



Morse inequalities for the Koszul complex of multi-persistence

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ABSTRACT

In this paper, we define the homological Morse numbers of a filtered cell complex in terms of relative homology of nested filtration pieces, and derive inequalities relating these numbers to the Betti tables of the multi-parameter persistence modules of the considered filtration. Using the Mayer-Vietoris spectral sequence we first obtain strong and weak Morse inequalities involving the above quantities, and then we improve the weak inequalities achieving a sharp lower bound for homological Morse numbers. Furthermore, we prove a sharp upper bound for homological Morse numbers, expressed again in terms of the Betti tables.

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1. Introduction

Topological data analysis [4] relies on algebraic topology to extract complex information from data, with the aim of recognizing complicated patterns, inferring the topological structure underlying the data or, more generally, complementing and enhancing standard techniques in data analysis. Persistent homology, one of the most successful methods of topological data analysis, summarizes the topology of the data at multiple scales, controlled by one measured parameter, and produces informative signatures based on homology [28]. As understanding complex correlations in multivariate data is one of the general goals of data analysis, the development of a multivariate version of persistent homology, called multi-parameter persistence or multi-persistence [7] is drawing increasing interest.

In persistent homology and multi-parameter persistence, algebraic objects called persistence modules are used to encode the data in a suitable way for the extraction of topological invariants. Typically, persistence modules are obtained from datasets endowed with measurements via an intermediate step, in which a com-

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binatorial or topological object (e.g., a simplicial complex or, more generally, a cell complex) is constructed from the data points. As data are discrete, without loss of generality from the algebraic point of view, we can use elements of $\mathbb{Z}^n$ ($n \geq 1$) to encode the values of n measurements. Filtering the given data according to increasing values of the measurements (in the coordinate-wise partial order of $\mathbb{Z}^n$), and applying homology, one obtains the corresponding n -parameter persistence module [28].

In the literature, the study of persistence modules has been tackled mainly from two different perspectives. In the first works appeared in the literature about persistence theory, a dataset was encoded as a manifold, and the measurements on it as a Morse function on it. In this perspective, persistence can be grounded in Morse theory, with persistence modules determined by pairs of critical points of functions that give *birth* and *death* to a topological feature [2,10,13,29]. This paradigm has also a combinatorial counterpart developed for accelerating computations and based on discrete Morse theory [1,26]. In the combinatorial setting the role of critical points is played by critical cells.

From a different standpoint, it was soon realized that the persistent homology of a filtered finite simplicial complex is simply a particular graded module over a polynomial ring [32]. Therefore, forgetting about the data and the measurements that yield them, persistence modules can be studied using tools of commutative algebra. In particular, new invariants for persistence modules can be obtained by considering the Betti tables¹ of a minimal free resolution of theirs [7]. There is, however, an important difference between the study of n -parameter persistence modules and the classical commutative algebra approach to graded modules: n -parameter persistence modules are usually obtained as the homology of a cell complex associated with data, and the connection between the algebraic object and the underlying cell complex is part of the investigation. The focus of the present article is precisely on the relation between the invariants of n -parameter persistence modules and the filtered cell complex from which it is obtained.

When the number of filtering parameters n is equal to 1, this dual perspective is easily interpreted by noticing that births are captured by the 0th Betti table and deaths by the 1st Betti table of persistence modules. Persistence theory is thus a canonical way of pairing births and deaths.

For $n \geq 2$ things get more complicated as there is no way of pairing births and deaths in a natural way, and Betti tables do not mirror entrance of critical cells in the filtration. In [19], this is heuristically explained by the presence of *virtual* critical cells.

Borrowing from the terminology of discrete Morse theory, where Morse numbers are defined as the number of critical cells in the various dimensions, we introduce the notion of a *homological Morse number* (see below and Section 2.3) to mean the dimension of the homology of a piece of the filtration relative to the union of all the pieces entered before.

The main goal of the present paper is to provide insights on the interplay between the values of the Betti tables of a persistence module in any number $n \geq 1$ of parameters, and the homological Morse numbers of an n -filtration inducing it.

In order to do so, we consider persistence modules obtained applying the q th homology functor to an n -parameter filtration $\{X^u\}_{u \in \mathbb{Z}^n}$ of a finite cell complex X . The corresponding p th Betti table $\xi_p^q: \mathbb{Z}^n \rightarrow \mathbb{N}$ is obtained as the p th homology of the Koszul complex associated with the persistence module, a strategy already used in [19] and later in [24]. As for the homological Morse numbers $\mu_q(u)$ of degree q at grade $u \in \mathbb{Z}^n$ of the filtration, we define them as the dimension of the homology at grade u relative to the previous grades: $\mu_q(u) := \dim H_q(X^u, \cup_j X^{u-e_j})$. Informally, $\mu_q(u)$ is the number of critical cells of $X^u - \cup_j X^{u-e_j}$ whose entrance is reflected in a change of q th homology. More formally, the homological Morse number $\mu_q(u)$ can be seen as the “natural” lower bound, in terms of homology, for the number of critical cells of dimension q entering at grade u (see Section 2.3).

¹ In commutative algebra Betti tables are often called (multi-graded) Betti numbers, but we prefer avoiding calling them so to avoid confusion with the Betti numbers of persistent homology groups.

Under these assumptions, we present inequalities relating the homological Morse numbers $\mu_q(u)$ of a multi-filtration of a cell complex and the homology invariants of the Koszul complex of the persistence module obtained from it. In other words, we obtain Morse-type inequalities for multi-parameter persistence.

We start with the *strong Morse inequalities* according to which an alternating sum of the entries of the Betti tables of the persistence modules of X at grade $u \in \mathbb{Z}^n$ (where the summation is on the homology degrees while the filtration grade is fixed) is bounded from above by an alternating sum of the homological Morse numbers of the filtration of X at grade u . This is Theorem 5.1:

Theorem. For each $q \geq 0$, and each fixed grade $u \in \mathbb{Z}^n$, we have

$$\sum_{i=0}^q (-1)^{q+i} \mu_i(u) \geq \sum_{i=0}^q (-1)^{q+i} \left(\xi_0^i(u) - \sum_{p=1}^{i+1} \xi_p^{i+1-p}(u) \right).$$

As usual, the strong inequalities imply the *weak Morse inequalities*, given in Corollary 5.2:

Corollary. For each $q \geq 0$, and each fixed grade $u \in \mathbb{Z}^n$, we have

$$\mu_q(u) \geq \xi_0^q(u) - \sum_{p=1}^{q+1} \xi_p^{q+1-p}(u).$$

Moreover, we can define the *Euler characteristic* of a filtration at grade u by considering the relative homology of $(X^u, \cup_j X^{u-e_j})$ and setting

$$\chi(X^u, \cup_j X^{u-e_j}) := \sum_q (-1)^q \dim H_q(X^u, \cup_j X^{u-e_j}).$$

It is interesting to see how this notion of Euler characteristic of a filtration relates to the Betti tables of its persistence modules. As shown in Theorem 6.2, for q large enough, the q th strong Morse inequality is actually an equality:

Theorem. $\chi(X^u, \cup_j X^{u-e_j}) = \sum_{i=0}^{\dim X+1} (-1)^i \sum_{p=0}^i \xi_p^{i-p}(u)$.

The weak Morse inequalities of Corollary 5.2 are too weak to be also sharp. In order to achieve sharpness, we improve them by proving Theorem 7.3, which can be summarized as follows.

Theorem. For an n -parameter filtration $\{X^u\}_{u \in \mathbb{Z}^n}$, for each grade $u \in \mathbb{Z}^n$, and for each $q \geq 0$,

$$\mu_q(u) \geq \xi_0^q(u) + \xi_1^{q-1}(u) - \sum_{p=1}^{n-1} \xi_{p+1}^{q-p}(u) + R,$$

where R is a non-negative integer.

Examples are provided to show that these estimates are sharp for every number of parameters n . In particular, these inequalities show that a non-trivial p th Betti table at u does not need the entrance of a critical cell at u as the presence of positive and negative terms in the right-hand side compensate each other. This is different from what happens when $n = 1$. Indeed, in the case of a single parameter, the above inequalities reduce to $\mu_q(u) \geq \xi_0^q(u) + \xi_1^{q-1}(u)$. Since in one-parameter persistence ξ_0^q counts the number of persistence births in homology of degree q , and ξ_1^{q-1} counts the number of persistence death in homology

of degree $q - 1$, these inequalities say that, when $n = 1$, in order to have a birth or a death we necessarily need the entrance of a critical cell, as is well known.

On the other hand, in Theorem 7.5 we also present the inverse inequalities, proven to be sharp as well, showing that a non-zero homological Morse number at grade u of the filtration necessarily causes some Betti table to become non-trivial:

Theorem. *For an n -parameter filtration $\{X^u\}_{u \in \mathbb{Z}^n}$, for each grade $u \in \mathbb{Z}^n$, and for each $q \geq 0$, we have*

$$\mu_q(u) \leq \sum_{i=0}^n \xi_i^{q-i}(u).$$

We observe that when $n = 1$ these new bounds reduce to $\mu_q(u) \leq \xi_0^q(u) + \xi_1^{q-1}(u)$. Here it is important to underline that these inequalities hold thanks to the fact that we are considering homological Morse numbers and not the numbers of all critical cells entering at u .

All the inequalities provided in this paper are obtained via the Mayer-Vietoris spectral sequence associated with a double complex built from the filtration of X . This strategy generalizes that of [22] where the particular case of $n = 2$ is studied applying the Mayer-Vietoris homology exact sequence. In particular, differently from papers like [5,17,23], where the Mayer-Vietoris spectral sequence is used in the context of single-parameter persistent homology to merge local data into global information, we work only locally at a fixed grade u of the multi-parameter filtration, but considering all the possible homology degrees.

At a basic level, our inequalities prove that a persistence module having “large” Betti tables at grade u does not necessarily come from a filtration with a large number of critical cells entering at u unless the p th Betti tables with $p \geq 2$ are trivial. On the other hand, a large homological Morse number necessarily implies large values in the Betti tables of specific indices.

This is a more diversified behavior than the case $n = 1$, in which we have $\mu_q(u) = \xi_0^q(u) + \xi_1^{q-1}(u)$ by combining the inequalities introduced above. In other words, the homological Morse number of degree q is equal to the number of births in degree q plus the number of deaths in degree $q - 1$.

