



DOTTORATO DI RICERCA IN MATEMATICA PURA E APPLICATA

XXXV Cycle

Università degli Studi di Torino

Dipartimento di Matematica "G. Peano"

Politecnico di Torino

Dipartimento di Matematica "G.L. Lagrange"

Doctoral Thesis

ON THE CLASSIFICATION OF FANO MANIFOLDS AND THEIR ANTICANONICAL BASE LOCUS

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Contents

Acknowledgements v

Introduction 1

Chapter 1. Preliminaries 3
1.1. Notation and conventions 3
1.2. Fano manifolds and Mori theory 4
1.3. Projective bundles 8
1.4. Cyclic coverings 12

Chapter 2. Fano fourfolds containing a prime divisor of Picard number 1 17
2.1. Introduction 17
2.2. The 28 families of Fano fourfolds 18
2.3. Numerical invariants 22
2.4. Deformations and automorphism groups 26
2.5. Base locus of $-K_X$ 28
2.6. Rationality 30
2.7. Conclusions and final tables 30

Chapter 3. Fano manifolds with Lefschetz defect 3 35
3.1. Introduction 35
3.2. Construction A 38
3.3. Construction B 40
3.4. Proof of Theorem 3.4 42
3.5. Geometry of the del Pezzo fibration $\sigma\colon X \to T$ 52
3.6. The case of dimension 4 57
3.7. The case $\rho_X = 5$ 59
3.8. Base locus of $-K_X$ 63

Chapter 4. Anticanonical base locus of Fano manifolds 71
4.1. Introduction 71
4.2. Examples 72
4.3. Preliminary technical results 75
4.4. The surface case 79
4.5. The threefold case 81

Bibliography 95

Acknowledgements

In memory of my dear friend, Brenno

The process of writing a doctoral thesis has been a very hard and enriching experience: it represents the conclusion of an important "journey", the doctorate, and symbolises my development as a scientist and researcher. I am very grateful to both the University of Turin and the Polytechnic University of Turin, and to their respective algebraic geometry groups, because they allowed me to grow mathematically in such a welcoming and instructive environment.

My first thanks go to my supervisor and coauthor, Professor Cinzia Casagrande, for her invaluable guidance and inspiration. I am very lucky to have been able to work under her supervision.

Furthermore, I would like to thank my other coauthor, Prof. Eleonora Romano, and the supervisor of my master's thesis, Prof. Angelo Lopez, for the very fruitful conversations and for the advice they gave me. A special mention must also go to Prof. Andreas Höring and to the Laboratoire de Mathématiques J.A. Dieudonné of the Université Côte d'Azur for hosting me during the Autumn of 2021 and for the thriving collaboration that began on that occasion.

Then, I would like to thank my family and friends. First and foremost my mother, father and siblings, who aided me and nurtured my passions with great love; Tiziana, who is very dear to me, for the encouragement and warmth she has always shown me; my doctoral colleagues and friends - Irene, Andrea, Elena, Francesca, Fabio and many others - for their continuous help and the many memories we share.

My final thanks go to D. Your support and affection pushed me to pursue my ambition with no fear: my life today would not be the same.

Ringraziamenti

In memoria del mio caro amico, Brenno

La stesura della tesi di dottorato è stata un'esperienza molto impegnativa e al tempo stesso arricchente: rappresenta la conclusione di un importante percorso, il dottorato, e simboleggia la mia crescita come scienziato e ricercatore. Sono molto grato all'Università di Torino e al Politecnico di Torino, e ai rispettivi gruppi di geometria algebrica, poiché mi hanno permesso di crescere matematicamente in un ambiente così accogliente e formativo.

Il mio primo ringraziamento va alla mia relatrice e coautrice, la Prof.ssa Cinzia Casagrande, per la sua preziosa guida e ispirazione. Sono molto fortunato ad aver potuto lavorare sotto la sua supervisione.

Inoltre, vorrei ringraziare l'altra mia coautrice, la Prof.ssa Eleonora Romano, e il relatore della mia tesi di laurea magistrale, il Prof. Angelo Lopez, per le conversazioni costruttive e per i consigli che mi hanno dato. Una menzione speciale va anche al Prof. Andreas Höring e al Laboratoire de Mathématiques J.A. Dieudonné dell'Université Côte d'Azur per avermi ospitato durante l'autunno del 2021 e per la fruttuosa collaborazione iniziata in quell'occasione.

Vorrei poi ringraziare la mia famiglia e i miei amici. Innanzitutto mia madre, mio padre e i miei fratelli, che mi hanno sostenuto e hanno coltivato le mie passioni con grande amore; Tiziana, a me molto cara, per l'incoraggiamento e il calore che mi ha sempre dimostrato; i miei colleghi e amici di dottorato - Irene, Andrea, Elena, Francesca, Fabio e molti altri - per il loro continuo aiuto e i tanti ricordi che condividiamo.

Il mio ultimo ringraziamento va a D. Il tuo sostegno e il tuo affetto mi hanno spinto a perseguire le mie ambizioni senza paura: la mia vita oggi non sarebbe la stessa.

Introduction

One of the main goals in many areas of Mathematics is to classify the studied "objects" up to some equivalence relation. In the case of algebraic geometry, this often results in the classification of projective varieties up to isomorphism. Here, we focus on a very natural class of projective varieties: complex Fano manifolds. They are defined as follows.

Definition 0.1. Let X be a smooth (complex) projective variety. We say that X is a *Fano manifold* if the anticanonical divisor $-K_X$ is ample.

These varieties have been widely studied both in classical and modern algebraic geometry, in particular due to their relation with the Minimal Model Program: they appear as the *general fibres of Mori fibre spaces*, one of the possible outcomes in the run of the MMP.

The classification of Fano manifolds has been completed up to dimension 3, and although many results have already been proven, we are still lacking a complete classification starting from dimension 4:

(i) there is only one Fano manifold of dimension 1, the projective line $\mathbb{P}^1$;
(ii) Fano manifolds of dimension 2, called del Pezzo surfaces, form 10 families: they are $\mathbb{P}^2$, the product $\mathbb{P}^1 \times \mathbb{P}^1$, and the blow-up of $\mathbb{P}^2$ along at most 8 points in general position;
(iii) in dimension 3, the classification is due to Mori and Mukai for the case of Picard number ≥ 2, and to Iskovskikh (and the Russian school) for the case of Picard number 1: Fano threefolds consist of 105 families.

In this text, we approach classification problems of higher-dimensional Fano manifolds via the use of birational geometry: the study of the Mori cone and extremal contractions appears to be a key tool for the description of their geometry. Specifically, this allows us to determine constructions that start from "simpler" varieties and that make use of well-known morphisms, such as blow-ups and projective bundles.

The text is divided into four chapters. In Chapter 1 we fix the notation and prove some preliminary results on Fano manifolds, base loci of linear systems, projective bundles and cyclic coverings.

Chapter 2 is devoted to the study of Fano manifolds that contain a prime divisor D of Picard number 1. These varieties were first studied in [**Tsu06**], where the author proved a classification result assuming that D is isomorphic to the projective space. Then, in [**CD15**], a characterisation result in arbitrary dimension was proved in the case of Picard number 3, and the above additional assumption was removed: the authors provided an explicit construction. Starting from this result, and the classification of Fano threefolds, in this chapter we classify Fano manifolds of Picard number 3 that

contain a prime divisor of Picard number 1 in the case of dimension 4. We study many of their properties in detail. The main result of the chapter is Theorem 2.1. Chapter 2 is based on [**Sec23**].

Theorem 0.2 (Thm. 2.1)**.** *There are* 28 *families of smooth Fano fourfolds X with $\rho_X = 3$ having a prime divisor D of Picard number 1.*

Among them, 22 *families consist of rational varieties, and the very general variety of* 4 *other families is not rational.*

Furthermore, the linear system $|-K_X|$ is free, with the exception of two families where the base locus consists of either 1 *or* 2 *points. In all cases, a general element of $|-K_X|$ is smooth.*

Chapter 3 collects the results of a joint work with C. Casagrande and E.A. Romano [**CRS22**], and is dedicated to the study of Fano manifolds with Lefschetz defect 3. The Lefschetz defect, denoted by δ_X, is a numerical invariant associated to a Fano manifold X. It was introduced by Casagrande in [**Cas12**], and it depends on the Picard number of prime divisors contained in X. After we prove a characterisation result (the structure Theorem 3.4), we provide several applications and study in detail some special cases. For instance, we complete the classification of Fano fourfolds with $\delta_X = 3$ started in [**CR22**], and we study in detail the case of Picard number 5. Finally, we point out that Section 3.8 is the result of a later study and is not contained in [**CRS22**].

Theorem 0.3 (Thm. 3.4)**.** *Every smooth Fano variety X with $\delta_X = 3$ is obtained with Construction A (Prop. 3.2) or Construction B (Prop. 3.3).*

More precisely, there exists a smooth Fano variety T with $\dim T = \dim X - 2$, $\rho_T = \rho_X - 4$ and $\delta_T \leq 3$, and a $\mathbb{P}^2$-bundle Z over T, such that X is obtained by blowing-up Z along three pairwise disjoint smooth, irreducible, codimension 2 subvarieties.

The two explicit constructions differ in the centre of the blow-up.

Finally, in Chapter 4 we study the base locus of the linear system $|-K_X|$. The anticanonical divisor, which is ample by definition, is not necessarily very ample. In fact, it is not even globally generated in general, so that $\mathrm{Bs}\,|-K_X|$ might be non-empty. We focus on Fano manifolds that satisfy the following condition: there exists a codimension 2 subvariety $B \subset X$ such that B is contained in $\mathrm{Bs}\,|-K_X|$; in particular, we aim to understand the relation between $\mathrm{Bs}\,|-K_X|$ and the birational geometry of X. In this chapter we address the low-dimensional case, avoiding results coming from the classification of Fano manifolds as much as possible, in order to pave the way for the study in higher dimension. Chapter 4 is based on an ongoing work with A. Höring, and its main result is Theorem 4.3.

Theorem 0.4 (Thm. 4.3)**.** *Let X be a smooth Fano threefold with $\rho_X \geq 2$ and let $\mu\colon X \to T$ be an elementary Mori contraction with fibres of dimension ≤ 1. Assume that there exists an irreducible curve $B \subseteq \mathrm{Bs}\,|-K_X|$, and that B intersects a non-trivial fibre of μ. Then μ contracts B to a point, T is a smooth Fano variety of index r, and $\mu(B)$ belongs to the base locus of $|-\frac{1}{r}K_T|$.*

We refer to Sections 2.1, 3.1 and 4.1 for a detailed introduction to each chapter.

Unless specified otherwise, a *scheme* is a separated scheme of finite type over $\mathbb{C}$, and a *variety* is an integral scheme. We will occasionally denote by *manifold* a smooth variety.

CHAPTER 1

Preliminaries

1.1. Notation and conventions

We work over the field of complex numbers. Let X be a smooth projective variety of arbitrary dimension n.

• We denote by K_X the canonical divisor of X, that is the divisor class corresponding to the invertible sheaf $\omega_X := \bigwedge^n \Omega_X$. When X is singular, by ω_X we mean the dualizing sheaf.

• $\mathcal{N}_1(X)$ $\big($respectively, $\mathcal{N}^1(X)\big)$ is the real vector space of 1-cycles (respectively, Cartier divisors) with real coefficients, modulo numerical equivalence. These vector spaces are dual to each other (via the intersection number) and we denote by ρ_X the Picard number of X, that is $\rho_X := \dim \mathcal{N}_1(X) = \dim \mathcal{N}^1(X)$.

• Let C be a 1-cycle of X, and D a divisor of X. We denote by $[C]$ (respectively, $[D]$) the numerical equivalence class in $\mathcal{N}_1(X)$ $\big($respectively, in $\mathcal{N}^1(X)\big)$.

• We denote by $C^\perp \subset \mathcal{N}^1(X)$ $\big($respectively, $D^\perp \subset \mathcal{N}_1(X)\big)$ the orthogonal hyperplane, that is

$$C^\perp := \{[D] \in \mathcal{N}^1(X) \,|\, D \cdot C = 0\} \quad \Big(\text{resp. } D^\perp := \{[C] \in \mathcal{N}_1(X) \,|\, D \cdot C = 0\}\Big).$$

Analogously, we say that a 1-cycle C is D-positive if $D \cdot C > 0$ (respectively, D-negative if $D \cdot C < 0$).

• A facet of a convex polyhedral cone $\mathcal{C}$ is a face of codimension 1; moreover $\mathcal{C}$ is simplicial when it can be generated by linearly independent elements.

• Let S be a subset of $\mathcal{N}_1(X)$. We denote by $\langle S \rangle$ the convex cone generated by S, and we set

$$S_{D \geq 0} := \{C \in S | D \cdot C \geq 0\},$$

where D is Cartier divisor on X. Analogously we have $S_{D \leq 0}$, $S_{D>0}$ and $S_{D<0}$.

• The symbol $\equiv$ stands for numerical equivalence (for both 1-cycles and divisors), while the symbol $\sim$ stands for linear equivalence of divisors.

Since we will be studying linear systems of divisors and their base locus several times in this text, we collect here two preliminary results that we will need in the sequel.

Lemma 1.1. *Let X be a smooth projective variety, and let D_1, D_2 be two divisors. Then:*

$$\mathrm{Bs}\,|D_1 + D_2| \subseteq \mathrm{Bs}\,|D_1| \cup \mathrm{Bs}\,|D_2|.$$

Proof. Suppose first that D_1, D_2 are effective divisors. Let $P \in X$ be a point in $\mathrm{Bs}\,|D_1 + D_2|$: then $s(P) = 0$ for all sections $s \in H^0(X, D_1 + D_2)$. In particular, given sections $s_1 \in H^0(X, D_1)$ and $s_2 \in H^0(X, D_2)$, their product $s_1 s_2$ belongs to

$H^0(X, D_1 + D_2)$. Therefore $s_1(P)s_2(P) = 0$ for all $s_i \in H^0(X, D_i)$, $i = 1, 2$, thus proving the statement.

If one of the divisors, say D_1, is not effective, then $\operatorname{Bs}|D_1| = X$ and the claim is trivial. Lastly, if $D_1 + D_2$ is not effective, then at least one of the D_i's is neither, and we reason as above. ■

The above Lemma appears to be a valid tool to study the set-theoretic base locus of divisors. That is, assume we have $D \sim D_1 + D_2 \sim D_1' + D_2'$: the above Lemma yields

$$\operatorname{Bs}|D| \subseteq \big(\operatorname{Bs}|D_1| \cup \operatorname{Bs}|D_2|\big) \cap \big(\operatorname{Bs}|D_1'| \cup \operatorname{Bs}|D_2'|\big).$$

For instance, if D_i, D_i' are prime divisors, then

$$\big(\operatorname{Bs}|D_1| \cup \operatorname{Bs}|D_2|\big) \cap \big(\operatorname{Bs}|D_1'| \cup \operatorname{Bs}|D_2'|\big) \subseteq \big(D_1 \cup D_2\big) \cap \big(D_1' \cup D_2'\big).$$

Once we have limited the base locus of a divisor D to a subvariety of X, we can verify which points of this subvariety are in fact basepoints of $|D|$ with the following Lemma.

Lemma 1.2. *Let X be a smooth projective variety, $Y \subset X$ a subvariety and let D be a divisor. Then:*

(i) $\operatorname{Bs}|D_{|Y}| \subseteq \operatorname{Bs}|D| \cap Y$*;*

(ii) if the restriction $H^0(X, D) \twoheadrightarrow H^0(Y, D_{|Y})$ is surjective, then $\operatorname{Bs}|D_{|Y}| = \operatorname{Bs}|D| \cap Y$.

Proof. Consider the restriction map

$$r\colon H^0(X, D) \to H^0(Y, D_{|Y})$$

and let $V := \operatorname{Im}(r)$. It follows immediately that $\operatorname{Bs}|V| = \operatorname{Bs}|D| \cap Y$. As $V \subseteq H^0(Y, D_{|Y})$ is a vector subspace, we obtain (i). When r is surjective, hence $V = H^0(Y, D_{|Y})$, we obtain (ii). ■

1.2. Fano manifolds and Mori theory

In this section, we set out the fundamentals of Mori theory, with the objective of understanding the properties of the Mori cone $\overline{\operatorname{NE}}(X)$ and its influence on the geometry of the ambient variety X. We will then see some applications to the case of Fano manifolds. Although the results in this section are stated for smooth varieties, they are also valid in a more general setting. Our main source will be [**Deb01**], but all the results on Mori theory collected in this section can also be found in other texts, such as [**KM98**] or [**Mat02**].

1.2.1. The Mori cone and extremal contractions. Let us begin with the following definition.

Definition 1.3. Let $\operatorname{NE}(X) \subset \mathcal{N}_1(X)$ be the cone generated by the classes of effective 1-cycles. An *extremal ray* R of X is a one-dimensional face of the *Mori cone* $\overline{\operatorname{NE}}(X)$, the closure of $\operatorname{NE}(X)$ in $\mathcal{N}_1(X)$.

It follows from its definition that $\operatorname{NE}(X)$ is a pointed convex cone, that is $\operatorname{NE}(X)$ does not contain any line and, for any $C_1, C_2 \in \operatorname{NE}(X)$ and positive $a, b \in \mathbb{R}$, the sum $aC_1 + bC_2 \in \operatorname{NE}(X)$. In general $\operatorname{NE}(X)$ it is not closed nor polyhedral, see for instance [**Deb01**, Example 1.35] and [**Deb01**, Example 6.3].

The cone theorem [**Deb01**, Theorem 6.1] describes the structure of the Mori cone: the half-cone $\overline{\mathrm{NE}}(X)_{K_X<0}$ is generated by a countable family of extremal rays, whereas $\overline{\mathrm{NE}}(X)_{K_X\geq 0}$ remains generally unknown. For the more general logarithmic version, see [**Deb01**, Theorem 7.38].

Theorem 1.4 (Cone Theorem)**.** *Let X be a smooth projective variety. There exists a countable family $\{\Gamma_i\}_{i\in I}$ of rational curves on X such that*

$$\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(X)_{K_X\geq 0} + \sum_{i\in I} \mathbb{R}^+[\Gamma_i],$$

where the $\mathbb{R}^+[\Gamma_i]$ are all the K_X-negative extremal rays of $\overline{\mathrm{NE}}(X)$. These rays are locally discrete in the half-space $\mathcal{N}_1(X)_{K_X<0}$.

The dual cone of $\overline{\mathrm{NE}}(X)$ in $\mathcal{N}^1(X)$ is the nef cone $\mathrm{Nef}(X) \subset \mathcal{N}^1(X)$, that is the closed convex cone generated by the classes of nef divisors. Recall that a divisor D is nef if $D \cdot C \geq 0$ for every irreducible curve C on X.

The Kleiman's ampleness criterion [**Deb01**, Theorem 1.27] provides a numerical condition for a divisor to be ample.

Theorem 1.5 (Kleiman's ampleness criterion)**.** *Let X be a smooth projective variety and let H be a Cartier divisor on X. Then, D is ample if and only if $D \cdot C > 0$ for all nonzero 1-cycles $C \in \overline{\mathrm{NE}}(X)$.*

It follows that the sum of a nef divisor and an ample divisor is ample. Thus, the interior of $\mathrm{Nef}(X)$ is non-empty and consists of the classes of ample divisors.

As mentioned at the beginning of the section, the Mori cone contains information on the geometry of X. Specifically, as we will see in Theorem 1.8, we can associate to any K_X-negative extremal ray $R \subseteq \overline{\mathrm{NE}}(X)$ a surjective morphism with connected fibres $c_R\colon X \to Y$ such that, for any irreducible curve C on X,

$$[C] \in R \iff c_R(C) = \{pt\}.$$

The morphism c_R is called the *Mori contraction* corresponding to the K_X-negative ray R of $\overline{\mathrm{NE}}(X)$ (in short, the contraction of R).

More generally, a *contraction* of X is a surjective morphism $\varphi\colon X \to Y$ with connected fibres, where Y is normal and projective. In particular, $\varphi_*\mathcal{O}_X \cong \mathcal{O}_Y$.

In order to prove Theorem 1.8, we need to state an important result, that is the basepoint-free theorem [**Deb01**, Theorem 7.32].

Theorem 1.6 (Basepoint-free Theorem)**.** *Let X be a smooth projective variety, and let D be a nef Cartier divisor on X. Assume that $aD - K_X$ is nef and big for some positive rational a. Then D is semiample, that is the linear system $|mD|$ is basepoint-free for all integers $m \gg 0$.*

If D is a semiample divisor, the morphism $\varphi\colon X \to Y$ associated to the linear system $|mD|$ for $m \gg 0$ is a contraction, and there exists an ample Cartier divisor A on Y such that $mD \sim \varphi^*A$ ([**Laz04a**, Theorem 2.1.27]).

We are now going to show that the contraction of a K_X-negative extremal ray always exists.

Lemma 1.7 ([**Deb01**], Corollary 6.9)**.** *Let X be a smooth projective variety and let R be a K_X-negative extremal ray. Then there exists a nef divisor M_R on X such that:*

(i) $(M_R)^\perp \cap \overline{\mathrm{NE}}(X) = R$, *that is* $M_R \cdot C = 0$ *for all* $[C] \in R$ *and* $M_R \cdot C > 0$ *for all* $[C] \in \overline{\mathrm{NE}}(X) \setminus R$;

(ii) the divisor $mM_R - K_X$ is ample for $m \gg 0$.

The divisor M_R is called a *supporting divisor* of R. Finally, Theorem 1.6 and the above Lemma yield the following result ([**Deb01**, Theorem 7.39]).

Theorem 1.8 (Contraction Theorem)**.** *Let X be a smooth projective variety and let $R := \mathbb{R}^+[\Gamma]$ be a K_X-negative extremal ray. Then:*

(i) the contraction $c_R \colon X \to Y$ of R exists;

(ii) there is an exact sequence

$$\begin{aligned} 0 \longrightarrow \mathrm{Pic}(Y) \xrightarrow{(c_R)^*} \mathrm{Pic}(X) &\longrightarrow \mathbb{Z} \\ [D] &\longmapsto D \cdot \Gamma \end{aligned}$$

Proof. Let $R := \mathbb{R}^+[\Gamma]$ be a K_X-negative extremal ray generated by the class of a rational curve Γ on X, and let M_R be the supporting divisor of R. Then $mM_R - K_X$ is ample for $m \gg 0$ and, by Theorem 1.6, we have that M_R is semiample. Therefore, the morphism associated to mM_R, $m \gg 0$, yields the contraction of R. This proves (i).

As for (ii), first note that the pull-back $(c_R)^*$ is injective: we have $(c_R)_*\mathcal{O}_X \cong \mathcal{O}_Y$, and so $(c_R)_*(c_R)^*\mathcal{L} \cong \mathcal{L}$ for any $\mathcal{L} \in \mathrm{Pic}(Y)$ by the projection formula. Now, let D be a Cartier divisor that vanishes on Γ: since M_R has positive intersection with all 1-cycles in $\overline{\mathrm{NE}}(X) \setminus R$, the divisor $mM_R + D$ is a supporting divisor of R for m sufficiently large. As mM_R and $mM_R + D$ are both in $(c_R)^* \mathrm{Pic}(Y)$, so it is D. This concludes the proof. ∎

Remark 1.9. Note that Lemma 1.7 and Theorem 1.8 hold for any K_X-negative extremal face $F \subsetneq \overline{\mathrm{NE}}(X)$ with the same proofs. Therefore, we can associate to any extremal face of $\overline{\mathrm{NE}}(X)$ a nef and not ample divisor (the supporting divisor of F) and a Mori contraction that contracts the irreducible curves whose class lays in F. Moreover, by Theorem 1.8(ii), we have $\dim(F) = \rho_X - \rho_Y$.

We conclude this subsection with a few more definitions. Let $c_R \colon X \to Y$ be the contraction of a K_X-negative extremal ray R: there are three different types, depending on the locus of contracted curves.

(i) If $\dim(X) > \dim(Y)$, we say that c_R is a *fibre contraction*;

(ii) if c_R is birational and $\mathrm{Exc}(c_R)$ has codimension 1, we say that c_R is *divisorial*[1];

(iii) if c_R is birational and $\mathrm{Exc}(c_R)$ has codimension ≥ 2, way say that c_R is *small*.

A divisorial contraction is, for instance, the blow-up of Y along a subvariety of codimension ≥ 2, while the natural projection associated to a projective bundle (Section 1.3) is an example of a fibre contraction.

[1] If c_R is birational and an irreducible component Z of $\mathrm{Exc}(c_R)$ has codimension 1, then $Z = \mathrm{Exc}(c_R)$ is a prime divisor by [**Deb01**, Prop. 6.10].

Definition 1.10. Let $\varphi\colon X \to Y$ be a contraction. The *relative cone* $\mathrm{NE}(\varphi)$ of φ is the convex subcone of $\overline{\mathrm{NE}}(X)$ generated by classes of curves contracted by φ. A contraction is called *elementary* if $\rho_X - \rho_Y = 1$, that is if $\mathrm{NE}(\varphi)$ is an extremal ray.

Note that $\mathrm{NE}(\varphi)$ is an extremal face of $\overline{\mathrm{NE}}(X)$ by [**Deb01**, Prop. 1.14(a)].

Definition 1.11. Let X be a smooth projective variety and let R be a K_X-negative extremal ray. The *length* of R is

$$\ell(R) = \min\{-K_X \cdot \Gamma \,|\, \Gamma \text{ a rational curve in } X,\, [\Gamma] \in R\}.$$

Definition 1.12. A *conic bundle* $\varphi\colon X \to Y$ is a K_X-negative fibre contraction where every fibre is one-dimensional. In particular, the φ-fibres are (non necessarily smooth) plane conics.

1.2.2. Fano manifolds. We now focus on Fano manifolds, and show how the above results apply in this case. Let us recall the following definition:

Definition 1.13. Let X be a smooth projective variety. We say that X is a *Fano manifold* if the anticanonical divisor $-K_X$ is ample.

Remark 1.14. When X is Fano, the convex cone $\mathrm{NE}(X)$ is closed, rational and polyhedral, that is $\mathrm{NE}(X)$ is generated by a finite number of extremal rays which contain the class of an effective 1-cycle with integer coefficients.

In fact, the cone theorem 1.4 implies that $\overline{\mathrm{NE}}(X) \setminus \{0\}$ is contained in the half-space $\mathcal{N}_1(X)_{K_X<0}$, so that the set of K_X-negative extremal rays is finite. Thus,

$$\overline{\mathrm{NE}}(X) = \mathrm{NE}(X) = \sum_{i=1}^{m} \mathbb{R}^+[\Gamma_i]$$

and all the contractions of extremal rays are Mori contractions.

In view of the above remark, the Kleiman's criterion 1.5 implies that, give a Fano manifold X, a divisor D is ample if and only if D has positive intersection with any curve on X. In particular, it suffices to compute the intersection numbers $D \cdot C_i$, where $\{C_i\}_{i=1}^m$ is a set of generators of the extremal rays of $\mathrm{NE}(X)$: D is ample if and only if $D \cdot C_i > 0$ for all $i = 1, \ldots, m$.

The following is a consequence of Theorem 1.6.

Corollary 1.15. *Let X be a Fano manifold. Then, every nef divisor is semiample.*

Proof. Let D be a nef divisor. Since $-K_X$ is ample, we have that $D - K_X$ is also ample, thus D is semiample by Theorem 1.6. ∎

Therefore, if D is a nef and *not ample* divisor on a Fano manifold X, the morphism defined by the linear system $|mD|$ for $m \gg 0$ corresponds to the contraction of the extremal face $F := D^\perp \cap \mathrm{NE}(X)$, and D is a supporting divisor. This shows that, in the above situation, we can "reverse" in some sense Lemma 1.7 and Theorem 1.8.

Lastly, we define the index of a Fano manifold and write an application of the Kodaira vanishing and Riemann-Roch theorems in the case of dimension 3.

Definition 1.16. The *index* of a Fano manifold, denoted by i_X, is the greatest positive integer r such that the anticanonical divisor $-K_X$ is linearly equivalent to rD for some ample Cartier divisor D in X. A divisor H such that $-K_X \sim i_X \cdot H$ is called a *fundamental divisor*.

Lemma 1.17. *Let X be a Fano threefold and let H be a fundamental divisor. Then, for any $d \geq 1$ we have*

$$h^0(X, dH) = 1 + \frac{2d}{i_X} + \frac{d \cdot H^3}{12}(i_X^2 + 3d \cdot i_X + 2d^2) > 0.$$

In particular, $h^0(X, -K_X) > 3$.

Proof. First note that the Kodaira vanishing theorem yields $h^i(X, dH) = 0$ for all $i > 0$, so that $\chi(X, dH) = h^0(X, dH)$. Then, the formula for $h^0(X, dH)$ follows from the Riemann-Roch theorem for smooth threefolds [**Har77**, Appendix A, Exercise 6.7], which in this case gives

$$\chi(X, dH) = \frac{1}{12} dH(dH - K_X)(2dH - K_X) + \frac{2d}{i_X} + 1.$$

■

We conclude by recalling that a Fano manifold X is rationally connected [**KMM92a**, Thm. 3.3], thus it is simply connected [**Cam91**, Thm. 3.5] and has $h^i(X, \mathcal{O}_X) = 0$ for $i > 0$ [**KMM92b**, Prop. 2.5]; moreover, linear and numerical equivalence of divisors coincide [**IP99**, Prop. 2.1.2].

1.3. Projective bundles

A projective bundle, or $\mathbb{P}^n$-bundle, over a projective manifold T is the projectivization of a rank $n+1$ vector bundle $\mathcal{E}$ over T and it is defined as $\mathbb{P}_T(\mathcal{E}) := \mathbf{Proj}_{\mathcal{O}_T}\big(\mathrm{Sym}(\mathcal{E})\big)$, where $\mathrm{Sym}(\mathcal{E})$ is the symmetric algebra of $\mathcal{E}$. It is equipped with a natural projection

$$\varphi \colon \mathbb{P}_T(\mathcal{E}) \longrightarrow T$$

and a tautological line bundle $\mathcal{O}_{\mathbb{P}_T(\mathcal{E})}(1)$. Similarly to vector bundles, there exists an affine open covering $\{U_i := \mathrm{Spec}(A_i)\}$ of T such that $\varphi^{-1}(U_i)$ is isomorphic to $\mathbb{P}^n_{A_i} \cong \mathbb{P}^n \times U_i$, and the fibres of φ over the closed points of T are isomorphic to $\mathbb{P}^n$. By construction, the tautological line bundle restricted to $\varphi^{-1}(U_i)$ (resp. to $\varphi^{-1}(t) \cong \mathbb{P}^n$) is isomorphic to $\mathcal{O}_{\mathbb{P}^n_{A_i}}(1)$ (resp. $\mathcal{O}_{\mathbb{P}^n}(1)$).

Let H be a tautological divisor, that is a divisor whose class corresponds to $\mathcal{O}_{\mathbb{P}_T(\mathcal{E})}(1)$. The canonical divisor formula for projective bundles ([**Deb01**, Example 6.14]) yields

$$K_{\mathbb{P}_T(\mathcal{E})} \sim \varphi^*\big(K_T + \det(\mathcal{E})\big) - (n+1)H. \tag{1.1}$$

Moreover, given a line bundle $\mathcal{L}$ on T and the $\mathbb{P}^n$-bundle $\overline{\varphi} \colon \mathbb{P}_T(\mathcal{E} \otimes \mathcal{L}) \to T$, there exists a natural isomorphism $\psi \colon \mathbb{P}_T(\mathcal{E} \otimes \mathcal{L}) \to \mathbb{P}_T(\mathcal{E})$ over T such that

$$\mathcal{O}_{\mathbb{P}_T(\mathcal{E}\otimes\mathcal{L})}(1) \cong \psi^*\mathcal{O}_{\mathbb{P}_T(\mathcal{E})}(1) \otimes \overline{\varphi}^*\mathcal{L}.$$

In general, a surjective morphism $\varphi \colon X \to T$ whose fibres are isomorphic to $\mathbb{P}^n$ does not necessarily have the structure of a $\mathbb{P}^n$-bundle over T. For a more detailed treatment, see [**Har77**, §II.7]; note that Hartshorne denotes the tautological line bundle by $\mathcal{O}(1)$.

We collect in the following remark some properties and notation on projective bundles that we are going to use throughout this section.

Remark 1.18. Let T be a smooth projective variety and $D_1, \dots, D_r$ divisors on T. Set $Z := \mathbb{P}_T\big(\mathcal{O}(D_1) \oplus \cdots \oplus \mathcal{O}(D_r)\big) \xrightarrow{\varphi} T$ and let H be a tautological divisor. For every $i = 1, \dots, r$ set

$$F_i := \mathbb{P}_T\Big(\bigoplus_{j \neq i} \mathcal{O}(D_j)\Big) \hookrightarrow Z$$

with the embedding given by the projection $\bigoplus_j \mathcal{O}_T(D_j) \twoheadrightarrow \bigoplus_{j\neq i} \mathcal{O}_T(D_j)$. Then:

$$F_i \sim H - \varphi^*(D_i) \ \text{ for every } i = 1, \dots, r, \quad \text{and} \quad K_Z - \varphi^* K_T \sim -F_1 - \cdots - F_r.$$

In fact, the former equivalence follows from [**Har77**, Prop. 1.6][2], while the latter follows from the former, (1.1) and $\det\big(\bigoplus_j \mathcal{O}_T(D_j)\big) \sim \sum_j D_j$.

Let $S_i \subset Z$ be the section of φ given by the projection $\bigoplus_j \mathcal{O}_T(D_j) \twoheadrightarrow \mathcal{O}_T(D_i)$: then S_i is the complete intersection of the divisors $\{F_j\}_{j\neq i}$ and $S_i \cap F_i = \emptyset$. Therefore $S_i \cap S_j = \emptyset$ for all $j \neq i$ and the normal bundle of S_i in Z is

$$\mathcal{N}_{S_i/Z} \cong \bigoplus_{j\neq i} \mathcal{O}_{S_i}\big(H - \varphi^*(D_j)\big) \cong \bigoplus_{j\neq i} \mathcal{O}_T\big(D_i - D_j\big). \tag{1.2}$$

Note that, when $r = 2$, we have $S_i = F_j$ for $i \neq j \in \{1, 2\}$.

We now study the base locus of the tautological divisor H. Although the following lemma is given for $r = 2$, the reasoning in its proof applies straightforwardly to any rank. For the reader's convenience, we choose not to state it in more generality, as the claim would become too intricate.

Lemma 1.19. *Notation as in Rem. 1.18 with $r = 2$. Then:*

$$\operatorname{Bs}|H| = \big(\varphi^{-1}\operatorname{Bs}|D_1| \cap S_1\big) \cup \big(\varphi^{-1}\operatorname{Bs}|D_2| \cap S_2\big) \cup \varphi^{-1}\big(\operatorname{Bs}|D_1| \cap \operatorname{Bs}|D_2|\big). \tag{1.3}$$

In particular, H is globally generated if and only if D_1, D_2 are globally generated.

Proof. Let $i \neq j \in \{1, 2\}$. By tensoring the exact sequence

$$0 \to \mathcal{O}_Z(-S_i) \to \mathcal{O}_Z \to \mathcal{O}_{S_i} \to 0$$

with the line bundle $\mathcal{O}_Z(H)$, we obtain the exact sequence

$$0 \to \mathcal{O}_Z(\varphi^* D_j) \to \mathcal{O}_Z(H) \to \mathcal{O}_T(D_i) \to 0. \tag{1.4}$$

Exercise III.8.4(a) in [**Har77**] yields isomorphisms $H^k(Z, \varphi^* D_i) \cong H^k(T, D_i)$ and $H^k(Z, H) \cong H^k(T, D_1) \oplus H^k(T, D_2)$ for $i = 1, 2$ and $k \geq 0$. Therefore, we can split the long exact sequence in cohomology given by (1.4) into short exact sequences

$$0 \to H^k(T, D_j) \to H^k(T, D_1) \oplus H^k(T, D_2) \to H^k(T, D_i) \to 0$$

for all $k \geq 0$. Indeed, by counting dimensions, we have that $H^0(Z, H) \to H^0(T, D_i)$ is surjective because $H^0(Z, \varphi^* D_j) \to H^0(Z, H)$ is injective, which implies the injectivity of $H^1(Z, \varphi^* D_j) \to H^1(Z, H)$, and so on.

[2] This result is given for ruled surfaces, that is $\mathbb{P}^1$-bundles over a smooth curve. Nonetheless, its proof can be easily extended to any $\mathbb{P}^n$-bundle $\mathbb{P}(\mathcal{E})$ by considering divisors of the form $\mathbb{P}(\mathcal{F}) \hookrightarrow \mathbb{P}(\mathcal{E})$ instead of sections of $\mathbb{P}(\mathcal{E})$, where the vector bundle $\mathcal{F}$ is the quotient of $\mathcal{E}$ by a line bundle.

In particular the restriction morphism

$$H^0(Z,H) \to H^0(S_i, H_{|S_i})$$

is surjective, thus $\operatorname{Bs}|H| \cap S_i = \varphi^{-1}(\operatorname{Bs}|D_i|) \cap S_i$ for $i = 1,2$ by Lemma 1.2.

Assume now that $\emptyset \neq \operatorname{Bs}|D_1| \cap \operatorname{Bs}|D_2| \ni t$. Let l_t be the fibre over t, and consider the restriction morphism to l_t

$$H^0(Z,H) \to H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) \cong \mathbb{C}^2.$$

The restriction is not surjective, otherwise $\operatorname{Bs}|H| \cap l_t = \emptyset$, a contradiction. If instead its image has dimension one, then any effective divisor in $|H|$ not containing l_t would intersect the latter in one point. This gives a contradiction, since we already know that $l_t \cap (S_1 \cup S_2)$ consists of two points belonging to $\operatorname{Bs}|H|$ (recall from Remark 1.18 that $S_1 \cap S_2 = \emptyset$). Therefore, the restriction morphism is trivial, and this concludes the proof of the inclusion

$$\operatorname{Bs}|H| \supseteq \big(\varphi^{-1}\operatorname{Bs}|D_1| \cap S_1\big) \cup \big(\varphi^{-1}\operatorname{Bs}|D_2| \cap S_2\big) \cup \varphi^{-1}\big(\operatorname{Bs}|D_1| \cap \operatorname{Bs}|D_2|\big).$$

For the converse, we may assume that $h^0(T,D_1), h^0(T,D_2) \neq 0$. Then, the linear system $|H|$ contains effective divisors $S_i + \varphi^* D_j$, for $i \neq j \in \{1,2\}$. It follows from Lemma 1.1 that

$$\operatorname{Bs}|H| \subseteq \big(S_1 \cup \varphi^{-1}\operatorname{Bs}|D_2|\big) \cap \big(S_2 \cup \varphi^{-1}\operatorname{Bs}|D_1|\big),$$

and this concludes the proof. ∎

Remark 1.20. In the proof of ($\subseteq$) we assume that $h^0(T,D_1), h^0(T,D_2) \neq 0$. Although this hypothesis is required to say that $S_i + \varphi^* D_j$ is effective, equation (1.3) is still valid. In fact, the exact sequence on global sections in the proof shows that if $h^0(T,D_i) = 0$, then $S_i \subseteq \operatorname{Bs}|H|$. Moreover, $h^0(T,D_1) = h^0(T,D_2) = 0$ if and only if $h^0(Z,H) = 0$. This is compatible with equation (1.3), since we have $\operatorname{Bs}|D_i| = T$ when $h^0(T,D_i) = 0$, and $\operatorname{Bs}|H| = Z$ when $h^0(Z,H) = 0$.

Let $\varphi\colon Z \to T$ be the projective bundle of Remark 1.18 with $r = 3$ and let $h\colon X \to Z$ be the blow-up of Z along one (or more) of the sections S_i. The following two Lemmata describe the behaviour of $-K_X$ along special divisors of X. We will apply these results in Chapter 3.

Lemma 1.21. *Notation as in Rem. 1.18 with $r = 3$. Let $h\colon X \to Z$ be the blow-up of S_3, with exceptional divisor $E_3 \subset X$.*

Then $E_3 \cong \mathbb{P}_T(\mathcal{O}(-K_T + D_3 - D_1) \oplus \mathcal{O}(-K_T + D_3 - D_2))$ and $(-K_X)_{|E_3}$ is the tautological line bundle.

Proof. We have $\mathcal{N}^\vee_{S_3/Z} \cong \mathcal{O}_T(D_1 - D_3) \oplus \mathcal{O}_T(D_2 - D_3)$ by (1.2), $E_3 \cong \mathbb{P}_T(\mathcal{N}^\vee_{S_3/Z})$ and $\mathcal{O}_X(-E_3)_{|E_3}$ is the tautological line bundle.

Let $E_i' \subset X$ be the transform of $F_i \subset Z$, for $i = 1,2,3$; note that $S_3 = F_1 \cap F_2$ and $S_3 \cap F_3 = \emptyset$. Set $\sigma := \varphi \circ h\colon X \to T$ and $\sigma_3 := \sigma_{|E_3}\colon E_3 \to T$. By Rem. 1.18 we have $-K_Z \sim \varphi^*(-K_T) + F_1 + F_2 + F_3$, so that

$$-K_X \sim h^*(-K_Z) - E_3 \sim \sigma^*(-K_T) + E_1' + E_2' + E_3' + E_3,$$
$$\text{and} \quad \mathcal{O}_X(-K_X)_{|E_3} \cong \mathcal{O}_{E_3}\big(\sigma_3^*(-K_T) + G_1 + G_2\big) \otimes \mathcal{O}_X(E_3)_{|E_3}$$

where $G_i := E_i' \cap E_3$ for $i = 1, 2$. Note that G_1 and G_2 are the sections of σ_3 corresponding to the two summands of $\mathcal{O}_T(D_1 - D_3) \oplus \mathcal{O}_T(D_2 - D_3)$, so by Rem. 1.18 we have $\mathcal{O}_{E_3}(G_1 + G_2) \cong \mathcal{O}_X(-2E_3 + \sigma^*(2D_3 - D_1 - D_2))_{|E_3}$. Finally we get

$$\mathcal{O}_X(-K_X)_{|E_3} \cong \mathcal{O}_X(-E_3)_{|E_3} \otimes \mathcal{O}_{E_3}\big(\sigma_3^*(-K_T + 2D_3 - D_1 - D_2)\big),$$

which yields the statement. ■

In a similar way one shows the following.

Lemma 1.22. *Notation as in Rem. 1.18 with $r = 3$. Let $h\colon X \to Z$ be the blow-up of S_2 and S_3. Let $E_1' \subset X$ be the transform of $F_1 \subset Z$.*

Then $E_1' \cong \mathbb{P}_T(\mathcal{O}(-K_T + D_2 - D_1) \oplus \mathcal{O}(-K_T + D_3 - D_1))$ and $(-K_X)_{|E_1'}$ is the tautological line bundle.

Again, let $\varphi\colon Z \to T$ be the projective bundle of Remark 1.18. The following Lemma provides a sufficient condition for Z to be a Fano manifold.

Lemma 1.23 ([**Deb01**], Example 3.3(2))**.** *Let T be a Fano manifold, and let*

$$\mathbb{Z} := \mathbb{P}_T\Big(\bigoplus_{j=1}^{r} \mathcal{O}(D_j)\Big) \xrightarrow{\varphi} T$$

be a projective bundle as in Remark 1.18. Assume that $D_1, \dots, D_r$ are nef divisors on T such that $-K_T - D_1 - \cdots - D_r$ is ample. Then Z is a Fano manifold.

Proof. Since the divisors D_i's are nef, we have that H (i.e. the tautological divisor of Z) is nef. Also, let $A := -K_T - D_1 - \cdots - D_r$: since A is ample and D_j is nef, the divisors $A + D_j$ are ample on T for all $j = 1, \dots, r$. Now observe that $H + \varphi^*A$ is the tautological divisor of $\mathbb{P}_T\big(\bigoplus_{j=1}^{r} \mathcal{O}(A + D_j)\big)$, thus it is ample on Z.

Finally we have

$$-K_Z \sim \varphi^*\big(-K_T - \sum_{j=1}^{r} D_j\big) + (n+1)H \sim (H + \varphi^*A) + (r-1)H,$$

where the first equivalence is given by (1.1), and so $-K_Z$ is the sum of an ample divisor and a nef divisor. Therefore $-K_Z$ is ample, thus concluding the proof. ■

Lastly, we conclude the section with a general result on $\mathbb{P}^1$-fibrations: in Lemma 1.25 and Corollary 1.26 we show that a fibration with fibre $\mathbb{P}^1$ that contains two disjoint prime divisors, horizontal for the fibration, is a $\mathbb{P}^1$-bundle.

Remark 1.24. Let X be a smooth projective variety. Then $\mathcal{N}_1(X)$ is generated, as a vector space, by classes of complete intersections of very ample divisors.

Proof. Let n be the dimension of X and H a very ample divisor. By the hard Lefschetz theorem, the linear map

$$\begin{aligned} \mathcal{N}^1(X) &\longrightarrow \mathcal{N}_1(X) \\ [D] &\mapsto [D \cdot H^{n-2}] \end{aligned}$$

is an isomorphism. Since the ample cone has maximal dimension in $\mathcal{N}^1(X)$, we can choose very ample divisors $H_1, \dots, H_{\rho_X}$ on X such that their classes generate $\mathcal{N}^1(X)$; then their images $[H_1 \cdot H^{n-2}], \dots, [H_{\rho_X} \cdot H^{n-2}]$ generate $\mathcal{N}_1(X)$. ■

Lemma 1.25. *Let X be a smooth Fano variety and $\varphi\colon X \to T$ a smooth morphism with fibre $\mathbb{P}^1$. Let $A_1, A_2 \subset X$ be disjoint prime divisors such that φ is finite on A_i for $i = 1, 2$. Then at least one among A_1, A_2 is a section of φ.*

Proof. We first show the following

Claim. *If $[A_1]$ and $[A_2]$ are multiples in $\mathcal{N}^1(X)$, then $X \cong \mathbb{P}^1 \times T$ and $A_i = \{pt\} \times T$.*

Let $\lambda \in \mathbb{Q}_{>0}$ be such that $A_1 \equiv \lambda A_2$. If $C \subset A_1$ is a curve, then $A_1 \cdot C = \lambda A_2 \cdot C = 0$ because $A_1 \cap A_2 = \emptyset$. This shows that A_1 is nef; since X is Fano, A_1 is semiample by Corollary 1.15, and the linear system $|mA_1|$ defines, for $m \gg 0$, a contraction $\alpha\colon X \to X_0$ which corresponds to the extremal ray $\mathrm{NE}(\alpha) = A_1^{\perp} \cap \mathrm{NE}(X)$.

