UNIVERSITÀ DEGLI STUDI DI TORINO

DIPARTIMENTO DI MATEMATICA GIUSEPPE PEANO

Dottorato in Matematica – XXXIV ciclo

The Universality of Forcing

Relatore: Professor Matteo Viale

Candidato: Tsvetlin Marinov

ANNO ACCADEMICO 2022/2023

Contents

Chapter 0

Introduction

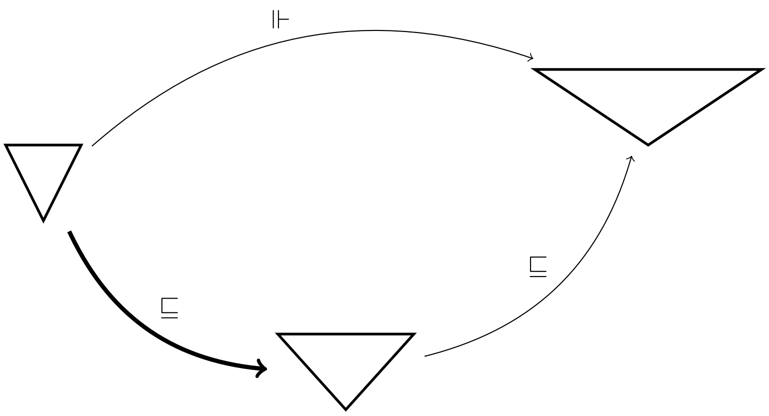
Forcing is a powerful method for constructing models of Set Theory. It was invented by Paul Cohen, who introduced it in his pivotal 1963 paper [5] to construct a model of $\mathsf{ZFC} + \neg\mathsf{CH}$. Since then it has been employed to produce a staggering variety of models of extensions of ZF. By now it has taken quite a central place in contemporary set theorists' arsenal.

But how exhaustive is the method of forcing? How rich is the class of structures it can produce? Would set theorists miss out on a lot if they don't look for other methods for constructing models of set theory and just used forcing?

This is obviously not a precise question. To move towards thinking about it more precisely, let's imagine the set-theoretic multiverse as an oriented multigraph whose vertices are models of set theories we might be interested in, and whose arrows denote relations like "is a substructure of", "is an elementary substructure of", "is a ground of"[1], etc. Then a question about the power of the forcing method becomes a question about the prevalence of forcing arrows in this multigraph.

The formally strongest conceivable answers like "every two models are connected by a forcing arrow" or "every substructure arrow is also a forcing arrow" are immediately seen to be impossible because of some obvious limitations of forcing – for example, forcing preserves the cardinality of models.

This suggests that the general question of how powerful forcing is will not have a single formalization but a variety of formalizable aspects. One such aspect that has long been established is contained in what we call the Intermediate Model Theorem [2], which states that a model of ZFC which sits as a substructure between a model $\mathcal{M}$ of ZFC and a forcing extension of $\mathcal{M}$ is itself a forcing extension of $\mathcal{M}$. I.e. whenever we have the following situation



[1] Saying that $\mathcal{M}$ is a ground of $\mathcal{N}$ is equivalent to saying that $\mathcal{N}$ is a forcing extension of $\mathcal{M}$.
[2] It can be found for example in Jech's [10, Lemma 15.43] or Kanamori's [11, Proposition 10.10]

then the thick substructure arrow is also a forcing arrow. In loose terms, forcing does not jump over models of ZFC.

In this thesis we explore another specific aspect of the general question of how powerful the forcing method is.

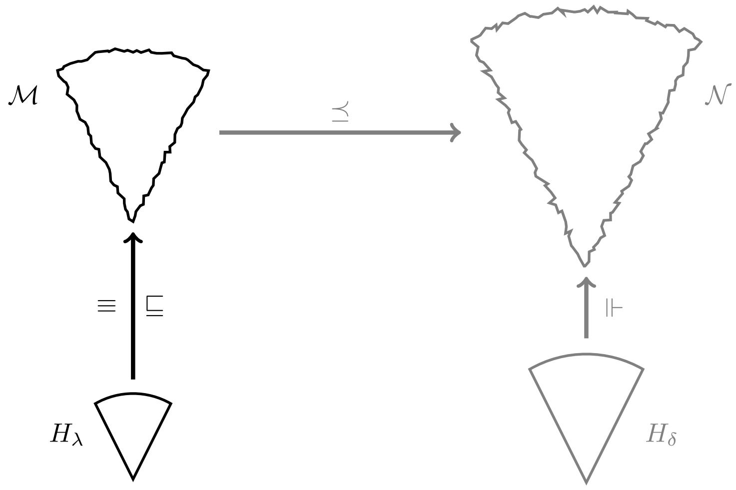
There are simple ways to construct models of ZFC. Ones which do not require elaborate methods to be produced. For example, V_δ for δ inaccessible or some other worldly cardinal.

Our dream goal is to argue that a large class of set theoretic worlds are accessible through forcing from a class of easily accessible structures. So we want to identify two classes – $\mathcal{V}$ and $\mathcal{W}$ – of models, such that every model in $\mathcal{W}$ is accessible through forcing from a model in $\mathcal{V}$. And we want $\mathcal{W}$ to be as versatile as possible and the models in $\mathcal{V}$ to be as simple to construct as possible.

Our choice of simple models that would comprise our $\mathcal{V}$ shall be standard structures[3] of the form $(H_\delta, \in)$, for δ being a certain kind of large cardinals. We consider those fairly easy to construct – they are separated from V by the simple property $|\operatorname{trcl}(x)| < \delta$. The reason for this choice are some remarkable properties of the stationary tower forcings and their elementary embeddings.

And the class $\mathcal{W}$ we want to be as diverse as possible. We'd be happy if we're able to include each structure that has even a vague set-theoretic flavour. We set out to include in this class all structures of the form $(H_\lambda, \in)$ (with possible restrictions on λ) and all possible elementary extensions of them. In fact it turns out we'll be able to extend it to all elementarily equivalent to $(H_\lambda, \in)$ superstructures of $(H_\lambda, \in)$.

So our dream goal becomes to prove that for a wide variety of cardinals λ, for any elementary equivalent to $(H_\lambda, \in)$ superstructure $\mathcal{M}$ of $(H_\lambda, \in)$, there is some H_δ which has a forcing extension $\mathcal{N}$ which absorbs $\mathcal{M}$ elementarily. Or to visualize it in the following diagram, every time we have the configuration in black, we have the configuration in grey as well.



The wiggly contours represent possible ill-foundedness.

Finally we add an additional technical detail. Since we are including ill-founded and non-transitive models in our considerations, we want to eliminate models which fail to interpret basic concepts properly. For that reason we consider the above-mentioned sets as domains of structures for a richer language – the language which has predicate symbols for all relations which are Δ_0-definable in terms of membership; constants for all Δ_0-definable sets; and function symbols for the basic

[3]A structure for the language of set-theory is called *standard* iff the interpretation of the membership predicate is the real membership relation $\in$.

set-theoretic operations known as the Gödel operations[4]. That way when we take a substructure of a standard structure, we know that it interprets basic concepts correctly, e.g. it agrees with the standard structures which set is the empty set, which sets are functions, etc.

In the first chapter we present the *algebraic* background for our investigation. We start from general basic preliminary facts about Boolean algebras, discuss the so-called iteration systems used for iterated forcing, and finally concentrate on a particular kind of ultrafilters, called *good ultrafilters*, which will make central use of later.

In the second chapter we present the *semantical* side of the background we need. We introduce the general model-theoretic notion of Boolean-valued structures for first order formal systems, then we discuss the crucial properties *fullness* and *mixing* of such structures. We extend the classical notion of *embedding* between structures to include Boolean-valued ones. We then focus on Boolean-valued models of Set Theory. We present the general construction of Boolean *ultrapowers* of any structure and its forcing formalization in the case of Set Theory. We discuss the model-theoretic applications of good ultrafilters for producing *saturated* structures.

In the third chapter we develop a method for transforming a certain kind of Boolean algebras in a way such that:

(1) we make them algebraically more *tame*

(2) we preserve their *forcing* properties

This is the main original contribution of the thesis whose purpose is to address the following issue.

We wish to obtain a saturated structure by taking a quotient by a good ultrafilter of a particular Boolean-valued structure $H_\delta \cap V^{\mathsf{B}}$, whose underlying Boolean algebra B is Woodin's stationary tower of inaccessible height δ. In order to argue for the saturation of the quotient, we need $H_\delta \cap V^{\mathsf{B}}$ to be full, and there's no reason to expect that. The devised method transforms B into a different Boolean algebra C, such that the corresponding structure $H_\delta \cap V^{\mathsf{C}}$ is, on the one hand, full, and, on the other hand, similar to $H_\delta \cap V^{\mathsf{B}}$.

The two features listed above as (1) and (2) are seen thanks to the fact that the devised method provides a way to view the resultant algebra C in two different ways – as a quotient $\mathsf{B}/_F \cong \mathsf{C}$ of the initial algebra B by a carefully constructed filter F, and as the direct limit $\varinjlim \mathcal{F} \cong \mathsf{C}$ of an iteration system $\mathcal{F} = \left\{ i_{\alpha\beta} : \mathsf{B}_\alpha \longrightarrow \mathsf{B}_\beta \,:\, \alpha \leq \beta < \delta \right\}$ of subalgebras B_α of restrictions $\mathsf{B} \restriction c_\alpha$ of the initial algebra B.

(1) The *algebraic tameness* properties of C are granted by Baumgartner's theorem[5]. We formalize the construction of $\mathcal{F}$ as a two-player game. We specify a strategy for player I and let player II win the game. The strategy of player I ensures that $\varinjlim \mathcal{F} \cong C$ is $<\delta$-disjointable, like B, and isomorphic to $\mathsf{B}/_F$. The strategy of player II ensures that we can apply Baumgartner's theorem to argue that $\mathcal{F}$ is $<\delta$-cc. The existence of such a winning counterstrategy for player II remains only a conjecture.

(2) The preservation of *forcing* properties of C is granted by the fact that the filter F is $< \delta$-complete and can be extended to a good ultrafilter $G \supseteq F$. It turns out that this suffices to show that the quotient $(H_\delta \cap V^{\mathsf{B}})/_G$ we wish to work with can equivalently be constructed in two steps, by taking two consecutive quotients – first forming the corresponding structure $H_\delta \cap V^{(\mathsf{B}/_F)}$ and then taking the quotient of that structure with the filter $G/_F$ on $\mathsf{B}/_F$. Obviously in general we don't have such freedom. That's in some sense the point of forcing – the quotient $V^{\mathsf{B}}/_G$, when G is

[4]Introduced for the first time in Gödel's original paper [7] on the consistency of the Continuum Hypothesis.
[5]The one stated as 1.2.7.

an ultrafilter, is nontrivially different from $V^{(\mathsf{B}/G)} \cong V^2 \cong V$.

In the fourth chapter we outline the application of our convergence theorem from chapter 3 to Woodin's stationary tower to argue that a large class of structures can be constructed by forcing. More precisely, elementary equivalent superstructures of a large class of structures of the form H_λ, can be elementarily embedded into forcing extensions of structures of the form H_δ.

Chapter 1

Boolean Algebras

1.1 Preliminaries

Here we briefly present basic facts about Boolean algebras which we will directly use. For a systematic and comprehensive treatment of the subject, we refer the reader to classic texts like the Handbook of Boolean algebras [17] and Sikorski's and Halmos' classic books on Boolean algebras [22] and [6].

Definition 1.1.1. Let $\mathsf{P} = \langle P, \leq \rangle$ be poset, $\mathsf{B} = \langle B, \neg, \vee, \wedge, 0_{\mathsf{B}}, 1_{\mathsf{B}}, \leq \rangle$ be a Boolean algebra, and κ be a cardinal.

(1) For the sake of clarity, let us make the convention that every time u and v are sets, the phrase "u contains v" will be used to express $u \ni v$, i.e. u contains v *as an element*. We shall express $u \supseteq v$ as "u *extends* v".

(2) We call the sets P and B the *domains* or *carriers* of P and B respectively. We shall sometimes conflate in our notation a structure, like a poset, with its domain, e.g. we shall feel free to write $a \in \mathsf{B}$ instead of $a \in B$. We shall call elements of P or B *conditions of* P or B, respectively. A condition of P which is not a least element (bottom element) of P is called *positive*. We denote the set of positive conditions of P by P^+.

(3) The least upper bound and the greatest lower bound of a subset H of a Boolean algebra, if they exists, are called respectively *the join* and *the meet of H*.

(4) We say that a condition q *refines* a condition p iff $q \leq p$. They are called *comparable* iff one of them refines the other. Two conditions are called *compatible* iff they have a common refinement; if they are conditions in a Boolean algebra, we can further say that they are incompatible iff their meet is zero. A set of incompatible conditions of P is called *an antichain on* P. A is called *a maximal antichain on* P iff no antichain on P properly extends A.

(5) A condition $p \in P$ is called an *atom of* P iff it doesn't have incompatible refinements. P is called *atomless* iff it has no atoms. B is called atomless iff B^+ is atomless.

(6) By $\triangle$ we shall denote the binary operation on B called *symmetric difference*, defined as

$$\langle a, b \rangle \longmapsto \big(a \wedge \neg b\big) \vee \big(b \wedge \neg a\big)$$

(7) For any subset H of P, by $\downarrow H$ we denote the set $\big\{ p \in P : (\exists q \in H)(q \leq p) \big\}$ called *the downward closure of H*, and by $\uparrow H$ we denote its *upward closure* $\big\{ p \in P : (\exists q \in H)(q \geq p) \big\}$. If p is a condition of P and is not a subsets of P, we feel free to write $\downarrow p$ and $\uparrow p$, instead of $\downarrow \{p\}$ and $\uparrow \{p\}$, respectively.

(8) Let H and H' be subsets of P. H is called:

- *dense in H'* iff below every condition in H' there is a condition in H; when we say just *dense* we mean dense in P;

- *predense* iff its *downward closure* is dense;
- *upward/downward closed* iff it coincides with its upward/downward closure;
- *downward directed* iff every finite subset of H has a common refinement in H;
- *upward directed* iff every finite subset of H has a common upper bound in H;
- a *prefilter* iff every finite subset has a positive common refinement (possibly not in H);
- a *filter* iff it's upward closed and downward directed.

(9) We say that an antichain A' *refines* an antichain A iff
- for each $a' \in A'$ there is (an unique) $a \in A$ such that $a' \leq a$
- $\downarrow A' \cap \downarrow A$ is dense in A and in A'

(10) Given $H \subseteq \mathsf{B}$, the set $\left\{ \neg a \,:\, a \in H \right\}$ is called *the dual of* H and is denoted by $\breve{H}$; H is:
- an *ultrafilter* iff it is a filter and for each condition a, either a or $\neg a$ belongs to it;
- a *(prime) ideal* iff its dual is an (ultra)filter.

(11) B is called:
- κ*-complete* iff every subset of B which has size at most κ has a join in B
- $<\kappa$*-complete* iff every subset of B which has size strictly less than κ has a join in B
- *complete* iff each of its subsets has a join in B; we abbreviate the term "complete Boolean algebra" as "cba".

(12) A prefilter G on B is called κ*-incomplete* iff there is a subset D of G of size at most κ such that $G \cup \{\bigwedge D\}$ is not a prefilter. It's called $<\kappa$-incomplete iff it fails to be λ-incomplete for some $\lambda < \kappa$. Or, equivalently, G is $(<)\kappa$-incomplete iff there is a subset of G of size $(<)\kappa$ whose meet is zero. An ideal is called $(<)\kappa$*-incomplete* iff its dual is a $(<)\kappa$-incomplete filter.

(13) Let a be a condition in B, J be an ideal on B and G be the dual filter of J. We shall denote the equivalence class $\left\{ b \in \mathsf{B} \,:\, a \triangle b \in J \right\}$ by either $[\,a\,]_J$ or $[\,a\,]_G$. When the ideal in question is clear from the context, we could omit the subscript altogether and write simply $[\,a\,]$. Also, we shall denote the quotient algebra $\left\{ [\,a\,]_J \,:\, a \in \mathsf{B} \right\}$ by either $\mathsf{B}/_J$ or $\mathsf{B}/_G$.

(14) Let a be a condition in B. By $\mathsf{B} \restriction a$ we shall denote the Boolean algebra with domain $\left\{ a \wedge b \,:\, b \in \mathsf{B} \right\}$ and operations inherited from B in the obvious way. It's easy to check that the maps $b \longmapsto [\,b\,]_{\uparrow a}$ is an isomorphism of $\mathsf{B} \restriction a$ onto $\mathsf{B}/_{\uparrow a}$. We call $\mathsf{B} \restriction a$ *the restriction of* B *to* a.

(15) A function $i : P \longrightarrow Q$ is called *a complete embedding of* P *into* Q iff i is order-preserving, incompatibility-preserving, and for every condition q in Q there is a condition in P all of whose refinements are mapped by i to compatible with q conditions. It's called *a dense embedding of* P *into* Q iff it is order-and-incompatibility-preserving and each condition $q \in Q$ is refined by some value $i(p)$ of i. To spell it out more formally, i is a complete embedding iff it satisfies (1), (2) and (3), and is a dense embedding iff it satisfies (1), (2) and (4), where:
  (1) $(\forall p, q \in P)(p \leq q \;\rightarrow\; i(p) \leq i(q))$
  (2) $(\forall p, q \in Q)(p \perp q \;\leftrightarrow\; i(p) \perp i(q))$
  (3) $(\forall q \in Q)(\exists p \in P)(\forall r \in \downarrow p)(i(r)$ is compatible with $q)$
  (3') $(\forall q \in Q)(\exists p \in P)(i(p) \leq q)$

(16) A partial order is called an *upper/lower semilattice* iff every two conditions have a join/meet. It is called a lattice iff it is an upper semilattice and it is a lower semilattice. Upper and lower semilattices are sometimes called *join semilattices* and *meet semilattices*, respectively.

(17) Every partial order P embeds densely into a unique (up to isomorphism) complete Boolean algebra, called *the Boolean completion of* P, which we denote by $\mathsf{RO}(\mathsf{P})$. This dense embedding $i_{\mathsf{P}} : \mathsf{P} \longrightarrow \mathsf{RO}(\mathsf{P})^+$ can be explicitly described as follows: the order topology on P considers as open sets the downward closed subsets of P, $\mathsf{RO}(\mathsf{P})$ is the complete Boolean algebra that consists of the regular open sets[1] of this topology, i_{P} is defined by $p \mapsto \mathrm{Reg}\,(\downarrow p)$ where $\mathrm{Reg}\,(X)$ is the interior of the closure of $X \subseteq \mathsf{P}$. $\mathsf{RO}(\mathsf{P})$ is the unique cba B (up to isomorphism) into which P embeds densely. For a detailed proof see the Handbook of Boolean algebras [17, Theorems 4.13 and 4.14]. When we feel no confusion can arise, we shall conflate a condition a in P with $i_{\mathsf{P}}(a)$. In other words, for the sake of technical simplicity, we may pretend the dense embedding i_{P} is the identity. Yet another way to frame this, by $\mathsf{RO}(\mathsf{P})$ we shall denote an isomorphic copy of the algebra defined above into which Id_P embeds P densely, instead of i_{P}.

(18) A function $i : \mathsf{B} \longrightarrow \mathsf{C}$ is called a *homomorphism* iff it preserves the Boolean operations, i.e. iff $i(\neg_{\mathsf{B}} a) = \neg_{\mathsf{C}} i(a)$ and $i(a \vee_{\mathsf{B}} b) = i(a) \vee_{\mathsf{C}} i(b)$ for each $a, b \in \mathsf{B}$.

Note that this ensures that $i(0_{\mathsf{B}}) = 0_{\mathsf{C}}$ and $i(1_{\mathsf{B}}) = 1_{\mathsf{C}}$.

The homomorphism e is further called *κ-complete* iff it preserves joins of size up to κ, i.e. iff $e(\bigvee_{\mathsf{B}} X) = \bigvee_{\mathsf{C}} e[X]$ for each subset X of B of size at most κ that has a join in B. A homomorphism is called *$<\kappa$-complete* iff it is λ-complete for every $\lambda < \kappa$; and it is called *complete* iff it is $|\mathsf{B}|$-complete. If there is a complete injective homomorphism of B into C, we say that C *absorbs* B.

(19) Let C again be a Boolean algebra as well and $e : \mathsf{B} \longrightarrow \mathsf{C}$ be a complete homomorphism. We define the kernel and co-kernel of e respectively as follows:

$$\begin{aligned} \ker(i) &= \bigvee \left\{ b \in \mathsf{B} \,:\, i(b) = 0_{\mathsf{C}} \right\} \\ \mathrm{coker}(i) &= \neg \ker(i) \end{aligned}$$

(20) A subalgebra C of B is called a *((<)λ-)complete subalgebra* of B iff for every subset X of C (of size (less than) λ), C agrees with B on the existence and value of a join of X.

(21) Let X be a subset of B and λ be a cardinal not greater than $|X|^+$. There exists a smallest $< \lambda$-complete subalgebra of B that extends[2] X. We denote it by $\mathsf{B}(X, \lambda)$ and call it *generated by $X < \lambda$-complete subalgebra of* B. For reference see for example the Handbook of Boolean algebra [17, Lemma and Definition 10.3].

(22) Let B be a Boolean algebra and $a \in \mathsf{B}$. Then by $\mathrm{Ult}(\mathsf{B})$ we shall denote the set

$$\left\{ G \subseteq \mathsf{B} \,:\, G \text{ is an ultrafilter on } \mathsf{B} \right\}$$

of ultrafilters on B. By N_a we shall denote the set $\left\{ G \in \mathrm{Ult}(\mathsf{B}) \,:\, a \in G \right\}$ of ultrafilters on B that contain a an element. By τ_{B} we shall denote the topology on $\mathrm{Ult}(\mathsf{B})$ generated by $\left\{ N_a \,:\, a \in \mathsf{B} \right\}$. By $St(\mathsf{B})$ we shall denote the resultant topological space $(\mathrm{Ult}(\mathsf{B}), \tau_{\mathsf{B}})$, called *the Stone space of* B. The following is true for the Stone space of any Boolean algebra B and for any $a, b \in \mathsf{B}$:

- $N_a \cap N_{\neg a} = \emptyset$
- $N_a \cup N_{\neg_a} = \mathrm{Ult}(\mathsf{B})$
- $N_{a \wedge b} = N_a \cap N_b$
- $N_{a \vee b} = N_a \cup N_b$
- $St(\mathsf{B})$ is Hausdorff, i.e. any two distinct points have disjoint neighbourhoods
- $St(\mathsf{B})$ is zero-dimensional, i.e. has a base of clopen sets (the base $\left\{ N_a \,:\, a \in \mathsf{B} \right\}$
- $St(\mathsf{B})$ is compact, i.e. each of its covers has a finite subcover

[1] A subset of a topological space is called *regular open* iff it coincides with the interior of its closure.
[2] 1.2.6(1)

Definition 1.1.2 (adjoint pairs; [26, Definition 1.2.1])**.** Let P, Q be partial orders and $i : P \longrightarrow Q$, $\pi : Q \longrightarrow P$ be order preserving maps between them. The pair $\langle\, i, \pi\,\rangle$ is called *an adjoint pair* if for all $p \in P$ and $q \in Q$

$$i(p) \geq q \quad \text{iff} \quad p \geq \pi(q).$$

If $\langle\, i, \pi\,\rangle$ is an adjoint pair, then we call π *the adjoint of* i.

Here we summarize the basic properties of adjoint pairs which we'll make use of:

Proposition 1.1.3. *Let $i : P \longrightarrow Q$ and $\pi : Q \longrightarrow P$ be maps between partial orders $(P, <_P)$, $(Q, <_Q)$ such that π is the adjoint of i. Then:*

(1) $i \circ \pi(q) \geq q$ *for all* $q \in Q$*;*

(2) $\pi(q) = \bigwedge \{\, p \,:\, i(p) \geq q \,\}$

(3) $i(p) = \bigvee \{\, q \,:\, \pi(q) \leq p \,\}$

(4) $i \circ \pi \circ i = i$*, and* $\pi \circ i \circ \pi = \pi$*;*

(5) *if i is injective, then $\pi \circ i$ is the identity map on P;*

Assume moreover i is a complete injective homomorphism from $P = \mathsf{B}$ to $Q = \mathsf{C}$ for B and C complete Boolean algebras. Then

(A) π *is a surjective map with* $\ker(\pi) = \{0_{\mathsf{C}}\}$*;*

(B) $\pi(c) \wedge b = \pi\left(c \wedge i(b)\right)$ *for all* $b \in \mathsf{B}$ *and* $c \in \mathsf{C}$*;*

Proof. (1) Since $\langle\, i, \pi\,\rangle$ is an adjoint pair we get that $i(p) \geq q$ iff $p \geq \pi(q)$, therefore for $p = \pi(q)$ we get $i(\pi(q)) \geq q$ iff $\pi(q) \leq \pi(q)$, which is clearly the case.

(2) Observe that $i \circ \pi(q) \geq q$ by (1), hence $\pi(q)$ is in the set $\{\, p \,:\, i(p) \geq q \,\}$ on the right-hand side. Moreover $p \geq \pi(q)$ iff $i(p) \geq q$ by the defining property of adjoint pairs. Hence $\pi(q)$ is the minimum of the set on the right-hand side.

(3) Observe that $\pi \circ i(p) \leq p$ since $i(p) \leq i(p)$ trivially holds and $\langle\, i, \pi\,\rangle$ is an adjoint pair, hence $i(p)$ is in the set on the right-hand side. Moreover $p \geq \pi(q)$ iff $i(p) \geq q$ since $\langle\, i, \pi\,\rangle$ is an adjoint pair. Therefore $i(p)$ is the maximum of the set on the right-hand side.

(4) $\pi(i(p)) \leq p$ iff $i(p) \leq i(p)$ since $\langle\, i, \pi\,\rangle$ is an adjoint pair. Therefore $i \circ \pi \circ i(p) \leq i(p)$ since i is order preserving. $i(\pi(i(p))) \geq i(p)$ iff $\pi(i(p)) \geq \pi(i(p))$ since $\langle\, i, \pi\,\rangle$ is an adjoint pair. Hence $i \circ \pi \circ i(p) = i(p)$. The other assertion is proved in exactly the same way.

(5) If i is injective $i \circ \pi \circ i(p) = i(p)$ iff $\pi \circ i(p) = p$ for all $p \in P$.

(A) The surjectivity of π is directly ensured by (5). If $\pi(c) = 0_{\mathsf{B}}$ for some $c \in \mathsf{C}$, then by (1) and the fact that i is a homomorphism we have $0_{\mathsf{C}} = i(0_{\mathsf{B}}) = i(\pi(c)) \geq c$.

(B) Notice that for any conditions $a, b \in \mathsf{B}$ and $c \in \mathsf{C}$ such that $\left(i(a) \geq c \ \text{ or } \ i(a) \geq i(b)\right)$, we have $i(a) \geq c \wedge i(b)$. Then we have

$$\{\, a \in \mathsf{B} \,:\, i(a) \geq c \wedge i(b) \,\} \supseteq \{\, a \in \mathsf{B} \,:\, i(a) \geq c \ \text{ or } \ i(a) \geq i(b) \,\},$$

and hence

$$\bigwedge \{\, a \in \mathsf{B} \,:\, i(a) \geq c \wedge i(b) \,\} \leq \bigwedge \{\, a \in \mathsf{B} \,:\, i(a) \geq c \ \text{ or } \ i(a) \geq i(b) \,\}.$$

$$
\begin{aligned}
\pi(c \wedge i(b)) &= \bigwedge \{\, a \in \mathsf{B} \,:\, i(a) \geq c \wedge i(b) \,\} && \text{(by(2))} \\
&\leq \bigwedge \{\, a \in \mathsf{B} \,:\, i(a) \geq c \;\text{ or }\; i(a) \geq i(b) \,\} && \text{(by the above remark)} \\
&= \bigwedge \Big(\{\, a \in \mathsf{B} \,:\, i(a) \geq c \} \cup \{\, a \in \mathsf{B} \,:\, i(a) \geq i(b) \,\} \Big) \\
&= \bigwedge \{\, a \in \mathsf{B} \,:\, i(a) \geq c \} \wedge \bigwedge \{\, a \in \mathsf{B} \,:\, i(a) \geq i(b) \,\} \\
&= \pi(c) \wedge \bigwedge \{\, a \in \mathsf{B} \,:\, a \geq b \} \\
&= \pi(c) \wedge b
\end{aligned}
$$

Now suppose towards a contradiction that the above inequality is strict, i.e.

$$
\begin{aligned}
\pi(c \wedge i(b)) \lneq \pi(c) \wedge b \quad &\text{i.e.} \quad \pi(c \wedge i(b)) \wedge \neg(\pi(c) \wedge b) \gneq 0 \\
&\text{i.e.} \quad \pi(c \wedge i(b)) \wedge (\neg\pi(c) \vee \neg b) \gneq 0 \\
&\text{i.e.} \quad \big(\pi(c \wedge i(b)) \wedge \neg\pi(c)\big) \vee \big(\pi(c \wedge i(b)) \wedge \neg b\big) \gneq 0
\end{aligned}
$$

Since π is order-preserving we have that $\pi(c \wedge i(b)) \leq \pi(c)$ and therefore the first disjunct $\pi(c \wedge i(b)) \wedge \neg\pi(c)$ is zero, thus it is the second disjunct that is positive. So we have

$$
\begin{aligned}
0 &\lneq \pi(c \wedge i(b)) \wedge \neg b \\
&= \bigwedge \{\, a \in \mathsf{B} \,:\, i(a) \geq c \wedge i(b) \,\} \wedge \neg b \\
&\leq \bigwedge \{\, a \in \mathsf{B} \,:\, i(a) \geq i(b) \,\} \wedge \neg b \\
&= \bigwedge \{\, a \in \mathsf{B} \,:\, a \geq b \} \wedge \neg b \\
&= b \wedge \neg b \\
&= 0
\end{aligned}
$$

□

Notation 1.1.4. Whenever i, possibly with indices, denotes a complete injective homomorphism between Boolean algebras, then by π with the same indices we'll denote the adjoint of i.

Definition 1.1.5 (Density of a partial order)**.** Let P be a partial order. The smallest cardinality of a dense in P set is called *the density of* P and is denoted by $\mathfrak{d}(\mathsf{P})$. If P has a least element, for example if it's a Boolean algebra, by $\mathfrak{d}(\mathsf{P})$ we actually mean $\mathfrak{d}(\mathsf{P}^+)$.

Notice that the density of P is the same as the density of $\mathsf{RO}(\mathsf{P})$, since the composition of dense embeddings is a dense embedding.

Definition 1.1.6 (The small completion property)**.** A $<\kappa$-complete Boolean algebra B of density κ has *the small completion property* iff for every strictly smaller that κ subset X of B, the smallest complete subalgebra of B containing X has size strictly below κ.

Definition 1.1.7 (Chain conditions)**.** Let κ be a cardinal. A partial order is said to satisfy [3] the κ-chain condition iff it has no antichain of size greater than κ. We use the abbreviation κ-cc. If all antichains are strictly smaller than κ, we say that it is $<\kappa$-cc. By $\mathrm{cc}(\mathsf{P})$ we shall denote the smallest cardinal λ, such that P has no antichain of size, i.e. the smallest λ such that P is not $<\lambda$-cc.

Proposition 1.1.8. *Let* P *be a poset,* B *be a Boolean algebra and* e *be a dense embedding of* P *into* B*. Then every positive condition of* B *is the join of the image* $e[A]$ *under* e *of an antichain* A *on* P*.*

Proof. Let b be a positive condition in B. Define

$$D = e[P] \cap \downarrow_{\mathsf{B}^+} b$$

[3]for convenience we sometimes use terms of the form κ-cc like adjectives, e.g. saying "B *is a* $<\kappa$-cc Boolean algebra"

By Zorn's lemma applied to the poset of antichains on D, ordered by inclusion, let A be a maximal antichain on D. Obviously b is an upper bound of A. Suppose it is not the least one, i.e. assume there is some condition a which is an upper bound of A but is not above b. Then $b \wedge \neg a$ is positive, and since the embedding e is dense, pick some $d \in D$ below $b \wedge \neg a$. Then $A \cup \{d\}$ is an antichain on D properly extending A, which contradicts the maximality of A. □

Corollary 1.1.9. *If a poset P is $<\kappa$-cc, then $\mathsf{RO}(P)$ has size at most $|P|^{<\kappa}$.*

Definition 1.1.10 (disjointability)**.** Let B be a Boolean algebra.

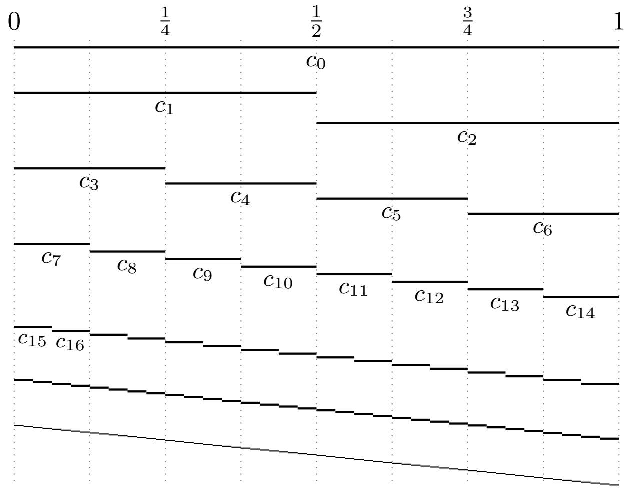
- A function $f : X \longrightarrow \mathsf{B}^+$ is said to be *disjoint* if it is injective and its range is an antichain.
- A function g is called a *refinement of f* iff they have the same domain and for each $u \in \mathrm{dom}(g) = \mathrm{dom}(f)$ we have $g(u) \leq_{\mathsf{B}} f(u)$.
- B is called *weakly κ-disjointable* iff below each positive condition there is an antichain of size κ
- B is called *κ-disjointable* iff every function $f : \kappa \longrightarrow \mathsf{B}^+$ has a disjoint refinement.
- B is called *(weakly) $<\kappa$-disjointable* iff it is (weakly) λ-disjointable for each $\lambda < \kappa$

Remark 1.1.11. Clearly $(<)\kappa$-disjointability implies weak $(<)\kappa$-disjointability. The converse is, however, not the case, witnessed for example by the cba $\mathsf{RO}(\mathbb{R})$ of regular open subsets of the real line, which is wealky countably disjointable but is not countably disjointable.

I.e. $\mathsf{RO}(\mathbb{R})$. Each regular open subset of $\mathbb{R}$ extends (contains as a subset) an open interval, and below each open interval (a, b) there is a countable antichain, say

$$\left\{ \left(b - \frac{b-a}{2^n} \,,\; b - \frac{b-a}{2^{n+1}} \right) \;:\; n < \omega \right\}.$$

However, consider a family $C = \langle\, c_n \;:\; n < \omega \,\rangle$ that contains arbitrarily fine partitions of, say, $(0, 1)$, for example the one obtained from iterated bisections, depicted in the following diagram:



Notice that every open subinterval (a, b) of $(0, 1)$ completely covers some element of C. So fixing any prospective refinement d_0 of c_0 would make it impossible to pick a disjoint from d_0 refinement d_n of some c_n. So C has no disjoint refinement, so $\mathsf{RO}(\mathbb{R})$ is not countably disjointable.