At a higher level, we believe our results are interesting from different perspectives. Firstly, in contrast to the state of the art in the literature where multi-parameter persistence modules are usually studied as obtained by applying homology in an arbitrary but fixed degree, our inequalities show the interplay among persistent homology modules at all the various homology degrees simultaneously. In particular, we can see that, contrary to one-parameter persistence where the entrance of a critical cell of dimension q can modify only the persistence modules in degree q or $q - 1$, for multi-parameter persistence the effect of its entrance may involve many more homology degrees.

Secondly, starting from Knudson’s observation in [19] about the fact that the Betti tables in multi-parameter persistence are determined not only by elements corresponding to real critical cells in the filtration, but also by elements corresponding to virtual cells, our inequalities allow for a measure of the gap between the number of real critical cells and that of virtual critical cells.

Organization of the article. In Section 2 we review the needed background on cell complexes, Morse inequalities and persistence modules, and provide a brief description of the Mayer-Vietoris spectral sequence. In Section 3 we describe the Koszul complex of a multi-parameter persistence module and its Betti tables. In Section 4 we introduce the Mayer-Vietoris spectral sequence of a multi-parameter filtration and show its relation with the Betti tables. In Section 5 we derive Morse inequalities for multi-parameter persistence modules, which are applied in Section 6 to obtain Euler characteristic formulas for the relative homology of the filtration. In Section 7 we improve our Morse inequalities using the Mayer-Vietoris spectral sequence and show the sharpness of our lower and upper bound for homological Morse numbers in terms of the Betti tables.

2. Preliminaries

2.1. Chain complexes, cell complexes and homology

Let $\mathbb{F}$ be a fixed field. In this work, we consider bounded finitely generated chain complexes $C_* = (C_q, \partial_q)_{q \in \mathbb{Z}}$ over $\mathbb{F}$, simply called *chain complexes*, meaning that $C_q = 0$ whenever $q < 0$ or $q \geq m$, for some $m \in \mathbb{N}$, and each C_q is a finite dimensional vector space over $\mathbb{F}$. Let us further assume that a distinguished (finite) $\mathbb{F}$ -basis X_q of each C_q is given, so that $C_q \cong \bigoplus_{\sigma \in X_q} \mathbb{F}\sigma$. A chain complex C_* endowed with such distinguished bases is called a *based chain complex*. We use the notation $C_*(X) = (C_q(X), \partial_q)_{q \in \mathbb{Z}}$ to explicitly recall the fixed bases X of C_* . We can express the differentials $\partial_q : C_q(X) \rightarrow C_{q-1}(X)$ with respect to the fixed bases as

$$\partial_q(\tau) = \sum_{\sigma \in X_{q-1}} \kappa(\tau, \sigma) \sigma$$

for each $\tau \in X_q$; in other words, for each $\tau \in X_q$ and $\sigma \in X_{q-1}$, we denote by $\kappa(\tau, \sigma)$ the coefficient with which σ appears in $\partial_q(\tau)$.

The distinguished bases of C_* inherit a combinatorial structure which coincides with the abstract notion of a cell complex as introduced by Lefschetz [21] (see also [18]). In topological data analysis, considering this equivalent combinatorial perspective is sometimes advantageous, since a cell complex is usually constructed from the data and hence interpretable in the concrete situation at hand. A *cell complex* is a finite graded set $X = \bigsqcup_{q \in \mathbb{Z}} X_q$, whose elements are called *cells*, endowed with an *incidence function* $\kappa : X \times X \rightarrow \mathbb{F}$. A cell $\sigma \in X_q$ is called a *q-cell* or a cell of *dimension q*, denoted $\dim \sigma = q$. The *dimension of X* is defined as the maximum dimension of its cells. The incidence function must satisfy the following conditions: (i) $\kappa(\tau, \sigma) \neq 0$ implies $\dim \tau = \dim \sigma + 1$, and (ii) for each τ and σ in X , it holds $\sum_{\rho \in X} \kappa(\tau, \rho) \cdot \kappa(\rho, \sigma) = 0$. We regard X as the graded poset endowed with the partial order $\leq$ generated by the covering relation $\sigma < \tau$ whenever $\kappa(\tau, \sigma) \neq 0$.

We underline that based chain complexes constitute a rather general setting, since chain complexes canonically associated with many combinatorial or topological objects (such as simplicial complexes, cubical complexes, finite CW complexes) fall within this definition.

As an example, an *abstract simplicial complex* Δ given by a collection of non-empty finite subsets of a given set S , with the property of being closed under taking subsets, can be regarded as a cell complex as follows: each $\sigma \in \Delta$ containing $q + 1$ elements can be viewed as a q -cell, and called a *q-simplex*, and in particular singletons are called vertices. Fixing an ordering for vertices induces an ordering on the elements of each simplex, and one can define the incidence function

$$\kappa(\tau, \sigma) := \begin{cases} (-1)^i & \text{if } \sigma \text{ is obtained from } \tau \text{ by removing its } (i+1)\text{th element} \\ 0 & \text{otherwise} \end{cases}$$

which induces the usual simplicial boundary map.

A collection of subsets $A_q \subseteq X_q$ freely generates a chain subcomplex $C_*(A) \subseteq C_*(X)$ if and only if $A = \bigsqcup_{q \in \mathbb{Z}} A_q$ is a *subcomplex* of X , meaning that, endowed with the restriction of the incidence function of X , it is a cell complex in its own right. Given a cell complex X and a subcomplex $A \subseteq X$, the relative chain complex $C_*(X, A)$ is defined as the chain complex $(C_q(X)/C_q(A), \partial'_q)_{q \in \mathbb{Z}}$, with ∂'_q being the differential induced by ∂_q on the quotient.

Applying q th homology to a chain complex C_* gives the $\mathbb{F}$ -module $H_q(C_*) = \ker \partial_q / \operatorname{im} \partial_{q+1}$, denoted $H_q(X)$ if the chain complex has a distinguished basis X . Analogously, the notation $H_q(X, A)$ is used for homology of a relative chain complex $C_*(X, A)$. In this paper, homology is always assumed to be over a

fixed field $\mathbb{F}$, so that taking homology or relative homology of a complex always gives (finite-dimensional) $\mathbb{F}$ -vector spaces.

2.2. Standard Morse inequalities

Given a (non-negatively graded) chain complex $C_* = (C_q, \partial_q)$ and setting $c_q := \dim C_q$, the *strong Morse inequalities* are:

$$\sum_{i=0}^q (-1)^{q+i} c_i \geq \sum_{i=0}^q (-1)^{q+i} \dim H_i(C_*), \quad (2.1)$$

for all $q \geq 0$. These inequalities are obtained via standard linear algebra by observing that the equations

$$\dim H_i(C_*) = \dim \ker \partial_i - \dim \operatorname{im} \partial_{i+1} = c_i - \dim \operatorname{im} \partial_i - \dim \operatorname{im} \partial_{i+1},$$

for all i , imply that the difference between the left-hand side and the right-hand side of (2.1) is $\dim \operatorname{im} \partial_{q+1}$. Strong inequalities imply *weak Morse inequalities*: $c_q \geq \dim H_q(C_*)$ for all $q \geq 0$. As is well known, they are obtained simply by observing that $c_q = \sum_{i=0}^q (-1)^{q+i} c_i + \sum_{i=0}^{q-1} (-1)^{q-1+i} c_i$ and applying the corresponding strong inequalities.

Moreover, if C_* is bounded, for values of q sufficiently large the strong inequalities are actually equalities involving the Euler characteristic $\chi(C_*) := \sum_{q \geq 0} (-1)^q c_q$ of C_* : it holds that

$$\chi(C_*) = \sum_{q \geq 0} (-1)^q \dim H_q(C_*).$$

Weak Morse inequalities represent constraints on the number of generators of a chain complex C_* , which can be improved by replacing C_* with a chain complex quasi-isomorphic to it with less generators. This strategy is used, for example, in [12] where, endowing a regular cell complex X with a discrete Morse function f , $C_*(X)$ is shown to be quasi-isomorphic to the Morse complex containing only the critical cells of f . Thus, c_q can be taken to coincide with the number of critical cells of f with dimension q . Similarly, in the case of a PL Morse function defined on a simplicial complex, strong and weak inequalities hold with c_q being the number of critical vertices of index q [8].

2.3. Multi-filtrations and multi-parameter persistence

Persistent homology was originally introduced as a method to encode in a single object the evolution of the homology of a family of nested cell complexes (usually simplicial complexes) parametrized by a linearly ordered set of indexes, such as the integers or the reals [2,10,13,29]. Later it became clear that families of nested complexes parametrized over other sets of indices can be equally relevant (see, e.g., [28] for a review). In particular, *multi-persistence* [7] treats the case of integer parameters along multiple directions, that is a grid. This is the setting we consider here.

For an integer $n \geq 1$, indicating the grid dimension, we denote by $[n]$ the set $\{1, 2, \dots, n\}$, by $\{e_j\}_{j \in [n]}$ the standard basis of $\mathbb{Z}^n$, and by $\preceq$ the coordinate-wise partial order on $\mathbb{Z}^n$: if $u = (u_i), v = (v_i) \in \mathbb{Z}^n$, we write $u \preceq v$ if and only if $u_i \leq v_i$, for all $1 \leq i \leq n$.

An *n-parameter persistence module* V consists in a collection $\{V^u\}_{u \in \mathbb{Z}^n}$ of $\mathbb{F}$ -vector spaces and a collection $\{\varphi^{u,v} : V^u \rightarrow V^v\}_{u \preceq v \in \mathbb{Z}^n}$ of linear maps such that $\varphi^{u,w} = \varphi^{v,w} \circ \varphi^{u,v}$ whenever $u \preceq v \preceq w$, and $\varphi^{u,u} = \operatorname{id}_{V^u}$, for all u .

In applications, persistence modules usually originate from filtrations of cell complexes. An *n-filtration* of a complex X is a family $\mathcal{X} = \{X^u\}_{u \in \mathbb{Z}^n}$ of subcomplexes of X such that $X^u \subseteq X^v$ whenever $u \preceq v$. If

$n > 1$ we refer to $\mathcal{X} = \{X^u\}_{u \in \mathbb{Z}^n}$ generically as a *multi-filtration*, as opposed to the case $n = 1$ that is called simply a (single-parameter) filtration. The index $u \in \mathbb{Z}^n$ is called a *filtration grade*. If $\sigma \in X^u - \bigcup_{j=1}^n X^{u-e_j}$, then u is called an *entrance grade* of σ in $\mathcal{X}$. The *dimension* of $\mathcal{X}$ is, by definition, the dimension of X .

