 | 13.41 | 8.8 | 6.72 | 13.02 | 8.54 | 6.46 | 13.64 | 8.94 | 6.88 | 13.51 | 8.84 | 6.76 | 13.6 | 8.82 | 6.67 |
| | NE | | | NV | | | NH | | | NJ | | | NM | | | NY | | |
| Age/Term | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 |
| 35 | 15.64 | 10.21 | 7.36 | 16.69 | 10.56 | 7.65 | 17.26 | 10.52 | 7.47 | 17.06 | 10.63 | 7.66 | 19.62 | 11.89 | 8.46 | 18.11 | 11.03 | 7.91 |
| 40 | 14.05 | 9.13 | 6.71 | 14.81 | 9.51 | 7.05 | 14.88 | 9.32 | 6.82 | 14.89 | 9.48 | 7 | 17.11 | 10.52 | 7.65 | 15.63 | 9.83 | 7.22 |
| 45 | 13.13 | 8.54 | 6.39 | 13.49 | 8.85 | 6.77 | 13.29 | 8.59 | 6.51 | 13.47 | 8.76 | 6.66 | 14.93 | 9.47 | 7.12 | 14.03 | 9.03 | 6.85 |
| | NC | | | ND | | | OH | | | OK | | | OR | | | PA | | |
| Age/Term | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 |
| 35 | 15.48 | 10.08 | 7.46 | 14.94 | 9.69 | 7.21 | 15.55 | 10.12 | 7.45 | 16.05 | 9.98 | 7.24 | 15.84 | 9.99 | 7.32 | 17.17 | 10.66 | 7.65 |
| 40 | 14.1 | 9.28 | 6.97 | 13.98 | 9.1 | 6.82 | 14.12 | 9.22 | 6.91 | 14.28 | 9.07 | 6.74 | 14.12 | 9.17 | 6.83 | 14.9 | 9.47 | 6.98 |
| 45 | 13.19 | 8.76 | 6.75 | 13.08 | 8.6 | 6.61 | 13.05 | 8.64 | 6.66 | 13.08 | 8.5 | 6.55 | 13.3 | 8.72 | 6.64 | 13.37 | 8.73 | 6.64 |
| | RI | | | SC | | | SD | | | TN | | | TX | | | UT | | |
| Age/Term | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 |
| 35 | 15.85 | 10.21 | 7.4 | 16.79 | 10.7 | 7.82 | 16.54 | 10.3 | 7.59 | 17.11 | 10.73 | 7.77 | 16.71 | 10.54 | 7.62 | 16.16 | 10.37 | 7.66 |
| 40 | 14.52 | 9.27 | 6.86 | 14.95 | 9.7 | 7.25 | 14.95 | 9.54 | 7.06 | 15.17 | 9.7 | 7.19 | 14.68 | 9.45 | 6.99 | 14.31 | 9.32 | 6.96 |
| 45 | 13.03 | 8.57 | 6.56 | 13.68 | 9.05 | 6.98 | 13.55 | 8.84 | 6.76 | 13.81 | 9.03 | 6.92 | 13.44 | 8.78 | 6.7 | 13.08 | 8.61 | 6.55 |
| | VT | | | VA | | | WA | | | WV | | | WI | | | WY | | |
| Age/Term | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 |
| 35 | 16.01 | 10.16 | 7.46 | 15.76 | 9.96 | 7.24 | 16.83 | 10.55 | 7.65 | 17.11 | 10.52 | 7.74 | 16.26 | 10.1 | 7.33 | 16.64 | 10.65 | 7.74 |
| 40 | 14.46 | 9.32 | 7.02 | 14.01 | 9.02 | 6.7 | 14.89 | 9.54 | 7.04 | 15.17 | 9.55 | 7.17 | 14.46 | 9.19 | 6.81 | 14.84 | 9.58 | 7.09 |
| 45 | 13.39 | 8.81 | 6.91 | 12.85 | 8.44 | 6.47 | 13.84 | 8.92 | 6.72 | 13.76 | 8.93 | 6.89 | 13.21 | 8.62 | 6.61 | 13.62 | 8.9 | 6.71 |

Figure 3: Cost of the optimal hedging strategy for an annuity issued in California using one single life insurance as a hedging instrument across different terms and cohorts from the 48 states.

The figure can be read as a tool to guide the selection of the best hedging instruments. Looking at the table in Figure 2, it is clear that one should hedge using life insurance policies sold in California, which guarantee an hedge effectiveness that ranges between around 38% to 85%. However, the capacity of the life insurance market in California may be limited and not sufficient to implement the strategy depicted in the table. Using hedging instruments from other states is indeed less effective. We notice remarkable heterogeneity in hedge effectiveness depending on the state, and within states, on the selected hedging instrument. In general, hedge effectiveness, within states, increases with the term and the proximity of the cohort to the one that needs to be hedged. The second-best hedging strategy, when using instruments sold outside California, consists in using 20-year term policies, rather surprisingly sold to 35-year old from Nevada. This strategy allows to reach the maximum second best hedge effectiveness (53.05%), even though it is accompanied by a moderate hedge cost/effectiveness (7.65), as Figure 3 depicts.

We then explore an example of a general hedge as in Section 3.2.4, where several hedging instruments can be optimally mixed together to hedge the annuity. First, we allow the use of hedging instruments from a single state. We consider hedging portfolios composed of policies sold to a single population, possibly a different state than the one to be hedged. We solve the constrained optimization program described by (15) with $h_i^{min} = 0, h_i^{max} = \infty$, allowing the company to sell an unrestricted number of policies, but not to re-insure itself (either through customized hedges or market-based solutions). The optimal hedges, their effectiveness state-by-state and their cost-efficiency measures are reported in Figures 4, 5 and 6, respectively. Analyzing the optimal constrained hedges, we find that, in most cases, it is optimal to sell only the policy which delivers the highest hedge effectiveness level as a single hedging instrument. In some cases, however, a mix of policies is optimal (e.g. when hedging using policies from Connecticut or Delaware). Shorter term (10 year) policies are almost never in the optimal mix. This happens because of their highest "duration mismatch" relative to the annuity.

Combining optimally hedging instruments from California allows to deliver a very good hedge (around 85% effectiveness). This is achieved using 20-year life insurance policies written on the lives of 45-year old only, under the assumption that the life insurance market capacity in California allows for such hedge. This strategy is the best one in terms of hedge effectiveness and cost-effectiveness. However, as discussed previously, the capacity of the Californian market for life insurance policies may be constrained. We thus analyze the effectiveness of second-best optimal policies, achieved using only hedging instruments from a different state. Such strategies have remarkably different effectiveness across states, ranging from 24% to 53%. Also, second-best strategies that achieve higher effectiveness are in general also less cost-effective, see Figure 6, highlighting a trade-off between coverage quality and cost.

As a theoretical benchmark, we compute the unconstrained optimal strategy (as described in (16)), allowing for negative positions in the life insurance policies, which we interpret as cost-less reinsurance bought by the company. Such hedging strategy obviously increases the hedging effectiveness.[11] In unreported simulations, we show that the effectiveness reaches 89% when using hedging instruments from California only. Differently from the constrained case, effectiveness increases when instruments from other states can be used, or combined with the original instruments. For instance, it goes up to 90.27% when policies from both

[11] Also, if the reinsurance is fairly priced, optimal strategies are also more cost-effective that the constrained ones. In practice, usually, reinsurance, when available, is costly. This would lead to a decrease in the strategy cost-effectiveness.