Proposition 1.1.12. *If B is a $<\kappa$-cc $<\kappa$-complete Boolean algebra, then B is a complete Boolean algebra.*

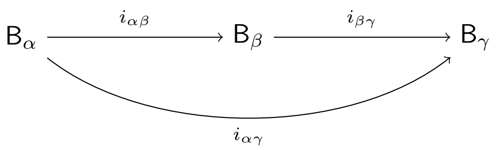
Proof. Let X be any subset of B. By Zorn's lemma, let A be a maximal antichain in the downward closure of X. Since B is $< \kappa$-cc, A is strictly smaller than κ, and since B is $< \kappa$-complete, let a be the join of A. Now clearly a is the join of X as well. If we assume some $x \in X$ is not below a, then we'd obtain that A is not maximal, witnessed by $A \cup \{x \wedge \neg a\}$. And any upper bound of X is an upper bound of A. □

1.2 Iteration Systems

Iteration systems are a tool to study chains[4] of Boolean algebras without a greatest element, and construct "limit" Boolean algebras, which absorb all algebras in the chain. The reference for this topic is Viale's [26].

Definition 1.2.1 (iteration system)**.** $\mathcal{F} = \left\{ i_{\alpha\beta} : \mathsf{B}_\alpha \longrightarrow \mathsf{B}_\beta \ : \ \alpha \leq \beta < \lambda \right\}$ is an *iteration system* of Boolean algebras iff for all $\alpha \leq \beta \leq \gamma < \lambda$:

1. B_α is a Boolean algebra[5] and $i_{\alpha\alpha}$ is the identity on it;
2. $i_{\alpha\beta}$ is a complete injective homomorphism with an associated adjoint $\pi_{\alpha\beta}$;
3. $i_{\beta\gamma} \circ i_{\alpha\beta} = i_{\alpha\gamma}$.



If $\gamma < \lambda$, we denote the restriction $\left\{ i_{\alpha\beta} \ : \ \alpha \leq \beta < \gamma \right\}$ of the iteration system $\mathcal{F}$ by $\mathcal{F} \restriction \gamma$.

Definition 1.2.2. Let $\mathcal{F} = \left\{ i_{\alpha\beta} : \mathsf{B}_\alpha \longrightarrow \mathsf{B}_\beta \ : \ \alpha \leq \beta < \lambda \right\}$ be an iteration system.

- An element f of $\prod_{\alpha<\kappa} \mathsf{B}_\alpha$ is called *a thread* iff each of its values $f(\alpha)$ can be computed from any of its succeeding values $f(\beta)$ for $\beta \geq \alpha$ via the projection $\pi_{\alpha\beta}$. I.e. iff

$$(\forall \alpha < \kappa)(\forall \beta \geq \alpha)(f(\alpha) = \pi_{\alpha\beta} \circ f(\beta))$$

  Notice that a thread is determined by any of its tails. Or by its restriction to any unbounded subset of its domain.

- A thread f is called *constant* iff for some $\alpha < \kappa$, the succeeding values $f(\beta)$ of f for $\beta > \alpha$ can be computed from $f(\alpha)$ via $i_{\alpha\beta}$. I.e. iff

$$(\exists \alpha < \kappa)(\forall \beta > \alpha)(f(\beta) = i_{\alpha\beta} \circ f(\alpha))$$

  The minimal such α is called *the support of* f and is denoted by $\mathrm{supp}(f)$. Notice that a constant thread f is determined by its restriction $f \restriction \left\{ \mathrm{supp}(f) \right\}$.

- The collection of threads, endowed with the order given by pointwise comparison, is called *the inverse limit of* $\mathcal{F}$ and is denoted by $\varprojlim \mathcal{F}$.

- The substructure of $\varprojlim \mathcal{F}$ consisting of the constant threads only, is called *the direct limit of* $\mathcal{F}$ and is denoted by $\varinjlim \mathcal{F}$.

Notation 1.2.3. Given an iteration system $\left\{ i_{\alpha\beta} : \mathsf{B}_\alpha \longrightarrow \mathsf{B}_\beta \ : \ \alpha \leq \beta < \kappa \right\}$, by t^a_α we shall denote the constant thread

$$\left\{ \left\langle \beta, \pi_{\beta\alpha}(a) \right\rangle \ : \ \beta < \alpha \right\} \cup \left\{ \left\langle \beta, i_{\alpha\beta}(a) \right\rangle \ : \ \alpha \leq \beta < \kappa \right\}$$

that has value a at α and has the greatest possible support not greater than α.

Note that the support of t^a_α is surely at most α but is not necessarily exactly α. With this piece of notation we can think of the direct limit as

$$\bigcup_{\alpha<\kappa} \left\{ t^a_\alpha \ : \ a \in \mathsf{B}_\alpha \right\}.$$

The direct limit of an iteration system inherits a natural Boolean structure from the operations on its constituent algebras. For any $\alpha \leq \beta < \kappa$, $a \in \mathsf{B}_\alpha$ and $b \in \mathsf{B}_\beta$:

[4] ordered by the relation "is (completely) embeddable in"

[5] Note that we allow the Boolean algebras to not be complete.

- $\neg t^a_\alpha = t^{\neg a}_\alpha$
- $t^a_\alpha \vee t^b_\beta = t^{i_{\alpha\beta}(a)\vee b}_\beta$

It is trivial but tedious to check that these are the operations induced by the pointwise ordering.

Definition 1.2.4. Let $\mathcal{F} = \left\{ i_{\alpha\beta} \,:\, \alpha \le \beta < \lambda \right\}$ be an iteration system. For all $\alpha < \lambda$, we define $i_{\alpha\lambda}$ and $\pi_{\alpha\lambda}$ as follows:

$$\begin{array}{rrcl} i_{\alpha\lambda} : & \mathsf{B}_\alpha & \longrightarrow & \varinjlim \mathcal{F} \\ & a & \longmapsto & t^a_\alpha \\ \pi_{\alpha\lambda} : & \varprojlim \mathcal{F} & \longrightarrow & \mathsf{B}_\alpha \\ & f & \longmapsto & f(\alpha) \end{array}$$

Fact 1.2.5. *Let $\varinjlim \mathcal{F} \subseteq Q \subseteq \varprojlim \mathcal{F}$. Then $i_{\alpha\lambda}$ and $\pi_{\alpha\lambda} \restriction Q$ form an adjoint pair between B_α and Q.*

Proof. Indeed for any $a \in \mathsf{B}_\alpha$ and $f \in Q$, we have $\pi_{\alpha\lambda}(f) \le a$ iff $f(\alpha) \le a$, by the definition of $\pi_{\alpha\lambda}$, and $f(\alpha) \le a$ iff $f \le t^a_\alpha = i_{\alpha\lambda}(a)$ by the definition of the order on $\varprojlim \mathcal{F}$. □

Lemma 1.2.6. *Let $\mathcal{F} = \left\{ i_{\alpha\beta} : \mathsf{B}_\alpha \longrightarrow \mathsf{B}_\beta \,:\, \alpha \le \beta < \lambda \right\}$ be an iteration system of inaccessible length λ. Let B_λ denote the direct limit of $\mathcal{F}$, and G_λ be an ultrafilter on B_λ. For each $\alpha < \lambda$, let G_α denote the set $\pi_{\alpha\lambda}[G] = \left\{ f(\alpha) \,:\, f \in G_\lambda \right\}$. Then the following hold for each $\alpha \le \beta < \lambda$, each $a \in \mathsf{B}_\alpha$, and each constant thread $f \in \mathsf{B}_\lambda$:*

(1) If $f(\alpha) = a$, then $f \le t^a_\alpha$.

(2) If $a \in G_\alpha$, then $t^a_\alpha \in G_\lambda$

(3) G_α is an ultrafilter on B_α.

Proof.

(1) Fix $f(\alpha) = a$ and $\alpha \le \beta < \lambda$. Then

$$f(\beta) \;\le\; i_{\alpha\beta} \circ \pi_{\alpha\beta}\big(f(\beta)\big) \;=\; i_{\alpha\beta}\big(f(\alpha)\big) \;=\; i_{\alpha\beta}(a) \;=\; i_{\alpha\beta}\big(t^a_\alpha(\alpha)\big) \;=\; t^a_\alpha(\beta).$$

The restrictions of f and t^a_α to $\alpha + 1$ obviously coincide.

(2) Fix $f \in G_\lambda$ such that $f(\alpha) = a$. By (1) and since G_λ is upward closed, $t^a_\alpha \in G_\lambda$.

(3) G_α is upward closed: Let $G_\alpha \ni a \le b \in \mathsf{B}_\alpha$. By (2), $t^a_\alpha \in G_\lambda$. Then $t^b_\alpha \ge_{\mathsf{B}_\lambda} t^a_\alpha$, and since G_λ is upward closed, $t^b_\alpha \in G_\lambda$. Then $b = t^b_\alpha(\alpha) \in G_\alpha$.

G_α is downward directed: Let $a, b \in G_\alpha$. Then by (2), $t^a_\alpha, t^b_\alpha \in G_\lambda$. G_λ is downward directed, thus $t^a_\alpha \wedge_{\mathsf{B}_\lambda} t^b_\alpha \in G_\lambda$. Then $G_\alpha \ni \left(t^a_\alpha \wedge_{\mathsf{B}_\lambda} t^b_\alpha\right)(\alpha) = t^a_\alpha(\alpha) \wedge t^b_\alpha(\alpha) = a \wedge b$.

G_α is ultra: If $\mathsf{B}_\alpha \ni a \notin G_\alpha$, then $t^a_\alpha \notin G_\lambda$. But G_λ is prime, thus $\neg_{\mathsf{B}_\lambda} t^a_\alpha = t^{\neg a}_\alpha \in G_\lambda$, thus $\neg a \in G_\alpha$.

□

Theorem 1.2.7 (Baumgartner; [2, Theorem 2.2]; [26, Theorem 7.2.2])**.** *Let λ be a regular cardinal and $\mathcal{F} = \left\{ i_{\alpha\beta} \,:\, \alpha \le \beta < \lambda \right\}$ be an iteration system such that B_α is $<\lambda$-cc for all α and $S = \left\{\alpha : \mathsf{B}_\alpha \cong \mathsf{RO}(\varinjlim(\mathcal{F} \restriction \alpha))\right\}$ is stationary. Then $\varinjlim(\mathcal{F})$ is $<\lambda$-cc.* □

1.3 Good ultrafilters

Definition 1.3.1 (good ultrafilter)**.** Let P be an upper semilattice and Q be a lower semilattice.

- A function $f : \mathsf{P} \longrightarrow \mathsf{Q}$ is *monotonically decreasing* iff for all $p \leq q \in \mathsf{P}$ we have $f(q) \leq f(p)$.
- A function $f : \mathsf{P} \longrightarrow \mathsf{Q}$ is *multiplicative* iff for all $p, q \in \mathsf{P}$ we have $f(p \vee q) = f(p) \wedge f(q)$.
- A filter G on P is $<\kappa$*-good* iff for every $\lambda < \kappa$ and every monotonically decreasing function $f : \mathcal{P}_\omega(\lambda) \longrightarrow G$ from the ideal of finite subsets of λ to G has a multiplicative refinement[6] with range again in G
- A filter on P is *good* iff it is $<\mathsf{sat}(\mathsf{P})$-good, where $\mathsf{sat}(\mathsf{P})$ is the smallest cardinal λ such that P has no antichain of size λ, called *the saturatedness of* P.

Lemma 1.3.2. [19, Theorem 3.2.1]. *Let κ be a regular cardinal such that $\kappa^{<\kappa} = \kappa$. Let B be a $<\kappa$-disjointable and $<\kappa$-complete Boolean algebra of density κ. Let F be a smaller than κ prefilter on B, and $f : \mathcal{P}_\omega(\lambda) \longrightarrow F$ be monotonically decreasing with $\lambda < \kappa$. Then F can be extended to a smaller than κ prefilter F' such that there is a multiplicative refinement of f with range in F'.*

Proof. WLoG assume that F is closed under finite meets.

Define the auxiliary function

$$\begin{array}{rcrcl} l & : & \mathcal{P}_\omega(\lambda) \times F & \longrightarrow & \mathsf{B}^+ \\ & & \langle S, d \rangle & \longmapsto & f(S) \wedge d \end{array}$$

and using the $<\kappa$-disjointability of B, take a disjoint refinement h of l.

We claim that all the desired properties are possessed by the function

$$\begin{array}{rcrcl} g & : & \mathcal{P}_\omega(\lambda) & \longrightarrow & \mathsf{B}^+ \\ & & S & \longmapsto & \bigvee \{ h(T, d) \,:\, T \supseteq S, d \in F \} \end{array}$$

First of all, g obviously refines f. Indeed,

$$\begin{aligned} g(S) &= \bigvee \{ h(T, d) \,:\, T \supseteq S, d \in F \} \\ &\leq \bigvee \{ l(T, d) \,:\, T \supseteq S, d \in F \} \\ &\leq \bigvee \{ f(T) \,:\, T \supseteq S \} = f(S). \end{aligned}$$

g is multiplicative. Fix any $S_1, S_2 \in \mathcal{P}_\omega(\lambda)$. We have

$$\begin{aligned} g(S_1) \wedge g(S_2) &= \bigvee \{ h(T, d) \,:\, T \supseteq S_1, d \in F \} \wedge \bigvee \{ h(T, d) \,:\, T \supseteq S_2, d \in F \} \\ &= \bigvee \{ h(T_1, d_1) \wedge h(T_2, d_2) \,:\, T_1 \supseteq S_1, T_2 \supseteq S_2, d_1 \in F, d_2 \in F \} \\ &= \bigvee \{ h(T, d) \,:\, T \supseteq S_1 \cup S_2, d \in F \} \end{aligned}$$

where the last equality follows from the fact that h is a disjoint function.

Since $\operatorname{ran}(h)$ is an antichain and h is injective, we have

$$h(T_1, d_1) \wedge h(T_2, d_2) > 0_B \ \text{ iff } \ T_1 = T_2 \text{ and } d_1 = d_2$$

therefore $h(T_1, d_1) \wedge h(T_2, d_2) > 0_B$ with $T_1 \supseteq S_1$ and $T_2 \supseteq S_2$ if and only if $d_1 = d_2$ and $T_1 = T_2 \supseteq S_1 \cup S_2$. We conclude that

[6] We say that a function g *refines* a function f iff they have the same domain X and at each argument $x \in X$ we have $g(x) \leq f(x)$ (where the order $\leq$ is clear from the context).

$$g(S_1) \wedge g(S_2) = \bigvee \left\{ h(T,d) \,:\, T \supseteq S_1 \cup S_2, d \in F \right\} = g(S_1 \cup S_2).$$

Finally we show that $F \cup \operatorname{ran}(g)$ is a prefilter: Fix $d_1, \ldots, d_n \in F$ and $S_1, \ldots, S_m \in \mathcal{P}_\omega(\lambda)$. Then since g is multiplicative and F is closed under finite conjunctions, we have

$$g(S_1) \wedge \cdots \wedge g(S_m) \wedge d_1 \wedge \cdots \wedge d_m = g(S) \wedge d \geq h(S,d) > 0_{\mathsf{B}},$$

where $S = S_1 \cup \cdots \cup S_m$ and $d = d_1 \wedge \cdots \wedge d_m$. Clearly the prefilter $F' = F \cup \operatorname{ran}(g)$ has the properties stated in the Lemma. □

Theorem 1.3.3. [19, Theorem 3.2.2]. *Assume κ is a regular cardinal such that $\kappa^{<\kappa} = \kappa$. Let B be a $<\kappa$-disjointable $<\kappa$-complete Boolean algebra of cardinality κ. Then every prefilter F on B of size less than κ can be extended to a κ-good ultrafilter.*

Proof. First of all, we fix an enumeration $\left\{ b_\alpha \,:\, \alpha < \kappa \right\}$ of B. Since $\kappa^{<\kappa} = \kappa$, we can also fix an enumeration $\left\{ f_\alpha \,:\, \alpha < \kappa \right\}$ of all the monotonically decreasing partial functions $\mathcal{P}_\omega(\lambda) \longrightarrow \mathsf{B}^+$, where λ is any ordinal less then κ. We will recursively construct an increasing chain $\left\{ F_\alpha \,:\, \alpha \leq \kappa \right\}$ of smaller than κ downward directed prefilters on B^+ extending F and also satisfying the following properties:

- For all $\alpha < \kappa$ there exists a multiplicative refinement g of f_α with range in some F_β;
- For all $\alpha < \kappa$ there is some $\beta < \kappa$ such that exactly one of b_α and $\neg b_\alpha$ belongs to F_β.

We proceed by recursion on α as follows:

- As base F_0 we take the closure of F under finite meets.
- If α is limit, we let $F_\alpha = \bigcup_{\beta<\alpha} F_\beta$.
- On even successor stages we make sure the filter we construct is ultra. If α is an even successor ordinal, we let ξ be the least ordinal such that neither b_ξ nor $\neg b_\xi$ belong to F_α and we let $F_{\alpha+1}$ be the closure under finite meets of F_α and exactly one of b_ξ or $\neg b_\xi$, so that $F_{\alpha+1}$ is a prefilter.
- On odd successor stages we ensure disjointability. If α is odd, we let ξ be the least ordinal such that f_ξ has no multiplicative refinement with range in F_α. Then by the previous lemma 1.3.2 let $F_{\alpha+1}$ be a downward directed prefilter extending F_α, with the property that some multiplicative g with range in $F_{\alpha+1}$ refines f_ξ.

F_κ is an ultrafilter by construction, since we included one of each pair b_ξ and $\neg b_\xi$ on the even successor stages of the construction. Also F_κ is κ-good. Indeed, assume ξ is the least ordinal such that $f_\xi : \mathcal{P}_\omega(\lambda) \longrightarrow F_\kappa$ is monotonically decreasing for some $\lambda < \kappa$, but no $g : \mathcal{P}_\omega(\lambda) \longrightarrow F_\kappa$ refines f_ξ . For each $\eta < \xi$, let $g_\eta : \mathcal{P}_\omega(\lambda) \longrightarrow F_\kappa$ be a multiplicative refinement of f_η . By the regularity of κ there exists a least limit β such that $\operatorname{ran}(g_\eta) \subseteq F_\beta$ for all $\eta < \xi$. Then at stage $\beta + 1$, f_ξ must be chosen to define $F_{\beta+1}$, which gives that some multiplicative $g : \mathcal{P}_\omega(\lambda) \longrightarrow F_{\beta+1} \subseteq F_\kappa$ refines f_ξ, a contradiction. □

Lemma 1.3.4. *Let B be a $<\kappa$-complete Boolean algebra and F be a $<\kappa$-complete filter on B. Let H be a good ultrafilter on $\mathsf{B}/_F$. Then $\cup H$ is a good ultrafilter on B.*

Proof. Fix some $\lambda < \kappa$ and some monotonically decreasing function $k : \mathcal{P}_\omega(\lambda) \longrightarrow \cup H$. Then the function

$$\begin{array}{rccl} K & : & \mathcal{P}_\omega(\lambda) & \longrightarrow & H \\ & & S & \longmapsto & \left[k(S)\right]_F \end{array}$$

is monotonically decreasing too. Indeed, for any $S, T \in \mathcal{P}_\omega(\lambda)$ we have

$$K(S) \leq K(T) \quad \text{iff} \quad k(S) \leq k(T) \quad \text{iff} \quad S \supseteq T.$$

Since H is good, let L be a multiplicative refinement of K. Let $l : \mathcal{P}_\omega(\lambda) \longrightarrow \cup H$ be such that $l(S) \in L(S)$, for each $S \in \mathcal{P}_\omega(\lambda)$. Then for each $S, T \in \mathcal{P}_\omega(\lambda)$ we have

$$
\begin{array}{llll}
L(S\cup T)=L(S)\wedge L(T) & \text{i.e.} & \big[\, l(S\cup T)\big]_F=\big[\, l(S)\wedge l(T)\big]_F \\
 & \text{i.e.} & l(S\cup T)\bigtriangleup \big(l(S)\wedge l(T)\big) \ \in \ \breve{F} \\
 & \text{i.e.} & \exists a\in F \ : \ l(S\cup T)\bigtriangleup (l(S)\wedge l(T)) \ \leq \ \neg a \\
 & \text{i.e.} & \exists a\in F \ : \ l(S\cup T)\wedge a = l(S)\wedge l(T)\wedge a
\end{array}
$$

So for each pair $S, T \in \mathcal{P}_\omega(\lambda)$ pick some $a_{ST} \in F$ such that $l(S \cup T) \wedge a_{ST} = l(S) \wedge l(T) \wedge a_{ST}$. Since F is $<\kappa$-complete, then $a = \bigwedge \big\{ a_{ST} : S, T \in \mathcal{P}_\omega(\lambda) \big\} \in F$. Then clearly we have $l(S \cup T) \wedge a = l(S) \wedge l(T) \wedge a$ for each $S, T \in \mathcal{P}_\omega(\lambda)$. Define

$$
\begin{array}{rccc}
l' & : & \mathcal{P}_\omega(\lambda) & \longrightarrow & \cup H \\
 & & S & \longmapsto & l(S)\wedge a
\end{array}
$$

Then clearly l' is a multiplicative refinement of k. □

Lemma 1.3.5. *Let* B *be a countably complete Boolean algebra and* F *be a countably complete filter on* B*. Let* H *be a countably incomplete filter on* $\mathsf{B}/_F$*. Then* $\cup H$ *is a countably incomplete filter on* B*.*

Proof. Let $D = \big\{ [\, d_n]_F : n < \omega \big\} \subseteq H$ have meet zero. Denote $\big\{ d_n : n < \omega \big\}$ by D' By the countable completeness of F we have $\big[\bigwedge D' \big]_F = \bigwedge D = [\, 0_\mathsf{B}]_F$, i.e. $\breve{F} \ni 0_\mathsf{B} \bigtriangleup \bigwedge D' = \bigwedge D'$. □

Lemma 1.3.6. *Let* B *be a countably disjointable Boolean algebra and* F *be a prefilter on* B *which is not ultra. Then* F *can be extended to a countably incomplete prefilter of size at most* $|F| + \aleph_0$*.*

Proof. Let A be a countable maximal antichain disjoint from $\uparrow F$. Such indeed exists because otherwise $\uparrow F$, being upward closed, would meet every two-element maximal antichain $\{a, \neg a\}$, i.e. would be ultra. Then $F \cup \breve{A}$ is clearly a countably incomplete prefilter. □

Chapter 2

Boolean-valued semantics

2.1 Preliminaries

Model-theoretic preliminaries

Notation 2.1.1. Let $\mathcal{L}$ be a first order language, $\mathcal{M}$ be a structure for $\mathcal{L}$ and X be a subset of the universe M of $\mathcal{M}$. By $\mathcal{L}_X$ we denote the first order language obtained from $\mathcal{L}$ by adding a constant symbol c_σ for each individual σ in X. By $\mathcal{M}_X$ we denote the structure for $\mathcal{L}_X$ obtained from $\mathcal{L}$ by further setting the interpretation of each new constant symbol c_σ to be σ itself. In situations we feel no confusion can arise, we shall conflate the notations and write simply σ instead of c_σ and $\mathcal{M}$ instead of $\mathcal{M}_M$.

Definition 2.1.2 (Elementary maps and embeddings)**.** Let $\mathcal{M}$ and $\mathcal{N}$ be structures for a first order language $\mathcal{L}$, n be a natural number, and f be a (partial) function from M to N. We say that j is a *(partial) n-elementary* map iff for any Σ_n-formula $\varphi(\vec{x})$ in $\mathcal{L}$ and every $\vec{m} \in \mathrm{dom}(j)$, we have

$$\mathcal{M} \vDash \varphi(\vec{m}) \qquad \text{iff} \qquad \mathcal{N} \vDash \varphi(j(\vec{m}))$$ [1]

We say that j is *elementary* iff it is n-elementary for each natural number n. If there is a(n n-)elementary map from $\mathcal{M}$ to $\mathcal{N}$, then we say that $\mathcal{M}$ *embeds (n-)elementarily into* $\mathcal{N}$, or, equivalently, that $\mathcal{N}$ *(n-)elementarily absorbs* $\mathcal{M}$, which we denote by $\mathcal{M} \preceq_n \mathcal{N}$. If the empty set is an elementary map from $\mathcal{M}$ to $\mathcal{N}$, we say that $\mathcal{M}$ is elementarily equivalent to $\mathcal{N}$, which we denote by $\mathcal{M} \equiv \mathcal{N}$.

Theorem 2.1.3 (Downward Löwenheim-Skolem; [16, Theorem 2.3.7]; [4, Corollary 2.4.1])**.** *Let $\mathcal{N}$ be an infinite structure for a first order language $\mathcal{L}$. Any subset A of N can be extended to an elementary substructure of $\mathcal{N}$ with cardinality at most $|L| + |A| + \aleph_0$.* □

Definition 2.1.4 (type, [15, Definition 5.3.1])**.** Let M be a structure for a first order language $\mathcal{L}$, X be a subset of M and n be a positive integer.

- A *(consistent) n-type over X in $Th(\mathcal{M}_X)$* is a set Γ of $\mathcal{L}_X$ formulas $\varphi(x_1, \ldots, x_n)$ such that every finite conjunction of formulas of Γ is satisfied in $\mathcal{M}$ - more precisely in $\mathcal{M}_X$ - by some tuple $\vec{u} = (u_1, \ldots, u_n) \in M^n$.
- A complete n-type over X in $Th(\mathcal{M}_X)$ is a consistent n-type maximal with respect to inclusion.

Definition 2.1.5 (Saturated and universal structures)**.** Let κ be an infinite cardinal. A first order structure is called

- κ-saturated iff it realizes every type with at most κ parameters;
- κ-universal iff it absorbs elementarily every elementarily equivalent to it structure of cardinality smaller than κ;

[1] More precisely: $\mathcal{M}_X \vDash \varphi(\vec{m})$ iff $\mathcal{N}_{j[X]} \vDash \varphi(j(\vec{m}))$, using notation 2.1.1.

- $<\kappa$ saturated iff it is λ-saturated for each λ below κ;
- $<\kappa$-universal iff it is λ-universal for each λ below κ. [2]

Theorem 2.1.6. [4, Theorem 5.1.12]. *$<\kappa$-saturated structures are κ-universal.* □

Set-theoretic preliminaries

Definition 2.1.7 (Hierarchy)**.** A hierarchy is a class $\langle Z_\alpha : \alpha \in \mathrm{Ord} \rangle$ of sets ordered by the ordinals such that $\alpha < \beta \to Z_\alpha \subseteq Z_\beta$ and $Z_\alpha = \bigcup \{ Z_\beta : \beta < \alpha \}$ for all limit α.

Theorem 2.1.8 (Reflection scheme;[12, Chapter IV, Theorem 7.5])**.** *Let $\langle Z_\alpha : \alpha \in \mathrm{Ord} \rangle$ be a hierarchy, $Z = \bigcup \{ Z_\alpha : \alpha \in \mathrm{Ord} \}$, and $\varphi(x_1, \dots, x_n)$ be a formula in the language $\{\in\}$. Then above each ordinal α there is some ordinal β such that for each $a_1, \dots, a_n \in Z_\alpha$ we have $Z_\beta \vDash \varphi(a_1, \dots, a_n)$ iff $Z \vDash \varphi(a_1, \dots, a_n)$.* □

Remark 2.1.9. The theorem can be immediately generalized to arbitrary finite sets of formulas, not just a single formula φ, by considering the conjunction of those formulas. The generalization to formulas in $\in_{\Delta_0}$ is also straightforward.

Definition 2.1.10 (Extensional and well-founded relations)**.** A binary relation E on a set M is *extensional* if distinct members of M have distinct immediate E-predecessors and *well-founded* if every non-empty subset of M has an E-minimal element.

Theorem 2.1.11 (Mostowski collapse, transitive collapse; [10], Theorem 6.15)**.** *If E is an extensional well-founded relation on a set M, then there is a unique transitive set N and an isomorphism of $\langle M, E \rangle$ onto $\langle N, \in \rangle$. The structure $\langle N, \in \rangle$ is called* the transitive collapse of $\langle M, E \rangle$.

We adopt the list of Gödel operations as in Jech's classic book [10, Definition 13.6], though we'll never need their explicit form. We shall make use of the following fact.

Proposition 2.1.12. [10, Lemma 13.7]. *For each Gödel operation G there is a bounded formula $\varphi_G(\vec{x}, y)$ which expresses that y is the result of the Gödel operation G on the input $\vec{x}$. We shall call this formula* the defining formula of G.

We shall consider transitive models of Set Theory. We deal with the usual caveat about them in a classical way by assuming the existence of large cardinals.

The right language for Set Theory

We formalize the complexity of formulas in any first order language by forming the hierarchy Σ_n, Π_n: the quantifier-free formulas are considered to be both Σ_0 and Π_0, and a formula is Σ_{n+1}/Π_{n+1} iff it can be obtained from a Π_n/Σ_n formula by prefixing it with some finite amount of existential/universal quantifiers.

We often then mod out by logical equivalence under some theory T, by saying that a property is Σ_n modulo T (or is Σ_n^T) iff the formula expressing it is equivalent under T to some Σ_n formula; analogically for Π_n^T. A property is Δ_n^T if it is both Σ_n^T and Π_n^T.

Although, to emphasize, this is a general scheme applicable to any first order language and theory, what one finds to be used in Set Theory appears to be a modified version of it, called the Lévy hierarchy.

In Set Theory we treat what's called *bounded quantifiers* in a special way, different from what the above definition of Σ_n implies. If a formula φ is of the form $\exists x(x \in y \ \& \ \psi)$, we abbreviate it as $(\exists x \in y)\psi$ and we say that the outermost quantifier is *bounded* by y. Bounded quantifiers make formal sense for any language with a (binary) predicate symbol, but in Set Theory in particular

[2] Be aware that what we've just defined as $<\kappa$-saturation and $<\kappa$-universal are in some books and papers called κ-saturation and κ-universality.

they get a special treatment because their contribution to the complexity of a formula, in terms of, say, computing it's truth, is decisively smaller than that of an unbounded quantifier.

To compute the truth value of an existential formula $\exists x\varphi(x)$ one needs to take into account the truth values of all instances $\varphi(a)$ of the quantified formula $\varphi(x)$. And those instances are in general class-many. In contrast, to account for a bounded quantifier when computing the truth value of a formula $(\exists x \in b)\varphi(x, b, \vec{c})$ we only need to compute the truth values of set-many instances $\varphi(a, b, \vec{c})$ of the quantified formula - one for each $a \in b$.

That's why in Set Theory we regard as simple – and place at the base level $\Sigma_0 = \Pi_0$ of the hierarchy of complexity – not only the quantifier-free formulas but all formulas which have only bounded quantifiers. The collection of those formulas is customarily referred to as Δ_0, rather than Σ_0 or Π_0. But strictly speaking they are not Σ_0 in the language for set theory whose only nonlogical symbol is the binary membership predicate. In order to be extremely clear we shall call these *bounded formulas*.

The formal way to have these formulas indeed sit at the base level $\Sigma_0 = \Pi_0 = \Delta_0$ without Set Theory being an exception of the general definition of the Σ_n complexity hierarchy is to enrich the language of Set Theory by adding new nonlogical symbols, e.g. a predicate p_φ for for each bounded formula φ together with the defining axiom $\varphi(\vec{x}) \leftrightarrow p_\varphi(\vec{x})$. This makes φ equivalent to an atomic formula.

Notation 2.1.13 (Languages for set theory and their interpretations)**.**

- **Core.** We shall refer to the first order language whose only nonlogical symbol is the binary membership predicate symbol $\in$ as the *core* language of set theory. This language is often (sloppily) denoted as $\{\in\}$.
- **Basic.** By $\in_{\Delta_0}$ we shall denote the first order language obtained from the core language $\{\in\}$ by adding an n-ary predicate symbol p_φ for each bounded formula $\varphi(x_1, \ldots, x_n)$ in the core language $\{\in\}$ with free variables exactly $x_1, ..., x_n$.
- **Extended.** By $\in_{\Delta_0^+}$ we shall denote the language obtained from $\in_{\Delta_0}$ by adding:

  ◦ a constant symbol c_φ for each set definable in $\mathsf{ZFC} - \mathsf{P}$ by a bounded formula, i.e. for each bounded formula $\varphi(x)$ in the core language $\{\in\}$ such that $\mathsf{ZFC} - \mathsf{P} \vdash \exists! x\varphi(x)$,

  ◦ a function symbol f_G (unary or binary) for each Gödel operation G.

  We shall call $\in_{\Delta_0}$ and $\in_{\Delta_0^+}$ respectively the *basic* language of set theory and the *extended* language of set theory.
- If $\mathcal{M} = (M, E)$ is a model of $\mathsf{ZFC} - \mathsf{P}$ in the core language, by $E^{\mathcal{M}}_{\Delta_0}$ and $E^{\mathcal{M}}_{\Delta_0^+}$ we shall denote the intended interpretations respectively of the basic language $\in_{\Delta_0}$ and the extended language $\in_{\Delta_0^+}$ in $\mathcal{M}$, namely:

  ◦ for each bounded formula $\varphi(x)$ in $\{\in\}$ such that $\mathsf{ZFC} - \mathsf{P} \vdash \exists! x\varphi(x)$, we set the interpretation of the constant c_φ to be the unique set $u \in M$ such that $\mathcal{M} \vDash \varphi(u)$

  ◦ for each bounded formula $\varphi(x_1, \ldots, x_n)$ in $\{\in\}$ with free variables exactly $x_1, ..., x_n$, we set the interpretation of the predicate symbol p_φ to be such that for all $u_1, ..., u_n \in M$

$$\mathcal{M} \vDash p_\varphi(u_1, \ldots, u_n) \qquad \text{iff} \qquad \mathcal{M} \vDash \varphi(u_1, \ldots, u_n)$$

  ◦ for each Gödel operation G, we set the interpretation of the function symbol f_G to be such that for each $\vec{x} \in M$ the value $f^{\mathcal{M}}_G(\vec{x})$ is the unique $y \in M$ such that $\mathcal{M} \vDash \varphi_G(\vec{x}, y)$, where φ_G is, as in proposition 2.1.12, a bounded formula in the core language which expresses that y is the value of the Gödel operation on input $\vec{x}$.

  We shall denote $(V, \in)$, $(V, \in_{\Delta_0})$ and $(V, \in_{\Delta_0^+})$ respectively by $V_\in$, V_{Δ_0} and V_+.