Throughout this article, we make the important assumption that all filtrations $\mathcal{X} = \{X^u\}_{u \in \mathbb{Z}^n}$ we consider arise from the sublevel sets of an order-preserving function $f : (X, \leq) \rightarrow (\mathbb{Z}^n, \preceq)$, where the partial order on X is defined in Section 2.1. This means that $X^u = \{\sigma \in X \mid f(\sigma) \preceq u\}$, for every $u \in \mathbb{Z}^n$. In topological data analysis, such a filtration of a cell complex X is usually called *one-critical* [6] because every cell $\sigma \in X$ admits exactly one entrance grade. In what follows, this assumption on filtrations will be crucial, as it ensures that, for each subset $\sigma \subseteq [n]$, setting $e_\sigma = \sum_{j \in \sigma} e_j$, we have

$$\bigcap_{j \in \sigma} X^{u-e_j} = X^{u-e_\sigma}. \quad (2.2)$$

Additionally, we assume all filtrations to be *bounded* by requiring that $X^u \neq \emptyset$ implies $0 \preceq u$, and that $X^u = X$ whenever u is sufficiently large.

Applying the q th homology functor to an n -filtration $\mathcal{X} = \{X^u\}_{u \in \mathbb{Z}^n}$ yields the n -parameter persistent homology module $V_q = \{V_q^u, \iota_q^{u,v}\}_{u \preceq v \in \mathbb{Z}^n}$, with $V_q^u = H_q(X^u)$ and $\iota_q^{u,v} : H_q(X^u) \rightarrow H_q(X^v)$ induced by the inclusion maps $X^u \hookrightarrow X^v$ for $u \preceq v$. We denote this persistence module by $V_q = H_q(\mathcal{X})$. Inspired by the one-parameter situation where a critical filtration grade is characterized by the property that the relative homology of the pair (X^u, X^{u-1}) is non-trivial (cf., e.g., [11]), a grade $u \in \mathbb{Z}^n$ of a multi-filtration $\mathcal{X} = \{X^u\}_{u \in \mathbb{Z}^n}$ will be said to be a *critical filtration grade of index q* if $H_q(X^u, \bigcup_j X^{u-e_j})$ is non-trivial. Clearly, critical filtration grades are a subset of entrance grades. We call the number

$$\mu_q(u) := \dim H_q(X^u, \bigcup_{j=1}^n X^{u-e_j}) \quad (2.3)$$

the *homological Morse number* of degree q at $u \in \mathbb{Z}^n$. Informally, we can think of $\mu_q(u)$ as the number of critical cells whose entrance at $u \in \mathbb{Z}^n$ has an effect on q th homology of the filtration. The connection with the number of critical cells with entrance grade u , as defined in discrete Morse theory [12,20] (with the notion of *critical* depending on the choice of a discrete gradient vector field, also called an acyclic matching), can be made rigorous. Let $m_q(u)$ denote the q th *Morse number* at u of the filtration $\mathcal{X}$, defined as the number of critical q -cells with entrance grade u ; then $m_q(u) \geq \mu_q(u)$ [22, Prop. 1], for every choice of an acyclic matching used to define $m_q(u)$.

2.4. The Mayer-Vietoris spectral sequence

In this subsection we provide a brief description of the Mayer-Vietoris spectral sequence. We follow [3, Ch. VII] and adapt the construction to the case of cell complexes. The Mayer-Vietoris spectral sequence is a particular case of a spectral sequence associated with a double complex, a standard construction that can be found in most books on homological algebra (see for example [25,30,31]). More details on the Mayer-Vietoris spectral sequence in our situation of interest are given in Section 4.

Let X be a cell complex (as defined in Section 2.1) and let $\{X^j\}_{j \in J}$ be a collection of subcomplexes of X , with J a totally ordered index set. The Mayer-Vietoris spectral sequence relates the homology of the union of the collection $\bigcup_{j \in J} X^j$ with the homology of the subcomplexes X^j and their intersections $\bigcap_{j \in \sigma} X^j$ for $\sigma \subseteq J$. The *nerve* Σ of the collection $\{X^j\}_{j \in J}$ is defined as the abstract simplicial complex of all $\sigma \subseteq J$ such that $\bigcap_{j \in \sigma} X^j \neq \emptyset$. For all $p \geq 0$ we denote Σ_p the set of p -simplices of Σ , which are of the form $\sigma = \{j_1 < \dots < j_{p+1}\}$. For all p , consider the chain complexes

$$C_{p,*} = \bigoplus_{\sigma \in \Sigma_p} C_*(\bigcap_{j \in \sigma} X^j)$$

with differential maps

$$\delta_{p,*} : C_{p,*} \rightarrow C_{p-1,*} \quad (2.4)$$

between them defined as follows: for $1 \leq \ell \leq p+1$ and $\sigma = \{j_1 < \dots < j_{p+1}\}$ consider $\partial_\ell \sigma := \{j_1, \dots, j_\ell, \dots, j_{p+1}\}$, obtained by removing j_ℓ ; then, for $p \geq 1$, observe that the inclusions $C_*(\cap_{j \in \sigma} X^j) \hookrightarrow C_*(\cap_{j \in \partial_\ell \sigma} X^j)$ induce chain maps $\delta_{p,*}^{(\ell)} : C_{p,*} \rightarrow C_{p-1,*}$ and define $\delta_{p,*} := \sum_{i=0}^p (-1)^i \delta_{p,*}^{(p+1-i)}$. A chain map $\varepsilon_* = \delta_{0,*} : C_{0,*} \rightarrow C_*(\cup_{j \in J} X^j)$ is induced in a similar way by the inclusions $C_*(X^j) \hookrightarrow C_*(\cup_{j \in J} X^j)$. To facilitate our manipulations in the following sections, in the definition of $\delta_{p,*}$ we made a different choice from [3] regarding the alternating signs, which leads however to an isomorphic construction of the double complex.

The following sequence of chain complexes is exact (see [3]):

$$0 \leftarrow C_*(\cup_{j \in J} X^j) \xleftarrow{\varepsilon_*} C_{0,*} \xleftarrow{\delta_{1,*}} C_{1,*} \leftarrow \dots \leftarrow C_{p-1,*} \xleftarrow{\delta_{p,*}} C_{p,*} \leftarrow \dots \quad (2.5)$$

We note that if the index set J is finite and $m = |J|$, then $C_{p,*} = 0$ for all $p \geq m$. We will henceforth refer to the sequence of chain complexes $C_{0,*} \xleftarrow{\delta_{1,*}} \dots \xleftarrow{\delta_{p,*}} C_{p,*} \leftarrow \dots$ as the *truncation* of the exact sequence (2.5). As a consequence of the definitions, this is a double complex

$$C_{p,q} = \bigoplus_{\sigma \in \Sigma_p} C_q(\cap_{j \in \sigma} X^j)$$

with the horizontal differential $\delta_{p,q} : C_{p,q} \rightarrow C_{p-1,q}$ we just introduced, and the vertical differential $\partial_{p,q} : C_{p,q} \rightarrow C_{p,q-1}$ induced by the differential of $C_*(X)$, with a sign change of $(-1)^p$ to ensure that squares are anticommutative (that is, $\partial_{p-1,q} \delta_{p,q} + \delta_{p,q-1} \partial_{p,q} = 0$), which is the convention we choose for double complexes in this article.

Let T_* denote the total complex of the double complex $\{C_{p,q}, \delta_{p,q}, \partial_{p,q}\}_{p,q \in \mathbb{Z}}$, which is the chain complex with chain groups T_k and differentials d_k^T are defined by

$$T_k := \bigoplus_{p+q=k} C_{p,q}, \quad d_k^T := \sum_{p+q=k} (\delta_{p,q} + \partial_{p,q}).$$

The *Mayer-Vietoris spectral sequence* of the collection $\{X^j\}_{j \in J}$ is defined as the spectral sequence associated with the *first filtration* F_p^I of T_* (see e.g. [30, Ch. 10] for details), given by $F_p^I T_k := \bigoplus_{i \leq p} C_{i,k-i}$, with differentials induced by d^T . The Mayer-Vietoris spectral sequence converges to the homology $H_*(\cup_{j \in J} X^j)$ of the union $\cup_{j \in J} X^j$, since it can be shown [3] that $H_*(T_*) \cong H_*(\cup_{j \in J} X^j)$.

3. The Koszul complex of persistence and its Betti tables

The direct analysis of an n -parameter persistence module when $n > 1$ is in general quite complicated due to the lack of a finite or at least tame family of indecomposable summands for such objects, as proved by Gabriel in [14]. Hence, one often resorts to simpler albeit incomplete algebraic invariants of a persistence module. In this paper we focus on the Betti tables (also called multi-graded Betti numbers) of a persistence module, calculated via the homology of its Koszul complex.

Betti tables have been studied since early works on multi-parameter persistence [7,19] where it was noted that there is an equivalence between the category of n -parameter persistence modules and the category of n -graded modules over the polynomial ring $S := \mathbb{F}[x_1, \dots, x_n]$. Explicitly, the correspondence takes a persistence module $\{V^u, \varphi^{u,v}\}$ to the n -graded S -module $\bigoplus_{u \in \mathbb{Z}^n} V^u$ with the action of S defined by $x_i \cdot z = \varphi^{u, u+e_i}(z)$, for all $z \in V^u$ and all $i \in [n]$. This correspondence allows for the use of tools from

commutative algebra to study persistence modules. We refer the reader to [27] for background on such invariants for n -graded modules, while here we adopt the point of view of persistence modules.