| | AL | | | AZ | | | AR | | | CA | | | CO | | | CT | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Age/Term | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 |
| 35 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1.7 | 0 | 0.96 |
| 40 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 45 | 0 | 0 | 1.98 | 0 | 0 | 3.94 | 0 | 0 | 1.24 | 0 | 0 | 4.58 | 0 | 0 | 3.64 | 0 | 0 | 4.01 |
| | DE | | | FL | | | GA | | | ID | | | IL | | | IN | | |
| Age/Term | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 |
| 35 | 0 | 0 | 4.01 | 0 | 0 | 1.98 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 40 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 45 | 0 | 0.36 | 1.51 | 0 | 0 | 3 | 0 | 0 | 3 | 0 | 0 | 2.14 | 0 | 0 | 2.86 | 0 | 0 | 3.11 |
| | IA | | | KS | | | KY | | | LA | | | ME | | | MD | | |
| Age/Term | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 |
| 35 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1.51 | 0 | 6.75 | 0 |
| 40 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 45 | 0 | 0 | 3.68 | 0 | 0 | 1.89 | 0 | 0 | 2.36 | 0 | 0 | 2.61 | 0 | 0 | 3.22 | 0 | 0 | 1.99 |
| | MA | | | MI | | | MN | | | MS | | | MO | | | MT | | |
| Age/Term | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 |
| 35 | 0 | 0.42 | 9.91 | 0 | 4.19 | 5.72 | 0 | 0 | 6.75 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 9.98 |
| 40 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 45 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 2.18 | 0 | 0 | 1.56 | 0 | 0 | 3.04 | 0 | 0 | 0 |
| | NE | | | NV | | | NH | | | NJ | | | NM | | | NY | | |
| Age/Term | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 |
| 35 | 0 | 0 | 4.68 | 0 | 0 | 8.73 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 40 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 5.58 | 0 | 0 | 3.4 | 0 | 0 | 0 | 0 | 0 | 0 |
| 45 | 0 | 0.67 | 1.77 | 0 | 0 | 0 | 0 | 0 | 0.53 | 0 | 0 | 0 | 0 | 0 | 1.9 | 0 | 0 | 1.92 |
| | NC | | | ND | | | OH | | | OK | | | OR | | | PA | | |
| Age/Term | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 |
| 35 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 4.28 | 0 | 0 | 2.48 |
| 40 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 4.45 |
| 45 | 0 | 0 | 3.39 | 0 | 0 | 3.87 | 0 | 0 | 3.13 | 0 | 0 | 1.46 | 0 | 0 | 2.39 | 0 | 0 | 0 |
| | RI | | | SC | | | SD | | | TN | | | TX | | | UT | | |
| Age/Term | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 |
| 35 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 40 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 2.49 | 0 | 0 | 0 | 0 | 0 | 4.06 | 0 | 0 | 0 |
| 45 | 0 | 0 | 3.71 | 0 | 0 | 2.58 | 0 | 0 | 2.38 | 0 | 0 | 2.04 | 0 | 0 | 1.23 | 0 | 0 | 2.87 |
| | VT | | | VA | | | WA | | | WV | | | WI | | | WY | | |
| Age/Term | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 |
| 35 | 0 | 0 | 6.41 | 0 | 0 | 0 | 0 | 15.85 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 10.03 |
| 40 | 0 | 0 | 0.03 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 3.15 | 0 | 0 | 0.05 |
| 45 | 0 | 2.74 | 0 | 0 | 0 | 2.41 | 0 | 0 | 0 | 0 | 0 | 1.88 | 0 | 0 | 2.4 | 0 | 0 | 0 |

Figure 4: Positions of the optimal hedging strategy for an annuity issued in California using all the insurance products sold in a single state as a hedging instrument across different terms and cohorts from the 48 states.

California and Nevada are used. Combining positions on all the possible policies from the 48 states allows to increase the effectiveness further, to 91.28%. This is the theoretical best effectiveness achievable. Notice that this is not too distant from the 85% effectiveness that can be achieved in the constrained case.

The Appendix reports, for illustrative purposes, the results from hedging a different annuity, sold to a 45-year old from Illinois. Similar comments to the ones above apply, highlighting a regular pattern across portfolios to be hedged.

| AL | AZ | AR | CA | CO | CT | DE | FL | GA | ID | IL | IN |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.32 | 0.51 | 0.2 | 0.85 | 0.42 | 0.52 | 0.47 | 0.52 | 0.47 | 0.25 | 0.38 | 0.44 |
| IA | KS | KY | LA | ME | MD | MA | MI | MN | MS | MO | MT |
| 0.44 | 0.25 | 0.38 | 0.43 | 0.48 | 0.46 | 0.47 | 0.45 | 0.52 | 0.28 | 0.43 | 0.51 |
| NE | NV | NH | NJ | NM | NY | NC | ND | OH | OK | OR | PA |
| 0.47 | 0.53 | 0.49 | 0.28 | 0.24 | 0.24 | 0.5 | 0.44 | 0.45 | 0.24 | 0.51 | 0.52 |
| RI | SC | SD | TN | TX | UT | VT | VA | WA | WV | WI | WY |
| 0.47 | 0.42 | 0.47 | 0.32 | 0.53 | 0.32 | 0.46 | 0.33 | 0.35 | 0.3 | 0.52 | 0.52 |

Figure 5: Effectiveness of the optimal hedging strategy for an annuity issued in California using all the insurance products sold in a single state as a hedging instrument across different terms and cohorts from the 48 states.

| AL | AZ | AR | CA | CO | CT | DE | FL | GA | ID | IL | IN |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 7.08 | 6.78 | 6.74 | 4.38 | 6.51 | 6.92 | 7.36 | 7.1 | 6.84 | 6.76 | 6.84 | 6.68 |
| IA | KS | KY | LA | ME | MD | MA | MI | MN | MS | MO | MT |
| 6.52 | 6.44 | 6.8 | 6.87 | 6.72 | 8.25 | 7.63 | 8.31 | 6.89 | 6.88 | 6.76 | 7.75 |
| NE | NV | NH | NJ | NM | NY | NC | ND | OH | OK | OR | PA |
| 7.02 | 7.65 | 6.78 | 7 | 7.12 | 6.85 | 6.75 | 6.61 | 6.66 | 6.55 | 6.92 | 7.15 |
| RI | SC | SD | TN | TX | UT | VT | VA | WA | WV | WI | WY |
| 6.56 | 6.98 | 6.88 | 6.92 | 6.9 | 6.55 | 7.94 | 6.47 | 10.55 | 6.89 | 6.7 | 7.73 |

Figure 6: Cost of the optimal hedging strategy for an annuity issued in California using all the insurance products sold in a single state as a hedging instrument across different terms and cohorts from the 48 states.

### 4.3.3 Multi State Optimal Hedging

As a final illustrative example, we consider optimal hedging in a more complex case in which an annuity portfolio written on the lives of policyholders from different cohorts (35,40,45) in different states (Arizona, California, Illinois) can be hedged using instruments underwritten in two states.

In Figure 7 we portray the hedging effectiveness of strategies which allow for the use of capacities from any two states (as in the previous section, using up to 9 hedging instruments for each state). This figure, coupled with Figure 8 exemplifies the use of our hedging framework as a guiding tool toward efficient spatial natural hedging. The optimization is constrained and no negative (reinsurance) positions on life insurance policies are allowed. Red colors indicate state combinations for which effectiveness is higher, while violet indicate lower effectiveness. We immediately notice that most of the pairs that include California, Illinois and Arizona have