We note that $\alpha(A_1)$ is a point, because $\alpha(C) = \{pt\}$ for every curve $C \subset A_1$; on the other hand $[A_1] \in \alpha^*\mathcal{N}^1(X_0)$, so that $\dim X_0 = 1$ and $X_0 \cong \mathbb{P}^1$. Moreover α is finite on the fibres f of $\varphi\colon X \to T$, because φ is finite on A_1, hence $A_1 \cdot f > 0$. Then $X \cong T \times \mathbb{P}^1$ and α is the projection, by [**Cas09**, Lemma 4.9]. This yields the claim. ■

Set $d_i := \deg \varphi_{|A_i}$ for $i = 1, 2$, so that $d_2A_1 - d_1A_2 \equiv \varphi^*(N_0)$ for some divisor N_0 on T. If $N_0 \equiv 0$, then $d_2A_1 \equiv d_1A_2$ and we can apply the claim.

If $N_0 \not\equiv 0$, then $N_0^{\perp} \subset \mathcal{N}_1(T)$ is a hyperplane, and by Rem. 1.24 there exists a curve $C \subset T$, complete intersection of very ample divisors $H_1, \ldots, H_{n-2}$, such that $N_0 \cdot C \neq 0$. Note that $n - 2 = \dim(T) - 1$. Moreover, we choose H_i general in their linear system, so that C is an irreducible smooth curve.

Set $S := \varphi^{-1}(C) \subset X$ the complete intersection of globally generated divisors $\varphi^*H_1, \ldots, \varphi^*H_{n-2}$, general in their linear systems; then, by Bertini's theorem, S is a smooth ruled surface.

Moreover, set $C_i := A_{i\,|S}$ for $i = 1, 2$; then $C_1 \cap C_2 = \emptyset$ and note that C_i is the complete intersection in X of A_i with $\varphi^*H_1, \ldots, \varphi^*H_{n-2}$, so that it is the complete intersection in A_i of $(\varphi_{|A_i})^*H_1, \ldots, (\varphi_{|A_i})^*H_{n-2}$. Again, by Bertini's theorem, C_1 and C_2 are irreducible curves.

Suppose that $N_0 \cdot C > 0$. Then

$$d_1(C_2)^2 = d_1A_2 \cdot C_2 = \big(d_2A_1 - \varphi^*(N_0)\big) \cdot C_2 = -\varphi^*(N_0) \cdot C_2 = -d_2N_0 \cdot C,$$

so that $(C_2)^2 < 0$. We deduce that C_2 is the negative section of the ruled surface S, hence $d_2 = 1$. Since $\varphi_{|A_2}$ is finite, A_2 is a section of φ.

In the case $N_0 \cdot C < 0$, we conclude in a similar way that A_1 is a section. ■

Corollary 1.26. *In the situation of Lemma 1.25, then $\varphi\colon X \to T$ is a $\mathbb{P}^1$-bundle.*

Proof. Follows from Lemma 1.25 and [**BS95**, Prop. 3.2.1]. ■

1.4. Cyclic coverings

The notion of cyclic covering in algebraic geometry is very classical, and is found in many manuals (e.g. [**Laz04a**, §4.1.B] and [**BHPdV04**, §Coverings], just to name a few). After recalling the definition and the basic properties of a cyclic covering, we prove the main result of this section, that is Proposition 1.32: the objective is to understand the relationship between the base locus of a divisor on a given variety and the base locus of its pull-back on a cyclic covering. This will be needed farther in the text.

Definition 1.27 (Definition-Proposition)**.** Let T be a variety and let $\mathcal{L} \in \mathrm{Pic}(T)$ be a line bundle. Fix an integer $d \geq 1$ and assume that $\mathcal{L}^d$ is effective, that is there exists a non-zero global section $\delta \in H^0(T, \mathcal{L}^d)$ defining an effective divisor Δ. Then there exists a variety S, a finite surjective morphism

$$\varphi \colon S \longrightarrow T$$

and a section $\delta' \in H^0(S, \varphi^*\mathcal{L})$ such that $(\delta')^d = \varphi^*\delta$. The variety S is called the *d-cyclic covering of T branched along Δ and determined by $\mathcal{L}$.*

The explicit construction of S and the morphism $\varphi \colon S \to T$ is given in the following remark.

Remark 1.28. Let $L := \mathbf{Spec}_{\mathcal{O}_T}\left(\mathrm{Sym}(\mathcal{L}^{-1})\right)$ be the total space of the line bundle $\mathcal{L}$, and let $\phi \colon L \to T$ be the natural projection (see [**Har77**, Example II.5.18]). There exists a tautological section $\sigma \in H^0(L, \phi^*\mathcal{L})$ so that the locus $S := \{\sigma^d - \phi^*\delta = 0\}$ verifies the conditions in Definition 1.27. That is, $\varphi := \phi_{|S} \colon S \to T$ is a finite surjective morphism and $\delta' := \sigma_{|S}$ satisfies $(\delta')^d = \varphi^*\delta$.

Proof. Let $\{U_i := \mathrm{Spec}(A_i)\}$ be an affine covering of T, so that $\mathcal{L}^{-1}(U_i) = f_i A_i$ with $f_i \in \mathrm{Quot}(A_i)$, i.e. the field of fractions of A_i. We have $U_i \cap U_j := \mathrm{Spec}(A_{ij})$, and the quotient $\frac{f_i}{f_j}$ is invertible in A_{ij}. Then, $\mathcal{O}_L(\phi^{-1}(U_i))$ is isomorphic to the polynomial ring $A_i[\sigma_i]$, and the transition maps on $\phi^{-1}(U_i \cap U_j)$ are given by *linear* isomorphisms of A_{ij}-algebras

$$A_{ij}[\sigma_i] \longrightarrow A_{ij}[\sigma_j], \qquad \sigma_i \mapsto \sigma_j = \frac{f_j}{f_i}\sigma_i.$$

Finally, the data $\left\{\left(\frac{\sigma_i}{f_i}, \phi^{-1}(U_i)\right)\right\}$ defines naturally a global section $\sigma \in H^0(L, \phi^*\mathcal{L})$. One can think of this section as the "vertical coordinate" of L, and in fact the locus $\{\sigma = 0\}$ gives the zero-section of L.

When we restrict to $S = \{\sigma^d - \phi^*\delta = 0\}$, it is clear that $\delta' = \sigma_{|S}$ satisfies the condition $(\delta')^d = \varphi^*\delta$ and that $\varphi = \phi_{|S} \colon S \to T$ is a finite morphism. ∎

When $\mathrm{Pic}(T)$ is torsion free, the line bundle $\mathcal{L}$ is uniquely determined by Δ. The divisor Δ' defined by δ' is called *ramification divisor* and is isomorphic to Δ. Moreover, if Δ and T are smooth, then so is S. The previous remark yields the following lemma.

Lemma 1.29 ([**BHPdV04**], Lemmata 17.1, 17.2)**.** *Let $\varphi \colon S \to T$ be the d-cyclic covering of T branched along a smooth divisor Δ and determined by $\mathcal{L}$. Then:*

(i) $\mathcal{O}_S(\Delta') \cong \varphi^*\mathcal{L}$;
(ii) $\varphi^*\Delta \sim d\Delta'$;
(iii) $\varphi_*\mathcal{O}_S \cong \mathcal{O}_T \oplus \mathcal{L}^{-1} \oplus \cdots \oplus \mathcal{L}^{1-d}$.

Moreover, one can see S as a divisor of the $\mathbb{P}^2$-bundle $\phi' \colon \mathbb{P}_T(\mathcal{O}_T \oplus \mathcal{L}^{-1}) =: P \to T$ in the following way. If $\{U_i := \mathrm{Spec}(A_i)\}$ is an affine covering of T, then the transition maps on $\phi'^{-1}(U_i \cap U_j)$ are given by *linear* isomorphisms of A_{ij}-algebras

$$A_{ij}[\xi_i, \eta_i] \longrightarrow A_{ij}[\xi_i, \eta_i], \qquad \xi_i \mapsto \xi_j = \xi_i, \qquad \eta_i \mapsto \eta_j = \frac{f_j}{f_i}\eta_i.$$

We show that L is an open subscheme of P: the affine scheme $\phi^{-1}(U_i) \cong \mathrm{Spec}(A_i[\sigma_i])$ can be identified with the affine open subscheme $D_+(\xi_i) \subset \mathrm{Proj}(A_i[\xi_i, \eta_i]) \cong \phi'^{-1}(U_i)$, so that the transition maps of P restrict to the transition maps of L.

$$\begin{array}{ccccc} S & \hookrightarrow & L & \hookrightarrow & P \\ & \searrow^{\varphi} & \downarrow{\scriptstyle \phi} & \swarrow_{\phi'} & \\ & & T & & \end{array}$$

Up to a tensor product by $\mathcal{L}$, we can assume that $S \subset \mathbb{P}_T(\mathcal{O}_T \oplus \mathcal{L})$.

Lemma 1.30 ([**LP21**], Lemma 1.2)**.** *Let $\varphi\colon S \to T$ be the d-cyclic covering of T branched along a smooth divisor Δ and determined by $\mathcal{L}$. Let P be the projective bundle $\mathbb{P}_T(\mathcal{O}_T \oplus \mathcal{L})$ over T, with tautological bundle $\mathcal{O}_P(1)$. Let $S' \subset P$ be the section corresponding to the projection $\mathcal{O}_T \oplus \mathcal{L} \twoheadrightarrow \mathcal{O}_T$ Then:*

(i) $\mathcal{O}_P(S) \cong \mathcal{O}_P(d)$, and $S \cap S' = \emptyset$;
(ii) the normal bundle $\mathcal{N}_{S/P}$ of S in P is isomorphic to $\varphi^*\mathcal{L}^d$;
(iii) $\mathcal{O}_S(K_S) = \varphi^*\big(\mathcal{O}_T(K_T) \otimes \mathcal{L}^{d-1}\big)$.

Lemma 1.31. *Let $\varphi\colon S \to T$ be the d-cyclic covering of T branched along a smooth divisor Δ and determined by $\mathcal{L}$. Then, for any $\mathcal{M} \in \mathrm{Pic}(T)$ there is an isomorphism*

$$H^0(S, \varphi^*\mathcal{M}) \cong \bigoplus_{k=0}^{d-1} H^0(T, \mathcal{M} \otimes \mathcal{L}^{-k}).$$

Precisely, for any $m \in H^0(S, \varphi^*\mathcal{M})$ there exist sections $m_k \in H^0(T, \mathcal{M} \otimes \mathcal{L}^{-k})$ such that

$$m = \sum_{i=0}^{d-1} \varphi^* m_k \cdot (\delta')^k,$$

where $\delta' \in H^0(S, \varphi^*\mathcal{L})$ is the section satisfying $(\delta')^d = \varphi^*\delta$.

Proof. The morphism $\varphi\colon S \to T$ is affine, so the first statement follows from the projection formula and Lemma 1.29(*iii*).

Now, let $\{U_i := \mathrm{Spec}(A_i)\}$ be an affine covering of T, so that $\mathcal{O}_L(\varphi^{-1}(U_i))$ is isomorphic to the polynomial ring $A_i[\delta_i{}']$, where $(\delta_i{}')^d = \delta_i \in A_i$. If $\mathcal{M}(U_i) = g_i A_i$ and $\mathcal{L}^{-1}(U_i) = f_i A_i$ for some $f_i, g_i \in \mathrm{Quot}(A_i)$, then $\varphi^*\mathcal{L}(\varphi^{-1}(U_i)) \cong \left(\frac{1}{f_i} A_i\right)[\delta_i{}']$. For any global section $m \in H^0(S, \varphi^*\mathcal{M})$, its restriction to $\varphi^{-1}(U_i)$ is

$$m_{|\varphi^{-1}(U_i)} = g_i \sum_{k=0}^{d-1} m_{k,i} \cdot (\delta_i{}')^k,$$

with $m_{k,i} \in A_i$ and $k = 0, \dots, d-1$. On $\varphi^{-1}(U_i \cap U_j)$ we have an isomorphism

$$g_i \sum_{k=0}^{d-1} m_{k,i} \cdot (\delta_i{}')^k \mapsto g_i \sum_{k=0}^{d-1} m_{k,i} \cdot \left(\frac{f_i}{f_j}\delta_j{}'\right)^k = g_j \sum_{k=0}^{d-1} m_{k,j} \cdot (\delta_j{}')^k,$$

which yields

$$g_i f_i^k \cdot m_{k,i} = g_j f_j^k \cdot m_{k,j}$$

for all $k = 0, \dots d-1$. Therefore we can write $m_{|\varphi^{-1}(U_i)}$ as $g_i \sum (m_{k,i} \cdot f_i^k) \left(\frac{\delta_i'}{f_i}\right)^k$, and the data $\left\{\left(g_i f_i^k \cdot m_{k,i}, U_i\right)\right\}$ and $\left\{\left(\left(\frac{\delta_i'}{f_i}\right)^k, \varphi^{-1}(U_i)\right)\right\}$ define, respectively, sections $m_k \in H^0(T, \mathcal{M} \otimes \mathcal{L}^{-k})$ and $(\delta')^k \in H^0(S, \varphi^* \mathcal{L}^k)$ for all $k = 0, \dots\ d-1$. ∎

Finally, we are ready to prove the main result of this section.

Proposition 1.32. *Let $\varphi\colon S \to T$ be the d-cyclic covering of T branched along a smooth divisor Δ and determined by $\mathcal{L}$. Then, for any $\mathcal{M} \in \mathrm{Pic}(T)$ we have*

$$\mathrm{Bs}\,|\varphi^* \mathcal{M}| = \varphi^{-1}\left(\mathrm{Bs}\,|\mathcal{M}| \cap \left(\Delta \cup \bigcap_{k=0}^{d-1} \mathrm{Bs}\,|\mathcal{M} \otimes \mathcal{L}^{-k}|\right)\right). \tag{1.5}$$

Proof. By Lemma 1.31, any global section $m \in H^0(S, \varphi^* \mathcal{M})$ can be written as $m = \varphi^* m_0 + \varphi^* m_1 \cdot \delta' + \dots + \varphi^* m_{d-1} \cdot (\delta')^{d-1}$, with $m_k \in H^0(T, \mathcal{M} \otimes \mathcal{L}^{-k})$, $k = 0, \dots d-1$. If $s \in S$ is a basepoint of the linear system $|\varphi^* \mathcal{M}|$, then

$$\varphi^* m_0(s) = \varphi^* m_k(s) \cdot (\delta')^k(s) = 0$$

for all $m_k \in H^0(T, \mathcal{M} \otimes \mathcal{L}^{-k})$, $k = 0, \dots, d-1$. Therefore $\varphi(s) \in \mathrm{Bs}\,|\mathcal{M}|$ and, for any fixed k, either $\varphi(s) \in \mathrm{Bs}\,|\mathcal{M} \otimes \mathcal{L}^{-k}|$ or $\delta'(s) = 0$. Since $\Delta' = \{\delta' = 0\}$, we have the inclusion ($\subseteq$). The opposite inclusion ($\supseteq$) is trivial, thus we have the statement. ∎

Note that Lemma 1.31 and Proposition 1.32 make sense even if some of the $\mathcal{M} \otimes \mathcal{L}^{-k}$ have no (non-trivial) global sections. In fact, assume $h^0(T, \mathcal{M} \otimes \mathcal{L}^{-k}) = 0$: the data $\left\{\left(g_i f_i^k \cdot m_{k,i}, U_i\right)\right\}$ still define a global section $m_k \in H^0(T, \mathcal{M} \otimes \mathcal{L}^{-k})$, but since there is only the trivial one, then $m_{k,i} = 0\ \forall i$. Moreover, under the same hypothesis we have $\mathrm{Bs}\,|\mathcal{M} \otimes \mathcal{L}^{-k}| = T$, and this does not invalidate equation (1.5).

CHAPTER 2

Fano fourfolds containing a prime divisor of Picard number 1

2.1. Introduction

Let X be a smooth, complex Fano variety. In this chapter we focus on Fano manifolds of dimension $n \geq 3$ that contain a prime divisor D with $\rho_D = 1$: in [**Tsu06**, Proposition 5], Tsukioka shows that if this condition is satisfied, then the Picard number of X is at most 3. Later on, these varieties have been studied also by Casagrande and Druel in [**CD15**]. To be more precise, for any prime divisor $\iota\colon D \hookrightarrow X$, we denote by $\mathcal{N}_1(D,X)$ the image of the push-forward $\iota_*\colon \mathcal{N}_1(D) \to \mathcal{N}_1(X)$ induced by the inclusion, which is generated by the classes in $\mathcal{N}_1(X)$ of curves contained in D. Note that $0 < \dim \mathcal{N}_1(D,X) \leq \rho_D$. Let us assume that there exists a prime divisor D such that $\dim \mathcal{N}_1(D,X) = 1$: then, in [**CD15**, Lemma 3.1], the authors prove that either $\rho_X = 1$, or $2 \leq \rho_X \leq 3$ and X has some contraction onto a smooth Fano variety of lower Picard number. This contraction is either a blow-up or a conic bundle if $\rho_X = 2$, and it is a conic bundle of relative Picard number 2 if $\rho_X = 3$. Furthermore, in [**CD15**, Theorem 3.8], the authors give a general characterisation of X when $\rho_X = 3$ (see Remark 2.2).

Our goal in this chapter is to classify all families of Fano fourfolds with $\rho_X = 3$ containing a prime divisor D with $\rho_D = 1$, or more generally with $\dim \mathcal{N}_1(D,X) = 1$. We compute the main numerical invariants of X, study the base locus of the linear system $|-K_X|$, and we also try to address the question of its rationality.

Theorem 2.1. *There are* 28 *families of smooth Fano fourfolds X with $\rho_X = 3$ having a prime divisor D such that $\dim \mathcal{N}_1(D,X) = 1$. They have index $i_X = 1$.*

Among them, 22 *families consist of rational varieties, and the very general variety of* 4 *other families is not rational.*

Furthermore, the linear system $|-K_X|$ is free, with the exception of two families where the base locus consists of either 1 *or* 2 *points. In all cases, a general element of $|-K_X|$ is smooth.*

The families, their numerical invariants and properties are described in Table II.2.

We also study their deformations in order to compute the dimension of the cohomology groups $H^0(\mathcal{T}_X)$ and $H^1(\mathcal{T}_X)$ of the tangent sheaf $\mathcal{T}_X$ of X. They correspond, respectively, to the dimension of the automorphism group $\mathrm{Aut}(X)$ and the dimension of the base of the Kuranishi family of a general member. We recall that deformations of smooth Fano varieties are unobstructed, since all obstruction classes of deformations of X vanish. These classes are defined as elements of the cohomology group $H^2(\mathcal{T}_X)$, which vanishes as a consequence of the Nakano vanishing theorem [**Laz04a**, Theorem 4.2.3].

The chapter is organised as follows. In Section 2.2 we recall some results from [**CD15**, Section 3], and we use them to identify the list of 28 families of Fano fourfolds of Theorem 2.1. In Section 2.3 we compute the Hodge numbers, the anticanonical degree K_X^4, and the dimension of the space of global sections $h^0(\mathcal{O}_X(-K_X))$ for all 28 families. In Section 2.4 we study the deformations of the varieties of Theorem 2.1 in order to estimate $h^0(\mathcal{T}_X)$ and $h^1(\mathcal{T}_X)$. In Section 2.5 we study the base locus of the linear system $|-K_X|$, while in Section 2.6 we determine whether the Fano fourfolds under consideration are rational or not (when possible). In Section 2.7 we display the final tables with all the results of the paper.

Related results appear in [**Bat99**] and [**Tsu06**]: in the former the author provides a complete classification of toric Fano fourfolds, while in the latter the author classifies Fano varieties of dimension ≥ 3 which admit a negative divisor isomorphic to the projective space. See Section 2.7 for the relations with these classifications.

2.2. The 28 families of Fano fourfolds

In this section we recall the needed results from [**CD15**, Section 3], and use them to determine the 28 families of Fano fourfolds that appear in Theorem 2.1.

Remark 2.2. The following is a summary of [**CD15**, Example 3.4 - Theorem 3.8], when $n = 4$. Firstly, we recall the construction of a manifold X of dimension 4 with $\rho_X = 3$ and containing a prime divisor of Picard number 1 ([**CD15**, Example 3.4]); then we describe the geometry of the Mori cone ([**CD15**, Lemma 3.5]) to determine the conditions for X to be Fano ([**CD15**, Remark 3.6]). Lastly, [**CD15**, Theorem 3.8] implies that all the varieties of Theorem 2.1 are constructed in this way.

Step. 1 *Given a smooth Fano threefold Z, we construct from Z a smooth Fano fourfold X with $\rho_X = 3$, containing a prime divisor with $\rho = 1$.*

Let Z be a smooth Fano threefold with Picard number $\rho_Z = 1$ and index i_Z, and let $\mathcal{O}_Z(1)$ be the ample generator of $\mathrm{Pic}(Z)$. Recall that the linear system $|\mathcal{O}_Z(1)|$ is non-empty by Lemma 1.17. The,n take an effective divisor $H \in |\mathcal{O}_Z(1)|$, so that $-K_Z \equiv i_Z H$. Moreover, fix integers $a \geq 0$ and $d \geq 1$, and assume that the linear system $|\mathcal{O}_Z(d)|$ contains smooth surfaces. Then, fix a smooth surface $A \in |\mathcal{O}_Z(d)|$.

Let $Y := \mathbb{P}\big(\mathcal{O}_Z \oplus \mathcal{O}_Z(a)\big)$ be the $\mathbb{P}^1$-bundle with projection $\pi\colon Y \to Z$. Take a section G_Y with normal bundle $\mathcal{N}_{G_Y/Y} \cong \mathcal{O}_Z(-a)$, corresponding to a surjection $\mathcal{O}_Z \oplus \mathcal{O}_Z(a) \twoheadrightarrow \mathcal{O}_Z$. If $a = 0$, then $Y \cong Z \times \mathbb{P}^1$ and take another section $\widehat{G}_Y$ disjoint from G_Y; if $a > 0$, take a section $\widehat{G}_Y$ with normal bundle $\mathcal{N}_{\widehat{G}_Y/Y} \cong \mathcal{O}_Z(a)$, which is also disjoint from G_Y, corresponding to a surjection $\mathcal{O}_Z \oplus \mathcal{O}_Z(a) \twoheadrightarrow \mathcal{O}_Z(a)$.

Set $S := \widehat{G}_Y \cap \pi^{-1}(A)$, and let $\sigma\colon X \to Y$ be the blow-up with centre S. Let us denote by E the exceptional divisor and, by G and $\widehat{G}$, the transforms of G_Y and $\widehat{G}_Y$. Note that $G \cap \widehat{G} = \emptyset$, $\mathcal{N}_{G/X} \cong \mathcal{O}_Z(-a)$ and $\mathcal{N}_{\widehat{G}/X} \cong \mathcal{O}_Z\big(-(d-a)\big)$.

The morphism $\varphi := \pi \circ \sigma\colon X \to Z$ is a conic bundle, that is a fibre-type contraction whose fibres are plane conics: the fibres over $Z \setminus A$ are smooth and isomorphic to $\mathbb{P}^1$, while the fibres over A are the union of two distinct lines. In particular, if we denote by $\widehat{E}$ the strict transform of $\pi^{-1}(A)$, we have that $\varphi^{-1}(A) = E \cup \widehat{E}$ is the locus of the singular φ-fibres.

Then X is a smooth variety of dimension 4, it has Picard number $\rho_X = 3$ and contains prime divisors (namely, G and $\widehat{G}$) with Picard number 1. In particular, we have that $\dim \mathcal{N}_1(G, X) = \dim \mathcal{N}_1(\widehat{G}, X) = 1$.

Step. 2 *We describe the geometry of X.*

The cone $\mathrm{NE}(\varphi)$ has dimension 2, thus φ admits a second factorisation $\varphi := \widehat{\pi} \circ \widehat{\sigma}$ into elementary contractions. We have that

$$\widehat{\pi} \colon \widehat{Y} := \mathbb{P}\big(\mathcal{O}_Z \oplus \mathcal{O}_Z(d-a)\big) \to Z$$

is a $\mathbb{P}^1$-bundle and that $\widehat{\sigma} \colon X \to \widehat{Y}$ is the blow-up with centre $\widehat{S} := \widehat{\sigma}(G) \cap \widehat{\pi}^{-1}(A)$. Indeed, $\mathrm{NE}(\varphi)$ is generated by the numerical class of a fibre F of $\varphi_{|E} \colon E \to A$ and of a fibre $\widehat{F}$ of $\varphi_{|\widehat{E}} \colon \widehat{E} \to A$. The statement then follows from [**Wiś91a**, Thm. 1.2] for $\widehat{\sigma}$ and from Corollary 1.26 for $\widehat{\pi}$.

Note that $\widehat{E}$ is the exceptional divisor of $\widehat{\sigma}$, while $\widehat{\sigma}(G)$ and $\widehat{\sigma}(\widehat{G})$ are disjoint sections of $\widehat{\pi}$ with normal bundles $\mathcal{N}_{\widehat{\sigma}(G)/\widehat{Y}} \cong \mathcal{O}_Z(d-a)$ and $\mathcal{N}_{\widehat{\sigma}(\widehat{G})/\widehat{Y}} \cong \mathcal{O}_Z(a-d)$.

$$\begin{array}{ccc} X & \xrightarrow{\sigma} & Y \\ {\scriptstyle\widehat{\sigma}}\downarrow & \searrow^{\varphi} & \downarrow{\scriptstyle\pi} \\ \widehat{Y} & \xrightarrow[\widehat{\pi}]{} & Z \end{array}$$

Now let C_Z be an irreducible curve in Z such that $\mathcal{O}_Z(1) \cdot C_Z$ is minimal, and let $C_G \subset G$ (resp. $C_{\widehat{G}} \subset \widehat{G}$) be the curve in G (resp. $\widehat{G}$) corresponding to C_Z. Then, by [**CD15**, Lemma 3.5], the Mori cone $\mathrm{NE}(X)$ is closed and polyhedral, and it is generated by the classes of

- F, $\widehat{F}$ and $C_{\widehat{G}}$, if $a = 0$;
- F, $\widehat{F}$, C_G and $C_{\widehat{G}}$, if $0 < a < d$;
- F, $\widehat{F}$ and C_G, if $a \geq d$.

All the relevant intersection numbers are collected in [**CD15**, Table 1], and one can see that

$$dG + a\widehat{E} \equiv d\widehat{G} + (d-a)E. \tag{2.1}$$

See Remark 2.4 for the description of the contractions of X, where the sections of $\mathrm{NE}(X)$ and $\mathrm{Nef}(X)$ are presented.

Moreover, by [**CD15**, Remark 3.7], the pairs (a, d) and $(d-a, d)$ give rise to isomorphic fourfolds when $d \geq a$, so we can assume that $a > d$ or $0 \leq a \leq \frac{d}{2}$.

Step. 3 *We show when X is a Fano manifold.*

By computing the intersection numbers between $-K_X$ and the generators of $\mathrm{NE}(X)$, we see that X is a Fano manifold if and only if

$$a \leq i_Z - 1, \qquad d - a \leq i_Z - 1,$$

see [**CD15**, Remark 3.6].

Step. 4 *Conclusions.*

Finally, by [**CD15**, Theorem 3.8], any Fano fourfold X of Picard number $\rho_X = 3$ containing a prime divisor D with $\dim \mathcal{N}_1(D, X) = 1$ is isomorphic to one of the varieties

X constructed above. Any such X, then, is given by a triplet (Z, a, d), where Z is a smooth Fano threefold with $\rho_Z = 1$ and:

$$\begin{aligned} d \geq 1, \qquad & a > d \text{ or } 0 \leq a \leq \frac{d}{2}, \\ a \leq i_Z - 1, \qquad & d - a \leq i_Z - 1; \end{aligned} \tag{2.2}$$

as we will see in Lemma 2.5, $|dH|$ contains a smooth surface. The inequalities in (2.2) imply that the index of the Fano threefold Z cannot be equal to 1.

Remark 2.3. Further observations:

(i) Since $-K_X \cdot F = 1$, we have that the index of X is $i_X = 1$.

(ii) The classes of φ^*H, $\widehat{G}$, E, and the classes of φ^*H, G, $\widehat{E}$ are both a basis of $\mathcal{N}^1(X)$. Moreover,

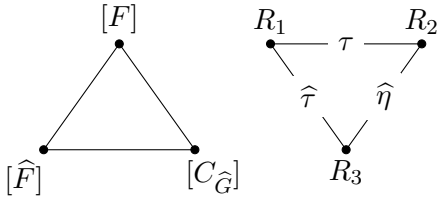
$$\alpha\varphi^*H + \beta\widehat{G} + (\beta+\gamma)E \equiv (\alpha + a\beta + d\gamma)\varphi^*H + \beta G - \gamma\widehat{E}$$

for all $\alpha, \beta, \gamma \in \mathbb{R}$, and

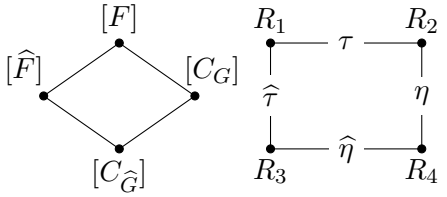
$$-K_X \equiv (i_Z - a)\varphi^*H + 2\widehat{G} + E \equiv (i_Z + a - d)\varphi^*H + 2G + \widehat{E}.$$

(iii) Under the conditions (2.2), Y and $\widehat{Y}$ are also Fano fourfolds by Lemma 1.23. The nef cone $\mathrm{Nef}(Y)$ is generated by the classes of π^*H and $\widehat{G}_Y$, while $\mathrm{Nef}(\widehat{Y})$ is generated by the classes of $\widehat{\pi}^*H$ and $\widehat{\sigma}(G)$. Note that the tautological line bundles $\mathcal{O}_Y(1)$ and $\mathcal{O}_{\widehat{Y}}(1)$ are isomorphic, respectively, to the line bundles $\mathcal{O}_Y(\widehat{G}_Y)$ and $\mathcal{O}_{\widehat{Y}}(\widehat{\sigma}(G))$.

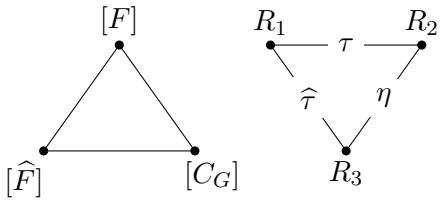
Remark 2.4. For any irreducible curve $C \subset X$, set $C^\perp := \{D \in \mathcal{N}^1(X) | D \cdot C = 0\}$. As X is smooth and Fano, the Mori cone $\mathrm{NE}(X)$ and the nef cone $\mathrm{Nef}(X)$ are closed and polyhedral, thus they are generated by finitely many extremal rays. The sections of $\mathrm{NE}(X)$ (on the left) and $\mathrm{Nef}(X)$ (on the right) are:



$$R_1 = \mathbb{R}^+[\varphi^*H],\ R_2 = \mathbb{R}^+[G],\ R_3 = \mathbb{R}^+[G + \widehat{E}],$$
$$\text{if } a = 0;$$



$$R_1 = \mathbb{R}^+[\varphi^*H],\ R_2 = \mathbb{R}^+[a\varphi^*H + G],$$
$$R_3 = \mathbb{R}^+[G + \widehat{E}],\ R_4 = \mathbb{R}^+[dG + a\widehat{E}],$$
$$\text{if } 0 < a \leq \tfrac{d}{2};$$



$$R_1 = \mathbb{R}^+[\varphi^*H],\ R_2 = \mathbb{R}^+[a\varphi^*H + G],\ R_3 = \mathbb{R}^+[\widehat{G}],$$
$$\text{if } a > d.$$

Note that $\tau = F^{\perp} \cap \mathrm{Nef}(X)$, $\widehat{\tau} = \widehat{F}^{\perp} \cap \mathrm{Nef}(X)$, $\eta = C_G{}^{\perp} \cap \mathrm{Nef}(X)$ and $\widehat{\eta} = C_{\widehat{G}}{}^{\perp} \cap \mathrm{Nef}(X)$. One can check that all the contractions are divisorial, apart from two cases: the contraction given by the face $\mathrm{NE}(\varphi) = \mathbb{R}^+[F] + \mathbb{R}^+[\widehat{F}]$, which is the conic bundle $\varphi\colon X \to Z$, and the contraction given by the face $\mathbb{R}^+[F] + \mathbb{R}^+[C_{\widehat{G}}]$ for $a = 0$, which is again fibre-type and corresponds to a fibration $X \to \mathbb{P}^1$. The contractions of $\mathbb{R}^+[F]$ and $\mathbb{R}^+[\widehat{F}]$ are $\sigma\colon X \to Y$ and $\widehat{\sigma}\colon X \to \widehat{Y}$, while the contractions of $\mathbb{R}^+[C_G]$ and $\mathbb{R}^+[C_{\widehat{G}}]$ send a divisor to a point, respectively G and $\widehat{G}$.

In [**IP99**, Table §12.1] we find the list of all smooth Fano threefolds with Picard number $\rho_Z = 1$. We are interested in those with index $i_Z \geq 2$, and we collect all the relevant varieties in Table II.1 for the reader's convenience. The last two columns display, respectively, the base locus of the linear system $|H|$, and whether these varieties are rational (+) or not (−). We also include the dimension of the cohomology groups $H^0(\mathcal{T}_Z)$ and $H^1(\mathcal{T}_Z)$ of the tangent sheaf $\mathcal{T}_Z$. The references for $h^0(\mathcal{T}_Z)$ are [**PS19**, Theorem 1.3] for Z_1, [**LP21**, Theorem 3.4] for Z_2, [**PdV99**, Theorem 7.5] for Z_5, and [**CPZ15**, Theorem 1.1, Theorem 1.3] for Z_3, Z_4 and Z_6 (see also [**Ben13**, Theorem 3.1]). For similar results on the automorphism group of smooth Fano threefolds with $\rho_Z = 1$, see also [**KPS18**, Theorem 1.1.2].

As for computing $h^1(\mathcal{T}_Z)$, we apply the Hirzebruch-Riemann-Roch theorem [**Har77**, Appendix A, Theorem 4.1] to the sheaves $\mathcal{T}_Z$ and Ω^1_Z, and by the Nakano vanishing theorem [**Laz04a**, Theorem 4.2.3] we get

$$\chi(\mathcal{T}_Z) = h^0(\mathcal{T}_Z) - h^1(\mathcal{T}_Z) = -\frac{1}{2}K_Z^3 - h^{1,2}(Z) - 17.$$

TABLE II.1. Fano threefolds of Picard number 1 and index $i_Z \geq 2$

Z	i_Z	H^3	$-K_Z^3$	$h^{1,2}$	$h^0(\mathcal{T}_Z)$	$h^1(\mathcal{T}_Z)$	Bs $\lvert H\rvert$	R
Z_1 = a hypersurface of degree 6 in the weighted projective space $\mathbb{P}(1,1,1,2,3)$	2	1	8	21	0	34	$\{P_0\}$	−
Z_2 = a cyclic cover of degree 2 of $\mathbb{P}^3$ ramified along a smooth surface of degree 4	2	2	$8 \cdot 2$	10	0	19	$\emptyset$	−
Z_3 = a smooth cubic in $\mathbb{P}^4$	2	3	$8 \cdot 3$	5	0	10	$\emptyset$	−
Z_4 = a smooth intersection of two quadrics in $\mathbb{P}^5$	2	4	$8 \cdot 4$	2	0	3	$\emptyset$	+
$Z_5 \subset \mathbb{P}^6$, a section of the Grassmannian $\mathrm{Gr}(2,5) \subset \mathbb{P}^9$ by a subspace of codimension 3	2	5	$8 \cdot 5$	0	3	0	$\emptyset$	+
Z_6 = a smooth quadric in $\mathbb{P}^4$	3	2	$27 \cdot 2$	0	10	0	$\emptyset$	+
$Z_7 = \mathbb{P}^3$	4	1	64	0	15	0	$\emptyset$	+

Lemma 2.5. *For $i = 2, \ldots, 7$, the line bundle $\mathcal{O}_{Z_i}(d)$ is globally generated for all $d \geq 1$, and $\mathcal{O}_{Z_1}(d)$ is globally generated for all $d \geq 2$. Moreover, a general surface $A \in |\mathcal{O}_{Z_1}(1)|$ is smooth.*

PROOF. Observe that, when $i \neq 1, 2$, the ample generator $\mathcal{O}_{Z_i}(1)$ of $\mathrm{Pic}(Z_i)$ is the restriction of the hyperplane line bundle $\mathcal{O}_{\mathbb{P}}(1)$ of some projective space $\mathbb{P}$, thus it is very ample. This is clear for $i = 3, 4, 6, 7$, while it follows from [**IP99**, Theorem 3.2.5(v)] that $\mathcal{O}_{Z_5}(1)$ is the restriction to Z_5 of the very ample divisor on the Grassmannian $\mathrm{Gr}(2, 5)$ defining the Plücker embedding $\mathrm{Gr}(2, 5) \hookrightarrow \mathbb{P}^9$. Furthermore, $\mathcal{O}_{Z_2}(1)$ is globally generated since it is the pull-back of $\mathcal{O}_{\mathbb{P}^3}(1)$. Therefore, any positive multiple of the ample generator is globally generated.

As for Z_1, by [**IP99**, Proposition 2.4.1(i), Theorem 2.4.5(i)] we have that $\mathcal{O}_{Z_1}(2)$ is globally generated, while $\mathcal{O}_{Z_1}(1)$ has a unique simple basepoint. Moreover, $\mathcal{O}_{Z_1}(3)$ is very ample (see [**KPS18**, Table 1]), so that $\mathcal{O}_{Z_1}(d)$ is globally generated for all $d \geq 2$. Nonetheless, by [**IP99**, Proposition 2.3.1], a general surface A in the linear system $|\mathcal{O}_{Z_1}(1)|$ is smooth. ■

Corollary 2.6. *For every Z_i, $i = 1, \ldots, 7$, and for every choice of integers a, d satisfying* (2.2)*, there exists a smooth Fano fourfold $X^i_{a,d}$ of Picard number 3 constructed as in Remark 2.2.*

Conversely, if X is a smooth Fano fourfold with $\rho_X = 3$ and containing a prime divisor D with $\dim \mathcal{N}_1(D, X) = 1$, then X is isomorphic to $X^i_{a,d}$ for some (Z_i, a, d) as above. This gives rise to 28 families, see Table 2*.*

PROOF. Assume that the integers a, d satisfy (2.2). This yields $d \leq 2i_Z - 2$. Therefore, by Lemma 2.5, a general surface $A \in |\mathcal{O}_{Z_i}(d)|$ is smooth for every $i = 1, \ldots, 7$, and so, given the triplet (Z_i, a, d), we can construct a Fano fourfold $X := X^i_{a,d}$ of Picard number 3. Notice that $X^i_{a,d}$ contains a prime divisor G such that $\rho_G = 1$, hence $\dim \mathcal{N}_1(G, X) = 1$ (see Remark 2.2).

Conversely, by [**CD15**, Theorem 3.8], any Fano fourfold X of Picard number $\rho_X = 3$ which admits a prime divisor D with $\dim \mathcal{N}_1(D, X) = 1$ is isomorphic to one of the $X^i_{a,d}$. ■

Corollary 2.6 is the first step towards the proof of Theorem 2.1, as it shows that there are 28 choices of integers a, d satisfying (2.2), each of which determines a family of smooth Fano varieties, and that every smooth Fano fourfold of Picard number 3 containing a prime divisor of Picard number 1 appears in one of these families. We will later see in Section 2.4 that the members of each family are all deformation equivalent, and that distinct families do not have common members (see Corollary 2.14).

Notation. If not specified otherwise, the varieties Z, Y, X, the corresponding maps and divisors will be the one described in Remark 2.2. When needed, we will use the notation Z_i for the varieties in Table II.1, and $X^i_{a,d}$ for the variety given by the triplet (Z_i, a, d).

2.3. Numerical invariants

In this section we compute the numerical invariants of the Fano fourfolds in Corollary 2.6; these invariants are listed in Table II.2.

2.3.1. Hodge numbers. Our first goal is to compute the Hodge numbers. To do so, we will use the Hodge polynomial of a smooth variety W

$$e(W) = e(W)(u, v) := \sum_{p,q} h^{p,q}(W) u^p v^q$$

and its properties with respect to $\mathbb{P}^n$-bundles and blow-ups. Namely:

Lemma 2.7. *Let W be a smooth variety, $\mathbb{P}(\mathcal{E})$ a $\mathbb{P}^n$-bundle over W and $\widetilde{W}$ the blow-up of W along a subvariety V of codimension c. Then:*

$$e(\mathbb{P}(\mathcal{E})) = e(W) \cdot e(\mathbb{P}^n)$$

and

$$e(\widetilde{W}) = e(W) + e(V) \cdot [e(\mathbb{P}^{c-1}) - 1].$$

Proof. By [**Che96**, Introduction 0.1] and quick computations. ∎

Since Y and X are Fano varieties of Picard number, respectively, 2 and 3, the only unknown Hodge numbers are $h^{1,2}$, $h^{1,3}$, and $h^{2,2}$. We begin by computing the Hodge numbers of A, which is the smooth surface in $|\mathcal{O}_Z(d)|$ fixed in Remark 2.2. Recall that (2.2) yields $d \leq 2i_Z - 2$, and that H is an effective divisor of the linear system $|\mathcal{O}_Z(1)|$.

Lemma 2.8. *Let Z be a smooth Fano threefold with $\rho_X = 1$ and $i_Z \geq 2$. Let A be a smooth surface in $|\mathcal{O}_Z(d)|$. Then all but the following Hodge numbers of A vanish:*

- $h^{0,0}(A) = 1$;
- $h^{0,2}(A) = 1$, *for* $d = i_Z$;
- $h^{0,2}(A) = 5$, *for* $d = 4$ *and* $i_Z = 3$ *(i.e. Z is a smooth quadric threefold);*
- $h^{0,2}(A) = \binom{d-1}{3}$, *for* $d = 5, 6$ *and* $i_Z = 4$ *(i.e $Z \cong \mathbb{P}^3$);*
- $h^{1,1}(A) = 10 + 10 \cdot h^{0,2}(A) - d(d - i_Z)^2\delta$, *where* $\delta = H^3$.

Proof. Firstly, $h^{0,0}(A) = 1$ is trivial, while the vanishing of $h^{0,1}(A)$ follows from the Lefschetz hyperplane theorem. We then carry out the computation for $h^{0,2}(A)$. By the exact sequence

$$0 \to \mathcal{O}_Z(-d) \to \mathcal{O}_Z \to \mathcal{O}_A \to 0$$

we have $H^2(\mathcal{O}_A) \cong H^3(\mathcal{O}_Z(-d))$. Moreover, $h^3(\mathcal{O}_Z(-d)) = 0$ for $d < i_Z$, and the Kodaira vanishing theorem yields $h^3(\mathcal{O}_Z(-d)) = 1$ for $d = i_Z$. This concludes the case $i_Z = 2$. For $i_Z = 3$ we have $1 \leq d \leq 4$ and $h^3(\mathcal{O}_Z(-4)) = h^4(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(-6)) = 5$; for $i_Z = 4$ we have $1 \leq d \leq 6$ and $h^3(\mathcal{O}_{\mathbb{P}^3}(-d)) = \binom{d-1}{3}$ for $d = 5, 6$.

To compute $h^{1,1}(A)$ it suffices to apply Noether's formula $\chi_{top}(A) = 12\chi(\mathcal{O}_A) - K_A^2$ [**Bea96**, Remark I.14] to get

$$h^{1,1}(A) = 10 + 10 \cdot h^{0,2}(A) - d(d - i_Z)^2\delta$$

where $\delta = H^3$. Indeed: by adjunction we have $K_A \sim (d - i_Z)H_{|A}$ and let $b_i(A)$ the i-th Betti number of A, so that

$$\chi_{top}(A) = 2b_0(A) - 2b_1(A) + b_2(A) = 2 + h^{1,1}(A) + 2h^{0,2}(A),$$
$$\chi(\mathcal{O}_A) = 1 + h^{0,2}(A), \qquad K_A^2 = d(d - i_Z)^2\delta.$$

∎

By Lemma 2.7 we get $e(Y) = e(Z) \cdot e(\mathbb{P}^1)$ and $e(X) = e(Y) + e(A) \cdot (e(\mathbb{P}^1) - 1)$, thus the Hodge numbers of Y and X are:

$$h^{p,q}(Y) = \begin{cases} 2 & (p,q) = (2,2) \\ h^{1,2}(Z) & (p,q) = (1,2) \\ 0 & (p,q) = (1,3) \end{cases}$$

and

$$h^{p,q}(X) = \begin{cases} h^{1,2}(Z) & (p,q) = (1,2) \\ h^{0,2}(A) & (p,q) = (1,3) \\ 2 + h^{1,1}(A) & (p,q) = (2,2)\ . \end{cases}$$

Notice that all the Hodge numbers of X only depend on d and Z, and not on a.

2.3.2. Anticanonical degree and $h^0(-K_X)$. Our next goal is to compute the anticanonical degree K_X^4 and the dimension of the space of global sections $H^0(\mathcal{O}_X(-K_X))$. We can do it simultaneously: the Hirzebruch-Riemann-Roch theorem [**Har77**, Appendix A, Theorem 4.1] provides a formula to compute the Euler characteristic of a locally free sheaf on a smooth projective variety. We apply it to the anticanonical sheaf $\mathcal{O}_W(-K_W)$ of a smooth fourfold W, yielding

$$\chi(\mathcal{O}_W(-K_W)) = \chi(\mathcal{O}_W) + \frac{1}{12}\Big(2K_W^4 + K_W^2 \cdot c_2(W)\Big). \tag{2.3}$$

Since X is Fano, Kodaira vanishing implies $\chi(\mathcal{O}_X(-K_X)) = h^0(\mathcal{O}_X(-K_X))$.