Given an n -parameter persistence module V , and regarding it as an n -graded S -module $V = \bigoplus_{u \in \mathbb{Z}^n} V^u$ via the equivalence of categories mentioned above, the i th Betti table (or *multi-graded Betti numbers*) of V is defined as $\xi_i: \mathbb{Z}^n \rightarrow \mathbb{N}$ with

$$\xi_i(u) = \dim_{\mathbb{F}}(\mathrm{Tor}_i^S(V, \mathbb{F})(u)),$$

for all $u \in \mathbb{Z}^n$ and all $i \in \{0, 1, \dots, n\}$, where $\mathrm{Tor}_i^S(V, \mathbb{F})(u)$ is the part of grade u of $\mathrm{Tor}_i^S(V, \mathbb{F})$ viewed as an n -graded S -module. By definition of $\mathrm{Tor}_i^S(V, \mathbb{F})$, the i th Betti table $\xi_i(u)$ of V at u can thus be calculated by applying the functor $-\otimes_S \mathbb{F}$ to a free resolution of V , taking i th homology of the resulting chain complex and considering the dimension over $\mathbb{F}$ of the part of grade u of the homology module.

The general property $\mathrm{Tor}_i^S(V, \mathbb{F}) \cong \mathrm{Tor}_i^S(\mathbb{F}, V)$ from homological algebra (see, e.g., [30, Theorem 7.1]) provides an alternative way to calculate the i th Betti table of V by applying the functor $-\otimes_S V$ to a free resolution of $\mathbb{F}$ and taking i th homology. This yields an equivalent definition of the Betti tables of V based on its Koszul complex. Given an n -graded S -module $V = \bigoplus_{u \in \mathbb{Z}^n} V^u$, the *Koszul complex of V at grade $u \in \mathbb{Z}^n$* , denoted $\mathbb{K}_*(x_1, \dots, x_n; V)(u)$, is the part of grade u of the (n -graded) chain complex $\mathbb{K}_* \otimes_S V$, where $\mathbb{K}_* = \mathbb{K}_*(x_1, \dots, x_n)$ is the classical Koszul complex of S , defined for example in [27, Def. 1.26] or [9, Ch. 17.2]. Below, we provide an explicit definition of $\mathbb{K}_*(x_1, \dots, x_n; V)(u)$. Since $\mathbb{K}_*$ is a (minimal) free resolution of $\mathbb{F} \cong S/\langle x_1, \dots, x_n \rangle$ [27, Prop. 1.28], as observed above the i th homology module of the chain complex $\mathbb{K}_* \otimes_S V$ has dimensions (over $\mathbb{F}$) in the various grades $u \in \mathbb{Z}^n$ coinciding with the Betti table ξ_i of V . In other words, for each $u \in \mathbb{Z}^n$,

$$\xi_i(u) = \dim_{\mathbb{F}} H_i(\mathbb{K}_*(x_1, \dots, x_n; V)(u)).$$

For our purposes, we focus on the Betti tables ξ_i^q of the persistent homology module V_q arising from the q th homology of a filtration $\{X^u\}_{u \in \mathbb{Z}^n}$. For each $i \in \{0, 1, \dots, n\}$, the module appearing in degree i in the chain complex $\mathbb{K}_*(x_1, \dots, x_n; V_q)(u)$ is

$$\mathbb{K}_i(x_1, \dots, x_n; V_q)(u) = \bigoplus_{|\sigma|=i} H_q(X^{u-e_\sigma}) \quad (3.1)$$

with $e_\sigma := \sum_{j \in \sigma} e_j$ and $\sigma \subseteq [n]$. The modules $\mathbb{K}_i(x_1, \dots, x_n; V_q)(u)$ are zero for all $i \notin \{0, 1, \dots, n\}$. The differentials of $\mathbb{K}_*(x_1, \dots, x_n; V_q)(u)$ are defined in terms of the maps $\iota_q^{v,w}: H_q(X^v) \rightarrow H_q(X^w)$ that define V_q as follows: the restriction of

$$d_i: \mathbb{K}_i(x_1, \dots, x_n; V_q)(u) \rightarrow \mathbb{K}_{i-1}(x_1, \dots, x_n; V_q)(u) \quad (3.2)$$

to each direct summand $H_q(X^{u-e_\sigma})$ of its domain, with $\sigma = \{j_1 < j_2 < \dots < j_i\}$, is

$$d_{i|} = \sum_{r=0}^{i-1} (-1)^r \iota_q^{u-e_\sigma, u-e_{\partial_{i-r}\sigma}},$$

where $\partial_{i-r}\sigma := \{j_1, \dots, \hat{j}_{i-r}, \dots, j_i\}$. As we said, $\xi_i^q(u)$ can be defined as the dimension (over $\mathbb{F}$) of the i th homology module of $\mathbb{K}_*(x_1, \dots, x_n; V_q)(u)$.

Let us examine the map d_1 in the Koszul complex. This map, sometimes called the *merge map* and denoted mer_q^u , is the composition of the available maps $\bar{\varepsilon}_q^u$ followed by i_q^u , making the diagram

$$\begin{array}{ccc}
 \bigoplus_j H_q(X^{u-e_j}) & \xrightarrow{\bar{\varepsilon}_q^u} & H_q(\bigcup_j X^{u-e_j}) \\
 & \searrow \text{mer}_q^u = d_1 & \downarrow i_q^u \\
 & & H_q(X^u)
 \end{array} \quad (3.3)$$

commute. Here, $\bar{\varepsilon}_q^u$ is the map induced by the obvious inclusions, whose restriction to each direct summand $H_q(X^{u-e_j})$ of the domain is the map $H_q(X^{u-e_j}) \rightarrow H_q(\bigcup_j X^{u-e_j})$ induced in homology by $X^{u-e_j} \hookrightarrow \bigcup_j X^{u-e_j}$. The map i_q^u is induced in homology by the inclusion $\bigcup_j X^{u-e_j} \hookrightarrow X^u$.

4. The Mayer-Vietoris spectral sequence of a multi-filtration

Let $\{X^u\}_{u \in \mathbb{Z}^n}$ be an n -parameter filtration of a cell complex X . For a fixed grade $u \in \mathbb{Z}^n$ of this filtration, consider the collection of cell subcomplexes $\{X^{u-e_j}\}_{j \in [n]}$. If $n = 2$, it is well-known that there is a short exact sequence

$$0 \rightarrow C_*(X^{u-e_1} \cap X^{u-e_2}) \rightarrow C_*(X^{u-e_1}) \oplus C_*(X^{u-e_2}) \rightarrow C_*(X^{u-e_1} \cup X^{u-e_2}) \rightarrow 0$$

inducing in homology the Mayer-Vietoris long exact sequence, which clarifies the relation between the homology of $X^{u-e_1} \cap X^{u-e_2}$, X^{u-e_1} , X^{u-e_2} and $X^{u-e_1} \cup X^{u-e_2}$. This can be generalized for $n > 2$ via the Mayer-Vietoris spectral sequence relating the homology of $\bigcap_{j \in \sigma} X^{u-e_j}$, for all $\sigma \subseteq [n]$, to the homology of $\bigcup_{j \in [n]} X^{u-e_j}$. Even if the Mayer-Vietoris spectral sequence can be defined for general collections of subcomplexes as seen in Section 2.4, in this article we will focus on the collection $\{X^{u-e_j}\}_{j \in [n]}$ for a fixed grade $u \in \mathbb{Z}^n$. Here, we provide more details on the Mayer-Vietoris spectral sequence associated with the collection of subcomplexes $\{X^{u-e_j}\}_{j \in [n]}$, in preparation to describe the connection with the Koszul complex.

Given an n -parameter filtration $\{X^u\}_{u \in \mathbb{Z}^n}$, fix a grade $u \in \mathbb{Z}^n$ and consider the collection of subcomplexes $\{X^{u-e_j}\}_{j \in [n]}$. Considering the intersections of all possible subcollections of $\{X^{u-e_j}\}$, we can define a double complex $C_{*,*} = \{C_{p,q}, \delta_{p,q}, \partial_{p,q}\}_{p,q \in \mathbb{Z}}$ with

$$C_{p,q} := \begin{cases} \bigoplus_{|\sigma|=p+1} C_q(\bigcap_{j \in \sigma} X^{u-e_j}) & \text{if } p, q \geq 0 \\ 0 & \text{otherwise,} \end{cases} \quad (4.1)$$

for $\sigma \subseteq [n]$, and with the two differentials $\delta_{p,q} : C_{p,q} \rightarrow C_{p-1,q}$ and $\partial_{p,q} : C_{p,q} \rightarrow C_{p,q-1}$ defined as in Section 2.4. The Mayer-Vietoris spectral sequence is the (first quadrant) spectral sequence associated with the filtration $\{F_p^I T_* = \bigoplus_{i \leq p} C_{i,*-i}\}_{p \in \mathbb{Z}}$ of the total complex T_* of $C_{*,*}$, and it converges to $H_*(T_*) \cong H_*(\bigcup_{j \in [n]} X^{u-e_j})$. Let us recall (see e.g. [30,31]) that convergence of the spectral sequence to $H_*(T_*)$ is expressed by isomorphisms $E_{p,q}^\infty \cong \mathcal{F}_p H_k(T_*) / \mathcal{F}_{p-1} H_k(T_*)$, for all p, q and $k = p+q$, where $\{\mathcal{F}_p H_*(T_*)\}_{p \in \mathbb{Z}}$ is the induced filtration on $H_*(T_*)$ defined by

$$\mathcal{F}_p H_k(T_*) := \text{im}(f_k^p : H_k(F_p^I T_*) \rightarrow H_k(T_*)), \quad (4.2)$$

with f_k^p being the map induced by the inclusion $F_p^I T_* \hookrightarrow T_*$.

A key observation for this work is that, as we showed in (2.2), the one-criticality assumption on the n -parameter filtration ensures that $\bigcap_{j \in \sigma} X^{u-e_j} = X^{u-e_\sigma}$, with $e_\sigma := \sum_{j \in \sigma} e_j$, for each $\sigma \subseteq [n]$. Keeping this in mind, we want to explicitly describe the low-degree pages of the spectral sequence, as is possible for spectral sequences associated with double complexes (see e.g. [30,31]).