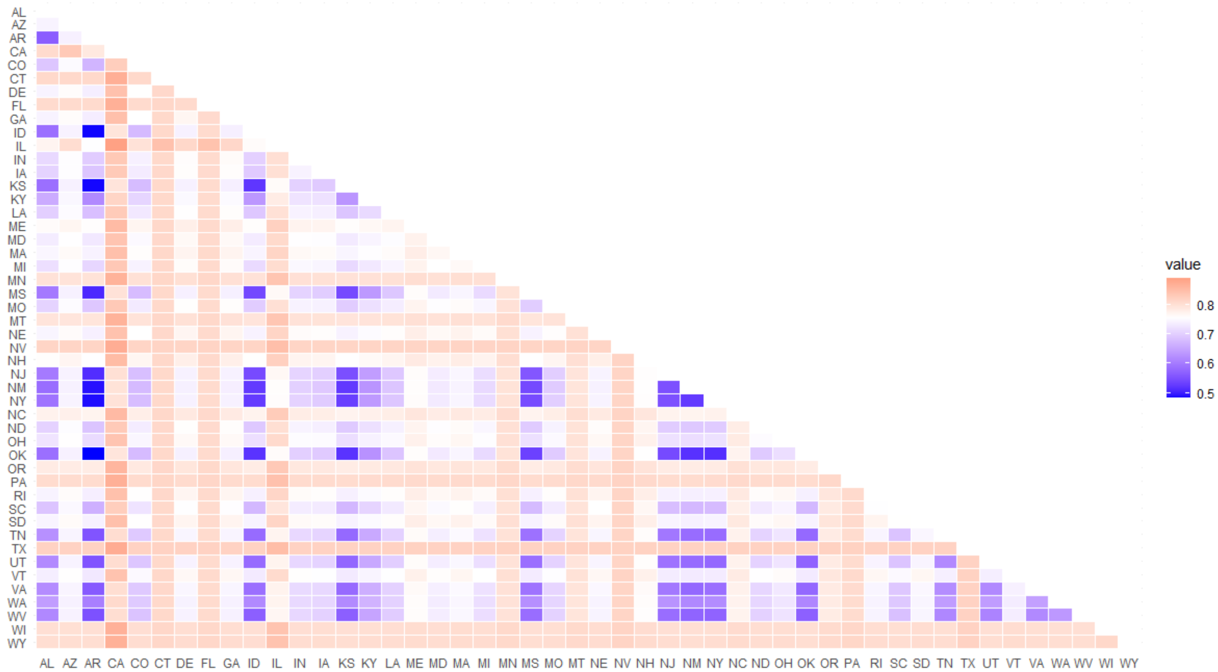


Figure 7: Effectiveness of the optimal hedging strategy for the complex annuity portfolio which allows for hedging using the capacities of two states.

high hedge effectiveness, which is intuitively understandable as such choice would reduce basis risk. However, most of the pairs that include Nevada and Texas, for example, as one of the pair members, show high hedging effectiveness. This is due to the similarity of the mortality experience of such states relative to that of the portfolio to be hedged. Ideally, obviously, best combinations of hedging instruments include pairs of states with high hedging effectiveness and high cost/effectiveness. Indeed, Figure 8 shows the cost/effectiveness of the optimal hedging strategies obtained combining life insurance policies from two states. With red colors indicating higher costs and violet indicating lower costs, we immediately notice that some two states' choices account for very high hedging costs. For example, the cases accounting for high cost are the Washington and Alabama pair, the Washington and Idaho pair, the Michigan and Kansas pair, etc. At the same time, for example, the cases accounting for low hedging costs are Iowa and Illinois pair, Wyoming and Illinois pair, etc. Also, it is easy to notice that in the case of California, when one of the two states in the pair is California, most other states in the pair will have the pair account for low hedging costs. However, when it comes to, for example, Michigan being one of the states in the pair, largely, irrespective of other members the hedging costs will be high.

# 5 Conclusion

In this paper, we develop a novel methodology for life insurers to apply static natural hedging when there is limited capacity of the hedging instruments. We do this in presence of mortality

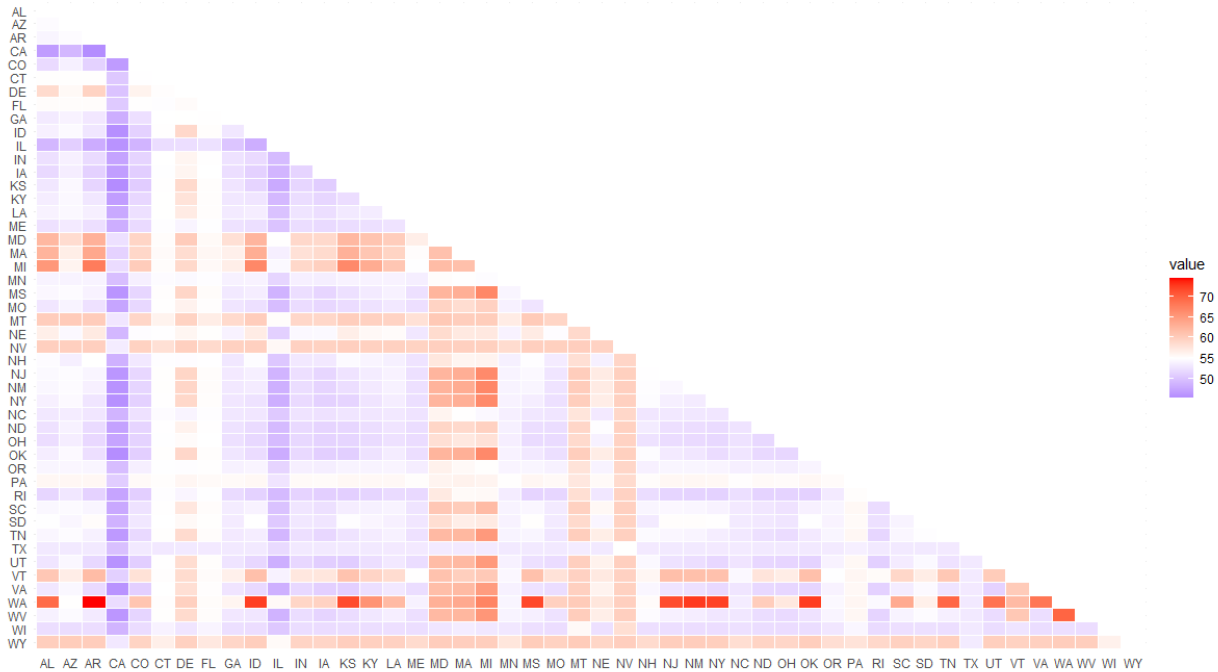


Figure 8: Cost of the optimal hedging strategy for the complex annuity portfolio which allows for hedging using the capacities of two states.

uncertainty across different populations, which allows us to model the situation of cross-border businesses. The methodology is versatile and allows for the evaluation of realistic choices of hedging portfolios, with the objective of reducing basis risk. In essence, this paper represents a theoretical cornerstone to comprehensively address the practical applications of natural hedging, thus aiding life insurance companies in reducing the cost of capital and improving their solvency.

Further research may combine spatial dimension, as introduced in this paper, but also account for the stochasticity of interest rates (such as Wong et al. [2017], Luciano et al. [2017]), dynamic natural hedging, the trade-off between longevity risk and overall company risk (see Gatzert and Wesker [2012]), or the impact of an alternative mortality models. In particular, mortality models that explicitly include spatial dependencies might be of special interest (see Cupido et al. [2020a]). Also, in the context of recent (and potentially future) pandemic(s), the impact of mortality shocks, on top of interest rate ones, would be interesting to investigate.

Finally, we acknowledge that insurance markets are dynamic, where risks are evolving due to changing demographics, economic conditions, and regulatory environments. Thus, further research might involve adapting the proposed methodology to account for the need of dynamic hedging. This can be achieved by periodically re-evaluating our proposed hedging strategy and adjusting the hedging portfolios by changing product composition, spatial placing, and marketing efforts. Alternatively, other more complex mathematical approaches based on dynamic optimization may be applied. We leave this to future research.

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# A Li-Lee model parameters and fit