The following two standard computations provide the necessary tools.

Proposition 2.9 ([**CR22**], Lemma 3.2)**.** *Let W be a smooth projective variety,* $\dim W = 4$*, and let $\alpha\colon \widetilde{W} \to W$ be the blow-up of W along a smooth irreducible surface V. Then:*

(i) $K_{\widetilde{W}}^4 = K_W^4 - 3(K_{W|V})^2 - 2K_V \cdot K_{W|V} + c_2(\mathcal{N}_{V/W}) - K_V^2$*;*

(ii) $K_{\widetilde{W}}^2 \cdot c_2(\widetilde{W}) = K_W^2 \cdot c_2(W) - 12\chi(\mathcal{O}_V) + 2K_V^2 - 2K_V \cdot K_{W|V} - 2c_2(\mathcal{N}_{V/W})$*;*

(iii) $\chi(\mathcal{O}_{\widetilde{W}}(-K_{\widetilde{W}})) = \chi(\mathcal{O}_W(-K_W)) - \chi(\mathcal{O}_V) - \frac{1}{2}\Big((K_{W|V})^2 + K_V \cdot K_{W|V}\Big)$.

Proposition 2.10. *Let W be a smooth projective variety of dimension* 3*, and $\mathcal{E}$ a rank* 2 *vector bundle on W. Let $\beta\colon \mathbb{P}(\mathcal{E}) \to W$ be a $\mathbb{P}^1$-bundle over W. Then:*

(i) $K_{\mathbb{P}(\mathcal{E})}^4 = -8K_W \cdot c_1(\mathcal{E})^2 + 32K_W \cdot c_2(\mathcal{E}) - 8K_W^3$*;*

(ii) $K_{\mathbb{P}(\mathcal{E})}^2 \cdot c_2(\mathbb{P}(\mathcal{E})) = -2K_W \cdot c_1(\mathcal{E})^2 + 8K_W \cdot c_2(\mathcal{E}) - 2K_W^3 - 4K_W \cdot c_2(W)$*;*

(iii) $\chi(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(-K_{\mathbb{P}(\mathcal{E})})) = \chi(\mathcal{O}_{\mathbb{P}(\mathcal{E})}) + 6K_W \cdot c_2(\mathcal{E}) - \frac{1}{2}\Big(3K_W^3 + 3K_W \cdot c_1(\mathcal{E})^2\Big) - \frac{1}{3}K_W \cdot c_2(W)$.

Proof. Let D be the divisor associated to $\det(\mathcal{E})$, and ξ be the divisor associated to $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$. We have $\sum_{i=0}^{2}(-1)^i \beta^* c_i(\mathcal{E}) \cdot \xi^{(2-i)} = 0$, which yields

$$\xi^2 = \beta^* D \cdot \xi - \beta^* c_2(\mathcal{E}).$$

Then:

- $\beta^*(K_W + D)^4 = 0$;
- $\beta^*(K_W + D)^3 \cdot \xi = (K_W + D)^3$;
- $\beta^*(K_W + D)^2 \cdot \xi^2 = \beta^*(K_W + D)^2 \cdot \Big(\beta^* D \cdot \xi - \beta^* c_2(\mathcal{E})\Big) = (K_W + D)^2 \cdot D$;
- $\beta^*(K_W + D) \cdot \xi^3 = \beta^*(K_W + D) \cdot \xi \cdot \Big(\beta^* D \cdot \xi - \beta^* c_2(\mathcal{E})\Big) = (K_W + D) \cdot D^2 - (K_W + D) \cdot c_2(\mathcal{E})$;
- $\xi^4 = \Big(\beta^* D \cdot \xi - \beta^* c_2(\mathcal{E})\Big)^2 = D^3 - 2D \cdot c_2(\mathcal{E})$.

We recall that $K_{\mathbb{P}(\mathcal{E})} = \beta^*(K_W + D) - 2\xi$, therefore

$$K^4_{\mathbb{P}(\mathcal{E})} = \sum_{i=0}^{4} \binom{4}{i} \beta^*(K_W + D)^i \cdot (-2\xi)^{(4-i)},$$

and we obtain (i).

To prove (ii), we need to compute $c_2(\mathbb{P}(\mathcal{E}))$. By [**Ful98**, Example 3.2.11], we get

$$c_2(\mathbb{P}(\mathcal{E})) = \beta^* K_W \cdot (\beta^* D - 2\xi) + \beta^* c_2(W).$$

So, $K^2_{\mathbb{P}(\mathcal{E})} \cdot c_2(\mathbb{P}(\mathcal{E})) = \Big(\beta^*(K_W + D) - 2\xi\Big)^2 \cdot \Big(\beta^* K_W \cdot (\beta^* D - 2\xi) + \beta^* c_2(W)\Big)$, which gives (ii). To get (iii), just apply (2.3). ■

We are now able to compute the numerical invariants of X under consideration.

Lemma 2.11. *Let X as in Corollary 2.6, and let $\delta = H^3$. Then:*

(i) $K^4_X = 8\delta i_Z(a^2 + i_Z^2) - 3d\delta(a + i_Z)^2 + 2d\delta(a + i_Z)(d - i_Z) + ad^2\delta - d\delta(d - i_Z)^2$,

(ii) $K^2_X \cdot c_2(X) = 84 + 2\delta i_Z(a^2 + i_Z^2) - 12h^{0,2}(A) + 2d\delta(d - i_Z)(a + d) - 2ad^2\delta$,

(iii) $\chi(\mathcal{O}_X(-K_X)) = 8 + \frac{3}{2}\delta i_Z(a^2 + i_Z^2) - h^{0,2}(A) - \frac{1}{2}d\delta(a + i_Z)(a - d + 2i_Z)$.

Proof. Recall that in the setting of Remark 2.2 $\mathcal{E} = \mathcal{O}_Z \oplus \mathcal{O}_Z(a)$, therefore $\det(\mathcal{E}) = \mathcal{O}_Z(a)$, $c_1(\mathcal{E}) = aH$, and $c_2(\mathcal{E}) = 0$. Since Z is a Fano threefold, the Riemann-Roch Theorem [**Har77**, Appendix A, Exercise 6.7] applied to the sheaf $\mathcal{O}_Z$ yields $K_Z \cdot c_2(Z) = -24$. Moreover, the exact sequence

$$0 \to \mathcal{N}_{S/\widehat{G}_Y} \to \mathcal{N}_{S/Y} \to \mathcal{N}_{\widehat{G}_Y/Y\,|S} \to 0$$

corresponds, under the natural isomorphism $S \cong A$, to

$$0 \to \mathcal{O}_A(dH_{|A}) \to \mathcal{N}_{S/Y} \to \mathcal{O}_A(aH_{|A}) \to 0,$$

and so the Chern classes of the normal bundle of S are

$$c_1(\mathcal{N}_{S/Y}) \sim (a + d)H_{|A}, \qquad c_2(\mathcal{N}_{S/Y}) = ad^2\delta.$$

Furthermore, it is not difficult to verify that $K_{Y|S} \sim -(a + i_Z)H_{|A}$, $K_S \sim (d - i_Z)H_{|A}$, thus

$$K^2_{Y|S} = d\delta(a + i_Z)^2, \quad K^2_S = d\delta(d - i_Z)^2,$$
$$K_S \cdot K_{Y|S} = -d\delta(a + i_Z)(d - i_Z).$$

Finally, Proposition 2.9 yields

(i) $K^4_Y = 8\delta i_Z(a^2 + i_Z^2)$,

(ii) $K^2_Y \cdot c_2(Y) = 2\delta i_Z(a^2 + i_Z^2) + 96$,

(iii) $\chi(\mathcal{O}_Y(-K_Y)) = 9 + \frac{3}{2}\delta i_Z(a^2 + i_Z^2)$,

and Proposition 2.10 gives the statement. ■

The explicit invariants for the varieties of Corollary 2.6 are collected in Table II.2, page 32.

2.4. Deformations and automorphism groups

In this section we show that the varieties of Corollary 2.6 form indeed 28 distinct families of deformations, and we provide some partial results on the dimension of their Kuranishi family by giving an upper bound of $h^1(\mathcal{T}_X)$. This also leads to an upper bound of $h^0(\mathcal{T}_X)$, and precise results for some of the families. Recall that $h^0(\mathcal{T}_X)$ is the dimension of $\mathrm{Aut}(X)$, and that $h^2(\mathcal{T}_X) = 0$ by the Nakano vanishing theorem [**Laz04a**, Theorem 4.2.3]. This implies that the base of the Kuranishi family of X is smooth of dimension $h^1(\mathcal{T}_X)$, i.e. deformations of X are unobstructed.

By a smooth family of Fano varieties we mean a smooth, projective morphism $f\colon \mathcal{X} \to S$ with connected fibres, and S an irreducible quasi-projective variety, such that the anticanonical divisor $-K_{\mathcal{X}}$ is Cartier and f-ample.

Remark 2.12. The construction of X, as in Remark 2.2, can be also done in families. Given $g\colon \mathcal{Z} \to B$ a smooth family of Fano threefolds with Picard number 1 and index $i_Z > 1$, and given $\mathcal{O}_{\mathcal{Z}}(\mathcal{H}) \in \mathrm{Pic}(\mathcal{Z})$ such that the restriction $\mathcal{O}_{Z_b}(\mathcal{H}_{|_{Z_b}})$ to the fibre Z_b is the generator of $\mathrm{Pic}(Z_b)$ for all $b \in B$, it is possible to construct a smooth family of Fano fourfolds $f\colon \mathcal{X} \to \widetilde{B}$, where $\widetilde{B}$ is the open subset of $\mathbb{P}_B\left(\left(g_*\mathcal{O}_{\mathcal{Z}}(d\mathcal{H})\right)^\vee\right)$ which parametrises the smooth surfaces $A_b \in |d\mathcal{H}_{|_{Z_b}}|$ for all $b \in B$, so that the fibre of f over $\widetilde{b} \in \widetilde{B}$ is obtained from (Z_b, A_b) as in Remark 2.2.

Let us fix a smooth Fano threefold Z_i, $i \in \{1, \dots, 7\}$, and integers a, d satisfying (2.2), so that the varieties $X^i_{a,d}$ of Corollary 2.6 are parametrised by pairs (Z, A), with Z smooth and deformation equivalent to Z_i, and $A \in |\mathcal{O}_Z(d)|$ a smooth surface. In this section we will use the notation $X(Z, A)$ to denote the Fano fourfold given by the pair (Z, A).

Our first goal is to prove that if $f\colon \mathcal{X} \to S$ is a smooth family of Fano varieties, and if there exists $s_0 \in S$ such that its fibre X_{s_0} is isomorphic to $X(Z_{s_0}, A_{s_0})$, then all the fibres X_s are isomorphic to $X^i_{a,d}$ for some (Z_s, A_s), that is $X_s \cong X(Z_s, A_s)$. We are going to prove this by looking at the behaviour of the nef cone $\mathrm{Nef}(X_s)$, and the corresponding contractions, under the deformation f. Wiśniewski [**Wiś09**, Theorem 1] proved that, in the case of a smooth family of Fano varieties, the nef cone is locally constant. Moreover, generalised results in the setting of singular varieties were proved in [**dFH11**] and [**CFST16**].

Proposition 2.13. *Let $f\colon \mathcal{X} \to S$ be a smooth family of Fano varieties. Assume that there exists $s_0 \in S$ such that the fibre X_{s_0} is isomorphic to $X^i_{a,d}$. Then, any fibre X_s is one of the varieties of Corollary 2.6, and belongs to the same family of X_{s_0}.*

Proof. ***Step 1.*** We can assume $\dim S = 1$, and up to pull-back to the normalisation of S, we can assume S smooth. Since the fibres of a smooth family are all diffeomorphic, we have that the second Betti number is constant on the fibres of f; moreover, for smooth Fano varieties, the second Betti number coincides with the rank of the Picard group. Therefore the fibres X_s are smooth Fano fourfolds of Picard number 3, for all $s \in S$.

Step 2. In the following we adopt an argument in [**CFST16**] to show that the fibres of f have the same type of contractions.

By [**CFST16**, Theorem 2.2], the monodromy action (see [**Voi03**, Section 3]) of $\pi_1(S,s)$ on $H^2(X_s,\mathbb{Q})$ is finite. As in the proof of [**CFST16**, Theorem 2.7], we may take a finite étale cover $g\colon V \to S$ and the pull-back $\mathcal{X}_V = \mathcal{X} \times_S V$ of f via g. The resulting family $f_V\colon \mathcal{X}_V \to V$, which is still a deformation of X_{s_0}, now has trivial monodromy action on $H^2(X_t,\mathbb{Q})$ for all $t \in V$. By [**CFST16**, Theorem 2.12], the restriction map

$$\mathcal{N}^1(\mathcal{X}_V) \to \mathcal{N}^1(X_t)$$

is surjective for all $t \in V$. As a consequence, the map

$$\mathcal{N}^1(\mathcal{X}_V/V) \to \mathcal{N}^1(X_t)$$

is an isomorphism for all $t \in V$, where $\mathcal{N}^1(\mathcal{X}_V/V)$ is the real vector space of divisor with real coefficients, modulo numerical equivalence on the curves contracted by f_V.

Now, it follows from [**Wiś91b**, Proposition 1.3] that every elementary contraction of a fibre $X_t \to W_t$ extends to a relative elementary contraction $\mathcal{X}_V \to \mathcal{W}$, and conversely every relative elementary contraction $\mathcal{X}_V \to \mathcal{W}$ restricts to an elementary contraction $X_t \to W_t$ on every fibre. Therefore, [**dFH11**, Theorem 4.1] implies that $X_{t_1} \to W_{t_1}$ is of fibre type (resp. divisorial, small) if and only if $X_{t_2} \to W_{t_2}$ of fibre type (resp. divisorial, small), for any $t_1, t_2 \in V$. By applying a similar technique as in the proof of [**CFST16**, Theorem 2.7], we see that the contractions on the fibres are deformation equivalent.

Step 3. Step 2 and Remark 2.4 imply that every fibre X_t of f_V has an elementary divisorial contraction sending a divisor to a point, that is every X_t contains a divisor G_t (the exceptional divisor of that contraction) with $\dim \mathcal{N}_1(G_t, X_t) = 1$. Then, by [**CD15**, Theorem 3.8], every fibre X_t is one of the varieties of Corollary 2.6, and since the numerical invariants in Table II.2 are constant under deformation and are different amongst the families, we can conclude that every X_t is in the same family of X_{s_0}. ■

Corollary 2.14. *The varieties of Corollary 2.6 form 28 families of deformations. They are all distinct, and do not have common members.*

The next step is to understand when the varieties of the same family are isomorphic.

Proposition 2.15. *Let $X = X(Z,A)$ and $X' = X(Z',A')$. Then: $X \cong X'$ if and only if there exists an isomorphism $\psi\colon Z \to Z'$ such that $\psi(A) = A'$.*

Proof. Assume first that $X \cong X'$. Let φ and φ' be the conic bundles of Remark 2.2, and let ρ be the composition of φ' with the isomorphism $X \cong X'$. The cones $\mathrm{NE}(\varphi)$ and $\mathrm{NE}(\rho)$, which are the convex cones in $\mathrm{NE}(X)$ generated by the curves contracted by φ and ρ, must coincide, since they are both fibre-type contractions of X, and X admits only one such contraction by Remark 2.4. Therefore, by [**Deb01**, Proposition 1.14(b)], there exists a unique isomorphism $\psi\colon Z \to Z'$ which makes the following diagram commute.

$$\begin{array}{ccc} X & \xrightarrow{\cong} & X' \\ {\scriptstyle\varphi}\downarrow & \overset{\rho}{\searrow} & \downarrow{\scriptstyle\varphi'} \\ Z & \xrightarrow[\psi]{} & Z' \end{array}$$

Since the diagram commutes, the discriminant divisor of φ

$$\Delta := \{z \in Z | \varphi^{-1}(z) \text{ is singular}\}$$

maps via ψ onto the discriminant divisor Δ' of φ'. In fact $\Delta = A$ and $\Delta' = A'$, and we get the statement,

Conversely, if there exists an isomorphism $\psi\colon Z \to Z'$ such that $\psi(A) = A'$, for smooth surfaces $A, A' \in |\mathcal{O}_Z(d)|$, then ψ lifts as an isomorphism between $X(Z, A)$ and $X(Z', A')$. ■

At this point it is clear, by the previous results, that the dimension of the base of the Kuranishi family of $X^i_{a,d}$ is at most $h^1(\mathcal{T}_{Z_i}) + h^0(\mathcal{O}_{Z_i}(d)) - 1$.

Proposition 2.16. *Let $X = X^i_{a,d}$ be a Fano fourfold of Corollary 2.6. Then:*

$$h^1(\mathcal{T}_X) \leq h^1(\mathcal{T}_{Z_i}) + h^0(\mathcal{O}_{Z_i}(d)) - 1. \tag{2.4}$$

Furthermore:

(i) if $i = 1, \dots, 4$, then equality holds;

(ii) if $i = 7$ and $d = 1, 2$, then X is rigid, that is $h^1(\mathcal{T}_X) = 0$.

Proof. Inequality (2.4) follows from Propositions 2.13 and 2.15. As for (i), just observe that if $i = 1, \dots, 4$ then $h^0(\mathcal{T}_{Z_i}) = 0$ (see Table II.1), so that $\mathrm{Aut}(Z_i)$ is finite. Therefore, for a given deformation Z of Z_i there are at most finitely many different choices of A which yield isomorphic Fano fourfolds.

On the other hand, if $i = 7$ and $d = 1, 2$, the smooth surface $A \in |\mathcal{O}_Z(d)|$ is either an hyperplane or a smooth quadric in $\mathbb{P}^3$. Since for a fixed d they are all projectively equivalent to each other, it follows that $X^7_{a,1}$ and $X^7_{a,2}$ are rigid. This gives (ii). ■

We note that $\chi(\mathcal{T}_X) = h^0(\mathcal{T}_X) - h^1(\mathcal{T}_X)$ can be computed from the other invariants of X (see Table II.2) by the Hirzebruch-Riemann-Roch theorem, see for instance [**CCF19**, Lemma 6.25], which yields

$$\chi(\mathcal{T}_X) = 27 - 5h^0(\mathcal{O}_X(-K_X)) + K_X^4 + 3b_2(X) - h^{1,2}(X) - h^{2,2}(X) + 3h^{1,3}(X).$$

We can then estimate also $h^0(\mathcal{T}_X)$; we give the estimates in Table II.3.

2.5. Base locus of $-K_X$

In this section we study the base locus of the linear system $|-K_X|$, and show that it is empty for 26 of the 28 families of Corollary 2.6, while it consists of at most two distinct points for the remaining cases. Moreover, we show that a general element of $|-K_X|$ is smooth. All the base loci are collected in Table II.2.

To compute the base locus of $|-K_X|$, we give special effective divisors in the linear system and then apply Lemmata 1.1 and 1.2. We already know that

$$-K_X \equiv (i_Z - a)\varphi^*H + 2\widehat{G} + E \equiv (i_Z + a - d)\varphi^*H + 2G + \widehat{E},$$

see Remark 2.3. Note that $d\varphi^*H \equiv E + \widehat{E}$, and so, by (2.1), we have

$$G + a\varphi^*H \equiv \widehat{G} + E, \quad \widehat{G} + (d - a)\varphi^*H \equiv G + \widehat{E},$$

which yields

$$-K_X \equiv i_Z\varphi^*H + G + \widehat{G}. \tag{2.5}$$

Proposition 2.17. *Let X be as in Corollary 2.6. If H is globally generated, then so is $-K_X$. Otherwise, either $X \cong X^1_{0,1}$ and $|-K_X|$ has a unique basepoint, or $X \cong X^1_{1,2}$ and $|-K_X|$ has two basepoints. In all cases, a general element of $|-K_X|$ is smooth.*

PROOF. ***Step 1.*** Assume that H is globally generated. Then, by Remark 2.3(ii) and (2.5), we have

$$\mathrm{Bs}\,|-K_X| \subseteq \big(G \cup \widehat{G}\big) \cap \big(\widehat{G} \cup E\big) \cap \big(G \cup \widehat{E}\big) = \emptyset.$$

Therefore, $|-K_X|$ has no basepoint and $-K_X$ is globally generated.

Conversely, assume that H is not globally generated. Then $Z = Z_1$, and either $a = 0$ and $d = 1$ or $a = 1$ and $d = 2$. The linear system $|H|$ has a unique simple basepoint P_0, while $|2H|$ is free; moreover, recall that in Remark 2.2 we fixed a smooth surface $A \in |dH|$, which exists for $Z = Z_1$ and $d = 1, 2$ (see Table II.1 and Lemma 2.5).

Step 2. Let $Z = Z_1$, $a = 0$ and $d = 1$: then, the smooth surface $A \in |H|$ contains P_0, and the fibre $T := \varphi^{-1}(P_0)$ over P_0 is equal to $F_0 \cup \widehat{F}_0$, with $F_0 \subset E$ and $\widehat{F}_0 \subset \widehat{E}$ irreducible rational curves. Therefore

$$\mathrm{Bs}\,|-K_X| \subseteq \big(G \cup \widehat{G}\big) \cap \big(\widehat{G} \cup E\big) \cap \big(T \cup G \cup \widehat{E}\big) = F_0 \cap \widehat{G},$$

which is the point $Q_0 \in \widehat{G}$ corresponding to P_0 under the isomorphism $\widehat{G} \cong Z$. Now, observe that

$$\mathcal{O}_{\widehat{G}}(-K_{X|\widehat{G}}) \cong \mathcal{O}_Z(H),$$

so that $\{Q_0\} = \mathrm{Bs}\,|-K_{X|\widehat{G}}| \subseteq \mathrm{Bs}\,|-K_X| \cap \widehat{G}$ by Lemma 1.2. Thus, Q_0 is a basepoint for $|-K_X|$ and $\mathrm{Bs}\,|-K_X| = \{Q_0\}$.

Note that in the class of $2\varphi^*H + G + \widehat{G}$, which is numerically equivalent to $-K_X$ by (2.5), we can choose an effective divisor that is smooth at Q_0: fix an effective divisor $D \equiv 2H$ such that $P_0 \notin D$, hence $\varphi^*D \in |2\varphi^*H|$ does not contain Q_0. Therefore, $\varphi^*D + G + \widehat{G}$ is smooth at Q_0 and a general effective divisor in $|-K_X|$ is smooth by Lemma 2.18.

Step 3. The last case to consider is $Z = Z_1$, $a = 1$, $d = 2$. Let T be the fibre $\varphi^{-1}(P_0)$ over P_0: then, similarly as above,

$$\mathrm{Bs}\,|-K_X| \subseteq \big(G \cup \widehat{G}\big) \cap \big(T \cup \widehat{G} \cup E\big) \cap \big(T \cup G \cup \widehat{E}\big) = \big(T \cap G\big) \cup \big(T \cap \widehat{G}\big),$$

where $\{Q_1\} = T \cap G$ and $\{Q_2\} = T \cap \widehat{G}$ are the points corresponding to P_0 in G and $\widehat{G}$. Again, similarly as above,

$$\mathcal{O}_G(-K_{X|G}) \cong \mathcal{O}_{\widehat{G}}(-K_{X|\widehat{G}}) \cong \mathcal{O}_Z(H),$$

so that $\{Q_1\} = \mathrm{Bs}\,|-K_{X|G}|$ and $\{Q_2\} = \mathrm{Bs}\,|-K_{X|\widehat{G}}|$ are both contained in $\mathrm{Bs}\,|-K_X|$ by Lemma 1.2. Thus Q_1 and Q_2 are basepoints for $|-K_X|$ and $\mathrm{Bs}\,|-K_X| = \{Q_1, Q_2\}$.

We now show that a general effective divisor in $|-K_X|$ is smooth. We have two possibilities: either $A \in |2H|$ contains P_0 or it does not. If we assume the latter, then the divisor $\varphi^*A + G + \widehat{G}$ is smooth at Q_1 and Q_2. If we assume the former, then we can choose a divisor $D \in |2H|$ such that $P_0 \notin D$, which implies that the divisor $\varphi^*D + G + \widehat{G} \in |-K_X|$ is smooth at Q_1 and Q_2 (as in Step 2). Thus the statement holds by Lemma 2.18. ■

Lemma 2.18. *Let X be a smooth projective variety of dimension ≥ 2 and let L be a linear system of divisors on X. Assume that* $\mathrm{Bs}(L) = \{x_1, \ldots, x_r\}$.

Then, either a general element of L is smooth or $\exists i \in \{1, \ldots, r\}$ such that D is singular at x_i for all $D \in L$.

PROOF. First note that the linear system L does not have fixed component, since $\operatorname{codim} \mathrm{Bs}(L) \geq 2$.

Set $S_i := \{D \in L | \text{ D is singular at } x_i\}$, $i = 1, \ldots, r$. If $S_i = L$ for some i, then every divisor in L is singular at x_i. Otherwise, $U := L \setminus \left(\bigcup_{i=1}^{r} S_i\right)$ is non-empty, open and such that every $D \in U$ is smooth at $\mathrm{Bs}(L) = \{x_1, \ldots, x_r\}$; by Bertini's theorem, L contains smooth divisors. ■

2.6. Rationality

By construction, X is birationally equivalent to $Z \times \mathbb{P}^1$, and so we can study rationality on X by what is known on Z. If $i = 4, 5, 6, 7$, Z_i is rational, while Z_j is not rational for $j = 1, 2, 3$ (see Table II.1). So $X^i_{a,d}$ is rational for $i = 4, 5, 6, 7$.

The remaining cases have been studied in [**HT19**] with respect to stable rationality. For $i = 1, 2$, the very general Z_i is not stably rational, which implies that $Z_i \times \mathbb{P}^1$ is not rational. Therefore, the very general $X^i_{a,d}$ is not rational for $i = 1, 2$. As for cubics in $\mathbb{P}^4$, i.e. Z_3, stable rationality is still an open problem, and so we have no information on $X^3_{a,d}$.

2.7. Conclusions and final tables

Remark 2.19. The proof of Theorem 2.1 is a consequence of the results of the previous sections: Corollary 2.6 gives the 28 families, in Section 2.3 we compute the numerical invariants, Proposition 2.17 provides the results on the base locus of $|-K_X|$, and in Section 2.6 we discuss rationality.

Toric Fano fourfolds have been classified by Batyrev [**Bat99**], and we observe that the varieties $X^i_{a,d}$ are toric when $Z = Z_7 = \mathbb{P}^3$ and $d = 1$; this gives the three cases $X^7_{0,1}$, $X^7_{2,1}$ and $X^7_{3,1}$. Keeping the notation in [**Bat99**], they correspond, respectively, to the varieties E_3, E_2 and E_1.

Furthermore, Tsukioka [**Tsu06**] gave a classification for Fano varieties of dimension ≥ 3 containing a negative divisor E isomorphic to the projective space. When $n = 4$, the varieties in [**Tsu06**, Theorem 1, 2.(b)] are isomorphic to $X^7_{a,d}$ for all the admitted values a, d given by (2.2). Specifically, we can choose $E = \widehat{G}$ when $0 \leq a \leq \frac{d}{2}$, and $E = G$ when $a > d$ (see Remark 2.2 and Corollary 2.6).

Remark 2.20. As a final note, we can ask whether the varieties $X^i_{a,d}$ of Theorem 2.1 appear as fibres of elementary fibre-type contractions, that is if they are "fibre-like". This property has been introduced and studied in [**CFST16**], where the authors give two criteria for fibre-likeness, one necessary and one sufficient. As a consequence of [**CFST16**, Corollary 3.9], by looking at the elementary contractions of $X^i_{a,d}$ (see Remark 2.4) we see that, if $a \neq \frac{d}{2}$, then $X^i_{a,d}$ cannot be fibre-like.

In Table II.2 we collect all the numerical invariants computed in Section 2.3, where $h^0(-K_X) := h^0(\mathcal{O}_X(-K_X))$, and in the last two columns we include the base locus of $|-K_X|$ and whether the variety is rational, not rational, toric, or unknown (?).

In Table II.3 we display the explicit invariants of Section 2.4. In order to do so, we are left to compute $h^0(\mathcal{O}_Z(d))$, as the rest of the ingredients are already in Table II.1 and Table II.2. The Riemann-Roch and Kodaira vanishing theorems (see Lemma 1.17) yield

$$\chi(\mathcal{O}_Z(d)) = h^0(\mathcal{O}_Z(d)) = 1 + \frac{2d}{i_Z} + \frac{d\delta}{12}(i_Z^2 + 3di_Z + 2d^2).$$

TABLE II.2. Fano fourfolds of Picard number 3 with a prime divisor of Picard number 1

$X^i_{a,d}$	K_X^4	$K_X^2 \cdot c_2(X)$	$h^0(-K_X)$	$h^{1,2}$	$h^{1,3}$	$h^{2,2}$	$\mathrm{Bs}\lvert -K_X\rvert$	
$X^1_{0,1}$	47	98	17	21	0	11	$\{Q_0\}$	the very general is not rational
$X^1_{1,2}$	30	84	13	21	1	22	$\{Q_1, Q_2\}$	the very general is not rational
$X^2_{0,1}$	94	112	26	10	0	10	$\emptyset$	the very general is not rational
$X^2_{1,2}$	60	96	19	10	1	22	$\emptyset$	the very general is not rational
$X^3_{0,1}$	141	126	35	5	0	9	$\emptyset$	?
$X^3_{1,2}$	90	108	25	5	1	22	$\emptyset$	?
$X^4_{0,1}$	188	140	44	2	0	8	$\emptyset$	rational
$X^4_{1,2}$	120	120	31	2	1	22	$\emptyset$	rational
$X^5_{0,1}$	235	154	53	0	0	7	$\emptyset$	rational
$X^5_{1,2}$	150	132	37	0	1	22	$\emptyset$	rational
$X^6_{0,1}$	346	184	74	0	0	4	$\emptyset$	rational
$X^6_{0,2}$	296	176	65	0	0	8	$\emptyset$	rational
$X^6_{1,2}$	260	164	58	0	0	8	$\emptyset$	rational
$X^6_{1,3}$	210	156	49	0	1	22	$\emptyset$	rational
$X^6_{2,1}$	430	208	90	0	0	4	$\emptyset$	rational
$X^6_{2,4}$	160	148	40	0	5	54	$\emptyset$	rational
$X^7_{0,1}$	431	206	90	0	0	3	$\emptyset$	toric
$X^7_{0,2}$	376	196	80	0	0	4	$\emptyset$	rational
$X^7_{0,3}$	341	194	74	0	0	9	$\emptyset$	rational
$X^7_{1,2}$	350	188	75	0	0	4	$\emptyset$	rational
$X^7_{1,3}$	295	178	65	0	0	9	$\emptyset$	rational
$X^7_{1,4}$	260	176	59	0	1	22	$\emptyset$	rational
$X^7_{2,1}$	489	222	101	0	0	3	$\emptyset$	toric
$X^7_{2,4}$	240	168	55	0	1	22	$\emptyset$	rational
$X^7_{2,5}$	205	166	49	0	4	47	$\emptyset$	rational
$X^7_{3,1}$	605	254	123	0	0	3	$\emptyset$	toric
$X^7_{3,2}$	454	220	95	0	0	4	$\emptyset$	rational
$X^7_{3,6}$	170	164	43	0	10	88	$\emptyset$	rational

TABLE II.3. Cohomology groups of the tangent sheaf

$X^i_{a,d}$	$h^0(\mathcal{T}_X)$	$h^1(\mathcal{T}_X)$	$\chi(\mathcal{T}_X)$
$X^1_{0,1}$	2	36	-34
$X^1_{1,2}$	1	40	-39
$X^2_{0,1}$	2	22	-20
$X^2_{1,2}$	1	29	-28
$X^3_{0,1}$	2	14	-12
$X^3_{1,2}$	1	24	-23
$X^4_{0,1}$	2	8	-6
$X^4_{1,2}$	1	21	-20
$X^5_{0,1}$	≤ 5	≤ 6	-1
$X^5_{1,2}$	≤ 4	≤ 22	-18
$X^6_{0,1}$	≤ 12	≤ 4	8
$X^6_{0,2}$	≤ 12	≤ 13	-1
$X^6_{1,2}$	≤ 11	≤ 13	-2
$X^6_{1,3}$	≤ 11	≤ 29	-18

$X^i_{a,d}$	$h^0(\mathcal{T}_X)$	$h^1(\mathcal{T}_X)$	$\chi(\mathcal{T}_X)$
$X^6_{2,1}$	≤ 16	≤ 4	12
$X^6_{2,4}$	≤ 11	≤ 54	-43
$X^7_{0,1}$	14	0	14
$X^7_{0,2}$	8	0	8
$X^7_{0,3}$	≤ 17	≤ 19	-2
$X^7_{1,2}$	7	0	7
$X^7_{1,3}$	≤ 16	≤ 19	-3
$X^7_{1,4}$	≤ 16	≤ 34	-18
$X^7_{2,1}$	17	0	17
$X^7_{2,4}$	≤ 16	≤ 34	-18
$X^7_{2,5}$	≤ 16	≤ 55	-39
$X^7_{3,1}$	23	0	23
$X^7_{3,2}$	11	0	11
$X^7_{3,6}$	≤ 16	≤ 83	-67

CHAPTER 3

Fano manifolds with Lefschetz defect 3

3.1. Introduction

Let X be a smooth, complex Fano variety. The *Lefschetz defect* δ_X is an invariant of X which depends on the Picard number of prime divisors in X. More precisely, consider the vector space $\mathcal{N}_1(X)$ of 1-cycles in X, with real coefficients, up to numerical equivalence; its dimension is the Picard number ρ_X of X. Given a prime divisor D in X, we consider the pushforward $\iota_*\colon \mathcal{N}_1(D) \to \mathcal{N}_1(X)$ induced by the inclusion $\iota\colon D \hookrightarrow X$, and set $\mathcal{N}_1(D,X) := \iota_*(\mathcal{N}_1(D)) \subseteq \mathcal{N}_1(X)$; finally:

$$\delta_X := \max\big\{\operatorname{codim}\mathcal{N}_1(D,X) \,|\, D \text{ a prime divisor in } X\big\}.$$

A very strong result involving the Lefschetz defect is the following:

Theorem 3.1 ([**Cas12**], Th. 3.3)**.** *Let X be a smooth Fano variety with $\delta_X \geq 4$. Then $X \cong S \times T$, where S is a del Pezzo surface with $\rho_S = \delta_X + 1$.*

In this chapter we consider Fano varieties X with $\delta_X = 3$. Even if such X does not need to be a product, it turns out that it still has a very rigid and explicit structure. More precisely, we show that there exist a smooth Fano variety T with $\dim T = \dim X - 2$, $\rho_T = \rho_X - 4$, and $\delta_T \leq 3$, and a $\mathbb{P}^2$-bundle Z over T, such that X is obtained by blowing-up Z along three pairwise disjoint smooth, irreducible, codimension 2 subvarieties S_1, S_2, S_3. The $\mathbb{P}^2$-bundle $\varphi\colon Z \to T$ is the projectivization of a suitable decomposable vector bundle on T, and S_2 and S_3 are sections of φ. Instead, $\varphi_{|S_1}\colon S_1 \to T$ is finite of degree 1 or 2.

To state our results, we first describe two explicit ways to obtain X from T; the two constructions differ in the degree of S_1 over T. Then we prove (Th. 3.4) that every Fano variety X with Lefschetz defect 3 is obtained by one of these two constructions; this is our main result.

Proposition 3.2 (Construction A)**.** *Let T be a smooth Fano variety with $\delta_T \leq 3$, and D_1, D_2, D_3 divisors on T such that $-K_T + D_i - D_j$ is ample for every $i, j \in \{1,2,3\}$. Set*

$$Z := \mathbb{P}_T\big(\mathcal{O}(D_1) \oplus \mathcal{O}(D_2) \oplus \mathcal{O}(D_3)\big) \xrightarrow{\varphi} T,$$

and let $S_i \subset Z$ be the section of φ corresponding to the projection onto the summand $\mathcal{O}_T(D_i)$, for $i = 1,2,3$. Finally let $h\colon X \to Z$ be the blow-up of S_1, S_2, S_3, and $\sigma := \varphi \circ h\colon X \to T$.

Then X is a smooth Fano variety with $\delta_X = 3$, $\dim X = \dim T + 2$, and $\rho_X = \rho_T + 4$.

Proposition 3.3 (Construction B)**.** *Let T be a smooth Fano variety with $\delta_T \leq 3$, and N a divisor on T such that $N \not\equiv 0$ and both $-K_T + N$ and $-K_T - N$ are ample. Set*

$$Z := \mathbb{P}_T\big(\mathcal{O}(N) \oplus \mathcal{O} \oplus \mathcal{O}\big) \xrightarrow{\varphi} T,$$

and let $S_2, S_3 \subset Z$ be the sections of φ corresponding to the projections onto the summands $\mathcal{O}$. Let H be a tautological divisor of Z, and assume that there exists a smooth, codimension 2 subvariety $S_1 \subset Z$ which is a complete intersection of general elements in the linear systems $|H|$ and $|2H|$. Finally let $h\colon X \to Z$ be the blow-up of S_1, S_2, S_3, and $\sigma := \varphi \circ h\colon X \to T$.

Then X is a smooth Fano variety with $\delta_X = 3$, $\dim X = \dim T + 2$, and $\rho_X = \rho_T + 4$.

Let us note that Constructions A and B give new, explicit ways to construct Fano varieties of any dimension.

Theorem 3.4 (Structure theorem)**.** *Every smooth Fano variety X with $\delta_X = 3$ is obtained with Construction A or B.*

When $\dim X = 4$ and $\rho_X = 5$, Th. 3.4 is proved in [**CR22**], and used to obtain the classification of Fano 4-folds with $\delta_X = 3$ and $\rho_X = 5$. To treat the general case, we apply the same strategy as in [**CR22**], adapting the proof step by step. Our starting point is the existence of a fibration in del Pezzo surfaces $\sigma\colon X \to T$ after [**Cas12**, Th. 3.3]. We give an outline of the proof at the end of the Introduction.

In particular, given a Fano variety X of dimension n with $\delta_X = 3$, by Th. 3.4 there exists a smooth Fano variety T with $\dim T = n - 2$, $\rho_T = \rho_X - 4$, and $\delta_T \leq 3$, and a morphism $\sigma\colon X \to T$, such that σ realises X as a blow-up of a $\mathbb{P}^2$-bundle Z over T, as in Construction A or B. We will denote X by X_A in the former case, and by X_B in the latter.

Let us now describe some applications of the structure theorem.

3.1.1. Fano 4-folds. In dimension 4, we complete the *classification of Fano 4-folds with Lefschetz defect 3*, started in [**CR22**]; note that all of them are rational, see [**MR21**, Cor. 1.3]. We treat the last case left open, that is $\delta_X = 3$ and $\rho_X = 6$, in which we show that there are 11 families, among which 8 are toric; we refer the reader to Section 3.6 for more details and for the description of the two non-toric families and their invariants. Note that the toric case is simpler, as toric Fano 4-folds have been classified by Batyrev [**Bat99, Sat00**]. The final classification is as follows.

Proposition 3.5. *Let X be a Fano 4-fold with $\delta_X = 3$. Then $5 \leq \rho_X \leq 8$ and there are 19 families for X, among which 14 are toric. More precisely:*
- *if $\rho_X = 8$, then $X \cong F \times F$, F the blow-up of $\mathbb{P}^2$ at 3 non-collinear points;*
- *if $\rho_X = 7$, then $X \cong F \times F'$, F' the blow-up of $\mathbb{P}^2$ at 2 points;*
- *if $\rho_X = 6$, there are 11 families for X, among which 8 are toric;*
- *if $\rho_X = 5$, there are 6 families for X, among which 4 are toric.*

We also apply our results to the study of conic bundles $\eta\colon X \to Y$ where X is a Fano 4-fold and $\rho_X - \rho_Y \geq 3$, see Cor. 3.26.

3.1.2. The case $\boldsymbol{\rho_X = 5}$. The second case that we consider in detail is that of Fano varieties with Lefschetz defect 3 and Picard number 5, the minimal one, in Section 3.7. After Th. 3.4 there are a smooth Fano variety T with $\rho_T = 1$ and a morphism $\sigma\colon X \to T$ that realises X as a blow-up of a $\mathbb{P}^2$-bundle over T as in Construction A or B. We show that in this case T and σ are uniquely determined, and we list explicitly the different X that are obtained from a given T. To state the result, let us fix some notation for the case $\rho_T = 1$: $\mathcal{O}_T(1)$ is the ample generator of $\mathrm{Pic}(T) \cong \mathbb{Z}$,

and $\mathcal{O}_T(a) := \mathcal{O}_T(1)^{\otimes a}$ for every $a \in \mathbb{Z}$. Moreover $\mathcal{O}_T(-K_T) \cong \mathcal{O}_T(i_T)$, where i_T is the index of T.

Proposition 3.6. *Let X be a smooth Fano variety with $\delta_X = 3$ and $\rho_X = 5$. Then T and the morphism $\sigma\colon X \to T$ as in Construction A or B are uniquely determined; we have $X = X_A$ if σ is smooth, and $X = X_B$ if σ is not smooth.*

Let $X = X_A$. Then there are uniquely determined integers a, b with

$$b \leq 0, \quad |b| \leq \frac{i_T - 1}{2}, \quad \text{and} \quad |b| \leq a \leq i_T - 1 - |b|,$$

such that X is obtained with Construction A from T with $Z = \mathbb{P}_T(\mathcal{O}(a) \oplus \mathcal{O} \oplus \mathcal{O}(b))$.

Let $X = X_B$. Then there is a uniquely determined integer a with $1 \leq a \leq i_T - 1$ such that X is obtained with Construction B from T with $Z = \mathbb{P}_T(\mathcal{O}(a) \oplus \mathcal{O} \oplus \mathcal{O})$.

3.1.3. Fano varieties containing a divisor with Picard number 2. We note that the assumptions $\rho_X = 5$ and $\delta_X = 3$ imply that X contains a prime divisor D with $\dim \mathcal{N}_1(D, X) = 2$; in fact it is easy to see that all the varieties as in Prop. 3.6 also contain a prime divisor D' with $\rho_{D'} = 2$. We obtain the following application to Fano varieties containing a prime divisor with $\rho = 2$; an analogous result for the case of a prime divisor with $\rho = 1$ is given in [**CD15**, Th. 3.8].

Corollary 3.7. *Let X be a smooth Fano variety containing a prime divisor D with $\rho_D = 2$, or more generally with $\dim \mathcal{N}_1(D, X) = 2$. Then either $X \cong S \times T$ where S is a del Pezzo surface and $\rho_T = 1$, or $\rho_X \leq 5$. Moreover, $\rho_X = 5$ if and only if X is as in Prop. 3.6.*

3.1.4. Elementary divisorial contractions. Corollary 3.7 is also related to the study of *Fano varieties having an elementary divisorial contraction $\tau\colon X \to X'$ where $\tau(\mathrm{Exc}(\tau))$ is a curve,* because then automatically $\dim \mathcal{N}_1(\mathrm{Exc}(\tau), X) = 2$. It follows from Th. 3.1 that $\rho_X \leq 5$, and Tsukioka [**Tsu10**] has classified the case $\rho_X = 5$ when τ is the blow-up of a smooth curve in a smooth variety. We generalise this classification to an arbitrary elementary divisorial contraction τ such that $\dim \tau(\mathrm{Exc}(\tau)) = 1$, as follows.

Theorem 3.8. *Let X be a smooth Fano variety of dimension $n \geq 4$ with $\rho_X = 5$. Then the following are equivalent:*

- *(i) there is an elementary divisorial contraction $\tau\colon X \to X'$ such that $\tau(\mathrm{Exc}(\tau))$ is a curve;*
- *(ii) X is obtained with Construction A or B from a smooth Fano variety T such that $\dim T = n-2$, $\rho_T = 1$, $i_T > 1$, with $Z = \mathbb{P}_T(\mathcal{O}(a) \oplus \mathcal{O} \oplus \mathcal{O})$ and $a \in \{1, \ldots, i_T - 1\}$.*

If these conditions hold we have $\mathrm{Exc}(\tau) \cong \mathbb{P}^1 \times T$, $\mathcal{N}_{\mathrm{Exc}(\tau)/X} \cong \pi_{\mathbb{P}_1}^* \mathcal{O}(-1) \otimes \pi_T^* \mathcal{O}(-a)$, and $\tau(\mathrm{Exc}(\tau)) \cong \mathbb{P}^1$.

When $\dim X = 4$, Cor. 3.7 and Th. 3.8 are already proved in [**CR22**].

3.1.5. Strategy of the proof of Theorem 3.4. Our starting point is the existence, from [**Cas12**, Th. 3.3], of a flat fibration $\sigma\colon X \to T$, where T is a smooth Fano variety with $\dim T = \dim X - 2$ and $\rho_T = \rho_X - 4$; moreover σ factors as $X \to Y \to T$, where the first map is a conic bundle, and the second one is smooth with fibre $\mathbb{P}^1$. We collect the properties of σ in Th. 3.16.

We show that the fibration $\sigma\colon X \to T$ has a different factorisation as $X \to Z \xrightarrow{\varphi} T$, where φ is a $\mathbb{P}^2$-bundle, and $X \to Z$ is the blow-up along three pairwise disjoint smooth, irreducible subvarieties $S_i \subset Z$ of codimension 2, horizontal for φ.

Then we prove that S_2 and S_3 are always sections of φ. When S_1 is a section as well, we show that $\sigma\colon X \to T$ is as in Construction A.