- Let ST be a first order set theory in the core language extending $\mathsf{ZFC}-\mathsf{P}$.
  - By ST_{Δ_0} we shall denote the set theory in $\in_{\Delta_0}$ obtained from ST by adding the defining axiom $p_\varphi(\vec{x}) \leftrightarrow \varphi(\vec{x})$ of each new predicate symbol p_φ, i.e. for each bounded formula $\varphi(\vec{x})$ in the core language.
  - By ST_+ we shall denote the first order set theory obtained from ST_{Δ_0} by adding
    - · the defining axiom $\varphi(c_\varphi)$ for each new constant symbol c_φ, i.e. for each bounded formula φ in the core language such that $\mathsf{ZFC}-\mathsf{P} \vdash \exists! x\varphi(x)$.
    - · the defining axiom $\varphi_G(\vec{x}, f(\vec{x}))$ of the function symbol for each Gödel operation G, where φ_G is as in 2.1.12.

Remark 2.1.14. Notice that there is a well-founded relation, a well-order in fact, of the formulas in $\in_{\Delta_0^+}$, such that any formula in which there is an occurrence of some of the additional symbols p_φ, c_φ or f_φ, is preceded by φ [3] We shall use this well-order when arguing inductively about all formulas in the basic language $\in_{\Delta_0^+}$.

We will present our inductive proofs on this well-order in the following way. Let's say that we want to prove that each term t and each formula φ in the extended language, possibly with parameters, have a property P (which, for simplicity's sake, let's say is read as "is nice") holds, we prove that:

- atomic formulas in the core language are nice
- negations and disjunctions of nice formulas are nice
- bounded quantifications $(\exists x \in \tau)\psi$ of nice formulas are nice
- the constants c_φ are nice
- a term obtained as a function symbol applied to nice terms is nice
- an atomic formula $p_\varphi(\vec{t})$, where $\vec{t}$ is a list of nice terms, is nice
- a quantification $\exists x\psi(x, \vec{t})$ of formula whose instances $\psi(\sigma, \vec{t})$ are nice, is nice

Notation 2.1.15 (translation [21, 4.6])**.** For each formula φ in the extended language $\in_{\Delta_0^+}$ there is a formula φ^* in the core language, called *the translation of* φ *into* $\{\in\}$, such that

$$(\mathsf{ZFC}-\mathsf{P})_+ \vdash \varphi \leftrightarrow \varphi^*$$

$$(\mathsf{ZFC}-\mathsf{P})_+ \vdash \varphi \quad \text{iff} \quad \mathsf{ZFC}-\mathsf{P} \vdash \varphi^*.$$

Theorem 2.1.16 (Levy's Absoluteness; [25, Lemma 3.1];[24, Theorem 5.1])**.** *Let* κ *be an uncountable cardinal. Then* $(H_\kappa, \in_{\Delta_0^+})$ *embeds* Σ_1*-elementarily into* $(V, \in_{\Delta_0^+})$. □

Proof. We prove it for the basic language $\in_{\Delta_0}$. Given that, the generalization to the extended language is trivial since the interpretations of the additional symbols is determined by the truth values of bounded formulas.

Given any Σ_1 formula $\varphi = \exists x\ \psi(x, p_1, \ldots, p_n)$ in $\in_{\Delta_0}$ with parameters $p_1, \ldots, p_n$ in H_κ, if $V \models \neg\varphi$ also $H_\kappa \models \neg\varphi$ since $H_\kappa \subseteq V$ and ψ is bounded and hence absolute for transitive models.

Suppose now that $V \models \varphi$, so there exists a q such that $V \models \psi(q, p_1, \ldots, p_n)$. Let λ be large enough so that $q \in H_\lambda$. By the downward Löwenheim-Skolem Theorem $\{q\} \cup \bigcup_{i<n} \mathrm{trcl}(\{p_i\})$ can be extended to an elementary substructure $\mathcal{M}$ of H_λ of size strictly below κ. Let $\mathcal{N}$ be the Mostowski collapse of $\mathcal{M}$. Let π denote the isomorphism of $\mathcal{M}$ onto $\mathcal{N}$. Notice that $\pi(p_i) = p_i$ for all $i < n$. Since $H_\lambda \models \psi(q, p_1, \ldots, p_n)$, the same holds for $\mathcal{M}$. Then $\mathcal{N} \models \psi(\pi(q), p_1, \ldots, p_n)$. Since N is transitive of cardinality less than κ, $N \subseteq H_\kappa$, so $\pi(q) \in H_\kappa$, thus $H_\kappa \models \varphi$. □

[3] Recall 2.1.12.

Corollary 2.1.17. *Every Σ_1 definable set is hereditarily countable.*

Proof. Let $\psi(x)$ be a Σ_1 formula and assume $\exists! x\psi(x)$. Then $\exists x\psi(x)$ and by the above theorem we have $H_{\aleph_1} \models \exists x\psi(x)$. Fix some $u \in H_{\aleph_1}$ such that $H_{\aleph_1} \models \psi(a)$. By upward absoluteness of Σ_1 formulas for transitive classes we have $\psi(a)$ (in V). But $\exists! x\psi(a)$. $\square$

2.2 Boolean-valued structures

Boolean-valued structures are a generalization of the standard Tarski structures for first order languages, in which statements' truth take value not necessarily in the two-element Boolean algebra, but in any Boolean algebra. Reference for the material in this section are Bell's [3], Jech's [10] and Viale's [26].

Definition 2.2.1 (Boolean-valued structure)**.** Let $\mathcal{L}$ be a first order language and B be a Boolean algebra. A B-*valued structure* $\mathcal{M}$ for $\mathcal{L}$ consists of:

1. A non-empty set M [4], called the *domain*, *carrier* or *universe* of $\mathcal{M}$. The elements of M are called *names*.
2. An interpretation of the equality symbol, i.e. a function $=^{\mathcal{M}}$
$$\begin{aligned} M^2 &\longrightarrow \mathsf{B} \\ \langle \tau, \sigma \rangle &\longmapsto [\![\tau = \sigma]\!]^{\mathcal{M}} \end{aligned}$$
3. Interpretations of the nonlogical symbols in $\mathcal{L}$. That is:
   - for each n-ary predicate symbol p of $\mathcal{L}$, a function $p^{\mathcal{M}}$
   $$\begin{aligned} M^n &\longrightarrow \mathsf{B} \\ \langle \tau_1, \dots, \tau_n \rangle &\longmapsto [\![p(\tau_1, \dots, \tau_n)]\!]^{\mathcal{M}} \end{aligned}$$
   - for each n-ary function symbol f of $\mathcal{L}$, a function $f^{\mathcal{M}}$
   $$f^{\mathcal{M}} : M^n \longrightarrow M$$
   - for each constant symbol c of $\mathcal{L}$, a name $c^{\mathcal{M}} \in M$.

We require that the following conditions are satisfied:

1. For all $\tau, \sigma, \rho \in M$,
$$[\![\tau = \tau]\!]^{\mathcal{M}} = 1_{\mathsf{B}}, \tag{2.1}$$
$$[\![\tau = \sigma]\!]^{\mathcal{M}} = [\![\sigma = \tau]\!]^{\mathcal{M}}, \tag{2.2}$$
$$[\![\tau = \sigma]\!]^{\mathcal{M}} \wedge [\![\sigma = \rho]\!]^{\mathcal{M}} \leq [\![\tau = \rho]\!]^{\mathcal{M}}. \tag{2.3}$$
2. If p is an n-ary predicate symbol in $\mathcal{L}$, for all $\langle \tau_1, \dots, \tau_n \rangle, \langle \sigma_1, \dots, \sigma_n \rangle \in M^n$,
$$\left(\bigwedge_{i=1}^{n} [\![\tau_i = \sigma_i]\!]^{\mathcal{M}}_{\mathsf{B}} \right) \wedge [\![p(\tau_1, \dots, \tau_n)]\!]^{\mathcal{M}}_{\mathsf{B}} \leq [\![p(\sigma_1, \dots, \sigma_n)]\!]^{\mathcal{M}}. \tag{2.4}$$
3. If f is an n-ary function symbol, for all $\mu, \nu \in M$ and $\langle \tau_1, \dots, \tau_n \rangle, \langle \sigma_1, \dots, \sigma_n \rangle \in M^n$,
$$[\![\mu = \nu]\!]^{\mathcal{M}} \wedge \left(\bigwedge_{i=1}^{n} [\![\tau_i = \sigma_i]\!]^{\mathcal{M}}_{\mathsf{B}} \right) \wedge [\![f(\tau_1, \dots, \tau_n) = \mu]\!]^{\mathcal{M}} \leq [\![f(\sigma_1, \dots, \sigma_n) = \nu]\!]^{\mathcal{M}}. \tag{2.5}$$

[4]We adopt the standard practice to denote the domain of a structure denoted by a calligraphic capital Latin letter by the same letter but not calligraphic.

Definition 2.2.2. A B-valued structure $\mathcal{M}$ is called *extensional* iff $[\![\, \sigma = \tau \,]\!]^{\mathcal{M}} = 1_{\mathsf{B}}$ entails that $\sigma = \tau$, for all $\sigma, \tau \in M$

The interpretation of more complicated terms is defined recursively in the expected way. If $u_1, \ldots, u_n$ are terms in $\mathcal{L}_M$ and their interpretations $u_1^{\mathcal{M}}, \ldots, u_n^{\mathcal{M}}$ have already been defined, then we set the interpretation $\big(f(u_1, \ldots, u_n)\big)^{\mathcal{M}}$ of the term $f(u_1, \ldots, u_n)$ to be $f^{\mathcal{M}}(u_1^{\mathcal{M}}, \ldots, u_n^{\mathcal{M}})$.

Definition 2.2.3 (Boolean truth value)**.** Let $\mathcal{L}$ be a first order language, B be a Boolean algebra and $\mathcal{M}$ be a B-valued structure for $\mathcal{L}$. We define the $\mathcal{M}$-truth value $[\![\, \varphi \,]\!]^{\mathcal{M}} \in \mathsf{RO}(\mathsf{B})$ of each closed formula in $\mathcal{L}_{\mathcal{M}}$ recursively as follows:

- $[\![\, p(u_1, \ldots, u_n) \,]\!]^{\mathcal{M}} = p^{\mathcal{M}}\left(u_1^{\mathcal{M}}, \ldots, u_n^{\mathcal{M}}\right)$
- $[\![\, \neg\varphi \,]\!]^{\mathcal{M}} = \neg\, [\![\, \varphi \,]\!]^{\mathcal{M}}$
- $[\![\, \varphi \vee \psi \,]\!]^{\mathcal{M}} = [\![\, \varphi \,]\!]^{\mathcal{M}} \vee [\![\, \psi \,]\!]^{\mathcal{M}}$
- $[\![\, \exists x \varphi \,]\!]^{\mathcal{M}} = \bigvee_{\mathsf{RO}(\mathsf{B})} \left\{ [\![\, \varphi(c_\sigma) \,]\!]^{\mathcal{M}} \,:\, \sigma \in M \right\}$

We say that a formula φ holds in $\mathcal{M}$ with *probability* or *certainty* $[\![\, \varphi \,]\!]$.

Remark 2.2.4. Note that the set defining the truth value of an existential formula may not have a least upper bound in B. For that reason we generally consider the truth values of formulas in the language of a B-valued model to be conditions in the completion $\mathsf{RO}(\mathsf{B})$ of B, and not just in B. Recall our convention from 1.1.1 that we take $\mathsf{RO}(\mathsf{B})$ to be a complete Boolean algebra into which *the identity* embeds B densely.

When we feel no confusion can arise we shall omit the superscript $\mathcal{M}$ in the notation of Boolean truth values.

Definition 2.2.5. Let $\mathcal{L}$ be a first order language, B be a Boolean algebra and $\mathcal{M}$ be a B-valued structure for $\mathcal{L}$. A sentence φ in $\mathcal{L}$ is *valid in* $\mathcal{M}$ iff $[\![\, \varphi \,]\!] = 1_{\mathsf{B}}$. $\mathcal{M}$ is *a model of* a first order theory T, iff every axiom of T is valid in $\mathcal{M}$. A formula $\varphi(\vec{x})$ is said to be valid in $\mathcal{M}$ iff its universal closure $\forall \vec{x} \varphi(\vec{x})$ is valid in $\mathcal{M}$. We shall say that a formula φ in $\mathcal{L}$ is valid in the theory T iff φ is valid in each Boolean-valued model of T, i.e. iff for each Boolean algebra B, φ is valid in each B-valued model of T.

We've just redefined the classical notion of a formula being valid in a theory, which refers to two-valued models only. This redefinition is justified by the soundness and completeness theorem for Boolean-valued semantics, which we present next:

Theorem 2.2.6 (Soundness and Completeness)**.** *First order logic is sound and complete with respect to Boolean-valued semantics. More precisely, for any first order theory T and sentence φ in the language of T, φ is a theorem of T iff φ is valid in T.*

Proof. WLoG, let's consider a first order theory as a formal system as in the classic book by Shoenfield [21] with the following rules of inference.

- The expansion rule: Infer $\psi \vee \varphi$ from φ
- The contraction rule: Infer φ from $\varphi \vee \varphi$
- The association rule: Infer $(\varphi \vee \psi) \vee \theta$ from $\varphi \vee (\psi \vee \theta)$
- The cut rule: Infer $\psi \vee \theta$ from $\varphi \vee \psi$ and $\neg\varphi \vee \theta$
- The $\exists$-introduction rule: If x is not free in ψ, infer $\exists x \varphi(x) \to \psi$ from $\varphi \to \psi$

It is more than obvious that the first three rules of inference preserve Boolean validity, as defined above in 2.2.5. For the cut rule, suppose $\varphi \vee \psi$ and $\neg\varphi \vee \theta$ are valid in $\mathcal{M}$. Then, $\neg [\![\varphi]\!] \vee [\![\theta]\!] = 1$, so $[\![\theta]\!] \geq [\![\varphi]\!]$. Then we have $[\![\psi \vee \theta]\!] = [\![\psi]\!] \vee [\![\theta]\!] \geq [\![\varphi]\!] \vee [\![\psi]\!] = [\![\varphi \vee \psi]\!] = 1_{\mathsf{B}}$.

For the $\exists$-introduction rule, suppose $\varphi(x) \rightarrow \psi$ is valid in $\mathcal{M}$. I.e. $\left[\!\left[\forall x(\varphi(x) \rightarrow \psi) \right]\!\right] = 1_{\mathsf{B}}$. So for each $\sigma \in M$ we have $[\![\varphi(\sigma) \rightarrow \psi]\!] = 1$, i.e. $[\![\varphi(\sigma)]\!] \leq [\![\psi]\!]$. Then $[\![\psi]\!] \geq \bigvee \left\{ [\![\varphi(\sigma)]\!] : \sigma \in M \right\} = [\![\exists x \varphi(x)]\!]$, thus $[\![\exists x \varphi(x) \rightarrow \psi]\!] = 1$.

Completeness of first order logic with respect to Boolean-valued semantics is a trivial consequence of the classical completeness theorem for first order logic and two-valued semantics. Indeed, if a formula in $\mathcal{L}$ is valid in all Boolean-valued models of a theory T in $\mathcal{L}$, then it is valid, in particular, in all 2-valued models of T, which, by the classical completeness theorem grants that T proves that formula. □

Definition 2.2.7. Let $\mathcal{L}$ be a first order language, B be a cba and $\mathcal{M}$ be a B-valued structure for $\mathcal{L}$. Let F be a filter on B. The quotient model $\mathcal{M}/_F$ is the $\mathsf{B}/_F$-valued model defined as follows:

- its domain $M/_F$ is the set $\left\{ [\sigma]_F : \sigma \in M \right\}$,
  where $[\sigma]_F = \left\{ \tau \in M : [\![\sigma = \tau]\!]^{\mathcal{M}} \in F \right\}$;
- the interpretation $p^{\mathcal{M}/_F}$ of each n-ary predicate symbol p in $\mathcal{L}$ has values
$$\left[\!\left[p\big([\sigma_1]_F, \ldots, [\sigma_n]_F\big) \right]\!\right]^{\mathcal{M}/_F} = \left[[\![p(\sigma_1, \ldots, \sigma_n)]\!]^{\mathcal{M}} \right]_F;$$
- $c^{\mathcal{M}/_F} = \left[c^{\mathcal{M}} \right]_F$ for every constant symbol c in $\mathcal{L}$.
- the interpretation $f^{\mathcal{M}/_F}$ of each n-ary function symbol f is the function
$$\begin{array}{rrcl} f^{\mathcal{M}/_F} : & (M/_F)^n & \longrightarrow & M/_F \\ & \big\langle [\sigma_1]_F, \ldots, [\sigma_n]_F \big\rangle & \longmapsto & \left[f^{\mathcal{M}}(\sigma_1, \ldots, \sigma_n) \right]_F \end{array}$$

Note that (2.4) and (2.5) guarantee that the interpretations $p^{\mathcal{M}/_F}$ and $f^{\mathcal{M}/_F}$ are well-defined, i.e. that the values $\left[\!\left[p\big([\sigma_1]_F, \ldots, [\sigma_n]_F\big) \right]\!\right]^{\mathcal{M}/_F}$ and $f^{\mathcal{M}/_G}\big([\sigma_1]_F, \ldots, [\sigma_n]_F\big)$ do not depend on our choice of representatives $\sigma_1 \in [\sigma_1]_F, \ldots, \sigma_n \in [\sigma_n]_F$ for computing $\left[[\![p(\sigma_1, \ldots, \sigma_n)]\!]^{\mathcal{M}} \right]_F$ and $\left[f^{\mathcal{M}}(\sigma_1, \ldots, \sigma_n) \right]_F$. In particular (for p being the equality symbol), for any Boolean-valued structure $\mathcal{M}$, the quotient $\mathcal{M}/_{\{1\}}$ with the trivial ideal $\{1\}$ is extensional.

Clearly such a quotient $\mathcal{M}/_F$ satisfies all the conditions for being a $\mathsf{B}/_F$-valued model.

Note that having defined the domain $M/_F$ and the interpretations $p^{\mathcal{M}/_F}$, $c^{\mathcal{M}/_F}$ and $f^{\mathcal{M}/_F}$ of the nonlogical symbols and equality, specifies the whole structure $\mathcal{M}/_F$. In particular, the truth values of non-atomic formulas, are determined by the general semantics rules stated in definition 2.2.3.

If F is an ultrafilter, the quotient $\mathsf{B}/_F$ is the two-element Boolean algebra and the structure $\mathcal{M}/_F$ is a two-valued Tarski[5] structure for $\mathcal{L}$. In this case we say that $\mathcal{M}/_F$ is the *Tarski quotient of $\mathcal{M}$ by F*.

One might expect that that $[\![\varphi(\tau)]\!]^{\mathcal{M}/_F} = \left[[\![\varphi([\tau]_F)]\!]^{\mathcal{M}} \right]_F$ for any formula φ in the language of $\mathcal{M}$. We indeed have

[5] We call classical two-valued structures for first order formal systems *Tarski* structures.

$$
\begin{aligned}
[\![\neg\varphi]\!]^{\mathcal{M}/F} &= \neg [\![\varphi]\!]^{\mathcal{M}/F} && \text{(by the semantics of negation in } \mathcal{M}/_F\text{)}\\
&= \neg \left[[\![\varphi]\!]^{\mathcal{M}} \right]_F && \text{(by the induction hypothesis)}\\
&= \left[\neg [\![\varphi]\!]^{\mathcal{M}} \right]_F && \text{(by simple algebraic properties of quotents of ba's)}\\
&= \left[[\![\neg\varphi]\!]^{\mathcal{M}} \right]_F && \text{(by the semantics of negation in } \mathcal{M}\text{)}
\end{aligned}
$$

$$
\begin{aligned}
[\![\varphi \vee \psi]\!]^{\mathcal{M}/F} &= [\![\varphi]\!]^{\mathcal{M}/F} \vee [\![\psi]\!]^{\mathcal{M}/F} && \text{(by the semantics in } \mathcal{M}/_F\text{)}\\
&= \left[[\![\varphi]\!]^{\mathcal{M}} \right]_F \vee \left[[\![\psi]\!]^{\mathcal{M}} \right]_F && \text{(by the induction hypothesis)}\\
&= \left[[\![\varphi]\!]^{\mathcal{M}} \vee [\![\psi]\!]^{\mathcal{M}} \right]_F && \text{(by operations in quotients of ba's)}\\
&= \left[[\![\varphi \vee \psi]\!]^{\mathcal{M}} \right]_F && \text{(by the semantics of disjunction in } \mathcal{M}\text{)}
\end{aligned}
$$

However, for existential formulas we have

$$
\begin{aligned}
[\![\exists x\varphi(x)]\!]^{\mathcal{M}/F} &= \bigvee_{\mathsf{B}/_F} \left\{ [\![\varphi(\sigma)]\!]^{\mathcal{M}/F} : \sigma \in M/_F \right\} && \text{(by the semantics in } \mathcal{M}/_F\text{)}\\
&= \bigvee_{\mathsf{B}/_F} \left\{ [\![\varphi([\sigma]_F)]\!]^{\mathcal{M}/F} : \sigma \in M \right\} && \text{(by the definition of } M/_F\text{)}\\
&= \bigvee_{\mathsf{B}/_F} \left\{ \left[[\![\varphi(\sigma)]\!]^{\mathcal{M}} \right]_F : \sigma \in M \right\} && \text{(by the induction hypothesis)}
\end{aligned}
$$

But the join does not necessarily commute with taking the quotient. I.e. the F-equivalence class $\left[\bigvee_{\mathsf{B}} A\right]_F$ of the join of a set A in the big algebra B does not necessarily equal the corresponding join $\bigvee_{\mathsf{B}/_F}(A/_F)$ in the quotient algebra $\mathsf{B}/_F$. That would be guaranteed if F were $|A|$-complete. So we do not necessarily have

$$[\![\exists x\varphi(x)]\!]^{\mathcal{M}/F} = \left[[\![\exists x\varphi(x)]\!]^{\mathcal{M}} \right]_F$$

and hence more generally we don't have

$$[\![\varphi]\!]^{\mathcal{M}/F} = \left[[\![\varphi]\!]^{\mathcal{M}} \right]_F .$$

In words – quotients do not necessarily preserve truth. This is a special property which not all Boolean-valued structures have. One particular weakening of this property – its restriction to ultrafilters only – is commonly referred to as *fullness.* Noticing that when G is an ultrafilter we have the following equivalences

$$\mathcal{M}/_G \vDash \varphi \quad \text{iff} \quad [\![\varphi]\!]^{\mathcal{M}/G} = 1$$

$$[\![\varphi]\!]^{\mathcal{M}} \in G \quad \text{iff} \quad \left[[\![\varphi]\!]^{\mathcal{M}} \right]_G = 1$$

we arrive at the common form of the definition of fullness:

Definition 2.2.8 (Fullness)**.** We say that a B-valued structure $\mathcal{M}$ is *full* iff for each ultrafilter G on B, each formula $\varphi(\vec{x})$ in the language of $\mathcal{M}$ and all $\vec{\sigma} \in M$, we have

$$\mathcal{M}/_G \vDash \varphi([\vec{\sigma}]_G) \qquad \text{iff} \qquad [\![\varphi(\vec{\sigma})]\!]^{\mathcal{M}} \in G$$

Theorem 2.2.9. [20, Theorem 4.11]. *The following are equivalent for a Boolean-valued structure* $\mathcal{M}$*:*

(1) $\mathcal{M}$ *is full*

(2) *the truth value* $\llbracket \exists x\varphi(x) \rrbracket$ *of each existential formula coincides with the join* $\bigvee \left\{ \llbracket \varphi(\sigma_i) \rrbracket : i < m \right\}$ *of the truth values of* finitely many *instances* $\varphi(\sigma_i)$ *of* $\varphi(x)$.

Proof. Assume first that $\mathcal{M}$ is a full B-valued structure. Fix some existential formula $\exists x\varphi(x,\tau)$. Simply by the semantics of existential formulas in classical two-valued structures, we have that: for each ultrafilter G on B such that $\mathcal{M}/_G$ models $\exists x\varphi(x,[\tau]_G)$, there is some witness σ_G such that $\mathcal{M}/_G$ models $\varphi([\sigma_G]_G,[\tau]_G)$. By fullness of $\mathcal{M}$, this amounts to saying that each ultrafilter G in $N_{\llbracket \exists x\varphi(x,\tau) \rrbracket}$ belongs to some $N_{\llbracket \varphi(\sigma_G,\tau) \rrbracket}$. Then, if we collect all these witnesses σ_G into a set Γ, we'll have

$$N_{\llbracket \exists x\varphi(x,\tau) \rrbracket} \subseteq \bigcup_{\sigma\in\Gamma} N_{\llbracket \varphi(\sigma_G,\tau) \rrbracket} \subseteq \bigcup_{\sigma\in M} N_{\llbracket \varphi(\sigma_G,\tau) \rrbracket}$$

But the Stone space of B is compact, so the open cover on the right-hand side has a finite subcover $\bigcup_{i<m} N_{\llbracket \varphi(\sigma_i,\tau) \rrbracket}$. Then

$$\llbracket \exists x\varphi(x,\tau) \rrbracket \leq \bigvee_{i<m} \llbracket \varphi(\sigma_i,\tau) \rrbracket$$

The converse inequality holds simply by the definition of Boolean truth-values of existential formulas. This concludes the proof of this direction of the claim.

Now assume (2). We prove $\mathcal{M}$ is full by induction on the construction of the formulas but we present only the nontrivial case when the formula is existential.

If an ultrafilter G contains the truth-value $\llbracket \exists x\varphi(x,\tau) \rrbracket$ of an existential formula, and this value can be represented as $\llbracket \exists x\varphi(x,\tau) \rrbracket = \bigvee_{i<m} \llbracket \varphi(\sigma_i,\tau) \rrbracket$, then since G is downward directed, we obtain that there is some $i < m$ such that $\llbracket \varphi(\sigma_i,\tau) \rrbracket \in G$. Then we have $\mathcal{M}/_G \vDash \varphi([\sigma_i]_G,[\tau]_G)$ by induction hypothesis.

Conversely, if $\mathcal{M}/_G$ models $\exists x\varphi(x,[\tau]_G)$ and $[\sigma]_G$ is a witness for it, then by the induction hypothesis and the upward closedness of G we have $\llbracket \varphi(\sigma,\tau) \rrbracket \leq \llbracket \exists x\varphi(x,\tau) \rrbracket \in G$. □

Fullness is often difficult to prove directly. A more tangible concept often used to infer fullness is the strictly stronger *mixing property*

Definition 2.2.10 (Mixing)**.** Let $\mathcal{M}$ be a B-valued structure for $\mathcal{L}$ and κ be a cardinal.

- We say that $\mathcal{M}$ has *the* κ*-mixing property* iff for every antichain $A \subseteq \mathsf{B}$ of size at most κ, and for every subset $\left\{ \tau_a : a \in A \right\} \subseteq M$, there exists a name $\tau \in M$ such that $a \leq \llbracket \tau = \tau_a \rrbracket$ for every $a \in A$.
- We say that $\mathcal{M}$ has *the* $<\kappa$*-mixing property* iff it has the λ-mixing property for each $\lambda < \kappa$.
- We say that $\mathcal{M}$ has *the mixing property* iff $\mathcal{M}$ has the $<\mathrm{cc}(\mathsf{B})$-mixing property.

And yet another natural property we often need in practice is the following strengthening of fullness, whose strength sits between fullness and mixing.

Definition 2.2.11. We shall say that a Boolean-valued structure *golden* iff the truth value $\llbracket \exists x\varphi(x) \rrbracket$ of each existential formula coincides with the truth value $\llbracket \varphi(\sigma) \rrbracket$ of a single instance $\varphi(\sigma)$ of the quantified formula $\varphi(x)$.

Theorem 2.2.12 (Łoś theorem; the maximality principle)**.** *Boolean-valued structures with the mixing property are golden, and hence full.*

Proof. Let B be a Boolean algebra and $\mathcal{M}$ be a B-valued model with the mixing property. Fix an existential formula $\exists x\varphi(x,\vec{\tau})$ in the language of $\mathcal{M}$, where $\vec{\tau}$ is a finite sequence of names.

By definition, the Boolean value $b = \llbracket \exists x\varphi(x,\vec{\tau}) \rrbracket$ is the join of the set $S = \left\{ \llbracket \varphi(\sigma,\vec{\tau}) \rrbracket : \sigma \in \mathcal{M} \right\}$ of the truth values of all instances of $\varphi(x,\vec{\tau})$. Let A be a maximal antichain below b contained in the downward closure of S, so that every element of A is below an element of S and $\bigvee A = b$. For each $a \in A$ choose σ_a such that $a \leq [\![\varphi(\sigma_a, \vec{\tau})]\!]$.

Using the mixing property of $\mathcal{M}$, find a name σ such that $a \leq [\![\sigma = \sigma_a]\!]$ for all $a \in A$. Then we have

$$a \leq [\![\sigma = \sigma_a]\!] \wedge [\![\varphi(\sigma_a, \vec{\tau})]\!] \leq [\![\varphi(\sigma, \vec{\tau})]\!]$$

for each $a \in A$, and so $b = \bigvee A \leq [\![\varphi(\sigma, \vec{\tau})]\!]$. Since $[\![\varphi(\sigma, \vec{\tau})]\!] \in S$, we must also have $[\![\varphi(\sigma, \vec{\tau})]\!] \leq \bigvee S = b$, and so $[\![\exists x \varphi(x, \vec{\tau})]\!] = [\![\varphi(\sigma, \vec{\tau})]\!]$. By the previous theorem 2.2.9, the proof is completed. □

Remark 2.2.13. Obviously a finitely-mixing full Boolean-valued structure is golden.

The Boolean notion of uniqueness for has a very tangible meaning in golden structures.

Proposition 2.2.14 (Boolean uniqueness)**.** *Let $\mathcal{M}$ be a golden* B*-valued structure, $\varphi(x, \vec{y})$ be a formula in $\mathcal{L}$ and $\vec{\rho}$ be a list of names in $\mathcal{M}$. Assume $[\![\exists! x \varphi(x, \vec{\rho})]\!]^{\mathcal{M}} = 1_{\mathsf{B}}$. Then there exists a name $\sigma \in M$ such that $[\![\varphi(\sigma, \vec{\rho})]\!]^{\mathcal{M}} = 1_{\mathsf{B}}$ and for each name $\tau \in M$ we have $[\![\varphi(\tau, \vec{\rho})]\!]^{\mathcal{M}} \leq [\![\tau = \sigma]\!]^{\mathcal{M}}$.*

Proof. We have

$$\left[\!\!\left[\exists x \big(\varphi(x, \vec{\rho}) \;\;\&\;\; \forall y (\varphi(y, \vec{\rho}) \rightarrow y = x) \big) \right]\!\!\right]^{\mathcal{M}} = 1_{\mathsf{B}}$$

Since $\mathcal{M}$ is golden, fix some $\sigma \in M$ such that

$$[\![\varphi(\sigma, \vec{\rho}) \;\;\&\;\; \forall y (\varphi(y, \vec{\rho}) \rightarrow y = \sigma)]\!]^{\mathcal{M}} = 1_{\mathsf{B}}$$

Then we have $[\![\varphi(\sigma, \vec{\rho})]\!]^{\mathcal{M}} = 1_{\mathsf{B}}$ and

$$\bigwedge \left\{ \neg [\![\varphi(\tau, \vec{\rho})]\!]^{\mathcal{M}} \vee [\![\tau = \sigma]\!]^{\mathcal{M}} \;:\; \tau \in M \right\} = 1_{\mathsf{B}}$$

Which happens only if for each name τ in M we have that

$$[\![\varphi(\tau, \vec{\rho})]\!]^{\mathcal{M}} \leq [\![\tau = \sigma]\!]^{\mathcal{M}}$$

□

2.3 Boolean-valued morphisms

We now extend the classical definition of elementarity between two-valued Tarski structures for Boolean-valued structures. In the case of Tarski structures we have the simplification that the truth values of sentences in the language of any structure are conditions in the same Boolean algebra - the trivial Boolean algebra 2. A map which relates truth between Boolean-valued structures has to presuppose some relationship between the Boolean algebras in which sentences take their truth value.

Definition 2.3.1. An ordered pair $\langle \mathsf{B}, \mathcal{M} \rangle$ is called a *Boolean couple* if its first component B is a Boolean algebra, and its second component $\mathcal{M}$ is a B-valued model.

Definition 2.3.2. [1, Definition 3.3.1]. Let $\mathcal{M}$ and $\mathcal{N}$ be respectively a B-valued and a C-valued structures for a language $\mathcal{L}$. Let e be a homomorphism from B and C, and $\Phi \subseteq M \times N$ be a binary relation. The couple $\langle e, \Phi \rangle$ is a *morphism from* $\langle \mathsf{B}, \mathcal{M} \rangle$ *to* $\langle \mathsf{C}, \mathcal{N} \rangle$ iff:

(1) $\text{dom}(\Phi) = M$;

(2) for each n-ary function symbol f and each $\langle \sigma_i, \tau_i \rangle, \langle \mu, \nu \rangle \in \Phi$, we have

$$e\left([\![f(\vec{\sigma}) = \mu]\!]^{\mathcal{M}} \right) \leq [\![f(\vec{\tau}) = \nu]\!]^{\mathcal{N}} \tag{2.6}$$

(3) in particular, for each constant symbol c, for each $\langle \mu, \nu \rangle \in \Phi$ we have

$$e\left(\llbracket c = \mu \rrbracket^{\mathcal{M}}\right) \leq \llbracket c = \nu \rrbracket^{\mathcal{N}}$$

(4) for each n-ary predicate symbol p and $(\sigma_i, \tau_i) \in \Phi$, we have

$$e\left(\llbracket p(\vec{\sigma}) \rrbracket^{\mathcal{M}}\right) \leq \llbracket p(\vec{\tau}) \rrbracket^{\mathcal{N}}, \tag{2.7}$$

(5) in particular we have

$$e\left(\llbracket \sigma_1 = \sigma_2 \rrbracket^{\mathcal{M}}\right) \leq \llbracket \tau_1 = \tau_2 \rrbracket^{\mathcal{N}}. \tag{2.8}$$

- An *injective morphism* is a morphism such that in (2.6) and (2.8) equality holds.
- An *embedding* of Boolean valued models is an injective morphism such that in (2.7) equality holds.
- Let Γ be Σ or Π. An embedding $\langle e, \Phi \rangle$ is called *(weakly)* Γ_n*-elementary* iff for any Γ_n-formula $\varphi(\vec{x})$ in $\mathcal{L}$ and any $\langle \sigma_i, \tau_i \rangle \in \Phi$, we have

$$e\left(\llbracket \varphi(\vec{\sigma}) \rrbracket^{\mathcal{M}}\right) \leq \llbracket \varphi(\vec{\tau}) \rrbracket^{\mathcal{N}} \tag{2.9}$$

  It is called Γ_n-elementary (without "weakly") iff in (2.9) we have equality. It is called simply an *elementary embedding* iff it is Σ_n-elementary for all $n < \omega$.