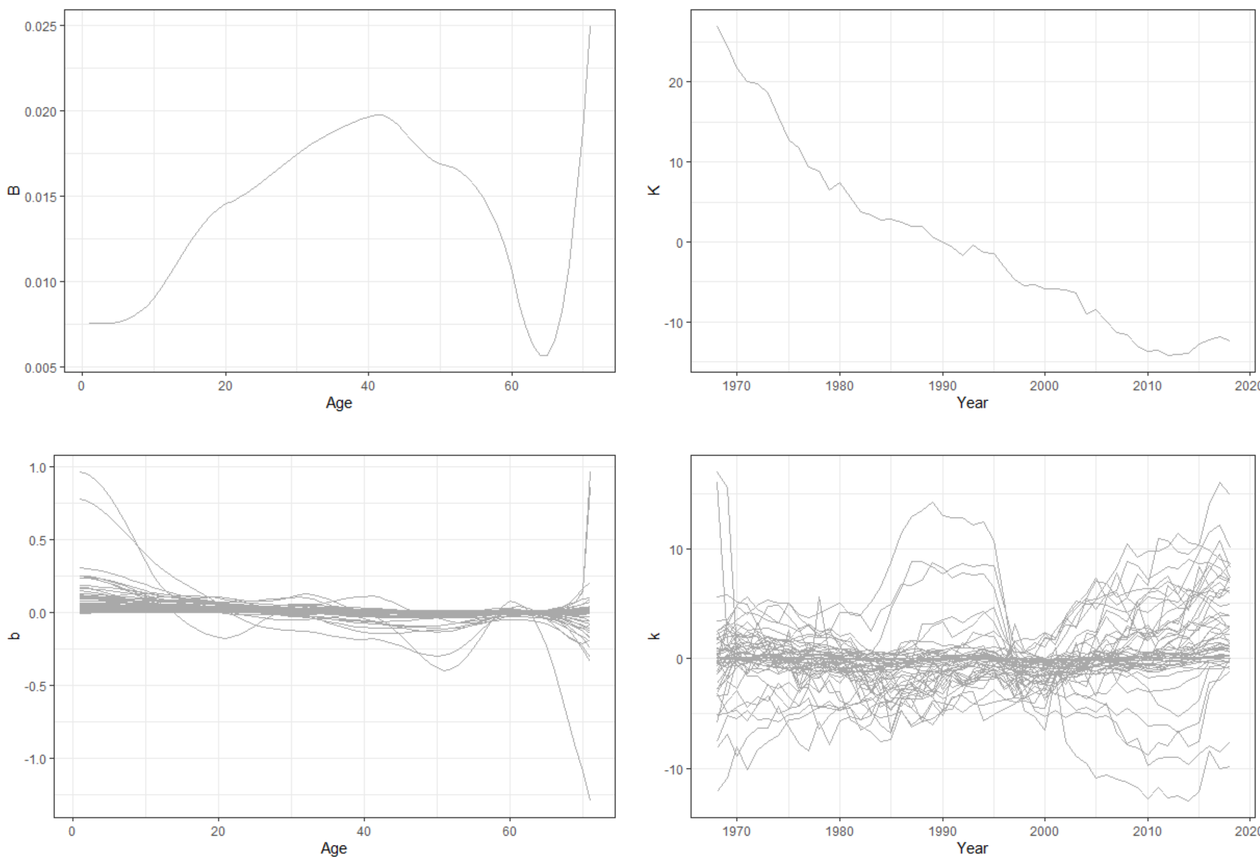


Figure 9: Parameter estimates of the Li-Lee mortality model. The Li-Lee model had an RSS equal to 845.0295, an AIC value of -906,987.6, and an $r^2$ statistic of 0.9984.

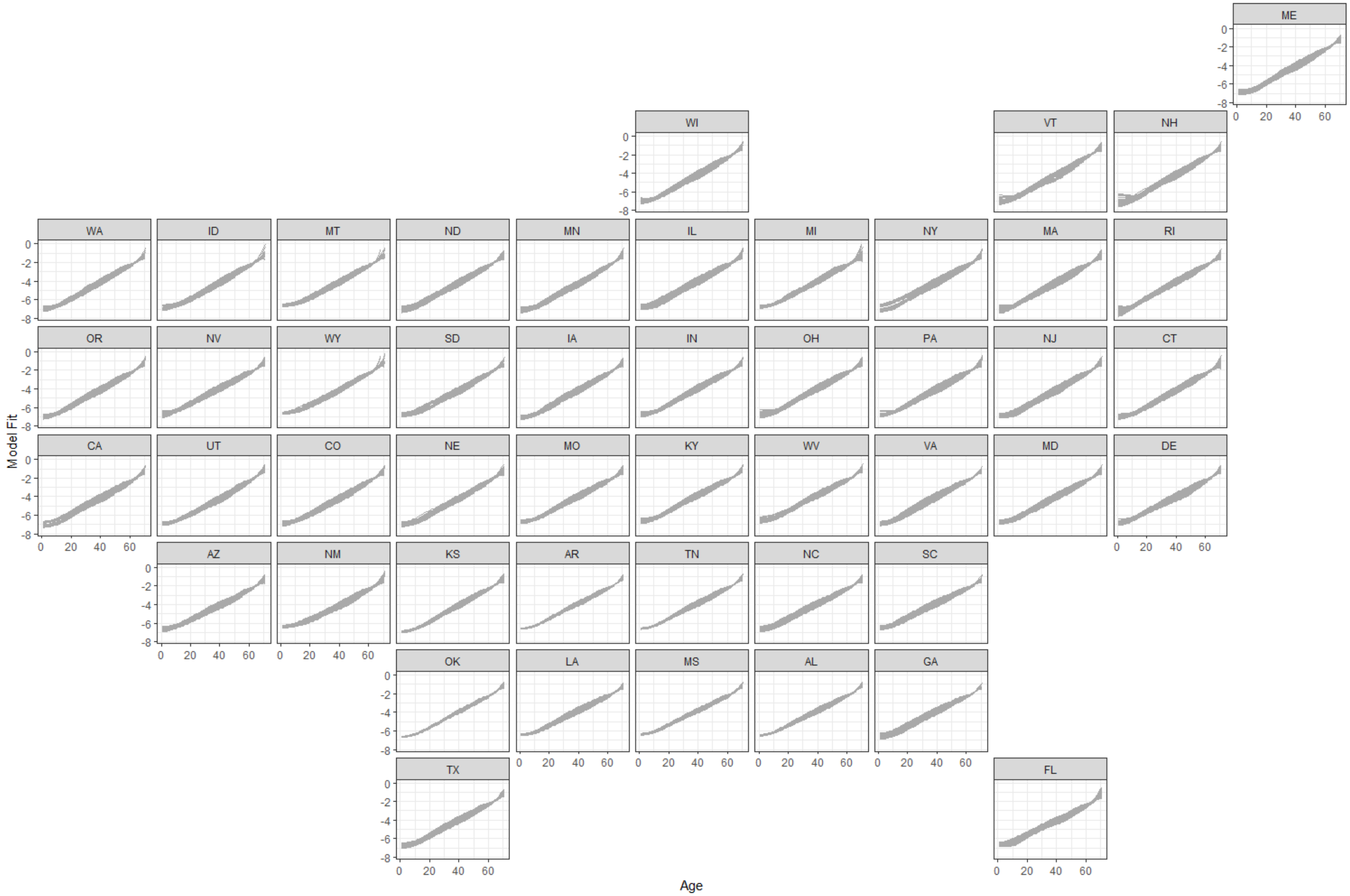


Figure 10: Central death rate estimates of the Li-Lee mortality model.