Otherwise, we study the degree $d > 1$ of S_1 over T. Fibre-wise, for $t \in T$ general, we have $Z_t := \varphi^{-1}(t) \cong \mathbb{P}^2$, $X_t := \sigma^{-1}(t)$ a smooth del Pezzo surface, and $X_t \to Z_t$ the blow-up of the $d+2$ points $(S_1 \cup S_2 \cup S_3) \cap Z_t$. This is the hardest part of the proof; we restrict to a general curve $C \subset T$, and construct in $\overline{Z} := \varphi^{-1}(C)$ a divisor which is a $\mathbb{P}^1$-bundle over C and contains $S_1 \cap \overline{Z}$. Therefore, for $t \in C$ general, the d points $S_1 \cap Z_t$ are collinear; since X_t is del Pezzo, this implies that $d = 2$. Finally we show that $\sigma\colon X \to T$ is as in Construction B.

3.1.6. Structure of the chapter. Sections 3.2, 3.3 and 3.4 represent the core of the chapter: in Sections 3.2 and 3.3 we present Constructions A and B respectively, and prove Propositions 3.2 and 3.3, while in Section 3.4 we prove Theorem 3.4.

In the second part of the chapter we consider the applications of the structure theorem. In Section 3.5 we study the del Pezzo fibration $\sigma\colon X \to T$; we describe its fibres, the relative cone $\mathrm{NE}(\sigma)$, the relative contractions, and the different factorisation of σ. We also give some conditions on T in order to perform Constructions A and B to obtain Fano varieties X different from the product $F \times T$, F the blow-up of $\mathbb{P}^2$ at three non-collinear points. In Section 3.6 we give the applications to Fano 4-folds, and in Section 3.7 the applications to Fano varieties with $\rho_X = 5$. Finally, in Section 3.8 we study the base locus of the anticanonical divisor $-K_X$ and provide some non-trivial examples.

We point out that Section 3.8 is the result of a later study and is not contained in [**CRS22**].

3.2. Construction A

Let $n \in \mathbb{Z}_{\geq 2}$ and T a smooth Fano variety of dimension $n-2$. We consider three divisors D_1, D_2, D_3 in T and set

$$Z := \mathbb{P}_T\big(\mathcal{O}(D_1) \oplus \mathcal{O}(D_2) \oplus \mathcal{O}(D_3)\big) \xrightarrow{\varphi} T,$$

so that Z is a smooth projective variety of dimension n with $\rho_Z = \rho_T + 1$. Let $S_i \subset Z$ be the section of φ corresponding to the projection $\mathcal{O}_T(D_1) \oplus \mathcal{O}_T(D_2) \oplus \mathcal{O}_T(D_3) \twoheadrightarrow \mathcal{O}_T(D_i)$, for $i = 1, 2, 3$. Finally let $h\colon X \to Z$ be the blow-up of Z along S_1, S_2, S_3, so that X is a smooth projective variety of dimension n with $\rho_X = \rho_T + 4$, and set $\sigma := \varphi \circ h\colon X \to T$.

Lemma 3.9. *The following are equivalent:*

(*i*) *X is Fano;*
(*ii*) *$-K_T + D_i - D_j$ is ample for every $i, j \in \{1, 2, 3\}$.*

Proof. For $i = 1, 2, 3$ let $E_i \subset X$ be the exceptional divisor over S_i, $F_i \subset Z$ as in Rem. 1.18, and $E_i' \subset X$ its transform.

The implication $(i) \Rightarrow (ii)$ follows directly from Lemma 1.21. In fact X is Fano and $-K_X$ is ample, so that $-K_{X|E_i}$ is an ample divisors on E_i for all $i \in \{1, 2, 3\}$; Lemma 1.21 implies that $-K_{X|E_i}$ is the tautological divisor of

$$E_i \cong \mathbb{P}_T\big(\mathcal{O}(-K_T + D_i - D_j) \oplus \mathcal{O}(-K_T + D_i - D_j)\big),$$

and so $\mathcal{O}(-K_T + D_i - D_j) \oplus \mathcal{O}(-K_T + D_i - D_k)$ is an ample vector bundle on T. Therefore, the line bundles $\mathcal{O}(-K_T + D_i - D_j)$ are ample for all $i, j \in \{1, 2, 3\}$.

Let us show the converse $(ii) \Rightarrow (i)$. We first show that $-K_X$ is strictly nef, namely that $-K_X \cdot \Gamma > 0$ for every irreducible curve $\Gamma \subset X$.

Again, it follows directly from Lemmata 1.21 and 1.22 that $(-K_X)_{|E_i}$ and $(-K_X)_{|E_i'}$ are ample divisors respectively on E_i and E_i', for every $i = 1, 2, 3$. Therefore if Γ is contained in one of these divisors, we have $-K_X \cdot \Gamma > 0$.

Assume that $\sigma(\Gamma)$ is a point. By construction, the sections S_i of φ are fibre-wise in general linear position, so that every fibre $X_t := \sigma^{-1}(t)$ is a smooth del Pezzo surface. We have $\Gamma \subset X_t$ for some $t \in T$, therefore $-K_X \cdot \Gamma = -K_{X_t} \cdot \Gamma > 0$.

Now assume that Γ is not contracted by σ, and that $\Gamma \not\subset E_i, E_i'$ for $i = 1, 2, 3$. It follows from Rem. 1.18 that:

$$-K_X \sim \sigma^*(-K_T) + \sum_{i=1}^{3}(E_i + E_i').$$

Then $\sigma(\Gamma)$ is an irreducible curve in T, and $-K_X \cdot \Gamma = -K_T \cdot \sigma_*(\Gamma) + \sum_{i=1}^{3}(E_i + E_i') \cdot \Gamma > 0$.

Now we show that $-K_X$ is big. Consider the divisor $A_Z := F_1 + \varphi^*(-K_T)$. By Rem. 1.18, $F_1 + \varphi^*(D_1)$ is a tautological divisor for $\mathbb{P}_T(\mathcal{O}(D_1) \oplus \mathcal{O}(D_2) \oplus \mathcal{O}(D_3))$, therefore $A_Z = F_1 + \varphi^*(D_1) + \varphi^*(-K_T - D_1)$ is a tautological divisor for

$$\mathbb{P}_T(\mathcal{O}(-K_T) \oplus \mathcal{O}(-K_T + D_2 - D_1) \oplus \mathcal{O}(-K_T + D_3 - D_1)),$$

hence A_Z is ample on Z.

Therefore there exist non-negative integers m, c_i such that the divisor

$$\begin{aligned} A_X := mh^*(A_Z) - \sum_{i=1}^{3} c_i E_i \;&=\; m\big(E_1' + E_2 + E_3 + \sigma^*(-K_T)\big) - \sum_{i=1}^{3} c_i E_i \\ &= m\sigma^*(-K_T) + mE_1' - c_1 E_1 + (m - c_2)E_2 + (m - c_3)E_3 \end{aligned}$$

is ample on X. Then

$$\begin{aligned} -mK_X &\sim m\sigma^*(-K_T) + m\sum_{i=1}^{3}(E_i + E_i') \\ &= A_X + (c_1 + m)E_1 + c_2 E_2 + c_3 E_3 + mE_2' + mE_3'. \end{aligned}$$

We have written a multiple of $-K_X$ as the sum of an ample divisor and an effective divisor, which implies that $-K_X$ is big.

Finally, as $-K_X$ is nef and big, it is ample by the basepoint-free theorem 1.6. In fact $-K_X$ is semiample and the morphism

$$\varphi_{|-mK_X|} \colon X \to X_0$$

corresponding to the linear system $|-mK_X|$ for $m \gg 0$ is finite, since $-K_X$ is strictly nef and so $\varphi_{|-mK_X|}$ does not contract any curve on X. Since $-mK_X$ is the pull-back of some ample divisor on X_0 and $\varphi_{|-mK_X|}$ is finite, it is ample. ■

Lemma 3.10. *Suppose that X is Fano. Then $\delta_X \geq 3$, and $\delta_X = 3$ if and only if $\delta_T \leq 3$.*

PROOF. The divisor $E_1 \subset X$ is a $\mathbb{P}^1$-bundle over T, so that $\dim \mathcal{N}_1(E_1, X) \leq \rho_{E_1} = \rho_T + 1 = \rho_X - 3$, which implies that $\delta_X \geq 3$.

If $\delta_X = 3$, then $\delta_T \leq \delta_X = 3$ by [**Cas12**, Rem. 3.3.18]. If instead $\delta_X \geq 4$, then by Th. 3.1 we have $X \cong S \times X'$ where S is a del Pezzo surface with $\rho_S = \delta_X + 1 \geq 5$; moreover σ must be a product morphism (see [**Rom19**, Lemma 2.10]). Since the fibre of σ is a del Pezzo surface with $\rho = 4$, we must have $T \cong S \times T'$. Then $\delta_T \geq \delta_S$ again by [**Cas12**, Rem. 3.3.18]; on the other hand since S is a surface, it is easy to see that $\delta_S = \rho_S - 1 = \delta_X$, hence $\delta_T \geq \delta_X \geq 4$. ■

Prop. 3.2 follows from Lemmata 3.9 and 3.10.

3.3. Construction B

Let $n \in \mathbb{Z}_{\geq 2}$, T a smooth Fano variety of dimension $n-2$, and N a divisor in T such that $N \not\equiv 0$. Set

$$Z := \mathbb{P}_T(\mathcal{O}(N) \oplus \mathcal{O} \oplus \mathcal{O}) \xrightarrow{\varphi} T$$

and let H be a tautological divisor of Z. Note that $h^0(Z, H) = h^0(T, \mathcal{O}(N) \oplus \mathcal{O} \oplus \mathcal{O}) \geq 2$, and in fact the constant surjections $\mathcal{O} \oplus \mathcal{O} \twoheadrightarrow \mathcal{O}$ yield a pencil of divisors $\mathbb{P}_T(\mathcal{O}(N) \oplus \mathcal{O}) \hookrightarrow Z$ in the linear system $|H|$ (see Rem. 1.18).

We assume that a complete intersection of general elements in the linear systems $|H|$ and $|2H|$ is smooth, and we fix such a complete intersection $S_1 \subset Z$.

We also consider the divisor $D := \mathbb{P}_T(\mathcal{O} \oplus \mathcal{O}) \hookrightarrow Z$ given by the projection $\mathcal{O}(N) \oplus \mathcal{O} \oplus \mathcal{O} \twoheadrightarrow \mathcal{O} \oplus \mathcal{O}$, so that $D \cong \mathbb{P}^1 \times T$ with $\mathcal{N}_{D/Z} \cong \pi^*_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}(1)) \otimes \pi^*_T(\mathcal{O}_T(-N))$, and the two sections $S_2, S_3 \subset D$, $S_i \cong \{pt\} \times T \subset D$, again corresponding to the two projections $\mathcal{O} \oplus \mathcal{O} \twoheadrightarrow \mathcal{O}$.

Remark 3.11. There exists a unique divisor H_0 in $|H|$ that contains S_1; H_0 is smooth, $H_0 \cong \mathbb{P}_T(\mathcal{O}(N) \oplus \mathcal{O})$, $H_{|H_0}$ is a tautological divisor, and $S_1 \sim 2H_{|H_0}$.

PROOF. Since $|H|$ contains smooth members, and S_1 is a general complete intersection, there exists $H_0 \in |H|$ smooth containing S_1, and $\varphi_{|H_0} \colon H_0 \to T$ is a $\mathbb{P}^1$-bundle. For general $t \in T$, $(H_0)_{|Z_t}$ is a line, and $S_1 \in |2H_{|H_0}|$, so that $S_1 \cap Z_t$ yields two points which determine uniquely the line $(H_0)_{|Z_t}$.

We note that H_0 must intersect D in a section S_0 of φ, given by a surjection $\lambda \colon \mathcal{O} \oplus \mathcal{O} \twoheadrightarrow \mathcal{O}$, so that $H_0 \cong \mathbb{P}_T(\mathcal{E})$ where $\mathcal{E}$ is a rank 2 vector bundle on T. We have commutative diagrams:

$$\begin{array}{ccc} Z & \hookleftarrow & H_0 \\ \uparrow & & \uparrow \\ D & \hookleftarrow & S_0 \end{array} \qquad\qquad \begin{array}{ccc} \mathcal{O}(N) \oplus \mathcal{O} \oplus \mathcal{O} & \twoheadrightarrow & \mathcal{E} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \sigma} \\ \mathcal{O} \oplus \mathcal{O} & \xrightarrow{\lambda} & \mathcal{O} \end{array}$$

Since λ and π have a section, this yields a section of σ, which implies that $\mathcal{E} \cong \mathcal{O}(N) \oplus \mathcal{O}$ and gives the rest of the statement. ■

Remark 3.12. We have $H_0 \cap S_2 = H_0 \cap S_3 = D \cap S_1 = \emptyset$; in particular S_1, S_2, S_3 are pairwise disjoint.

PROOF. Every divisor in $|H|$ intersects D in $\{pt\} \times T$, and so it follows that H_0 is disjoint from S_2 and S_3 by generality.

Furthermore, the section $H_0 \cap D$ of H_0 has normal bundle $\mathcal{N}_{H_0 \cap D/H_0} \cong \mathcal{O}_T(-N)$, and is given by a surjection $\mathcal{O}(N) \oplus \mathcal{O} \twoheadrightarrow \mathcal{O}$.

Note that S_1 does not intersect any section of H_0 with normal bundle $\mathcal{O}_T(-N)$, and it follows that S_1 is disjoint from D. ■

Remark 3.13. S_1 is irreducible and $\varphi_{|S_1} \colon S_1 \to T$ is finite of degree 2; moreover $2N$ is linearly equivalent to the branch divisor $\Delta \subset T$, $h^0(T, 2N) > 0$, and $H_{|S_1} \sim (\varphi_{|S_1})^*(N) \sim R$ where $R \subset S_1$ is the ramification divisor.

PROOF. The restriction $\varphi_{|S_1} \colon S_1 \to T$ is finite of degree 2, because if $\dim(S_1 \cap Z_t) > 0$ for some $t \in T$, we should have $S_1 \cap D \neq \emptyset$, contradicting Rem. 3.12. Let $R \subset S_1$ be the ramification divisor and $\Delta \subset T$ the branch divisor. By adjunction

$$-K_{S_1} = (-K_Z - 3H)_{|S_1} = (\varphi_{|S_1})^* \mathcal{O}_T(-K_T - N),$$

so, since S_1 is smooth, $(\varphi_{|S_1})^*(N) \sim R$ and $2N \sim \Delta$ by the Hurwitz formula. In particular $\varphi_{|S_1}$ is not étale because $N \not\equiv 0$, and we deduce that S_1 is irreducible. We also note that $h^0(T, 2N) > 0$ and that S_1 is Fano if and only if $-K_T - N$ is ample. Finally we have $D \sim H - \varphi^*(N)$ (see Rem. 1.18) and $D \cap S_1 = \emptyset$ by Rem. 3.12, thus $H_{|S_1} \sim \varphi^*_{|S_1}(N)$. (For the last statement, see also Lemma 1.30) ■

Let $h \colon X \to Z$ be the blow-up of S_1, S_2, S_3, so that X is a smooth projective variety with $\rho_X = \rho_T + 4$, and set $\sigma := \varphi \circ h \colon X \to T$.

Lemma 3.14. *The following are equivalent:*

(*i*) *X is Fano;*
(*ii*) *$-K_T \pm N$ is ample on T.*

PROOF. Let $E_i \subset X$ be the exceptional divisor over S_i, and let $\widetilde{H}_0$ and $\widetilde{D}$ be the transforms of H_0 and D.

We show (*i*) $\Rightarrow$ (*ii*). If X is Fano, by restricting $-K_X$ to E_2 (or E_3) and using Lemma 1.21 we see that $-K_T - N$ is ample. Then, restricting $-K_X$ to E_1 and using Lemma 3.13 we see that $-K_T + N$ is ample: indeed, $-K_{X|E_1}$ is the tautological divisor of $\mathbb{P}_{S_1}\Big(\varphi^*_{|S_1}\mathcal{O}_T(-K_T) \oplus \varphi^*_{|S_1}\mathcal{O}_T(-K_T + N)\Big)$.

Let us show the converse (*ii*) $\Rightarrow$ (*i*). Again, it follows directly from Lemmata 1.21 and 1.22 that $(-K_X)_{|E_i}$ for $i = 2, 3$ and $(-K_X)_{|\widetilde{D}}$ are ample divisors respectively on E_i and $\widetilde{D}$.

We show that $(-K_X)_{|\widetilde{H}_0}$ is ample on $\widetilde{H}_0$. Since $\widetilde{H}_0 = h^*H_0 - E_1$ by Rem. 3.11 and 3.12, we have $\mathcal{O}_X(\widetilde{H}_0)_{|\widetilde{H}_0} \cong \mathcal{O}_Z(H_0)_{|H_0} \otimes \mathcal{O}_{H_0}(-S_1) \cong \mathcal{O}_{H_0}(-H_{|H_0})$ under the isomorphism $\widetilde{H}_0 \cong H_0$. Set $\varphi_0 := \varphi_{|H_0} \colon H_0 \to T$. Using Rem. 3.11 we get:

$$\begin{aligned}\mathcal{O}_X(-K_X)_{|\widetilde{H}_0} &\cong \mathcal{O}_{\widetilde{H}_0}(-K_{\widetilde{H}_0}) \otimes \mathcal{O}_X(\widetilde{H}_0)_{|\widetilde{H}_0} \cong \mathcal{O}_{H_0}(-K_{H_0} - H_{|H_0}) \\ &\cong \mathcal{O}_{H_0}\big(\varphi_0^*(-K_T - N) + H_{|H_0}\big)\end{aligned}$$

which is the tautological line bundle of $\mathbb{P}_T(\mathcal{O}(-K_T) \oplus \mathcal{O}(-K_T - N))$, and hence it is ample.

We show that $-K_X \cdot \Gamma > 0$ for every irreducible curve $\Gamma \subset X$. If Γ is contained in one of the divisors $E_2, E_3, \widetilde{D}, \widetilde{H}_0$, then $-K_X \cdot \Gamma > 0$ by what precedes.

Now assume that $\Gamma \not\subset E_2, E_3, \widetilde{H}_0, \widetilde{D}$. We have $D \sim H_0 - \varphi^*(N)$ (see Rem. 1.18) and $-K_Z \sim \varphi^*(-K_T - N) + 3H_0 \sim \varphi^*(-K_T + N) + H_0 + 2D$. Then it follows from Rem. 3.12 that:

$$-K_X \sim h^*(-K_Z) - E_1 - E_2 - E_3 \sim \sigma^*(-K_T + N) + \widetilde{H}_0 + 2\widetilde{D} + E_2 + E_3.$$

If Γ is not contracted by σ, then $\sigma(\Gamma)$ is an irreducible curve in T, and $-K_X \cdot \Gamma = (-K_T + N) \cdot \sigma_*(\Gamma) + (E_2 + E_3 + \widetilde{H}_0 + 2\widetilde{D}) \cdot \Gamma > 0$.

If instead $\sigma(\Gamma)$ is a point, then either Γ is a fibre of h, and so $-K_X \cdot \Gamma = 1$, or $h(\Gamma)$ is an irreducible curve in a fibre Z_t of φ, for some $t \in T$. Assume the latter: then $-K_X \cdot \Gamma = (\widetilde{H}_0 + \widetilde{D} + h^*(D)) \cdot \Gamma \geq h^*(D) \cdot \Gamma = D \cdot h_*(\Gamma) > 0$, because $D_{|Z_t}$ is a line in $Z_t \cong \mathbb{P}^2$.

Now we show that $-K_X$ is big. Consider the divisor $A_Z := H_0 + \varphi^*(-K_T)$, which is a tautological divisor for $\mathbb{P}_T(\mathcal{O}(-K_T + N) \oplus \mathcal{O}(-K_T) \oplus \mathcal{O}(-K_T))$, and hence it is ample on Z. Therefore there exist non-negative integers m, c_i such that the divisor

$$\begin{aligned} A_X &:= mh^*(A_Z) - \sum_{i=1}^{3} c_i E_i \;=\; m\big(\widetilde{H}_0 + E_1 + \sigma^*(-K_T)\big) - \sum_{i=1}^{3} c_i E_i \\ &= m\sigma^*(-K_T) + m\widetilde{H}_0 + (m - c_1)E_1 - c_2 E_2 - c_3 E_3 \end{aligned}$$

is ample on X. Finally

$$-mK_X \sim m\big(\sigma^*(-K_T) + E_1 + 2\widetilde{H}_0 + \widetilde{D}\big) = A_X + m\widetilde{H}_0 + m\widetilde{D} + \sum_{i=1}^{3} c_i E_i$$

which implies that $-K_X$ is big and then ample, as in the proof of Lemma 3.9. ∎

The proof of the following lemma is very similar to the one of Lemma 3.10, thus we omit it.

Lemma 3.15. *Suppose that X is Fano. Then $\delta_X \geq 3$, and $\delta_X = 3$ if and only if $\delta_T \leq 3$.*

Prop. 3.3 follows from Lemmata 3.14 and 3.15.

3.4. Proof of Theorem 3.4

Let X be a Fano manifold with Lefschetz defect $\delta_X = 3$. In this section we prove Theorem 3.4: we follow [**CR22**, proof of Th. 1.1], where the case $\dim X = 4$, $\rho_X = 5$, and $T \cong \mathbb{P}^2$ is considered, generalising the proof step by step.

We first recall [**Cas12**, Th. 3.3], which shows the existence of a flat fibration

$$\sigma \colon X \longrightarrow T \ ,$$

where T is a smooth Fano variety with $\dim T = \dim X - 2$ and $\rho_T = \rho_X - 4$; σ factors as $X \to Y \to T$, where $X \to Y$ is a conic bundle and $Y \to T$ is a smooth fibration with fibre $\mathbb{P}^1$. We collect the properties of σ in Th. 3.16.

Then, in Steps 1, 2, 3 we further study the fibration $\sigma \colon X \to T$ and we show that it has a different factorisation as $X \xrightarrow{h} Z \xrightarrow{\varphi} T$, where φ is a $\mathbb{P}^2$-bundle over T and h is

the blow-up of Z along three pairwise disjoint smooth, irreducible subvarieties $S_i \subset Z$ of codimension 2, horizontal for φ. We prove that S_2 and S_3 are always sections of φ.

$$\begin{array}{ccccc} X & \xrightarrow{f} & X_2 & \longrightarrow & Y \\ & \searrow^{h} & \downarrow{\scriptstyle g} & & \downarrow \\ & & Z & \xrightarrow{\varphi} & T \end{array}$$

Here g is the blow-up along S_3 and f is the blow-up along $g^{-1}(S_1), g^{-1}(S_2)$.

When S_1 is a section as well, we show that $\sigma\colon X \to T$ is as in Construction A, see Steps 4, 5.

Otherwise, we study the degree $d > 1$ of S_1 over T. Fibre-wise, for $t \in T$ general, we have $Z_t := \varphi^{-1}(t) \cong \mathbb{P}^2$, $X_t := \sigma^{-1}(t)$ a smooth del Pezzo surface and $X_t \to Z_t$ the blow-up of $\mathbb{P}^2$ along the $d+2$ points $(S_1 \cup S_2 \cup S_3) \cap Z_t$. Our objective is to show that $d = 2$, so that $\sigma\colon X \to T$ is as in Construction B.

This is the hardest part of the proof. In Step 6 we construct $H_0 \subset Z$, the relative secant variety of S_1 in Z: we show that it has pure codimension 1 and that it is disjoint from S_2 and S_3 (Step 7). In Steps 8, 9, 10, 11, 12 we study and describe the varieties and morphisms that appear in the following diagram

$$\begin{array}{ccc} X_2 & \longrightarrow & Y \\ \downarrow & & \downarrow \\ Z & \xrightarrow{\varphi} & T \end{array}$$

and then we restrict to a general curve $C \subset T$ (Steps 13, 14) in order to construct in $\overline{Z} := \varphi^{-1}(C)$ a divisor which is a $\mathbb{P}^1$-bundle over C and contains $S_1 \cap \overline{Z}$ (Step 15). Therefore, for $t \in C$ general, the d points $S_1 \cap Z_t$ are collinear; since X_t is del Pezzo, this implies that $d = 2$. Finally, through Steps 16, 17, 18 and 19, we show that $\sigma\colon X \to T$ is as in Construction B.

We begin by recalling a result on the structure of Fano varieties X with $\delta_X = 3$.

Theorem 3.16. *Set $n := \dim X$. There exists a diagram:*

$$X \xrightarrow{f} X_2 \xrightarrow[\psi]{} Y \xrightarrow[\xi]{} T, \qquad X \xrightarrow{\sigma} T$$

with the following properties:

(a) *all varieties are smooth and projective;*
(b) *Y is Fano of dimension $n-1$ and T is Fano of dimension $n-2$;*
(c) *ψ and ξ are smooth morphisms with fibre $\mathbb{P}^1$;*
(d) *$\sigma\colon X \to T$ is a flat contraction of relative dimension 2;*
(e) *$\psi \circ f\colon X \to Y$ is a conic bundle;*
(f) *f is the blow-up of two disjoint smooth, irreducible subvarieties $B_1, B_2 \subset X_2$ of codimension 2;*
(g) *for $i = 1, 2$ set $A_i := \psi(B_i) \subset Y$ and $W_i := \psi^{-1}(A_i) \subset X_2$; A_1 and A_2 are disjoint smooth divisors in Y, and B_i is a section of $\psi_{|W_i}\colon W_i \to A_i$, for $i = 1, 2$;*
(h) *ξ is finite on A_i for $i = 1, 2$.*

PROOF. See [**Cas12**], Prop. 3.3.1 and its proof, in particular paragraphs 3.3.15-3.3.17; see also [**Rom19**, Prop. 3.4 and 3.5] for (g). ■

We keep the same notation as in Th. 3.16.

Step 1. *By Lemma 1.25, up to switching A_1 and A_2, we can assume that A_2 is a section of $\xi\colon Y \to T$.*

Step 2. *We can assume that there exists a commutative diagram:*

$$\begin{array}{ccc} X_2 & \xrightarrow{\psi} & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \xi} \\ Z & \xrightarrow{\varphi} & T \end{array}$$

where φ is a $\mathbb{P}^2$-bundle, ψ and ξ are $\mathbb{P}^1$-bundles, g is the blow-up along a section $S_3 \subset Z$ of φ, $E := \mathrm{Exc}(g)$ is a section of ψ, and $E \cap (B_1 \cup B_2) = \emptyset$.

PROOF. Consider the natural factorisation of f as a sequence of two blow-ups (see Th. 3.16(f)):

$$X \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2$$

where f_2 is the blow-up of B_2 and f_1 is the blow-up of the transform of B_1.

Let us consider the morphism $\zeta := \xi \circ \psi \circ f_2\colon X_1 \to T$. Since both ψ and ξ are smooth by Th. 3.16(c), the composition $\xi \circ \psi\colon X_2 \to T$ is smooth. Moreover, since $A_2 \subset Y$ is a section of ξ by Step 1, and the centre B_2 of the blow-up $f_2\colon X_1 \to X_2$ is a section over A_2 (see Th. 3.16(g)), we conclude that B_2 is a section of $\xi \circ \psi\colon X_2 \to T$. This implies that $\zeta\colon X_1 \to T$ is a smooth contraction of relative Picard number 3.

As in [**CR22**, proof of Step 2.3] one shows that $-K_{X_1}$ is ζ-ample; every fibre of ζ is isomorphic to the blow-up F of $\mathbb{P}^2$ in two points. Moreover, again as in [**CR22**, proof of Step 2.3] we see that every contraction of the fibre F extends to a global contraction of X_1 over T. Therefore the sequence of elementary contractions:

$$\begin{array}{ccccc} F & \longrightarrow & \mathbb{F}_1 & \longrightarrow & \mathbb{P}^1 \\ & & \downarrow & & \downarrow \\ & & \mathbb{P}^2 & \longrightarrow & \{pt\} \end{array}$$

yields a corresponding factorisation of ζ:

$$\begin{array}{ccccc} X_1 & \xrightarrow{f_2'} & X_2' & \xrightarrow{\psi'} & Y' \\ & & {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \xi'} \\ & & Z & \xrightarrow{\varphi} & T \end{array}$$

We have:

- $\xi'\colon Y' \to T$ and $\psi'\colon X_2' \to Y'$ are smooth with fibre $\mathbb{P}^1$, and $\varphi\colon Z \to T$ is smooth with fibre $\mathbb{P}^2$;
- g is the blow-up of a smooth irreducible subvariety $S_3 \subset Z$ of codimension 2, which is a section of φ;

- f_2' is the blow-up of a smooth irreducible subvariety $B_2' \subset X_2'$ of codimension 2, which is a section of $\varphi \circ g \colon X_2' \to T$, and is disjoint from $E := \mathrm{Exc}(g)$;
- $A_2' := \psi'(B_2')$ is a section of ξ';
- E is a section of $\psi' \colon X_2' \to Y'$;
- since φ, ξ', and ψ' all have a section, they are projectivizations of vectors bundles.

Notice that ζ is finite on $f_1(\mathrm{Exc}(f_1))$, because f_2 is an isomorphism on it, ψ is finite on B_1, and ξ is finite on A_1 (see Th. 3.16(f),(g),(h)). In particular $f_1(\mathrm{Exc}(f_1))$ cannot contain any (-1)-curve in the fibre F, therefore it cannot meet any such curve, otherwise X would not be Fano. Hence $B_1' := f_2'(f_1(\mathrm{Exc}(f_1)))$ is disjoint from B_2' and from E.

We show that properties of Th. 3.16 still hold, so that we can replace the original factorisation of ζ with the new one. Properties (a),(c),(f) are clear. Since ζ is finite on $f_1(\mathrm{Exc}(f_1))$, ξ' is finite on $A_1' := \psi'(B_1')$, and we have (h).

The composition $\psi' \circ f_2' \circ f_1 \colon X \to Y'$ is a fibre-type contraction with one-dimensional fibres, hence a conic bundle, which yields (e) and (d). Moreover this conic bundle has only reduced fibres; this implies that Y' is Fano by [**Wiś91a**, Prop. 4.3], and we have (b). Finally (g) follows from [**Rom19**, Prop. 3.4 and 3.5]. ■

Set $S_i := g(B_i) \subset Z$ for $i = 1, 2$. Then S_1, S_2, and S_3 are pairwise disjoint smooth irreducible subvarieties of codimension 2, and X is the blow-up of Z along $S_1 \cup S_2 \cup S_3$. We set $Z_t := \varphi^{-1}(t)$ for every $t \in T$. Moreover we denote by d the degree of the finite morphism $\xi_{|A_1} \colon A_1 \to T$ (see Th. 3.16(h)).

Step 3. *S_2 is a section of φ, and $\varphi_{|S_1} \colon S_1 \to T$ is finite of degree d.*

Proof. For $i = 1, 2$, since B_i is a section over A_i by Th. 3.16(g), the degree of S_i over T is equal to the degree of A_i over T; in particular S_2 is a section of φ by Step 1. ■

Step 4. *For every $t \in T$ and for every line $\ell \subset Z_t \cong \mathbb{P}^2$, the scheme-theoretic intersection $\ell \cap (S_1 \cup S_2 \cup S_3)$ has length ≤ 2.*

In particular, the points $(S_1 \cup S_2 \cup S_3) \cap Z_t$ (with the reduced structure) are in general linear position in Z_t.

Proof. Let a be the length of scheme-theoretic intersection $\ell \cap (S_1 \cup S_2 \cup S_3)$, and let $\widetilde{\ell} \subset X$ be the transform of ℓ. Then $1 \leq -K_X \cdot \widetilde{\ell} = 3 - a$, thus $a \leq 2$. ■

Step 5. *If $d = 1$, then $X \xrightarrow{h} Z \xrightarrow{\varphi} T$ is as in Construction A (Prop. 3.2).*

Proof. If $d = 1$, then $\varphi \colon Z \to T$ has three pairwise disjoint sections S_i, which are fibre-wise in general linear position, by Steps 2, 3, and 4. This implies that $Z \cong \mathbb{P}_T(\mathcal{O}(D_1) \oplus \mathcal{O}(D_2) \oplus \mathcal{O}(D_3))$ in such a way that the three sections S_i correspond to the projections onto the summands $\mathcal{O}(D_i)$. Moreover $-K_T + D_i - D_j$ is ample for every $i, j \in \{1, 2, 3\}$ by Lemma 3.9, and $\delta_T \leq 3$ by Lemma 3.10. ■

From now on we assume that $d \geq 2$. For $q_1, q_2 \in Z_t$ distinct points, we denote by $\overline{q_1 q_2} \subset Z_t$ the line spanned by q_1 and q_2.

Step 6. *Let $H_0 \subset Z$ be the relative secant variety of S_1 in Z, namely the closure in Z of the set:*

$$\bigcup_{q_1,q_2 \in S_1 \cap Z_t, q_1 \neq q_2, t \in T} \overline{q_1 q_2}.$$

Then H_0 has pure dimension $n-1$, and for every $t \in T$, $H_0 \cap Z_t$ is a union of lines ℓ such that the scheme-theoretic intersection $\ell \cap S_1$ has length ≥ 2.

Proof. For t general we have $|S_1 \cap Z_t| = d \geq 2$, so that H_0 is non-empty. We first consider the fibre product:

$$\begin{array}{ccc} S_1 \times_T S_1 & \longrightarrow & S_1 \\ \downarrow & & \downarrow \\ S_1 & \longrightarrow & T \end{array}$$

Since the morphism $S_1 \to T$ is finite between smooth varieties, it is flat. Therefore also $S_1 \times_T S_1 \to T$ is finite and flat, so that $S_1 \times_T S_1$ has pure dimension $n-2$ and every irreducible component dominates T.

We also note that the morphism $S_1 \times_T S_1 \to S_1$ has a natural section, whose image is the diagonal Δ, which is an irreducible component of the fibre product. We denote by $(S_1 \times_T S_1)_0$ the union of the remaining irreducible components of the fibre product, so that $(S_1 \times_T S_1)_0 \smallsetminus \Delta$ is a dense open subset in $(S_1 \times_T S_1)_0$.

Now we consider the dual projective bundle $Z^* \to T$. We have a morphism over T:

$$\alpha \colon (S_1 \times_T S_1)_0 \smallsetminus \Delta \longrightarrow Z^*$$

given by $\alpha(q_1, q_2) = [\overline{q_1 q_2}]$. If $[\ell] \in \mathrm{Im}(\alpha)$, then $\ell \cap S_1$ is finite, hence $\alpha^{-1}([\ell])$ is finite. Moreover

$$\mathrm{Im}(\alpha) = \{[\overline{q_1 q_2}] \in Z^*_t \mid q_1, q_2 \in S_1 \cap Z_t, q_1 \neq q_2, t \in T\}.$$

Let $\mathcal{S} \subset Z^*$ be the closure of $\mathrm{Im}(\alpha)$. If K is an irreducible component of $(S_1 \times_T S_1)_0$, set $\mathcal{S}_K := \overline{\alpha(K \smallsetminus \Delta)}$. Then $\alpha \colon K \smallsetminus \Delta \to \mathcal{S}_K$ is a dominant morphism between irreducible varieties, hence its image $\alpha(K \smallsetminus \Delta)$ contains an open subset of $\mathcal{S}_K$; moreover $\dim \mathcal{S}_K = \dim K = n-2$ and $\mathcal{S}_K$ dominates T, because K does.

We note that $\mathcal{S}$ is the union of $\mathcal{S}_K$ when K varies among the finitely many irreducible components of $(S_1 \times_T S_1)_0$, therefore all the irreducible components of $\mathcal{S}$ are of type $\mathcal{S}_K$.

We conclude that $\mathcal{S}$ has pure dimension $n-2$ and every irreducible component dominates T. Moreover, every irreducible component of $\mathcal{S}$ contains an open subset contained in $\mathrm{Im}(\alpha)$. We also note that, for every $[\ell] \in \mathcal{S}$, the scheme-theoretical intersection $\ell \cap S_1$ has length ≥ 2.

Now we consider the universal family over T:

$$\begin{array}{ccc} & \mathcal{U} & \\ \swarrow & & \searrow^{\pi} \\ Z^* & & Z \end{array}$$

where $\mathcal{U} = \{([\ell], p) \in Z^* \times_T Z \mid p \in \ell\}$.

The inverse image $\widehat{\mathcal{S}}$ of $\mathcal{S}$ in $\mathcal{U}$ has pure dimension $n-1$; moreover every irreducible component of $\widehat{\mathcal{S}}$ is the inverse image of an irreducible component of $\mathcal{S}$, and contains an open subset contained in the inverse image $\widehat{\mathcal{S}}_0$ of $\mathrm{Im}(\alpha)$. In particular $\widehat{\mathcal{S}}_0$ is dense in $\widehat{\mathcal{S}}$.

Finally we consider the images $\pi(\widehat{\mathcal{S}}_0)$ and $\pi(\widehat{\mathcal{S}})$ in Z; note that $\pi(\widehat{\mathcal{S}}_0)$ is dense in $\pi(\widehat{\mathcal{S}})$. By construction we have

$$\pi(\widehat{\mathcal{S}}_0) = \bigcup_{q_1,q_2 \in S_1 \cap Z_t, q_1 \neq q_2, t \in T} \overline{q_1 q_2} \subset Z,$$

so that $H_0 = \overline{\pi(\widehat{\mathcal{S}}_0)} = \pi(\widehat{\mathcal{S}})$.

Let W be an irreducible component of $\widehat{\mathcal{S}}$, and let W_0 be an open subset of W contained in $\widehat{\mathcal{S}}_0$. Then $\overline{\pi(W_0)} = \overline{\pi(W)}$, so that $\pi_0 \colon W_0 \to \overline{\pi(W)}$ is a dominant morphism between irreducible varieties, and its image contains an open subset U of $\overline{\pi(W)}$. It is clear that $\pi_0^{-1}(p)$ is finite for every $p \in U$, so that $\dim \overline{\pi(W)} = \dim W = n-1$. This shows that H_0 has pure dimension $n-1$.

Moreover, since H_0 is the locus of lines parameterised by $\mathcal{S}$, we also have that for every $t \in T$, $H_0 \cap Z_t$ is a union of finitely many lines ℓ, and the scheme-theoretic intersection $\ell \cap S_1$ has length ≥ 2. ■

Step 7. *We have $H_0 \cap (S_2 \cup S_3) = \emptyset$.*

Proof. Let ℓ be a line in $H_0 \cap Z_t$ for some $t \in T$; $\ell \cap (S_1 \cup S_2 \cup S_3)$ has length ≤ 2 by Step 4 and $\ell \cap S_1$ has length ≥ 2 by Step 6. Then, $\ell \cap S_1$ has length 2 and $\ell \cap (S_2 \cup S_3)$ has length 0, so $\ell \cap (S_2 \cup S_3) = \emptyset$. Therefore $H_0 \cap (S_2 \cup S_3) = \emptyset$. ■

Recall that $W_2 = \psi^{-1}(A_2) \subset X_2$ and that $E = \mathrm{Exc}(g) \subset X_2$ (see Th. 3.16(g) and Step 2).

Step 8. *Set $D := g(W_2) \subset Z$. Then $W_2 \cong D \cong \mathbb{P}^1 \times T$, and $D \cap Z_t$ is a line in Z_t for every $t \in T$. Moreover D contains S_2 and S_3 (as $\{pt\} \times T$), while $D \cap S_1 = \emptyset$.*

Proof. We have a commutative diagram:

$$\begin{array}{ccc} W_2 & \xrightarrow{\psi_{|W_2}} & A_2 \\ {\scriptstyle g_{|W_2}}\downarrow & & \downarrow{\scriptstyle \xi_{|A_2}} \\ D & \xrightarrow{\varphi_{|D}} & T \end{array}$$

where the vertical maps are isomorphisms by Steps 1 and 2, and the horizontal maps are $\mathbb{P}^1$-bundles.

We also have $S_3 = g(E \cap W_2) \subset D$; moreover $B_2 \subset W_2$ (see Th. 3.16(g)), hence $S_2 = g(B_2) \subset D$. By Steps 2 and 3, S_2 and S_3 are disjoint sections of the $\mathbb{P}^1$-bundle $\varphi_{|D} \colon D \to T$. Thus we can assume that $D \cong \mathbb{P}_T(\mathcal{O} \oplus M)$ for some $M \in \mathrm{Pic}(T)$, such that the section S_2 corresponds to the projection $\mathcal{O} \oplus M \twoheadrightarrow \mathcal{O}$, so that $\mathcal{N}_{S_2/D}^\vee \cong M$.

Now we have $H_0 \cap D \neq \emptyset$, because both contain a line in Z_t, so that $H_0 \cap D$ yields a non-zero effective divisor in D. On the other hand, this divisor is disjoint from both sections S_2 and S_3 by Step 7. Write $\mathcal{O}_D(H_0 \cap D) \cong \mathcal{O}_D(mS_2 + (\varphi_{|D})^* G)$, with $m \in \mathbb{Z}$ and G divisor on T.

We have $\mathcal{O}_{S_3} \cong \mathcal{O}_D(H_0 \cap D)_{|S_3} \cong \mathcal{O}_D((\varphi_{|D})^* G)_{|S_3}$, which gives $G \sim 0$ and $\mathcal{O}_D(H_0 \cap D) \cong \mathcal{O}_D(mS_2)$. Note that $m \neq 0$ because $H_0 \cap D$ is non-zero.

Then $\mathcal{O}_{S_2} \cong \mathcal{O}_D(H_0 \cap D)_{|S_2} \cong \mathcal{O}_D(mS_2)_{|S_2} \cong M^{\otimes(-m)}$, thus $M^{\otimes m} \cong \mathcal{O}_T$. Since T is a Fano variety, $\mathrm{Pic}(T)$ is finitely generated and free, so that $M \cong \mathcal{O}_T$ and $D \cong \mathbb{P}^1 \times T$.

Finally, we note that for every $t \in T$, $D \cap Z_t$ is the line spanned by the points $S_2 \cap Z_t$ and $S_3 \cap Z_t$, so that $D \cap S_1 = \emptyset$ by Step 4. ■

Recall from Steps 1 and 2 that $\xi\colon Y \to T$ is a $\mathbb{P}^1$-bundle and $A_2 \subset Y$ is a section. Let $\mathcal{E}$ be a rank 2 vector bundle on T such that $Y = \mathbb{P}_T(\mathcal{E})$; up to tensoring $\mathcal{E}$ with a line bundle, we can assume that the section A_2 corresponds to a surjection $\mathcal{E} \twoheadrightarrow \mathcal{O}_T$, and we denote by $L \in \mathrm{Pic}(Y)$ the tautological line bundle, so that $L_{|A_2} \cong \mathcal{O}_{A_2}$. Moreover let N be a divisor on T such that $\mathcal{O}_T(N) \cong \mathcal{N}^\vee_{A_2/Y}$ under the isomorphism $T \cong A_2$, so that we have an exact sequence on T:

$$0 \longrightarrow \mathcal{O}_T(N) \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_T \longrightarrow 0. \tag{3.1}$$

Step 9. *We have $\mathcal{O}_Y(A_1) \cong L^{\otimes d}$, $\mathcal{O}_Y(A_2 + \xi^*(N)) \cong L$, and $H^1(Y, L) = 0$.*

Proof. Both L and $\mathcal{O}_Y(A_1)$ belong to the kernel of the restriction

$$r\colon \mathcal{N}^1(Y) \to \mathcal{N}^1(A_2);$$

moreover r is surjective, because for $G \in \mathrm{Pic}(A_2)$ the line bundle $\xi^*\Big(\big((\xi_{|A_2})^{-1}\big)^* G\Big)$ restricts to G, and $\rho_{A_2} = \rho_T = \rho_Y - 1$, so that $\dim(\ker r) = 1$ and the classes of L and A_1 must be proportional in $\mathcal{N}^1(Y)$; intersecting with a fibre of ξ we deduce that $\mathcal{O}_Y(A_1) \cong L^{\otimes d}$.

Similarly, we have $\mathcal{O}_Y(A_2 + \xi^*(N))_{|A_2} \cong \mathcal{O}_{A_2}$, and again intersecting with a fibre we get $\mathcal{O}_Y(A_2 + \xi^*(N)) \cong L$.

Finally we have

$$-K_Y + L \equiv -K_Y + \frac{1}{d} A_1$$

where $-K_Y$ is ample, $1/d < 1$, and A_1 is a smooth divisor, therefore $H^1(Y, L) = 0$ by Kawamata-Viehweg vanishing (see [**Laz04b**, Th. 9.1.18]). ■

Step 10. *We have $X_2 \cong \mathbb{P}_Y(\mathcal{O} \oplus L)$, and E corresponds (as a section of ψ) to the projection $\mathcal{O} \oplus L \twoheadrightarrow \mathcal{O}$. Moreover $\mathcal{N}^\vee_{E/X_2} \cong L$ and $\mathcal{N}^\vee_{S_3/Z} \cong \mathcal{E}$.*

Proof. By Step 2 we have a commutative diagram:

$$\begin{array}{ccc} E & \xrightarrow{\psi_{|E}} & Y \\ \downarrow & & \downarrow \xi \\ S_3 & \xrightarrow{\varphi_{|S_3}} & T \end{array}$$

where the horizontal maps are isomorphisms, and the vertical maps are $\mathbb{P}^1$-bundles. Since $g\colon X_2 \to Z$ is the blow-up of S_3, we get:

$$\mathbb{P}_{S_3}(\mathcal{N}^\vee_{S_3/Z}) \cong E \cong Y \cong \mathbb{P}_T(\mathcal{E}),$$

therefore $\mathcal{N}^\vee_{S_3/Z} \cong \mathcal{E} \otimes M$ with $M \in \mathrm{Pic}(T)$. Moreover $\mathcal{N}^\vee_{E/X_2}$ is the tautological line bundle of $\mathbb{P}_T(\mathcal{E} \otimes M)$, which gives $\mathcal{N}^\vee_{E/X_2} \cong L \otimes \xi^*(M)$.