- A Boolean embedding is called Δ_n-elementary iff for every formula $\varphi(\vec{x})$, for which there exist a Σ_n formula $\psi(\vec{x})$ and a Π_n formula $\theta(\vec{x})$ such that the formula

$$\forall \vec{x}\Big(\big(\varphi(\vec{x}) \leftrightarrow \psi(\vec{x})\big) \quad \& \quad \big(\varphi(\vec{x}) \leftrightarrow \theta(\vec{x})\big)\Big)$$

  is valid[6] both in $\mathcal{M}$ and $\mathcal{N}$, then for each $\langle \sigma_i, \tau_i \rangle \in \Phi$ we have

$$e\left(\llbracket \varphi(\vec{\sigma}) \rrbracket^{\mathcal{M}}\right) = \llbracket \varphi(\vec{\tau}) \rrbracket^{\mathcal{N}}$$

- An embedding $\langle e, \Phi \rangle$ from $\mathcal{M}$ to $\mathcal{N}$ is called an *isomorphism* of Boolean valued models if e is an isomorphism of Boolean algebras Φ is surjective up to Boolean equality, i.e. for every $\tau \in N$ there is a $\langle \sigma, \tau' \rangle \in \Phi$ such that $\llbracket \tau = \tau' \rrbracket^{\mathcal{N}} = \mathsf{1}$.

Remark 2.3.3. Notice that

- (not-weakly) Σ_n-elementarity, Π_n-elementarity and Δ_n elementarity coincide, and we shall refer to is simply as *n-elementarity*;
- an elementary embedding cannot be properly weak because a weakly $(n+1)$-elementary embedding is an n-elementary embedding.

  Indeed, if φ is Σ_n, then $\neg\varphi$ is Π_n, and hence $\neg\varphi$ is Σ_{n+1}, and hence $\neg e\left(\llbracket \varphi \rrbracket^{\mathcal{M}}\right) = e\left(\neg \llbracket \varphi \rrbracket^{\mathcal{M}}\right) = e\left(\llbracket \neg\varphi \rrbracket^{\mathcal{M}}\right) \leq \llbracket \neg\varphi \rrbracket^{\mathcal{N}} = \neg \llbracket \varphi \rrbracket^{\mathcal{N}}$, and hence $e\left(\llbracket \varphi \rrbracket^{\mathcal{M}}\right) \geq \llbracket \varphi \rrbracket^{\mathcal{N}}$.

- weak Γ_n elementarity clearly implies Δ_n elementarity. More precisely, if Γ is Σ or Π and $\langle e, \Phi \rangle$ is a weakly Γ_n-elementary embedding of $\langle \mathsf{B}, \mathcal{M} \rangle$ into $\langle \mathsf{C}, \mathcal{N} \rangle$, then $\langle e, \Phi \rangle$ is also a (not-weakly) Δ_n-elemenary embedding of $\langle \mathsf{B}, \mathcal{M} \rangle$ into $\langle \mathsf{C}, \mathcal{N} \rangle$.

[6] Recall definition 2.2.5.

Remark 2.3.4. $\langle e, \Phi \rangle$ is a morphism between the Boolean couples $\langle \mathsf{B}, \mathcal{M} \rangle$ and $\langle \mathsf{C}, \mathcal{N} \rangle$ iff letting H be the trivial filter $\{1_{\mathsf{C}}\}$ the map Φ' defined by $\sigma \mapsto [\tau]_H$ for $(\sigma, \tau) \in \Phi$ is a function and is also a well defined morphism of the Boolean couple $\langle \mathsf{B}, \mathcal{M} \rangle$ with the Boolean couple $\langle \mathsf{C}, \mathcal{N}/_H \rangle$: observe that if Φ is a morphism, then Φ' is a function. Indeed, given $(\sigma, \tau_1), (\sigma, \tau_2) \in \Phi$,

$$1_{\mathsf{C}} = e(1_{\mathsf{B}}) = e\Big(\llbracket \sigma = \sigma \rrbracket^{\mathcal{M}}\Big) \leq \llbracket \tau_1 = \tau_2 \rrbracket^{\mathcal{N}}$$

hence $[\tau_1]_H = [\tau_2]_H$, since $\mathcal{N}/_H$ is extensional. The rest of the claim is trivial to check.

In particular morphisms between extensional Boolean valued models are maps and not just binary relations.

Proposition 2.3.5. *Let $\langle \mathsf{B}, \mathcal{M} \rangle$ and $\langle \mathsf{C}, \mathcal{N} \rangle$ be Boolean couples for $\mathcal{L}$ and $\langle e, \Phi \rangle$ be a Boolean morphism from the former to the latter. Let $u(\vec{x})$ be a term in $\mathcal{L}$ and $\langle \sigma_i, \tau_i \rangle \in \Phi$. Then Φ can relate $u^{\mathcal{M}}(\vec{\sigma})$ only to elements of $\Big[u^{\mathcal{N}}(\vec{\tau})\Big]_{\{1_{\mathsf{C}}\}}$. In other words, for each $\nu \in N$ we have*

$$\Big\langle u^{\mathcal{M}}(\vec{\sigma}), \nu \Big\rangle \in \Phi \quad \rightarrow \quad \llbracket u(\vec{\tau}) = \nu \rrbracket^{\mathcal{N}} = 1_{\mathsf{C}}$$

Proof. Induction on the length of terms. If u is a parameter τ, then the witness ν is τ itself. If it is a constant c, then $\llbracket c = \nu \rrbracket^{\mathcal{N}} = e\Big(\llbracket c = c \rrbracket^{\mathcal{M}}\Big) = e(1_{\mathsf{B}}) = 1_{\mathsf{C}}$. Now assume the claim is true for $u_1, \ldots, u_n$. If $\Big\langle f^{\mathcal{M}}(\vec{\sigma}), \nu \Big\rangle \in \Phi$, then by (2.6) we have

$$\llbracket f(\vec{\tau}) = \nu \rrbracket^{\mathcal{N}} \geq e\Big(\llbracket f(\vec{\sigma}) = f(\vec{\sigma}) \rrbracket^{\mathcal{M}}\Big) = e(1_{\mathsf{B}}) = 1_{\mathsf{C}}$$

□

Corollary 2.3.6. *If $\mathcal{N}$ is extensional, then for every function symbol f and $\langle \sigma_i, \tau_i \rangle \in \Phi$ we have $\Big\langle f^{\mathcal{M}}(\vec{\sigma}), f^{\mathcal{N}}(\vec{\tau}) \Big\rangle \in \Phi$.*

Definition 2.3.7 (Boolean extension)**.** Let $\mathcal{M}$ be a B-valued structure and $\mathcal{N}$ be a C-valued structure for a first order language $\mathcal{L}$, such that B is a complete subalgebra of C, $M \subseteq N$, and for any $\sigma_i \in M$ we have

$$c^{\mathcal{M}} = c^{\mathcal{N}} \tag{2.10}$$

$$f^{\mathcal{M}}(\sigma_1, \ldots, \sigma_n) = f^{\mathcal{N}}(\sigma_1, \ldots, \sigma_n) \tag{2.11}$$

$$\llbracket p(\sigma_1, \ldots, \sigma_n) \rrbracket^{\mathcal{M}} = \llbracket p(\sigma_1, \ldots, \sigma_n) \rrbracket^{\mathcal{N}} \tag{2.12}$$

for all predicate symbols p, function symbols f and constant symbols c in $\mathcal{L}$. Then $\langle \mathrm{Id}_{\mathsf{B}}, \mathrm{Id}_M \rangle$ is an embedding of $\langle \mathsf{B}, \mathcal{M} \rangle$ into $\langle \mathsf{C}, \mathcal{N} \rangle$ and $\mathcal{N}$ is said to be a *Boolean extension* of $\mathcal{M}$.

Definition 2.3.8. Let $\mathcal{L}$ be a first order language, B be a Boolean algebra, $\mathcal{M}$ be a B-valued structure for $\mathcal{L}$ and $N \subseteq M$ be closed under the interpretations $f^{\mathcal{M}}$ in $\mathcal{M}$ of the function symbols f in $\mathcal{L}$. Then we define *the induced by N substructure of* $\mathcal{M}$, as the B-valued structure $\mathcal{M} \restriction N$ with domain N and interpretations of the predicate symbols directly inherited from $\mathcal{M}$ – namely, for each n-ary symbol k in $\mathcal{L}$ (be it predicate or function) we define its interpretation $k^{\mathcal{M} \restriction N}$ in $\mathcal{M} \restriction N$ as $k^{\mathcal{M}} \restriction N$.

Obviously $\langle \mathrm{Id}_{\mathsf{B}}, \mathrm{Id}_N \rangle$ is a Boolean embedding of $\langle \mathsf{B}, \mathcal{M} \restriction N \rangle$ into $\langle \mathsf{B}, \mathcal{M} \rangle$.

Lemma 2.3.9 (Intermediate quotient)**.** *Let $\mathcal{M}$ be a B-valued model and $F \subseteq G$ be filters on B. Then the union function*

$$\begin{aligned} \bigcup \ : \ (\mathsf{B}/_F)/_{(G/_F)} &\longrightarrow \mathsf{B}/_F \\ \big[[a]_F\big]_{(G/_F)} &\longmapsto [a]_G \end{aligned}$$

$$\begin{aligned} \bigcup \ : \ (\mathcal{M}/_F)/_{(G/_F)} &\longrightarrow \mathcal{M}/_G \\ \big[[\tau]_F\big]_{(G/_F)} &\longmapsto [\tau]_G \end{aligned}$$

provides an isomorphism from $(\mathcal{M}/_F)/_{(G/_F)}$ *onto* $\mathcal{M}/_G$*. More precisely, the tuple*

$$\left\langle \bigcup\restriction(\mathsf{B}/_F)/_{(G/_F)}\,,\, \bigcup\restriction(\mathcal{M}/_F)/_{(G/_F)} \right\rangle$$

is an isomorphism from the Boolean couple $\left\langle (\mathsf{B}/_F)/_{(G/_F)}\,,(\mathcal{M}/_F)/_{(G/_F)} \right\rangle$ *to* $\langle\, \mathsf{B}/_G\,, \mathcal{M}/_G \,\rangle$*.*

Proof. Let f be any function symbol (including a constant) in the language of $\mathcal{M}$ and let $\tau_i \mu \in M$. Then using the quotient semantics 2.2.7 and the fact that $F \subseteq G$ we have:

$$\begin{aligned}
\bigcup \left[\!\!\left[f\Big([\,[\vec{\tau}]_F\,]_{(G/_F)} \Big) = [\,[\mu]_F\,]_{(G/_F)} \right]\!\!\right]^{(\mathcal{M}/_F)/_{(G/_F)}} &= \bigcup \left[\left[\!\!\left[f\big([\vec{\tau}]_F\big) = [\mu]_F \right]\!\!\right]^{(\mathcal{M}/_F)} \right]_{(G/_F)} \\
&= \bigcup \left[\left[[\![\, f(\vec{\tau}) = \mu \,]\!]^{\mathcal{M}} \right]_F \right]_{(G/_F)} \\
&= \left[[\![\, f(\vec{\tau}) = \mu \,]\!]^{\mathcal{M}} \right]_G \\
&= \left[\!\!\left[f\big([\vec{\tau}]_G\big) = [\mu]_G \right]\!\!\right]^{(\mathcal{M}/_G)} \\
&= \left[\!\!\left[f\Big(\bigcup\Big([\,[\vec{\tau}]_F\,]_{(G/_F)} \Big)\Big) = \bigcup\, [\,[\mu]_F\,]_{(G/_F)} \right]\!\!\right]^{(\mathcal{M}/_G)}
\end{aligned}$$

Let p be a predicate symbol in the language of $\mathcal{M}$ and let $\tau_i \in M$. Then

$$\begin{aligned}
\bigcup \left[\!\!\left[p\Big([\,[\vec{\tau}]_F\,]_{(G/_F)} \Big) \right]\!\!\right]^{(\mathcal{M}/_F)/_{(G/_F)}} &= \bigcup \left[\left[\!\!\left[p\big([\vec{\tau}]_F\big) \right]\!\!\right]^{\mathcal{M}/_F} \right]_{(G/_F)} && \text{by the quotient semantics 2.2.7} \\
&= \bigcup \left[\left[[\![\, p(\vec{\tau}) \,]\!]^{\mathcal{M}} \right]_F \right]_{(G/_F)} && \text{by the quotient semantics 2.2.7} \\
&= \left[[\![\, p(\vec{\tau}) \,]\!]^{\mathcal{M}} \right]_G && \text{since } F \subseteq G \\
&= \left[\!\!\left[p\big([\vec{\tau}]_G\big) \right]\!\!\right]^{\mathcal{M}/_G} && \text{by the quotient semantics 2.2.7} \\
&= \left[\!\!\left[p\Big(\bigcup\Big([\,[\vec{\tau}]_F\,]_{(G/_F)} \Big)\Big) \right]\!\!\right]^{\mathcal{M}/_G}
\end{aligned}$$

□

Proposition 2.3.10. *Let* $\mathcal{M}$ *be a* B*-valued structure and* $\mathcal{N}$ *be a* C*-valued structure for the same relational language* $\mathcal{L}$*. Assume* $\langle\, e, \Phi \,\rangle$ *is an isomorphism of* $\langle\, \mathsf{B}, \mathcal{M} \,\rangle$ *onto* $\langle\, \mathsf{C}, \mathcal{N} \,\rangle$*. Then for each term* $u(x_1, \ldots, x_n)$ *and formula* $\varphi(x_1, \ldots, x_n)$ *in* $\mathcal{L}$ *and each* $\langle\, \sigma_1\,, \tau_1 \,\rangle, \ldots, \langle\, \sigma_n\,, \tau_n \,\rangle \in \Phi$*, we have*

$$e\left([\![\, \varphi(\sigma_1, \ldots, \sigma_n) \,]\!]^{\mathcal{M}} \right) = [\![\, \varphi(\tau_1, \ldots, \tau_n) \,]\!]^{\mathcal{N}}$$

Proof. Induction on the length of φ. Let φ be an atomic formula $p(u_1, \ldots, u_n)$ where p is an n-ary predicate symbol (including $=$) and $u_i(x_1, \ldots, x_k)$ are any terms. Fix $\langle\, \sigma_1\,, \tau_1 \,\rangle, \ldots, \langle\, \sigma_k\,, \tau_\kappa \,\rangle \in \Phi$. By proposition 2.3.5 fix some names $\rho_1, \ldots, \rho_k \in N$ such that $\left\langle u_i^{\mathcal{M}}(\vec{\sigma}), \rho_i \right\rangle \in \Phi$ and $[\![\, u_i(\vec{\tau}) = \rho_i \,]\!]^{\mathcal{N}} = 1_{\mathsf{C}}$. Then we have

$$\begin{aligned}
e\Big([\![\, \varphi(\vec{\sigma}) \,]\!]^{\mathcal{M}} \Big) &= e\Big([\![\, p(u_1(\vec{\sigma}), \ldots, u_n(\vec{\sigma})) \,]\!]^{\mathcal{M}} \Big) \\
&= [\![\, p(\rho_1, \ldots, \rho_n) \,]\!]^{\mathcal{N}} \\
&= [\![\, p(\rho_1, \ldots, \rho_n) \,]\!]^{\mathcal{N}} \wedge 1_{\mathsf{C}} \\
&= [\![\, p(\rho_1, \ldots, \rho_n) \,]\!]^{\mathcal{N}} \wedge \bigwedge_{i=1}^{n} [\![\, u_i(\vec{\tau}) = \rho_i \,]\!]^{\mathcal{N}} \\
&= [\![\, p(u_1(\vec{\tau}), \ldots, u_n(\vec{\tau})) \,]\!]^{\mathcal{N}} = [\![\, \varphi(\vec{\tau}) \,]\!]^{\mathcal{N}}
\end{aligned}$$

If φ is a negation or disjunction, then the result follows easily from the induction hypothesis. Now suppose φ is $\exists x\psi$. Since Φ is surjective and has domain M, for each name $\pi \in M$ choose a name ρ_π in $\mathcal{N}$ such that $\langle \pi, \rho_\pi \rangle \in \Phi$ and, conversely, for each name $\rho \in N$ choose a name π_ρ in $\mathcal{M}$ such that $\langle \pi_\rho, \rho \rangle \in \Phi$. For the sake of notational simplicity, assume WLoG that e is the identity. Then

$$\begin{aligned}
[\![\varphi(\vec{\tau})]\!]^{\mathcal{N}} &= [\![\exists x\psi(x,\vec{\tau})]\!]^{\mathcal{N}} \\
&= \bigvee \left\{ [\![\psi(\rho,\vec{\tau})]\!]^{\mathcal{N}} : \rho \in N \right\} && \text{(by the semantics)} \\
&= \bigvee \left\{ [\![\psi(\pi_\rho,\vec{\sigma})]\!]^{\mathcal{M}} : \rho \in N \right\} && \text{(by i.h.)} \\
&\leq \bigvee \left\{ [\![\psi(\pi,\vec{\sigma})]\!]^{\mathcal{M}} : \pi \in M \right\} && \text{(since } \{ \pi_\rho : \rho \in N \} \subseteq M) \\
&= [\![\exists x\psi(x,\vec{\sigma})]\!]^{\mathcal{M}} = [\![\varphi(\vec{\sigma})]\!]^{\mathcal{M}} && \text{(by the semantics)}
\end{aligned}$$

$$\begin{aligned}
[\![\varphi(\vec{\sigma})]\!]^{\mathcal{M}} &= [\![\exists x\psi(x,\vec{\sigma})]\!]^{\mathcal{M}} \\
&= \bigvee \left\{ [\![\psi(\pi,\vec{\sigma})]\!]^{\mathcal{M}} : \pi \in M \right\} && \text{(by the semantics)} \\
&= \bigvee \left\{ [\![\psi(\rho_\pi,\vec{\tau})]\!]^{\mathcal{N}} : \pi \in M \right\} && \text{(by i.h.)} \\
&\leq \bigvee \left\{ [\![\psi(\rho,\vec{\tau})]\!]^{\mathcal{N}} : \rho \in N \right\} && \text{(since } \{ \rho_\pi : \pi \in M \} \subseteq N) \\
&= [\![\exists x\psi(x,\vec{\tau})]\!]^{\mathcal{N}} = [\![\varphi(\vec{\tau})]\!]^{\mathcal{N}} && \text{(by the semantics)}
\end{aligned}$$

□

Morphisms of Boolean valued models are preserved by quotients.

Proposition 2.3.11. *Let $\langle \mathsf{B}, \mathcal{M} \rangle$ and $\langle \mathsf{C}, \mathcal{N} \rangle$ be two Boolean couples for a relational language $\mathcal{L}$. Let F be a filter on B, G be a filter on C and $e : \mathsf{B} \longrightarrow \mathsf{C}$ be a homomorphism such that $G \supseteq e[F]$. Assume now $\Phi \subseteq M \times N$ is such that $\langle e, \Phi \rangle$ is a morphism from $\langle \mathsf{B}, \mathcal{M} \rangle$ to $\langle \mathsf{C}, \mathcal{N} \rangle$. Define*

$$\begin{array}{rccc} e/_{F,G} & : & \mathsf{B}/_F & \longrightarrow & \mathsf{C}/_G \\ & & [b]_F & \longmapsto & [e(b)]_G \end{array}$$

and

$$\Phi/_{F,G} = \left\{ \langle [\sigma]_F, [\tau]_G \rangle : \langle \sigma, \tau \rangle \in \Phi \right\} \tag{2.13}$$

Then $\langle e/_{F,G}, \Phi/_{F,G} \rangle$ is a morphism between the extensional Boolean valued models $\mathcal{M}/_F$ and $\mathcal{N}/_G$. Moreover, if $\langle e, \Phi \rangle$ is an injective morphism, embedding, or isomorphism of Boolean valued models, and $e/_{F,G}$ is injective, then $\langle e/_{F,G}, \Phi/_{F,G} \rangle$ is respectively an injective morphism, embedding, or isomorphism of Boolean valued models.

Proof. Given $\langle \sigma, \tau \rangle \in \Phi/_{F,G}$, we let $\pi_\sigma \in M$ and $\rho_\tau \in N$ be two elements such that $\langle \pi_\sigma, \rho_\tau \rangle \in \Phi$ and $\sigma = [\pi_\sigma]_F$, $\tau = [\rho_\tau]_G$.

1. Since $\mathrm{dom}(\Phi) = M$, it follows that $\Phi/_{F,G}$ is everywhere defined.
2. Let f be an n-ary function symbol in $\mathcal{L}$ and $\langle \sigma_i, \tau_i \rangle, \langle \mu, \nu \rangle \in \Phi_{F,G}$. Then

$$\begin{aligned}
e/_{F,G}\left([\![f(\vec{\sigma}) = \mu]\!]^{\mathcal{M}/_F} \right) &= e/_{F,G}\left(\left[[\![f(\vec{\pi_\sigma}) = \pi_\mu]\!]^{\mathcal{M}} \right]_F \right) && \text{(by the semantics in quotients)} \\
&= \left[e\left([\![f(\vec{\pi_\tau}) = \pi_\nu]\!]^{\mathcal{M}} \right) \right]_G && \text{(by the defining property of } e/_{F,G}) \\
&\leq \left[[\![f(\vec{\rho_\tau})]\!]^{\mathcal{N}} \right]_G && \text{(since } \langle e, \Phi \rangle \text{ is a Boolean morphism)} \\
&= [\![f(\vec{\tau}) = \nu]\!]^{\mathcal{N}/_G} && \text{(by the semantics in quotients)}
\end{aligned}$$

3. Let p be an n-ary predicate symbol in $\mathcal{L}$ and $\langle \sigma_i, \tau_i \rangle, \langle \mu, \nu \rangle \in \Phi/_{F,G}$. Then by an exactly analogous reasoning we have

$$\begin{aligned} e/_{F,G}\Big([\![p(\vec{\sigma})]\!]^{\mathcal{M}/_F} \Big) &= e/_{F,G}\Big(\Big[[\![p(\vec{\pi_\sigma})]\!]^{\mathcal{M}} \Big]_F \Big) \\ &= \Big[e\Big([\![p(\vec{\pi_\sigma})]\!]^{\mathcal{M}} \Big) \Big]_G \\ &\le \Big[[\![p(\vec{\rho_\tau})]\!]^{\mathcal{N}} \Big]_G \\ &= [\![p(\vec{\tau})]\!]^{\mathcal{N}/_G} . \end{aligned}$$

It can be easily checked that whenever equality holds in (2.7) and 4 of Definition 2.3.2, equality holds as well in the above equations, which completes the proof. □

2.4 Boolean ultrapowers

Ultrapowers are a general method for constructing elementary extensions of first order structures. The classic construction which uses ultrafilters on powerset algebras has been generalized for arbitrary complete[7] Boolean algebras by Mansfield in [14].

Definition 2.4.1 (Solid sets, tops and coherent functions)**.** Let P be a poset.

- If a subset of P is the downward closure of a (maximal) antichain on P, we shall call it *a (maximal) solid set on* P.

  Notice that there is a natural one-to-one correspondence between antichains and solid sets – each antichain determines a unique solid set (its downward closure) and every solid set is the downward closure of a unique antichain – the set of its maximal elements.

- We shall call the set A of maximal elements of a solid set $\downarrow A$ the *top* of the solid set.

- We shall call a partial function on P *a coherent function on* P iff its values at comparable conditions are equal.

  Notice that a coherent function $\sigma : \downarrow A \longrightarrow W$ on a solid set is determined by its restriction $\sigma \restriction A$ to the top of its domain.

- If σ is a coherent function on a solid set, by A_σ we shall denote the top of $\mathrm{dom}(\sigma)$. If $\vec{\sigma}$ is a finite list $\sigma_1,\ldots,\sigma_n$ of coherent functions on solid sets on a Boolean algebra B, then by $A_{\vec{\sigma}}$ or $A_{\sigma_1\ldots\sigma_n}$ we shall denote the set $\{ a_1 \wedge \cdots \wedge a_n \ : \ a_1 \in A_{\sigma_1}, \ldots, a_n \in A_{\sigma_n} \} \setminus \{0_{\mathsf{B}}\}$. Notice that this set is a common refinement of $A_{\sigma_1},\ldots,A_{\sigma_n}$.

Definition 2.4.2. Let $\mathcal{L}$ be a first order language and $\mathcal{W}$ be a two-valued (class) structure for it. We define *the* B*-power of* $\mathcal{W}$ as the B-valued structure with:

- domain the set $W^{\downarrow\mathsf{B}}$ of coherent functions on maximal solid sets on B^+ with ranges in W.

- for each n-ary function symbol f, an n-ary operation $f^{\mathcal{W}^{\downarrow\mathsf{B}}}$ on $W^{\downarrow\mathsf{B}}$

$$\begin{array}{rccl} f^{\mathcal{W}^{\downarrow\mathsf{B}}} & : (W^{\downarrow\mathsf{B}})^n & \longrightarrow & W^{\downarrow\mathsf{B}} \\ & \vec{\sigma} & \longmapsto & f^{\mathcal{W}^{\downarrow\mathsf{B}}}(\vec{\sigma}) \end{array}$$

  such that

$$\begin{array}{rccl} f^{\mathcal{W}^{\downarrow\mathsf{B}}}(\vec{\sigma}) & : \downarrow A_{\vec{\sigma}}{}^+ & \longrightarrow & W \\ & a & \longmapsto & f^{\mathcal{W}}(\vec{\sigma}(a)) \end{array}$$

[7]Just as in the general construction of Boolean-valued models in the previous section, full completeness of the algebra is not always necessary. The sets whose joins need to be computed are given by the truth values of instances of existential formulas.

- in particular, for each constant symbol c

$$\begin{array}{rccl} c^{\mathcal{W}\downarrow\mathsf{B}} & : \mathsf{B}^+ & \longrightarrow & W \\ & a & \longmapsto & c^{\mathcal{W}} \end{array}$$

- for each n-ary predicate symbol p (including equality):

$$[\![\, p(\vec{\sigma}) \,]\!]^{\mathcal{W}\downarrow\mathsf{B}} = \bigvee \{ \, a \in A_{\vec{\sigma}} \, : \, \mathcal{W} \vDash p(\vec{\sigma}(a)) \, \}$$

The interpretation of more complicated terms and formulas is as it is defined for all Boolean valued structures in definition 2.2.1. It is immediate to see inductively that the domain of the interpretation of each term of the form $f(u_1, ..., u_n)$ is

$$\begin{array}{rcl} (f(u_1,...,u_n))^{\mathcal{W}\downarrow\mathsf{B}} = f^{\mathcal{W}\downarrow\mathsf{B}}\left(u_1^{\mathcal{W}\downarrow\mathsf{B}},...,u_n^{\mathcal{W}\downarrow\mathsf{B}}\right) \; : \; \downarrow A_{u_1^{\mathcal{W}\downarrow\mathsf{B}},...,u_n^{\mathcal{W}\downarrow\mathsf{B}}}{}^+ & \longrightarrow & W \\ a & \longmapsto & f\left(u_1^{\mathcal{W}\downarrow\mathsf{B}}(a),...,u_n^{\mathcal{W}\downarrow\mathsf{B}}(a)\right) \end{array}$$

Proposition 2.4.3. *Let $\varphi(\vec{x})$ be a formula in $\mathcal{L}$ and σ_1,...,$\sigma_n \in W^{\downarrow\mathsf{B}}$. Then:*

$$[\![\, \varphi(\vec{\sigma}) \,]\!]^{\mathcal{W}\downarrow\mathsf{B}} = \bigvee \left\{ a \in A_{\vec{\sigma}} \, : \, \mathcal{W} \vDash \varphi\left(\vec{\sigma}(a)\right) \right\}. \tag{2.14}$$

Proof. Induction on the complexity of φ. If φ is an atomic formula it is immediate.

Let $\varphi = \neg\psi$. Notice that if A_1, A_2 is a partition of a maximal antichain, then $\bigvee A_1 = \neg \bigvee A_2$. Using this we get

$$\begin{aligned} [\![\, \varphi(\vec{\sigma}) \,]\!] &= \neg \, [\![\, \psi(\vec{\sigma}) \,]\!] && \text{(by general Boolean semantics)} \\ &= \neg \bigvee \left\{ a \in A_{\vec{\sigma}} \, : \, \mathcal{W} \vDash \psi\big(\vec{\sigma}(a)\big) \right\} && \text{(by induction hypothesis)} \\ &= \bigvee \left\{ a \in A_{\vec{\sigma}} \, : \, \mathcal{W} \vDash \neg\psi\big(\vec{\sigma}(a)\big) \right\} && \text{(by the above remark)} \\ &= \bigvee \left\{ a \in A_{\vec{\sigma}} \, : \, \mathcal{W} \vDash \varphi\big(\vec{\sigma}(a)\big) \right\} \end{aligned}$$

If $\varphi = \psi \vee \theta$, then:

$$\begin{aligned} [\![\, \varphi(\vec{\sigma}) \,]\!] &= [\![\, \psi(\vec{\sigma}) \,]\!] \vee [\![\, \theta(\vec{\sigma}) \,]\!] \\ &= \bigvee \left\{ a \in A_{\vec{\sigma}} \, : \, \mathcal{W} \vDash \psi\big(\vec{\sigma}(a)\big) \right\} \vee \bigvee \left\{ a \in A_{\vec{\sigma}} \, : \, \mathcal{W} \vDash \theta\big(\vec{\sigma}(a)\big) \right\} \\ &= \bigvee \left\{ a \in A_{\vec{\sigma}} \, : \, \mathcal{W} \vDash \psi\big(\vec{\sigma}(a)\big) \text{ or } \mathcal{W} \vDash \theta\big(\vec{\sigma}(a)\big) \right\} \\ &= \bigvee \left\{ a \in A_{\vec{\sigma}} \, : \, \mathcal{W} \vDash \Big(\psi\big(\vec{\sigma}(a)\big) \vee \theta\big(\vec{\sigma}(a)\big)\Big) \right\}. \\ &= \bigvee \left\{ a \in A_{\vec{\sigma}} \, : \, \mathcal{W} \vDash \varphi\big(\vec{\sigma}(a)\big) \right\}. \end{aligned}$$

Finally, if $\varphi(\vec{x}) = \exists y \psi(y, \vec{x})$. Denote $[\![\, \exists y \psi(y, \vec{\sigma}) \,]\!]^{\mathcal{W}\downarrow\mathsf{B}}$ by m and $\bigvee \{ \, a \in A_{\vec{\sigma}} \, : \, \mathcal{W} \vDash \exists y \psi(y, \vec{\sigma}(a)) \, \}$ by n.

It is easy to see that $m \geq n$. Indeed, pick some $a \in A_{\vec{\sigma}}$ such that $\mathcal{W} \vDash \exists y \psi(y, \vec{\sigma}(a))$ (if there is no such a, the inequality is trivially true, because the right-hand term equals 0_B. Fix a witness $u \in W$ such that $\mathcal{W} \vDash \psi(u, \vec{\sigma}(a))$. Let $\tau = \; \downarrow A_{\vec{\sigma}}{}^+ \times \{u\}$. Then $\mathcal{W} \vDash \psi(\tau(a), \vec{\sigma}(a))$ and $A_{\vec{\sigma}} = A_{\tau\vec{\sigma}}$, so by induction hypothesis, we have $a \leq [\![\, \psi(\tau, \vec{\sigma}) \,]\!] \leq [\![\, \exists y \psi(y, \vec{\sigma}) \,]\!] = m$. By taking a join over all such a, we obtain $n \leq m$.

Now we prove the converse inequality, i.e. that $m \leq n$. Suppose the contrary. Then $0_\mathsf{B} \lneq m \wedge \neg n = b$.

Pick $a' \in A_\sigma$ such that $\mathcal{W} \vDash \neg\exists y \psi(y, \vec{\sigma}(a'))$ and $a' \wedge b \gneq 0_{\mathsf{B}}$. Denote $a' \wedge b$ by a''. Now, $0_{\mathsf{B}} \nleq a'' = b \wedge a' \leq b \leq m$, so $a'' \leq m = \bigvee \left\{ [\![\psi(\tau, \vec{\sigma})]\!] : \tau \in W^{\downarrow \mathsf{B}} \right\}$. Then there is some $\tau \in W^{\downarrow \mathsf{B}}$ such that $0_{\mathsf{B}} \nleq a'' \wedge [\![\psi(\tau, \vec{\sigma})]\!] = a'''$. By the induction hypothesis $\bigvee \left\{ a \in A_{\tau \vec{\sigma}} : \mathcal{W} \vDash \psi(\tau(a), \vec{\sigma}(a)) \right\} = [\![\psi(\tau, \vec{\sigma})]\!] \geq a'''$. Then there exists some $a'''' \in A_{\tau\vec{\sigma}}$ such that $0_{\mathsf{B}} \nleq a''' \wedge a'''' = a$. Then $\mathcal{W} \vDash \psi(\tau(a), \vec{\sigma}(a))$. But τ and $\vec{\sigma}$ are coherent function and $a \leq a'$, so $\tau(a) = \tau(a')$ and $\sigma(a) = \sigma(a')$, so $\mathcal{W} \vDash \psi(\tau(a), \vec{\sigma}(a'))$. A contradiction. □

Proposition 2.4.4. [14, Theorem 1.3]. *If* B *is a complete Boolean algebra, the* B*-power* $\mathcal{W}^{\downarrow \mathsf{B}}$ *satisfies the mixing property and hence is full by 2.2.12.*

Proof. Let A' be an antichain on B and let $\left\{ \sigma_a : a \in A' \right\} \subseteq \mathcal{W}^{\downarrow \mathsf{B}}$ be a family of names. We'll construct a new name σ witnessing the mixing property.

Let A be a maximal antichain on B extending A' and extend (in any way) the family of names to a new one $\left\{ \sigma_a : a \in A \right\}$ indexed by A. For $a \in A$, we define the following auxiliary functions

$$r_a = \left\{ \langle a \wedge b, a \rangle : b \in A_{\sigma_a} \right\} \restriction \mathsf{B}^+$$
$$r = \bigcup \left\{ r_a : a \in A \right\}$$

The domain of r is the refinement of A which looks like A_{σ_a} below each $a \in A$. To phrase the last statement more formally, for each $a \in A$, $\left\{ a \wedge c : c \in \operatorname{dom}(r) \right\}$ is a refinement of $\left\{ a \wedge c : c \in A_{\sigma_a} \right\}$. And for each $c \in \operatorname{dom}(r)$, $r(c)$ is the unique $a \in A$ above c.

Let σ be the name with domain $\downarrow\operatorname{dom}(r)$ and values $\sigma(c) = \sigma_{r(c)}(c)$ for each $c \in \operatorname{dom}(r)$. We want to prove that, for every $a \in A'$, we have $a \leq [\![\sigma = \sigma_a]\!]$, so fix some $a \in A' \subseteq A$.