# B Hedging results for an annuity sold to a 45-year old Illinois age

| | AL | | | AZ | | | AR | | | CA | | | CO | | | CT | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Age/Term | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 |
| 35 | 8.35 | 5.24 | 3.69 | 25.7 | 14.91 | 9.25 | 4.8 | 3.08 | 2.16 | 6.88 | 6.73 | 5.62 | 18.36 | 10.1 | 5.56 | 8.75 | 14.76 | 11.05 |
| 40 | 6.02 | 3.93 | 2.68 | 16.93 | 9.62 | 5.93 | 3.6 | 2.35 | 1.59 | 9.82 | 6.82 | 4.68 | 11.31 | 5.61 | 3.65 | 20.84 | 10.95 | 6.63 |
| 45 | 4.61 | 2.84 | 2.11 | 10.81 | 6.1 | 4.12 | 2.77 | 1.72 | 1.28 | 8.45 | 5.08 | 3.1 | 6.1 | 3.74 | 3.76 | 11.14 | 6.51 | 4.63 |
| | DE | | | FL | | | GA | | | ID | | | IL | | | IN | | |
| Age/Term | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 |
| 35 | 2.87 | 3.14 | 4.82 | 4.22 | 5.87 | 6.75 | 5.73 | 4.52 | 3.99 | 6.59 | 5.7 | 3.21 | 12.46 | 9.63 | 7.41 | 5.12 | 3.57 | 3.14 |
| 40 | 5.1 | 7.12 | 4.98 | 11.17 | 8.92 | 5.72 | 5.86 | 4.64 | 3.79 | 7.82 | 3.45 | 2.05 | 12.42 | 8.41 | 5.8 | 4.43 | 3.66 | 3.22 |
| 45 | 10.17 | 4.4 | 2.04 | 10.39 | 5.61 | 3.93 | 5.99 | 4.27 | 3.17 | 3.64 | 2.02 | 2.22 | 10.02 | 6.17 | 4.03 | 4.83 | 3.78 | 3.26 |
| | IA | | | KS | | | KY | | | LA | | | ME | | | MD | | |
| Age/Term | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 |
| 35 | 4.89 | 2.63 | 2.31 | 4.51 | 2.54 | 1.96 | 2.78 | 2.05 | 1.95 | 10.51 | 7 | 5.13 | 2.71 | 2.5 | 3.11 | 5.49 | 9.53 | 7.85 |
| 40 | 2.88 | 2.55 | 2.75 | 2.84 | 2.15 | 1.77 | 2.62 | 2.35 | 2.24 | 8.47 | 5.65 | 3.75 | 3.61 | 4.37 | 4.85 | 16.67 | 6.99 | 3.73 |
| 45 | 3.45 | 3.47 | 3.83 | 2.75 | 2.06 | 1.96 | 3.23 | 2.75 | 2.46 | 6.72 | 4 | 2.73 | 7.24 | 6.2 | 3.7 | 5.9 | 3.31 | 2.71 |
| | MA | | | MI | | | MN | | | MS | | | MO | | | MT | | |
| Age/Term | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 |
| 35 | 13.97 | 16.52 | 10.58 | 24.33 | 15.65 | 7.92 | 15.78 | 10.78 | 9.98 | 11.16 | 5.76 | 3.48 | 4.61 | 3.46 | 3.16 | 32.62 | 18.03 | 10.46 |
| 40 | 21.12 | 9.89 | 5.18 | 17.96 | 8.01 | 4.2 | 12.69 | 11.14 | 7.8 | 6.21 | 3.55 | 2.22 | 4.28 | 3.71 | 3.22 | 20.06 | 10.61 | 6.36 |
| 45 | 9.45 | 4.76 | 2.7 | 8.24 | 4.04 | 2.97 | 13.91 | 7.85 | 3.49 | 3.95 | 2.29 | 1.61 | 4.94 | 3.78 | 3.19 | 11.57 | 6.42 | 3.49 |
| | NE | | | NV | | | NH | | | NJ | | | NM | | | NY | | |
| Age/Term | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 |
| 35 | 2.71 | 3.06 | 5.4 | 27.77 | 15.04 | 9.11 | 2.57 | 3.35 | 5.04 | 3.28 | 4.24 | 3.96 | 2.15 | 1.35 | 1.16 | 2.42 | 2.57 | 2.46 |
| 40 | 4.55 | 7.79 | 6.17 | 16.35 | 9.23 | 5.43 | 5.81 | 7.83 | 6.66 | 7.58 | 5.49 | 3.55 | 1.64 | 1.36 | 1.37 | 3.95 | 3.27 | 2.61 |
| 45 | 12.26 | 5.8 | 2.33 | 9.96 | 5.34 | 3.58 | 12.31 | 6.91 | 2.91 | 7.15 | 3.9 | 1.99 | 1.88 | 1.77 | 2.04 | 4.39 | 3.04 | 2.06 |
| | NC | | | ND | | | OH | | | OK | | | OR | | | PA | | |
| Age/Term | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 |
| 35 | 4.62 | 4.07 | 3.94 | 0.63 | 0.66 | 0.88 | 4.4 | 3.63 | 3.21 | 7.27 | 3.88 | 2.55 | 10.01 | 4.55 | 5.83 | 10.49 | 10.86 | 9.2 |
| 40 | 5.41 | 4.79 | 4.01 | 0.95 | 1.31 | 1.93 | 4.95 | 3.89 | 3.25 | 4.45 | 2.72 | 1.87 | 4.22 | 6.24 | 6.11 | 15.17 | 10.2 | 6.16 |
| 45 | 6.42 | 4.63 | 3.51 | 2.48 | 3.4 | 4 | 5.17 | 3.78 | 3.3 | 3.26 | 2.03 | 1.5 | 9.39 | 6.43 | 3.1 | 11.37 | 6.05 | 3.44 |
| | RI | | | SC | | | SD | | | TN | | | TX | | | UT | | |
| Age/Term | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 |
| 35 | 15.04 | 14.3 | 7.39 | 6.23 | 4.95 | 4.09 | 1.03 | 1.4 | 2.3 | 5.85 | 3.74 | 2.9 | 17.2 | 14.27 | 9.49 | 3.01 | 1.55 | 1.24 |
| 40 | 19.49 | 7.94 | 4.66 | 6.25 | 4.7 | 3.4 | 2.38 | 4.13 | 4.85 | 4.42 | 3.23 | 2.47 | 17.89 | 10.12 | 6.21 | 1.74 | 1.36 | 1.52 |
| 45 | 8.05 | 4.53 | 3.88 | 5.92 | 3.76 | 2.72 | 9.31 | 6.89 | 3.66 | 4.01 | 2.78 | 2.12 | 11.26 | 6.33 | 4.03 | 1.76 | 1.94 | 2.98 |
| | VT | | | VA | | | WA | | | WV | | | WI | | | WY | | |
| Age/Term | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 |
| 35 | 3.68 | 5.11 | 7.94 | 11.16 | 6.51 | 4.28 | 34.59 | 16.8 | 6.7 | 1.73 | 1.43 | 1.36 | 10.95 | 8.55 | 7.44 | 21.9 | 15.96 | 10.53 |
| 40 | 9.36 | 10.89 | 4.98 | 7.13 | 4.45 | 3.11 | 17.71 | 6.3 | 3.44 | 1.87 | 1.69 | 1.62 | 10.99 | 8.6 | 6.61 | 19.17 | 10.83 | 6.24 |
| 45 | 12.16 | 3.92 | 1.15 | 4.99 | 3.28 | 2.47 | 5.89 | 3.25 | 2.53 | 2.3 | 2.01 | 2 | 10.86 | 7.16 | 4.42 | 11.73 | 6.17 | 3.2 |

Figure 11: Positions of the optimal hedging strategy for an annuity issued in Illinois using one single life insurance as a hedging instrument across different terms and cohorts from the 48 states.