The 0-page of the Mayer-Vietoris spectral sequence associated with the collection $\{X^{u-e_j}\}_{j \in [n]}$ has terms $E_{p,q}^0 = C_{p,q} = \bigoplus_{|\sigma|=p+1} C_q(X^{u-e_\sigma})$ and differentials $d_{p,q}^0 = \partial_{p,q} : C_{p,q} \rightarrow C_{p,q-1}$ induced by the differentials of $C_*(X)$, up to a $(-1)^p$ sign change (see Section 2.4). The terms of the 1-page are therefore

$$E_{p,q}^1 = H_q(C_{p,*}) = H_q\left(\bigoplus_{|\sigma|=p+1} C_*(X^{u-e_\sigma})\right) \cong \bigoplus_{|\sigma|=p+1} H_q(X^{u-e_\sigma}).$$

Let us explicitly write the 1-page of the Mayer-Vietoris spectral sequence in our setting:

$$\begin{array}{ccccccc} \vdots & & \vdots & & \vdots & & \\ E_{0,q}^1 = \bigoplus_j H_q(X^{u-e_j}) & \xleftarrow{\bar{\delta}_{1,q}} & E_{1,q}^1 = \bigoplus_{j < h} H_q(X^{u-e_j-e_h}) & \xleftarrow{\bar{\delta}_{2,q}} & \dots & \xleftarrow{\bar{\delta}_{n-1,q}} & E_{n-1,q}^1 = H_q(X^{u-e_{[n]}}) \\ \vdots & & \vdots & & \vdots & & \\ E_{0,1}^1 = \bigoplus_j H_1(X^{u-e_j}) & \xleftarrow{\bar{\delta}_{1,1}} & E_{1,1}^1 = \bigoplus_{j < h} H_1(X^{u-e_j-e_h}) & \xleftarrow{\bar{\delta}_{2,1}} & \dots & \xleftarrow{\bar{\delta}_{n-1,1}} & E_{n-1,1}^1 = H_1(X^{u-e_{[n]}}) \\ E_{0,0}^1 = \bigoplus_j H_0(X^{u-e_j}) & \xleftarrow{\bar{\delta}_{1,0}} & E_{1,0}^1 = \bigoplus_{j < h} H_0(X^{u-e_j-e_h}) & \xleftarrow{\bar{\delta}_{2,0}} & \dots & \xleftarrow{\bar{\delta}_{n-1,0}} & E_{n-1,0}^1 = H_0(X^{u-e_{[n]}}) \end{array}$$

We display only the first quadrant $p, q \geq 0$, since elsewhere the terms $E_{p,q}^1$ are null. Moreover, the columns of indices $p = 0, 1, \dots, n-1$ we showed in the diagram are the only (possibly) non-null ones. The differentials $d_{p,q}^1 : E_{p,q}^1 \rightarrow E_{p-1,q}^1$ are the maps induced in homology by the horizontal differentials $\delta_{p,q}$ of the double complex, which we denote $\bar{\delta}_{p,q}$. Explicitly, the differential

$$d_{p,q}^1 = \bar{\delta}_{p,q} : E_{p,q}^1 \cong \bigoplus_{|\sigma|=p+1} H_q(X^{u-e_\sigma}) \longrightarrow E_{p-1,q}^1 \cong \bigoplus_{|\tau|=p} H_q(X^{u-e_\tau}) \quad (4.3)$$

is the linear map acting on each direct summand $H_q(X^{u-e_\sigma})$ of the domain by

$$d_{p,q}^1 = \sum_{i=0}^p (-1)^i \iota_q^{u-e_\sigma, u-e_{\partial_{p+1-i}\sigma}},$$

where $\iota_q^{u-e_\sigma, u-e_{\partial_\ell\sigma}}$ denotes the map induced in q th homology by the inclusion $C_*(X^{u-e_\sigma}) \hookrightarrow C_*(X^{u-e_{\partial_\ell\sigma}})$, for each for $1 \leq \ell \leq p+1$. Let us recall that, if $\sigma = \{j_1 < \dots < j_{p+1}\}$, we denote $\partial_\ell\sigma := \{j_1 < \dots < \hat{j}_\ell < \dots < j_{p+1}\}$. We observe that each row in the 1-page $\{E_{p,q}^1, d_{p,q}^1\}_{p,q \in \mathbb{Z}}$ is the truncation of a Koszul complex, for each degree q of homology. More precisely, the q th row $\{E_{p,q}^1, d_{p,q}^1\}_{p \in \mathbb{Z}}$ is a truncated version of the Koszul complex $\mathbb{K}_*(x_1, \dots, x_n; V_q)(u)$, with the chain group $\mathbb{K}_0(x_1, \dots, x_n; V_q)(u) = H_q(X^u)$ replaced by the zero vector space. We will prove the details of this claim in Proposition 4.1.

We obtain the 2-page of the Mayer-Vietoris spectral sequence by taking homology of the horizontal chain complexes in the 1-page. For our purposes, we are not as interested in its terms as we are in their dimensions (as vector spaces), which we express as follows in terms of the Betti tables $\xi_i^q(u)$, dropping in the notation the dependence on u for readability's sake:

$$\begin{array}{ccccccc}
& \vdots & & \vdots & & & \vdots \\
\dim E_{0,q}^2 = \dim \operatorname{im} \operatorname{mer}_q + \xi_1^q & & \dim E_{1,q}^2 = \xi_2^q & \cdots & & \dim E_{n-1,q}^2 = \xi_n^q & \\
& \vdots & & \vdots & & & \vdots \\
\dim E_{0,1}^2 = \dim \operatorname{im} \operatorname{mer}_1 + \xi_1^1 & & \dim E_{1,1}^2 = \xi_2^1 & \cdots & & \dim E_{n-1,1}^2 = \xi_n^1 & \\
\dim E_{0,0}^2 = \dim \operatorname{im} \operatorname{mer}_0 + \xi_1^0 & & \dim E_{1,0}^2 = \xi_2^0 & \cdots & & \dim E_{n-1,0}^2 = \xi_n^0 &
\end{array}$$

As before, we have only n (possibly) non-null columns, corresponding to $p = 0, \dots, n-1$. For $1 \leq p \leq n-1$, it is clear why the multi-graded Betti numbers appear in the table, since they are defined as the dimension of the homology groups of the Koszul complex. In Proposition 4.1 we prove the equalities in the column $p = 0$, upon rigorously checking the claims we made regarding the 1-page.

Proposition 4.1. *For each $q \geq 0$, the q th row $\{E_{p,q}^1, d_{p,q}^1\}_{p \in \mathbb{Z}}$ of the 1-page of the Mayer-Vietoris spectral sequence associated with $\{X^{u-e_j}\}_{j \in [n]}$ coincides with the truncation of the Koszul complex $\mathbb{K}_{*+1}(x_1, \dots, x_n; V_q)(u)$. The terms of the 2-page have dimension*

$$\dim E_{p,q}^2 = \begin{cases} \dim \operatorname{im} \operatorname{mer}_q^u + \xi_1^q(u) & \text{if } p = 0 \\ \xi_{p+1}^q(u) & \text{if } 1 \leq p \leq n-1 \\ 0 & \text{otherwise} \end{cases}$$

Proof. For all $p, q \geq 0$, it is clear that $E_{p,q}^1 = \bigoplus_{|\sigma|=p+1} H_q(X^{u-e_\sigma})$ coincides with $\mathbb{K}_{p+1}(x_1, \dots, x_n; V_q)(u)$ as defined in (3.1). Comparing the explicit description (4.3) of the differentials $d_{p,q}^1 = \bar{\delta}_{p,q}$ with the differentials d_p of the Koszul complexes $\mathbb{K}_*(x_1, \dots, x_n; V_q)(u)$, defined in (3.2), we observe that they coincide up to a shift in grading: $\bar{\delta}_{p,q} = d_{p+1}$, for all $p \geq 1$.

Since $E_{p,q}^2 \cong \ker d_{p,q}^1 / \operatorname{im} d_{p+1,q}^1$ for all $p, q \geq 0$, it follows that $\dim E_{p,q}^2 = \xi_{p+1}^q(u)$ for all $p \geq 1$. We remark that $\dim E_{p,q}^2 = \xi_{p+1}^q(u) = 0$ if $p \geq n$. It is also clear that $\dim E_{p,q}^2 = 0$ if $p < 0$. If $p = 0$, as an effect of the truncation of the Koszul complex (which clearly does not affect the other columns) we have

$$\begin{aligned}
\dim E_{0,q}^2 &= \dim (\mathbb{K}_1(x_1, \dots, x_n; V_q)(u) / \operatorname{im} d_2) \\
&= \dim (\oplus_j H_q(X^{u-e_j}) / \operatorname{im} \bar{\delta}_{1,q}) \\
&= \dim (\oplus_j H_q(X^{u-e_j})) - \dim \operatorname{im} \bar{\delta}_{1,q} \\
&= \dim \operatorname{im} \operatorname{mer}_q^u + \dim \ker \operatorname{mer}_q^u - \dim \operatorname{im} \bar{\delta}_{1,q} \\
&= \dim \operatorname{im} \operatorname{mer}_q^u + \xi_1^q(u),
\end{aligned}$$

where the last two equalities follow from the existence of the differential

$$d_1 = \operatorname{mer}_q^u : \mathbb{K}_1(x_1, \dots, x_n; V_q)(u) = \oplus_j H_q(X^{u-e_j}) \longrightarrow \mathbb{K}_0(x_1, \dots, x_n; V_q)(u) = H_q(X^u)$$

in the non-truncated Koszul complex. $\square$

Remark 4.2. Since $\dim \operatorname{im} \operatorname{mer}_q^u = \dim H_q(X^u) - \xi_0^q(u)$, if $p = 0$ the statement of Proposition 4.1 can be equivalently expressed as $\dim E_{0,q}^2 = \dim H_q(X^u) - \xi_0^q(u) + \xi_1^q(u)$.

Let us now focus on convergence and on the ∞ -page of the Mayer-Vietoris spectral sequence.

Proposition 4.3. *Let $\{X^u\}_{u \in \mathbb{Z}^n}$ be an n -parameter filtration. The Mayer-Vietoris spectral sequence of $\{X^{u-e_j}\}_{j \in [n]}$, for a fixed grade $u \in \mathbb{Z}^n$, has $E_{p,q}^n = E_{p,q}^\infty$, for all p, q , and*

$$H_k(\cup_j X^{u-e_j}) \cong \bigoplus_{p+q=k} E_{p,q}^n = \bigoplus_{i=0}^k E_{i,k-i}^n, \quad (4.4)$$

for all $k \geq 0$.