Notice that for each $a \in A$ and each $c \in A_{\sigma_a \sigma}$ such that $c \leq a$, we have

$$\sigma(c) = \sigma_{r(c)}(c) = \sigma_a(c)$$

And hence

$$\begin{aligned}
[\![\sigma = \sigma_a]\!]^{\mathcal{W}^{\downarrow \mathsf{B}}} &= \bigvee \left\{ c \in A_{\sigma_a \sigma} : \mathcal{W} \vDash \sigma(c) = \sigma_a(c) \right\} \\
&\geq \bigvee \left\{ c \in A_{\sigma_a \sigma} \restriction a : \mathcal{W} \vDash \sigma(c) = \sigma_a(c) \right\} \\
&\bigvee A_{\sigma_a \sigma} \restriction a \\
&= a
\end{aligned}$$

□

Lemma 2.4.5. *The* 2*-valued structure* $\mathcal{W}$ *embeds elementarily into its* B*-power* $\mathcal{W}^{\downarrow \mathsf{B}}$*. More precisely, if* $e = \left\{ \langle 0, 0 \rangle, \langle 1, 1 \rangle \right\}$ *is the unique homomorphism from the trivial Boolean algebra* 2 *into* B*, and*

$$\begin{aligned} j \ : \ W &\longrightarrow W^{\downarrow \mathsf{B}} \\ u &\longmapsto c_u \end{aligned}$$

where $c_u : \mathsf{B} \longrightarrow \{u\}$*, then* $\langle e, j \rangle$ *is an elementary embedding of* $\langle 2, \mathcal{W} \rangle$ *into* $\left\langle \mathsf{B}, \mathcal{W}^{\downarrow \mathsf{B}} \right\rangle$*.*

Proof. $\operatorname{dom}(j) = W$ by definition. If f is an n-ary function symbol and $\vec{x}$ in an n-tuple of elements of W, then we do have $j(f^{\mathcal{W}}(\vec{x})) = f^{\mathcal{W}^{\downarrow \mathsf{B}}}(\vec{c_x})$, i.e. (2.6) in the case where $\Phi = j$ is a function. Indeed

$$\operatorname{dom}\left(f^{\mathcal{W}^{\downarrow \mathsf{B}}}(\vec{c_x}) \right) = \downarrow A_{\vec{c_x}} = \mathsf{B}$$

and its value is $f^{\mathcal{W}}(\vec{x})$ everywhere, thus $f^{\mathcal{W}^{\downarrow \mathsf{B}}}$ coincides with

$$j(f^{\mathcal{W}}(\vec{x})) = c_{f^{\mathcal{W}}(\vec{x})}$$

In light of proposition 2.4.3, for any formula $\varphi(\vec{x})$ in $\mathcal{L}$ and parameters $\vec{u}$ in W we have

$$\begin{aligned}
[\![\, \varphi(j(\vec{u})) \,]\!]^{\mathcal{W}^{\downarrow \mathsf{B}}} &= [\![\, \varphi(\vec{c_u})) \,]\!]^{\mathcal{W}^{\downarrow \mathsf{B}}} \\
&= \bigvee \{\, a \in \mathsf{B} \,:\, \mathcal{W} \vDash \varphi(\vec{u}) \,\} \\
&= [\![\, \varphi(\vec{x}) \,]\!]^{\mathcal{W}} = \begin{cases} 1 & , \text{if } \; \mathcal{W} \vDash \varphi(\vec{x}) \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

□

Definition 2.4.6. Let $\mathcal{W}$ be a first order structure, B be a complete Boolean algebra, and G be an ultrafilter on B. Then the Tarski quotient $\mathcal{W}^{\downarrow \mathsf{B}}/_G$ by G of the B-power of $\mathcal{W}$ is called the *(Tarski) ultrapower* of $\mathcal{W}$ by G.

Corollary 2.4.7. *In view of lemma 2.4.5, Every structure is isomorphic to any of its Tarski ultrapowers.*

2.5 Boolean-valued models of Set Theory

We'll now describe the standard way to construct Boolean-valued models of set theories such as ZFC. A name in such a model is a function on a set of (previously constructed) names which takes values in the underlying Boolean algebra. The intuition behind this is that the set whose name is the name $\sigma \in \mathrm{dom}(\tau)$ is an element of the set whose name is τ "with probability" at least $\tau(\sigma)$. Meaning that if an ultrafilter G on the underlying Boolean algebra contains the condition $\tau(\sigma)$, then in the quotient structure we'll have $[\,\tau\,]_G \in^{\mathcal{M}/_G} [\,\sigma\,]_G$.

We first define the standard Boolean-valued structure $V^{\mathsf{B}}_{\in} = \left(V^{\mathsf{B}}, \in^{V^{\mathsf{B}}}, =^{V^{\mathsf{B}}}\right)$ for the core language $\{\in\}$, which we shall later extend to a structure V^{B}_{+} for the extended language $\in_{\Delta_0^+}$.

Definition 2.5.1 ($V^{\mathsf{B}}_{\in}$; [3, (1.6)]; [10, (14.15)])**.** Let V model some set theory ST extending ZFC and $\mathsf{B} \in V$ be a complete Boolean algebra in V. *The* B*-extension of* V is the B-valued structure $V^{\mathsf{B}}_{\in}$ with

- domain:

$$\begin{aligned}
V^{\mathsf{B}} \;&=\; \bigcup \left\{\, V^{B}_{\alpha} \,:\, \alpha \in \mathrm{Ord} \,\right\} = \left\{\, \sigma \,:\, \exists \alpha \left(\sigma \in V^{\mathsf{B}}_{\alpha}\right) \right\} \;, \quad \text{where} \\
V^{\mathsf{B}}_{\alpha} \;&=\; \left\{\, \sigma \,:\, (\exists \beta < \alpha)(\sigma \text{ is a partial function from } V^{\mathsf{B}}_{\beta} \text{ to } \mathsf{B}) \,\right\}
\end{aligned}$$

- equality function $=^{\mathsf{B}}$:

$$\begin{aligned}
[\![\, \sigma = \tau \,]\!]^{V^{\mathsf{B}}_{\in}} \;&=\; [\![\, \sigma \subseteq \tau \,]\!]^{V^{\mathsf{B}}_{\in}} \wedge \; [\![\, \sigma \supseteq \tau \,]\!]^{V^{\mathsf{B}}_{\in}} \;, \quad \text{where} \\
[\![\, \sigma \subseteq \tau \,]\!]^{V^{\mathsf{B}}_{\in}} \;&=\; \bigwedge \left\{ \neg\sigma(\rho) \vee [\![\, \rho \in \tau \,]\!]^{V^{\mathsf{B}}_{\in}} \,:\, \rho \in \mathrm{dom}(\sigma) \right\}
\end{aligned}$$

- membership $\in^{\mathsf{B}}$:

$$[\![\, \sigma \in \tau \,]\!]^{V^{\mathsf{B}}_{\in}} = \bigvee \left\{ [\![\, \rho = \sigma \,]\!]^{V^{\mathsf{B}}_{\in}} \wedge \tau(\rho) \,:\, \rho \in \mathrm{dom}(\tau) \right\}$$

The elements of V^{B} are called B-names.

Remark 2.5.2. What appear as circular definitions above are in fact recursive definition: the interpretations of the predicate symbols are defined by recursion on the so called *square order* $<^2$ on Ord^2, which is defined as follows: $\langle\, \alpha_0 \,, \beta_0 \,\rangle <^2 \langle\, \alpha_1 \,, \beta_1 \,\rangle$ iff $\max\{\alpha_0, \beta_0\} < \max\{\alpha_1, \beta_1\}$ or $\max\{\alpha_0, \beta_0\} = \max\{\alpha_1, \beta_1\}$ and $\langle\, \alpha_0 \,, \beta_0 \,\rangle$ is lexicographically below $\langle\, \alpha_1 \,, \beta_1 \,\rangle$.

When working with this kind of structure it is customary to denote truth values in it as $[\![\varphi]\!]_{\mathsf{B}}$, instead of $[\![\varphi]\!]^{V_\in^{\mathsf{B}}}$. Or, in situations we feel no confusion can arise, simply by $[\![\varphi]\!]$.

Note that, indeed, whenever $\sigma \in \mathrm{dom}(\tau)$, we have that the "probability" $[\![\sigma \in \tau]\!]$ that (the set named by) σ belongs to (the set named by) τ is at least $\tau(\sigma)$. At the same time the value $[\![\sigma \in \tau]\!]$ may be strictly positive even if σ is not in the domain of τ. This seems to raise an issue about bounded quantifiers.

In a classical two-valued Tarski structure, to compute the truth value of a formula $(\exists x \in u)\varphi(x, u, v)$ starting with a bounded quantifier, we need to consider only the truth values of those instances $\varphi(w, u, v)$ of $\varphi(x, u, v)$ where $w \in u$, i.e. "few" ones, set-many ones. In Boolean-valued structures, this restriction amounts to those instances $\varphi(\sigma, \tau, \rho)$ of $\varphi(x, \tau, \rho)$ such that $[\![\sigma \in \tau]\!]$ is positive. However, there may very well be class-many names $\sigma \in V^{\mathsf{B}}$ such that $[\![\sigma \in \tau]\!] \geq 0_{\mathsf{B}}$. But, as intuition would suggest, it turns out we can further restrict the computation not only to those σ that belong to τ with positive probability, but to just the elements of the domain of τ. In other words, we have:

Proposition 2.5.3. [3, Corollary 1.18] .

$$[\![(\exists x \in \tau)\varphi(x, \tau, \rho)]\!]_{\mathsf{B}} = \bigvee \left\{ \tau(\sigma) \wedge [\![\varphi(\sigma, \tau, \rho)]\!]_{\mathsf{B}} \;:\; \sigma \in \mathrm{dom}(\tau) \right\}$$

Theorem 2.5.4 ([3, Theorem 1.33], [10, Theorem 14.24])**.** *$V_\in^{\mathsf{B}}$ is a B-valued model of ZFC, for any complete Boolean algebra $\mathsf{B} \in V$.* □

Definition 2.5.5 (Mixtures)**.** Let A be an antichain on B and $\langle \tau_a : a \in A \rangle$ be an A-indexed family of B-names. We define *the mixture* $\sum_{a\in A} a \cdot \tau_a$ *of* $\langle \tau_a : a \in A \rangle$ *with respect to* A to be the B-name τ with domain $\bigcup \left\{ \mathrm{dom}(\tau_a) : a \in A \right\}$ and values

$$\tau(\sigma) = \bigvee \left\{ a \wedge [\![\sigma \in \tau_a]\!] \;:\; a \in A \right\}$$

Remark 2.5.6. Mixtures are absolute between transitive models of ZFC. More precisely, the sentence stating that τ is the mixture of the function $f = \left\{ \langle a, \tau_a \rangle : a \in A \right\}$, where the underlying Boolean algebra is B, is a Δ_0 sentence with parameters τ, f and B.

Remark 2.5.7 (Redundancy for mixtures). Notice that if A' is a refinement of A and we have two families $\langle \sigma_{a'} : a' \in A' \rangle$ and $\langle \tau_a : a \in A \rangle$ indexed by A' and A respectively, such that for each $A' \ni a' \leq a \in A$ we have $\sigma_{a'} = \tau_a$, then we have

$$\left[\!\!\left[\sum_{a\in A'} a \cdot \sigma_a \;=\; \sum_{a\in A} a \cdot \tau_a \right]\!\!\right] \;=\; 1_{\mathsf{B}}$$

In loose terms, non-injectivities of mixing families are redundant.

Theorem 2.5.8 (Mixing and fullness for V^{B}; [3, Mixing Lemma 1.25]; [10, Lemma 14.18])**.** *If $\mathsf{B} \in V$ is a cba, then $V_\in^{\mathsf{B}}$ has the mixing property, witnessed by the mixtures defined above in 2.5.5, and hence is golden by 2.2.12.* □

Remark 2.5.9. $V_\in^{\mathsf{B}}$ has the finite mixing property ($<\aleph_0$-mixing property) for any Boolean algebra B.

Definition 2.5.10 (Canonical names)**.** Let B be a Boolean algebra. For each set x in the ground model V, by $\check{x}$ we denote *the canonical* B*-name for* x defined recursively as:

$$\check{x} = \left\{ \langle \check{y}, 1_{\mathsf{B}} \rangle \;:\; y \in x \right\}.$$

We now extend $V_\in^{\mathsf{B}}$ to a structure for $\in_{\Delta_0^+}$. Recall definition 2.1.13.

Definition 2.5.11. Let B be a Boolean algebra and $\mathcal{M} = (M, E)$ be a golden B-valued model of $\mathsf{ZFC} - \mathsf{P}$. We extend $\mathcal{M}$ to the structures $\mathcal{M}_{\Delta_0}$ and $\mathcal{M}_+$ respectively for the basic and extended languages for Set Theory by defining the interpretations of the additional symbols as follows:

- **Predicates.** For each bounded formula $\varphi(x_1, \ldots, x_n)$ in the core language $\{\in\}$, define the interpretation of the predicate symbol p_φ to be

$$[\![p_\varphi(\sigma_1, \ldots, \sigma_n)]\!] = [\![\varphi(\sigma_1, \ldots, \sigma_n)]\!]$$

- **Constants.** Let $\varphi(x)$ be a bounded formula in the core language $\{\in\}$ such that $\mathsf{ZFC} \vdash \exists! x \varphi(x)$. Appealing to the properties of Boolean uniqueness 2.2.14, we let the interpretation $c_\varphi^{\mathcal{M}_+}$ of c_φ in $\mathcal{M}_+$ be a name such that $\left[\!\!\left[\varphi(c_\varphi^{\mathcal{M}_+}) \right]\!\!\right] = 1_\mathsf{B}$ and for each name $\sigma \in M$ we have $[\![\varphi(\sigma)]\!] \leq \left[\!\!\left[\sigma = c_\varphi^{\mathcal{M}_+} \right]\!\!\right]$.

  Let $c_\varphi^{V^\mathsf{B}}$ be the canonical name $\check{u}$ of the unique set u in V that has the property φ (according to V).

  In V^B for example, using 2.5.15, we can choose $c_\varphi^{V_+^\mathsf{B}}$ to be the canonical B-name $\check{u}$ of the set u in V that has the property φ. And when in the place of $\mathcal{M}$ we have $\mathcal{W}^{\downarrow\mathsf{B}}$, we can choose $c_\varphi^{\mathcal{W}_+^{\downarrow\mathsf{B}}}$ to be c_u.

- **Functions.** Similarly for the the function symbols of positive arity. Let G be a Gödel operation (unary or binary) and $\varphi_G(\vec{x}, y)$ be a Δ_0 formula in the core language which states that y is the result of the operation G on $\vec{x}$. Then $\mathsf{ZFC} - \mathsf{P} \vdash \forall \vec{y} \exists! x \varphi_G(\vec{x}, y)$. But then since $V^\mathsf{B} \vDash \mathsf{ZFC}$, we have

$$\begin{aligned} 1_\mathsf{B} &= [\![\forall \vec{x} \exists! y \varphi_G(\vec{x}, y)]\!]_\mathsf{B} \\ &= \bigwedge_{\sigma_1 \in V^\mathsf{B}} \cdots \bigwedge_{\sigma_n \in V^\mathsf{B}} [\![\exists! y \varphi_G(\sigma_1, \ldots, \sigma_n, y)]\!]_\mathsf{B} \end{aligned}$$

  Again by proposition 2.2.14 on the meaning of Boolean uniqueness, for all names $\tau_1, \ldots, \tau_n \in M$ we have a name σ such that for each name $\rho \in V^\mathsf{B}$ we have $[\![\varphi(\rho, \vec{\tau})]\!] \leq [\![\rho = \sigma]\!]$. Pick one such name of minimal rank and denote it by $\sigma_{\varphi_G, \vec{\tau}}$.

  Collect those and set the interpretation $f_G^\mathcal{M}$ of f_G to be

$$\begin{array}{rcccl} f_G^{\mathcal{M}_+} & : & M^n & \longrightarrow & M \\ & & \vec{\tau} & \longmapsto & \sigma_{\varphi_G, \vec{\tau}} \end{array}$$

Notation 2.5.12. We shall denote the structures for the basic language $\in_{\Delta_0}$ and for the extended language $\in_{\Delta_0^+}$ obtained from $(V^\mathsf{B}, \in^{V^\mathsf{B}})$ in the way described above respectively by $V_{\Delta_0}^\mathsf{B}$ and V_+^B.

Lemma 2.5.13. *Let f_φ be an n-ary function symbol with interpretation in V^B defined as above for a bounded formula φ in the core language (like the described above constants and Gödel functions). Then for every $\sigma_i, \mu \in V^\mathsf{B}$ it holds that*

$$[\![f_\varphi(\vec{\sigma}) = \mu]\!]^{V_+^\mathsf{B}} = [\![\varphi(\vec{\sigma}, \mu)]\!]^{V_+^\mathsf{B}}$$

Proof. The direction $\geq$ we have from the definition 2.5.11 of the interpretation $f_\varphi^{V^\mathsf{B}}$ of f in V^B. Moreover we have

$$\left[\!\!\left[f_\varphi^{V_+^\mathsf{B}}(\vec{\sigma}) = \mu \right]\!\!\right]^{V_+^\mathsf{B}} = \left[\!\!\left[f_\varphi^{V_+^\mathsf{B}}(\vec{\sigma}) = \mu \right]\!\!\right]^{V_+^\mathsf{B}} \wedge 1_\mathsf{B} = \left[\!\!\left[f_\varphi^{V_+^\mathsf{B}}(\vec{\sigma}) = \mu \right]\!\!\right]^{V_+^\mathsf{B}} \wedge \left[\!\!\left[\varphi\Big(\vec{\sigma}, f_\varphi^{V_+^\mathsf{B}}(\vec{\sigma})\Big) \right]\!\!\right]^{V_+^\mathsf{B}} \leq [\![\varphi(\vec{\sigma}, \mu)]\!]^{V_+^\mathsf{B}}$$

□

Corollary 2.5.14. *In particular, for each constant c_φ in the extended language and every name $\mu \in V^\mathsf{B}$ we have*

$$[\![c_\varphi = \mu]\!]^{V_+^\mathsf{B}} = [\![\varphi(\mu)]\!]^{V_\in^\mathsf{B}}$$

Lemma 2.5.15. *$(V, \in_{\Delta_0})$ embeds weakly Σ_1-elementarily into $V^{\mathsf{B}}_{\Delta_0}$ via the Boolean embedding $\langle e, j \rangle$ where $e = \{\langle 0_2, 0_{\mathsf{B}} \rangle, \langle 1_2, 1_{\mathsf{B}} \rangle\}$ is the only homomorphism of the trivial Boolean algebra 2 into B, j is the map which sends each set $u \in V$ to its canonical B-name $\check{u}$.*

Proof. The claim is proved by induction on the construction of φ, according to the well-order mentioned in remark 2.1.14. The atomic case, negation and disjunction are trivial, so consider a bounded formula φ be $(\exists y \in x)\varphi(y, x, z_1, \dots, z_n)$ and $u, v_1..., v_n \in V$ in the core language. Then by proposition 2.5.3 we have

$$\begin{aligned}
\left[\!\left[(\exists y \in \check{u})\varphi(y, \check{u}, \vec{\check{v}}) \right]\!\right]_{\mathsf{B}} &= \bigvee \left\{ \check{u}(\sigma) \wedge \left[\!\left[\varphi(\sigma, \check{u}, \vec{\check{v}}) \right]\!\right]_{\mathsf{B}} : \sigma \in \mathrm{dom}(\check{u}) \right\} \\
&= \bigvee \left\{ \left[\!\left[\varphi(\check{w}, \check{u}, \vec{\check{v}}) \right]\!\right]_{\mathsf{B}} : w \in u \right\} \\
&= \bigvee \left\{ [\![\varphi(w, u, \vec{v})]\!]^V : w \in u \right\} && \text{(by ih)} \\
&= [\![(\exists y \in u)\varphi(y, u, \vec{v})]\!]^V
\end{aligned}$$

We now consider the additional nonlogical symbols in the extended language:

$$\left[\!\left[p_\varphi(\vec{\check{u}}) \right]\!\right]^{V^{\mathsf{B}}_{+}} = \left[\!\left[\varphi(\vec{\check{u}}) \right]\!\right]^{V^{\mathsf{B}}_{\in}} = [\![\varphi(\vec{u})]\!]^{(V,\in)} = [\![p_\varphi(\vec{u})]\!]^{V_+}$$

$$[\![c_\varphi = u]\!]^{V_+} = [\![\varphi(u)]\!]^{(V,\in)} = [\![\varphi(\check{u})]\!]^{V^{\mathsf{B}}_{\in}} = [\![c = \check{u}]\!]^{V^{\mathsf{B}}_{+}}$$

$$[\![f_G(\vec{u}) = v]\!]^{V_+} = [\![\varphi_G(\vec{u}, v)]\!]^{(V,\in)} = \left[\!\left[\varphi_G\big(\vec{\check{u}}, \check{v}\big) \right]\!\right]^{V^{\mathsf{B}}_{\in}} = \left[\!\left[f_G\big(\vec{\check{u}}\big) = \check{v} \right]\!\right]^{V^{\mathsf{B}}_{+}}$$

Finally, for unbounded existential quantifiers we have

$$\begin{aligned}
\left[\!\left[\exists x\varphi(x, \vec{\check{u}}) \right]\!\right]^{V^{B}_{+}} &= \bigvee \left\{ \left[\!\left[\varphi(\sigma, \vec{\check{u}}) \right]\!\right]^{V^{\mathsf{B}}_{+}} : \sigma \in V^{\mathsf{B}} \right\} \\
&\geq \bigvee \left\{ \left[\!\left[\varphi(\check{v}, \vec{\check{u}}) \right]\!\right]^{V^{\mathsf{B}}_{+}} : v \in V \right\} \\
&= \bigvee \left\{ [\![\varphi(v, u)]\!]^{V_+} : v \in V \right\} \\
&= [\![\exists x\varphi(x, \vec{u})]\!]^{V}
\end{aligned}$$

□

Corollary 2.5.16. *By remark 2.3.3 and 2.5.4 we have that V_+ embeds Δ_0-elementarily into V^{B}_{+}.*

2.5.1 Genericity and Ill-foundedness

Definition 2.5.17. Let $\mathcal{M} = \langle M, \in, = \rangle$ be a transitive model of ZFC. Let $\mathsf{B} \in M$ be a complete Boolean algebra according to $\mathcal{M}$. By M^{B} we shall denote the relativization $(V^{\mathsf{B}})^M$ of V^{B} to M; and by $\mathcal{M}^{\mathsf{B}}$ – the structure $\left(M^{\mathsf{B}}, (\in^{\mathsf{B}})^{\mathcal{M}}, (=^{\mathsf{B}})^{\mathcal{M}}\right)$.

Theorem 2.5.18 (Cohen-Solovay-Scott-Vopenka [3, Theorem 4.1])**.** *Let $\mathcal{M} = \langle M, \in, = \rangle$ be a transitive standard model of* ZFC *and $\mathsf{B} \in M$ be a complete Boolean algebra according to $\mathcal{M}$. Let $\mathcal{M}^{\mathsf{B}}$ denote the Boolean-valued structure that M thinks is V^{B}. Let G be any ultrafilter on B. Then the quotient $\mathcal{M}^{\mathsf{B}}/_G$ defined as in 2.2.7, is a Tarski model of* ZFC. □

Remark 2.5.19. A model $\mathcal{M}^{\mathsf{B}}$ of the kind mentioned above is full according to V as well, despite the fact that the mixing property is not absolute. Indeed, with antichains that belong to $V \setminus M$ we can build in V mixtures of names in M^{B} that do not belong to M^{B}. However, the mixtures relevant to the issue of fullness are induced by the truth values of existential formulas $\exists x\varphi(x)$ and their instances $\varphi(\sigma)$, as in 2.2.12, which makes those mixtures definable in M.

There is a way to ensure that certain quotients of Boolean-valued models of set theory of the form V^{B} are well-founded and hence isomorphic to transitive $\in$-structures. This is achieved by using a special kind of ultrafilters, called *generic* ultrafilters.

Definition 2.5.20. Let M be a model of ZFC and $P \in M$ be a partial order. A filter G on P is called *M-generic on P* iff it meets [8] every dense subset of P, which is an element of M.

Remark 2.5.21. We could equivalently define genericity by requiring that the filter meet every *maximal antichain*. This is due to a natural correspondence between maximal antichains and dense sets: the downward closure of a maximal antichain is dense, and every dense set has, by Zorn's lemma, a subset which is a maximal antichain.

Lemma 2.5.22. *Let $\mathcal{M}$ be a countable standard* [9] *model of* ZFC *and* $\mathsf{P} = \langle P, \leq \rangle \in M$ *be a partial order in $\mathcal{M}$. Then for each condition p of* P*, there exists an M-generic filter G on* P *that contains p. Moreover, if* P *is atomless, then such a filter is guaranteed to not belong to M. Moreover, if* $\mathsf{P} = \mathsf{B}^+$ *for some Boolean algebra* B*, then G is an ultrafilter on* B. □

Proof. Fix some condition p. Since M is countable, only countably many dense subsets of P belong to M. We can enumerate them as $\{ D_n : n < \omega \}$. Then construct recursively a decreasing ω-sequence $p = q_0 \geq q_1 \geq q_2 \geq \dots$ such that for each $n < \omega$ we have $q_n \in D_n$. Then the upward closure of this sequence is a filter on P that meets every dense subset of P that belongs to M.

Notice that since all conditions of a filter G on P are mutually compatible, if P is atomless, then the complement $\{ p \in P : p \notin G \}$ of G to P is dense in P. If G belongs to M, then D belongs to M. But if G is M-generic over P, then G meets D, which is an obvious contradiction.

Suppose some filter G on a Boolean algebra $\mathsf{B} \in M$ is not an ultrafilter, hence does not meet $\{a, \neg a\}$ for some $a \in \mathsf{B}^+$. Take any dense in B set E. Then the set $D = E \cap \downarrow\{a, \neg a\}$ is dense, and G, being upward closed, misses it. Thus G is not M-generic. □

Corollary 2.5.23. *For a countable transitive model M, the set of M-generic ultrafilters for* $\mathsf{B} = \mathsf{RO}^{\mathcal{M}}(\mathsf{P})$ *are dense in $St(\mathsf{B})$.*

Definition 2.5.24. Let M be a transitive model of set theory, $\mathsf{B} \in M$ be a cba according to $\mathcal{M}$ and G be an ultrafilter on B. We define by recursion on the rank of names $\sigma \in M$:

$$\tau_G = \left\{ \sigma_G : (\exists a \in G)\big(\langle \sigma, a \rangle \in \tau\big) \right\}$$

and

$$M[G] = \{ \tau_G : \tau \in M \}$$

Theorem 2.5.25. [3, Lemma 4.7]. *Assume $\mathcal{M} = \langle M, \in, = \rangle$ is a transitive model of* ZFC. *Let* $\mathsf{B} \in M$ *be a complete Boolean algebra in $\mathcal{M}$ and G be an M-generic filter for* B. *Then:* $\mathcal{M}^{\mathsf{B}}/_G = ((V^{\mathsf{B}})^{\mathcal{M}})/_G = \left\langle M^{\mathsf{B}}/_G, \in^{M^{\mathsf{B}}/_G}, =^{M^{\mathsf{B}}/_G} \right\rangle$*, defined as in 2.2.7, is a well-founded structure isomorphic to the transitive model $\langle M[G], \in, = \rangle$, via the map $[\sigma]_G \longmapsto \sigma_G$.* □

Theorem 2.5.26 (Cohen's forcing theorem; [3, Lemma 4.11], [10, Theorem 14.6, Theorem 14.29])**.** *Assume $\langle M, \in, =, \subseteq \rangle$ models* ZFC *and M is transitive. Let* B *be a complete Boolean algebra in $\langle M, \in, =, \subseteq \rangle$, G be a M-generic ultrafilter on* B*. Then:*

---

[8] We say that a set *meets* a set iff they have a nonempty intersection

[9] A model of set theory $\langle M, E \rangle$ is called *standard* if its membership relation E is the real membership relation $\in$ of the ambient set-theoretic universe V, restricted to M.

1. $\langle M[G], \in, \subseteq, = \rangle$ *is isomorphic to* $\left\langle M^{\mathsf{B}}/_G, \in^{M^{\mathsf{B}}}/_G, \subseteq^{M^{\mathsf{B}}}/_G, =^{M^{\mathsf{B}}}/_G \right\rangle$ *via the map* $\tau_G \longmapsto [\tau]_G$.

2. $M[G] \models \varphi(\vec{\tau}_G)$ *iff* $[\![\varphi(\vec{\tau})]\!] \in G$.

3. $M \models b \leq_{\mathsf{B}} [\![\varphi(\vec{\tau})]\!]$ *iff* $M[G] \models \varphi(\vec{\tau}_G)$, *for all M-generic filters G on B such that $b \in G$.*

Conversely, forcing with a non-generic ultrafilter yields an ill-founded model. In fact, all we need is an infinite antichain, not necessarily maximal, not met by the ultrafilter, to have in the forcing extension an infinite decreasing sequence.

Theorem 2.5.27. *Let B be a Boolean algebra, A be an infinite antichain and G be an ultrafilter disjoint from A but containing*[10] *its join $\bigvee A$. Then $V^{\mathsf{B}}/_G$ is ill-founded.*

Proof. Let $\langle a_n : n < \omega \rangle$ be an enumeration[11] in type ω of countably many elements of A. For each $n < \omega$, denote $\bigvee \{ a_m : m \leq n \}$ by A_n, $\bigvee A$ by a, and $A_n \wedge a$ by A^*_n. $\breve{G}$ is upward directed, so it contains each A_n. Then $G \ni \neg A_n$ and $G \ni A^*_n$.

Now we want to define a name τ_n for each $n < \omega$ such that each

$$a_m \leq [\![\tau_n = (m{-}n)\check{}]\!] \tag{2.15}$$

where $m - n$ equals 0 whenever $n > m$.

By 2.2.10, this is satisfied if we define each τ_n to be the mixture

$$\tau_n = \sum_{k<\omega} a_k \cdot (k-n)\check{}$$

Let's calculate the values of τ_n explicitly using the definition of mixtures:

$$\begin{aligned}
\tau_n(\check{m}) &= \bigvee \left\{ a_k \wedge [\![m \in (k-n)\check{}]\!] \,:\, k < \omega \right\} \\
&= \bigvee \{ a_k \,:\, m < k - n \} \\
&= \bigvee \{ a_{k+n} \,:\, k > m \} \\
&= \bigvee \{ a_{m+n+1}, a_{m+n+2}, \dots \} \\
&= A^*_{m+n}
\end{aligned}$$

Let's now present in a more convenient form our defining forcing conditions (2.15) for τ_n

$$\begin{aligned}
a_m \leq [\![\tau_n = (m-n)\check{}]\!] \quad &\text{iff} \quad a_{m+n} \leq [\![\tau_n = ((m+n)-n)\check{}]\!] \\
&\text{iff} \quad a_{m+n} \leq [\![\tau_n = \check{m}]\!] \\
&\text{iff} \quad a_{m+n+1} \leq [\![\tau_{n+1} = \check{m}]\!]
\end{aligned}$$

$$\begin{aligned}
[\![\tau_{n+1} \in \tau_n]\!] &= \bigvee \left\{ \tau_n(\sigma) \wedge [\![\sigma = \tau_{n+1}]\!] \,:\, \sigma \in \operatorname{dom}(\tau_n) \right\} \\
&= \bigvee \left\{ \tau_n(\check{m}) \wedge [\![\tau_{n+1} = \check{m}]\!] \,:\, m < \omega \right\} \\
&\geq \bigvee \left\{ A^*_{m+n} \ \wedge\ a_{m+n+1} \,:\, m < \omega \right\} \\
&= \bigvee \{ a_{m+n+1} \,:\, m < \omega \} \\
&= \bigvee \{ a_{n+1}, a_{n+2}, \dots \} \\
&= A^*_n \in G
\end{aligned}$$

[10] Convention 1.1.1(1)

[11] An enumeration is understood to be injective.