| | AL | | | AZ | | | AR | | | CA | | | CO | | | CT | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Age/Term | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 |
| 35 | 0.14 | 0.22 | 0.32 | 0.33 | 0.49 | 0.61 | 0.07 | 0.12 | 0.18 | 0.07 | 0.19 | 0.32 | 0.19 | 0.27 | 0.31 | 0.09 | 0.38 | 0.58 |
| 40 | 0.16 | 0.27 | 0.36 | 0.36 | 0.51 | 0.61 | 0.09 | 0.15 | 0.21 | 0.18 | 0.32 | 0.44 | 0.19 | 0.25 | 0.32 | 0.34 | 0.47 | 0.57 |
| 45 | 0.2 | 0.3 | 0.41 | 0.36 | 0.5 | 0.63 | 0.11 | 0.18 | 0.25 | 0.25 | 0.38 | 0.44 | 0.17 | 0.27 | 0.52 | 0.31 | 0.46 | 0.64 |
| | DE | | | FL | | | GA | | | ID | | | IL | | | IN | | |
| Age/Term | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 |
| 35 | 0.04 | 0.11 | 0.34 | 0.06 | 0.21 | 0.48 | 0.09 | 0.17 | 0.31 | 0.06 | 0.15 | 0.17 | 0.31 | 0.53 | 0.74 | 0.06 | 0.12 | 0.22 |
| 40 | 0.12 | 0.4 | 0.54 | 0.26 | 0.51 | 0.63 | 0.15 | 0.29 | 0.47 | 0.13 | 0.15 | 0.18 | 0.41 | 0.65 | 0.84 | 0.09 | 0.2 | 0.36 |
| 45 | 0.37 | 0.39 | 0.33 | 0.37 | 0.49 | 0.63 | 0.24 | 0.43 | 0.59 | 0.1 | 0.14 | 0.3 | 0.48 | 0.7 | 0.86 | 0.17 | 0.34 | 0.55 |
| | IA | | | KS | | | KY | | | LA | | | ME | | | MD | | |
| Age/Term | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 |
| 35 | 0.05 | 0.07 | 0.12 | 0.05 | 0.08 | 0.13 | 0.04 | 0.08 | 0.16 | 0.18 | 0.3 | 0.44 | 0.03 | 0.08 | 0.19 | 0.07 | 0.32 | 0.52 |
| 40 | 0.05 | 0.11 | 0.24 | 0.05 | 0.11 | 0.19 | 0.07 | 0.15 | 0.28 | 0.23 | 0.39 | 0.5 | 0.07 | 0.22 | 0.48 | 0.35 | 0.37 | 0.39 |
| 45 | 0.09 | 0.25 | 0.54 | 0.09 | 0.17 | 0.31 | 0.13 | 0.28 | 0.47 | 0.29 | 0.43 | 0.54 | 0.23 | 0.5 | 0.56 | 0.2 | 0.28 | 0.44 |
| | MA | | | MI | | | MN | | | MS | | | MO | | | MT | | |
| Age/Term | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 |
| 35 | 0.14 | 0.45 | 0.59 | 0.3 | 0.51 | 0.54 | 0.14 | 0.25 | 0.49 | 0.2 | 0.27 | 0.32 | 0.06 | 0.12 | 0.23 | 0.39 | 0.54 | 0.64 |
| 40 | 0.37 | 0.45 | 0.47 | 0.38 | 0.44 | 0.46 | 0.19 | 0.45 | 0.64 | 0.18 | 0.26 | 0.32 | 0.09 | 0.21 | 0.37 | 0.38 | 0.52 | 0.62 |
| 45 | 0.28 | 0.35 | 0.38 | 0.29 | 0.36 | 0.5 | 0.36 | 0.52 | 0.46 | 0.18 | 0.26 | 0.34 | 0.18 | 0.34 | 0.54 | 0.36 | 0.51 | 0.53 |
| | NE | | | NV | | | NH | | | NJ | | | NM | | | NY | | |
| Age/Term | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 |
| 35 | 0.02 | 0.07 | 0.29 | 0.38 | 0.54 | 0.66 | 0.02 | 0.09 | 0.27 | 0.04 | 0.12 | 0.24 | 0.03 | 0.05 | 0.09 | 0.03 | 0.08 | 0.16 |
| 40 | 0.07 | 0.34 | 0.56 | 0.37 | 0.54 | 0.62 | 0.1 | 0.35 | 0.6 | 0.14 | 0.27 | 0.34 | 0.04 | 0.08 | 0.15 | 0.08 | 0.17 | 0.26 |
| 45 | 0.35 | 0.43 | 0.34 | 0.37 | 0.5 | 0.62 | 0.35 | 0.51 | 0.42 | 0.22 | 0.31 | 0.3 | 0.07 | 0.15 | 0.32 | 0.14 | 0.25 | 0.31 |
| | NC | | | ND | | | OH | | | OK | | | OR | | | PA | | |
| Age/Term | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 |
| 35 | 0.06 | 0.14 | 0.28 | 0.01 | 0.02 | 0.05 | 0.06 | 0.13 | 0.23 | 0.11 | 0.16 | 0.22 | 0.1 | 0.12 | 0.33 | 0.12 | 0.34 | 0.59 |
| 40 | 0.12 | 0.28 | 0.46 | 0.02 | 0.06 | 0.17 | 0.11 | 0.23 | 0.37 | 0.11 | 0.18 | 0.25 | 0.07 | 0.29 | 0.58 | 0.3 | 0.53 | 0.64 |
| 45 | 0.24 | 0.43 | 0.61 | 0.07 | 0.24 | 0.54 | 0.19 | 0.35 | 0.57 | 0.14 | 0.21 | 0.3 | 0.28 | 0.5 | 0.47 | 0.38 | 0.51 | 0.55 |
| | RI | | | SC | | | SD | | | TN | | | TX | | | UT | | |
| Age/Term | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 |
| 35 | 0.16 | 0.41 | 0.44 | 0.1 | 0.21 | 0.34 | 0.01 | 0.04 | 0.14 | 0.09 | 0.15 | 0.23 | 0.21 | 0.46 | 0.62 | 0.03 | 0.04 | 0.07 |
| 40 | 0.35 | 0.38 | 0.45 | 0.17 | 0.32 | 0.44 | 0.04 | 0.19 | 0.45 | 0.11 | 0.21 | 0.31 | 0.37 | 0.54 | 0.66 | 0.03 | 0.06 | 0.13 |
| 45 | 0.25 | 0.36 | 0.59 | 0.26 | 0.4 | 0.53 | 0.28 | 0.51 | 0.51 | 0.16 | 0.28 | 0.4 | 0.38 | 0.55 | 0.66 | 0.05 | 0.13 | 0.39 |
| | VT | | | VA | | | WA | | | WV | | | WI | | | WY | | |
| Age/Term | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 |
| 35 | 0.04 | 0.14 | 0.43 | 0.13 | 0.2 | 0.27 | 0.34 | 0.44 | 0.37 | 0.03 | 0.06 | 0.12 | 0.11 | 0.23 | 0.41 | 0.26 | 0.49 | 0.65 |
| 40 | 0.16 | 0.48 | 0.44 | 0.14 | 0.23 | 0.32 | 0.3 | 0.28 | 0.32 | 0.05 | 0.12 | 0.22 | 0.18 | 0.38 | 0.59 | 0.37 | 0.53 | 0.61 |
| 45 | 0.34 | 0.28 | 0.16 | 0.16 | 0.28 | 0.4 | 0.17 | 0.24 | 0.37 | 0.1 | 0.21 | 0.39 | 0.31 | 0.52 | 0.62 | 0.37 | 0.48 | 0.49 |

Figure 12: Effectiveness of the optimal hedging strategy for an annuity issued in Illinois using one single life insurance as a hedging instrument across different terms and cohorts from the 48 states.