Recall from Step 2 that $E \subset X_2$ is a section of the $\mathbb{P}^1$-bundle $\psi\colon X_2 \to Y$. Let $\mathcal{F}$ be a rank 2 vector bundle on Y such that $X_2 = \mathbb{P}_Y(\mathcal{F})$; up to tensoring $\mathcal{F}$ with a line bundle, we can assume that the section E corresponds to a surjection $\sigma\colon \mathcal{F} \twoheadrightarrow \mathcal{O}_Y$, so that $\ker\sigma \cong \mathcal{N}^\vee_{E/X_2} \cong L \otimes \xi^*(M)$. We obtain the following exact sequence over Y:

$$0 \longrightarrow \ker\sigma \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_Y \longrightarrow 0. \tag{3.2}$$

Now let us consider $A_2 \subset Y$; we have $\ker \sigma_{|A_2} \cong L_{|A_2} \otimes \xi^*(M)_{|A_2} \cong M$, so by restricting to A_2 the above exact sequence we get:

$$0 \longrightarrow M \longrightarrow \mathcal{F}_{|A_2} \longrightarrow \mathcal{O} \longrightarrow 0.$$

On the other hand $\mathbb{P}_{A_2}(\mathcal{F}_{|A_2}) = W_2 \cong \mathbb{P}^1 \times T$ by Step 8, and we deduce that $M = \mathcal{O}_T$ and $\ker \sigma \cong L$.

Now we have $\mathrm{Ext}^1(\mathcal{O}_Y, \ker \sigma) \cong H^1(Y, L) = 0$ by Step 9, hence the sequence (3.2) splits, so that $\mathcal{F} \cong \mathcal{O}_Y \oplus L$. ■

Step 11. *We have $-K_T+N$ ample on T, $H^1(T,N) = 0$, $H^0(T, d(d-1)N) \neq 0$, $N \not\equiv 0$, $\mathcal{E} \cong \mathcal{O}_T(N) \oplus \mathcal{O}_T$, and $Y \cong \mathbb{P}_T(\mathcal{O}(N) \oplus \mathcal{O})$.*

Proof. Set $\xi_1 := \xi_{|A_1} \colon A_1 \to T$, so that ξ_1 is finite of degree d. By Step 9 we have $L \cong \mathcal{O}_Y(A_2 + \xi^*(N))$ and $\mathcal{O}_Y(A_1) \cong L^{\otimes d}$, so that $L_{|A_1} \cong \mathcal{O}_{A_1}(\xi_1^*(N))$ and $\mathcal{O}_Y(A_1)_{|A_1} \cong \mathcal{O}_{A_1}(\xi_1^*(dN))$. We have $\det(\mathcal{E}) \cong \mathcal{O}_T(N)$ by (3.1), therefore $\mathcal{O}_Y(K_Y) \cong \mathcal{O}_Y(\xi^*(K_T + N)) \otimes L^{\otimes(-2)}$ and

$$-K_{Y|A_1} \sim \xi_1^*(-K_T + N).$$

Since Y is Fano and ξ_1 is finite, we deduce that $-K_T + N$ is ample on T, so that $H^1(T,N) = H^1(T, K_T - K_T + N) = 0$ by Kodaira vanishing. Moreover we have $\mathrm{Ext}^1(\mathcal{O}_T, \mathcal{O}_T(N)) = 0$, so that the sequence (3.1) splits.

Finally:

$$K_{A_1} \sim (K_Y + A_1)_{|A_1} \sim \xi_1^*\big(K_T + (d-1)N\big).$$

This shows that the ramification divisor R of ξ_1 is linearly equivalent to $(d-1)\xi_1^*(N)$; by taking the pushforward we get $(\xi_1)_*(R) \sim d(d-1)N$. On the other hand R is effective and non-zero (because T is simply connected), therefore $H^0(T, d(d-1)N) \neq 0$ and $N \not\equiv 0$. ■

Recall from Step 8 that $D \cong \mathbb{P}^1 \times T$ and $S_2, S_3 \cong \{\mathrm{pt}\} \times T$.

Step 12. *We have $Z \cong \mathbb{P}_T(\mathcal{O}(N) \oplus \mathcal{O} \oplus \mathcal{O})$ with the two sections S_2, S_3 corresponding to the projections onto the trivial summands; moreover $(\mathcal{N}_{D/Z})_{|\{pt\}\times T} \cong \mathcal{O}_T(-N)$.*

Proof. We know by Steps 2, 10, and 11 that S_3 is a section of $\varphi \colon Z \to T$ with conormal bundle $\mathcal{E} \cong \mathcal{O}_T(N) \oplus \mathcal{O}_T$. As in the proof of Step 10, using that $H^1(T,N) = H^1(T, \mathcal{O}_T) = 0$ by Step 11, one shows that $Z \cong \mathbb{P}_T(\mathcal{O}(N) \oplus \mathcal{O} \oplus \mathcal{O})$, with the section S_3 corresponding to the projection p_3 onto the last summand.

The inclusions $S_2, S_3 \hookrightarrow D \hookrightarrow Z$ yield a diagram:

$$\begin{array}{ccccc} \mathcal{O}_T(N) \oplus \mathcal{O}_T \oplus \mathcal{O}_T & \xrightarrow{\alpha} & \mathcal{O}_T \oplus \mathcal{O}_T & \xrightarrow{\pi_1} & \mathcal{O}_T \\ & & \downarrow{\scriptstyle \pi_2} & & \\ & & \mathcal{O}_T & & \end{array}$$

where π_i are the projections, π_1 corresponds to $S_2 \subset D$, π_2 corresponds to $S_3 \subset D$, and $\pi_2 \circ \alpha = p_3$.

By Step 11 we have $H^0(T, -N) = 0$, thus $\mathrm{Hom}(\mathcal{O}_T(N), \mathcal{O}_T) = 0$ and α factors through the projection $p_{23} \colon \mathcal{O}_T(N) \oplus \mathcal{O}_T \oplus \mathcal{O}_T \to \mathcal{O}_T \oplus \mathcal{O}_T$. Then up to changing the isomorphism $Z \cong \mathbb{P}_T(\mathcal{O}(N) \oplus \mathcal{O} \oplus \mathcal{O})$ we can assume that $\alpha = p_{23}$ and $S_2 \subset Z$ corresponds to the projection p_2 onto the second summand. ■

Since $N \not\equiv 0$ by Step 11, as in the proof of Lemma 1.25 we see that there exists a curve $C \subset T$ which is a complete intersection of very ample divisors (general in their linear system) and such that $N \cdot C \neq 0$. Set $S := \xi^{-1}(C) \subset Y$ and $C_1 := A_{1|S}$.

Step 13. *We have $\mathcal{O}_S(C_1) = (L_{|S})^{\otimes d}$, $\deg(L_{|C_1}) > 0$, and $L_{|S}$ is nef and big in S.*

Proof. Recall that $\mathcal{O}_Y(A_1) \cong L^{\otimes d} \cong \mathcal{O}_Y(d(A_2 + \xi^*(N)))$ by Step 9, in particular $\mathcal{O}_S(C_1) = (L_{|S})^{\otimes d}$. We have

$$(C_1)^2 = A_1 \cdot C_1 = d\big(A_2 + \xi^*(N)\big) \cdot C_1 = d\xi^*(N) \cdot C_1 = d^2 N \cdot C \neq 0.$$

If $(C_1)^2 < 0$, C_1 should be the negative section of the ruled surface S, which is impossible because $\xi_{|C_1}$ has degree $d \geq 2$; therefore $(C_1)^2 > 0$, C_1 is nef and big in S, and $L_{|S}$ too. Moreover $\deg(L_{|C_1}) = C_1 \cdot L_{|S} > 0$. ∎

Set $\overline{X}_2 := \psi^{-1}(S) \subset X_2$, $\overline{Z} := \varphi^{-1}(C) \subset Z$, and let $\overline{\xi}$, $\overline{\psi}$, $\overline{\varphi}$, $\overline{g}$ be the respective restricted morphisms, so that we have a diagram:

$$\begin{array}{ccc} \overline{X}_2 & \xrightarrow{\overline{\psi}} & S \\ {\scriptstyle \overline{g}}\downarrow & & \downarrow{\scriptstyle \overline{\xi}} \\ \overline{Z} & \xrightarrow{\overline{\varphi}} & C \end{array}$$

where all varieties are smooth and projective, and $\dim \overline{X}_2 = \dim \overline{Z} = 3$. We also set $\overline{W}_1 := (\overline{\psi})^{-1}(C_1) = W_1 \cap \overline{X}_2 \subset \overline{X}_2$ and $\overline{B}_1 := B_1 \cap \overline{X}_2$, so that $\overline{B}_1$ is a section of $\overline{\psi}_{|\overline{W}_1} : \overline{W}_1 \to C_1$ (see Th. 3.16(g)). Specifically, the diagram

$$\begin{array}{ccc} X_2 & \xrightarrow{\psi} & Y \\ \cup & & \cup \\ W_1 = \psi^{-1}(A_1) & \xrightarrow{\psi_{|W_1}} & A_1 \\ \cup & & \\ B_1 & \xleftarrow[\cong]{} & \end{array} \qquad \text{restricts to} \qquad \begin{array}{ccc} \overline{X}_2 & \xrightarrow{\overline{\psi}} & S \\ \cup & & \cup \\ \overline{W}_1 = \psi^{-1}(C_1) & \xrightarrow{\overline{\psi}_{|\overline{W}_1}} & C_1 \\ \cup & & \\ \overline{B}_1 & \xleftarrow[\cong]{} & \end{array} .$$

Step 14 below shows that the section $\overline{B}_1$ of $\overline{\psi}_{|\overline{W}_1}$ extends to a section K of $\overline{\psi}$ and that $K \cap E = \emptyset$. Then, we apply Step 14 to prove that $\varphi_{|S_1} : S_1 \to T$ has degree $d = 2$ (Step 15); finally, we show that X is obtained as in Construction B (Prop. 3.3).

Step 14. *There exists a section K of $\overline{\psi} : \overline{X}_2 \to S$ containing $\overline{B}_1$ and disjoint from E.*

Proof. By Step 10 we have $\overline{W}_1 = \mathbb{P}_{C_1}(\mathcal{O}_{C_1} \oplus L_{|C_1})$. Moreover by Steps 2 and 10 we deduce that $\overline{W}_1 \cap E$ is a section of $\overline{\psi}_{|\overline{W}_1}$, disjoint from $\overline{B}_1$, and corresponding to the projection $\mathcal{O}_{C_1} \oplus L_{|C_1} \to \mathcal{O}_{C_1}$. Then it is not difficult to see that $\overline{B}_1$ corresponds, as a section, to a surjection $\tau : \mathcal{O}_{C_1} \oplus L_{|C_1} \twoheadrightarrow L_{|C_1}$.

Let us consider the restriction $r : \mathrm{Hom}(\mathcal{O}_S \oplus L_{|S}, L_{|S}) \to \mathrm{Hom}(\mathcal{O}_{C_1} \oplus L_{|C_1}, L_{|C_1})$. We have

$$\mathrm{Hom}(\mathcal{O}_{C_1} \oplus L_{|C_1}, L_{|C_1}) \cong \mathrm{Hom}(L^{\vee}_{|C_1} \oplus \mathcal{O}_{C_1}, \mathcal{O}_{C_1}) \cong H^0(C_1, L_{|C_1}) \oplus H^0(C_1, \mathcal{O}_{C_1}),$$

and similarly $\mathrm{Hom}(\mathcal{O}_S \oplus L_{|S}, L_{|S}) \cong H^0(S, L_{|S}) \oplus H^0(S, \mathcal{O}_S)$. Since the restriction $H^0(S, \mathcal{O}_S) \to H^0(C_1, \mathcal{O}_{C_1})$ is an isomorphism, if $H^0(S, L_{|S}) \to H^0(C_1, L_{|C_1})$ is surjective, then so is r.

We have an exact sequence of sheaves on S:

$$0 \longrightarrow L_{|S} \otimes \mathcal{O}_S(-C_1) \longrightarrow L_{|S} \longrightarrow L_{|C_1} \longrightarrow 0.$$

Using Step 13, Serre duality and Kawamata-Viehweg vanishing:

$$H^1(S, L_{|S} \otimes \mathcal{O}_S(-C_1)) = H^1(S, L_{|S}^{\otimes(1-d)}) = H^1(S, K_S \otimes L_{|S}^{\otimes(d-1)}) = 0$$

because $d \geq 2$.

We conclude that r is surjective, so that τ extends to a morphism $\tau_S \colon \mathcal{O}_S \oplus L_{|S} \to L_{|S}$.

We show that τ_S is surjective. Under the isomorphism $\mathrm{Hom}(\mathcal{O}_S \oplus L_{|S}, L_{|S}) \cong \mathrm{Hom}(\mathcal{O}_S, L_{|S}) \oplus \mathrm{Hom}(L_{|S}, L_{|S}) \cong \mathrm{Hom}(\mathcal{O}_S, L_{|S}) \oplus \mathbb{C}$, τ_S corresponds to (α, λ).

If $\lambda = 0$, then τ_S factors through the projection $\mathcal{O}_S \oplus L_{|S} \twoheadrightarrow \mathcal{O}_S$, and the same happens by restricting to C_1. This is impossible, because $\tau \colon \mathcal{O}_{C_1} \oplus L_{|C_1} \to L_{|C_1}$ is surjective, and $\deg(L_{|C_1}) > 0$ by Step 13.

We conclude that $\lambda \neq 0$ and τ_S is surjective, so it yields a section $K \subset \overline{X}_2$ extending $\overline{B}_1$.

We show that $K \cap E = \emptyset$. Let us consider the projection $\mathcal{O}_S \oplus L_{|S} \twoheadrightarrow L_{|S}$ and the corresponding section $\widetilde{K} \subset \overline{X}_2$. Since $E \cap \overline{X}_2$ is a section corresponding to the projection onto the other summand, we have $\widetilde{K} \cap E = \emptyset$. On the other hand, it is easy to check that $K \sim \widetilde{K}$ in $\overline{X}_2$, hence for every curve $C \subset E \cap \overline{X}_2$ we have $K \cdot C = 0$. Since $K \neq E \cap \overline{X}_2$, this implies that $K \cap E = \emptyset$. ■

Step 15. *We have $d = 2$.*

Proof. Recall that $K \cap E = \emptyset$ and $K \supset \overline{B}_1$ by Step 14. Consider $\overline{g}(K) \subset \overline{Z}$, so that $K \cong \overline{g}(K)$ and $\overline{g}(K) \supset S_1 \cap \overline{Z}$. If $t \in C$ is general, then $g^{-1}(Z_t) \cong \mathbb{F}_1$, and $K \cap g^{-1}(Z_t)$ is a section of $\mathbb{F}_1 \to \mathbb{P}^1$, disjoint from the (-1)-curve $E \cap g^{-1}(Z_t)$. Thus $\overline{g}(K) \cap Z_t$ is a line in $Z_t \cong \mathbb{P}^2$, and this line contains the d points $S_1 \cap Z_t$. Since these points are in general linear position (see Step 4), we conclude that $d = 2$. ■

Step 16. *The divisor H_0 is a tautological divisor for $Z = \mathbb{P}_T(\mathcal{O}(N) \oplus \mathcal{O} \oplus \mathcal{O})$.*

Proof. Since $d = 2$ by Step 15, for a general fibre Z_t of φ the restriction $(H_0)_{|Z_t}$ is a line (see Step 6), therefore if H is a tautological divisor, we have $H_0 \sim H + \varphi^*(G)$, G a divisor on T. Restricting to S_3 we have $H_0 \cap S_3 = \emptyset$ by Step 7 and $\mathcal{O}_Z(H)_{|S_3} \cong \mathcal{O}_{S_3}$ by Step 12, thus $G \sim 0$ and $H_0 \sim H$. ■

Step 17. *We have $H^1(Z, H_0) = 0$.*

Proof. By Step 16 the divisor $A := \varphi^*(-K_T) + H_0$ is the tautological divisor for $\mathbb{P}_T(\mathcal{O}(-K_T + N) \oplus \mathcal{O}(-K_T) \oplus \mathcal{O}(-K_T))$, hence it is ample by Th. 3.16(b) and Step 11.

Consider the divisors $H_0', H_0'' \subset Z$ corresponding to the two projections $\mathcal{O}_T(N) \oplus \mathcal{O}_T \oplus \mathcal{O}_T \twoheadrightarrow \mathcal{O}_T(N) \oplus \mathcal{O}_T$, such that $H_0' \cap D = S_2$ and $H_0'' \cap D = S_3$ (see Step 12); we note that the divisor $H_0' + H_0'' + D$ is simple normal crossing in Z.

By Step 16 and Rem. 1.18 we have $H_0' \sim H_0'' \sim H_0$ and $D \sim H_0 - \varphi^*(N)$, hence $H_0' + H_0'' + D \sim 3H_0 - \varphi^*(N)$. Moreover $-K_Z \sim \varphi^*(-K_T - N) + 3H_0$ and

$$-K_Z + H_0 \sim \varphi^*(-K_T - N) + 4H_0 \sim A + H_0' + H_0'' + D.$$

By Norimatsu's Lemma [**Laz04a**, Lemma 4.3.5] we get $H^1(Z, H_0) = H^1(Z, K_Z - K_Z + H_0) = 0$. ■

Step 18. *S_1 is a complete intersection of H_0 and a divisor in $|2H_0|$.*

PROOF. We note first of all that H_0 is a smooth $\mathbb{P}^1$-bundle over T. Indeed, since $d = 2$ by Step 15, for every $t \in T$ we have $\mathcal{O}_Z(H_0)_{|Z_t} \cong \mathcal{O}_{\mathbb{P}^2}(1)$ (see Step 6). Note that no fibre Z_t can be contained in H_0, because $H_0 \cap S_2 = \emptyset$ by Step 7; therefore $(H_0)_{|Z_t}$ is a line for every $t \in T$, and $\varphi_{|H_0} \colon H_0 \to T$ is a $\mathbb{P}^1$-bundle.

Now we show that $\mathcal{O}_{H_0}(S_1) \cong \mathcal{O}_Z(2H_0)_{|H_0}$. We consider the divisor H_0' as in the proof of Step 17, so that $H_0' \sim H_0$ and $H_0' \cap D = S_2$.

We have that $D \cap H_0$ is $\{pt\} \times T$ in $D \cong \mathbb{P}^1 \times T$, hence $D \cap H_0 \subset H_0$ is a section of φ, disjoint from both S_1 and H_0', because $D \cap S_1 = \emptyset$ and $D \cap H_0' \cap H_0 = S_2 \cap H_0 = \emptyset$ (see Steps 7 and 8). Since S_1 has degree 2 over T, in H_0 we have

$$S_1 \sim_{H_0} 2(H_0')_{|H_0} + (\varphi_{|H_0})^*(G),$$

where G is a divisor in T. By restricting to the section $D \cap H_0$, we get $G \sim 0$ and $\mathcal{O}_{H_0}(S_1) \cong \mathcal{O}_Z(2H_0)_{|H_0}$.

Finally we consider the exact sequence in Z:

$$0 \longrightarrow \mathcal{O}_Z(H_0) \longrightarrow \mathcal{O}_Z(2H_0) \longrightarrow \mathcal{O}_{H_0}(2H_0) \longrightarrow 0.$$

By Step 17 we have a surjection $H^0(Z, \mathcal{O}_Z(2H_0)) \twoheadrightarrow H^0(H_0, \mathcal{O}_Z(2H_0)_{|H_0})$, and this yields the statement. ∎

Step 19. *$X \xrightarrow{h} Z \xrightarrow{\varphi} T$ is as in Construction B (Prop. 3.3).*

PROOF. We just note that $-K_T \pm N$ is ample by Lemma 3.14, and $\delta_T \leq 3$ by Lemma 3.15. ∎

3.5. Geometry of the del Pezzo fibration $\sigma \colon X \to T$

Let X be a smooth Fano variety with $\delta_X = 3$. By Th. 3.4 there are a smooth Fano variety T, and a morphism $\sigma \colon X \to T$, as in Construction A or B; let us denote X by X_A in the former case, and by X_B in the latter.

In this section we study the geometry of the fibration in del Pezzo surfaces σ, starting with the description of its fibres in §3.5.1. Then, in §3.5.2 and §3.5.3, we describe the 4-dimensional cone $\mathrm{NE}(\sigma)$ and the associated relative contractions, respectively for X_A and X_B. In particular this allows to determine whether the same $\sigma \colon X \to T$ can be obtained with Construction A or B in different ways, or for different choices of divisors on T.

Finally in §3.5.4 we show that, if $X \ncong F \times T$ (F the blow-up of $\mathbb{P}^2$ at 3 non-collinear points), then T must satisfy some non-trivial condition.

We keep the same notation as in Sections 3.2 and 3.3; in particular we recall that σ factors as $X \xrightarrow{h} Z \xrightarrow{\varphi} T$, where φ is a $\mathbb{P}^2$-bundle, and h is the blow-up along 3 horizontal, pairwise disjoint, codimension 2 smooth subvarieties $S_1, S_2, S_3 \subset Z$.

3.5.1. Fibres of σ. Let X be as above, and X_t the fibre of σ over $t \in T$.

For X_A, the morphism σ is smooth, and X_t is isomorphic to the blow-up of $\mathbb{P}^2$ at non-collinear 3 points for every $t \in T$.

For X_B, recall that $\varphi_{|S_1} \colon S_1 \to T$ is a finite cover of degree 2; let $\Delta \subset T$ be the branch divisor (note that $\Delta \neq \emptyset$). Then σ is smooth over $T \smallsetminus \Delta$, where X_t is isomorphic to the blow-up of $\mathbb{P}^2$ at 4 points in general linear position. Instead, for $t \in \Delta$, X_t is

singular and isomorphic to the blow-up of $\mathbb{P}^2$ at 3 non-collinear points, one of which is a double point contained in a line (see Rem. 3.11); thus X_t is irreducible and has one rational double point of type A_1 (see for instance [**EH00**, §IV.2.3]).

Remark 3.17. Let X be a smooth Fano variety with $\delta_X = 3$. Then X is toric if and only if X is obtained using Construction A from a toric Fano variety T.

If X is toric, then $\sigma\colon X \to T$ is smooth (so that $X = X_A$) and T is toric. In fact, the whole construction of σ is toric: in particular Z and T are toric varieties and the centres of the blow-up $X \to Z$ are invariant under the torus action. Let $t \in T$ be a fixed point by the torus action on T, so that $Z_t \subset Z$ is an invariant fibre of φ under the torus action on Z. Then, S_1 and Z_t are invariant, they have non-empty intersection and $\varphi_{|S_1}\colon S_1 \to T$ is finite, so they must intersect in a unique fixed point by the torus action on Z. Thus S_1 is a section of φ and σ is smooth.

Conversely, if we apply Construction A to a toric Fano variety T, then Z is toric, and the three sections $S_1, S_2, S_3 \subset Z$ are invariant for the torus action, so that X is toric too.

3.5.2. The cone NE(σ) and relative contractions for X_A.

Remark 3.18. Consider $X = X_A$, let X_t be a fibre of σ, and $\iota\colon X_t \hookrightarrow X$ the inclusion. Since σ is smooth, it follows from [**Wiś91b**, Prop. 1.3] that the pushforward $\iota_*\colon \mathcal{N}_1(X_t) \to \mathcal{N}_1(X)$ yields an isomorphism among $\mathcal{N}_1(X_t)$ and $\ker \sigma_*$, and among the cones $\mathrm{NE}(X_t)$ and $\mathrm{NE}(\sigma)$. In particular, every relative elementary contraction of X/T restricts to an elementary contraction of X_t, and viceversa. In fact, as X_t is a smooth del Pezzo surface of Picard number 4 and $\rho_X - \rho_T = 4$, we have

$$\dim \mathcal{N}_1(X_t) = \dim(\ker \sigma_*) = 4;$$

if R is an extremal ray of $\mathrm{NE}(\sigma)$ then the locus of R dominates T by [**Wiś91b**, Prop. 1.3], so $R \subset \iota_*\big(\mathcal{N}_1(X_t)\big)$. This implies that the map

$$\iota_*\colon \mathcal{N}_1(X_t) \longrightarrow \ker \sigma_*$$

is surjective, thus an isomorphism, and it identifies the cones $\mathrm{NE}(X_t)$ and $\mathrm{NE}(\sigma)$.

Since X_t is the blow-up of $\mathbb{P}^2$ at 3 non-collinear points p_1, p_2, p_3, the cone $\mathrm{NE}(X_t)$ is generated by the classes of the six (-1)-curves of X_t, given by the exceptional curve e_i over p_i, and the transform e_i' of the line $\overline{p_j p_k}$. In X we have $e_i = E_i \cap X_t$ and $e_i' = E_i' \cap X_t$; the exceptional divisors E_i, E_i' are $\mathbb{P}^1$-bundles over T.

Figure III.1 shows the 3-dimensional polytope obtained as a hyperplane section of the 4-dimensional cone $\mathrm{NE}(\sigma)$.

Lemma 3.19. *Let $X = X_A$. Then there is a commutative diagram:*

$$\begin{array}{ccc} X & \xrightarrow{\widehat{h}} & \widehat{Z} = \mathbb{P}_T(\mathcal{O}(-D_1) \oplus \mathcal{O}(-D_2) \oplus \mathcal{O}(-D_3)) \\ {\scriptstyle h}\downarrow & \searrow^{\sigma} & \downarrow{\scriptstyle \widehat{\varphi}} \\ Z & \xrightarrow[\varphi]{} & T \end{array}$$

where $\mathrm{Exc}(\widehat{h}) = E_1' \cup E_2' \cup E_3'$, so that performing Construction A from T with divisors D_1, D_2, D_3, or with divisors $-D_1, -D_2, -D_3$, yields the same Fano variety X.

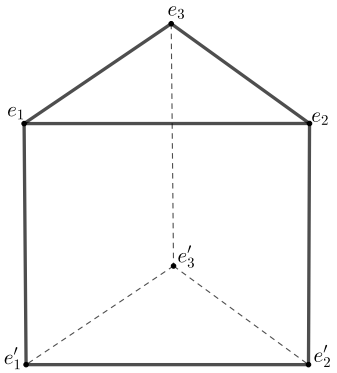


FIGURE III.1. A section of $\mathrm{NE}(\sigma)$ for X_A

PROOF. By Rem. 3.18 the face of $\mathrm{NE}(\sigma)$ spanned by the classes of e'_1, e'_2, e'_3 yields a contraction $\widehat{h}\colon X \to \widehat{Z}$ such that $\widehat{\varphi}\colon \widehat{Z} \to T$ is a $\mathbb{P}^2$-fibration, and $\widehat{h}$ is the blow-up of three sections that are fibre-wise in general linear position. Moreover by Lemmata 1.21 and 1.22 we have $E_i \cong \mathbb{P}_T(\mathcal{O}(-D_j) \oplus \mathcal{O}(-D_k))$ and $E'_i \cong \mathbb{P}_T(\mathcal{O}(D_j) \oplus \mathcal{O}(D_k))$ for every i, j, k with $\{i, j, k\} = \{1, 2, 3\}$; this implies the statement. ■

Observe that the birational contractions h and $\widehat{h}$ correspond to the two simplicial facets of $\mathrm{NE}(\sigma)$, see Fig. III.1. Instead the three non-simplicial facets yield conic bundles $X \to Y$, where Y is a $\mathbb{P}^1$-bundle over T.

3.5.3. The cone $\mathrm{NE}(\sigma)$ and relative contractions for X_B.

Lemma 3.20. *Let $X = X_B$, X_{t_0} a smooth fibre of σ, and $\iota\colon X_{t_0} \hookrightarrow X$ the inclusion. Then every relative elementary contraction of X/T restricts to a non-trivial contraction of X_{t_0}, and $\iota_* \mathrm{NE}(X_{t_0}) = \mathrm{NE}(\sigma)$.*

PROOF. Clearly $\iota_* \mathrm{NE}(X_{t_0}) \subseteq \mathrm{NE}(\sigma)$. For the converse, let R be an extremal ray of $\mathrm{NE}(\sigma)$, and $c_R\colon X \to X_R$ the associated elementary contraction of X. We show that X_{t_0} must contain some curve contracted by c_R; this implies that R is in the image of $\mathrm{NE}(X_{t_0})$ via ι_*, so that $\iota_* \mathrm{NE}(X_{t_0}) = \mathrm{NE}(\sigma)$.

The statement is clear if c_R is of fibre type, so let us assume that c_R is birational. Every fibre X_t of σ is irreducible (see §3.5.1), and $c_R(X_t) \subset X_R$ is the fibre of $X_R \to T$ over t. Since $\dim X_R = n$, we have $\dim c_R(X_t) = 2$ and hence $X_t \not\subset \mathrm{Exc}(c_R)$.

On the other hand every fibre of c_R is contained in some X_t, and we conclude that every fibre of c_R has dimension ≤ 1. By [**Wiś91a**, Thm. 1.2] the contraction c_R is a blow-up and $\dim \mathrm{Exc}(c_R) = n - 1$, so that $\dim \sigma(\mathrm{Exc}(c_R)) = n - 2$, $\sigma(\mathrm{Exc}(c_R)) = T$ and $\mathrm{Exc}(c_R)$ meets every fibre X_t. ■

Remark 3.21. Let $X = X_B$; we use Lemma 3.20 to describe the cone $\mathrm{NE}(\sigma)$.

Let X_t be a smooth fibre of σ and $\{p_1, p'_1, p_2, p_3\} \in Z_t$ be the points blown-up by $h_{|X_t}\colon X_t \to Z_t$, where $p_i = S_i \cap Z_t$ for $i = 2, 3$, and $\{p_1, p'_1\} = S_1 \cap Z_t$. The 5-dimensional cone $\mathrm{NE}(X_t)$ is generated by the classes of the ten (-1)-curves in X_t,

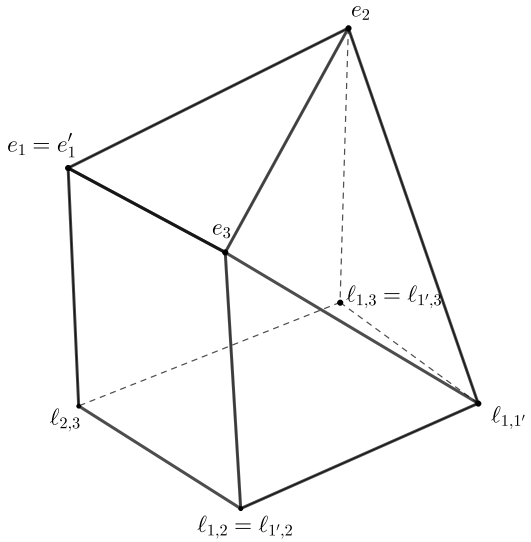


FIGURE III.2. A section of $\mathrm{NE}(\sigma)$ for X_B

given by the exceptional curves and the transforms of the lines through two blown-up points. We denote by e_i (respectively e_1') the exceptional curve over p_i (respectively p_1'), and $\ell_{i,j}$ (respectively $\ell_{1,1'}$, $\ell_{1',i}$ for $i = 2, 3$) the transform of the line $\overline{p_i p_j}$ (respectively $\overline{p_1 p_1'}$, $\overline{p_1' p_i}$ for $i = 2, 3$).

Consider as above $\iota\colon X_t \hookrightarrow X$ and $\iota_*\colon \mathcal{N}_1(X_t) \twoheadrightarrow \ker \sigma_*$. The kernel of ι_* has dimension 1, and is generated by the numerical class in $\mathcal{N}_1(X_t)$ of the 1-cycle $(e_1 - e_1')$. By looking at the numerical relations in $\mathcal{N}_1(X_t)$, we provide a complete description of the cone $\mathrm{NE}(\sigma)$: Figure III.2 shows the 3-dimensional polytope obtained as a hyperplane section of the 4-dimensional cone $\mathrm{NE}(\sigma)$. We see that $\mathrm{NE}(\sigma)$ has 7 extremal rays, and the figure shows their generators.

It follows from [**Wiś91a**, Thm. 1.2] that every relative elementary contraction of $\mathrm{NE}(\sigma)$ is the blow-up of a smooth variety along a smooth codimension 2 subvariety. The contraction corresponding to $[e_1] = [e_1']$ (resp. $[e_2], [e_3]$) is the blow-down of E_1 (resp. E_2, E_3), while the contractions corresponding to $[\ell_{1,1'}]$ and $[\ell_{2,3}]$ have respectively exceptional divisors $\widetilde{H}_0$ and $\widetilde{D}$ (the strict transforms of H_0 and D from Z). Lastly, for $i = 2, 3$ we denote by G_i the exceptional divisor of the contraction corresponding to $[\ell_{1,i}] = [\ell_{1',i}]$.

Observe that $\mathrm{NE}(\sigma)$ has 4 simplicial facets and 3 non-simplicial facets; $\mathrm{NE}(h)$ is the facet generated by $[e_1], [e_2], [e_3]$.

Proposition 3.22. *Let $X = X_B$, and consider the facet $\langle [\ell_{1,2}], [\ell_{1,1'}], [e_3] \rangle$ (respectively $\langle [\ell_{1,3}], [\ell_{1,1'}], [e_2] \rangle$) of $\mathrm{NE}(\sigma)$. The associated contraction $\widehat{h}$ yields a commutative diagram:*

$$\begin{array}{ccc} X & \xrightarrow{\widehat{h}} & Z \\ {\scriptstyle h}\downarrow & \searrow^{\sigma} & \downarrow{\scriptstyle \varphi} \\ Z & \xrightarrow[\varphi]{} & T \end{array}$$

where $\mathrm{Exc}(\widehat{h}) = G_2 \cup \widetilde{H}_0 \cup E_3$ *(respectively* $G_3 \cup \widetilde{H}_0 \cup E_2$*), and* $X \xrightarrow{\widehat{h}} Z \xrightarrow{\varphi} T$ *is as described in Construction B.*

Proof. We consider the face $\langle [\ell_{1,2}], [\ell_{1,1'}], [e_3] \rangle$, the other case being analogous. Let $\widehat{h}\colon X \to \widehat{Z}$ be the associated contraction, and $\widehat{\varphi}\colon \widehat{Z} \to T$ the map over T, so that $\widehat{\varphi}$ is an elementary contraction and $\sigma = \widehat{\varphi} \circ \widehat{h}$.

By Rem. 3.21, $\widehat{Z}$ is smooth and $\widehat{h}$ is the blow-up of three pairwise disjoint smooth, codimension 2, irreducible subvarieties, with $\mathrm{Exc}(\widehat{h}) = G_2 \cup \widetilde{H}_0 \cup E_3$.

We show that $\widehat{\varphi}\colon \widehat{Z} \to T$ is a $\mathbb{P}^2$-bundle. First note that all the fibres $\widehat{Z}_t$ of $\widehat{\varphi}$ are surfaces; in particular, since $\widehat{h}(\widetilde{H}_0)$ and $\widehat{h}(E_3)$ are sections of $\widehat{\varphi}$ and $\widehat{h}(G_2)$ is a bisection of $\widehat{\varphi}$, the general fibre of $\widehat{\varphi}$ is isomorphic to $\mathbb{P}^2$. Let $A := \widehat{h}(E_2)$, and $\widehat{Z}_t$ be general, so that $\mathcal{O}_{\widehat{Z}}(A)_{|\widehat{Z}_t} \cong \mathcal{O}_{\mathbb{P}^2}(1)$ and $A \cdot \ell = 1$ for a line $\ell \subset \widehat{Z}_t$. Since $\mathrm{NE}(\widehat{\varphi})$ has dimension 1, the relative Kleiman's criterion implies that A is $\widehat{\varphi}$-ample, thus there exists an ample divisor M on T such that $A' := A + m\widehat{\varphi}^*(M)$ is ample on $\widehat{Z}$ for $m \gg 0$. Note that $\mathcal{O}_{\widehat{Z}}(A')_{|\widehat{Z}_t} \cong \mathcal{O}_{\mathbb{P}^2}(1)$, so we apply [**BS95**, Prop. 3.2.1] and deduce that $\widehat{Z} = \mathbb{P}_T(\mathcal{G})$ for some rank 3 vector bundle $\mathcal{G}$ on T.

By construction $\widehat{S}_3 := \widehat{h}(E_3)$ is a section of $\widehat{\varphi}$, and moreover $\mathcal{N}^\vee_{\widehat{S}_3/\widehat{Z}} \cong \mathcal{N}^\vee_{S_3/Z} \cong \mathcal{O}(N) \oplus \mathcal{O}$. Up to tensoring with some line bundle on T, we may assume that the section $\widehat{S}_3$ gives an exact sequence on T:

$$0 \longrightarrow \mathcal{O}(N) \oplus \mathcal{O} \longrightarrow \mathcal{G} \longrightarrow \mathcal{O} \longrightarrow 0.$$

Since $-K_T + N$ is ample (see Lemma 3.14), we have $h^1(T, N) = 0$ and the above sequence splits, hence $\widehat{Z} \cong \mathbb{P}_T(\mathcal{O}(N) \oplus \mathcal{O} \oplus \mathcal{O}) = Z$.

Finally it is not difficult to see that $\widehat{h}(\widetilde{H}_0)$ is a section of φ corresponding to the projection onto $\mathcal{O}_T$, and that $\widehat{h}(G_2)$ is a complete intersection of elements in $|H|$ and $|2H|$. ■

Remark 3.23. Let $X = X_B$, and consider the facet $\langle [e_2], [e_3], [\ell_{1,1'}] \rangle$ of $\mathrm{NE}(\sigma)$ (see Fig. III.2). The associated contraction h' yields a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{h'} & W \\ {\scriptstyle h}\downarrow & \searrow^{\sigma} & \downarrow{\scriptstyle \alpha} \\ Z & \xrightarrow[\varphi]{} & T \end{array}$$

where $\mathrm{Exc}(h') = E_2 \cup E_3 \cup \widetilde{H}_0$, W is smooth, and $\alpha\colon W \to T$ is a quadric bundle, singular over $\Delta \subset T$.

Instead the three non-simplicial facets of $\mathrm{NE}(\sigma)$ yield conic bundles $X \to Y$, where Y is a $\mathbb{P}^1$-bundle over T.

We also note that, among the seven exceptional divisors associated to the extremal rays of $\mathrm{NE}(\sigma)$, E_2, E_3, $\widetilde{H}_0$, and $\widetilde{D}$ are $\mathbb{P}^1$-bundles over T, while E_1, G_2, and G_3 are $\mathbb{P}^1$-bundles over S_1, the double cover of T ramified along Δ.

3.5.4. Conditions on T. Given a smooth Fano variety T, Constructions A and B give algorithms to construct from T a smooth Fano variety X with $\dim X = \dim T + 2$ and $\rho_X = \rho_T + 4$. For every T one can get $X_A = F \times T$, where F is the blow-up of $\mathbb{P}^2$ at 3 non-collinear points, and $\sigma\colon X \to T$ the projection; this corresponds to the choice $D_1 \sim D_2 \sim D_3$. In order to get an X different from $F \times T$, the variety T must satisfy some conditions, as follows.

Lemma 3.24. *Let T be a smooth Fano variety. Suppose that there exists a Fano variety X obtained from T as in Construction A or B, and such that $\sigma\colon X \to T$ is not isomorphic to the projection $F \times T \to T$.*

Then there exists a hyperplane $\Lambda \subset \mathcal{N}_1(T)$ containing all the classes $[C]$ where $C \subset T$ is a curve with $-K_T \cdot C = 1$.

In particular, every extremal ray of length 1 of $\mathrm{NE}(T)$ is contained in Λ, and $\mathrm{NE}(T)$ has at least one extremal ray R with length $\ell(R) \geq 2$.

Proof. Let $C \subset T$ be a curve with $-K_T \cdot C = 1$. Suppose first that $X = X_A$. By Lemma 3.9 for every $i, j = 1, 2, 3$ we have $1 \leq (-K_T + D_i - D_j) \cdot C = 1 + (D_i - D_j) \cdot C$, so that $D_i \cdot C = D_j \cdot C$. We exclude by assumption the case $D_1 \sim D_2 \sim D_3$, hence for some $i \neq j$ we must have $D_i \not\sim D_j$; set $\Lambda := (D_i - D_j)^\perp$, so that Λ is a hyperplane in $\mathcal{N}_1(T)$, and contains $[C]$.

The case where $X = X_B$ is similar, we set $\Lambda := N^\perp$. ∎

In the next section we apply these conditions to the case where T is a del Pezzo surface.

3.6. The case of dimension 4

Let X be a Fano 4-fold with $\delta_X = 3$. By Th. 3.4, X is obtained with Construction A or B from a del Pezzo surface T with $\rho_X - \rho_T = 4$ and $\delta_T \leq 3$. Since T is a surface, it is easy to see that $\delta_T = \rho_T - 1$, thus $\rho_T \in \{1, 2, 3, 4\}$ and $\rho_X \in \{5, 6, 7, 8\}$.

If $\rho_T \geq 3$, then $\mathrm{NE}(T)$ is generated by classes of (-1)-curves, in particular there are no extremal rays of length > 1. Then it follows from Lemma 3.24 that $X \cong F \times T$, where F is the blow-up of $\mathbb{P}^2$ in 3 non-collinear points.

If instead $\rho_T = 1$, then $\rho_X = 5$; this case is completely classified in [**CR22**, Th. 1.1], and there are 6 families.

Finally suppose that $\rho_T = 2$, so that $\rho_X = 6$. If X is toric, after the classification of toric Fano 4-folds by Batyrev [**Bat99, Sat00**], we see that X has combinatorial type U (in the notation of [**Bat99**]), and there are 8 possibilities for X. More precisely, we have $T \cong \mathbb{F}_1$ in the two cases U_2 and $U_4 \cong F \times \mathbb{F}_1$, and $T \cong \mathbb{P}^1 \times \mathbb{P}^1$ in the remaining six cases, including $U_5 \cong F \times \mathbb{P}^1 \times \mathbb{P}^1$.

In the non-toric case we get the following result, that together with the previous discussion implies Prop. 3.5.

Proposition 3.25. *There are three families of non-toric Fano 4-folds X with $\delta_X = 3$ and $\rho_X = 6$.*

PROOF. By Th. 3.4 and Rem. 3.17, X is obtained with Construction B from a del Pezzo surface T with $\rho_T = \rho_X - 4 = 2$, namely $T \cong \mathbb{P}^1 \times \mathbb{P}^1$ or $T \cong \mathbb{F}_1$. The divisor N on T is such that the class of N is effective and non-zero, and $-K_T \pm N$ is ample.

It is not difficult to see that there are three choices of N satisfying these conditions: $N \in |\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(0,1)|$, $N \in |\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1)|$, and N the pullback of a general line in $\mathbb{P}^2$ in the case of $\mathbb{F}_1$.

In all cases N is nef, hence the tautological divisor H of $Z = \mathbb{P}_T(\mathcal{O}(N) \oplus \mathcal{O} \oplus \mathcal{O})$ is nef. On the other hand Z is toric, thus H is globally generated: we conclude that the general complete intersection of elements in $|H|$ and $|2H|$ is smooth, and we can apply Construction B. Finally we get three families of Fano 4-folds, respectively X_{B_0}, X_{B_1}, and X_{B_2}. ■

Description of X_{B_0}. We have $X_{B_0} \cong \mathbb{P}^1 \times Y$, where Y is the Fano 3-fold obtained by blowing-up $\mathbb{P}^3$ along a line, a conic disjoint from the line, and two non-trivial fibres of the blow-up of the line. In fact Y has $\delta_Y = 3$ and is obtained via Construction B, del Pezzo fibration $\sigma_Y \colon Y \to T_Y \cong \mathbb{P}^1$ and $N_Y \in |\mathcal{O}_{\mathbb{P}^1}(1)|$.

Description of X_{B_1}. Set $Z := \mathbb{P}_{\mathbb{P}^1\times\mathbb{P}^1}(\mathcal{O}(1,1) \oplus \mathcal{O}^{\oplus 2}) \xrightarrow{\varphi} \mathbb{P}^1 \times \mathbb{P}^1$; the 4-fold X_{B_1} is the blow-up of Z along three pairwise disjoint smooth surfaces S_1, S_2, S_3. The surfaces S_2 and S_3 are sections of φ, they are isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ and have normal bundle $\mathcal{O}(-1,-1) \oplus \mathcal{O}$. The blow-up X_1 of Z along S_2 and S_3 is a toric Fano 4-fold of combinatorial type Q_1, following [**Bat99**].

On the other hand the surface S_1 is a double cover of $\mathbb{P}^1 \times \mathbb{P}^1$ with $-K_{S_1} = \varphi^*\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1)_{|S_1}$ (see Rem. 3.13), so that S_1 is a del Pezzo surface of degree 4.

Description of X_{B_2}. Here $T = \mathbb{F}_1$ and N is the pullback of a general line in $\mathbb{P}^2$. Then Z is a $\mathbb{P}^2$-bundle over $\mathbb{F}_1$, and S_2, S_3 are two sections; the blow-up X_1 of Z along S_2 and S_3 is a toric Fano 4-fold of combinatorial type Q_2, following [**Bat99**]. Moreover $\varphi_{|S_1} \colon S_1 \to \mathbb{F}_1$ is a double cover with $-K_{S_1} = \varphi^*_{|S_1}(2N - E)$, where $E \subset \mathbb{F}_1$ is the exceptional curve, so that S_1 is a del Pezzo surface of degree 6, the blow-up of $\mathbb{P}^2$ at 3 points.

We give in Table III.1 some numerical invariants of the Fano 4-folds X_{B_0}, X_{B_1}, and X_{B_2}; they are computed using standard methods, see [**CR22**, Lemmata 3.2 and 3.3]. In the last column T denotes the tangent bundle.

TABLE III.1. Numerical invariants

4-fold	b_3	$h^{2,2}$	$h^{1,3}$	K^4	$K^2 \cdot c_2$	$h^0(-K)$	$\chi(T)$
X_{B_0}	0	10	0	224	152	51	4
X_{B_1}	0	14	0	222	156	51	-2
X_{B_2}	0	12	0	223	154	51	1

Finally we apply our results to the study of conic bundles on Fano 4-folds. Let X be a Fano 4-fold and $\eta\colon X \to Y$ a conic bundle such that $\rho_X - \rho_Y \geq 3$. Let us denote by $\triangle := \{y \in Y | \eta^{-1}(y) \text{ is singular}\}$ the discriminant divisor.