Notice that for every $\imath, m \leq \omega$ we have

$$\begin{aligned}\llbracket\, \hat{\imath} \in \tau_m \,\rrbracket &= \bigvee \left\{ \llbracket\, \sigma = \hat{\imath} \,\rrbracket \wedge \tau_m(\sigma) \,:\, \sigma \in \operatorname{dom}(\tau_m) \right\} \\ &= \bigvee \left\{ \llbracket\, \hat{\jmath} = \hat{\imath} \,\rrbracket \wedge \tau_m(\hat{\jmath}) \,:\, \jmath \leq \omega \right\} \\ &= \tau_m(\hat{\imath}) = A^*_{m+\imath}\end{aligned}$$

Now, for any $n < m$ we have $A^*_m \leq A^*_n$ and

$$\begin{aligned}\llbracket\, \tau_n = \tau_m \,\rrbracket \leq \llbracket\, \tau_n \subseteq \tau_m \,\rrbracket &= \bigwedge \left\{ \neg\tau_n(\check{\imath}) \vee \llbracket\, \check{\imath} \in \tau_m \,\rrbracket \,:\, \imath < \omega \right\} \\ &= \bigwedge \left\{ \neg A^*_{n+\imath} \vee A^*_{m+\imath} \,:\, \imath < \omega \right\} \\ &= 0_{\mathsf{B}} \notin G\end{aligned}$$

thus indeed all $[\tau_n]_G$ are distinct and form an infinite decreasing $\in^{\mathsf{B}/G}$-chain. □

2.5.2 Boolean morphisms for V^{B}

Notation 2.5.28. Let B and C be Boolean algebras such that V^{B} and V^{C} are full. Let $e : \mathsf{B} \longrightarrow \mathsf{C}$ be a complete homomorphism. By $\hat{e}$ we shall denote the map:

$$\begin{aligned}\hat{e} \;:\; V^{\mathsf{B}} &\longrightarrow V^{\mathsf{C}} \\ \tau &\longmapsto \left\{ \left\langle \hat{e}(\sigma), \bigvee \left\{ e(\tau(\sigma')) \,:\, \sigma' \in \operatorname{dom}(\tau) \;\&\; \hat{e}(\sigma') = \hat{e}(\sigma) \right\} \right\rangle \,:\, \sigma \in \operatorname{dom}(\tau) \right\}\end{aligned}$$

If e is injective, then the above definition can be presented by the more readable:

$$\begin{aligned}\hat{e} \;:\; V^{\mathsf{B}} &\longrightarrow V^{\mathsf{C}} \\ \tau &\longmapsto \left\{ \left\langle \hat{e}(\sigma), e(a) \right\rangle \,:\, \langle \sigma, a \rangle \in \tau \right\}\end{aligned}$$

Lemma 2.5.29. [26, Proposition 4.1.12]. *Let $e : \mathsf{B} \longrightarrow \mathsf{C}$ be a complete homomorphism. Then $(e, \hat{e})$ is a weakly Σ_1-elementary from $\left(\mathsf{B}, V^{\mathsf{B}}_+\right)$ to $\left(\mathsf{C}, V^{\mathsf{C}}_+\right)$. Moreover, if e is onto C and V^C_+ is golden, then $(e, \hat{e})$ is Σ_1-elementary.*

Proof. Induction on the construction of φ. For atomic formulas we proceed by further induction on the rank of σ, τ.

$$\begin{aligned}
[\![\, \hat{e}(\sigma) \in \hat{e}(\tau) \,]\!]_{\mathsf{C}} &= \bigvee_{\pi \in \mathrm{dom}(\hat{e}(\tau))} (\hat{e}(\tau))(\pi) \wedge [\![\, \pi = \hat{e}(\sigma) \,]\!]_{\mathsf{C}} \\
&= \bigvee_{\rho \in \mathrm{dom}(\tau)} (\hat{e}(\tau))(\hat{e}(\rho)) \wedge [\![\, \hat{e}(\rho) = \hat{e}(\sigma) \,]\!]_{\mathsf{C}} \\
&= \bigvee_{\rho \in \mathrm{dom}(\tau)} \left(\bigvee_{\rho' \sim \rho} e(\tau(\rho')) \right) \wedge [\![\, \hat{e}(\rho) = \hat{e}(\sigma) \,]\!]_{\mathsf{C}} \;, \\
&\text{where by } \rho' \sim \rho \text{ we've denoted } \rho' \in \mathrm{dom}(\tau) \;\&\; \hat{e}(\rho') = \hat{e}(\rho) \\
&= \bigvee_{\rho \in \mathrm{dom}(\tau)} \left(\bigvee_{\rho' \sim \rho} e(\tau(\rho')) \;\wedge\; [\![\, \hat{e}(\rho) = \hat{e}(\sigma) \,]\!]_{\mathsf{C}} \right) \\
&= \bigvee_{\rho \in \mathrm{dom}(\tau)} \left(\bigvee_{\rho' \sim \rho} e(\tau(\rho')) \;\wedge\; [\![\, \hat{e}(\rho') = \hat{e}(\sigma) \,]\!]_{\mathsf{C}} \right) \\
&= \bigvee_{\rho \in \mathrm{dom}(\tau)} e(\tau(\rho)) \;\wedge\; [\![\, \hat{e}(\rho) = \hat{e}(\sigma) \,]\!]_{\mathsf{C}} \\
&= \bigvee_{\rho \in \mathrm{dom}(\tau)} e(\tau(\rho)) \;\wedge\; [\![\, \rho = \sigma \,]\!]_{\mathsf{B}} \qquad \text{(by ih)} \\
&= [\![\, \sigma \in \tau \,]\!]_{\mathsf{B}}
\end{aligned}$$

and similarly for equality. For φ a negation or disjunction the proof is immediate since e is a homomorphism and hence preserves negation and join. Now we prove the induction step for bounded quantifiers utilizing 2.5.3:

$$\begin{aligned}
[\![\, (\exists x \in \hat{e}(\tau))\varphi(x, \hat{e}(\tau), \hat{e}(\vec{\sigma})) \,]\!]_{\mathsf{C}} &= \bigvee_{\pi \in \mathrm{dom}(\hat{e}(\tau))} (\hat{e}(\tau))(\pi) \wedge [\![\, \varphi(\pi, \hat{e}(\tau), \hat{e}(\vec{\sigma})) \,]\!]_{\mathsf{C}} \\
&= \bigvee_{\rho \in \mathrm{dom}(\tau)} (\hat{e}(\tau))(\hat{e}(\rho)) \wedge [\![\, \varphi(\hat{e}(\rho), \hat{e}(\tau)) \,]\!]_{\mathsf{C}} \\
&= \bigvee_{\rho \in \mathrm{dom}(\tau)} \left(\bigvee_{\rho' \sim \rho} e(\tau(\rho')) \right) \wedge [\![\, \varphi(\hat{e}(\rho), \hat{e}(\tau), \hat{e}(\vec{\sigma})) \,]\!]_{\mathsf{C}} \,, \\
&\text{where by } \rho' \sim \rho \text{ we've denoted } \rho' \in \mathrm{dom}(\tau) \;\&\; \hat{e}(\rho') = \hat{e}(\rho) \\
&= \bigvee_{\rho \in \mathrm{dom}(\tau)} \left(\bigvee_{\rho' \sim \rho} e(\tau(\rho')) \;\wedge\; [\![\, \varphi(\hat{e}(\rho), \hat{e}(\tau), \hat{e}(\vec{\sigma})) \,]\!]_{\mathsf{C}} \right) \\
&= \bigvee_{\rho \in \mathrm{dom}(\tau)} \left(\bigvee_{\rho' \sim \rho} e(\tau(\rho')) \;\wedge\; [\![\, \varphi(\hat{e}(\rho'), \hat{e}(\tau), \hat{e}(\vec{\sigma})) \,]\!]_{\mathsf{C}} \right) \\
&= \bigvee_{\rho \in \mathrm{dom}(\tau)} e(\tau(\rho)) \;\wedge\; [\![\, \varphi(\hat{e}(\rho), \hat{e}(\tau), \hat{e}(\vec{\sigma})) \,]\!]_{\mathsf{C}} \\
&= \bigvee_{\rho \in \mathrm{dom}(\tau)} e(\tau(\rho)) \;\wedge\; [\![\, \varphi(\rho, \tau, \hat{e}(\vec{\sigma})) \,]\!]_{\mathsf{B}} \qquad \text{(by ih)} \\
&= [\![\, (\exists x \in \tau)\varphi(x, \tau, \vec{\sigma}) \,]\!]_{\mathsf{B}}
\end{aligned}$$

The clause for the constants and other function symbols is directly granted by the induction hypothesis and 2.5.13 (and 2.5.14).

If $\exists x\varphi(x, \vec{\tau})$ is a Σ_1 formula, by 2.5.8, there exists some $\sigma \in V^{\mathsf{B}}$ such that $[\![\, \exists x\varphi(x, \vec{\tau}) \,]\!]_{\mathsf{B}} =$

$[\![\varphi(\sigma,\vec{\tau})]\!]_{\mathsf{B}}$ hence

$$\begin{aligned} e\Big([\![\exists x\varphi(x,\vec{\tau})]\!]_{\mathsf{B}} \Big) &= e\Big([\![\varphi(\sigma,\vec{\tau})]\!]_{\mathsf{B}} \Big) \\ &= \Big[\!\!\Big[\varphi\big(\hat{e}(\sigma),\hat{e}(\vec{\tau})\big) \Big]\!\!\Big]^{V^{\mathsf{C}}} \\ &\leq \Big[\!\!\Big[\exists x\varphi\big(x,\hat{e}(\vec{\tau})\big) \Big]\!\!\Big]^{V^{\mathsf{C}}} \end{aligned}$$

Now assume that e is also onto C. Then clearly $\hat{e}$ is onto V^{C}. Let φ be Σ_1 and let $\psi(x)$ be a Δ_0 formula such that $\exists x\psi(x)$ is equivalent to φ. Since V^{C} is golden, let $\sigma' \in V^{\mathsf{C}}$ be such that $[\![\psi(\sigma',\hat{e}(\tau_1),\dots,\hat{e}(\tau_n))]\!]_{\mathsf{C}} = [\![\exists x\psi(x,\hat{e}(\tau_1),\dots,\hat{e}(\tau_n))]\!]_{\mathsf{C}}$. Since $\hat{e}$ is onto V^{C}, let $\sigma \in V^{\mathsf{B}}$ such that $\hat{e}(\sigma) = \sigma'$. Then by what we already proved for Δ_0 formulas we obtain

$$\begin{aligned} e\left([\![\varphi(\tau_1,\dots,\tau_n)]\!]_{\mathsf{B}} \right) &= e\left([\![\exists x\psi(x,\tau_1,\dots,\tau_n)]\!]_{\mathsf{B}} \right) \\ &\geq e\left([\![\psi(\sigma,\tau_1,\dots,\tau_n)]\!]_{\mathsf{B}} \right) \\ &= \Big[\!\!\Big[\psi\left(\hat{e}(\sigma),\hat{e}(\tau_1),\dots,\hat{e}(\tau_n)\right) \Big]\!\!\Big]_{\mathsf{C}} \\ &= \Big[\!\!\Big[\psi\left(\sigma',\hat{e}(\tau_1),\dots,\hat{e}(\tau_n)\right) \Big]\!\!\Big]_{\mathsf{C}} \\ &= \Big[\!\!\Big[\exists x\psi\left(x,\hat{e}(\tau_1),\dots,\hat{e}(\tau_n)\right) \Big]\!\!\Big]_{\mathsf{C}} \\ &= \Big[\!\!\Big[\varphi\left(\hat{e}(\tau_1),\dots,\hat{e}(\tau_n)\right) \Big]\!\!\Big]_{\mathsf{C}} \end{aligned}$$

□

Theorem 2.5.30 (Σ_1 forcing absoluteness)**.** *Forcing cannot change the Σ_1 theory of V in the language for set theory with parameters for all subsets of ω. More precisely, let T be any theory extending* ZFC*, and φ be any Σ_1 formula with parameters $r_1,\dots,r_n$ such that $T \vdash r_i \subseteq \omega$. Then TFAE:*

(1) $T \vdash \exists \mathsf{B}\left([\![\varphi(\check{r})]\!]^{V^{\mathsf{B}}} \gneq 0_{\mathsf{B}} \right)$ [12]

(2) $T \vdash \exists \mathsf{B}\left([\![\varphi(\check{r})]\!]^{V^{\mathsf{B}}} = 1_{\mathsf{B}} \right)$

(3) $T \vdash \varphi(r)$

(4) $T \vdash \forall \mathsf{B}\left([\![\varphi(\check{r})]\!]^{V^{\mathsf{B}}} = 1_{\mathsf{B}} \right)$

(5) $T \vdash \forall \mathsf{B}\left([\![\varphi(\check{r})]\!]^{V^{\mathsf{B}}} \gneq 0_{\mathsf{B}} \right)$

Proof. We will prove the equivalence following the loop of implications (1) → (2) → (3) → (4) → (5) → (1). Suppose that V models T.

(1) → (2) : Fix one such cba $\mathsf{B} \in V$. Let b, F, C and e denote respectively the truth value $[\![\varphi(\check{r})]\!]_{\mathsf{B}} = b$, the filter $\uparrow b = F$, the quotient $\mathsf{B}/_F = \mathsf{C}$ and the map $e : a \mapsto [a]_F$ from B to C. Then e is a complete homomorphism from B onto C. Now, $\hat{e}$, defined as in 2.5.29, maps the canonical B-name for r to the canonical C-name for r, so by 2.5.29 we have

$$[\![\varphi(\check{r})]\!]^{V^{\mathsf{C}}} = e\left([\![\varphi(\check{r})]\!]^{V^{\mathsf{B}}} \right) = e(b) = [b]_F = [1_{\mathsf{B}}]_F = 1_{\mathsf{C}}$$

[12]For the sake of notational convenience here we are writing just one parameter, instead of a finite list, and we are using B as a special syntactic variable for a complete Boolean algebra. I.e. in (1), for example, we mean $T \vdash \exists \mathsf{B}\Big(\mathsf{B} \text{ is a cba } \ \& \ [\![\varphi(\check{r_1},\dots,\check{r_n})]\!]_{\mathsf{B}} = 1_{\mathsf{B}} \Big)$

(2) $\to$ (3) : Using the reflection theorem 2.1.8, find $\theta \geq \omega$ such that V_θ satisfies a finite fragment of T large enough to prove basic ZFC facts, and reflects $\exists \mathsf{B} \left(1_\mathsf{B} = [\![\varphi(\check{r})]\!]_\mathsf{B} \right)$. By the downward Löwenheim-Skolem theorem 2.1.3, let M be a countable elementary substructure of V_θ that extends $\mathrm{trcl}(\{r\})$. Let $N = \pi[M]$ be the transitive collapse of M. Then $\pi(r) = r$ and $N \models \exists \mathsf{B} \left(1_\mathsf{B} = [\![\varphi(\check{r})]\!]_\mathsf{B} \right)$. Let $\mathsf{B} \in N$ be such that $N \models \left(1_\mathsf{B} = [\![\varphi(\check{r})]\!]_\mathsf{B} \right)$. By 2.5.22, since N is countable, let G be N-generic for B. Then $N[G] \vDash \varphi(r)$. But since φ is Σ_1, φ is upward absolute for transitive classes, hence $V \vDash \varphi(r)$.

(3) $\to$ (4) : Let B be any cba in V. Now, φ is Σ_1 so let $\psi(x, r)$ be a Δ_0 formula such that $\varphi(r)$ is equivalent to $\exists x \psi(x, r)$ modulo T. Pick a witness $u \in V$ such that $V \models \psi(u, r)$. Then by lemma 2.5.29, taking e to be the unique homomorphism from 2 into B, we have $[\![\varphi]\!]_\mathsf{B} = [\![\exists x \psi(x)]\!]_\mathsf{B} \geq [\![\psi(\check{u})]\!]_\mathsf{B} = e\left([\![\psi(\check{u})]\!]_2 \right) = 1_\mathsf{B}$

(4) $\to$ (5) : trivial

(5) $\to$ (1) : trivial

Since we took V to be an arbitrary model of T, the claim follows from the completeness of first order logic. □

Remark 2.5.31. The above theorem also holds for Π_1 formulas. That follows from the theorem, using trivial Boolean computation.

2.5.3 Canonical Substructures of V^B

Substructures of the kind $H_\kappa \cap V^\mathsf{B}$

Definition 2.3.8 gives us a general way to define substructures of $V^\mathsf{B}_{\Delta_0}$. Such substructures are always well-defined for relational languages, like $\in_{\Delta_0}$. Notice, however, that in general, if $M \subseteq V^\mathsf{B}$, we may very well have

$$[\![p_\varphi(\vec{\tau})]\!]^{V^\mathsf{B}_{\Delta_0} \restriction M} \neq [\![\varphi(\vec{\tau})]\!]^{V^\mathsf{B}_{\Delta_0} \restriction M}$$

because the former is directly inherited from V^B, as computed there, and the latter is determined by the general semantics in Boolean-valued structures.

Moreover, $[\![p_\varphi(\vec{\sigma})]\!]^{V^\mathsf{B}_{\Delta_0} \restriction M}$ may not be computable in $V^\mathsf{B}_{\Delta_0} \restriction M$. For example if $\mathcal{M}$ is not *Boolean-transitive*, i.e. if some element σ of the domain of a name τ in $\mathcal{M}$ does not belong to M, then to compute $[\![\rho \in \tau]\!]$ we need the value of the interpretation of $\in$ at $\langle \rho, \sigma \rangle$, which is not in the domain of $\in^{V^\mathsf{B}_{\Delta_0} \restriction M}$. However:

Proposition 2.5.32. *If a Boolean-valued model $\mathcal{M}$ is Boolean-transitive, then for each bounded formula $\varphi(\vec{x})$ and each $\tau_i \in M$ we have*

$$[\![\varphi(\vec{\tau})]\!]^{V^\mathsf{B}_{\Delta_0} \restriction M} = [\![\varphi(\vec{\tau})]\!]^{V^\mathsf{B}}$$

Proof. Induction on the construction of φ. The atomic case, negation and disjunction are trivial, so let ψ be of the form $(\exists x \in \tau)\psi(x, \tau, \vec{\rho})$. Then

$$
\begin{aligned}
[\![\, \varphi(\tau,\vec{\rho}) \,]\!]^{V^{\mathsf{B}}_{\Delta_0}\restriction M} &= [\![\, (\exists x \in \tau)\psi(x,\tau,\vec{\rho}) \,]\!]^{V^{\mathsf{B}}_{\Delta_0}\restriction M} \\
&= [\![\, \exists x(x \in \tau \ \ \& \ \ \psi(x,\tau,\vec{\rho})) \,]\!]^{V^{\mathsf{B}}_{\Delta_0}\restriction M} \\
&= \bigvee_{\pi \in M} [\![\, \pi \in \tau \ \ \& \ \ \psi(\pi,\tau,\vec{\rho})) \,]\!]^{V^{\mathsf{B}}_{\Delta_0}\restriction M} \\
&= \bigvee_{\pi \in M} [\![\, \pi \in \tau \,]\!]^{V^{\mathsf{B}}_{\Delta_0}\restriction M} \wedge [\![\, \psi(\pi,\tau,\vec{\rho})) \,]\!]^{V^{\mathsf{B}}_{\Delta_0}\restriction M} \\
&= \bigvee_{\pi \in M} \bigvee_{\sigma \in \mathrm{dom}(\tau)} \tau(\sigma) \wedge [\![\, \pi = \sigma \,]\!] \wedge [\![\, \psi(\pi,\tau,\vec{\rho}) \,]\!]^{V^{\mathsf{B}}_{\Delta_0}\restriction M} \\
&= \bigvee_{\sigma \in \mathrm{dom}(\tau)} \bigvee_{\pi \in M} \tau(\sigma) \wedge [\![\, \pi = \sigma \,]\!] \wedge [\![\, \psi(\pi,\tau,\vec{\rho}) \,]\!]^{V^{\mathsf{B}}_{\Delta_0}\restriction M} \\
&= \bigvee_{\sigma \in \mathrm{dom}(\tau)} \tau(\sigma) \wedge \bigvee_{\pi \in M} [\![\, \pi = \sigma \,]\!] \wedge [\![\, \psi(\pi,\tau,\vec{\rho}) \,]\!]^{V^{\mathsf{B}}_{\Delta_0}\restriction M} \\
&= \bigvee_{\sigma \in \mathrm{dom}(\tau)} \tau(\sigma) \wedge \bigvee_{\pi \in M} [\![\, \pi = \sigma \ \ \& \ \ \psi(\pi,\tau,\vec{\rho}) \,]\!]^{V^{\mathsf{B}}_{\Delta_0}\restriction M} \\
&= \bigvee_{\sigma \in \mathrm{dom}(\tau)} \tau(\sigma) \wedge [\![\, \exists x(x = \sigma \ \ \& \ \ \psi(x,\tau,\vec{\rho}) \,]\!]^{V^{\mathsf{B}}_{\Delta_0}\restriction M} \\
&= \bigvee_{\sigma \in \mathrm{dom}(\tau)} \tau(\sigma) \wedge [\![\, \psi(\sigma,\tau,\vec{\rho}) \,]\!]^{V^{\mathsf{B}}_{\Delta_0}\restriction M} \\
&= \bigvee_{\sigma \in \mathrm{dom}(\tau)} \tau(\sigma) \wedge [\![\, \psi(\sigma,\tau,\vec{\rho}) \,]\!]^{V^{\mathsf{B}}} && \text{(by ih)} \\
&= [\![\, (\exists x \in \tau)\psi(x,\tau,\vec{\rho}) \,]\!]^{V^{\mathsf{B}}} && \text{(by 2.5.3)} \\
&= [\![\, \varphi(\tau,\vec{\rho}) \,]\!]^{V^{\mathsf{B}}}
\end{aligned}
$$

□

For the extended language $\in_{\Delta_0^+}$, however, we have to make sure we pick a subclass of V^{B} closed under the interpretations of the function symbols. We'll be interested in substructures of $V^{\mathsf{B}}_{\in}$, $V^{\mathsf{B}}_{\Delta_0}$ and V^{B}_{+} with domain $H_\kappa \cap V^{\mathsf{B}}$, where $\mathsf{B} \subseteq H_\kappa$. We now show that with the interpretations of all symbols in the extended language we specified in 2.5.11, definition 2.3.8 gives well-defined structures for the extended language $\in_{\Delta_0^+}$.

Constants. By corollary 2.1.17 to Levy's absoluteness, the interpretations of all constant symbols in the extended language $\in_{\Delta_0^+}$ can be taken, in accordance with definition 2.5.11, to be canonical names of hereditarily countable sets. So if κ is uncountable, the induced from $(V^{\mathsf{B}}, \in^{\mathsf{B}}_{\Delta_0})$ to $H_\kappa \cap V^{\mathsf{B}}$ interpretations of the constant symbols are well-defined.

Functions. To see that the symbols for the Gödel operations can be interpreted such that $H_\kappa \cap V^{\mathsf{B}}$ is closed under their interpretations, we provide explicit definitions for them and see they don't change hereditary cardinality of hereditarily infinite names.

- $f^{V^{\mathsf{B}}}_{G_1}(\sigma,\tau) = \mathrm{up}(\sigma,\tau) = \big\{ \langle\, \sigma \,,\, 1_{\mathsf{B}} \,\rangle \,,\, \langle\, \tau \,,\, 1_{\mathsf{B}} \,\rangle \big\}$;
- $f^{V^{\mathsf{B}}}_{G_2}(\sigma,\tau) = \Big\{ \big\langle\, \mathrm{op}(\pi,\rho) \,,\, \sigma(\pi) \wedge \tau(\rho) \,\big\rangle \ : \ \pi \in \mathrm{dom}(\sigma), \rho \in \mathrm{dom}(\tau) \Big\}$,

  where $\mathrm{op}(\pi,\rho) = \Big\{ \big\langle\, \mathrm{up}(\pi,\pi) \,,\, 1 \,\big\rangle \,,\, \big\langle\, \mathrm{up}(\pi,\rho) \,,\, 1 \,\big\rangle \Big\}$;
- $f^{V^{\mathsf{B}}}_{G_3}(\sigma,\tau) = \Big\{ \big\langle\, \mathrm{op}(\pi,\rho) \ ,\ \sigma(\pi) \wedge \tau(\rho) \wedge [\![\, \pi \in \rho \,]\!] \,\big\rangle \ : \ \pi \in \mathrm{dom}(\sigma), \rho \in \mathrm{dom}(\tau) \Big\}$;

- $f_{G_4}^{V^{\mathsf{B}}}(\sigma,\tau) = \Big\{ \langle\, \rho\,,\, \sigma(\rho) \wedge \neg [\![\, \rho \in \tau \,]\!] \,\rangle \;:\; \rho \in \operatorname{dom}(\sigma) \Big\}$;
- $f_{G_5}^{V^{\mathsf{B}}}(\sigma,\tau) = \Big\{ \langle\, \rho\,,\, \sigma(\rho) \wedge [\![\, \rho \in \tau \,]\!] \,\rangle \;:\; \rho \in \operatorname{dom}(\sigma) \Big\}$;
- $f_{G_6}^{V^{\mathsf{B}}}(\tau) = \Big\{ \Big\langle \rho, \bigvee \{\, [\![\, \rho \in \sigma \,]\!] \;:\; \sigma \in \operatorname{dom}(\tau) \,\} \Big\rangle \;:\; \rho \in \operatorname{dom}\big[\operatorname{dom}(\tau)\big] \Big\}$;
- $f_{G_7}^{V^{\mathsf{B}}}(\sigma,\tau) = \Bigg\{ \Bigg\langle \rho, \bigvee \Big\{ [\![\, \operatorname{op}(\rho,\sigma) \in \tau \,]\!] \;:\; \sigma \in f_{G_6}^{V^{\mathsf{B}}}\Big(f_{G_6}^{V^{\mathsf{B}}}(\tau)\Big) \Big\} \Bigg\rangle \;:\; \rho \in f_{G_6}^{V^{\mathsf{B}}}\Big(f_{G_6}^{V^{\mathsf{B}}}(\tau)\Big) \Bigg\}$;
- $f_{G_8}^{V^{\mathsf{B}}}(\tau) = \Big\{ \Big\langle \operatorname{op}(\rho,\pi),\, [\![\, \operatorname{op}(\pi,\rho) \in \tau \,]\!] \Big\rangle \;:\; \rho,\pi \in f_{G_6}^{V^{\mathsf{B}}}\Big(f_{G_6}^{V^{\mathsf{B}}}(\tau)\Big) \Big\}$;
- $f_{G_9}^{V^{\mathsf{B}}}(\sigma,\tau) = \Big\{ \Big\langle \operatorname{otr}(\pi,\rho,\sigma),\, [\![\, \operatorname{otr}(\pi,\sigma,\rho) \in \tau \,]\!] \Big\rangle \;:\; \pi,\rho,\sigma \in f_{G_6}^{V^{\mathsf{B}}}\Big(f_{G_6}^{V^{\mathsf{B}}}(\tau)\Big) \Big\}$;
  where $\operatorname{otp}(\pi,\rho,\sigma) = \operatorname{op}(\operatorname{op}(\pi,\rho),\sigma)$.
- $f_{G_{10}}^{V^{\mathsf{B}}}(\sigma,\tau) = \Big\{ \Big\langle \operatorname{otr}(\pi,\rho,\sigma),\, [\![\, \operatorname{otr}(\rho,\sigma,\pi) \in \tau \,]\!] \Big\rangle \;:\; \pi,\rho,\sigma \in f_{G_6}^{V^{\mathsf{B}}}\Big(f_{G_6}^{V^{\mathsf{B}}}(\tau)\Big) \Big\}$.

It is painstakingly tedious to check that those interpretations of the function symbols for the Gödel operations satisfy the requirements spelled out in definition 2.5.11.

It is evident that if κ is infinite, then $H_\kappa \cap V^{\mathsf{B}}$ is closed under all $f_{G_i}^{V^{\mathsf{B}}}$. Thus $V_+^{\mathsf{B}} \restriction (H_\kappa \cap V^{\mathsf{B}})$ is well defined. We shall denote it also by $\Big(H_\kappa \cap V^{\mathsf{B}}, \in_{\Delta_0^+}^{\mathsf{B}}\Big)$ or $V_+^{\mathsf{B}} \restriction H_\kappa$.

Remark 2.5.33. Notice that, although completeness of B grants the mixing property for V^{B}, $H_\kappa \cap V^{\mathsf{B}}$ may not have the mixing property, even if B is a complete Boolean algebra. If B has an antichain of size κ, then a mixture of a family $\big\langle\, \tau_a \;:\; a \in A \,\big\rangle \subseteq H_\kappa \cap V^{\mathsf{B}}$ of names indexed by such an antichain may have hereditary size κ and thus be outside of $H_\kappa \cap V^{\mathsf{B}}$.

Substructures of the kind W^{B} and $\check{W}^{\mathsf{B}}$

Now we turn our attention to two other kinds of substructures of V^{B}. If we have a class W in the ground model, then we can introduce a predicate for belonging to that class, which behaves like a canonical name for the class and describes the way the class W sits inside the Boolean extension. If $\check{W} = \big\{\, \langle \check{a}, 1_{\mathsf{B}} \rangle \;:\; a \in W \,\big\}$ *were* a B-name, then B-valued membership to it would look like this:

$$\Big[\!\!\Big[\, \tau \in \check{W} \,\Big]\!\!\Big]^{V^{\mathsf{B}}} = \bigvee \Big\{ [\![\, \tau = \sigma \,]\!] \wedge \check{W}(\sigma) \;:\; \sigma \in \operatorname{dom}(W) \Big\} = \bigvee \big\{\, [\![\, \tau = \check{x} \,]\!] \;:\; x \in W \,\big\}$$

We can take this as the definition of the interpretation of a predicate symbol $\check{W}$.

Notice that Boolean truth values of existential formulas relativised to $\check{W}$ are computed in a simple way:

$$\begin{aligned}
\left[\!\left[(\exists x\varphi(x))^{\check{W}} \right]\!\right]^{V^{\mathsf{B}}} &= \left[\!\left[\exists x\Big(x \in \check{W} \;\&\; \varphi(x)\Big) \right]\!\right]^{V^{\mathsf{B}}} \\
&= \bigvee_{\sigma\in V^{\mathsf{B}}} \left[\!\left[\sigma \in \check{W} \right]\!\right]^{V^{\mathsf{B}}} \wedge [\![\varphi(\sigma)]\!]^{V^{\mathsf{B}}} \\
&= \bigvee_{\sigma\in V^{\mathsf{B}}} \bigvee_{u\in W} [\![\sigma = \check{u}]\!]^{V^{\mathsf{B}}} \wedge [\![\varphi(\sigma)]\!]^{V^{\mathsf{B}}} \\
&\leq \bigvee_{\sigma\in V^{\mathsf{B}}} \bigvee_{u\in W} [\![\varphi(\check{u})]\!]^{V^{\mathsf{B}}} \\
&= \bigvee_{u\in W} [\![\varphi(\check{u})]\!]^{V^{\mathsf{B}}}
\end{aligned}$$

The converse inequality is immediate by the semantics of existential quantifier, so we have

$$\left[\!\left[(\exists x\varphi(x))^{\check{W}} \right]\!\right]^{V^{\mathsf{B}}} = \bigvee_{u\in W} [\![\varphi(\check{u})]\!]^{V^{\mathsf{B}}} \tag{2.16}$$

Definition 2.5.34. If B is a Boolean algebra in V and $W \subseteq V$ is a class defined by a formula $\psi_W(x, \vec{u})$, we define:

- W^{B} as the collection of those B-names $\sigma \in V^{\mathsf{B}}$ which in V^{B} have the property ψ with probability 1_{B}, i.e.

$$W^{\mathsf{B}} = \left\{ \sigma \in V^{\mathsf{B}} \;:\; \left[\!\left[\psi\Big(\sigma, \vec{\check{u}}\Big) \right]\!\right]^{V^{\mathsf{B}}} = 1_{\mathsf{B}} \right\}$$

- $\check{W}^{\mathsf{B}}$ as the collection of those B-names $\sigma \in V^{\mathsf{B}}$ which with probability 1_{B} belong to $\check{W}$ (i.e. satisfy the predicate $\check{W}$). More precisely:

$$\check{W}^{\mathsf{B}} = \left\{ \sigma \in V^{\mathsf{B}} \;:\; \bigvee_{u\in W} [\![\sigma = \check{u}]\!]^{V^{\mathsf{B}}} = 1_{\mathsf{B}} \right\}$$

Remark 2.5.35. Notice that each element of $\check{W}^{\mathsf{B}}$ is equal with probability one to a mixture of canonical names. Indeed, $\sigma \in \check{W}^{\mathsf{B}}$ means that $\bigvee \{ [\![\sigma = \check{u}]\!] \,:\, u \in W \} = 1_{\mathsf{B}}$. Obviously the canonical names of distinct sets equal each other with probability zero, thus for each name $\sigma \in \check{W}^{\mathsf{B}}$, we have that $A^\sigma = \{ [\![\sigma = \check{u}]\!] \,:\, u \in W \} = \Big\{ a \in \mathsf{B}^+ \,:\, (\exists u \in W)(a = [\![\sigma = \check{u}]\!]) \Big\}$ is an antichain. For each element a of that antichain A^σ pick some $u_a \in W$ such that $a = [\![\sigma = \check{u_a}]\!]$. Let τ be the mixture $\sum_{a\in A} a \cdot \check{u_a}$. Then by the defining property of mixtures we have

$$1_{\mathsf{B}} \;=\; \bigvee_{a\in A^\sigma} a \wedge a \;\leq\; \bigvee_{a\in A^\sigma} [\![\sigma = \check{u_a}]\!] \wedge [\![\tau = \check{u_a}]\!] \;=\; \bigvee_{a\in A^\sigma} [\![\sigma = \tau]\!] \;=\; [\![\sigma = \tau]\!]$$

W^{B} and $\check{W}^{\mathsf{B}}$ are subclasses of V^{B} but we are interested in sub*structures* of V^{B} in the languages we choose study V^{B} with. Definition 2.3.8 already provides a way to define substructures of V^{B} with such domains. And 2.5.32 points to a way to do so.

Lemma 2.5.36. *If the class W is defined by a Σ_1 formula with parameters in V, then $\check{W}^{\mathsf{B}}$ is a substructure of W^{B} in the basic language $\in_{\Delta_0}$.*

Proof. Let $\varphi_W(x, y)$ be a Σ_1 formula in the basic language $\in_{\Delta_0}$ and let u be any set in V. Denote the class $\{ a \,:\, \psi_W(a, u) \}$ by W. Then by lemma 2.5.15 we have that for each $a \in W$ we have $[\![\varphi_W(\check{a}, \check{u})]\!]^{V^{\mathsf{B}}} = 1_{\mathsf{B}}$. Then for any $\tau \in \check{W}^{\mathsf{B}}$ we have

$$1_{\mathsf{B}} = \bigvee_{a\in W} [\![\tau = \check{a}]\!]^{V^{\mathsf{B}}} = \bigvee_{a\in W} \Big([\![\tau = \check{a}]\!]^{V^{\mathsf{B}}} \wedge [\![\varphi_W(\check{a}, \check{u})]\!]^{V^{\mathsf{B}}} \Big) \leq [\![\varphi_W(\tau, \check{u})]\!]^{V^{\mathsf{B}}}$$

□

Theorem 2.5.37. *If V^{B} has the κ-mixing property, then W^{B} has the κ-mixing property.*

Proof. Let A be an antichain in B of cardinality κ and let, for every $a \in A$, $\tau_a \in W^{\mathsf{B}}$. By further extending A, we can assume A to be maximal. V^{B} satisfies the mixing property, so let $\tau \in V^{\mathsf{B}}$ be such that $[\![\tau = \tau_a]\!] \geq a$ for every $a \in A$. We only have to check that $\tau \in W^{\mathsf{B}}$. Since, for every $a \in A$ we have $\tau_a \in W^{\mathsf{B}}$, we have that $[\![\varphi_W(\tau_a)]\!] = 1_{\mathsf{B}}$. Therefore

$$a \leq [\![\tau = \tau_a]\!] = [\![\tau = \tau_a]\!] \wedge [\![\varphi_W(\tau_a)]\!] \leq [\![\varphi_W(\tau)]\!] .$$

Then,

$$[\![\varphi_W(\tau)]\!] \geq \bigvee A = 1_{\mathsf{B}}$$

by maximality of A. □

The proof of the following results can be found in Parente's thesis [18, Theorem 2.5.3, Proposition 2.5.4], else see Theorem 2.5.42 below.

Theorem 2.5.38. *If V^{B} has the κ-mixing property, then $\check{W}^{\mathsf{B}}$ has the κ-mixing property.*

Proof. The proof is similar to the previous one. We use the same mixture τ after extending the desired antichain to a maximal one. Then for each condition a of the antichain A and for each $x \in W$ we have $[\![\tau = \check{x}]\!] \geq [\![\tau = \tau_a]\!] \wedge [\![\tau_a = \check{x}]\!]$ and hence

$$\begin{aligned}
\bigvee_{x \in W} [\![\tau = \check{x}]\!] &\geq \bigvee_{x \in W} \bigvee_{a \in A} [\![\tau = \tau_a]\!] \wedge [\![\tau_a = \check{x}]\!] \\
&= \bigvee_{a \in A} [\![\tau_a = \tau]\!] \wedge \bigvee_{x \in W} [\![\tau_a = \check{x}]\!] \\
&= \bigvee_{a \in A} [\![\tau_a = \tau]\!] \wedge 1_{\mathsf{B}} \\
&= \bigvee_{a \in A} [\![\tau_a = \tau]\!] \geq \bigvee A = 1_{\mathsf{B}}
\end{aligned}$$ □

Theorem 2.5.39. [18, Proposition 2.5.4]. *If W is a class, $\varphi(x_1, \dots, x_n)$ is a formula in $\in_{\Delta_0}$ and $a_1, \dots, a_n \in W$, then*

$$\varphi^W(a_1, \dots, a_n) \quad \textit{iff} \quad [\![\varphi(\check{a_1}, \dots, \check{a_n})]\!]^{\check{W}^B} = 1_{\mathsf{B}}.$$

Proof. Induction on the construction of φ.

If φ is an atomic formula, then

$$(x = y) \iff [\![\check{x} = \check{y}]\!] = 1_{\mathsf{B}}, \quad \text{and} \quad (x \in y) \iff [\![\check{x} \in \check{y}]\!] = 1_{\mathsf{B}}$$

are clear from the definition of canonical names. The inductive step for $\neg$ and $\wedge$ is straightforward.