| | AL | | | AZ | | | AR | | | CA | | | CO | | | CT | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Age/Term | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 |
| 35 | 14.41 | 9.04 | 6.55 | 14 | 8.92 | 6.5 | 14.15 | 8.85 | 6.42 | 14.59 | 9 | 6.48 | 13.96 | 8.82 | 6.38 | 14.24 | 8.97 | 6.44 |
| 40 | 12.76 | 8.17 | 6.06 | 12.37 | 8.01 | 5.96 | 12.53 | 7.99 | 5.93 | 12.61 | 8.03 | 5.93 | 12.3 | 7.84 | 5.8 | 12.41 | 7.93 | 5.82 |
| 45 | 11.62 | 7.61 | 5.84 | 11.28 | 7.44 | 5.69 | 11.41 | 7.45 | 5.73 | 11.4 | 7.41 | 5.67 | 11.1 | 7.22 | 5.51 | 11.12 | 7.27 | 5.49 |
| | DE | | | FL | | | GA | | | ID | | | IL | | | IN | | |
| Age/Term | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 |
| 35 | 12.86 | 8.52 | 6.35 | 13.25 | 8.67 | 6.42 | 13.2 | 8.56 | 6.3 | 14.93 | 9.3 | 6.63 | 6.54 | 4.96 | 4.05 | 13.53 | 8.57 | 6.25 |
| 40 | 11.81 | 7.87 | 5.95 | 12.19 | 8.01 | 6.01 | 11.89 | 7.81 | 5.86 | 12.8 | 8.25 | 6.03 | 7.41 | 5.32 | 4.24 | 12.04 | 7.76 | 5.78 |
| 45 | 11.17 | 7.44 | 5.75 | 11.4 | 7.56 | 5.81 | 11.04 | 7.35 | 5.68 | 11.7 | 7.65 | 5.69 | 7.85 | 5.5 | 4.25 | 11.03 | 7.25 | 5.58 |
| | IA | | | KS | | | KY | | | LA | | | ME | | | MD | | |
| Age/Term | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 |
| 35 | 14.1 | 8.93 | 6.39 | 13.56 | 8.33 | 5.99 | 13.49 | 8.65 | 6.34 | 13.66 | 8.81 | 6.49 | 11.84 | 7.87 | 5.94 | 13.62 | 8.8 | 6.41 |
| 40 | 12.59 | 7.96 | 5.81 | 12.04 | 7.58 | 5.59 | 12.05 | 7.88 | 5.9 | 12.18 | 7.98 | 5.99 | 11.01 | 7.4 | 5.62 | 12.3 | 7.97 | 5.88 |
| 45 | 11.34 | 7.3 | 5.5 | 10.98 | 7.1 | 5.45 | 11.11 | 7.39 | 5.71 | 11.19 | 7.45 | 5.77 | 10.64 | 7.14 | 5.52 | 11.34 | 7.4 | 5.62 |
| | MA | | | MI | | | MN | | | MS | | | MO | | | MT | | |
| Age/Term | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 |
| 35 | 14.07 | 8.79 | 6.34 | 14.03 | 8.85 | 6.38 | 13.41 | 8.49 | 6.14 | 13.98 | 8.96 | 6.54 | 13.97 | 8.71 | 6.32 | 14.21 | 9 | 6.48 |
| 40 | 12.33 | 7.85 | 5.79 | 12.34 | 7.97 | 5.9 | 11.89 | 7.66 | 5.65 | 12.4 | 8.1 | 6.06 | 12.35 | 7.88 | 5.85 | 12.43 | 7.96 | 5.87 |
| 45 | 11.09 | 7.22 | 5.5 | 11.25 | 7.44 | 5.68 | 10.95 | 7.16 | 5.42 | 11.35 | 7.57 | 5.85 | 11.23 | 7.34 | 5.64 | 11.24 | 7.34 | 5.55 |
| | NE | | | NV | | | NH | | | NJ | | | NM | | | NY | | |
| Age/Term | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 |
| 35 | 15.03 | 9.16 | 6.37 | 13.95 | 8.84 | 6.43 | 13.9 | 8.64 | 6.21 | 14.55 | 8.97 | 6.45 | 13.81 | 9.05 | 6.75 | 13.8 | 8.73 | 6.37 |
| 40 | 12.74 | 7.86 | 5.69 | 12.36 | 7.97 | 5.93 | 12.17 | 7.75 | 5.72 | 12.57 | 7.97 | 5.88 | 12.2 | 8.12 | 6.14 | 12.36 | 7.93 | 5.88 |
| 45 | 11.08 | 7.12 | 5.33 | 11.26 | 7.42 | 5.68 | 11.06 | 7.21 | 5.49 | 11.31 | 7.34 | 5.59 | 11.15 | 7.5 | 5.79 | 11.28 | 7.36 | 5.6 |
| | NC | | | ND | | | OH | | | OK | | | OR | | | PA | | |
| Age/Term | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 |
| 35 | 14.39 | 9.01 | 6.5 | 13.04 | 8.1 | 5.98 | 12.5 | 8.18 | 6.06 | 13.31 | 8.5 | 6.22 | 14.15 | 8.96 | 6.35 | 14.51 | 8.96 | 6.43 |
| 40 | 12.62 | 8.08 | 5.98 | 12.09 | 7.66 | 5.72 | 11.48 | 7.54 | 5.68 | 11.82 | 7.7 | 5.77 | 12.58 | 7.88 | 5.77 | 12.51 | 7.95 | 5.87 |
| 45 | 11.42 | 7.49 | 5.72 | 11.13 | 7.25 | 5.6 | 10.7 | 7.13 | 5.55 | 10.83 | 7.2 | 5.59 | 11.19 | 7.23 | 5.48 | 11.18 | 7.32 | 5.6 |
| | RI | | | SC | | | SD | | | TN | | | TX | | | UT | | |
| Age/Term | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 |
| 35 | 13.36 | 8.58 | 6.19 | 13.69 | 8.77 | 6.47 | 13.98 | 8.7 | 6.39 | 14.18 | 8.98 | 6.54 | 14.17 | 8.88 | 6.41 | 13.73 | 8.89 | 6.49 |
| 40 | 12.14 | 7.75 | 5.73 | 12.31 | 8.01 | 6.02 | 12.57 | 8.01 | 5.93 | 12.59 | 8.12 | 6.05 | 12.29 | 7.92 | 5.86 | 12.16 | 7.94 | 5.88 |
| 45 | 10.9 | 7.17 | 5.5 | 11.32 | 7.5 | 5.81 | 11.34 | 7.41 | 5.67 | 11.48 | 7.56 | 5.81 | 11.23 | 7.35 | 5.62 | 11.02 | 7.27 | 5.52 |
| | VT | | | VA | | | WA | | | WV | | | WI | | | WY | | |
| Age/Term | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 |
| 35 | 13.03 | 8.43 | 6.19 | 13.85 | 8.66 | 6.27 | 13.96 | 8.72 | 6.25 | 13.43 | 8.46 | 6.25 | 13.83 | 8.6 | 6.22 | 14.34 | 9.03 | 6.51 |
| 40 | 11.93 | 7.75 | 5.78 | 12.27 | 7.81 | 5.78 | 12.26 | 7.77 | 5.73 | 12.07 | 7.76 | 5.85 | 12.19 | 7.75 | 5.73 | 12.47 | 8 | 5.9 |
| 45 | 11.17 | 7.32 | 5.6 | 11.19 | 7.26 | 5.54 | 11.23 | 7.24 | 5.5 | 11.11 | 7.33 | 5.7 | 11.08 | 7.22 | 5.51 | 11.25 | 7.35 | 5.54 |

Figure 13: Cost of the optimal hedging strategy for an annuity issued in Illinois using one single life insurance as a hedging instrument across different terms and cohorts from the 48 states.