Proof. Since $E_{p,q}^r = 0$ for all q and r whenever $p < 0$ or $p \geq n$, for each term $E_{p,q}^n$ both the incoming differential $d_{p+n,q-n+1}^n : E_{p+n,q-n+1}^n \rightarrow E_{p,q}^n$ and the outgoing differential $d_{p,q}^n : E_{p,q}^n \rightarrow E_{p-n,q+n-1}^n$ are trivial, so $E_{p,q}^n = E_{p,q}^{n+1} = \dots = E_{p,q}^\infty$. We saw in Section 2.4 that the spectral sequence converges to $H_*(T_*) \cong H_*(\cup_j X^{u-e_j})$. Recall that $E_{p,q}^\infty \cong \mathcal{F}_p H_k(T_*) / \mathcal{F}_{p-1} H_k(T_*)$, for every p, q and $k = p + q$, where $\{\mathcal{F}_p H_*(T_*)\}_{p \in \mathbb{Z}}$ is the filtration on $H_*(T_*)$ defined by (4.2). Since the spectral sequence is in the first quadrant, we have $\bigoplus_{i=0}^k E_{i,k-i}^n = \bigoplus_{i=0}^k E_{i,k-i}^\infty \cong H_k(T_*)$. $\square$

Let us now consider diagram (3.3) and observe that, for the Mayer-Vietoris spectral sequence associated with $\{X^{u-e_j}\}_{j \in [n]}$, the map $\bar{\varepsilon}_q^u$ is induced in homology by the chain map $\varepsilon_* : \bigoplus_j C_*(X^{u-e_j}) \rightarrow C_*(\cup_j X^{u-e_j})$ induced by the inclusions $X^{u-e_j} \hookrightarrow \cup_j X^{u-e_j}$. We end this section by showing in Theorem 4.5 that the image of $\bar{\varepsilon}_q^u$ is isomorphic to the term $E_{0,q}^\infty$ of the spectral sequence. We first need to state a general result in homological algebra [3, p. 165–166], a proof of which can be found, in a cohomological setting, in [16, III.7, Lemma 12].

Lemma 4.4. *Let $C_{*,*} = \{C_{p,q}, \delta_{p,q}, \partial_{p,q}\}$ be a first quadrant double complex with associated total complex T_* , let K_* be a chain complex and let $\varepsilon_* : C_{0,*} \rightarrow K_*$ be a chain map. Assume that*

$$0 \leftarrow K_* \xleftarrow{\varepsilon_*} C_{0,*} \xleftarrow{\delta_{1,*}} C_{1,*} \leftarrow \dots \leftarrow C_{p-1,*} \xleftarrow{\delta_{p,*}} C_{p,*} \leftarrow \dots$$

is an exact sequence of chain complexes. Consider the induced chain map $\tilde{\varepsilon}_ : T_* \rightarrow K_*$ defined by the maps $\tilde{\varepsilon}_k : T_k = C_{0,k} \oplus \dots \oplus C_{k,0} \rightarrow K_k$ sending $(c_0, \dots, c_k)$ to $\varepsilon_k(c_0)$. Then $\tilde{\varepsilon}_*$ induces isomorphisms*

$$H_k(\tilde{\varepsilon}_*) : H_k(T_*) \rightarrow H_k(K_*)$$

in homology, for each $k \in \mathbb{Z}$.

Theorem 4.5. *The terms of the ∞ -page of the Mayer-Vietoris spectral sequence of $\{X^{u-e_j}\}_{j \in [n]}$ having index $p = 0$ satisfy*

$$E_{0,q}^\infty \cong \operatorname{im} \left(\bar{\varepsilon}_q^u : \bigoplus_j H_q(X^{u-e_j}) \rightarrow H_q(\cup_j X^{u-e_j}) \right). \quad (4.5)$$

Proof. The Mayer-Vietoris spectral sequence is the spectral sequence associated with the filtration $\{F_p^I T_* = \bigoplus_{i \leq p} C_{i,*-i}\}_{p \in \mathbb{Z}}$ of the total complex T_* of the double complex $C_{*,*}$ introduced in (4.1). Consider the induced filtration $\{\mathcal{F}_p H_*(T_*)\}_{p \in \mathbb{Z}}$ on $H_*(T_*)$ defined as in (4.2). Since $C_{*,*}$ is a first quadrant double complex, $\mathcal{F}_p H_k(T_*) = 0$ whenever $p < 0$. In particular, for $p = 0$ we have

$$E_{0,q}^\infty \cong \frac{\mathcal{F}_0 H_q(T_*)}{\mathcal{F}_{-1} H_q(T_*)} = \mathcal{F}_0 H_q(T_*) = \text{im}(f_q^0 : H_q(F_0^I T_*) \rightarrow H_q(T_*)). \quad (4.6)$$

We complete the proof by showing that $\text{im } f_q^0 \cong \text{im } \bar{\varepsilon}_q^u$. Since $F_0^I T_* = C_{0,*}$ and the chain map $\varepsilon_* : C_{0,*} \rightarrow C_*(\cup_j X^{u-e_j})$ fits into the exact sequence (2.5), we can apply Lemma 4.4 and conclude that the induced map $\bar{\varepsilon}_* : T_* \rightarrow C_*(\cup_j X^{u-e_j})$, which makes the triangle

$$\begin{array}{ccc} C_{0,*} & \xhookrightarrow{\quad} & T_* \\ & \searrow \varepsilon_* & \downarrow \bar{\varepsilon}_* \\ & & C_*(\cup_j X^{u-e_j}) \end{array}$$

commutative, induces isomorphisms in homology. By applying q th homology and observing that $H_q(C_{0,*}) = \bigoplus_j H_q(X^{u-e_j})$ we obtain the commutative triangle

$$\begin{array}{ccc} \bigoplus_j H_q(X^{u-e_j}) & \xrightarrow{f_q^0} & H_q(T_*) \\ & \searrow \bar{\varepsilon}_q^u & \downarrow \cong \\ & & H_q(\cup_j X^{u-e_j}) \end{array}$$

which combined with (4.6) completes the proof. $\square$

5. Morse inequalities for persistence modules

Inspired by the standard Morse inequalities reviewed in Section 2.2, our goal in this section is to prove analogous inequalities for persistence modules obtained from an underlying (multi)-filtered cell complex $\mathcal{X} = \{X^u\}_{u \in \mathbb{Z}^n}$, $n \geq 1$:

Theorem 5.1. *For each $q \geq 0$, and each fixed grade $u \in \mathbb{Z}^n$, we have*

$$\sum_{i=0}^q (-1)^{q+i} \mu_i(u) \geq \sum_{i=0}^q (-1)^{q+i} \left(\xi_0^i(u) - \sum_{p=1}^{i+1} \xi_p^{i+1-p}(u) \right),$$

with $\mu_q(u) = \dim H_q(X^u, \cup_j X^{u-e_j})$, $\xi_p^q(u) = \dim H_p(\mathbb{K}_*(x_1, \dots, x_n; V_q)(u))$, and $V_q = H_q(\mathcal{X})$.

In our setting, the right-hand side of the inequality involves the Betti tables of the Koszul complex $\mathbb{K}_*(x_1, \dots, x_n; V_q)$ of the persistence modules V_q associated with a multi-filtration $\mathcal{X}$ of X . They play the same role as the Betti numbers in standard Morse inequalities. Similarly, the left-hand side involves the homological Morse numbers $\mu_q(u)$, defined (see Section 2.3) as the dimension of the q -th relative homology group of the pair $(X^u, \cup_j X^{u-e_j})$.

Before proving the theorem, let us state as a consequence an analogue of the weak Morse inequalities which follow from the strong ones of Theorem 5.1 in the usual way (see Section 2.2).

Corollary 5.2. *For each $q \geq 0$, and each fixed grade $u \in \mathbb{Z}^n$, we have*

$$\mu_q(u) \geq \xi_0^q(u) - \sum_{p=1}^{q+1} \xi_p^{q+1-p}(u).$$

Proof. Using the inequality of Theorem 5.1 for both q and $q - 1$ we obtain

$$\begin{aligned}\mu_q(u) &= \sum_{i=0}^q (-1)^{q+i} \mu_i(u) + \sum_{i=0}^{q-1} (-1)^{q-1+i} \mu_i(u) \\ &\geq \sum_{i=0}^q (-1)^{q+i} \left(\xi_0^i(u) - \sum_{p=1}^{i+1} \xi_p^{i+1-p}(u) \right) + \sum_{i=0}^{q-1} (-1)^{q-1+i} \left(\xi_0^i(u) - \sum_{p=1}^{i+1} \xi_p^{i+1-p}(u) \right) \\ &= \xi_0^q(u) - \sum_{p=1}^{q+1} \xi_p^{q+1-p}(u). \quad \square\end{aligned}$$

Let us now prove Theorem 5.1. First, let us recall a simple but useful fact.

Proposition 5.3. *In an exact sequence of finite-dimensional vector spaces with a final zero*

$$A_d \xrightarrow{f_d} B_d \xrightarrow{g_d} C_d \xrightarrow{h_d} A_{d-1} \xrightarrow{f_{d-1}} \cdots \xrightarrow{h_1} A_0 \xrightarrow{f_0} B_0 \xrightarrow{g_0} C_0 \xrightarrow{h_0} 0$$

we have

$$\sum_{i=0}^d (-1)^{d+i} \dim A_i + \sum_{i=0}^d (-1)^{d+i} \dim C_i \geq \sum_{i=0}^d (-1)^{d+i} \dim B_i.$$

Proof. Consider the exact sequence

$$0 \rightarrow \ker f_d \rightarrow A_d \xrightarrow{f_d} B_d \xrightarrow{g_d} C_d \xrightarrow{h_d} A_{d-1} \xrightarrow{f_{d-1}} \cdots \xrightarrow{h_1} A_0 \xrightarrow{f_0} B_0 \xrightarrow{g_0} C_0 \xrightarrow{h_0} 0.$$

The fact that the alternating sum of the dimensions vanishes can be expressed as

$$\sum_{i=0}^d (-1)^{d+i} \dim A_i + \sum_{i=0}^d (-1)^{d+i} \dim C_i = \dim \ker f_d + \sum_{i=0}^d (-1)^{d+i} \dim B_i,$$

which implies our claim. $\square$

We are now ready to prove our strong Morse inequalities.