Suppose the claim is true for $\varphi(x_1, \dots, x_n, y)$. If

$$(\exists y \in W)(\varphi^W(a_1, \dots, a_n, y)),$$

then there exists $b \in W$ such that $\varphi^W(a_1, \dots, a_n, b)$. By the inductive hypothesis, this implies $\left[\!\!\left[\varphi(\check{a_1}, \dots, \check{a_n}, \check{b}) \right]\!\!\right]^{\check{W}^{\mathsf{B}}} = 1_{\mathsf{B}}$. Since $\check{b} \in \check{W}^{\mathsf{B}}$, we can conclude that $[\![\exists y \varphi(\check{a_1}, \dots, \check{a_n}, y)]\!]^{\check{W}^{\mathsf{B}}} = 1$, as desired. Conversely, assume

$$[\![\exists y \varphi(\check{a_1}, \dots, \check{a_n}, y)]\!]^{\check{W}^{\mathsf{B}}} = 1_{\mathsf{B}}.$$

For every $\tau \in \check{W}^{\mathsf{B}}$, we have

$$[\![\varphi(\check{a_1}, \dots, \check{a_n}, \tau)]\!]^{\check{W}^{\mathsf{B}}} = [\![\varphi(\check{a_1}, \dots, \check{a_n}, \tau)]\!]^{\check{W}^{\mathsf{B}}} \wedge \bigvee_{b \in W} \left[\!\!\left[\tau = \check{b} \right]\!\!\right] \leq \bigvee_{b \in W} \left[\!\!\left[\varphi(\check{a_1}, \dots, \check{a_n}, \check{b}) \right]\!\!\right]^{\check{W}^{\mathsf{B}}},$$

hence

$$1_{\mathsf{B}} = [\![\exists y\varphi(\check{a_1},\dots,\check{a_n},y)]\!]^{\check{W}^{\mathsf{B}}} = \bigvee_{\tau\in\check{W}^{\mathsf{B}}} [\![\varphi(\check{a_1},\dots,\check{a_n},\tau)]\!]^{\check{W}^{\mathsf{B}}} \leq \bigvee_{b\in W} \left[\!\!\left[\varphi(\check{a_1},\dots,\check{a_n},\check{b}) \right]\!\!\right]^{\check{W}^{\mathsf{B}}}.$$

As a consequence, there is $b \in W$ such that $\left[\!\!\left[\varphi(\check{a_1},\dots,\check{a_n},\check{b}) \right]\!\!\right]^{\check{M}^{\mathsf{B}}} > 0$. By inductive hypothesis, it cannot happen that $\neg\varphi^W(a_1,\dots,a_n,b)$, otherwise we would have $\left[\!\!\left[\varphi(\check{a_1},\dots,\check{a_n},\check{b}) \right]\!\!\right]^{\check{W}^{\mathsf{B}}} = 0$. Therefore, $\varphi^W(a_1,\dots,a_n,b)$, which means that $\exists y\varphi(a_1,\dots,a_n,y)$, as desired. □

Corollary 2.5.40. *If the class $W \subseteq V$ is closed under a definable class-function f_φ, then we can interpret a function symbol for it in the intended way, i.e. as described in 2.1.13. Indeed, if $W \vDash \forall\vec{x}\exists! y\varphi(\vec{x},y)$, then for all $\tau_i \in W^{\mathsf{B}}$ there exists a name $\sigma \in W^{\mathsf{B}}$ such that $[\![\varphi(\vec{\tau},\sigma)]\!]^{W^{\mathsf{B}}} = 1_{\mathsf{B}}$ and for each $\rho \in W^{\mathsf{B}}$ we have $[\![\varphi(\vec{\tau},\rho)]\!]^{W^{\mathsf{B}}} \leq [\![\rho = \sigma]\!]^{W^{\mathsf{B}}}$.*

Another immediate consequence of proposition 2.5.39 is:

Corollary 2.5.41. *If G is an ultrafilter on B, the map*

$$\begin{array}{rcrcl} i & : & W & \longrightarrow & \check{W}^{\mathsf{B}}/_G \\ & & x & \longmapsto & [\check{x}]_G \end{array}$$

is an elementary embedding.

Theorem 2.5.42. [8, Theorem 30]. *Let $H_{\aleph_1} \subseteq W \subseteq V$ be a transitive class. Then $\mathcal{W}^{\downarrow\mathsf{B}}$ and $\check{W}^{\mathsf{B}}$ are isomorphic*[13] *B-valued structures for the language $\in_{\Delta_0^+}$.*

Proof. For the sake of notational elegance of canonical names, for any element $\sigma : \downarrow A_\sigma{}^+ \longrightarrow W$ of the ultrapower $\mathcal{W}^{\downarrow\mathsf{B}}$ we shall denote the value of σ at a by σ_a, rather than $\sigma(a)$. Denote

$$\begin{array}{rcrcl} j & : & \mathcal{W}^{\downarrow\mathsf{B}} & \longrightarrow & \check{W}^{\mathsf{B}} \\ & & \sigma & \longmapsto & \sum_{a\in A_\sigma} a\cdot\check{\sigma_a} \end{array}$$

We claim that $\langle \mathrm{Id}_{\mathsf{B}}, j \rangle$ is an isomorphism of $\left\langle \mathsf{B}, \mathcal{W}^{\downarrow\mathsf{B}} \right\rangle$ onto $\left\langle \mathsf{B}, \check{W}^{\mathsf{B}} \right\rangle$.

We prove the claim by induction on the well-order mentioned in 2.1.14 and further induction on the square order mentioned in 2.5.2. Let $\sigma, \tau \in W^{\downarrow\mathsf{B}}$ and denote $\mathrm{ran}(\tau)$ by T. Note that

$$\mathrm{dom}(j(\tau)) = \bigcup_{a\in A_\tau} \mathrm{dom}(\check{\tau_a}) = \check{}\left[\bigcup_{a\in A_\tau} \tau_a\right] = \check{}\left[\bigcup \mathrm{ran}(\tau)\right] = \check{}[\cup T] \tag{2.17}$$

$$(j(\tau))(\check{u}) = \bigvee_{a\in A_\tau} a \wedge [\![\check{u} \in \check{\tau_a}]\!] = \bigvee \{ a \in A_\tau \,:\, u \in \tau_a \} = [\![c_u \in \tau]\!]^{\mathcal{W}^{\downarrow\mathsf{B}}} \tag{2.18}$$

$$j(c_u) = \check{u} \tag{2.19}$$

[13] Recall from 2.3.2 the definition of Boolean-valued structures being isomorphic.

$$
\begin{aligned}
[\![\, j(\sigma) \in j(\tau) \,]\!]^{\check{W}^{\mathsf{B}}} &= \bigvee \left\{ (j(\tau))(\rho) \;\wedge\; [\![\, \rho = j(\sigma) \,]\!]^{\check{W}^{\mathsf{B}}} \;:\; \rho \in \mathrm{dom}(j(\tau)) \right\} && \text{(by 2.5.1)} \\
&= \bigvee \left\{ (j(\tau))(\check{u}) \;\wedge\; [\![\, \check{u} = j(\sigma) \,]\!]^{\check{W}^{\mathsf{B}}} \;:\; u \in \cup T \right\} && \text{(by (2.17))} \\
&= \bigvee \left\{ [\![\, c_u \in \tau \,]\!]^{\mathcal{W}{\downarrow}\mathsf{B}} \;\wedge\; [\![\, j(c_u) = j(\sigma) \,]\!]^{\check{W}^{\mathsf{B}}} \;:\; u \in \cup T \right\} && \text{(by (2.18) and (2.19))} \\
&= \bigvee \left\{ [\![\, c_u \in \tau \,]\!]^{\mathcal{W}{\downarrow}\mathsf{B}} \;\wedge\; [\![\, c_u = \sigma \,]\!]^{\mathcal{W}{\downarrow}\mathsf{B}} \;:\; u \in \cup T \right\} && \text{(by ih)} \\
&= \bigvee \left\{ [\![\, c_u \in \tau \;\;\&\;\; c_u = \sigma \,]\!]^{\mathcal{W}{\downarrow}\mathsf{B}} \;:\; u \in \cup T \right\} \\
&= \bigvee \left\{ \bigvee \{\, a \in A_{\sigma\tau} \;:\; \sigma_a = u \in \tau_a \,\} \;:\; u \in \cup T \right\} && \text{(by B-power semantics)} \\
&= \bigvee \{\, a \in A_{\sigma\tau} \;:\; (\exists u \in \cup T)(\sigma_a = u \in \tau_a) \,\} \\
&= \bigvee \{\, a \in A_{\sigma\tau} \;:\; \exists u (u \in \cup T \;\;\&\;\; \sigma_a = u \in \tau_a) \,\} \\
&= \bigvee \{\, a \in A_{\sigma\tau} \;:\; \exists u (\sigma_a = u \in \tau_a) \,\} && \text{(since } \tau_a \subseteq \cup T) \\
&= \bigvee \{\, a \in A_{\sigma\tau} \;:\; \sigma_a \in \tau_a \,\} \\
&= [\![\, \sigma \in \tau \,]\!]^{\mathcal{W}{\downarrow}\mathsf{B}}
\end{aligned}
$$

The induction step for negation and disjunction is trivial. For bounded quantifiers we have the very similar:

$$
\begin{aligned}
[\![\, (\exists x \in j(\tau))\varphi(x, j(\tau)) \,]\!]^{\check{W}^{\mathsf{B}}} &= \bigvee \left\{ (j(t))(\rho) \;\wedge\; [\![\, \varphi(\rho, j(\tau)) \,]\!]^{\check{W}^{\mathsf{B}}} \;:\; \rho \in \mathrm{dom}(j(\tau)) \right\} \\
&= \bigvee \left\{ (j(t))(\check{u}) \;\wedge\; [\![\, \varphi(\check{u}, j(\tau)) \,]\!]^{\check{W}^{\mathsf{B}}} \;:\; u \in \cup T \right\} \\
&= \bigvee \left\{ [\![\, c_u \in \tau \,]\!]^{\mathcal{W}{\downarrow}\mathsf{B}} \;\wedge\; [\![\, \varphi(j(c_u), j(\tau)) \,]\!]^{\check{W}^{\mathsf{B}}} \;:\; u \in \cup T \right\} \\
&= \bigvee \left\{ [\![\, c_u \in \tau \,]\!]^{\mathcal{W}{\downarrow}\mathsf{B}} \;\wedge\; [\![\, \varphi(c_u, \tau) \,]\!]^{\mathcal{W}{\downarrow}\mathsf{B}} \;:\; u \in \cup T \right\} \\
&= \bigvee \left\{ [\![\, c_u \in \tau \;\;\&\;\; \varphi(c_u, \tau) \,]\!]^{\mathcal{W}{\downarrow}\mathsf{B}} \;:\; u \in \cup T \right\} \\
&= \bigvee \left\{ \bigvee \{\, a \in A_\tau \;:\; \mathcal{W} \vDash (u \in \tau_a \;\;\&\;\; \varphi(u, \tau_a)) \,\} \;:\; u \in \cup T \right\} \\
&= \bigvee \{\, a \in A_\tau \;:\; (\exists u \in \cup T)(\mathcal{W} \vDash u \in \tau_a \;\;\&\;\; \varphi(u, \tau_a)) \,\} \\
&= \bigvee \{\, a \in A_\tau \;:\; (\exists u \in \cup T)(u \in \tau_a \;\;\&\;\; \varphi(u, \tau_a)) \,\} \\
&= \bigvee \{\, a \in A_\tau \;:\; \exists u (u \in \tau_a \;\;\&\;\; \varphi(u, \tau_a)) \,\} \\
&= \bigvee \{\, a \in A_\tau \;:\; (\exists u \in \tau_a)(\varphi(u, \tau_a)) \,\} \\
&= \bigvee \{\, a \in A_\tau \;:\; (\exists u \in \tau_a)\varphi(u, \tau_a) \,\} \\
&= \bigvee \{\, a \in A_\tau \;:\; \mathcal{W} \vDash (\exists u \in \tau_a)\varphi(u, \tau_a) \,\} \\
&= [\![\, (\exists x \in \tau)\varphi(x, \tau) \,]\!]^{\mathcal{W}{\downarrow}\mathsf{B}}
\end{aligned}
$$

The proof for the additional symbols in the basic and extended language is trivial using the defining formula φ of the symbol in question (p_φ, c_φ or $f_\varphi = f_{\varphi_G} = f_G$) and the equalities above. We spell out just the case for function symbols.

$$
[\![\, f_G(j(\vec{\sigma})) = j(\mu) \,]\!]^{\check{W}^{\mathsf{B}}} = [\![\, \varphi_G(j(\vec{\sigma}), j(\mu)) \,]\!]^{\check{W}^{\mathsf{B}}} = [\![\, \varphi_G(\vec{\sigma}, \mu) \,]\!]^{\mathcal{W}{\downarrow}\mathsf{B}} = [\![\, f_G(\vec{\sigma}) = \mu \,]\!]^{\mathcal{W}{\downarrow}\mathsf{B}}
$$

What remains is to prove that j is surjective. Fix any $\tau \in \check{W}^{\mathsf{B}}$. By the definition of $\check{W}^{\mathsf{B}}$ we have $\bigvee_{u \in W} [\![\tau = \check{u}]\!] = 1_{\mathsf{B}}$. By 2.5.15, we have that $A = \{ [\![\tau = \check{u}]\!] : u \in W \}$ is an antichain. For each $a \in A$ choose some u_a such that $[\![\tau = \check{u_a}]\!] = a$. Now it's obvious that τ equals the mixture $\sum_{a \in A} a \cdot \check{u_a}$ with probability 1_{B}. And that mixture is the value of j at the name σ such that

$$\begin{array}{rcccc} \sigma & : & \downarrow A & \longrightarrow & W \\ & & a & \longmapsto & u_a \end{array}$$

□

Corollary 2.5.43. *For any ultrafilter G on B, by 2.3.11 we have $W^{\downarrow \mathsf{B}}/_G \;\cong\; \check{W}^{\mathsf{B}}/_G$.* □

2.6 Saturation by Goodness

We describe how to obtain saturated two-valued Tarski structures as Boolean quotients by good ultrafilters. For reference see Ulrich's thesis [23], Pierobon's thesis [19] or Parente's thesis [18].

Theorem 2.6.1. [19, Theorem 3.1.2]. *Let B be a $<\kappa$-complete Boolean algebra and $\mathcal{M}$ be a golden B-valued structure for a first order language $\mathcal{L}$ satisfying the $<\kappa$-mixing property for some cardinal κ such that $\aleph_0 + |\mathcal{L}| < \kappa$. Assume G is countably incomplete $<\kappa$-good ultrafilter on B. Then $\mathcal{M}/_G$ is $<\kappa$-saturated.*

Proof. Let $A \subseteq \mathcal{M}/_G$ be a subset of size $\lambda < \kappa$ and fix a complete 1-type $p(x)$ over A every finite part of which is realized in $\mathcal{M}/_G$. We'll show that $p(x)$ is realized in $\mathcal{M}/_G$. Our assumptions grant that $|p(x)| = \lambda + |\mathcal{L}| = \lambda$. Therefore we can fix an enumeration $p(x) = \{ \varphi_\alpha(x) : \alpha < \lambda \}$ of $p(x)$. Since $p(x)$ is finitely satisfied in $\mathcal{M}/_G$, and $\mathcal{M}$ is full, for every $S \in \mathcal{P}_\omega(\lambda)$ we have

$$\left[\!\!\left[\exists x \bigwedge_{\alpha \in S} \varphi_\alpha(x) \right]\!\!\right] \in G$$

Now, by the countable incompleteness of G, there exists $\{ a_n : n < \omega \} \subseteq G$ such that $\bigwedge_{n<\omega} a_n = b \notin G$, refining each a_n to $\neg b \wedge \bigwedge_{i \leq n} a_i$, we may further assume that $\bigwedge_{n<\omega} a_n = 0$ and $a_i \geq a_j$ if $i \leq j$. Define the monotonically decreasing map

$$\begin{array}{rcccc} f & : & \mathcal{P}_\omega(\lambda) & \longrightarrow & G \\ & & S & \longmapsto & a_{|S|} \wedge \left[\!\!\left[\exists x \bigwedge_{\alpha \in S} \varphi_\alpha(x) \right]\!\!\right] \end{array}$$

By assumption G is κ-good, hence there exists a multiplicative refinement $g : \mathcal{P}_\omega(\lambda) \longrightarrow G$ of f. Consider the map

$$\begin{array}{rcccc} h & : & \mathcal{P}_\omega(\lambda) & \longrightarrow & \mathsf{B} \\ & & S & \longmapsto & g(S) \wedge \bigwedge \{ \neg g(T) : |T| > |S| \} \end{array}$$

We will prove later that h is not the constant map $S \longmapsto 0_{\mathsf{B}}$. The following observation is crucial in what follows:

Claim 1. *For all $S, T \in \mathcal{P}_\omega(\lambda)$, if $g(S) \wedge h(T) > 0_{\mathsf{B}}$, then $S \subseteq T$.*

Proof. Suppose not. Then $|T| < |T \cup S|$; hence (since g is multiplicative)

$$g(S) \wedge h(T) \;\leq\; g(S) \wedge g(T) \wedge \neg g(S \cup T) \;=\; g(S) \wedge g(T) \wedge \neg \big(g(S) \wedge g(T) \big) \;=\; 0,$$

against our assumption. □

We get the following:

Claim 2. $\operatorname{ran}(h) \setminus \{0_{\mathsf{B}}\}$ *is an antichain.*

Proof. Assume that for some $S, T \in \mathcal{P}_\omega(\lambda)$ we have $h(S) \wedge h(T) > 0_{\mathsf{B}}$. We must show that $S = T$. By the definition of h, we immediately observe that $h(S) \wedge g(T) > 0$ and $g(S) \wedge h(T) > 0$. Now apply the previous claim to conclude that $S = T$. $\square$

By the fullness of $\mathcal{M}$, we can find a subset $\{ \sigma_S : S \in \mathcal{P}_\omega(\lambda) \} \subseteq M$ such that

$$\left[\!\!\left[\exists x \bigwedge_{\alpha \in S} \varphi_\alpha(x) \right]\!\!\right] = \left[\!\!\left[\bigwedge_{\alpha \in S} \varphi_\alpha(\sigma_S) \right]\!\!\right]$$

for every $S \in \mathcal{P}_\omega(\lambda)$. Since $\mathcal{M}$ has the $<\kappa$-mixing property, we can find $\tau \in \mathcal{M}$ such that

$$h(S) \leq [\![\tau = \sigma_S]\!]$$

for every $S \in \mathcal{P}_\omega(\lambda)$. This means that, for every fixed $S \in \mathcal{P}_\omega(\lambda)$ and $\mathcal{P}_\omega(\lambda) \ni T \supseteq S$, we have

$$\begin{aligned}
\left[\!\!\left[\bigwedge_{\alpha \in S} \varphi_\alpha(\tau) \right]\!\!\right] &\geq \left[\!\!\left[\bigwedge_{\alpha \in T} \varphi_\alpha(\tau) \right]\!\!\right] \\
&\geq \left[\!\!\left[\bigwedge_{\alpha \in T} \varphi_\alpha(\sigma_T) \right]\!\!\right] \wedge [\![\tau = \sigma_T]\!] \\
&\geq \left[\!\!\left[\bigwedge_{\alpha \in T} \varphi_\alpha(\sigma_T) \right]\!\!\right] \wedge h(T) \\
&= \left[\!\!\left[\exists x \bigwedge_{\alpha \in T} \varphi_\alpha(x) \right]\!\!\right] \wedge h(T) \\
&\geq \left[\!\!\left[\exists x \bigwedge_{\alpha \in T} \varphi_\alpha(x) \right]\!\!\right] \wedge a_{|T|} \wedge h(T) \\
&= h(T) \wedge f(T) \\
&\geq h(T) \wedge g(T) \\
&\geq h(T) \wedge h(T) \\
&= h(T)
\end{aligned}$$

The above holds for arbitrary $T \supseteq S$ in $\mathcal{P}_\omega(\lambda)$, so by taking a join over all of them we have

$$\left[\!\!\left[\bigwedge_{\alpha \in S} \varphi_\alpha(\tau) \right]\!\!\right] \geq \bigvee \left\{ h(T) : T \supseteq S \right\} \tag{2.20}$$

If we can prove that, for every $S \in \mathcal{P}_\omega(\lambda)$ we have that the right-hand side of 2.20 is in G, then also the left- hand side is in G; this means that $[\tau]_G$ realizes the type $p(x)$ in $\mathcal{M}/_G$, as desired.

Claim 3. $\bigvee \{ h(T) : \mathcal{P}_\omega(\lambda) \ni T \supseteq S \} \in G$ *for every* $S \in \mathcal{P}_\omega(\lambda)$. *In particular,* h *is not identically* 0_{B}.

Proof. Fix S and denote $b = g(S) \wedge \bigwedge\{\neg h(T) : T \supseteq S\}$. We notice that

$$\begin{aligned}
& b \vee \bigvee \{ h(T) : T \supseteq S \} \\
&= \left(g(S) \wedge \bigwedge \{ \neg h(T) : T \supseteq S \} \right) \vee \bigvee \{ h(T) : T \supseteq S \} \\
&= \left(g(S) \vee \bigvee \{ h(T) : T \supseteq S \} \right) \wedge \left(\bigwedge \{ \neg h(T) : T \supseteq S \} \vee \bigvee \{ h(T) : T \supseteq S \} \right) \\
&= \left(g(S) \vee \bigvee \{ h(T) : T \supseteq S \} \right) \wedge \left(\neg \bigvee \{ h(T) : T \supseteq S \} \vee \bigvee \{ h(T) : T \supseteq S \} \right) \\
&\geq g(S) \wedge 1_{\mathsf{B}} = g(S) \in G.
\end{aligned}$$

Since G is an ultrafilter, either $b \in G$ or $\bigvee \{ h(T) : T \supseteq S \} \in G$. Suppose towards a contradiction that $b \in G$. For each $n < \omega$, let us define $c_n = \bigvee \{ g(T) : |T| = n \}$. Clearly, since g is monotonically decreasing, for every $n < \omega$ we have $c_{n+1} \leq c_n$ and $b \leq c_{|S|}$. Since $g(T) \leq a_{|T|}$ for every T, by taking a join over all T of size n, we get that $c_n \leq a_n$ for all $n < \omega$. This gives that

$$b \not\leq \bigwedge_{n<\omega} c_n$$

since $b \in G$, while

$$\bigwedge_{n<\omega} c_n \leq \bigwedge_{n<\omega} a_n = 0 \notin G.$$

Hence there exists $m < \omega$ such that $b \wedge c_m \wedge \neg c_{m+1} > 0_{\mathsf{B}}$. Notice that $m \geq |S|$, since $b \leq g(S) \leq c_{|S|}$. This gives that for some R of cardinality m, since $b \leq c_{|S|}$ and $c_m = \bigvee \{ g(T) : |T| = m \}$

$$0_{\mathsf{B}} < b \wedge g(R) \wedge \neg c_{m+1}.$$

Now observe that $g(R) \wedge \neg c_{m+1} = h(R)$, since

$$\begin{aligned} h(R) &= g(R) \wedge \bigwedge \{ \neg g(T) : |T| > |R| \} \\ &= g(R) \wedge \neg \bigvee \{ g(T) : |T| > |R| \} = g(R) \wedge \neg \bigvee_{n>|R|} c_n \end{aligned}$$

and, since $|R| = m$ and $c_{n+1} \leq c_n$ for every $n < \omega$, we obtain that $\bigvee_{n>|R|} c_n = c_{|R|+1} = c_{m+1}$. We conclude that $0_{\mathsf{B}} < b \wedge h(R)$. Since $b \leq g(S)$ we can apply again Claim 1, to get that $R \supseteq S$. Therefore

$$0_{\mathsf{B}} < b \wedge h(R) = g(S) \wedge \bigwedge \{ \neg h(T) : T \supseteq S \} \wedge h(R) = 0_{\mathsf{B}},$$

a contradiction. □

This completes the proof. □

Lemma 2.6.2. *Let* B *be a* $<\kappa$*-complete* $<\kappa$*-disjointable Boolean algebra of inaccessible size* κ *with the small completion property. Let* F *be a* $<\kappa$*-complete filter on* B *and* H *be a countably incomplete good ultrafilter on* $\mathsf{B}/_F$*. Then* $G = \cup H$ *is a countably incomplete good ultrafilter on* B.

Proof. G is clearly an ultrafilter on B. The countable incompleteness of G follows directly from the countable completeness of F. Notice that since H is a filter on the quotient $\mathsf{B}/_F$, then $F = 1_{\mathsf{B}/_F} \subseteq \cup H$.

To show that G is good, fix some $\lambda < \kappa$ and a monotonically decreasing function $f : \mathcal{P}_\omega(\lambda) \longrightarrow G$. Since H is good, let g be a multiplicative refinement of $f_F : S \longmapsto \left[f(S) \right]_F$. Let $h' : \mathcal{P}_\omega(\lambda) \longrightarrow G$ be such that $h'(S) \in g(S)$ for each $S \in \mathcal{P}_\omega(\lambda)$. Let $h : S \mapsto h'(S) \wedge f(S)$ for $S \in \mathcal{P}_\omega(\lambda)$. Clearly for each $S \in \mathcal{P}_\omega(\lambda)$ we have $h(S) \sim_F h'(S)$.

Let $S, T \in \mathcal{P}_\omega(\lambda)$. Clearly $h(S \cup T) \sim_F h(S) \wedge h(T)$, i.e. there exists some $a_{ST} \in F$ such that $h(S \cup T) \wedge a_{ST} = h(S) \wedge h(T) \wedge a_{ST}$. Since F is $<\kappa$-complete we have that $F \ni a = \bigwedge \{ a_{ST} : S, T \in \mathcal{P}_\omega(\lambda) \}$

Clearly the function $S \longmapsto h(S) \wedge a$ for $S \in \mathcal{P}_\omega(\lambda)$ is a multiplicative refinement of f with range in G. □

Chapter 3

The Convergence theorem

Definition 3.0.1 (iteration game)**.** Let κ be a regular cardinal and B be a Boolean algebra of density κ. The *iteration game* $\mathcal{G}_{\mathsf{B}}$ for B is the two-player game of length κ defined as follows:

- Player I plays at odd stages, and player II plays at even stages[1].
- At each stage $\beta < \kappa$, a valid move is a Boolean algebra B_β such that
  1. B_β is a subalgebra of $\mathsf{B} \restriction c_\beta$ for some condition $c_\beta \in \mathsf{B}^+$. Note that this implies that a move B_β determines a unique such c_β, which is the top element of the played algebra B_β
  2. B_β has density strictly smaller than κ.
  3. For each $\alpha < \beta$, the map $i_{\alpha\beta} : a \longmapsto a \wedge c_\beta$ is an injective homomorphism from B_α into B_β admitting an adjoint map $\pi_{\alpha\beta} : \mathsf{B}_\beta \longrightarrow \mathsf{B}_\alpha$.
- The game is considered to start at stage 0 with player II playing the trivial move $\{0_{\mathsf{B}}, 1_{\mathsf{B}}\}$ Notice that on successor stages there is always at least one valid move - copying the opponent's last move. In particular, player I always has a valid move.
- Player I wins by leaving player II with no valid move to play; equivalently, player II wins iff the game lasts κ-many moves.

Note that whenever $\left\langle \mathsf{B}_\alpha \,:\, \alpha < \gamma \right\rangle$, for some $\gamma \leq \kappa$, is a valid partial run of the game, the above rules force that the sequence $\langle c_\alpha \,:\, \alpha < \gamma \rangle = \left\langle 1_{\mathsf{B}_\alpha} \,:\, \alpha < \gamma \right\rangle$ formed by the top elements of the played algebras is a decreasing sequence of positive elements of B.

Definition 3.0.2 (game iterations, filters and quotients)**.** Let B be a Boolean algebra of density κ and $\left\langle \mathsf{B}_\alpha \,:\, \alpha < \gamma \right\rangle$ be a partial run of $\mathcal{G}_{\mathsf{B}}$, with $\gamma \leq \kappa$.

- We identify with it the uniquely associated iteration system $\mathcal{F} = \left\{ i_{\alpha\beta} : \mathsf{B}_\alpha \longrightarrow \mathsf{B}_\beta \,:\, \alpha \leq \beta < \gamma \right\}$, where $i_{\alpha\beta} : a \longmapsto a \wedge c_\beta$ for all $\alpha \leq \beta$ and $a \in \mathsf{B}_\alpha$. We shall call such an iteration system a *a (partial) game iteration of* B (we remove *partial* if $\gamma = \kappa$).
- the set $\left\{ 1_{\mathsf{B}_\alpha} \,:\, \alpha < \kappa \right\}$ is clearly a prefilter on B, so its upward closure $F = \uparrow_{\mathsf{B}} \left\{ 1_{\mathsf{B}_\alpha} \,:\, \alpha < \kappa \right\}$ is a filter on B. We shall refer to F and $\mathsf{B}/_F$ respectively as the *game filter* and *game quotient* associated to $\mathcal{F}$.

Remark 3.0.3. Note that if $\mathcal{F}$ has length γ and B is $<|\gamma|$-complete, then F is $<|\gamma|$-complete.

Definition 3.0.4. Let B, again, be a Boolean algebra of density κ. We shall call a strategy for player I for $\mathcal{G}_{\mathsf{B}}$

- *disjointing* iff for every won by player II run of $\mathcal{G}_{\mathsf{B}}$ in which player I has used such a strategy, the direct limit of the resultant game iteration is $<\kappa$-disjointable.

[1] Limit ordinals are even

- *exhaustive* iff for every won by player II run of $\mathcal{G}_{\mathsf{B}}$ in which player I has used such a strategy, we have that for each condition a of B there is some stage $\alpha < \kappa$ such that $a \wedge c_\alpha \in \mathsf{B}_\alpha$. In loose terms, a current projection of every condition of B has been added at some stage – B has been exhausted.

Definition 3.0.5. A Boolean algebra B is called *iterable* iff player II has a winning counterstratgy against some exhaustive disjointing strategy for player I

Lemma 3.0.6. *Let κ be a strong limit cardinal, B be a $<\kappa$-complete $<\kappa$-disjointable Boolean algebra of size κ with the small completion property. Then player I has a disjointing exhaustive strategy for $\mathcal{G}_{\mathsf{B}}$.*

Proof. Note that whenever C is a complete subalgebra of $\mathsf{B} \restriction c$ of size less than κ, the identity $\mathrm{Id}_{\mathsf{C}} : \mathsf{C} \longrightarrow \mathsf{B} \restriction c$ on C is a complete injective homomorphism with an adjoint: for each $d \in \mathsf{B} \restriction c$, we have that $\pi_{\mathsf{C}}(d) = \bigwedge_{\mathsf{C}} \left\{ b \in \mathsf{C} \,:\, b \geq d \right\}$ exists in C. Hence the map π_{C} is the adjoint of Id_{C}.

Now we'll describe a strategy which is simultaneously exhaustive and disjointing. First we enumerate $\mathsf{B} = \left\{ b_\alpha \,:\, \alpha \in \kappa \cap \mathsf{Even} \right\}$ in type κ using only the even ordinals. Then on each odd stage $\alpha + 1$, if B_α is what player II has played on stage α for some partial play of $\mathcal{G}_{\mathsf{B}}$ of length $\alpha + 1$, player I replies by doing the following:

- First of all WLoG we may assume B_α is complete, otherwise player I just takes the smallest complete subalgebra of $\mathsf{B} \restriction c_\alpha = \mathsf{B} \restriction 1_{\mathsf{B}_\alpha}$ containing B_α and replies assuming B_α is this complete superalgebra[2].
- Sets $c_{\alpha+1} = \neg_{\mathsf{B}_\alpha} \pi_{\mathsf{B}_\alpha}(b_\alpha \wedge c_\alpha) \vee (b_\alpha \wedge c_\alpha)$ (we will see below that $\pi_{\mathsf{B}_\alpha}(c_{\alpha+1}) = c_\alpha$ and $b \wedge c_{\alpha+1} > 0_{\mathsf{B}}$ for all $b \in \mathsf{B}_\alpha^+$).
- Considers the set $F_\alpha = {}^{|\alpha|}\mathsf{B}_\alpha^+$ of all functions from $|\alpha|$ to B_α^+; for each $f \in F_\alpha$, finds a disjoint refinement h_f of f with range in $\mathsf{B} \restriction c_{\alpha+1}$ (these exist since B is $<\kappa$-disjointable and $f(\xi) \wedge c_{\alpha+1} > 0_{\mathsf{B}}$ for all $\xi < \alpha$). Then lets $D_\alpha = \bigcup \left\{ \mathrm{ran}(h_f) \,:\, f \in F_\alpha \right\}$. Since κ is strong limit we have that F_α is strictly smaller than κ and hence D_α is as well.
- Defines $\mathsf{B}_{\alpha+1}$ to be the complete subalgebra of $\mathsf{B} \restriction c_{\alpha+1}$ generated by the strictly smaller than κ set
$$\left\{ b \wedge c_{\alpha+1} \,:\, b \in \mathsf{B}_\alpha \right\} \cup D_\alpha \cup \left\{ b_\alpha \wedge c_{\alpha+1} \right\}.$$
By the small completion property of B, we have that $\mathsf{B}_{\alpha+1}$ is strictly smaller than κ.

This completes the description of the strategy. Now we shall prove that the map $i_{\alpha\alpha+1} : a \longmapsto a \wedge c_{\alpha+1}$ is indeed a complete injective homomorphism of the cba B_α into the cba $\mathsf{B}_{\alpha+1}$.

$i_{\alpha\alpha+1}$ **preserves arbitrary joins:** obvious.

$i_{\alpha\alpha+1}$ **preserves negation:**
$$\begin{aligned}
\neg_{\mathsf{B}_{\alpha+1}} \big(i_{\alpha\alpha+1}(a)\big) &= c_{\alpha+1} \wedge \neg(c_{\alpha+1} \wedge a) \\
&= c_{\alpha+1} \wedge (\neg c_{\alpha+1} \vee \neg a) \\
&= 0_{\mathsf{B}} \vee (c_{\alpha+1} \wedge \neg a) \\
&= c_{\alpha+1} \wedge (c_\alpha \wedge \neg a) \\
&= c_{\alpha+1} \wedge \neg_{\mathsf{B}_\alpha}(a) \\
&= i_{\alpha\alpha+1}(\neg_{\mathsf{B}_\alpha} a).
\end{aligned}$$

$\pi_\alpha(c_{\alpha+1}) = c_\alpha$**:** We have
$$c_{\alpha+1} = \big(\neg \pi_\alpha(b_\alpha \wedge c_\alpha)\big) \vee (c_\alpha \wedge b_\alpha).$$

[2]Note that the rules of the game grant that II has played B_α so that $\mathrm{Id}_{\mathsf{B}_\alpha} : \mathsf{B}_\alpha \longrightarrow \mathsf{B} \restriction c_\alpha$ has an adjoint; this adjoint has to be the restriction of the adjoint of the identity map on the smallest complete subalgebra of $\mathsf{B} \restriction c_\alpha$ containing B_α.