If $X \cong S \times T$ is a product of two del Pezzo surfaces, then it follows easily that $Y \cong \mathbb{P}^1 \times T$ and η is induced by a conic bundle $S \to \mathbb{P}^1$ (see for instance [**Rom19**, Lemma 2.10]); in particular all the connected components of $\triangle$ are isomorphic to T.

Let us assume that X is not a product of surfaces. Then we have $\rho_X - \rho_Y = \delta_X = 3$ by [**Rom19**, Th. 4.2(1)] and [**MR21**, Th. 1.1], and $\rho_X \in \{5, 6\}$ by Prop. 3.5, so that the possible X are classified. The case where $\rho_X = 5$ has been studied in [**CR22**, Cor. 2.18]. As an application of our results, we describe the case $\rho_X = 6$.

Corollary 3.26. *Let $\eta\colon X \to Y$ be a conic bundle, where X is a Fano 4-fold with $\rho_X = 6$, and $\rho_X - \rho_Y = 3$. Let $\triangle$ be the discriminant divisor. Then one of the following holds:*

(i) $\triangle \cong \mathbb{F}_1 \sqcup \mathbb{F}_1$ and either $X \cong U_2$ or $X \cong U_4$;
(ii) $\triangle \cong \mathbb{P}^1 \times \mathbb{P}^1 \sqcup \mathbb{P}^1 \times \mathbb{P}^1$ and X is isomorphic to one of the following varieties: U_1, U_3, U_5, U_6, U_7, U_8, X_{B_0};
(iii) $\triangle \cong \mathbb{P}^1 \times \mathbb{P}^1 \sqcup S_1$ with S_1 a del Pezzo surface of degree 4 and $X \cong X_{B_1}$;
(iv) $\triangle \cong \mathbb{F}_1 \sqcup S_1$ with S_1 a del Pezzo surface of degree 6 and $X \cong X_{B_2}$.

Proof. If X is a product of two del Pezzo surfaces, the statement is easy, so let us assume this is not the case; then we have $\delta_X = 3$ by [**MR21**, Th. 1.1].

Now, replacing $\psi \circ f$ by η in Th. 3.16, we check that all the properties $(a) - (h)$ are satisfied. Indeed, by [**Rom19**, Prop. 3.5 (1)], η factors as a composition $X \to X_2 \to Y$ where the first map satisfies (f); and in view of [**Rom19**, Th. 4.2 (2)] also (a), (b), (c), (d) hold. Finally, (g) follows by the proof of [**Rom19**, Prop. 3.4], while (h) is shown in Step 2 of the proof of [**Rom19**, Th. 4.2 (2)]. Therefore, we may run the arguments of the proof of Th. 3.4 with η instead of $\psi \circ f$. We keep the notation as in that theorem.

Then $\triangle = A_1 \sqcup A_2$, and by Step 2 it follows that $A_i \cong B_i \cong S_i$ for $i = 1, 2$. Moreover, using Step 1 we know that $A_2 \cong T$, where in our case either $T \cong \mathbb{P}^1 \times \mathbb{P}^1$ or $T \cong \mathbb{F}_1$. If A_1 is a section of $\xi\colon Y \to T$ then using Steps 3 and 5 we get (i) or (ii). Otherwise, by Step 19 we deduce that X is obtained as in Construction B, and finally that X is isomorphic to X_{B_0}, X_{B_1}, or X_{B_2}, following the proof of Prop. 3.25. As observed in the above descriptions of these non-toric families, S_1 is respectively $\mathbb{P}^1 \times \mathbb{P}^1$, a del Pezzo surface of degree 4, or a del Pezzo of degree 6, hence we get (ii), (iii), or (iv). ∎

Remark 3.27. Let X be a Fano 4-fold with $\delta_X = 3$. By Th. 3.16, we know that $\psi \circ f\colon X \to Y$ is a conic bundle with $\rho_X - \rho_Y = 3$. All the possible targets Y have been classified in [**MR21**], but in [**MR21**, Prop. 1.2(b)] there is a missing case, that is $Y \cong \mathbb{P}_{\mathbb{F}_1}(\mathcal{O} \oplus \mathcal{O}(N))$, with N being the pullback of a general line through the blow-up $\mathbb{F}_1 \to \mathbb{P}^2$. In this case, Construction A gives $X = U_2$; while performing Construction B we obtain the non-toric Fano family X_{B_2}.

3.7. The case $\rho_X = 5$

In this section we consider Fano varieties with $\delta_X = 3$ and $\rho_X = 5$, the minimal Picard number, and prove Prop. 3.6, Cor. 3.7, and Th. 3.8. By Th 3.4, X is obtained as in Construction A or B from a smooth Fano variety T with $\rho_T = 1$. We denote by $\mathcal{O}_T(1)$

the ample generator of $\mathrm{Pic}(T) \cong \mathbb{Z}$, and $\mathcal{O}_T(a) := \mathcal{O}_T(1)^{\otimes a}$ for every $a \in \mathbb{Z}$. Recall that $\mathcal{O}_T(-K_T) \cong \mathcal{O}_T(i_T)$, where i_T is the index of T.

Proof of Prop. 3.6. We show the uniqueness of $\sigma\colon X \to T$. Let us assume that there exists another $\bar{\sigma}\colon X \to \overline{T}$ as in Construction A or B. We show that σ and $\bar{\sigma}$ coincide up to an isomorphism $T \cong \overline{T}$.

Let R be an extremal ray of the cone $\mathrm{NE}(\sigma)$. By Remarks 3.18, 3.21, and 3.23, the contraction associated to R is the blow-up of a smooth subvariety S which is an irreducible codimension 2 complete intersection, and either $S \cong T$ or $S \cong S_1$. Let $E \subset X$ be the exceptional divisor; since the normal bundle of S is decomposable, we have $E \cong \mathbb{P}_S(\mathcal{O} \oplus L)$ with $L \in \mathrm{Pic}(S)$.

We first assume that $n \geq 5$, where $n := \dim X$. Recall that S_1 is a ramified double cover of T, and $\rho_T = 1$, so that the ramification divisor is ample; since $\dim S_1 = n-2 > 2$, [**Cor81**] yields $\rho_{S_1} = 1$ (see also [**LS89**, Prop. 1.11]). Therefore in any case we have $\rho_S = 1$ and $\rho_E = 2$.

Since $\rho_E = 2$, E has at most two elementary contractions, one being the $\mathbb{P}^1$-bundle $E \to S$. If $L \cong \mathcal{O}_S$, then $E \cong \mathbb{P}^1 \times S$, and the second elementary contraction is $E \to \mathbb{P}^1$. Otherwise we can assume that L is ample; in this case E has an elementary divisorial contraction sending a divisor to a point (see for instance [**CD15**, p. 10768]).

Consider now the restriction $\bar{\sigma}_{|E}\colon E \to \bar{\sigma}(E) \subseteq \overline{T}$. The Stein factorisation gives

$$E \xrightarrow{\psi} B \xrightarrow{\nu} \bar{\sigma}(E),$$

where ψ is a contraction of E, and ν is finite. Thus $\dim B = \dim \bar{\sigma}(E) \leq \dim \overline{T} = n-2$; on the other hand every fibre of $\bar{\sigma}$ is a surface, thus $\dim B \geq \dim E - 2 = n-3 \geq 2$. By the previous observation on the possible contractions of E, we deduce that $B \cong S$ and ψ is the $\mathbb{P}^1$-bundle $E \to S$, so that $\bar{\sigma}$ contracts the fibres of $E \to S$ and $R \subset \mathrm{NE}(\bar{\sigma})$.

This holds for every extremal ray R of $\mathrm{NE}(\sigma)$, so that $\mathrm{NE}(\sigma) \subseteq \mathrm{NE}(\bar{\sigma})$. Since these two cones are both 4-dimensional faces of $\mathrm{NE}(X)$, we conclude that they are the same, and by [**Deb01**, Prop. 1.14] σ and $\bar{\sigma}$ coincide up to an isomorphism $T \cong \overline{T}$.

Suppose now that $n = 4$, so that $T \cong \overline{T} \cong \mathbb{P}^2$. We repeat the same argument as above with an extremal ray R such that E is a $\mathbb{P}^1$-bundle over T. We still have $\dim B \geq \dim E - 2 = 1$, and if $\dim B = 1$, then $E \cong \mathbb{P}^1 \times \mathbb{P}^2$ and ψ the projection onto $\mathbb{P}^1$. Thus the general fibre of $\bar{\sigma}$ over $\bar{\sigma}(E)$ is isomorphic to $\mathbb{P}^2$, which is impossible by the description of the fibres in §3.5.1. We conclude that $\dim B \geq 2$, and as before that $R \subset \mathrm{NE}(\bar{\sigma})$. If $X = X_A$ we get $\mathrm{NE}(\sigma) \subseteq \mathrm{NE}(\bar{\sigma})$ and hence $\mathrm{NE}(\sigma) = \mathrm{NE}(\bar{\sigma})$. If $X = X_B$, we see from Rem. 3.21 and Fig. III.2 that the extremal rays R of $\mathrm{NE}(\sigma)$ such that E is a $\mathbb{P}^1$-bundle over T generate a 4-dimensional cone. We conclude that $\dim(\mathrm{NE}(\sigma) \cap \mathrm{NE}(\bar{\sigma})) = 4 = \dim \mathrm{NE}(\sigma) = \dim \mathrm{NE}(\bar{\sigma})$, so that again the two cones coincide.

Finally suppose that $n = 3$, so that $T \cong \overline{T} \cong \mathbb{P}^1$, and repeat the same argument as above. We conclude that either $R \subset \mathrm{NE}(\bar{\sigma})$, or E has two different $\mathbb{P}^1$-bundle structures, hence $E \cong \mathbb{P}^1 \times \mathbb{P}^1$ with normal bundle $\mathcal{O}(-1,-1)$. We show that there is at most one extremal ray R with this property, so we conclude that $\mathrm{NE}(\sigma) = \mathrm{NE}(\bar{\sigma})$ as in the case $n = 4$. In fact, if $X = X_A$ and $E \cong \mathbb{P}^1 \times \mathbb{P}^1$ is an exceptional divisor corresponding to an extremal ray $R \subset \mathrm{NE}(\sigma)$, by Lemmata 1.21 and 1.22 we see that, up to reordering the indices and tensoring with a line bundle, $Z = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O} \oplus \mathcal{O})$ and E is either E_1 or E_1' with normal bundle $\mathcal{O}(-1,a)$ or $\mathcal{O}(-1,-a)$ respectively. By Lemma 3.9 $|a| \leq 1$,

and E has normal bundle $\mathcal{O}(-1,-1)$ iff $a = 1$ and $E = E_1'$ or $a = -1$ and $E = E_1$. If instead $X = X_B$, by Lemma 3.14 $\mathcal{O}_T(N) \cong \mathcal{O}_{\mathbb{P}^1}(1)$, and the only exceptional divisor associated to an extremal ray of $\mathrm{NE}(\sigma)$ that is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ is $\widetilde{D}$, which in this case has normal bundle $\mathcal{O}(-1,-1)$.

Let $X = X_A$. By Rem. 3.18 and Lemma 3.19, $\sigma\colon X \to T$ has exactly two different factorisations as in Construction A, through $\mathbb{P}_T(\mathcal{O}(D_1) \oplus \mathcal{O}(D_2) \oplus \mathcal{O}(D_3))$ and $\mathbb{P}_T(\mathcal{O}(-D_1) \oplus \mathcal{O}(-D_2) \oplus \mathcal{O}(-D_3))$. Moreover $\mathcal{O}_T(D_1), \mathcal{O}_T(D_2), \mathcal{O}_T(D_3)$ are determined up to reordering and tensoring with a line bundle.

We may assume that $\mathcal{O}_T(D_1) \cong \mathcal{O}_T(a)$, $\mathcal{O}_T(D_2) \cong \mathcal{O}_T$, and $\mathcal{O}_T(D_3) \cong \mathcal{O}_T(b)$, with $a, b \in \mathbb{Z}$. By Lemma 3.9, we see that (a, b) must satisfy the following conditions:

$$|a|,\ |b|,\ |a-b| \leq i_T - 1. \tag{3.3}$$

We say that two pairs of integers (a, b) and (a', b') are equivalent (denoted by $\sim$) if they satisfy (3.3) and give isomorphic X_A's. By the previous discussion, we see that all the pairs equivalent to (a, b) can be obtained by the following relations:

$$(a, b) \sim (b, a),\ (-b, a-b),\ (-a, -b).$$

Hence up to equivalence we can subsequently assume:

- $a \geq 0$: indeed if $a < 0$, we replace (a, b) with $(-a, -b)$;
- $a \geq b$: if $a < b$, we replace (a, b) with (b, a);
- $b \leq 0$: if $b > 0$, we replace (a, b) with $(a - b, -b)$;
- $a \geq -b$: if $a < -b$, we replace (a, b) with $(-b, -a)$,

so that in the end: $b \leq 0$ and $a \geq |b|$. These conditions, together with (3.3), are equivalent to the conditions in the statement:

$$b \leq 0, \quad |b| \leq \frac{i_T - 1}{2}, \quad \text{and} \quad |b| \leq a \leq i_T - 1 - |b|. \tag{3.4}$$

Finally it is not difficult to see that if $(a, b), (a', b')$ satisfy (3.4) and $(a, b) \sim (a', b')$, then $a = a'$ and $b = b'$.

Let $X = X_B$. By Rem. 3.21, Prop. 3.22, and Rem. 3.23, $\sigma\colon X \to T$ has exactly three different factorisations as in Construction B, all through $Z = \mathbb{P}_T(\mathcal{O}(N) \oplus \mathcal{O} \oplus \mathcal{O})$, where $N \not\equiv 0$ and $2N$ is effective. Therefore $\mathcal{O}_T(N)$ is ample and isomorphic to $\mathcal{O}_T(a)$ for some integer $a \geq 1$. By Lemma 3.14 we see that a must satisfy $a \leq i_T - 1$. ∎

Proof of Cor. 3.7. We have $\delta_X \geq \operatorname{codim} \mathcal{N}_1(D, X) = \rho_X - 2$. If $\rho_X \geq 6$, then $\delta_X \geq 4$, so that by Th. 3.1 we have $X \cong S \times T$ where S is a del Pezzo surface with $\rho_S = \delta_X + 1 \geq 5$. If $D_T \subset T$ is a prime divisor then $\dim \mathcal{N}_1(S \times D_T, X) \geq \rho_S$, therefore D must dominate T under the projection $\pi_T\colon S \times T \to T$. Moreover π_T is not finite on D, so that when we consider the pushforward $(\pi_T)_*\colon \mathcal{N}_1(S \times T) \to \mathcal{N}_1(T)$, we have $\ker(\pi_T)_* \cap \mathcal{N}_1(D, X) \neq \{0\}$, and we conclude that $\rho_T = 1$.

Suppose that $\rho_X = 5$, and note that the assumptions imply that $\dim X \geq 3$. If $\delta_X > \rho_X - 2$, then $\delta_X = \rho_X - 1$, which means that X contains a prime divisor D' with $\dim \mathcal{N}_1(D', X) = 1$. By [**CD15**, Lemma 3.1] this would imply that $\rho_X \leq 3$, a contradiction. Therefore $\delta_X = \rho_X - 2 = 3$, and X is as in Prop. 3.6. ∎

Proof of Th. 3.8. $(ii) \Rightarrow (i)$ In $Z = \mathbb{P}_T(\mathcal{O}(a) \oplus \mathcal{O} \oplus \mathcal{O})$ let S_2, S_3 be the sections of $Z \to T$ corresponding to the projections onto the trivial summands, and $F_1 := \mathbb{P}_T(\mathcal{O} \oplus \mathcal{O}) \hookrightarrow Z$ given by the projection $\mathcal{O}(a) \oplus \mathcal{O} \oplus \mathcal{O} \twoheadrightarrow \mathcal{O} \oplus \mathcal{O}$ (this is D in the

notation of Construction B), so that $S_2, S_3 \subset F_1$. Then, in both Constructions A and B, X is the blow-up of Z along S_2, S_3, and a third subvariety S_1 disjoint from F_1.

If $E_1' \subset X$ is the transform of F_1, we have $E_1' \cong \mathbb{P}^1 \times T$ with normal bundle $\mathcal{N}_{E_1'/X} \cong \pi_{\mathbb{P}_1}^*\mathcal{O}(-1) \otimes \pi_T^*\mathcal{O}(-a)$. Let $\Gamma_0 \subset T$ be a curve, and let $\Gamma \subset E_1'$ be the curve corresponding to $\{pt\} \times \Gamma_0$. We show that $[\Gamma]$ generates an extremal ray of $\mathrm{NE}(X)$; the associated contraction $\tau\colon X \to X'$ is elementary divisorial with $\mathrm{Exc}(\tau) = E_1'$ and $\tau(\mathrm{Exc}(\tau)) \cong \mathbb{P}^1$.

Set $c := \mathcal{O}_T(1) \cdot \Gamma_0$, and consider $e_1' \subset E_1'$ corresponding to $\mathbb{P}^1 \times \{pt\}$. We have:

$$E_1' \cdot e_1' = -1,\ -K_X \cdot e_1' = 1,\ E_1' \cdot \Gamma = -ac,\ -K_X \cdot \Gamma = c(i_T - a).$$

If $2a > i_T$, consider $H := a(-K_X) + (i_T - a)E_1'$. Then $H \cdot C \geq 0$ for every curve $C \subset X$ not contained in E_1'. Moreover $H \cdot e_1' = 2a - i_T > 0$ and $H \cdot \Gamma = 0$, so that H is nef and $H^\perp \cap \mathrm{NE}(X) = \mathbb{R}^+[\Gamma]$.

If instead $2a \leq i_T$, consider the exceptional divisor $E_2 \cong \mathbb{P}_T(\mathcal{O} \oplus \mathcal{O}(a))$ over S_2, and $e_2 \subset E_2$ a fibre of the $\mathbb{P}^1$-bundle $E_2 \to T$. We have:

$$E_2 \cdot e_2 = -1,\ E_2 \cdot e_1' = 1,\ E_2 \cdot \Gamma = 0,\ -K_X \cdot e_2 = 1,\ E_1' \cdot e_2 = 1,$$

and $E_2 \cdot R \geq 0$ for every extremal ray R of $\mathrm{NE}(X)$ not containing $[e_2]$. Consider $H' := a(-K_X) + (i_T - a)E_1' + (i_T - 2a + 1)E_2$. We have $H' \cdot e_1' = 1$, $H' \cdot \Gamma = 0$, $H' \cdot e_2 = 2a - 1 > 0$, and $H' \cdot R > 0$ for every extremal ray of $\mathrm{NE}(X)$ not containing $[e_1'], [\Gamma], [e_2]$. We conclude that H' is nef and $(H')^\perp \cap \mathrm{NE}(X) = \mathbb{R}^+[\Gamma]$.

(i) ⇒ (ii) If $n = 4$ the statement is shown in [**CR22**, Cor. 1.4], so that we can assume $n \geq 5$.

Consider the pushforward $\tau_*\colon \mathcal{N}_1(X) \to \mathcal{N}_1(X')$. We have $\tau_*(\mathcal{N}_1(\mathrm{Exc}(\tau), X)) = \mathbb{R}[\tau(\mathrm{Exc}(\tau))]$, thus $\dim \mathcal{N}_1(\mathrm{Exc}(\tau), X) = 2$, and it follows from Cor. 3.7 that X is as described in Prop. 3.6. We note first of all that $X \ncong F \times T$ (F the blow-up of $\mathbb{P}^2$ at three non-collinear points), as $F \times T$ has no elementary divisorial contraction sending a divisor to a curve, hence the case $X = X_A$ with $a = b = 0$ is excluded, and $i_T > 1$.

Let $\Gamma \subset X$ be a curve contracted by τ. We show that at least one of the exceptional divisors of the extremal rays of $\mathrm{NE}(\sigma)$ has non-zero intersection with Γ. We keep the same notation as in Rem. 1.18 and Sections 3.2 and 3.3.

By contradiction, suppose otherwise: then the classes of these divisors in $\mathcal{N}^1(X)$ all belong to the hyperplane $\Gamma^\perp$. This in turn implies in $\mathcal{N}^1(Z)$ that, if $X = X_A$ (respectively $X = X_B$), the linear span of the classes of F_1, F_2, F_3 (respectively the classes of D and H_0) has dimension 1. This is possible only if $X = X_A$ and $a = b = 0$, that we have already excluded.

Hence there exists some extremal ray of $\mathrm{NE}(\sigma)$ such that the associated exceptional divisor E satisfies $E \cdot \Gamma \neq 0$. We show that $E = \mathrm{Exc}(\tau)$.

With the same notation as in the proof of Prop. 3.6, E is a $\mathbb{P}^1$-bundle over S, where $S \cong T$ or $S \cong S_1$, and $\rho_S = 1$. If $T_0 \subset \mathrm{Exc}(\tau)$ is a non-trivial fibre of τ, we have $E \cap T_0 \neq \emptyset$ and hence $\dim(E \cap T_0) \geq n - 3 > 0$, and the Stein factorisation of $\tau_{|E}$ induces a non-trivial contraction $\psi\colon E \to B$. Since E meets every non-trivial fibre T_0 of τ, $\psi(\mathrm{Exc}(\psi))$ is a curve, and by the analysis of the elementary contractions of E in the proof of Prop. 3.6 we conclude that $E \cong \mathbb{P}^1 \times S$ and ψ is the projection onto $\mathbb{P}^1$. This means that $\dim \tau(E) = 1$, hence $E = \mathrm{Exc}(\tau)$.

If $X = X_A$, it is not difficult to check that $\mathbb{P}^1 \times T$ appears among the divisors $E_1, E_2, E_3, E_1', E_2', E_3'$ if and only if $b = 0$, so that $Z = \mathbb{P}_T(\mathcal{O}(a) \oplus \mathcal{O} \oplus \mathcal{O})$. Then

we have $E_1 \cong E_1' \cong \mathbb{P}^1 \times T$ with normal bundles $\mathcal{N}_{E_1/X} \cong \pi_{\mathbb{P}_1}^*\mathcal{O}(-1) \otimes \pi_T^*\mathcal{O}(a)$ and $\mathcal{N}_{E_1'/X} \cong \pi_{\mathbb{P}_1}^*\mathcal{O}(-1) \otimes \pi_T^*\mathcal{O}(-a)$. Since $a > 0$, we have $\mathrm{Exc}(\tau) = E_1'$, and we get the statement.

If $X = X_B$, we show that $S \cong T$. Indeed, if $S \cong S_1$, E should be one of the divisors E_1, G_2, G_3 (we know the exceptional divisors corresponding to the extremal rays of $\mathrm{NE}(X)$, see Rem. 3.21 and Rem. 3.23), and by Prop. 3.22 we have $E_1 \cong G_2 \cong G_3 \cong \mathbb{P}_{S_1}(\mathcal{O} \oplus \mathcal{O}(H_{|S_1}))$. On the other hand $H_{|S_1}$ is linearly equivalent to the ramification divisor of the non-trivial double cover $\varphi_{|S_1} \colon S_1 \to T$ by Rem. 3.13, so that $H_{|S_1} \not\sim 0$. Therefore we get $S \cong T$, and it is not difficult to check that $\mathrm{Exc}(\tau) = \widetilde{D}$. ■

3.8. Base locus of $-K_X$

Let X be a smooth Fano variety with $\delta_X = 3$. In this section we describe the base locus of the anticanonical divisor $-K_X$, and treat the cases X_A and X_B separately. We keep the same notation as in Sections 3.2 and 3.3.

We follow the same reasoning of Section 2.5. That is, if $h^0(X, -K_X) = 0$ [1], then $\mathrm{Bs}\,|-K_X| = X$. Otherwise, we find special (not necessarily) effective divisors linearly equivalent to $-K_X$ and apply Lemma 1.1 in order to limit $\mathrm{Bs}\,|-K_X|$ in a subvariety X' of X. Then, we study the restriction morphisms of $-K_X$ to the irreducible components Y' of X': if the restriction at the level of global sections is surjective, then $\mathrm{Bs}\,|-K_X| \cap Y' = \mathrm{Bs}\,|-K_{X|Y'}|$ by Lemma 1.2.

It turns out that the description of $\mathrm{Bs}\,|-K_X|$ is quite complex, and it is not possible to give simple statements in such a generality. We will relate the geometry of $\mathrm{Bs}\,|-K_X|$ to the exceptional divisors of the elementary divisorial contractions given by $\mathrm{NE}(\sigma)$, and to the base locus of the ample divisors $-K_T$, $-K_T + D_i - D_j$ for Construction A and $-K_T \pm N$ for Construction B.

3.8.1. Anticanonical base locus for X_A.

Remark 3.28. It follows from Construction A that the exceptional divisors E_i, E_i' satisfy, for $i, j \in \{1, 2, 3\}$,

$$E_i + E_j' \sim E_j + E_i' + \sigma^*(D_i - D_j).$$

Therefore, $-K_X$ can be written as

$$\begin{aligned} -K_X &\sim \sigma^*(-K_T) + \sum_{i=1}^{3}(E_i + E_i') \\ &\sim \sigma^*(-K_T + D_i - D_j) + 2(E_i' + E_j) + E_k + E_k', \quad i \neq j \neq k. \end{aligned}$$

Before proceeding, we fix the following notation: $B_{i,j} := \sigma^{-1}\,\mathrm{Bs}\,|-K_T + D_i - D_j|$, $E_{i,j} := E_j \cup E_k \cup E_i' \cup E_k'$ and $S_{i,j} := E_i \cap E_j'$ for $i \neq j \neq k$, and $B := \sigma^{-1}\,\mathrm{Bs}\,|-K_T|$, while we denote by E the union of all six exceptional divisors E_i, E_i', $i = 1, 2, 3$. Then,

[1] If the dimension of a Fano manifold X is low, then $h^0(X, -K_X) > 0$: this follows from Riemann-Roch for $\dim(X) \leq 3$ and from [**Kaw00**] for $\dim(X) = 4$. Although the non-vanishing of $h^0(X, -K_X)$ is believed to hold in general, it is still unknown in higher dimension.

by Lemma 1.1, we obtain

$$\operatorname{Bs}|-K_X| \subseteq \left(\bigcap_{i\neq j}(B_{i,j}\cup E_{i,j})\right)\cap(B\cup E)\,.$$

Moreover, it is not difficult to show that

$$\bigcap_{i\neq j,\ i\neq a} E_{i,j} = E_a, \qquad \bigcap_{i\neq j,\ j\neq b} E_{i,j} = E'_b, \qquad \bigcap_{i\neq j,\ (i,j)\neq(a,b)} E_{i,j} = S_{a,b},$$

thus we obtain the inclusion

$$\operatorname{Bs}|-K_X| \subseteq \left(B\cap\bigcap_{i\neq j} B_{i,j}\right)\cup E. \tag{3.5}$$

By the above Remark, we understand that the base locus of $-K_X$ is contained either in the exceptional divisors E_i, E'_i, or in the fibres over the points in the intersection $\bigcap_{i,j=1,2,3}\operatorname{Bs}|-K_T+D_i-D_j|$ (here $i=j$ is admitted). Lastly, $S_{i,j}$ is a section of both E_i and E'_j, and as a section of $E_i\cong\mathbb{P}_T(\mathcal{O}(-K_T+D_i-D_j)\oplus\mathcal{O}(-K_T+D_i-D_k))$, for instance, it corresponds to the projection onto the first summand.

The following results are stated for a fixed index, as the statements and proofs for the remaining cases are symmetrical.

Lemma 3.29. *Let X be a smooth Fano variety with $\delta_X=3$ obtained via Construction A. Then*

$$\operatorname{Bs}|-K_X|\cap E_1 = (B_{1,2}\cap S_{1,2})\cup(B_{1,3}\cap S_{1,3})\cup\big(B_{1,2}\cap B_{1,3}\cap E_1\big).$$

Proof. Consider the exact sequence

$$0\to\mathcal{O}_X(-K_X-E_1)\to\mathcal{O}_X(-K_X)\to\mathcal{O}_{E_1}(-K_{X|E_1})\to 0.$$

The divisor $-K_X-E_1$ can be written as $K_X+(-K_X+\sigma^*(-K_T))+(E-E_1)$, where $(-K_X+\sigma^*(-K_T))$ is ample and $(E-E_1)$ is simple normal crossing. Therefore $H^1(X,-K_X-E_1)=0$ by Norimatsu's lemma [**Laz04a**, Lemma 4.3.5], so that $\operatorname{Bs}|-K_X|\cap E_1=\operatorname{Bs}|-K_{X|E_1}|$.

By Lemma 1.21, we have that $-K_{X|E_1}$ is the tautological divisor of $E_1\cong\mathbb{P}_T(\mathcal{O}(-K_T+D_1-D_2)\oplus\mathcal{O}(-K_T+D_1-D_3))$, and we apply Lemma 1.19 to conclude. ∎

In other words, if $\sigma_1\colon E_1\to T$ is the $\mathbb{P}^1$-bundle obtained by restricting $\sigma\colon X\to T$ to E_i, then $\operatorname{Bs}|-K_X|$ intersects E_1 exactly at the points of the sections $S_{1,2}$, $S_{1,3}$ corresponding respectively to the points of the base loci $\operatorname{Bs}|-K_T+D_1+D_2|$, $\operatorname{Bs}|-K_T-D_1+D_3|$, and at the fibres of σ_1 over $\operatorname{Bs}|-K_T+D_1+D_2|\cap\operatorname{Bs}|-K_T-D_1+D_3|$.

Similarly, we have

Lemma 3.30. *Let X be a smooth Fano variety with $\delta_X=3$ obtained via Construction A. Then*

$$\operatorname{Bs}|-K_X|\cap E'_1 = (B_{2,1}\cap S_{2,1})\cup(B_{3,1}\cap S_{3,1})\cup\big(B_{2,1}\cap B_{3,1}\cap E'_1\big).$$

Corollary 3.31. *Let X be a smooth Fano variety with $\delta_X=3$ obtained via Construction A. Then:*

(i) $S_{1,2}\subset\operatorname{Bs}|-K_X| \iff h^0(T,-K_T+D_1-D_2)=0$;

(ii) *E_1 is a fixed component of* $|-K_X| \iff h^0(T,-K_T+D_1-D_j)=0$ *for* $j=2,3$;

(iii) *if* $h^0(T,-K_T+D_i-D_j)=0$ *for all* $i,j=1,2,3$, *then* $H^0(X,-K_X)\cong H^0(T,-K_T)$ *and* $\mathrm{Bs}\,|-K_X|=E\cup B$;
(iv) $h^0(X,-K_X)=0 \iff h^0(T,-K_T)=0$ *and* $h^0(T,-K_T+D_i-D_j)=0$ *for all* $i,j=1,2,3$.

Proof. At first we observe that items (i) and (ii) follow from Lemma 3.29. Moreover, if we assume $h^0(X,-K_X)=0$, then $h^0(T,-K_T+D_i-D_j)=0$ for all $i,j=1,2,3$ by (ii). Therefore, (iv) follows from (iii).

As for (iii), by (3.5) and (ii) it suffices to show that $B\subseteq \mathrm{Bs}\,|-K_X|$. In fact, consider the exact sequence of sheaves

$$0\to\mathcal{O}_X(-K_X-E_1)\to\mathcal{O}_X(-K_X)\to\mathcal{O}_{E_1}(-K_{X|E_1})\to 0,$$

which gives the exact sequence in cohomology

$$0\to H^0(X,-K_X-E_1)\to H^0(X,-K_X)\to H^0(E_1,-K_{X|E_1}).$$

We deduce from the hypothesis and Lemma 1.21 that $h^0(E_1,-K_{X|E_1})=0$, so that $H^0(X,-K_X-E_1)\cong H^0(X,-K_X)$. In the same fashion, consider the exact sequence

$$0\to H^0(X,-K_X-E_1-E_2)\to H^0(X,-K_X-E_1)\to H^0(E_2,(-K_X-E_1)_{|E_2}).$$

Since $E_1\cap E_2=\emptyset$, we have $H^0(E_2,(-K_X-E_1)_{|E_2})=H^0(E_2,-K_{X|E_2})$. Therefore, as above, $h^0(E_2,-K_{X|E_2})=0$ and it follows that $H^0(X,-K_X-E_1-E_2)\cong H^0(X,-K_X-E_1)$. Similarly we obtain $H^0(X,-K_X-E_1-E_2-E_3)\cong H^0(X,-K_X-E_1-E_2)$.

Now we restrict to the divisors E_i'. Take the exact sequence

$$0\to H^0(X,-K_X-\sum_{i=1}^3E_i-E_1')\to H^0(X,-K_X-\sum_{i=1}^3E_i)\to H^0(E_1',\big(-K_X-\sum_{i=1}^3E_i\big)_{|E_1'})$$

and recall that the divisor E_1' intersects E_2 and E_3 in $S_{2,1}$ and $S_{3,1}$ respectively, while it is disjoint from E_1. Therefore, $\sum_{i=1}^3E_{i|E_1'}\sim S_{2,1}+S_{3,1}$, and we want to show that the sheaf $\mathcal{O}_{E_1'}(-K_{X|E_1'}-S_{2,1}-S_{3,1})$ has no global sections. Since $-K_{X|E_1'}$ is the tautological divisor of $E_1'\cong\mathbb{P}_T(\mathcal{O}(-K_T+D_2-D_1)\oplus\mathcal{O}(-K_T+D_3-D_1))$, it is not difficult to show that $K_{X|E_1'}+S_{2,1}+S_{3,1}$ is the tautological divisor of $\mathbb{P}_T(\mathcal{O}(K_T+D_1-D_2)\oplus\mathcal{O}(K_T+D_1-D_3))$. Therefore $H^0(E_1',-K_{X|E_1'}-S_{2,1}-S_{3,1})=0$ by [**Har77**, Exercise III.8.4(a)], yielding $H^0(X,-K_X-\sum_{i=1}^3E_i-E_1')\cong H^0(X,-K_X-\sum_{i=1}^3E_i)$. Finally, we can repeat the same reasoning to the subtraction of E_2' and E_3' and we obtain $H^0(X,-K_X)\cong H^0(X,-K_X-E)$. To conclude the proof, recall that $-K_X-E\sim\sigma^*(-K_T)$ and observe that $H^0(X,\sigma^*(-K_T))\cong H^0(T,-K_T)$. ■

It remains to understand the behaviour of $\mathrm{Bs}\,|-K_X|$ along the fibres over the intersection $\bigcap_{i,j=1,2,3}\mathrm{Bs}\,|-K_T+D_i-D_j|$.

Lemma 3.32. *Let* X *be a smooth Fano variety with* $\delta_X=3$ *obtained via Construction A. Assume that there exists a point* $t\in\bigcap_{i,j=1,2,3}\mathrm{Bs}\,|-K_T+D_i-D_j|$ *and let* X_t *be the fibre over* t. *Then either* $X_t\subset\mathrm{Bs}\,|-K_X|$ *or* $\mathrm{Bs}\,|-K_X|\cap X_t=E\cap X_t$.

Proof. Recall that the σ-fibre X_t is a del Pezzo surfaces of Picard number 4, the blow-up of $\mathbb{P}^2$ along 3 non-collinear points. Consider the restriction morphism in the exact sequence on cohomology

$$0\to H^0(X,-K_X\otimes\mathcal{I}_{X_t/X})\to H^0(X,-K_X)\to H^0(X_t,-K_{X_t})\cong\mathbb{C}^7:$$

it cannot be surjective, since $-K_{X_t}$ is globally generated and $\operatorname{Bs}|-K_X| \cap X_t \neq \emptyset$ by Corollary 3.31.

If the restriction morphism is trivial, then $X_t \subset \operatorname{Bs}|-K_X|$. Otherwise, it follows from Lemma 3.29 that $\operatorname{Bs}|-K_X| \cap X_t$ contains all six exceptional curves of X_t, whose sum yields an effective divisor in $|-K_{X_t}|$. Therefore, the image of the restriction morphism has dimension 1 and $\operatorname{Bs}|-K_X| \cap X_t$ is exactly the union of the exceptional curves of X_t.

■

3.8.2. Anticanonical base locus for X_B.

Remark 3.33. It follows from Construction B and Proposition 3.22 that the exceptional divisors E_1, E_2, E_3, $\widetilde{H}_0$, $\widetilde{D}$, G_2, G_3 satisfy

$$\widetilde{H}_0 + E_1 - \sigma^* N \sim \widetilde{D} + E_2 + E_3; \qquad E_2 + G_2 - \sigma^* N \sim \widetilde{D} + \widetilde{H}_0 + E_3;$$
$$E_3 + G_3 - \sigma^* N \sim \widetilde{D} + \widetilde{H}_0 + E_2.$$

Recall that $\sigma\colon X \to T$ has three factorisations of the form $X \xrightarrow{h} Z \xrightarrow{\varphi} T$, where $h\colon X \to Z$ is the contraction of the exceptional divisors $\{E_1, E_2, E_3\}$, $\{G_2, \widetilde{H}_0, E_3\}$ or $\{G_3, \widetilde{H}_0, E_2\}$ (see Proposition 3.22). Moreover, the exceptional divisors E_2, E_3, $\widetilde{H}_0$ and $\widetilde{D}$ are $\mathbb{P}^1$-bundles over T, while E_1, G_2 and G_3 are $\mathbb{P}^1$-bundles over S_1, where $S_1 \subset Z$ is a double cover over T with branch divisor $\Delta \sim 2N$. Therefore, $-K_X$ can be written as

$$\begin{aligned} -K_X &\sim \sigma^*(-K_T + N) + \widetilde{H}_0 + 2\widetilde{D} + E_2 + E_3 \\ &\sim \sigma^*(-K_T) + 2\widetilde{H}_0 + \widetilde{D} + E_1 \sim \sigma^*(-K_T) + 2E_2 + \widetilde{D} + G_2 \\ &\sim \sigma^*(-K_T) + 2E_3 + \widetilde{D} + G_3. \end{aligned}$$

Set $B := \sigma^{-1}\operatorname{Bs}|-K_T|$, $B_+ := \sigma^{-1}\operatorname{Bs}|-K_T+N|$ and $B_- := \sigma^{-1}\operatorname{Bs}|-K_T-N|$. Then, by Lemma 1.1, we obtain

$$\begin{aligned} \operatorname{Bs}|-K_X| &\subseteq \left(\widetilde{H}_0 \cup \widetilde{D} \cup E_2 \cup E_3 \cup B_+\right) \cap \left(\widetilde{H}_0 \cup \widetilde{D} \cup E_1 \cup B\right) \cap \\ &\quad \cap \left(E_2 \cup \widetilde{D} \cup G_2 \cup B\right) \cap \left(E_3 \cup \widetilde{D} \cup G_3 \cup B\right) \\ &\subseteq \left(\widetilde{H}_0 \cup \widetilde{D} \cup E_2 \cup E_3 \cup B_+\right) \cap \left(\widetilde{D} \cup B\right). \end{aligned}$$

Eventually we get

$$\operatorname{Bs}|-K_X| \subseteq \widetilde{D} \cup \widetilde{H}_0 \cup E_2 \cup E_3 \cup (B \cap B_+). \tag{3.6}$$

Therefore, similarly to the case of X_A, the base locus of $-K_X$ is contained either in the exceptional divisors or in the fibres over the intersection $\operatorname{Bs}|-K_T| \cap \operatorname{Bs}|-K_T+N|$. We have that:

(i) $E_2 \cong \mathbb{P}_T(\mathcal{O}_T(-K_T) \oplus \mathcal{O}_T(-K_T-N))$ and $-K_{X|E_2}$ is the tautological divisor; the same holds for E_3 and $\widetilde{H}_0$.
(ii) $\widetilde{D} \cong \mathbb{P}_T(\mathcal{O}_T(-K_T-N) \oplus \mathcal{O}_T(-K_T-N))$ and $-K_{X|\widetilde{D}}$ is the tautological divisor.
(iii) $E_1 \cong \mathbb{P}_{S_1}(\varphi^*_{|S_1}\mathcal{O}_T(-K_T+N) \oplus \varphi^*_{|S_1}\mathcal{O}_T(-K_T))$ and $-K_{X|E_1}$ is the tautological divisor; the same holds for G_2 and G_3.

Items (i) and (ii) follow from Lemmata 1.21 and 1.22, and (iii) follows similarly. Note that the conormal bundle of S_1 in Z is $\mathcal{O}_{S_1}(-H) \oplus \mathcal{O}_{S_1}(-2H)$, $H_{S_1} \sim \varphi^*_{|S_1} N$, and $E_1 \cap \widetilde{H}_0$ is the section of E_1 corresponding to the projection onto the second summand.

Therefore, to understand the intersection of $\mathrm{Bs}\,|-K_X|$ and the exceptional divisors we proceed as in the case of Construction A: Lemma 1.19 describes the base locus of these tautological divisors, and we show that the restriction of $-K_X$ to the exceptional divisors is surjective on the global sections. This follows from the Lemma below. Note that by Proposition 1.32 we have

$$\mathrm{Bs}\,|\varphi^*_{|S_1}(-K_T+N)| = \varphi^{-1}_{|S_1}\Big(\mathrm{Bs}\,|-K_T+N| \cap \big(\Delta \cup \mathrm{Bs}\,|-K_T|\big)\Big);$$
$$\mathrm{Bs}\,|\varphi^*_{|S_1}(-K_T)| = \varphi^{-1}_{|S_1}\Big(\mathrm{Bs}\,|-K_T| \cap \big(\Delta \cup \mathrm{Bs}\,|-K_T-N|\big)\Big),$$

where Δ is the branching divisor of the double covering $\varphi_{|S_1}\colon S_1 \to T$.

Lemma 3.34. *Let X be a smooth Fano variety with $\delta_X = 3$ obtained via Construction B, and let $\widehat{E}$ be any of the exceptional divisors in Remark 3.33. Then $H^1(-K_X - \widehat{E}) = 0$.*

Proof. By Remark 3.33, it is not difficult to verify that $-2K_X - \widetilde{D}$ is the sum of an ample divisor and a simple normal crossing divisor, so that we can apply Norimatsu's lemma [**Laz04a**, Lemma 4.3.5]. This holds similarly for $\widetilde{H}_0$, E_2 and E_3.

As for E_1, we show that $-2K_X - E_1$ is nef and big. Then $H^1(X, -K_X - E_1) = 0$ by Kawamata-Viehweg vanishing (see [**Laz04a**, Th. 4.3.1]). This holds similarly for G_2 and G_3.

The restriction of $(-2K_X - E_1)_{|\widetilde{H}_0}$ is the nef divisor $2\sigma^*_{|\widetilde{H}_0}(-K_T - N)$, while the restriction $(-2K_X - E_1)_{|\widetilde{D}}$ is the ample divisor $-2K_{X|D}$. Let Γ be an irreducible curve not contained in $\widetilde{H}_0$ nor $\widetilde{D}$. Then,

$$(-2K_X - E_1)\cdot\Gamma = -K_X\cdot\Gamma + -K_T\cdot\sigma_*\Gamma + (2\widetilde{H}_0 + \widetilde{D})\cdot\Gamma \geq 0$$

as the first summand is strictly positive and the other summands are non-negative. Thus $-2K_X - E_1$ is nef.

To show that $-2K_X - E_1$ is big, we follow the same reasoning as in Lemma 3.14. The divisor $A_Z \sim H_0 + \varphi^*(-K_T - N) \sim D + \varphi^*(-K_T)$ is ample on Z. So, there exist non-negative integers m, c_i such that the divisor

$$\begin{aligned} A_X &:= mh^*(A_Z) - \sum_{i=1}^{3} c_i E_i \\ &= m\big(\widetilde{D} + E_2 + E_3 + \sigma^*(-K_T)\big) - \sum_{i=1}^{3} c_i E_i \\ &= m\sigma^*(-K_T) + m\widetilde{D} - c_1E_1 + (m-c_2)E_2 + (m-c_3)E_3 \end{aligned}$$

is ample on X. We have

$$\begin{aligned} -mK_X &\sim m\big(\sigma^*(-K_T+N) + \widetilde{H}_0 + 2\widetilde{D} + E_2 + E_3\big) \\ &\sim m\big(\sigma^*(-K_T) + 2\widetilde{H}_0 + \widetilde{D} + E_1\big) \\ &\sim A_X + 2m\widetilde{H}_0 + (m+c_1)E_1 + (c_2-m)E_2 + (c_3-m)E_3, \end{aligned}$$

and finally

$$-2mK_X - mE_1 \sim \big(A_X + m\sigma^*(-K_T)\big) + 3m\widetilde{H}_0 + 2m\widetilde{D} + \sum_{i=1}^{3} c_i E_i.$$

We conclude that $-2K_X - E_1$ is big. ■

Finally, we have a similar result to Corollary 3.31.

Corollary 3.35. *Let X be a smooth Fano variety with $\delta_X = 3$ obtained via Construction B. Then:*

(i) $\widetilde{D}$ is a fixed component of $|-K_X| \iff h^0(T, -K_T - N) = 0$;
(ii) $\widetilde{H}_0$, E_2 and E_3 are a fixed component of $|-K_X| \iff h^0(T, -K_T - N) = 0$ and $h^0(T, -K_T) = 0$;
(iii) E_1, G_2 and G_3 are a fixed component of $|-K_X| \iff h^0(T, -K_T - N) = 0$, $h^0(T, -K_T) = 0$ and $h^0(T, -K_T + N) = 0$;
(iv) if $h^0(X, -K_X) = 0$, then $h^0(T, -K_T - N) = h^0(T, -K_T) = h^0(T, -K_T + N) = 0$.

Proof. Similarly as in the proof of Corollary 3.31, the statements follow from the above discussion. ■

Although we have followed so far the description of $\operatorname{Bs}|-K_X|$ for Construction A to describe $\operatorname{Bs}|-K_X|$ in the case of Construction B, it remains unclear what happens on the fibres over the intersection $\operatorname{Bs}|-K_T| \cap \operatorname{Bs}|-K_T+N|$.

3.8.3. Examples with non-empty base locus. In this subsection we present some examples of Fano manifolds X with $\delta_X = 3$ and $\operatorname{Bs}|-K_X| \neq \emptyset$. By the following Proposition we see that this does not happen in low dimensions.