| | AL | | | AZ | | | AR | | | CA | | | CO | | | CT | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Age/Term | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 |
| 35 | -42.92 | -212.11 | 244.31 | -9.39 | -52.43 | 62.01 | -16.74 | 3.39 | 7.81 | -78.21 | -16.76 | 95.76 | 9.12 | -9.96 | 16.66 | 1.7 | -127.08 | 129.13 |
| 40 | 94.03 | 131.54 | -228.07 | -4.96 | 36.22 | -36.31 | 59.31 | -26.96 | -40.24 | 159.27 | -200.52 | 36.79 | 22.56 | 2.17 | -11.08 | 1.83 | 82.16 | -89.32 |
| 45 | -42.95 | -3.82 | 54.03 | 7.9 | -8.38 | 5.25 | -30.94 | 9.32 | 30.94 | -68.35 | 129.09 | -56.96 | -16.74 | -5.02 | 9.32 | -0.52 | -10.45 | 14.89 |
| | DE | | | FL | | | GA | | | ID | | | IL | | | IN | | |
| Age/Term | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 |
| 35 | -5.68 | 4.24 | 3.52 | -5.85 | -2.89 | 10.38 | -41.95 | 82.4 | -32.65 | -13.49 | -12.02 | 16.55 | 18.69 | -41.08 | 22.49 | 10.64 | -11.02 | 16.26 |
| 40 | 11.18 | -52.61 | 40.06 | 11.59 | -10.26 | -3.57 | 62.1 | -123.39 | 47.83 | 10.43 | 16.71 | -15.5 | -28.47 | -17.73 | 46.63 | -9.76 | 5.96 | -17.69 |
| 45 | -4.54 | 27.39 | -21.53 | -4.19 | 1.19 | 5.61 | -21.99 | 40.93 | -11.83 | 0.22 | -13.83 | 12.21 | 12.19 | 22.22 | -33.21 | 3.15 | -6.08 | 12.24 |
| | IA | | | KS | | | KY | | | LA | | | ME | | | MD | | |
| Age/Term | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 |
| 35 | 17.88 | -10.79 | 4.57 | 6.33 | -59.52 | 63.81 | 1.41 | -0.08 | -0.72 | 0.1 | -68.61 | 58.63 | 9.72 | -12.01 | 8.05 | 2.49 | -4.01 | 7.14 |
| 40 | -15.63 | 3.19 | 0.75 | -5.74 | 65.68 | -76.98 | -5.12 | 5.68 | -5.04 | -34.43 | 113.51 | -71.92 | -14.62 | 1.94 | 4.82 | -8.93 | 14.84 | -13.62 |
| 45 | 4.46 | -5.4 | 6.12 | 2.36 | -26.78 | 33.17 | 3.51 | -9.59 | 10.38 | 22.51 | -51.48 | 29.79 | 6.42 | -3.11 | 1.31 | 5.72 | -14.5 | 13.46 |
| | MA | | | MI | | | MN | | | MS | | | MO | | | MT | | |
| Age/Term | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 |
| 35 | -14 | -254.05 | 274.95 | -5.74 | -9.06 | 11.61 | 14.67 | -33.7 | 28.93 | 126.7 | -241.59 | 131.15 | -8.75 | -19.29 | 25.53 | -24.06 | -59.12 | 77.55 |
| 40 | 63.46 | 108.68 | -178.91 | -8.29 | 23.46 | -15.57 | -14.94 | 12.95 | -6.88 | -53.39 | 154 | -130.08 | 10.02 | 16.07 | -29.19 | -10.79 | 47.57 | -27.77 |
| 45 | -32.49 | 22.19 | 13.61 | 8.26 | -17.83 | 12.88 | 5.41 | -3.3 | 2.16 | -16.32 | -6.27 | 36.92 | -2.01 | -9.37 | 16.13 | 16.93 | -12.85 | -5.98 |
| | NE | | | NV | | | NH | | | NJ | | | NM | | | NY | | |
| Age/Term | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 |
| 35 | 4.73 | -20.78 | 21.74 | 2 | -25.33 | 22.72 | 4.95 | -30.77 | 27.17 | 11.72 | -63.47 | 49.98 | 7.02 | -8.45 | 8.99 | -47.95 | 100.45 | -54.82 |
| 40 | -6.69 | -6.16 | 6.2 | -5.68 | 22.57 | -15.28 | -9.92 | 14.37 | -6.44 | -31.7 | 31.93 | 1.35 | -7.93 | 10.68 | -14.61 | 87.48 | -211.13 | 123.92 |
| 45 | 3.35 | 4.98 | -4.41 | 3.81 | -10.37 | 7.53 | 5.22 | -4.15 | 1.59 | 16.07 | -1.16 | -13.74 | 2.88 | -10.7 | 14.22 | -35.57 | 89.69 | -51.89 |
| | NC | | | ND | | | OH | | | OK | | | OR | | | PA | | |
| Age/Term | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 |
| 35 | 15.94 | -47.18 | 36.31 | 3.67 | 1.92 | -1.56 | -9.44 | -13.85 | 16.5 | -84.21 | -0.8 | 92.7 | -4.16 | 16.74 | -8.87 | -12.59 | -145.36 | 164.02 |
| 40 | -26.35 | 54.53 | -37.92 | -4.95 | -15.3 | 12.82 | 8.73 | 10.53 | -15.1 | 135.41 | -19.6 | -137.96 | 16.41 | -57.24 | 39.76 | 26.31 | 108.57 | -144.22 |
| 45 | 11.42 | -22.41 | 16.93 | 2.51 | 1.74 | 0.88 | -0.93 | -7.61 | 10.35 | -52.68 | 4.75 | 61.21 | -7.41 | 23.36 | -14.7 | -10.64 | -19.17 | 35.1 |
| | RI | | | SC | | | SD | | | TN | | | TX | | | UT | | |
| Age/Term | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 |
| 35 | -2.82 | -6.74 | 7.57 | -12.18 | -201.98 | 226.43 | 2.16 | -12.99 | 13.36 | -64.03 | 64.37 | 10.71 | -55.15 | 61.53 | -12.9 | 9.6 | 13.29 | -1.75 |
| 40 | 7.7 | 0.37 | -3.81 | 63.51 | 154.31 | -239.47 | -1.32 | -6.47 | 3.9 | 122.38 | -100.5 | -40.51 | 56.4 | -91.21 | 53.69 | -2.02 | -24.07 | 7.26 |
| 45 | -2.42 | -4.69 | 7.42 | -35.49 | -23.82 | 70.18 | 0.89 | 2.25 | 0.24 | -51.86 | 31.6 | 30.12 | -13.17 | 28.82 | -23.27 | -1.96 | 2.48 | 5.01 |
| | VT | | | VA | | | WA | | | WV | | | WI | | | WY | | |
| Age/Term | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 | 10 | 15 | 20 |
| 35 | 12.42 | -45.98 | 43.89 | -102.95 | -3.89 | 116.94 | -9.67 | -3.31 | 5.09 | 4.65 | -15.03 | 13.44 | -25.77 | 7.55 | 13.25 | -12.07 | -177.53 | 172.43 |
| 40 | -17.9 | 23.73 | -18.22 | 79.75 | 39.97 | -160.11 | 18.77 | 14.55 | -20.33 | -9.42 | 22.94 | -22.03 | 31.32 | -30.59 | 1.17 | 7.24 | 142.61 | -124.25 |
| 45 | 7.56 | -4.39 | 2.6 | -10.67 | -31.85 | 65.5 | -6.48 | -15.55 | 19.57 | 4.86 | -14.7 | 15.7 | -8.56 | 8.92 | 2.43 | 1.07 | -25.83 | 16.91 |

Figure 14: Positions of the optimal hedging strategy for an annuity issued in Illinois using all the insurance products sold in a single state as a hedging instrument across different terms and cohorts from the 48 states.

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| **AL** | **AZ** | **AR** | **CA** | **CO** | **CT** | **DE** | **FL** | **GA** | **ID** | **IL** | **IN** |
| 0.69 | 0.69 | 0.68 | 0.69 | 0.69 | 0.69 | 0.69 | 0.69 | 0.69 | 0.69 | 0.9 | 0.69 |
| **IA** | **KS** | **KY** | **LA** | **ME** | **MD** | **MA** | **MI** | **MN** | **MS** | **MO** | **MT** |
| 0.69 | 0.69 | 0.69 | 0.69 | 0.69 | 0.69 | 0.69 | 0.69 | 0.69 | 0.69 | 0.69 | 0.69 |
| **NE** | **NV** | **NH** | **NJ** | **NM** | **NY** | **NC** | **ND** | **OH** | **OK** | **OR** | **PA** |
| 0.69 | 0.69 | 0.69 | 0.69 | 0.69 | 0.69 | 0.69 | 0.69 | 0.69 | 0.69 | 0.69 | 0.69 |
| **RI** | **SC** | **SD** | **TN** | **TX** | **UT** | **VT** | **VA** | **WA** | **WV** | **WI** | **WY** |
| 0.69 | 0.69 | 0.69 | 0.69 | 0.69 | 0.69 | 0.69 | 0.69 | 0.69 | 0.69 | 0.69 | 0.69 |

Figure 15: Effectiveness of the optimal hedging strategy for an annuity issued in Illinois using all the insurance products sold in a single state as a hedging instrument across different terms and cohorts from the 48 states.

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| **AL** | **AZ** | **AR** | **CA** | **CO** | **CT** | **DE** | **FL** | **GA** | **ID** | **IL** | **IN** |
| 6.04 | 5.25 | 3.15 | 5.15 | 5.56 | 5.57 | 4.71 | 4.63 | 4.22 | 4.6 | 3.91 | 4.7 |
| **IA** | **KS** | **KY** | **LA** | **ME** | **MD** | **MA** | **MI** | **MN** | **MS** | **MO** | **MT** |
| 4.54 | 4.88 | 4.41 | 5.39 | 4.61 | 4.67 | 6.69 | 4.61 | 4.83 | 3.38 | 4.47 | 6.04 |
| **NE** | **NV** | **NH** | **NJ** | **NM** | **NY** | **NC** | **ND** | **OH** | **OK** | **OR** | **PA** |
| 4.76 | 4.87 | 4.73 | 5.27 | 4.51 | 3.93 | 4.86 | 4.37 | 4.48 | 4.99 | 4.53 | 5.99 |
| **RI** | **SC** | **SD** | **TN** | **TX** | **UT** | **VT** | **VA** | **WA** | **WV** | **WI** | **WY** |
| 4.54 | 6.36 | 4.59 | 4.23 | 5.31 | 4.71 | 4.99 | 5.3 | 4.4 | 4.56 | 4.45 | 6.3 |

Figure 16: Cost of the optimal hedging strategy for an annuity issued in Illinois using all the insurance products sold in a single state as a hedging instrument across different terms and cohorts from the 48 states.