Hence:

$$\begin{aligned}
\pi_\alpha(c_{\alpha+1}) &= \pi_{\mathsf{B}_\alpha}(\neg_{\mathsf{B}_\alpha}\pi_{\mathsf{B}_\alpha}(b_\alpha \wedge c_\alpha) \vee (b_\alpha \wedge c_\alpha)) \\
&= \pi_{\mathsf{B}_\alpha}(\neg_{\mathsf{B}_\alpha}\pi_{\mathsf{B}_\alpha}(b_\alpha \wedge c_\alpha)) \vee \pi_{\mathsf{B}_\alpha}(c_\alpha \wedge b_\alpha) && \text{(since } \pi_{\mathsf{B}_\alpha} \text{ is suprema preserving)} \\
&= \neg_{\mathsf{B}_\alpha}\pi_{\mathsf{B}_\alpha}(b_\alpha \wedge c_\alpha) \vee \pi_{\mathsf{B}_\alpha}(c_\alpha \wedge b_\alpha) && \text{(since } \pi_{\mathsf{B}_\alpha}(b_\alpha \wedge c_\alpha) \in \mathsf{B}_\alpha) \\
&= c_\alpha && \text{(since } c_\alpha = 1_{\mathsf{B}_\alpha}).
\end{aligned}$$

$i_{\alpha\alpha+1}$ **is injective** First, notice that the only condition in B_α that is strictly above $c_{\alpha+1}$ is c_α: if we assume that $c_{\alpha+1} \leq a \in \mathsf{B}_\alpha$, then

$$1_{\mathsf{B}_\alpha} = c_\alpha = \pi_{\mathsf{B}_\alpha}(c_{\alpha+1}) = \bigwedge_{\mathsf{B}_\alpha}\left\{ b \in \mathsf{B}_\alpha : b \geq c_{\alpha+1} \right\} \leq \bigwedge_{\mathsf{B}_\alpha}\{a\} = a .$$

Hence for all $a \in \mathsf{B}_\alpha$, we have

$$c_{\alpha+1} \leq a \text{ implies } c_\alpha = a \tag{3.1}$$

Therefore if $i_{\alpha\alpha+1}(a) = 0_{\mathsf{B}_{\alpha+1}}$, then $i_{\alpha\alpha+1}(\neg_{\mathsf{B}_\alpha} a) = 1_{\mathsf{B}_{\alpha+1}} = c_{\alpha+1}$; hence $\neg_{\mathsf{B}_\alpha} a = c_\alpha$, giving that $a = 0_{\mathsf{B}_\alpha}$.

The adjoint of $i_{\alpha\alpha+1}$ clearly can be defined appealing to the completeness of B_α. It is clear that the described strategy is exhaustive. To show that it is disjointing, fix some $\lambda < \kappa$ and a family $\left\{ t^{a_\alpha}_{\zeta_\alpha} : \alpha < \lambda \right\}$ of constant threads in a game iteration of B of length κ in which player I has used the strategy described above. Let ζ be the smallest even ordinal above λ and all ζ_α for $\alpha < \lambda$. Consider the function

$$\begin{aligned} g \ : \ \lambda &\longrightarrow \mathsf{B}_\zeta \\ \alpha &\longmapsto i_{\zeta_\alpha \zeta}(a_\alpha) . \end{aligned}$$

Then g is the restriction to λ of some function f in F_ζ, so there is a disjoint refinement h of f whose range player I added to D_ζ and thus to $\mathsf{B}_{\zeta+1}$ on step $\zeta + 1$. Then clearly the function

$$\begin{aligned} k \ : \ \lambda &\longrightarrow \varinjlim \mathcal{F} \\ \alpha &\longmapsto t^{h(\alpha)}_{\zeta+1} \end{aligned}$$

is a disjoint refinement of f. This completes the proof of the lemma. □

Theorem 3.0.7 (Convergence)**.** *Let* B *be a* $<\kappa$*-complete* $<\kappa$*-disjointable Boolean algebra of inaccessible density* κ *with the small completion property. The direct limit of an exhaustive game iteration of* B *is isomorphic to its respective game quotient.*

Proof. We'll show that the map $j : f \longmapsto \left[f\big(\mathrm{supp}(f)\big) \right]_F$ is an isomorphism of $\varinjlim \mathcal{F}$ onto $\mathsf{B}/_F$.

j **is injective:** Fix two distinct constant threads $f = t^a_\alpha$ and $g = t^b_\beta$ in $\varinjlim \mathcal{F}$. Let γ be the least ordinal at which their values differ. Suppose towards a contradiction that $a \sim_F b$. By the structure of F, this occurs if and only if there is some $\delta < \kappa$ such that $a \wedge c_\delta = b \wedge c_\delta$. Fix one such δ.

Notice that the defining properties of γ and δ persist, namely $f(\gamma') \neq g(\gamma')$ for each $\gamma' \geq \gamma$, and $a \wedge c_{\delta'} = b \wedge c_{\delta'}$ for each $\delta' \geq \delta$: the latter is a trivial consequence of the antimonotonicity of $\left\langle c_\zeta : \zeta < \kappa \right\rangle$, and the former is the case because any condition in $\mathsf{B}_{\gamma'}$ determines a unique γ'-initial segment of a thread, thus a common value at $\gamma' \geq \gamma$ would imply a common value at γ.

Fix any ζ larger than all of α, β, γ and δ. Then

$$f(\zeta) = i_{\alpha\zeta}\left(f(\alpha)\right) = c_\zeta \wedge a = c_\zeta \wedge b = i_{\beta\zeta}\left(g(\beta)\right) = g(\zeta),$$

which is a contradiction.

j **is onto** $\mathsf{B}/_F$**:** Fix any $[a]_F \in \mathsf{B}/_F$. Since $\mathcal{F}$ is exhaustive, let β be such that $a \wedge c_\beta \in \mathsf{B}_\beta$. Consider the constant thread $f = t_\beta^{a\wedge c_\beta}$. Denote its support by α. Then $a \wedge c_\beta = f(\beta) = i_{\alpha\beta}\left(f(\alpha)\right) = c_\beta \wedge f\left(\operatorname{supp}(f)\right)$. Thus $a \sim_F f\left(\operatorname{supp}(f)\right)$, so $[a]_F = j(f)$.

j **preserves negation:** Fix some $t \in \mathsf{B}_\kappa$. Then $t = t_\alpha^a$ for some $a \in \mathsf{B}$ and $j(t) = [a]_F$. Hence

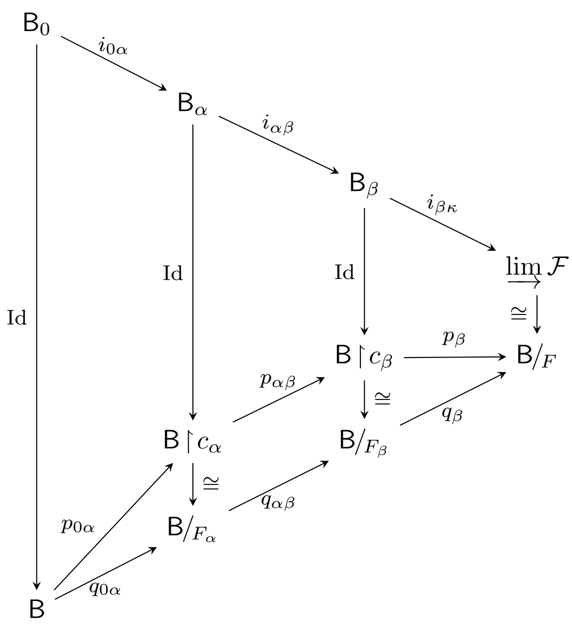
$$j\left(\neg_{\mathsf{B}_\kappa} t\right) = j\left(t_\alpha^{\neg_{\mathsf{B}_\alpha} a}\right) = [\neg_{\mathsf{B}_\alpha} a]_F = [c_\alpha \wedge \neg a]_F = [\neg a]_F = \neg_{\mathsf{B}/_F}[a]_F = \neg_{\mathsf{B}/_F} j(t).$$

j **preserves meets:** Given constant threads $f = t_\alpha^a$ and $g = t_\beta^b$, with α and β their exact supports, WLoG suppose $\alpha \leq \beta$. Then

$$\begin{aligned}
j(f \wedge g) &= j\left(t_\alpha^a \wedge t_\beta^b\right) \\
&= j\left(t_\beta^{i_{\alpha\beta}(a)\wedge b}\right) \\
&= j\left(t_\beta^{a\wedge c_\beta \wedge b}\right) \\
&= j\left(t_\beta^{a\wedge b}\right) = [a \wedge b]_F \\
&= [a]_F \wedge [b]_F \\
&= j(f) \wedge j(g).
\end{aligned}$$

□

What converges in the theorem above, giving its name, are the two sequences of Boolean algebras $\left\langle \mathsf{B}_\alpha : \alpha < \kappa \right\rangle$ and $\left\langle \mathsf{B}/_{F_\alpha} : \alpha < \kappa \right\rangle$, where $F_\alpha = \uparrow_{\mathsf{B}} c_\alpha$. They start from the smallest and biggest algebras in the construction - B_0 and B respectively - and they both converge to the same algebra $\varinjlim \mathcal{F} \cong \mathsf{B}/_F$, which is depicted in the following diagram



where $i_\beta\kappa$ is defined as in 1.2.4, and

$$\begin{aligned}
q_{\alpha\beta} &: [a]_{F_\alpha} \longmapsto [a]_{F_\beta}, \\
q_\beta &: [a]_{F_\beta} \longmapsto [a]_F, \\
p_{\alpha\beta} &: a \longmapsto a \wedge c_\beta, \\
p_\beta &: a \longmapsto [a]_F.
\end{aligned}$$

Definition 3.0.8. Let B be a Boolean algebra of density κ. We call a strategy for player II for $\mathcal{G}_{\mathsf{B}}$ *regular* iff, in short, it involves playing stationarily often the direct limit of the already constructed game iteration. More precisely, a strategy for player II for $\mathcal{G}_{\mathsf{B}}$ is called regular, iff it is a winning strategy and using it guarantees that the resultant game iteration $\mathcal{F}$ is such that the set $\left\{ \alpha < \kappa \,:\, \mathsf{B}_\alpha \cong \varinjlim(\mathcal{F} \restriction \alpha) \right\}$ is a stationary subset of κ. We shall call a game iteration $\mathcal{F}$ which can be constructed by a run of the game in which player II uses a regular strategy *a regular game iteration*. B is called *regularly iterable* iff in $\mathcal{G}_{\mathsf{B}}$ player II has a regular counterstrategy to some exhaustive disjointing strategy for player I.

Remark 3.0.9. Baumgartner's theorem 1.2.7 makes the usefulness of regular strategies clear. If B is a $<\kappa$-complete Boolean algebra of density κ and $\mathcal{F}$ is a regular game iteration of B, then its direct limit $\varinjlim \mathcal{F}$ is $<\kappa$-cc and complete.

Chapter 4

Applications to Forcing

4.1 The Levy Collapse

We now introduce a particular Boolean algebra called the Levy collapse, to illustrate that the assumptions on Boolean algebras we've considered in the previous chapter can be realized. For reference we suggest Jech's [10] and [9], Pierobon's thesis [19] or Parente's thesis [18].

For the rest of this section, fix an inaccessible cardinal κ. For every $\alpha < \kappa$, define the poset P_α with domain

$$P_\alpha = \left\{ p \subseteq \omega \times \alpha \, : \, p \text{ is a finite partial function} \right\},$$

and ordering: $p \leq q$ iff $p \supseteq q$.

If $p = \langle \, p_\alpha \, : \, \alpha < \kappa \, \rangle \in \prod_{\alpha<\kappa} P_\alpha$, define the *support* $\operatorname{supp}(p)$ of p to be the set $\left\{ \, \alpha < \kappa \, : \, p_\alpha \neq \emptyset \, \right\}$. Now define the Levy Collapse poset

$$\mathsf{Lv}_\kappa = \left\{ p \in \prod_{\alpha<\kappa} P_\alpha \, : \, \operatorname{supp}(p) \text{ is finite} \right\}; \tag{4.1}$$

with the order $p \leq q$ iff $p_\alpha \leq q_\alpha$ for all $\alpha < \kappa$.

Remark 4.1.1. Clearly Lv_κ has size κ. Indeed, for each $\alpha < \kappa$ we have

$$|P_\alpha| \leq |\mathcal{P}_\omega(\omega \times \alpha)| = \aleph_0 + |\mathcal{P}_\omega(\alpha)| \leq \aleph_0 + |\alpha|$$

And then $\mathsf{Lv}_\kappa \precsim \prod_{\alpha<\kappa} P_\alpha \precsim \kappa \cdot \kappa = \kappa$. On the other hand, Lv_κ obviously has at least κ distinct elements, so we have $\mathsf{Lv}_\kappa \sim \kappa$.

Lemma 4.1.2. [19, Theorem 3.2.9]. Lv_κ *is* $<\kappa$*-cc.*

Proof. Let $W \subseteq \mathsf{Lv}_\kappa$ be an antichain. We construct by recursion a sequence $\langle A_n : n < \omega \rangle$ such that $A_n \subseteq A_{n+1} \subseteq \kappa$ for all $n < \omega$, and another sequence $\langle W_n : n < \omega \rangle$ such that $W_n \subseteq W_{n+1} \subseteq W$ for all $n < \omega$. Let $A_0 = W_0 = \emptyset$. Suppose A_n and W_n are constructed. For every $p \in P$ such that $\operatorname{supp}(p) \subseteq A_n$, choose $q_p \in W$ such that $q_p \restriction A_n = p$, whenever it exists.[1] Then define

$$W_{n+1} = W_n \cup \left\{ q_p : p \in \mathsf{Lv}_\kappa, \ \operatorname{supp}(p) \subseteq A_n \right\}, \quad A_{n+1} = \bigcup \left\{ \operatorname{supp}(q) : q \in W_{n+1} \right\}, \quad A = \bigcup_{n<\omega} A_n.$$

We now prove that $W = \bigcup_{n<\omega} W_n$. Let $q \in W$; since $\operatorname{supp}(q)$ is finite, we can choose $n < \omega$ such that $\operatorname{supp}(q) \cap A = \operatorname{supp}(q) \cap A_n$. By construction, there is $q' \in W_{n+1}$ such that $q' \restriction A_n = q \restriction A_n$. But $\operatorname{supp}(q') \subseteq A$, hence

$$\operatorname{supp}(q) \cap \operatorname{supp}(q') = \operatorname{supp}(q) \cap \operatorname{supp}(q') \cap A = \operatorname{supp}(q) \cap \operatorname{supp}(q') \cap A_n \subseteq A_n,$$

[1] Recall that every $p \in \mathsf{Lv}_\kappa$ is a function $\langle p_\alpha : \alpha < \kappa \rangle$, so it makes sense to consider the restriction of p to a subset of κ.

therefore we have proved that q and q' are compatible. Since W is an antichain, we conclude that $q = q'$ and $q \in W_{n+1}$.

We finish the proof by showing that $|W_n| < \kappa$ by induction on $n < \omega$. Suppose that $|W_n| < \kappa$. First, note that

$$|A_n| = \left| \bigcup \left\{ \operatorname{supp}(q) : q \in W_n \right\} \right| \leq \aleph_0 \cdot |W_n| < \kappa.$$

It follows easily that $\left| \left\{ p \in \mathsf{Lv}_\kappa : \operatorname{supp}(p) \subseteq A_n \right\} \right| < \kappa$ and so $|W_{n+1}| < \kappa$. □

Corollary 4.1.3. *Combining the above result with 1.1.9, we conclude that* $\mathsf{RO}(\mathsf{Lv}_\kappa)$ *has size* κ.

Notation 4.1.4. Since $\mathsf{RO}(\mathsf{Lv}_\kappa)$ has size κ, there is an isomorphic copy of $\mathsf{RO}(\mathsf{Lv}_\kappa)$ which is a subset[2] of H_κ. We fix one such and denote it by $\mathrm{Coll}(\omega, {<}\kappa)$. Both Lv_κ and $\mathrm{Coll}(\omega, {<}\kappa)$ are commonly referred to as the *Levy collapse* (for any κ). The idea behind the name is that forcing with this algebra makes all ordinals below κ countable.

Proposition 4.1.5. *Coll*$(\omega, {<}\kappa)$ *has density* κ.

Proof. Let e be a dense embedding of Lv_κ into $\mathrm{Coll}(\omega, {<}\kappa)$ and X be a subset of $\mathrm{Coll}(\omega, {<}\kappa)$ of size $\lambda < \kappa$. Let Y denote $e^{-1}[X]$. Since e is injective, Y has size λ as well. Each condition in Y has finite support, thus there is some ordinal α below κ which belongs to the support of no condition in Y. Fix some condition p with support $\{\alpha\}$. Clearly no condition in Y refines p. Then no condition in X refines $e(p)$, thus X is not dense in $\mathrm{Coll}(\omega, {<}\kappa)$. □

Proposition 4.1.6. *Coll*$(\omega, {<}\kappa)$ *is* $<\kappa$*-disjointable.*

Proof. It suffices to prove that Lv_κ is $<\kappa$-disjointable. Fix some $\lambda < \kappa$ and some function $f : \lambda \longrightarrow \mathrm{Coll}(\omega, {<}\kappa)^+$.

By definition, $\operatorname{supp}(f(\alpha))$ is finite for every $\alpha < \lambda$, hence

$$\left| \bigcup_{\alpha < \lambda} \operatorname{supp}(f(\alpha)) \right| \leq \lambda < \kappa.$$

Thus we can find a cardinal μ, with $\lambda < \mu < \kappa$, such that $f(\alpha)_\mu = \emptyset$ for all $\alpha < \lambda$. Now define a function $h \colon \lambda \longrightarrow \mathsf{Lv}_\kappa$ as follows: for every $\alpha < \kappa$,

$$(h(\alpha))(\beta) = \begin{cases} \{\langle 0, \alpha \rangle\} & , \text{for } \beta = \mu \\ \emptyset & , \text{for } \beta \neq \mu \end{cases}$$

Observe that $\operatorname{ran}(h)$ is an antichain. By construction, for all $\alpha < \lambda$, $f(\alpha)$ and $h(\alpha)$ are compatible, hence we can find $g(\alpha) \in \mathsf{Lv}_\kappa$ such that $g(\alpha) \leq f(\alpha)$ and $g(\alpha) \leq h(\alpha)$. The function $g \colon \lambda \longrightarrow \mathsf{Lv}_\kappa$ has the desired properties. □

Proposition 4.1.7. *Coll*$(\omega, {<}\kappa)$ *has the small completion property.*

Proof. Let X be a smaller than κ subset of $\mathrm{Coll}(\omega, {<}\kappa)$. Let e be a dense embedding of Lv_κ into $\mathrm{Coll}(\omega, {<}\kappa)$. For each $x \in X$ choose some $p^x \in \mathsf{Lv}_\kappa$ such that $e(p^x)$ is below x. Denote the set $\left\{ p^x : x \in X \right\}$ by Y. The cardinal κ is regular, so the supremum α of the supports of all conditions in Y is strictly below κ and hence so do all of their meets and joins. Then all meets and joins of elements of X have support strictly below κ. Then clearly the generated by X complete subalgebra of $\mathrm{Coll}(\omega, {<}\kappa)$ is a subset of $e[\mathsf{Lv}_\kappa \restriction \alpha]$, where $\mathsf{Lv}_\kappa \restriction \alpha$ denotes $\mathsf{Lv}_\kappa \cap \prod_{\beta < \alpha} P_\alpha$. Finally, using that κ is strong limit, each P_β has small size:

$$P_\beta = \left\{ p \subseteq \omega \times \beta \,:\, p \text{ is a finite partial function} \right\} \subseteq \mathcal{P}(\omega \times \beta) = 2^{\aleph_0 \cdot |\beta|} \lneq \kappa$$

Hence $\mathsf{Lv}_\kappa \restriction \alpha \lneq |\alpha| \cdot |\beta| \lneq \kappa$, which completes the proof. □

Corollary 4.1.8. *Coll*$(\omega, {<}\kappa)$ *satisfies all the conditions of the convergence theorem 3.0.7.*

Corollary 4.1.9. *There are countably incomplete good ultrafilters on Coll*$(\omega, {<}\kappa)$. [3]

[2] More precise, whose domain is a subset of H_κ.

[3] Recall lemma 1.3.6

4.2 The Universality of Forcing

Lemma 4.2.1. *Let* $\mathsf{B} \subseteq H_\kappa$ *be a* $<\kappa$*-complete Boolean algebra and* F *be a* $<\kappa$*-complete filter on* B*. Let*

$$\begin{array}{cccc} p_F & : & \mathsf{B} & \longrightarrow & \mathsf{B}/_F \\ & & a & \longmapsto & [a]_F \end{array}$$

and $\hat{p_F}$ *be as defined as in 2.5.28. Let* Γ *be a choice function for* $V^{\mathsf{B}}/_F$ [4]*. Then* $\left\langle \mathrm{Id}_{(\mathsf{B}/_F)}, \hat{p_F} \circ \Gamma \right\rangle$ *is an isomorphism of* $\left\langle \mathsf{B}/_F, (V_+^{\mathsf{B}} \restriction H_\kappa)/_F \right\rangle$ *onto* $\left\langle \mathsf{B}/_F, \left(V_+^{(\mathsf{B}/_F)} \restriction H_\kappa\right) \right\rangle$*.*

Proof. Let $[\sigma_i]_F \in (V^{\mathsf{B}} \cap H_\kappa)/_F$ and denote each $\Gamma([\sigma_i]_F)$ by σ_i respectively. The for every atomic formula $p_\varphi(\vec{x})$ we have

$$\begin{aligned}
\left[\!\left[p_\varphi\left([\vec{\sigma}]_F \right) \right]\!\right]^{(V_+^{\mathsf{B}} \restriction H_\kappa)/_F} &= \left[[\![p_\varphi(\vec{\sigma})]\!]^{V_+^{\mathsf{B}} \restriction H_\kappa} \right]_F && \text{(by the general quotient semantics 2.2.7)} \\
&= p_F\left([\![p_\varphi(\vec{\sigma})]\!]^{V_+^{\mathsf{B}} \restriction H_\kappa} \right) && \text{(by the definition of } p_F\text{)} \\
&= p_F\left([\![p_\varphi(\vec{\sigma})]\!]^{V_+^{\mathsf{B}}} \right) && \text{(by the definition 2.3.8 of induced substructure)} \\
&= [\![p_\varphi(p_F(\vec{\sigma}))]\!]^{V_+^{\mathsf{B}/_F}} && \text{(by 2.5.29)}
\end{aligned}$$

$$\begin{aligned}
[\![c_\varphi = [\sigma]_F]\!]^{(V_+^{\mathsf{B}} \restriction H_\kappa)/_F} &= \left[\!\left[c_\varphi^{(V_+^{\mathsf{B}} \restriction H_\kappa)/_F} = [\sigma]_F \right]\!\right]^{(V_+^{\mathsf{B}} \restriction H_\kappa)/_F} && \text{(by the semantics 2.2.1 for function symbols)} \\
&= \left[\!\left[\left[c_\varphi^{V_+^{\mathsf{B}} \restriction H_\kappa} \right]_F = [\sigma]_F \right]\!\right]^{(V_+^{\mathsf{B}} \restriction H_\kappa)/_F} && \text{(by the quotient semantics 2.2.7)} \\
&= \left[\left[\!\left[c_\varphi^{V_+^{\mathsf{B}} \restriction H_\kappa} = \sigma \right]\!\right] \right]_F && \text{(by the quotient semantics 2.2.7)} \\
&= \left[[\![c_\varphi = \sigma]\!]^{V_+^{\mathsf{B}} \restriction H_\kappa} \right]_F && \text{(by the semantics 2.2.1 for function symbols)} \\
&= \left[\left([\![c_\varphi = \sigma]\!]^{V_+^{\mathsf{B}}} \right) \right]_F && \text{(by the definition 2.3.8 of induced substructure)} \\
&= p_F\left([\![c_\varphi = \sigma]\!]^{V_+^{\mathsf{B}}} \right) && \text{(by the definition of } p_F\text{)} \\
&= [\![c_\varphi = \hat{p_F}(\sigma)]\!]^{V_+^{\mathsf{B}/_F}} && \text{(by 2.5.29)} \\
&= [\![c_\varphi = \hat{p_F}(\sigma)]\!]^{V_+^{\mathsf{B}/_F} \restriction H_\kappa} && \text{(by the definition 2.3.8 of induced substructure)}
\end{aligned}$$

And analogically for the function symbols. Finally to prove that $\hat{p_F} \circ \Gamma$ is surjective (up to Boolean equality), fix some name $\tau \in V^{(\mathsf{B}/_F)} \cap H_\kappa$. Fix any choice function γ such that

$$\begin{array}{cccc} \gamma & : & \mathsf{B}/_F & \longrightarrow & \mathsf{B} \\ & & [a]_F & \longmapsto & a \end{array}$$

Then $\left[\hat{\gamma}(\tau)\right]_F \in H_\kappa \cap V^{\mathsf{B}}/_F$, with $\hat{\gamma}$ defined as in 2.5.28, is clearly mapped by $\hat{p_F} \circ \Gamma$ to τ.

Corollary 4.2.2. *Using the notation from the above lemma, by the intermediate quotient lemma 2.3.9 and 2.3.11, for every filter* G *extending* F *we have that*

$$\left(V_+^{\mathsf{B}} \restriction H_\kappa\right) \big/ G \;\cong\; \left(\left(V_+^{\mathsf{B}} \restriction H_\kappa\right) \big/ F\right) \big/ (G/_F) \;\cong\; \left(V_+^{\mathsf{B}/_F} \restriction H_\kappa\right) \big/ (G/_F)$$

[4]I.e. such that for each $[\tau]_F$ in $V^{\mathsf{B}}/_F$ we have $\Gamma([\tau]_F) \in [\tau]_F$

□

Proposition 4.2.3. *Let* $\mathcal{M} = \left(M, E^{\mathcal{M}}\right)$ *and* $\mathcal{N} = \left(N, E^{\mathcal{N}}\right)$ *be elementarily equivalent models of* $\mathsf{ZFC} - \mathsf{P}$. *Then* $\mathcal{M}_+ = \left(M, E^{\mathcal{M}}_{\Delta_0^+}\right) \equiv \left(N, E^{\mathcal{N}}_{\Delta_0^+}\right) = \mathcal{N}_+$.

Proof. Let ψ be any sentence in the extended language. Then

$$\begin{aligned}
[\![\, \psi \,]\!]^{\mathcal{M}_+} &= [\![\, \psi^* \,]\!]^{\mathcal{M}_+} && \text{(by 2.1.15)}\\
&= [\![\, \psi^* \,]\!]^{\mathcal{M}} && \text{(since } \mathcal{M} = \mathcal{M}_+ \restriction \{\in\})\\
&= [\![\, \psi^* \,]\!]^{\mathcal{N}} && \text{(by assumption)}\\
&= [\![\, \psi^* \,]\!]^{\mathcal{N}_+} && \text{(since } \mathcal{N} = \mathcal{N}_+ \restriction \{\in\})\\
&= [\![\, \psi \,]\!]^{\mathcal{N}_+} && \text{(by 2.1.15)}
\end{aligned}$$

□

Proposition 4.2.4. [13]. *For each successor cardinal* λ *and each Woodin cardinal* $\kappa > \lambda$ *there is a* $<\kappa$*-complete Boolean algebra of size and density* κ *with the small completion property, denoted by* $\mathcal{T}_\kappa^\lambda$ *and called* the stationary tower of height κ and critical point λ, *such that the two-valued Tarski structure* $(H_\lambda, \in)$ *is elementarily equivalent to the Tarski quotients* $\left(V_\in^{\mathcal{T}_\kappa^\lambda} \restriction H_\kappa\right)/_G$ *of* $V_\in^{\mathcal{T}_\kappa^\lambda} \restriction H_\kappa$.

Conjecture 4.2.5. *The stationary towers of Woodin height* κ *and with regular critical points are regularly iterable and* $<\kappa$*-disjointable.*

Conjecture 4.2.6 (weaker form)**.** *For every successor cardinal* λ *there is a Woodin cardinal* κ *above* λ *such that the stationary tower of height* κ *and critical point* λ *is regularly iterable and* $<\kappa$*-disjointable.*

Theorem 4.2.7 (under conjecture 4.2.5 or 4.2.6)**.** *Let* λ *be a successor cardinal. Then every elementary equivalent extension of* $\left\langle H_\lambda, \in_{\Delta_0^+} \right\rangle$ *embeds elementarily into a forcing extension of some* H_κ.

Proof. Let $\mathcal{M} = \langle M, E \rangle$ be an elementarily equivalent extension of $\langle H_\lambda, \in \rangle$. Let κ be a Woodin cardinal above λ and $|M|$. Using conjecture 4.2.5, let $\mathcal{F}$ be an exhaustive disjointing regular game iteration of $\mathcal{T}_\kappa^\lambda$ and F be its associated game filter. By remark 3.0.3, F is $<\kappa$-complete.

$\mathcal{F}$ is disjointing, so $\varinjlim \mathcal{F}$ is $<\kappa$-disjointable. By the convergence theorem 3.0.7 we have $\mathcal{T}_\kappa^\lambda/_F \cong \varinjlim \mathcal{F}$. Let A be a countable maximal antichain on $\mathcal{T}_\kappa^\lambda/_F$. Then $\breve{A}$ is a countably incomplete countable prefilter on $\mathcal{T}_\kappa^\lambda/_F$ with meet 0_{B}. By 1.3.3, let H be a good ultrafilter on $\mathcal{T}_\kappa^\lambda/_F$ extending $\breve{A}$.

Since $\mathcal{T}_\kappa^\lambda/_F$ is $<\kappa$-c.c. and $<\kappa$-complete, $V_+^{(\mathcal{T}_\kappa^\lambda/_F)} \restriction H_\kappa$ has the mixing property and by 2.2.12, is golden. Then by 2.6.1 and 4.2.2 we have that

$$\left(V_+^{\mathsf{B}} \restriction H_\kappa\right) \Big/ G \;\cong\; \left(\left(V_+^{\mathcal{T}_\kappa^\lambda} \restriction H_\kappa\right) \Big/ F\right) \Big/ (G/_F) \;\cong\; \left(V_+^{\mathcal{T}_\kappa^\lambda/_F} \restriction H_\kappa\right) \Big/ (G/_F)$$

is $<\kappa$-saturated, where $G/_F = H$.

By 4.2.4, we have $(H_\lambda, \in) \equiv \left(V_\in^{\mathcal{T}_\kappa^\lambda} \restriction H_\kappa\right)/_G$. By 4.2.3 we have that $(M, E_{\Delta_0^+}) \equiv (H_\lambda, \in_{\Delta_0}) \equiv \left(V_+^{\mathcal{T}_\kappa^\lambda} \restriction H_\kappa\right)/_G$. But by 2.1.6 we have that $(M, E_{\Delta_0^+}) \preceq \left(V_+^{\mathcal{T}_\kappa^\lambda} \restriction H_\kappa\right)/_G$. □

Index

Bibliography

[1] Giorgio Audrito, Raphäel Carroy, Silvia Steila, and Matteo Viale, *Iterated forcing, category forcings, generic ultrapowers, generic absoluteness*, Book in preparation, 2017.

[2] James Earl Baumgartner, *Iterated forcing*, Surveys in set theory, London Math. Soc. Lecture Note Ser., vol. 87, Cambridge Univ. Press, Cambridge, 1983, pp. 1–59. MR 823775

[3] J. L. Bell, *Set theory: boolean-valued models and independence proofs*, Oxford University Press, Oxford, 2005.

[4] Chen Chung Chang and H. Jerome Keisler, *Model theory*, North-Holland Pub. Co.; American Elsevier Amsterdam, New York, 1973.

[5] Paul Joseph Cohen, *The independence of the continuum hypothesis*, Proc. Nat. Acad. Sci. U.S.A. **50** (1963), 1143–1148. MR 0157890 (28 #1118)

[6] Steven R. Givant and Paul Richard Halmos, *Introduction to Boolean algebras*, Springer Verlag, 2009.

[7] Kurt Gödel, *The Consistency of the Continuum Hypothesis*, Annals of Mathematics Studies, No. 3, Princeton University Press, Princeton, N. J., 1940. MR 0002514

[8] Joel David Hamkins and Daniel Evan Seabold, *Well-founded boolean ultrapowers as large cardinal embeddings*, 2012.

[9] Thomas J. Jech, *Multiple forcing*, Cambridge Tracts in Mathematics, vol. 88, Cambridge University Press, Cambridge, 1986. MR 895139

[10] Thomas J. Jech, *Set theory*, Springer Monographs in Mathematics, Springer, Berlin, 2003, The third millennium edition, revised and expanded. MR 1940513

[11] Akihiro Kanamori, *The higher infinite*, second ed., Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2003, Large cardinals in set theory from their beginnings. MR 1994835 (2004f:03092)

[12] Kenneth Kunen, *Set theory*, Studies in Logic and the Foundations of Mathematics, vol. 102, North-Holland, Amsterdam, 1980, An introduction to independence proofs. MR 597342

[13] Paul Bradley Larson and W Hugh Woodin, *The stationary tower: Notes on a course by w. hugh woodin*, vol. 32, American Mathematical Soc., 2004.

[14] Richard Mansfield, *The theory of boolean ultrapowers*, Annals of Mathematical Logic **2** (1971), 297–323.

[15] Annalisa. Marcja and Carlo Toffalori, *A guide to classical and modern model theory*, Trends in Logic, Springer Netherlands, 2003.

[16] David Marker, *Model theory : An introduction*, Graduate Texts in Mathematics, Springer New York, 2013.

[17] James Donald Monk, *Handbook of boolean algebras*, vol. 1, Academic Press, London, 1991.

[18] Francesco Parente, *Boolean valued models, saturation, forcing axioms*, (2015), Unpublished Master Thesis, University of Pisa 2015.

[19] Moreno Pierobon, *Saturated structures constructed using forcing and applications*, (2019), Master Thesis, University of Turin 2019.

[20] Moreno Pierobon and Matteo Viale, *The duality between boolean valued models and topological presheaves*, 2022.

[21] Joseph R. Shoenfield, *Mathematical logic*, Reading, MA, USA: Reading, Mass., Addison-Wesley Pub. Co., 1967.

[22] Roman Sikorski, *Boolean algebras*, Third edition. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 25, Springer-Verlag New York Inc., New York, 1969. MR 0242724

[23] Douglas Ulrich, *Some applications of set theory to model theory*, Bull. Symb. Log. **25** (2019), no. 2, 214–215. MR 3984977

[24] Giorgio Venturi and Matteo Viale, *The model companions of set theory*, 2019.

[25] Matteo Viale, *Absolute model companionship, forcibility, and the continuum problem*, 2021.

[26] Matteo Viale, Giorgio Audrito, Silvia Steila, and Raphaël Carroy, *Iterated forcing, category forcings, generic ultrapowers, generic absoluteness*, Book in preparation.