Proposition 3.36. *Let X be a Fano manifold with $\delta_X = 3$ and dimension ≤ 4. Then $-K_X$ is globally generated.*

Proof. If $\dim(X) = 2$, then X is the del Pezzo surface of Picard number 4, thus the statement is trivial.

If $\dim(X) \in \{3, 4\}$, then T is either $\mathbb{P}^1$ or a del Pezzo surface of Picard number $\in \{1, 2, 3, 4\}$. It follows that the ample divisors $-K_T + D_i - D_j$ for Construction A and $-K_T \pm N$ for Construction B (see Proposition 3.25 for $\dim(X) = 4$) are globally generated. Therefore $-K_X$ is globally generated by the discussion in Subsections 3.8.1 and 3.8.2. ■

Therefore, in order to find non-trivial examples, we must look at dimension ≥ 5.

Example 3.37. Let T_1 be the del Pezzo threefold Z_1 (see Table II.1). If $H \sim -\frac{1}{2}K_{T_1}$, then $\operatorname{Bs}|H| = \{t_0\}$ and $\operatorname{Bs}|kH| = \emptyset$ for all $k \geq 2$.

(i). Let $X = X_A$ be the Fano manifold with $\delta_X = 3$ and $\rho_X = 5$ determined by the pair $(a, b) = (1, 0)$ (see Prop. 3.6). Let X_0 be the fibre of $\sigma\colon X \to T_1$ over the point t_0. Then it follows from Lemma 3.30 that $\operatorname{Bs}|-K_X| = E_1' \cap X_0$. In particular $\operatorname{Bs}|-K_X| \cong \mathbb{P}^1$.

(ii). Let $X = X_B$ be the Fano manifold with $\delta_X = 3$ and $\rho_X = 5$ determined by the pair $a = 1$ (see Prop. 3.6). Let X_0 be the fibre of $\sigma\colon X \to T_1$ over the point t_0. Then by Remark 3.33 we have $\operatorname{Bs}|-K_X| = \widetilde{D} \cap X_0$, and again $\operatorname{Bs}|-K_X| \cong \mathbb{P}^1$.

Example 3.38. Let $T_2 \to T_1$ be the blow-up of T_1 along an elliptic curve $\Gamma = H_1 \cap H_2$, for general $H_i \in |-\frac{1}{2}K_{T_1}|$. Then T_2 is a Fano threefold of Picard number 2, and $\mathrm{Bs}\,|-K_{T_2}|$ is the fibre $f_0 \cong \mathbb{P}^1$ of the blow-up over t_0. It is not difficult to verify that the other elementary contraction of T_2 is a fibration in degree 1 del Pezzo surfaces, and the corresponding extremal ray has length 1. Therefore, Lemma 3.5.4 implies that the only Fano manifold X of $\delta_X = 3$ that can be obtained from T_2 is the product $S \times T_2$, where S id the blow-up of $\mathbb{P}^2$ at 3 non-collinear points. In this case, $\mathrm{Bs}\,|-K_X| = S \times f_0$.

Example 3.39. Let T_3 be the product $\mathbb{P}^1 \times T_2$. Then, $-K_{T_3} \sim \pi_1^*\mathcal{O}_{\mathbb{P}^1}(2) + \pi_2^*(-K_{T_2})$ and $\mathrm{Bs}\,|-K_{T_3}| = \mathbb{P}^1 \times f_0$, where π_1, π_2 are the projections. Let $L \sim \pi_1^*\mathcal{O}_{\mathbb{P}^1}(1)$ and $Z := \mathbb{P}_{T_3}(L \oplus \mathcal{O}^{\oplus 2})$. Since Z is the projectivization of the pull-back of a vector bundle over $\mathbb{P}^1$, we have that $Z \cong \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}^{\oplus 2}) \times T_2$. Observe, moreover, that $\mathrm{Bs}\,|-K_{T_3}|$ coincides with $\mathrm{Bs}\,|-K_{T_3} \pm L|$.

(i). Let $X = X_A$ be the Fano manifold with $\delta_X = 3$ and $\rho_X = 7$ determined by $D_1 \sim L$ and $D_2 \sim D_3 \sim 0$. Then $X \cong X'_A \times T_2$, where X'_A is the Fano threefold with $\delta = 3$ obtained by $\mathbb{P}^1$ and determined by $D'_1 \sim \mathcal{O}_{\mathbb{P}^1}(1)$ and $D'_2 \sim D'_3 \sim 0$. In this case, $\mathrm{Bs}\,|-K_X| = X'_A \times f_0$.

(ii). Let $X = X_B$ be the Fano manifold with $\delta_X = 3$ and $\rho_X = 7$ determined by $N \sim L$. Then $X \cong X'_B \times T_2$, where X'_B is the Fano threefold with $\delta = 3$ obtained by $\mathbb{P}^1$ and determined by $N' \sim \mathcal{O}_{\mathbb{P}^1}(1)$. In this case, $\mathrm{Bs}\,|-K_X| = X'_B \times f_0$.

Example 3.40. Let T be the Fano fourfold $X^1_{0,1}$ (resp. $X^1_{1,2}$) of Theorem 2.1. Recall by Prop. 2.17 that $\mathrm{Bs}\,|-K_T| = \{t_0\}$ (resp. $\{t_1, t_2\}$) and observe that all its extremal rays have length 1, see Remark 2.4. Lemma 3.5.4 implies that the only Fano manifold X of $\delta_X = 3$ that can be obtained from T is the product $S \times T$. Therefore, $\mathrm{Bs}\,|-K_X| = S \times \{t_0\}$ (resp. $S \times \{t_1\} \cup S \times \{t_2\}$).

Remark 3.41. Lastly, let $\sigma\colon X \to T$ be a Fano manifold with $\delta_X = 3$ obtained as in Construction A or B, and assume $\dim(X) \geq 5$. To the author's knowledge there are not known examples that satisfy any of the properties in Corollaries 3.31 or 3.35.

In fact, since T is smooth and Fano, the Ambro-Kawamata conjecture[2] (see [**Amb99**] and [**Kaw00**]) implies that $h^0(T, H) > 0$ for any ample divisor H on T. If this holds, then we have

$$h^0(T, -K_T),\ h^0(T, -K_T + D_i - D_j),\ h^0(T, -K_T \pm N) > 0.$$

Therefore, if there exists a Fano manifold with $\delta_X = 3$ that satisfy some of the properties in Corollaries 3.31 or 3.35, it should be very special.

[2]In general, the Ambro-Kawamata conjecture states the following: let (X, Δ) be a klt pair and H be an ample Cartier divisor on X; if $H - (K_X + \Delta)$ is ample, then $h^0(X, H) > 0$.

CHAPTER 4

Anticanonical base locus of Fano manifolds

4.1. Introduction

Let X be a smooth, complex Fano variety of dimension $n \geq 2$. There are not many known examples where the base locus of the anticanonical system $\mathrm{Bs}\,|-K_X|$ is not empty. This chapter is based on an ongoing work with A. Höring and its goal is to study the relation between $\mathrm{Bs}\,|-K_X|$ and the birational geometry of X, at least in low dimension.

The background is as follows: let $\mu\colon X \to T$ be a an elementary Mori contraction. Then, in "many" cases the anticanonical bundle is relatively globally generated, i.e. the natural map

$$\mu^*\mu_*\mathcal{O}_X(-K_X) \longrightarrow \mathcal{O}_X(-K_X)$$

is surjective, see [**AW93**, Thm. 5.1] (or [**AW98**, Prop. 1.3.3] for this special case). Thus, if X_t is a φ-fibre, the restriction of $-K_X$ to X_t is globally generated. So it looks unlikely that a point of $\mathrm{Bs}\,|-K_X|$ is contained in such a fibre. But of course this is not true, since it is not clear if the restriction map

$$H^0(X, -K_X) \to H^0(X_t, -K_X)$$

is surjective (see Lemma 1.2): this is exactly the point that we have to understand. Let us start by recalling an important result:

Theorem 4.1 ([**Sho79**],[**Kaw00**] Thm. 5.2,[**HV11**] Thm. 1.7,[**Heu15**] Thm. 2)**.** *Let X be a smooth Fano variety of dimension at most four. Then a general element $Y \in |-K_X|$ is a prime divisor with at most terminal singularities.*

It is expected that this statement holds in general, so it is a reasonable assumption in arbitrary dimension. In particular, this shows that the linear system $|-K_X|$ does not have fixed part, so that $\mathrm{codim}\big(\mathrm{Bs}\,|-K_X|\big) \geq 2$. Moreover, if X is a threefold, then a general $Y \in |-K_X|$ is smooth.

Technical setup 4.2. Apart from Section 4.4, we will work under the following

Assumption. Let X be a Fano manifold of dimension $n \geq 3$ such that:

(*i*) $h^0(X, -K_X) \geq 3$;
(*ii*) there exists a subvariety $B \subset X$ of codimension 2 such that $B \subset \mathrm{Bs}\,|-K_X|$;
(*iii*) X has Picard number $\rho_X \geq 2$.

The first item holds for any Fano threefold (Lemma 1.17) and there are no known examples of Fano fourfolds with $h^0(X, -K_X) = 2$ (the minimal possible number, by [**Kaw00**]).

By classification, we know that there are only three examples in dimension ≤ 3 that satisfy assumption (ii): the del Pezzo surface of degree 1 and the Fano threefolds of

Examples 4.6 and 4.7. Starting from the above setup, the objective is to retrieve these examples using techniques that do not involve the classification of Fano manifolds, and then extend these results to higher dimension. In the long run, we hope to understand if we can describe the global geometry of anticanonical base loci and, more importantly, if there is a more systematic way to construct Fano manifolds with non-empty $\operatorname{Bs}|-K_X|$. In this chapter, we will treat the low-dimensional case.

The main result of this chapter is the following:

Theorem 4.3. *Let X be a smooth Fano threefold with $\rho_X \geq 2$ and let $\mu\colon X \to T$ be an elementary Mori contraction with fibres of dimension ≤ 1. Assume that there exists an irreducible curve $B \subseteq \operatorname{Bs}|-K_X|$, and that B intersects a non-trivial fibre of μ. Then μ contracts B to a point, T is a smooth Fano variety of index r, and $\mu(B)$ belongs to the base locus of $|-\frac{1}{r}K_T|$.*

To the author's awareness, the only known examples of Fano fourfolds with anticanonical base locus of dimension 2 are the varieties of Examples 4.11 and 4.12. Here we can already see a fundamental difference between these examples and the examples in dimension 3: a general anticanonical divisor is smooth in the latter, while in the former it is singular and *not* $\mathbb{Q}$-factorial.

The structure of this chapter is as follows. In Section 4.2 we collect some examples of Fano manifolds of dimension ≥ 3 with non-trivial anticanonical base locus. In Section 4.3 we provide the preliminaries for the following sections and some technical results. In Section 4.4 we treat the case of dimension 2, while the case of dimension 3 is discussed in Section 4.5.

4.2. Examples

We start by recalling the following useful fact:

Lemma 4.4. *Let X_1, X_2 be Fano manifolds, and let $X = X_1 \times X_2$ with natural projections $\pi_i\colon X \to X_i$. Then $\operatorname{Bs}|-K_X| = \pi_1^{-1}\operatorname{Bs}|-K_{X_1}| \cup \pi_2^{-1}\operatorname{Bs}|-K_{X_2}|$.*

Proof. This is a consequence of the Künneth formula, which gives a natural isomorphism $H^0(X, \pi_1^*(-K_{X_1}) + \pi_2^*(-K_{X_2})) \cong H^0(X_1, -K_{X_1}) \otimes_{\mathbb{C}} H^0(X_2, -K_{X_2})$. ■

Let us note that it is not true that $\operatorname{Bs}|-K_X|$ is disjoint from any fibre of an elementary Mori contraction:

Example 4.5. Let S be a del Pezzo surface of degree 1, and denote by p the unique basepoint of $|-K_S|$. Set $X := S \times S$. The base locus $\operatorname{Bs}|-K_X|$ is equal to $\{p\} \times S \cup S \times \{p\}$. Let now $S \to S'$ be the contraction of a (-1)-curve E, and denote by

$$\varphi\colon X \to S \times S'$$

the Mori contraction of $S \times E$. By Proposition 4.24, the curve E is disjoint from the basepoint p, so the intersection of the exceptional locus $S \times E$ with $S \times \{p\}$ is empty. However, the intersection with $\{p\} \times S$ is exactly the curve $\{p\} \times E$.

Note that all the contractions of $S \times S$ are of type $S \times S \to S' \times S''$, where $S \to S'$ and $S \to S''$ are contractions of S (see [**Rom19**, Lemma 2.10]). Thus if the base locus intersects the exceptional locus of some Mori contractions, the intersection is geometrically significant.

Thus we see that the intersection is not empty in general, but probably very special. We now collect some relevant examples.

Example 4.6. Let X be the Fano threefold T_2 of Example 3.38, that is the blow-up $\mu\colon X \to T_1$ of a del Pezzo threefold T_1 of degree 1 along an elliptic curve. Then X is a Fano threefold of Picard number 2 and $\operatorname{Bs}|-K_X|$ is a fibre of μ.

Example 4.7. Let X be the product $S \times \mathbb{P}^1$, where S is a del Pezzo surface of degree 1. Then X is a Fano threefold of Picard number 10 and $\operatorname{Bs}|-K_X|$ is a fibre of the projection $X \to S$.

Example 4.8. Let X be the Fano fourfold $X^1_{0,1}$ appearing in Chapter 2 (see Corollary 2.6). We keep the same notation as in Section 2.2. Then $\operatorname{Bs}|-K_X| = \{Q_0\}$ and:

(i) $Q_0 \in E$, $\widehat{G}$, while $Q_0 \notin \widehat{E}$;
(ii) $\operatorname{Bs}|-K_Y| = \emptyset$, $\operatorname{Bs}|-K_{\widehat{Y}}| = \widehat{\pi}^{-1}(P_0) \cap \widehat{\sigma}(\widehat{G})$;
(iii) $\operatorname{Bs}|-K_{X|E}| = \{Q_0\}$, $\operatorname{Bs}|-K_{X|\widehat{E}}| = \emptyset$, $\operatorname{Bs}|-K_{X|\widehat{G}}| = \{Q_0\}$;
(iv) let F, $\widehat{F}$ be non-trivial fibres of σ and $\widehat{\sigma}$. Then $-K_{X|F} \cong -K_{X|\widehat{F}} \cong \mathcal{O}_{\mathbb{P}^1}(1)$, which are globally generated.

A similar result holds for $X^1_{1,2}$. Thus $X^1_{0,1}$ and $X^1_{1,2}$ have finitely many points in the base locus of the anticanonical system, and these points can be detected via extremal contractions of X.

Example 4.9. Let X be a Fano manifold with Lefschetz defect 3, and let $\sigma\colon X \to T$ be the del Pezzo fibration of Construction A or Construction B (see Sections 3.2 and 3.3). Let X_t be a general fibre, i.e. the blow-up of $\mathbb{P}^2$ along 3 or 4 points in general position. Then $-K_{X|X_t} \sim -K_{X_t}$ is globally generated.

Nonetheless, from Examples 3.37, 3.38, 3.39 and 3.40 we see that the base locus $\operatorname{Bs}|-K_X|$ is contracted by some divisorial elementary contraction.

Example 4.10 ([**Xie22**])**.** Let $X := M_{S,-K_S}$ be the moduli space of semistable torsion-free sheaves of rank 2 with Chern classes $c_1 = -K_S$, $c_2 = 2$ on a polarised del Pezzo surface $(S, -K_S)$ of degree 1. By [**CCF19**, Prop. 1.6], X is a smooth, rational Fano fourfold of index 1 and $\rho_X = 9$, and it is related to the blow-up of $\mathbb{P}^4$ at 8 general points through a series of flips (see [**CCF19**, Lemma 5.18]). By [**Xie22**, Thm. 1.1], the base scheme of $|-K_X|$ is a reduced smooth rational curve. This is an example of a Fano fourfold with high Picard number which is not a product.

Example 4.11. Let S be a del Pezzo surface of degree 1, and let p be the basepoint of $|-K_S|$. Let T be any del Pezzo surface. Set $X := S \times T$, and let $Y \in |-K_X|$ be a general anticanonical divisor. Then Y has $K_T^2 > 0$ singular points along the base locus $\operatorname{Bs}|-K_X| = \{p\} \times T$. In particular, Y is not $\mathbb{Q}$-factorial.

Proof. Indeed we have

$$H^0(X, -K_X) \simeq H^0(S, -K_S) \otimes_{\mathbb{C}} H^0(T, -K_T).$$

Since $h^0(S, -K_S) = 2$, a general divisor $Y \in |-K_X|$ is given by a section of the form $s_1 \otimes t_1 + s_2 \otimes t_2$ where s_1, s_2 is a basis of $H^0(S, -K_S)$ and t_1, t_2 are general elements of $H^0(T, -K_T)$. Let $q \in T$ be an intersection point of the divisors defined by t_1 and t_2. Then we can choose local coordinates x_1, x_2 around p such that $s_i = \{x_i = 0\}$. Thus the divisor Y has local equation $x_1 \cdot t_1 + x_2 \cdot t_2 = 0$ around the point (p, q). Since t_1 and

t_2 vanish in q it is not difficult to see that the gradient vanishes in (p, q). Moreover, by Remark 4.14 this implies that Y is not $\mathbb{Q}$-factorial. ■

This generalises [**HV11**, Ex.2.12] in an interesting way: if T has degree at least 2, the singular point (p, q) depends on the choice of the divisor Y. Thus we have a first example of a Fano fourfold with "moving singularities" in the anticanonical system.

Example 4.12. Let X be the Fano fourfold T_3 of Example 3.39, that is the product $\mathbb{P}^1 \times T_2$, where $\mu\colon T_2 \to T_1$ is the blow-up of a del Pezzo threefold of degree 1 along an elliptic curve. Then, $\operatorname{Bs}|-K_X| \cong \mathbb{P}^1 \times \mathbb{P}^1$ and a general $Y \in |-K_X|$ is not $\mathbb{Q}$-factorial.

Proof. The variety T_2 has two elementary contractions: the blow-up μ and the fibration $\varphi\colon T_2 \to \mathbb{P}^1$ whose general fibre is a del Pezzo surface of degree 1. Therefore, X has three elementary contractions

$$\mathrm{id} \times \mu\colon X \to \mathbb{P}^1 \times T_1, \qquad \mathrm{id} \times \varphi\colon X \to \mathbb{P}^1 \times \mathbb{P}^1, \qquad \psi\colon X \to T_2,$$

where ψ is the projection on the second factor.

The base locus of $|-K_{T_2}|$ is a μ-exceptional curve f_0, and we note that $-K_{T_2} \cdot f_0 = 1$. Let $l \subset T_2$ be an elliptic curve that is an anticanonical divisor in a general fibre of φ, then $-K_{T_2} \cdot l = 1$. We identify the curve f_0 (resp. l) with a copy $\{p\} \times f_0$ (resp. $\{p\} \times l$) for some point $p \in \mathbb{P}^1$. Denote by f a fibre of ψ, then we have

$$-K_X \cdot f_0 = 1, \qquad -K_X \cdot l = 1, \qquad -K_X \cdot f = 2.$$

The base locus of the anticanonical system $|-K_X|$ is the surface $B = \mathbb{P}^1 \times f_0$.

Assume now that a general element $Y \in |-K_X|$ is factorial, more precisely assume that $B \subset Y$ is a Cartier divisor. By the Grothendieck Lefschetz Theorem [**Laz04a**, Remark 3.1.26], there exists a (not necessarily effective) Cartier divisor B_X on X such that $(B_X)|_Y \simeq B$. We will now compute the intersection numbers: since Y has trivial canonical bundle and B is Cartier, we have $K_B \sim B|_B$. Since

$$B \cong f \times f_0 \cong \mathbb{P}^1 \times \mathbb{P}^1,$$

we obtain $B_X \cdot f_0 = -2$ and $B_X \cdot f = -2$. In order to compute the intersection number with l, we observe first that B is a section of the fibration $\mathrm{id} \times \varphi$. The restriction of this fibration to Y gives an elliptic fibration $Y \to \mathbb{P}^1 \times \mathbb{P}^1$ such that the general fibre is an anticanonical divisor of a fibre of $\mathrm{id} \times \varphi$, i.e. a curve that is numerically equivalent to l. Since B is a section of the fibration $Y \to \mathbb{P}^1 \times \mathbb{P}^1$, we obtain $B_X \cdot l = 1$.

We set $R := -K_X - B_X$. The intersection numbers above imply that

$$R \sim (\mathrm{id} \times \varphi)^* \mathcal{O}(3, 4).$$

Since $\mathcal{O}(-3, -4)$ is antiample, we see by Kodaira vanishing (and $R^i(\mathrm{id} \times \varphi)_* \mathcal{O}_X = 0$ for $i > 0$) that $H^1(X, -R) = 0$. Consider now the restriction sequence for Y twisted by B_X:

$$0 \to \mathcal{O}_X(B_X - Y) \to \mathcal{O}_X(B_X) \to \mathcal{O}_Y(B_X) \to 0.$$

Since $B_X - Y \sim B_X + K_X \sim -R$, the restriction map

$$H^0(X, \mathcal{O}_X(B_X)) \to H^0(Y, \mathcal{O}_Y(B_X)) \simeq H^0(Y, \mathcal{O}_Y(B))$$

is surjective, hence B_X is effective. Yet the curve f deforms in a covering family of X, so $B_X \cdot f = -2$ yields a contradiction. Therefore, Y is not $\mathbb{Q}$-factorial by Remark 4.14. ■

Example 4.13. Let X be a smooth Fano fourfold, and let $B \subset \mathrm{Bs}\,|-K_X|$ be a smooth surface such that the conormal bundle $\mathcal{N}^\vee_{B/X}$ is nef. Then every anticanonical divisor $Y \in |-K_X|$ has at least one singular point along B.

Proof. Consider the natural morphism

$$H^0(X, -K_X) \simeq H^0(X, \mathcal{I}_B \otimes K_X^\vee) \to H^0(B, \mathcal{N}^\vee_{B/X} \otimes K_X^\vee).$$

Let Y be an anticanonical divisor defined by the zero locus of a section $s \in H^0(X, -K_X)$. By [**BS95**, Lemma 1.7.4], if the section $s_{|B} \in H^0(B, \mathcal{N}^\vee_{B/X} \otimes K_X^\vee)$ is nowhere vanishing, then Y is smooth along B. Since $\mathcal{N}^\vee_{B/X} \otimes K_X^\vee$ is a tensor product of a nef vector bundle and an ample line bundle, it is an ample vector bundle. Since $\dim B = \mathrm{rk}(\mathcal{N}^\vee_{B/X} \otimes K_X^\vee)$, every section vanishes in at least one point.

It is not difficult to show that this example generalises [**HV11**, Ex.2.12] and Examples 4.11 and 4.12. ■

Remark 4.14. Let Y be a normal projective threefold with terminal Gorenstein singularities. If Y is $\mathbb{Q}$-factorial, then it is factorial (see [**Cut88**, Lemma 2], [**Kaw88**, Lemma 5.1]).

Thus, if Y contains a smooth surface B that meets the singular locus of Y, then Y is not $\mathbb{Q}$-factorial. In fact, if B is a Cartier divisor, then $\mathrm{Sing}(Y) \cap B \subseteq \mathrm{Sing}(B)$.

4.3. Preliminary technical results

Let X be a projective manifold. Let $Y \in |-K_X|$ be any prime divisor. Then Y satisfies Serre's S_2 condition, is Gorenstein and $\omega_Y \cong \mathcal{O}_Y$. Let $\nu\colon \widetilde{Y} \to Y$ be the normalisation, then by [**Rei94**, p.704, Thm.] we have an inclusion $\nu_*\omega_{\widetilde{Y}} \subset \omega_Y$ and the cokernel has support on the non-normal locus. Since $\omega_Y \cong \mathcal{O}_Y$ we see that $\nu_*\omega_{\widetilde{Y}}$ is an ideal sheaf defining a subscheme C having support on the non-normal locus.

Let now A be an ample line bundle on Y. Tensoring with A and using the projection formula, we get

$$0 \to \nu_*(\omega_{\widetilde{Y}} \otimes \nu^* A) \to A \to A \otimes \mathcal{O}_C \to 0.$$

If we can show that $h^1(Y, \nu_*(\omega_{\widetilde{Y}} \otimes \nu^* A)) = 0$, the sequence above will give an exact sequence for global sections. Since $H^1(Y, \nu_*(\omega_{\widetilde{Y}} \otimes \nu^* A)) \cong H^1(\widetilde{Y}, \omega_{\widetilde{Y}} \otimes \nu^* A)$, this is true when $\widetilde{Y}$ is not "too singular" by applying Kodaira-type vanishing theorems.

Assume now that $B \subset Y$ is a prime divisor such that $B \not\subset Y_{\mathrm{sing}}$, and denote by $\widetilde{B} \subset \widetilde{Y}$ its strict transform.

Remark 4.15. We must be careful when taking the normalisation: even if $B \subset \mathrm{Bs}\,|A|$, this does not imply $h^0(\widetilde{Y}, \widetilde{B}) = 1$. Take for instance a smooth point B on a nodal cubic Y, and $A = \mathcal{O}_Y(B)$. By Serre duality and Riemann-Roch we have $h^0(Y, A) = 1$, so that B is a basepoint. Yet $\widetilde{B} = B$ is a point on a smooth rational curve, so it has $h^0 > 1$.

However we expect better properties for adjoint divisors, i.e. divisors of the form $\omega \otimes \text{nef}$. Indeed we have an injection

$$H^0(\widetilde{Y}, \omega_{\widetilde{Y}} \otimes \widetilde{B}) \hookrightarrow H^0(Y, \omega_Y \otimes B).$$

In the example of the nodal cubic, the left-hand space is zero.

We refer to [**KM98**] for the definitions and properties of *terminal*, *canonical*, *log terminal* (klt) and *log canonical* (lc) singularities of a normal projective variety. We explicitly recall the definition of *semi log canonical singularities* (slc) (see [**Kol13**], [**Fuj14**]).

Definition 4.16. Let X be an equidimensional variety that satisfies Serre's S_2 condition and is normal crossing in codimension 1, and let $\nu\colon \widetilde{X} \to X$ be the normalisation of X. The *conductor* of X is the subscheme $\mathcal{C}_X$ given by the *conductor ideal*

$$\mathcal{I}_{\mathcal{C}_X} := \mathcal{H}om_{\mathcal{O}_X}(\nu_*\mathcal{O}_{\widetilde{X}}, \mathcal{O}_X) \subseteq \mathcal{O}_X.$$

The scheme $\mathcal{C}_X$ has support on the non-normal locus of X. See [**Eis95**, Exercise 11.16] for the corresponding definition in commutative algebra.

The conductor ideal is the largest ideal sheaf on X that is also an ideal sheaf on $\widetilde{X}$: we denote by $\widetilde{\mathcal{C}_X}$ the subscheme of $\widetilde{X}$ given by the conductor ideal, as a sheaf on $\widetilde{X}$.

Definition 4.17. Let X be an equidimensional variety that satisfies Serre's S_2 condition and is normal crossing in codimension 1, and let $\nu\colon \widetilde{X} \to X$ be the normalisation of X. Let Δ be an effective $\mathbb{Q}$-divisor of X whose support does not contain any irreducible component of $\mathcal{C}_X$, and let $\widetilde{\Delta}$ be the divisorial part of $\nu^{-1}(\Delta)$.
We say that the pair (X, Δ) is *semi log canonical* (*slc* for short) if it satisfies the following conditions:

(i) $K_X + \Delta$ is $\mathbb{Q}$-Cartier;
(ii) the pair $(\widetilde{X}, \widetilde{\mathcal{C}_X} + \widetilde{\Delta})$ is log canonical.

We say that a variety X is *semi log canonical*, if the pair $(X, 0)$ is a semi log canonical pair.

Note that, for an slc pair (X, Δ), we have $K_{\widetilde{X}} + \widetilde{\mathcal{C}_X} + \widetilde{\Delta} \sim_{\mathbb{Q}} \nu^*(K_X + \Delta)$, see [**Kol13**, (5.7.5)].

4.3.1. Extending sections. Let X be a Fano manifold, and let $Y \in |-K_X|$ be an arbitrary anticanonical divisor. By Kodaira vanishing, the restriction morphism

$$H^0(X, -K_X) \to H^0(Y, -K_X)$$

is surjective. If Y is reducible, but not too singular, we can say something more.

Lemma 4.18. *Let X be a Fano manifold, and let $Y := D + E \in |-K_X|$ be an anticanonical divisor such that D and E do not have common components. If E has at most slc singularities, then the restriction morphism*

$$H^0(X, -K_X) \to H^0(D, -K_X)$$

is surjective.

Proof. By what precedes, it is sufficient to show that the restriction morphism $H^0(Y, -K_X) \to H^0(D, -K_X)$ is surjective.

Let R be the local ring $\mathcal{O}_{X,x}$. By hypothesis, the ideal sheaves $\mathcal{I}_{D/X}$ and $\mathcal{I}_{E/X}$ are locally generated by elements f, $g \in R$ satisfying $\gcd(f, g) = 1$, while $\mathcal{I}_{Z/X}$ is locally generated by the product fg. Therefore, we have an exact sequence

$$0 \to \big(f + (g)\big)\frac{R}{(g)} \xrightarrow{\alpha} \frac{R}{(fg)} \xrightarrow{\beta} \frac{R}{(f)} \to 0,$$

where $\alpha\big(rf+(g)\big) := \big(rf+(fg)\big)$ and $\beta\big(r+(fg)\big) = \big(r+(g)\big)$ for all $r \in R$. Globally, this is the exact sequence

$$0 \to \mathcal{O}_E(-D) \to \mathcal{O}_Y \to \mathcal{O}_D \to 0.$$

Twisting by $-K_X$, we see that it is sufficient to prove that $h^1(E, -D-K_X) = 0$. Yet by adjunction we have $\omega_E \simeq \mathcal{O}_E(-D)$. Since $-K_X$ is ample, we have

$$h^1(E, \mathcal{O}_E(-D-K_X)) = h^1(E, \omega_E \otimes \mathcal{O}_E(-K_X)) = 0$$

by Kodaira vanishing for slc varieties [**Fuj14**, Thm. 1.8] ([**LR14**, Prop. 16] for the surface case). ■

4.3.2. General remark on limiting singularities. We begin by recalling the following:

Proposition 4.19. [**Kaw00**, Prop.4.2] *Let Y be a normal projective variety with canonical singularities such that $\omega_Y \cong \mathcal{O}_Y$. Let A be an ample divisor on Y, and let $Z \in |A|$ be a general element.*

If $\dim Y \leq 3$, the pair (Y, Z) is lc, thus Z has at most slc singularities. In particular Z is reduced.

As a consequence, we have the following:

Lemma 4.20. *Let X be a Fano manifold of dimension at most four. Let $L \subset H^0(X, -K_X)$ be a linear subspace of codimension 1. Let $B \subset \operatorname{Bs}|-K_X|$ be an irreducible component of the base locus that has codimension 2 in X. Let $Y \in |L|$ be a general element. Then $B \not\subset Y_{\mathrm{sing}}$.*

Proof. Let $Y_g \in |-K_X|$ be a general anticanonical divisor. Then $Y_g \notin |L|$ and Y_g has at most canonical singularities [**Kaw00**, Thm.5.1]. Since Y_g is an anticanonical divisor, the restriction map

$$H^0(X, -K_X) \to H^0(Y_g, -K_X)$$

is surjective and its kernel is generated by a section vanishing on Y_g. Thus, since $Y_g \notin |L|$ and $L \subset H^0(X, -K_X)$ has codimension 1, the composition

$$L \subset H^0(X, -K_X) \to H^0(Y_g, -K_X)$$

is surjective. Therefore, a general element $Y \in |L|$ gives a general element $Y \cap Y_g \in |H^0(Y_g, -K_X)|$. By Proposition 4.19, the intersection is reduced. Since $B \subset Y \cap Y_g$ is an irreducible component, this shows that Y_g is smooth in the generic point of B. ■

4.3.3. Non-vanishing for slc curves.

Lemma 4.21. *Let B be a connected slc curve, and let M be an ample line bundle on B. If $\omega_B \otimes M$ is nef, then*

$$H^0(B, \omega_B \otimes M) \neq 0.$$

Proof. An slc curve has Gorenstein singularities, so by Serre duality

$$H^1(B, \omega_B \otimes M) \simeq H^0(B, M^\vee) = 0.$$

Thus $h^0(B, \omega_B \otimes M) = \chi(B, \omega_B \otimes M)$. The Riemann-Roch formula for a Gorenstein curve yields

$$\chi(B, \omega_B \otimes M) = 1 - p_a(B) + \deg(\omega_B \otimes M) = p_a(B) - 1 + \deg M \geq 0,$$

and equality holds if and only if $B \simeq \mathbb{P}^1$. Yet in the latter case the non-vanishing is trivial. ■

4.3.4. Fibred K3 surfaces. Although the Proposition below follows from the study of $K3$ surfaces in [**SD74**], we provide a proof.

Proposition 4.22. *Let Y be a smooth projective surface that is Calabi-Yau, i.e. $\omega_Y \cong \mathcal{O}_Y$ and $h^1(Y, \mathcal{O}_Y) = 0$. Assume that Y admits an elliptic fibration $\varphi\colon Y \to C$ onto a curve C, and an ample divisor A such that the following holds: $A \sim B + \varphi^* M$, where M is a nef and big divisor on C and B is a prime effective divisor such that $B \subset \operatorname{Bs}|A|$. Then $B \cong \mathbb{P}^1$ and B is a section of the fibration.*

Remark 4.23. Let B be an irreducible Gorenstein curve, and let $\nu\colon \widetilde{B} \to B$ be the normalisation. Then we have an exact sequence

$$0 \to \nu_* \omega_{\widetilde{B}} \to \omega_B \to T \to 0$$

where T is a torsion-sheaf with support on the non-normal locus of B. If we tensor the exact sequence by a line bundle M_B on B, taking cohomology we obtain the inequality

$$h^0(\widetilde{B}, \omega_{\widetilde{B}} \otimes \nu^* M_B) = h^0(B, \nu_* \omega_{\widetilde{B}} \otimes M_B) \leq h^0(B, \omega_B \otimes M_B), \tag{4.1}$$

which is very often strict (i.e. when we can show that $h^0(B, T \otimes M_B) \neq 0$).

Proof of Proposition 4.22. We show first that B is not contracted by φ. Otherwise $0 < A \cdot B = B \cdot B$, but B is a fixed part of $|A|$ and $h^0(Y, B) = 1$, a contradiction.

By taking Stein's factorisation, we may assume that C is smooth and that φ has connected fibres, i.e. $\varphi_* \mathcal{O}_Y \cong \mathcal{O}_C$. Moreover, the Leray spectral sequence for $\mathcal{O}_Y$ gives an inclusion

$$H^1(C, \mathcal{O}_C) \hookrightarrow H^1(Y, \mathcal{O}_Y) = 0,$$

so that C is rational. Consider the exact sequence

$$0 \to \mathcal{O}_Y(\varphi^* M) \to \mathcal{O}_Y(A) \to \mathcal{O}_B(A) \to 0$$

and the long exact sequence in cohomology

$$\begin{aligned} 0 \to H^0(Y, \mathcal{O}_Y(\varphi^* M)) \to H^0(Y, \mathcal{O}_Y(A)) \to H^0(B, \mathcal{O}_B(A)) \to \\ \to H^1(Y, \mathcal{O}_Y(\varphi^* M)) \to H^1(Y, \mathcal{O}_Y(A)) = 0, \end{aligned}$$

where $h^1(Y, \mathcal{O}_Y(A)) = 0$ by Kodaira vanishing. Since $B \subset \operatorname{Bs}|A|$, the restriction map $H^0(X, \mathcal{O}_X(A)) \to H^0(B, \mathcal{O}_B(A))$ is zero, so we have an isomorphism

$$H^0(B, \mathcal{O}_B(A)) \simeq H^1(X, \mathcal{O}_X(\varphi^* M)). \tag{4.2}$$

By the Leray spectral sequence for $\varphi^* \mathcal{O}_C(M)$ we have an exact sequence

$$0 \to H^1(C, \mathcal{O}_C(M)) \to H^1(Y, \mathcal{O}_Y(\varphi^* M)) \to H^0(C, M \otimes R^1 \varphi_* \mathcal{O}_Y).$$

Since $C \cong \mathbb{P}^1$ and M has positive degree, we have $h^1(C, \mathcal{O}_C(M)) = 0$. (This also follows more generally from Kollár's vanishing theorem [**Laz04a**, Rem.4.3.8], since $\omega_Y \cong \mathcal{O}_Y$.) Moreover $R^1 \varphi_* \mathcal{O}_Y \cong R^1 \varphi_* \omega_Y$ is torsion-free and by relative duality

$$(R^1 \varphi_* \mathcal{O}_Y)^\vee \cong \varphi_* \omega_{Y/C} \cong (\omega_C)^\vee,$$

where for the last isomorphism we used again $\omega_Y \cong \mathcal{O}_Y$. In particular we get an injection $R^1 \varphi_* \mathcal{O}_Y \hookrightarrow \omega_C$. Thus we see that (4.2) yields an injection

$$H^0(B, \mathcal{O}_B(A)) \hookrightarrow H^0(C, M \otimes \omega_C).$$

Actually, since $\omega_B \simeq \mathcal{O}_B(B)$, we can rewrite this as

$$H^0(B, \mathcal{O}_B(\varphi^* M) \otimes \omega_B) \hookrightarrow H^0(C, M \otimes \omega_C). \tag{4.3}$$

Our goal is to show that this never holds if the morphism $\varphi|_B \colon B \to C$ has degree $d \geq 2$. Denote by $\nu \colon \widetilde{B} \to B$ the normalisation, so that $\widetilde{B}$ is smooth. Then by (4.1) we have an injection

$$H^0(\widetilde{B}, \mathcal{O}_{\widetilde{B}}(\nu^* \varphi^* M) \otimes \omega_{\widetilde{B}}) \hookrightarrow H^0(C, M \otimes \omega_C).$$

Thus we can assume without loss of generality that both B and C are smooth curves.

Since M (and so $\varphi^*_{|B} M$) has positive degree, we have $h^1(B, \mathcal{O}_B(\varphi^* M) \otimes \omega_B) = h^1(C, M \otimes \omega_C) = 0$. Therefore, Riemann-Roch and (4.3) yield

$$d \cdot \deg(M) + g(B) - 1 \leq \deg(M) - 1,$$

so that $(d-1) \cdot \deg(M) \leq 0$. This implies that $d = 1$ and that B is rational. ■

4.4. The surface case

Let S be a smooth del Pezzo surface. By classification, it is known that the anticanonical divisor $-K_S$ is globally generated except for the case of degree 1. In this situation, S is the blow-up $\pi \colon S \to \mathbb{P}^2$ of $\mathbb{P}^2$ along 8 points in general position, and the linear system $|-K_S|$ consists of the strict transforms of the plane cubics that pass through the 8 blown-up points. All these cubics pass through a ninth point, whose preimage via π is exactly the base locus of $|-K_S|$.

In the following Proposition, we show that $\operatorname{Bs}|-K_S|$ does not meet any exceptional curve. For the proof we will not use classification results, but try to use arguments that might work also in higher dimension.

Proposition 4.24. *Let S be a smooth del Pezzo surface. Let $p \in \operatorname{Bs}|-K_S|$ be a basepoint of the anticanonical system, and let $E \subset S$ be a (-1)-curve. Then $p \notin E$.*

Proof. We argue by contradiction: assume that $p \in \operatorname{Bs}|-K_S|$ is contained in some (-1)-curve E.

Step 1. *Reducible anticanonical divisors.*

We know that $-K_S|_E \simeq \mathcal{O}_{\mathbb{P}^1}(1)$, which is globally generated. Therefore, the restriction map

$$H^0(E, -K_S) \simeq H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) \to H^0(p, -K_S) \simeq \mathbb{C}$$

is surjective. Since $p \in \operatorname{Bs}|-K_S|$, the composition

$$H^0(S, -K_S) \to H^0(E, -K_S) \to H^0(p, -K_S)$$

is the zero map. Now, Lemma 4.21 and Kodaira vanishing imply that $h^0(S, -K_S) \geq 2$. Since $h^0(E, -K_S) = 2$, it follows that

(i) the restriction map $H^0(S, -K_S) \to H^0(E, -K_S)$ is not surjective; $(*)$
(ii) the kernel of $H^0(S, -K_S) \to H^0(E, -K_S)$ is not zero.

In particular, the kernel of the above restriction map is exactly

$$H^0(S, -K_S \otimes \mathcal{I}_E)$$

and it has codimension 1. Thus, there exists an effective divisor R and a positive integer $m \in \mathbb{N}$ such that $E \not\subset R$ and

$$mE + R \in |-K_S|,$$

i.e. an anticanonical divisor that contains E. Note that $R \neq 0$, since $-K_S$ is ample and E is not. We set

$$Y := mE + R.$$

Remark. At this point the proof could end: by classification we have $-K_S^2 = 1$, so that the linear system $|-K_S|$ contains only irreducible members, a contradiction. We do not want to use classification, so we have to make a different argument.

Step 2. *The multiplicity.*

The linear space $H^0(S, -K_S \otimes \mathcal{I}_E) \subset H^0(S, -K_S)$ has codimension 1, so by Lemma 4.20, for a general choice of $Y = mE + R \in |H^0(S, -K_S \otimes \mathcal{I}_E)|$, the basepoint p is not contained in the singular locus of Y. Since $p \in E$, we see that $m = 1$.

Step 3. *The irreducible components of Y.*

The divisor $-2K_S - E \sim -K_S + R$ is not ample: otherwise $H^1(S, -K_S - E)$ vanishes, which contradicts $(*)$. Therefore, there exists an extremal ray $\mathbb{R}^+[\Gamma]$ such that $(-K_S + R) \cdot \Gamma \leq 0$. Since the divisor $-K_S$ is ample, we have $R \cdot \Gamma < 0$. Moreover R is effective, so that the corresponding contraction is birational and contracts a (-1)-curve E' onto a point, and we can write $R = R' + mE'$ with $m \geq 1$.

We have $R \cdot E' \leq -1$ and $-K_S \sim R + E$, so that $E \cdot E' \geq 2$. Since $R \cdot E = 2$ ($-K_S \cdot E = 1$ and $E^2 = -1$), we obtain that $E \cdot E' = 2$, E' has coefficient one in R (i.e. $m = 1$), and so $R \cdot E' = -1$. Yet writing $R = R' + E'$, we see that $R' \cdot E' = 0$, that is E' is a connected component of R. Since $-K_S$ is ample, so that $Y = E' + E + R'$ is connected, we obtain $E \cap R' \neq \emptyset$ unless $R' = 0$. Yet $E \cdot R = E \cdot E'$, so $R' = 0$.

Thus we have

$$R = E', \quad -K_S + R \sim E + 2E',$$

which is easily seen to be a nef and big divisor that has degree zero on E'.

Remark. With all these intersection numbers we can now compute that $-K_S^2 = 2$ and it is interesting to observe that a del Pezzo surface of degree 2 has an anticanonical divisor of the form $E' + E$, so we are not far from the reality. In order to get a contradiction we have to use the existence of the basepoint.

Step 4. *Using again the basepoint.*

Since $Y = E' + E$ and E' is a (-1)-curve, hence it has at most slc singularities, we can apply Lemma 4.18 to see that the restriction map

$$H^0(S, -K_S) \to H^0(Y, -K_S) \to H^0(E, -K_S)$$

is surjective, the final contradiction. ■

Remark 4.25. (Variant of Step 3.) The divisor $-2K_S - E \sim -K_S + R$ is not nef and big, otherwise $H^1(S, -K_S - E) = 0$ and again this contradicts $(*)$. Since $-K_S + R$ is certainly big, it is not nef. Arguing as in Step 3, there exists an extremal ray $\mathbb{R}^+[\Gamma]$ such that $(-K_S + R) \cdot \Gamma < 0$ and the corresponding contraction is again birational and contracts a (-1)-curve E' onto a point. At the end we show that $-K_S + R \sim E + 2E'$ has degree zero on E', but $[E'] \in \mathbb{R}^+[\Gamma]$, giving the final contradiction.

4.5. The threefold case

Let X be a smooth Fano threefold with $\rho_X \geq 2$: we begin by recalling the following result from [**Wiś91a**].

Proposition 4.26. *Let X be a smooth Fano threefold with $\rho_X \geq 2$. Then X has at least one elementary Mori contraction $\mu\colon X \to T$ with fibres of dimension ≤ 1.*

Proof. Let $\mu\colon X \to T$ be an elementary contraction of X: the fibres of μ have dimension $\leq 2 = \dim(X) - 1$. If the μ-fibres have dimension ≤ 1, we are done. Otherwise, μ sends a surface to a point and, by [**Wiś91a**, Corollary 1.3], there exists another elementary Mori contraction $\mu'\colon X \to T'$ such that the μ'-fibres have dimension ≤ 1. ■

Therefore, let $\mu\colon X \to T$ be an elementary contraction of X with one-dimensional non-trivial fibres. By [**Wiś91a**, Theorem 1.2] we have that T is smooth and that μ is either a conic bundle or the blow-up along a codimension 2 subvariety.

Assume that there exists an irreducible curve $B \subseteq \mathrm{Bs}\,|-K_X|$: the objective of this section is to show that, under the above assumption, X is one of Examples 4.6 and 4.7.

We argue as follows. First, we consider the blow-up $\nu\colon M \to X$ of X along B and we show that M admits an elliptic fibration $\varphi\colon M \to S$ onto a normal surface S. Once we understand the general geometry of $\varphi\colon M \to S$, we fix an elementary Mori contraction $\mu\colon X \to T$ with fibres of dimension ≤ 1, and we treat separately the cases where μ is birational or a fibration. In both cases, T is a smooth Fano variety.