Proof of Theorem 5.1. As usual, we denote by $\{E_{p,q}^r, d_{p,q}^r\}$ the Mayer-Vietoris spectral sequence associated with $\{X^{u-e_j}\}_{j \in [n]}$ for the fixed grade $u \in \mathbb{Z}^n$ of the statement. For each r , we will be interested only in the terms $E_{p,q}^r$ indexed by $(p, q) \in I_k$, where $I_k := \{(p, q) \in \mathbb{Z}^2 \mid p, q \geq 0 \text{ and } p + q \leq k\}$, for a fixed $k \geq 0$. It is however convenient for the sake of bookkeeping to consider all $(p, q) \in \mathbb{Z}^2$ such that $p \geq 0$ and $0 \leq p + q \leq k$, keeping in mind that $E_{p,q}^r = 0$ if $q < 0$. For fixed $r, p, k \geq 0$ consider the chain complex

$$\cdots \rightarrow E_{p,k-p}^r \xrightarrow{d_{p,k-p}^r} E_{p-r,k-p+r-1}^r \xrightarrow{d_{p-r,k-p+r-1}^r} \cdots \rightarrow 0. \quad (5.1)$$

It is clear that every such chain complex, built using the appropriate terms and differentials of the r -page, eventually ends with zero terms. Even if in the Mayer-Vietoris spectral sequence the chain complex (5.1) may extend on the left with non-zero terms, we now consider only the portion displayed in (5.1), restricting to terms of total degree not larger than k . By the standard strong Morse inequalities (2.1) we have

$$\sum_{\ell \geq 0} (-1)^\ell \dim E_{p-\ell r, k-p+\ell(r-1)}^r \geq \sum_{\ell \geq 0} (-1)^\ell \dim E_{p-\ell r, k-p+\ell(r-1)}^{r+1}.$$

Allowing $p \geq 0$ to vary, it is easy to observe that each term $E_{p',q'}^r$ with $(p', q') \in I_k$ appears in one (and only one) of the chain complexes (5.1). Keeping k fixed, we can sum over all $p \geq 0$ and obtain

$$\sum_{p \geq 0} \sum_{\ell \geq 0} (-1)^\ell \dim E_{p-\ell r, k-p+\ell(r-1)}^r \geq \sum_{p \geq 0} \sum_{\ell \geq 0} (-1)^\ell \dim E_{p-\ell r, k-p+\ell(r-1)}^{r+1}.$$

The choice of signs in the alternating sums is such that the terms of total degree $k-i$ have sign $(-1)^i$, for each $i \geq 0$. We can therefore write this inequality as

$$\sum_{i=0}^k (-1)^{k+i} \sum_{p+q=i} \dim E_{p,q}^r \geq \sum_{i=0}^k (-1)^{k+i} \sum_{p+q=i} \dim E_{p,q}^{r+1}. \quad (5.2)$$

Let us recall now that for the n -page, by convergence of the spectral sequence (see Proposition 4.3) we have

$$\sum_{i=0}^k (-1)^{k+i} \sum_{p+q=i} \dim E_{p,q}^n = \sum_{i=0}^k (-1)^{k+i} \dim H_i(\cup_j X^{u-e_j}).$$

By (repeatedly) applying (5.2) we obtain

$$\sum_{i=0}^k (-1)^{k+i} \dim H_i(\cup_j X^{u-e_j}) = \sum_{i=0}^k (-1)^{k+i} \sum_{p+q=i} \dim E_{p,q}^n \leq \sum_{i=0}^k (-1)^{k+i} \sum_{p+q=i} \dim E_{p,q}^2, \quad (5.3)$$

and since by Proposition 4.1 we know that $\sum_{p+q=i} \dim E_{p,q}^2 = \dim H_i(X^u) - \xi_0^i(u) + \sum_{p=1}^{i+1} \xi_p^{i+1-p}(u)$, we have

$$\sum_{i=0}^k (-1)^{k+i} \dim H_i(\cup_j X^{u-e_j}) \leq \sum_{i=0}^k (-1)^{k+i} \dim H_i(X^u) + \sum_{i=0}^k (-1)^{k+i} \left(\sum_{p=1}^{i+1} \xi_p^{i+1-p}(u) - \xi_0^i(u) \right). \quad (5.4)$$

On the other hand, applying Proposition 5.3 to the long exact sequence of relative homology of the pair $(X^u, \cup_j X^{u-e_j})$,

$$H_k(\cup_j X^{u-e_j}) \rightarrow H_k(X^u) \rightarrow H_k(X^u, \cup_j X^{u-e_j}) \rightarrow H_{k-1}(\cup_j X^{u-e_j}) \rightarrow \dots \rightarrow H_0(X^u, \cup_j X^{u-e_j}) \rightarrow 0,$$

yields the inequality

$$\sum_{i=0}^k (-1)^{k+i} \dim H_i(\cup_j X^{u-e_j}) + \sum_{i=0}^k (-1)^{k+i} \dim H_i(X^u, \cup_j X^{u-e_j}) \geq \sum_{i=0}^k (-1)^{k+i} \dim H_i(X^u),$$

which combined with (5.4) yields

$$\sum_{i=0}^k (-1)^{k+i} \dim H_i(X^u, \cup_j X^{u-e_j}) \geq - \sum_{i=0}^k (-1)^{k+i} \left(\sum_{p=1}^{i+1} \xi_p^{i+1-p}(u) - \xi_0^i(u) \right). \quad \square$$

6. Euler characteristic for persistence modules

In this section we derive, using our strong Morse inequalities, Euler characteristic formulas for the relative homology of a multi-filtration $\mathcal{X} = \{X^u\}_{u \in \mathbb{Z}^n}$ involving the Betti tables. Euler characteristic formulas are ubiquitous in homological algebra, as they are based on a general and well-known result valid for any free

and bounded chain complex. For n -parameter persistence modules, for example, the Euler characteristic of a minimal free resolution is considered in [15]. Here, we consider instead the Euler characteristic of the chain complex $C_*(X^u, \cup_j X^{u-e_j})$ for any fixed grade $u \in \mathbb{Z}^n$.

Firstly, it is worth observing that, in our setting, the Euler characteristic of $C_*(X^u)$ can be expressed in terms of Betti tables as follows.

Proposition 6.1. *For persistence modules $V_q = H_q(\mathcal{X})$ obtained from an n -parameter filtration $\mathcal{X} = \{X^u\}_{u \in \mathbb{Z}^n}$, it holds that*

$$\chi(X^u) := \sum_{p,q \geq 0} (-1)^{p+q} \sum_{v \preceq u} \xi_p^q(v),$$

with $\chi(X^u) := \sum_{q \geq 0} (-1)^q \dim H_q(X^u)$.

Proof. Proposition 2.3 of [24] states the following relation between the point-wise dimension at $u \in \mathbb{Z}^n$ of a (finitely presented) n -parameter persistence module V and its Betti tables ξ_j , which is an easy consequence of Hilbert's Syzygy theorem:

$$\dim V^u = \sum_{j=0}^n (-1)^j \sum_{v \preceq u} \xi_j(v).$$

Applying this formula in the particular case of a persistence module $V_q = H_q(\mathcal{X})$, we obtain

$$\sum_{q \geq 0} (-1)^q \dim H_q(X^u) = \sum_{p,q \geq 0} (-1)^{p+q} \sum_{v \preceq u} \xi_p^q(v). \quad \square$$

We consider now the Euler characteristic of the pair $(X^u, \cup_j X^{u-e_j})$, defined as

$$\chi(X^u, \cup_j X^{u-e_j}) := \sum_{q \geq 0} (-1)^q \dim H_q(X^u, \cup_j X^{u-e_j}),$$

which in our notation is equal to $\sum_{q \geq 0} (-1)^q \mu_q(u)$. We derive the following result on the Euler characteristic $\chi(X^u, \cup_j X^{u-e_j})$ as a corollary of our strong Morse inequalities (Theorem 5.1).

Theorem 6.2. *Given an n -parameter filtration $\mathcal{X} = \{X^u\}_{u \in \mathbb{Z}^n}$, for each fixed grade $u \in \mathbb{Z}^n$ the Euler characteristic of the pair $(X^u, \cup_j X^{u-e_j})$ is related to the Betti tables of persistent homology of $\mathcal{X}$ by*

$$\chi(X^u, \cup_j X^{u-e_j}) = \sum_{0 \leq p+q \leq d+1} (-1)^{p+q} \xi_p^q(u) = \sum_{p,q} (-1)^{p+q} \xi_p^q(u),$$

where d is the dimension of $\mathcal{X}$.

Remark 6.3. The first equality of the statement can be written also as $\chi(X^u, \cup_j X^{u-e_j}) = \sum_{i=0}^{d+1} (-1)^i \sum_{p=0}^i \xi_p^{i-p}(u)$. The second equality of the statement is not a simple rewriting of the alternating sum, as the sum in the right-hand side ranges over all $p, q \in \mathbb{Z}$. Some possibly non-zero $\xi_p^q(u)$ are thus involved, which do not appear in the first alternating sum. Let us recall that in our setting the possibly non-zero $\xi_p^q(u)$ have indices $0 \leq p \leq n$ and $0 \leq q \leq d$.