The proof of Theorem 4.3 goes as follows: Theorem 4.34 and Step 1 of Theorem 4.36 show that B is contracted by μ, while the statement on the base locus of the fundamental divisor of T is given by Step 7 of Theorem 4.35 and Step 2 of Theorem 4.36.

Lastly, note that Assumption (i) of the Technical setup 4.2 is always satisfied when X is a smooth Fano threefold: the inequality $h^0(X, -K_X) \geq 3$ follows from the Riemann-Roch theorem (Lemma 1.17).

4.5.1. The elliptic fibration.

Proposition 4.27. *Let X be a smooth Fano threefold and assume that there exists an irreducible curve $B \subseteq \mathrm{Bs}\,|-K_X|$. Then, up to replacing B by another curve in $\mathrm{Bs}\,|-K_X|$, we can assume that $B \cong \mathbb{P}^1$ and that $-K_{X|Y} - B$ is a nef and not big divisor on Y, with $Y \in |-K_X|$ general.*

Proof. By Theorem 4.1, Y is a smooth $K3$ surface: $h^1(Y, \mathcal{O}_Y) = 0$ and $\omega_Y \cong \mathcal{O}_Y$. Let $A \in |-K_{X|Y}|$ be a general element, which is reduced by Proposition 4.19. Since B is a fixed component of the linear system $|A|$, we know that $A - B$ is an effective divisor. Moreover, every element of $|A|$ is connected (A is ample on Y), thus we have $(A - B) \cdot B > 0$.

First, we observe that the divisor $A - B$ is not nef and big. Otherwise the restriction map

$$H^0(Y, A) \to H^0(B, A),$$

which is the zero map as B is a fixed component of $|A|$, is surjective by Kawamata-Viehweg. Thus $h^0(B, A) = 0$. Now, observe that

$$\mathcal{O}_Y(A) \otimes \mathcal{O}_B \cong \mathcal{O}_Y(B + A - B) \otimes \mathcal{O}_B \cong \omega_B \otimes \mathcal{O}_B(A - B).$$

Since B has slc singularities, by Proposition 4.19 and

$$\deg_B(A-B) = (A-B)\cdot B > 0$$

we can apply Lemma 4.21 to obtain $h^0(B, A) \neq 0$, a contradiction.

Step 1. *We may assume that $B \cong \mathbb{P}^1$ is a (-2)-curve in Y.*

1st case. If $A-B$ is not nef, there exists a curve $C \subset Y$ such that $(A-B)\cdot C < 0$. Thus C is an irreducible component of the support of $A-B$ and $C^2 < 0$. By adjunction, this shows that $C \cong \mathbb{P}^1$ is a (-2)-curve. Since $C \subseteq \operatorname{Bs}|A-B| \subset \operatorname{Bs}|A|$, we can replace B by C.

2nd case. If $A-B$ is nef and not big, some multiple of $A-B$ defines an elliptic fibration on Y. Thus we can apply Proposition 4.22 to see that $B \cong \mathbb{P}^1$.

Step 2. *We claim that the divisor $A-B$ is nef and not big.*

By the first step we can assume $B \cong \mathbb{P}^1$. Arguing as above, we see that if $A-B$ is nef, then it is not big.

Assume now that $A-B$ is not nef, so that we obtain another (-2)-curve C such that $(A-B)\cdot C < 0$. Since $A\cdot C \geq 1$, we get $B\cdot C \geq 2$. Thus, the divisor $B+C$ is nef and the curve $B\cup C$ has arithmetic genus at least one (see [**Har77**, Exercise V.1.3]). The vanishing $h^1(Y, \mathcal{O}_Y) = 0$ implies that the restriction map

$$H^0(Y, B+C) \to H^0(B\cup C, B+C)$$

is surjective. Moreover, by adjunction we obtain an isomorphism

$$\mathcal{O}_{B\cup C}(B+C) \cong \omega_{B\cup C}.$$

Since $B\cup C$ has arithmetic genus at least one, we have $h^0(B\cup C, \omega_{B\cup C}) \geq 1$ and hence $h^0(Y, B+C) \geq 2$. Yet by construction $B+C$ is in the fixed part of the linear system $|A|$, so $h^0(Y, B+C) = 1$, a contradiction. ∎

By Proposition 4.27, we can assume that $B \cong \mathbb{P}^1$ is a smooth rational curve and that $-K_{X|Y} - B$ is a nef and not big divisor on Y. Denote by

$$\nu\colon M \longrightarrow X$$

the blow-up of X along the curve B, and by $E \subset M$ the exceptional divisor. Observe that, since $B \cong \mathbb{P}^1$ by assumption, the exceptional divisor E is isomorphic to a Hirzebruch surface $\mathbb{F}_d$. We have $-K_M \sim \nu^*(-K_X) - E$, and the strict transform $Y' \in |-K_M|$ of a general $Y \in |-K_X|$ is smooth and isomorphic to Y. Thus we obtain that $-K_{M|Y'} \sim -K_{X|Y} - B$ is nef and not big, hence $-K_M$ is nef and not big.

Lemma 4.28. *Let X be a smooth Fano threefold. Assume that there exists an irreducible curve $B \subseteq \operatorname{Bs}|-K_X|$, and that for a general $Y \in |-K_X|$ the restriction $-K_{X|Y} - B$ is nef and not big.*

Denote by $\nu\colon M \to X$ the blow-up along B, and by E the exceptional divisor. Then $-K_M$ is semiample.

Proof. Since $-K_M \sim \nu^*(-K_X) - E$, our assumption implies that $-K_M$ is nef and not big. Indeed, let $Y' \in |-K_M|$ be the transform of a smooth general $Y \in |-K_X|$. The divisor $-K_M$ is nef if it has non-negative intersections with all the irreducible curves in Y', that is if $-K_{M|Y'}$ is nef on Y'. Since $-K_{M|Y'} \sim -K_{X|Y} - B$, this follows by hypothesis. Therefore, we have that $-K_M$ is not big if $-K_M^3 = 0$. Similarly as above, this follows by the equality $-K_M^3 = (-K_{X|Y} - B)^2 = 0$.

Moreover, since $H^0(X, -K_X) \cong H^0(M, -K_M)$, $-K_M$ contains a prime divisor Y' in its linear system. By [**Xie20**, Theorem 1.1], this yields the statement. ■

By Lemma 4.28 and $-K_{X|Y} - B \neq 0$, we obtain that $-K_M$ is semiample with numerical dimension 2, so that we can consider the elliptic fibration

$$\varphi\colon M \longrightarrow S$$

defined by some multiple of $-K_M$. Therefore, there exists an ample $\mathbb{Q}$-Cartier Weil divisor H on S such that

$$-K_M \equiv \varphi^* H. \tag{4.4}$$

By the following Lemma 4.29, $H \sim \varphi(Y')$ for a general $Y' \in |-K_M|$. Clearly the pull-back

$$\nu^*(-K_X) \sim -K_M + E$$

is nef and big, so it has positive degree on the general φ-fibre. In particular E maps surjectively onto S, hence S has Picard number at most 2.

More is true: $\nu^*(-K_X)$ is trivial exactly on the fibres of the blow-up $\nu\colon M \to X$, so if $\nu^*(-K_X)$ is not φ-ample, then φ contracts one fibre of the blow-up. Yet this implies that φ contracts all the fibres of the blow-up, hence E maps onto a proper subvariety of S, a contradiction. Thus we see that $\nu^*(-K_X)$, and hence E, is φ-ample.

Restricting to Y' and using Proposition 4.22, we see that

$$\varphi_{|E}\colon E \longrightarrow S$$

is birational, so that S is either a Hirzebruch surface $\mathbb{F}_d$ or a cone obtained by contracting the negative section of $\mathbb{F}_d$.

Lemma 4.29. *Let M be a projective manifold, and let $\varphi\colon M \to S$ be a fibration onto a surface. Let L be a Cartier divisor on M such that $L \equiv \varphi^* H$ for some $\mathbb{Q}$-Cartier Weil divisor H on S. Then $\varphi_* \mathcal{O}_M(L)$ is a reflexive sheaf.*

Proof. If φ has equidimensional fibres, this holds without any assumption by [**Har80**, Cor.1.7].

Since S is a surface, we have at most finitely many higher-dimensional φ-fibres consisting of exceptional divisors. By [**Nak04**, III, Lemma 5.10.(2)], it is sufficient to check that if $\sum a_i E_i$ is an effective φ-exceptional divisor mapping onto a point $p \in S$, then there exists an irreducible component G of $\operatorname{Supp} \sum a_i E_i$ such that

$$(L + \sum a_i E_i)_{|G} \equiv (\sum a_i E_i)_{|G}$$

is not pseudo-effective. Let now A be a general hyperplane section of S that passes through p, so that $\varphi^* A$ is an effective divisor whose support contains the support of $\sum a_i E_i$. Then we can choose $c > 0$ such that

$$R := \varphi^* A - c(\sum a_i E_i)$$

is effective, but there exist an irreducible component G of $\operatorname{Supp} \sum a_i E_i$ that appears with coefficient zero. Note that since φ has connected fibres, we can choose G such that the intersection $(\operatorname{Supp} R) \cap G$ is not empty. Thus

$$(\sum a_i E_i)_{|G} \equiv \frac{1}{c}(\varphi^* A - R)_{|G} \equiv -\frac{1}{c} R_{|G},$$

which is an antieffective non-zero divisor on G. In particular it is not pseudo-effective. ■

We now want to understand in more depth the geometry of the elliptic fibration $\varphi\colon M \to S$.

Lemma 4.30. *Let M be a projective manifold, and let $\varphi\colon M \to S$ be a fibration onto a normal variety S with general fibre of dimension 1. Let $E \subset M$ be a φ-ample prime divisor such that $E \twoheadrightarrow S$ has degree 1 (so that E is a rational section of φ). Then every φ-fibre of pure dimension 1 is an integral curve. If $\varphi^{-1}(s)$ does not have pure dimension 1, then $\varphi|_E^{-1}(s)$ has positive dimension.*

Proof. The relatively ample divisor E has degree 1 on every fibre of dimension 1, so the first statement is clear.

For the second statement, observe that the relatively ample divisor E meets every irreducible component G of $\varphi^{-1}(s)$. Since E is a Cartier divisor, the intersection has codimension at most one in G. Thus if $\dim G \geq 2$, the fibre $\varphi_{|E}^{-1}(s)$ has dimension at least $\dim G - 1 \geq 1$. ■

In our situation, Lemma 4.30 implies that every φ-fibre is an integral curve of arithmetic genus 1, unless it contains the negative section $N \subset E \cong \mathbb{F}_d$.

By Proposition 4.22, a general $Y' \in |-K_M|$ is contracted by φ onto a rational curve, but this can be generalised for any anticanonical divisor. In fact:

Lemma 4.31. *In the situation above, let $Y \in |-K_X|$ be an arbitrary anticanonical divisor. Then $B \not\subset Y_{\mathrm{sing}}$.*

Proof. Denote by m the multiplicity of Y along B, our goal is to show that $m = 1$. We have $\nu^*Y - E \in |-K_M|$ and $-K_M \equiv \varphi^*H$. Thus, if we denote by $Y = \sum m_i Y_i$ the decomposition in the irreducible components, then

$$\sum m_i Y_i + (m-1)E \equiv \varphi^*H.$$

Let now C be a general φ-fibre. Then we have $Y_i \cdot C \geq 0$ for all i and $E \cdot C = 1$. Since $\varphi^*H \cdot C = 0$, we obtain $m = 1$. ■

Corollary 4.32. *In the situation above, let $Y \in |-K_X|$ be an arbitrary anticanonical divisor, and let $Y' \subset M$ be its strict transform. Then every irreducible component of Y' is contracted by φ onto a curve.*

Proof. Denote by $Y = \sum m_i Y_i$ the decomposition in the irreducible components. Since Y has multiplicity 1 along B by Lemma 4.31, we have

$$Y' = \sum m_i Y_i' \in |-K_M|$$

and hence

$$\sum m_i Y_i' \equiv \varphi^*H.$$

Let now C be a general φ-fibre. Then we have $Y_i \cdot C \geq 0$ for all i. Since $\varphi^*H \cdot C = 0$, we obtain that $Y_i \cdot C = 0$ for all i. Thus Y_i does not surject onto S. ■

As a last preliminary result, we prove the following Lemma. This will be need when working with the ample $\mathbb{Q}$-Cartier Weil divisor H on S.

Lemma 4.33. *Let $\mathbb{F}_d := \mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(-d))$ be the d-th Hirzebruch surface, where $d \geq 0$.*

(i) If H is an ample Cartier divisor on $\mathbb{F}_d$, then $h^0(\mathbb{F}_d, H) \geq 4$. If H is a nef and big Cartier divisor on $\mathbb{F}_d$, then $h^0(\mathbb{F}_d, H) \geq 4$ unless $d = 1$.

(ii) Let $d \geq 3$ and C_d be the cone obtained by contracting the negative section $N \subset \mathbb{F}_d$. If H is an effective Weil divisor on C_d such that the linear system $|H|$ contains a prime divisor, then $h^0(C_d, H) \geq 4$ or $h^0(C_d, H) \geq 2$ and H is given by the ruling of the cone.
When $d = 2$, $h^0(C_2, H) \geq 3$ or $h^0(C_2, H) \geq 2$ and H is given by the ruling of the cone.

Proof. Let f be a fibre of the ruling on $\mathbb{F}_d$, and let $\zeta \sim N + (d+1)f$, that is ζ is the tautological divisor class for

$$\mathbb{P}_{\mathbb{P}^1}\big(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(d+1)\big).$$

If H is ample, we can write $H \sim \zeta + T$, with T a nef divisor. Since a nef divisor on $\mathbb{F}_d$ is effective, it is sufficient to show the statement for ζ, and this is elementary. This, moreover, concludes the case $d = 0$. The nef and big case is analogous.

Consider now the case of the cone for $d \geq 3$, and denote by $\mu\colon \mathbb{F}_d \to C_d$ the contraction of the negative section N. The class group of C_d is generated by the Weil divisor $\mu_* f$. Since $h^0(\mathbb{F}_d, f) = 2$, we have $h^0(C_d, \mu_* f) \geq 2$, so the second statement is clear. The first statement follows analogously from $h^0(\mathbb{F}_d, kf) = k + 1$, unless we are in the case $k = 2$. Yet, when $d \geq 3$, the linear system $|\mu_* 2f|$ does not contain a prime divisor, so this case is excluded by our assumption.

When $d = 2$, the linear system $|\mu_* 2f|$ does contain a prime divisor, but the statements follow as above. ∎

4.5.2. The birational case. Let X be a smooth Fano threefold satisfying the Assumption 4.2. We keep the same notation as in §4.5.1.

Assume that the elementary Mori contraction $\mu\colon X \to T$ is birational and contracts a divisor E_Γ onto a curve Γ. Our objective is to understand the relation between a curve $B \subseteq \operatorname{Bs}|-K_X|$ and $\operatorname{Exc}(\mu)$, and we show that X is the Fano threefold of Example 4.6.

Theorem 4.34. *Let X be a smooth Fano threefold and assume that there exists an irreducible curve $B \subseteq \operatorname{Bs}|-K_X|$. Assume moreover that X admits an elementary Mori contraction $\mu\colon X \to T$ that contracts a divisor E_Γ onto a curve Γ. If $E_\Gamma \cap B \neq \emptyset$, then B is a fibre of the contraction.*

Proof. We argue by contradiction, so assume that B is not a fibre of the Mori contraction.

Let $F_p \subset E_\Gamma$ be a fibre of μ that meets B, and consider the codimension 1 linear system $|\mathcal{I}_{F_p/X} \otimes -K_X| \subset |-K_X|$. Since by assumption $h^0(X, -K_X) \geq 3$, we can choose two general elements Y_1, $Y_2 \in |\mathcal{I}_{F_p/X} \otimes -K_X|$.

By Lemma 4.20 we know that the divisors Y_i are smooth in the generic point of B, so if Y_i' is the strict transform, then $\mu^* Y_i \sim Y_i' + E$. In particular $Y_i' \in |-K_M|$, so that these divisors are mapped onto an effective divisor of S by φ (see Corollary 4.32).

Step 1. *The linear system $|\mathcal{I}_{F_p/X} \otimes -K_X|$ does not have any fixed component.*

We argue by contradiction. The strict transform G' of a fixed component G would map onto a curve $\varphi(G') \subset S$ that is a fixed component of the linear system

$$\varphi_*\big(\mu^* |\mathcal{I}_{F_p/X} \otimes -K_X| - E\big) \subset \varphi_* |-K_M|.$$

In particular the curve $\varphi(G')$ would be a Weil divisor on S such that $h^0(S, \varphi(G')) = 1$. If S is a cone C_d, Lemma 4.33(*ii*) gives a contradiction. Thus we can assume that $S \simeq \mathbb{F}_d$ and E maps isomorphically onto S. The unique curve on $\mathbb{F}_d$ with a zero dimensional linear system is the negative section N. Since $-K_M \sim \varphi^* H$ (4.4) has degree 1 on the fibres f of the ruling $\nu_{|E}\colon E \to B$, we have

$$H \sim N + \lambda f$$

with $\lambda > 0$. Since we assume that N is a fixed component, we have

$$\varphi_*\big(\mu^* |\mathcal{I}_{F_p/X} \otimes -K_X| - E\big) \subset N + |\lambda f|.$$

Thus the linear system $|\mathcal{I}_{F_p/X} \otimes -K_X|$ has dimension at most $\lambda + 1$. On the other hand, we have

$$\dim |-K_X| = \dim |-K_M| = \dim |H| = \dim |N + \lambda f| = 2\lambda + 2 - d.$$

Since $|\mathcal{I}_{F_p/X} \otimes -K_X| \subset |-K_X|$ has codimension 1, we conclude that $d = \lambda$. Yet this implies that $H = N + \lambda f$ is nef and big and not ample, so φ contracts the negative section, a contradiction.

Step 2. *The general element $Y_i \in |\mathcal{I}_{F_p/X} \otimes -K_X|$ is a prime divisor.*

Since the linear system $|\mathcal{I}_{F_p/X} \otimes -K_X|$ has positive dimension and does not have fixed components, either the general element Y_i is irreducible or the index of X is at least 2. Indeed, if Y_i is a reducible divisor, then the rational map corresponding to $|\mathcal{I}_{F_p/X} \otimes -K_X|$ sends X onto a curve (Bertini's theorem). So we can compare the proof of [**Xie20**, Lemma 2.3] to see that $Y_i \sim mD$, with $m \geq 2$ and D a prime divisor. Yet this last case is excluded since the contraction μ has length 1, and so X has index 1. This proves the claim.

Step 3. *We claim that $E \cong \mathbb{F}_1$ and $S \cong \mathbb{P}^2$.*

From Step 2, we see that the Y_1 and Y_2 are irreducible. Since $F_p \subset Y_1 \cap Y_2$ and F_p is not contained in the blown-up locus of ν (here we use the assumption that B is not contracted by μ), its strict transform $F'_p \subset M$ is contained in $Y'_1 \cap Y'_2$. Moreover, since $-K_X \cdot F_p = 1$ and $E \cdot F'_p > 0$, we obtain $-K_M \cdot F'_p = 0$, i.e. F'_p is contracted by φ.

Recall that every φ-fibre is an integral curve of arithmetic genus 1, unless it contains the negative section $N \subset E \cong \mathbb{F}_d$. Since F'_p is rational, it must be contained in a higher-dimensional fibre, and so by Lemma 4.30 we see that the birational morphism $\varphi_{|E}\colon E \to S$ is not an isomorphism. We denote by $0 \in S$ the image of the negative section, so that $F'_p \subseteq \varphi^{-1}(0)$.

Consider now the restricted elliptic fibration $\varphi_{|Y'_i}\colon Y'_i \to \varphi(Y'_i)$. If Y'_i does not contain the negative section N, then the effective divisor $E \cap Y'_i \subset Y'_i$ is relatively ample, does not contain any fibre component and has degree 1 on the general fibre. Thus $\varphi_{|Y'_i}$ has irreducible fibres and F'_p is one of them, so that F'_p has arithmetic genus one, a contradiction.

Therefore $N \subset Y'_i$. Since $Y'_i \in |-K_M|$ and $-K_M$ is nef, the effective divisor $E \cap Y'_i \subset E$ is nef and contains the negative section. Thus we obtain

$$E \cap Y'_i = N + \sum \lambda_i f_i \ ,$$

where f_i are fibres of the ruling $\nu_{|E}\colon E \to B$, and each of the curves f_i maps surjectively onto the curve $\varphi(Y'_i)$. Since E is a φ-section over $S - \{0\}$, we see that $\sum \lambda_i f_i$ consists

of just one fibre, with multiplicity 1. Yet $N + f$ is a nef divisor, so if $d := -N^2$, we have $d \leq 1$. This shows the claim.

Step 4. *The final contradiction.*

By Step 3 and equation (4.4), we have $-K_M \sim \varphi^*\mathcal{O}_{\mathbb{P}^2}(k)$ with $k \geq 1$. If $k \geq 2$, the anticanonical divisor $-K_M$ is divisible by k. Since $-K_X \sim \nu_*(-K_M)$, the same holds for the anticanonical divisor of X. Yet X admits a Mori contraction of length 1, so this implies that $k = 1$.

Let now $Y \in |-K_X|$ be a general element, and denote by $Y' \subset M$ its strict transform. Recall that Y, and so its strict transform $Y' \in |-K_M|$, is smooth. Since $-K_M \sim \varphi^*\mathcal{O}_{\mathbb{P}^2}(1)$, we have

$$-K_{X|Y} \sim \nu^*(-K_X)_{|Y'} \sim -K_{M|Y'} + E \cap Y' = e + B,$$

where e is a fibre of the elliptic fibration $Y \to \mathbb{P}^1$ with section B, so that $e \cdot B = 1$. Yet $B^2 = -2$, so we obtain that $-K_X \cdot B < 0$, a contradiction. ■

Theorem 4.35. *In the situation of Theorem 4.34, X is isomorphic to the Fano threefold of Example 4.6.*

Proof. By Theorem 4.34, B is a fibre of the blow-up $\mu\colon X \to T$, hence its conormal bundle is

$$\mathcal{N}^\vee_{B/X} \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1).$$

Therefore, $E \cong \mathbb{F}_1$ and $-E_{|E} \sim f + N$, where f is a fibre of the ruling $\nu_{|E}\colon E \to B$ and N is the negative section. Moreover, it is not difficult to compute the anticanonical degree of X: since $-K_M$ is nef and not big (i.e. $-K_M^3 = 0$), we have $-K_X^3 = 4$.

Step 1. *The map $\varphi_{|E}\colon E \to S$ is an isomorphism.*

We show that φ does not contract the negative section $N \subset E$. If E'_Γ is the strict transform of E_Γ, then

$$E'_\Gamma \cap E = N,$$

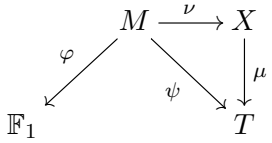
and this follows by the surjection $\mathcal{N}^\vee_{B/X} \to \mathcal{N}^\vee_{B/E_\Gamma} \cong \mathcal{O}_{\mathbb{P}^1}$. In the surface $E'_\Gamma \cong E_\Gamma$, the curve N is a fibre of the ruling. Let ψ be the composition of the two blow-ups μ and ν, so that

$$-K_M \sim \psi^*(-K_T) - \nu^* E_\Gamma - E.$$

Since $\nu(N) = B$ is contracted by μ, we have $-K_M \cdot N = -E_\Gamma \cdot B - E \cdot N = 1$. This yields the claim. Moreover, Lemma 4.30 implies that φ is equidimensional, hence flat.

Step 2. *$-K_M$ is basepoint-free and* $\operatorname{Bs}|-K_X| \cong B$.

Recall that $-K_M \sim \varphi^*H$, with H an ample $\mathbb{Q}$-Cartier divisor on S. By Step 1, we have $S \cong E \cong \mathbb{F}_1$, so S is smooth and H is Cartier. It is not difficult to verify that any ample divisor on the surface $\mathbb{F}_1$ is basepoint-free, thus $-K_M$ is basepoint-free and the second claim follows by the isomorphism $H^0(M, -K_M) \cong H^0(X, -K_X)$.



Moreover, it is possible to determine H. Under the isomorphism $S \cong E$, H corresponds to the divisor $-K_{M|E}$ on E, that is

$$H \sim -K_{M|E} \sim \nu_{|E}^*(-K_{X|B}) - E_{|E} \sim 2f + N.$$

Step 3. *The surface E_Γ is isomorphic to $\Gamma \times \mathbb{P}^1$, and $\Gamma \subset T$ is an elliptic curve.*

Recall from Step 1 that $E'_\Gamma \cap E = N$ and $-K_M \cdot N = 1$. We claim that $-K_M$ has degree 1 on all the curve of the ruling $\psi_{|E'_\Gamma} : E'_\Gamma \to \Gamma$. Indeed, let $l \subset E'_\Gamma$ be a fibre of the ruling distinct from N: then, as above, $-K_M \cdot l = -E_\Gamma \cdot \nu(l) = 1$. Thus, l is not contracted by φ. Since $l \equiv_M (E'_\Gamma \cap E)$, we have

$$\varphi_*(l) \equiv_{\mathbb{F}_1} \varphi_*(E'_\Gamma \cap E) = N.$$

Therefore $\varphi_*(l) = N$, as N is the negative section. This implies that the ruled surface $\mu_{|E_\Gamma} : E_\Gamma \to \Gamma$ admits a second fibration $\varphi_{|E'_\Gamma} : E'_\Gamma \to N \cong \mathbb{P}^1$. Hence, E_Γ is isomorphic to the product $\Gamma \times \mathbb{P}^1$. Finally, the fibres of $E'_\Gamma \to \mathbb{P}^1$ are smooth elliptic curves and, as curves in E_Γ, they are sections of $E_\Gamma \to \Gamma$. Thus, we obtain the second claim.

Step 4. *T is Fano and has index $i_T = 2$. Moreover, a general element in $|-\frac{1}{2}K_T|$ is smooth along Γ.*

Since X is Fano, we know that $-K_T$ has positive degree on all the curves $C \subset T$ other than Γ. Let e be a fibre of $\varphi_{|E'_\Gamma} : E'_\Gamma \to \mathbb{P}^1$. Then

$$-K_T \cdot \Gamma = \psi^*(-K_T) \cdot e = (E'_\Gamma + 2E) \cdot e = 2.$$

This shows that T is Fano of index $i_T \leq 2$.

On the other hand, we know from Step 2 that $-K_M \sim \varphi^*(2f+N)$. Since $E'_\Gamma = \varphi^* N$ is contracted by $\psi = \mu \circ \nu$, we see that the reducible divisor $\varphi^*(2f + N) \in |-K_M|$ maps onto the reducible divisor

$$2\psi_*(\varphi^* f) \in |-K_T|.$$

Thus the index i_T of T is 2. Finally, since $\varphi^* f$ is smooth and meets the exceptional locus of ψ transversely, the divisor $\psi_*(\varphi^* f) \in |-\frac{1}{2}K_T|$ contains Γ, is irreducible and smooth (along Γ).

Let H_T be a fundamental divisor of T, that is $H_T \sim -\frac{1}{2}K_T$: Step 4 shows that $h^0(T, \mathcal{O}_T(H_T)) \neq 0$. Moreover, it follows from Kodaira vanishing that $\operatorname{Bs}|H_{T|H_T}| = \operatorname{Bs}|H_T|$ and that $h^1(H_T, \mathcal{O}_{H_T}) = 0$.

Step 5. *$(-K_T)^3 = 8$ and $H_T^3 = 1$.*

Since $E_\Gamma \cong \Gamma \times \mathbb{P}^1$, we know that the restriction of $-K_X$ to E_Γ is numerically equivalent to $\lambda l + \Gamma$ for some $\lambda > 0$ (here by Γ we denote a section of $\nu_{|E_\Gamma} : E_\Gamma \to \Gamma$ of the form $\Gamma \times \{pt\}$). From Step 2 and Step 3, we see that the restriction of $-K_M$ to E'_Γ is numerically equivalent to Γ. Since the restrictions differ by $B \sim l$, we obtain $\lambda = 1$.

In the same spirit, recall that the restriction of E_Γ to itself is numerically equivalent to $\lambda l - \Gamma$ with $\lambda \in \mathbb{Z}$. We know that the restriction of E'_Γ to itself is numerically equivalent to $-\Gamma$, since it is the pull-back of a (-1)-curve (see Step 3). Since the restrictions differ by $B \sim l$, we obtain again $\lambda = 1$.

By assumption, we have $-K_X \sim \mu^*(-K_T) - E_\Gamma$. Since $-K_X^3 = 4$, it is not difficult to show the claim.

Remark. By Steps 4 and 5, we obtain that T is a smooth del Pezzo threefold of degree 1. By classification (see [**IP99**, Prop. 3.2.4]), we know that the linear system $|H|$ has a unique base point $\{P_0\}$. With this information, is it not difficult to show that $\mu^{-1}(P_0) \subset \mathrm{Bs}\,|-K_X|$, so that $\mu^{-1}(P_0) = B$ by Step 2, and this would conclude the proof. But we do not not want to use the classification, so we argue differently.

Step 6. *$|\mathcal{I}_{\Gamma/T} \otimes \mathcal{O}_T(H_T)| \subset |H_T|$ has codimension 1.*

Recall that $-K_T \cdot \Gamma = 2$ (Step 4), so that $\deg(\mathcal{O}_\Gamma(H_T)) = 1$, $h^0(\Gamma, H_{T|\Gamma}) = 1$, and $h^1(\Gamma, H_{T|\Gamma}) = 0$, since Γ is elliptic. The short exact sequence

$$0 \to \mathcal{I}_{\Gamma/T} \otimes \mathcal{O}_T(H_T) \to \mathcal{O}_T(H_T) \to \mathcal{O}_\Gamma(H_T) \to 0$$

gives the exact sequence in cohomology

$$0 \to H^0(T, \mathcal{I}_{\Gamma/T} \otimes \mathcal{O}_T(H_T)) \to \mathbb{C}^s \to \mathbb{C} \to H^1(T, \mathcal{I}_{\Gamma/T} \otimes \mathcal{O}_T(H_T)) \to 0,$$

where $s = h^0(T, \mathcal{O}_T(H_T))$. This implies $s - 1 \leq h^0(T, \mathcal{I}_{\Gamma/T} \otimes \mathcal{O}_T(H_T)) \leq s$.

Let us assume that $h^0(T, \mathcal{I}_{\Gamma/T} \otimes \mathcal{O}_T(H_T)) = s$, so that every effective divisor in $|H_T|$ contains Γ. Therefore we have $H_{T|H_T} \sim \Gamma + R$, with $R := \sum a_i R_i$, $a_1 \geq 0$, $\{R_i\}$ irreducible curves, and $H^0(H_T, \mathcal{O}_{H_T}(H_T)) \cong H^0(H_T, \mathcal{O}_{H_T}(R))$. We now show that either $R = 0$ or R is nef and big, so that $H^1(H_T, \mathcal{O}_{H_T}(R)) = 0$. If $R \neq 0$, then we have $\Gamma^2 = 1$ (by adjunction) and $\Gamma \cdot R = 0$, and $1 \leq H_{T|H_T} \cdot R_i = R \cdot R_i$. Then R is nef and $R^2 > 0$. Finally, the exact sequence

$$0 \to \mathcal{O}_{H_T}(R) \to \mathcal{O}_{H_T}(H_T) \to \mathcal{O}_\Gamma(H_T) \to 0$$

gives a short exact sequence for global sections

$$0 \to H^0(H_T, \mathcal{O}_H(R)) \to H^0(H_T, \mathcal{O}_{H_T}(H_T)) \to \mathbb{C} \to 0,$$

thus a contradiction.

Step 7. *$h^0(T, \mathcal{O}_T(H_T)) = 3$ and $\mathrm{Bs}\,|H_T| = \mu(B)$.*

Let $H_\Gamma \in |H_T|$ be a smooth prime divisor containing Γ (Step 4). Let moreover $H'_\Gamma = \mu^* H_\Gamma - E_\Gamma$ be the strict transform of H_Γ in X, and $H''_\Gamma = \nu^* H'_\Gamma$ be the strict transform of H'_Γ in M. Then

$$h^0(T, \mathcal{I}_{\Gamma/T} \otimes \mathcal{O}_T(H_T)) = h^0(X, H'_\Gamma) = h^0(M, H''_\Gamma).$$

Moreover, we have $H''_\Gamma = \varphi^* f$, where f is a fibre of the ruling of $S \cong \mathbb{F}_1$. Since φ has connected fibres, we conclude that

$$h^0(M, H''_\Gamma) = h^0(\mathbb{F}_1, l) = 2.$$

Let $H'_\Gamma \subset X$ be the strict transform of H_Γ. Then $\mu^* H_T + H'_\Gamma \in |-K_X|$. By the exact sequence

$$0 \to \mathcal{O}_X(\mu^* H_T) \to \mathcal{O}_X(-K_X) \to \mathcal{O}_{H'_\Gamma}(-K_X) \to 0,$$

we have that $\mathrm{Bs}\,|-K_{X|H'_\Gamma}| = B \cap H'_\Gamma$. Moreover, $-K_{X|H'_\Gamma} \sim H_{T|H_\Gamma}$ under the isomorphism $H'_\Gamma \cong H_\Gamma$. This shows that $\mathrm{Bs}\,|H_T| = \mu(B)$. ■

4.5.3. The fibration case. Let X be a smooth Fano threefold satisfying the Assumption 4.2. We keep the same notation as in §4.5.1.

We will now consider the case where X has an elementary contraction $\mu\colon X \to T$ onto a surface. It follows from [**Wiś91a**, Th. 1.2, Prop. 4.3] that $\mu\colon X \to T$ is a conic bundle, and that T is smooth and Fano. Our objective is to understand the relation between a curve $B \subseteq \mathrm{Bs}\,|-K_X|$ and the μ-fibres, and we show that X is the Fano threefold of Example 4.7.

Theorem 4.36. *Let X be a smooth Fano threefold and assume that there exists an irreducible curve $B \subseteq \mathrm{Bs}\,|-K_X|$. Assume moreover that X admits an elementary fibration $\mu\colon X \to T$ onto a surface. Then X is the Fano threefold of Example 4.7, that is $X \cong T \times \mathbb{P}^1$, where T is a smooth del Pezzo surface of degree 1 and μ is the projection. Moreover, $B \cong \mathbb{P}^1$ is a μ-fibre, $B = \mathrm{Bs}\,|-K_X|$, and $\mu(B) = \mathrm{Bs}\,|-K_T|$.*

Before proceeding with the proof, we give some preliminary results.

Remark 4.37. Let $\nu\colon M \to X$ be the blow-up of X along B introduced in §4.5.1, and let us denote by $\psi := \mu \circ \nu\colon M \to T$ the composition of the conic bundle and the blow-up. Then ψ is a contraction of relative Picard number 2, and the extremal cone $\mathrm{NE}(\psi) \subset \mathrm{NE}(M)$ generated by the numerical classes of curves contracted by ψ has dimension 2, that is $\mathrm{NE}(\psi) = \mathbb{R}^+[f] + \mathbb{R}^+[\Gamma]$, where $f \subset E$ is a line of the ruling $\nu_{|E}\colon E \to B$ and $\nu(\Gamma) \subset X$ is a curve contained in a μ-fibre that depends on B. Note that $-K_M \cdot f = 1$.

Lemma 4.38. *The birational morphism $\varphi_{|E}\colon E \to S$ is an isomorphism.*

Proof. We argue by contradiction and assume that φ contracts the negative section $N \subset E \cong \mathbb{F}_d$.

Step 1. *We claim that $d \geq 3$.*

Let a, b be integers such that $b - a = d \geq 1$ and $\mathcal{N}^\vee_{B/X} \cong \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b)$. The negative section $N \subset E$ corresponds to the projection

$$\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b) \twoheadrightarrow \mathcal{O}_{\mathbb{P}^1}(a),$$

so that $E \cdot N = -a$. Since N is contracted by φ and $-K_M \cdot N = 0$, we get $a = K_X \cdot B$. Moreover, by adjunction, we have $-K_X \cdot B = 2 - a - b$. This implies that $b = 2$ and $d = 2 - K_X \cdot B \geq 3$.

Step 2. *The contradiction.*

The morphism φ is defined by a multiple of $-K_M$ and contracts the negative section $N \subset E$, so that we have $-K_{M|E} \sim N + df$. Therefore $H \equiv \mathcal{O}_S(1)$, where $O_S(1)$ is the hyperplane section of the embedding $C_d \hookrightarrow \mathbb{P}^{d+1}$ given by the linear system $|N + df|$. By Lemma 4.29, the direct image sheaf $\varphi_*\mathcal{O}_M(-K_M)$ is reflexive. Since it has rank 1 (the anticanonical is trivial, not only numerically trivial, on the general fibre), it coincides with $\mathcal{O}_S(1)$.

Now observe that if $H \in |\mathcal{O}_S(1)|$ is a divisor that passes through the summit of the vertex, it consists of d lines $\sum f_i$. Thus, its pull-back $\varphi^* \sum f_i \in |-K_M|$ has at least d irreducible components Y_i given by the unique irreducible component of $\varphi^* f_i$ that dominates f_i.

If $l' \subset M$ is the strict transform of a general fibre l of $\mu\colon X \to T$, it is not contracted by the elliptic fibration φ. Since $f_i \subset C_d$ is an ample $\mathbb{Q}$-Cartier Weil divisor, we see

that $Y_i \cdot l' > 0$. Thus we have $\nu_* Y_i \cdot l > 0$. Since

$$-K_X \equiv \nu_*(\varphi^* \sum f_i)$$

and l is movable, we obtain that $2 = -K_X \cdot l \geq d \geq 3$, a contradiction. ■

Lemma 4.39. *Assume that B is contracted by μ. Then B is a smooth fibre, and the divisor $-K_M$ is ψ-ample.*

Proof. If B is the reduction of a non-reduced fibre, we know that the conormal bundle of B in X is $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(2)$, so that $E \cong \mathbb{F}_3$ and the negative section N is contracted by φ (see Step 1 of the proof above). Thus we have $-K_{M|E} \sim N + 3f$, $S \cong C_3$ and $H \sim \mathcal{O}_S(1)$. Yet this contradicts Lemma 4.38.

If instead B is an irreducible component of a reduced fibre, let l be the other component. The conormal bundle of B in X is $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)$, so that $E \cong \mathbb{F}_1$ and N is not contracted by φ by Lemma 4.38. In particular, $\varphi_{|E}$ in an isomorphism and the φ-fibres are integral curves of arithmetic genus 1. On the other hand, if l' is the strict transform of l, then $-K_M \cdot l' = 0$. This implies that $l' \cong \mathbb{P}^1$ is a φ-fibre, a contradiction.

Therefore B is a smooth fibre. Let $l \neq B$ be another curve contracted by μ, with strict transform l', then $-K_M \cdot l' > 0$. Moreover, the conormal bundle of B in X is $\mathcal{O}_{\mathbb{P}^1}^{\oplus 2}$, so that E is isomorphic to the product $\mathbb{P}^1 \times \mathbb{P}^1$, $\varphi_{|E}$ is an isomorphism, and $-K_{M|E} \sim \mathcal{O}(1,2)$ is an ample divisor on E. This concludes the proof. ■

Proof of Theorem 4.36.

Step 1. *We show that $\mu(B)$ is a point.*

We argue by contradiction. Assume that B is not contracted by μ. Let l be an irreducible component of a μ-fibre that intersects B, and let l' be its strict transform. Since $-K_M$ is nef, we have

$$2 \geq -K_X \cdot l \geq E \cdot l' \geq 1,$$

where the inequality in the middle is sharp if and only if l' is not contracted by φ. In particular, B and l intersect at most in two points counting multiplicity, and $-K_{X|l}$ is globally generated.

1st case: assume that $-K_M$ is not ψ-ample. Then there exists an irreducible component l of a μ-fibre that intersects B such that the strict transform $l' \subset M$ satisfies $-K_M \cdot l' = 0$. This $l' \cong \mathbb{P}^1$ is contracted by φ. Yet $E \subset M$ is a relatively ample divisor that is a section, so by Lemma 4.30 every fibre of φ is an integral curve of arithmetic genus 1. This contradicts $p_a(l') = 0$.

2nd case: assume that $-K_M$ is ψ-ample. Consider the exact sequence

$$0 \to H^0(\mathcal{I}_{l/X} \otimes -K_X) \to H^0(X, -K_X) \xrightarrow{res_l} H^0(l, -K_X).$$

The restriction morphism is not surjective, since $B \cap l \neq \emptyset$, and we can assume that its image is not trivial. Otherwise $l \subset \mathrm{Bs}\,|-K_X|$, so that we can substitute B with l and M with the blow-up M' of X along l.

Since $-K_M$ is ψ-ample, we have $-K_X \cdot l = 2$ and B intersects l in a single point. This implies that l is a smooth μ-fibre. Note that by Lemma 4.29 we have

$$h^0(X, -K_X) = h^0(M, -K_M) = h^0(S, H).$$

Yet $S \cong \mathbb{F}_d$ by Lemma 4.38 and H is an ample Weil divisor on $\mathbb{F}_d$, so $h^0(S,H) \geq 4$ by Lemma 4.33(i). Since

$$H^0(X,-K_X) \xrightarrow{res_l} H^0(l,-K_X) \cong H^0(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(2))$$

is not surjective, we obtain that $h^0(\mathcal{I}_{l/X} \otimes -K_X) \geq 2$.

Assume first that the fixed part of the linear system $|\mathcal{I}_{l/X} \otimes -K_X|$ does not contain $\mu^*\mu(B)$. Let G be the fixed part of $|\mathcal{I}_{l/X} \otimes -K_X|$. Replace l by a fibre over $\mu(B)$ that is not contained in G. Let L be the mobile part of $|\mathcal{I}_{l/X} \otimes -K_X|$. Then two general elements $L_1, L_2 \in L$ have no irreducible component in common, but contain l. Since $l \neq B$, the strict transforms L_1', L_2' contain the strict transform $l' \subset M$. Let

$$L_1'' \subset L_1', L_2'' \subset L_2'$$

be the irreducible component that contains l'. Since l' is not contracted by φ (here we use that $-K_M$ is ψ-ample), and L_1'', L_2'' map onto curve by Corollary 4.32, we have

$$\varphi(L_1'') = \varphi(l') = \varphi(L_2'').$$

Since φ has integral fibres, this implies that $L_1'' = L_2''$. Thus $L_1, L_2 \in L$ have a common component, a contradiction.

Thus we can now suppose that fixed part of the linear system $|\mathcal{I}_{l/X} \otimes -K_X|$ contains $\mu^*\mu(B)$. Note that, by what precedes, the curve B maps isomorphically onto its image in T and every fibre of $\mu^*\mu(B) \to B$ is smooth: thus $X_B := \mu^*\mu(B)$ is a Hirzebruch surface, in particular it is smooth and isomorphic to its strict transform $X_B' \subset M$. By Corollary 4.32 the strict transform X_B' maps onto a curve in S. Yet the fibrations on Hirzebruch surface are known, they are the rulings. Thus φ contracts some smooth rational curves. Yet once again, this contradicts the fact that the fibres are irreducible of arithmetic genus one.

Step 2. *The second factorisation.*

Therefore, B is contracted by μ, and we can apply Lemma 4.39, so that B is a smooth fibre, $\mathcal{N}_{B/X}^\vee \cong \mathcal{O}_{\mathbb{P}^1}^{\oplus 2}$, $E \cong \mathbb{P}^1 \times \mathbb{P}^1$, $\varphi_{|E}\colon E \to S$ is an isomorphism, and $-K_M$ is ψ-ample. There exists another factorisation of ψ

$$\begin{array}{ccccc} & & M & \xrightarrow{\nu} & X \\ & \overset{\varphi}{\swarrow} & \downarrow{\scriptstyle \alpha} & \overset{\psi}{\searrow} & \downarrow{\scriptstyle \mu} \\ \mathbb{P}^1 \times \mathbb{P}^1 & & Z & \xrightarrow[\beta]{} & T \end{array}$$

as $\beta \circ \alpha$, where α and β are the Mori contractions corresponding respectively to the extremal rays $\mathbb{R}^+[\Gamma] \subset \mathrm{NE}(M)$ and $\mathbb{R}^+[\alpha_*(f)] \subset \mathrm{NE}(Z)$.

It is not difficult to show that Z is the blow-up of T along $\mu(B)$, and that M is a conic bundle over Z. We have $\alpha(E) = \mathrm{Exc}(\beta)$, and $\Delta_\alpha = \beta^{-1}\Delta_\mu$, where Δ_α and Δ_μ are the discriminant divisors of the two conic bundles.

Let B' be a section of $\nu_{|E}\colon E \to B$ of the form $B \times \{p\}$. We denote by $D := \varphi^*(\varphi(f))$, so that $-K_M \sim a\varphi^*(\varphi(B')) + bD$, for integers $a, b > 0$. By intersecting $-K_M$ with f and B', we see that $a = 1$ and $b = 2$. Therefore D has degree 1 on the general fibre of α, and it is an α-ample divisor. This implies that M is a $\mathbb{P}^1$-bundle over Z, and that D is a section of α for any line f of the ruling $\nu_{|E}\colon E \to B$. Moreover, X is a $\mathbb{P}^1$-bundle over T. Finally, $D \cap D' = \emptyset$ for any $f \neq f'$. Therefore $M \cong Z \times \mathbb{P}^1$ and $X \cong T \times \mathbb{P}^1$.

Since $-K_M \sim \varphi^*\mathcal{O}(1,2)$, it is globally generated and in particular $B = \mathrm{Bs}\,|-K_X|$. Moreover, since $X \cong T \times \mathbb{P}^1$, it follows from Lemma 4.4 that $\mu(B) = \mathrm{Bs}\,|-K_T|$.

Step 3. *T is a del Pezzo surface of degree 1.*

It remains to prove that the del Pezzo surface T has degree 1. Since $-K_M^3 = 0$ and $-K_M \sim \nu^*(-K_X) - E$, it is not difficult to show that $-K_X^3 = 6$. In a similar way, we see that $-K_T^2 = 1$. ■

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