Remark 6.4. We can consider the Euler characteristic of an n -parameter persistence module as defined in [15] for the persistence module $V_q = H_q(\mathcal{X})$, for any fixed $q \geq 0$, which corresponds in our notations to $\chi(V_q) := \sum_{p=0}^n (-1)^p \sum_{u \in \mathbb{Z}^n} \xi_p^q(u)$. Theorem 6.2 clarifies its relation with $\chi(X^u, \cup_j X^{u-e_j})$:

$$\sum_q (-1)^q \chi(V_q) = \sum_{u \in \mathbb{Z}^n} \chi(X^u, \cup_j X^{u-e_j}).$$

Proof. For each fixed grade $u \in \mathbb{Z}^n$, it is clear that $\xi_p^q(u) = 0$ whenever $q > d$. By the standard argument on the last strong inequality of a bounded chain complex (see Section 2.2), Theorem 5.1 yields the equality

$$\sum_{i=0}^d (-1)^{d+i} \mu_i(u) = \sum_{i=0}^d (-1)^{d+i} \left(\xi_0^i(u) - \sum_{p=0}^i \xi_{p+1}^{i-p}(u) \right). \quad (6.1)$$

The left-hand side of (6.1) is $(-1)^d \chi(X^u, \cup_j X^{u-e_j})$ and the right-hand side can be rearranged as the sum

$$\begin{aligned} \sum_{i=0}^d (-1)^{d+i} \xi_0^i(u) - \sum_{i=0}^d (-1)^{d+i} \sum_{p=0}^i \xi_{p+1}^{i-p}(u) &= \sum_{i=0}^{d+1} (-1)^{d+i} \xi_0^i(u) - \sum_{i=0}^d (-1)^{d+i} \sum_{p=0}^i \xi_{p+1}^{i-p}(u) \\ &= \sum_{i=0}^{d+1} (-1)^{d+i} \sum_{p=0}^i \xi_p^{i-p}(u) \\ &= \sum_{i=0}^{d+1} (-1)^{d+i} \sum_{p+q=i} \xi_p^q(u) \\ &= (-1)^d \sum_{i=0}^{d+1} (-1)^i \sum_{p+q=i} \xi_p^q(u), \end{aligned}$$

where the first equation is obtained by subtracting $\xi_0^{d+1}(u) = 0$ and the following ones are obtained formally. This yields the first equality of the statement.

The second equality of the statement is obtained by repeating the proof with $m := d + n$ in place of d , observing that $\xi_p^q(u) = 0$ whenever $p + q > m$, as a consequence of the facts described in Section 4. $\square$

7. Improving Morse inequalities

In this section we improve the weak Morse inequalities given in Corollary 5.2. Theorem 7.3 gives a new lower bound for the homological Morse number $\mu_q(u) = \dim H_q(X^u, \cup_j X^{u-e_j})$ in terms of Betti tables. Reciprocally, Theorem 7.5 will provide an upper bound for the homological Morse number $\mu_q(u)$. Finally, we will show that all these new inequalities are sharp.

7.1. A new lower bound for homological Morse numbers

We now derive Theorem 7.3 improving the lower bound of Corollary 5.2 for homological Morse numbers as a function of the Betti tables of the persistent homology modules.

Our strategy is based again on the interplay between the long exact sequence of relative homology of $(X^u, \cup_j X^{u-e_j})$ and the Mayer-Vietoris spectral sequence. The connection between them is made via commutative triangles as in (3.3). More precisely, we leverage Theorem 4.5. The difference with Theorem 5.1 is that we now track the Betti tables $\xi_p^q(u)$ (which appear as dimensions of the terms of the 2-page of the spectral sequence) all the way to the n -page (which coincides with the ∞ -page), to use then Theorem 4.5 and convergence of the spectral sequence to $H_*(\cup_j X^{u-e_j})$.

Before proving the lower bound inequality, let us show what we mean by “tracking” the Betti tables $\xi_p^q(u)$ by presenting the case of $n = 3$ parameters as an example. Since the case of bifiltrations ($n = 2$) is treated in [22] using the Mayer-Vietoris long exact sequence, this represents the case with the smallest

number of parameters that requires the Mayer-Vietoris spectral sequence instead. As usual, we suppress in the notation of the spectral sequence the dependence on the fixed grade $u \in \mathbb{Z}^3$.

Case n = 3. In this case, the 1-page $\{E_{p,q}^1, d_{p,q}^1\}$ of the Mayer-Vietoris spectral sequence of $\{X^{u-e_j}\}_{j=1,2,3}$, for a fixed grade $u \in \mathbb{Z}^3$, consists of three non-null columns. As we said in Section 4, the rows correspond to truncated Koszul complexes, for each degree of homology q :

$$\begin{array}{ccc} \vdots & & \vdots \\ E_{0,2}^1 = \bigoplus_j H_2(X^{u-e_j}) & \xleftarrow{\bar{\delta}_{1,2}} & E_{1,2}^1 = \bigoplus_{j < h} H_2(X^{u-e_j-e_h}) \xleftarrow{\bar{\delta}_{2,2}} E_{2,2}^1 = H_2(X^{u-\sum_{i=1}^3 e_i}) \\ E_{0,1}^1 = \bigoplus_j H_1(X^{u-e_j}) & \xleftarrow{\bar{\delta}_{1,1}} & E_{1,1}^1 = \bigoplus_{j < h} H_1(X^{u-e_j-e_h}) \xleftarrow{\bar{\delta}_{2,1}} E_{2,1}^1 = H_1(X^{u-\sum_{i=1}^3 e_i}) \\ E_{0,0}^1 = \bigoplus_j H_0(X^{u-e_j}) & \xleftarrow{\bar{\delta}_{1,0}} & E_{1,0}^1 = \bigoplus_{j < h} H_0(X^{u-e_j-e_h}) \xleftarrow{\bar{\delta}_{2,0}} E_{2,0}^1 = H_0(X^{u-\sum_{i=1}^3 e_i}) \end{array}$$

By taking homology we obtain the terms $E_{p,q}^2$ of the 2-page. Differentials $d_{p,q}^2 : E_{p,q}^2 \rightarrow E_{p-2,q+1}^2$ between them are defined:

$$\begin{array}{ccc} \vdots & & \vdots \\ E_{0,2}^2 = \text{coker } \bar{\delta}_{1,2} & E_{1,2}^2 = \frac{\ker \bar{\delta}_{1,2}}{\text{im } \bar{\delta}_{2,2}} & E_{2,2}^2 = \ker \bar{\delta}_{2,2} \\ & \nwarrow d_{2,1}^2 & \\ E_{0,1}^2 = \text{coker } \bar{\delta}_{1,1} & E_{1,1}^2 = \frac{\ker \bar{\delta}_{1,1}}{\text{im } \bar{\delta}_{2,1}} & E_{2,1}^2 = \ker \bar{\delta}_{2,1} \\ & \nwarrow d_{2,0}^2 & \\ E_{0,0}^2 = \text{coker } \bar{\delta}_{1,0} & E_{1,0}^2 = \frac{\ker \bar{\delta}_{1,0}}{\text{im } \bar{\delta}_{2,0}} & E_{2,0}^2 = \ker \bar{\delta}_{2,0} \end{array}$$

By Proposition 4.1, the dimensions as vector spaces of the terms of the 2-page are as follows:

$$\begin{array}{ccc} \vdots & & \vdots \\ \dim E_{0,2}^2 = \dim \text{im mer}_2^u + \xi_1^2(u) & \dim E_{1,2}^2 = \xi_2^2(u) & \dim E_{2,2}^2 = \xi_3^2(u) \\ \dim E_{0,1}^2 = \dim \text{im mer}_1^u + \xi_1^1(u) & \dim E_{1,1}^2 = \xi_2^1(u) & \dim E_{2,1}^2 = \xi_3^1(u) \\ \dim E_{0,0}^2 = \dim \text{im mer}_0^u + \xi_1^0(u) & \dim E_{1,0}^2 = \xi_2^0(u) & \dim E_{2,0}^2 = \xi_3^0(u) \end{array}$$

In the 2-page the only non-trivial differentials are of the form $d_{2,q}^2$, for $q \geq 0$, since for $p \neq 2$ either the domain or the target of the differentials $d_{p,q}^2$ is zero. The terms of the 3-page can be expressed as:

$$\begin{array}{ccc} \vdots & \vdots & \vdots \\ E_{0,2}^3 = \text{coker } d_{2,1}^2 & E_{1,2}^3 = E_{1,2}^2 & E_{2,2}^3 = \ker d_{2,2}^2 \\ E_{0,1}^3 = \text{coker } d_{2,0}^2 & E_{1,1}^3 = E_{1,1}^2 & E_{2,1}^3 = \ker d_{2,1}^2 \\ E_{0,0}^3 = E_{0,0}^2 & E_{1,0}^3 = E_{1,0}^2 & E_{2,0}^3 = \ker d_{2,0}^2 \end{array}$$

We observe that some terms, and in particular all terms $E_{1,q}^3$, have already stabilized at the 2-page, meaning that taking homology with respect to differentials $d_{p,q}^2$ does not affect them. The dimensions of the terms of the 3-page can be derived from the previous arguments:

$$\begin{array}{ccc} \vdots & \vdots & \vdots \\ \dim E_{0,2}^3 = (\dim \text{im mer}_2^u + \xi_1^2(u)) - \dim \text{im } d_{2,1}^2 & \dim E_{1,2}^3 = \xi_2^2(u) & \dim E_{2,2}^3 = \xi_3^2(u) - \dim \text{im } d_{2,2}^2 \\ \dim E_{0,1}^3 = (\dim \text{im mer}_1^u + \xi_1^1(u)) - \dim \text{im } d_{2,0}^2 & \dim E_{1,1}^3 = \xi_2^1(u) & \dim E_{2,1}^3 = \xi_3^1(u) - \dim \text{im } d_{2,1}^2 \\ \dim E_{0,0}^3 = \dim \text{im mer}_0^u + \xi_1^0(u) & \dim E_{1,0}^3 = \xi_2^0(u) & \dim E_{2,0}^3 = \xi_3^0(u) - \dim \text{im } d_{2,0}^2 \end{array}$$

Let us recall that, for $n = 3$, the 3-page of the Mayer-Vietoris spectral sequence coincides with the ∞ -page (Proposition 4.3). We have thus kept track of the Betti tables $\xi_p^q(u)$ within the 3-page (that is, ∞ -page), meaning that we have found expressions for the dimensions of the terms $E_{p,q}^3$ involving the Betti tables. Below, we will detail in the general case how this can be used to derive the lower bound inequality for $\mu_q(u)$.

General case $n \geq 2$. In order to generalize our argument for multi-filtrations with any number $n \geq 2$ of parameters, let us prove the following general fact:

Proposition 7.1. *For a spectral sequence $\{E_{p,q}^r, d_{p,q}^r\}$ of finite dimensional vector spaces, the following statements hold for all p, q and for all $r \geq 2$:*

1. $\dim E_{p,q}^{r+1} = \dim E_{p,q}^2 - \sum_{i=2}^r \dim \text{im } d_{p,q}^i - \sum_{i=2}^r \dim \text{im } d_{p+i,q-i+1}^i$;
2. $\dim E_{p,q}^2 - \sum_{i=2}^r \dim E_{p-i,q+i-1}^2 - \sum_{i=2}^r \dim E_{p+i,q-i+1}^2 \leq \dim E_{p,q}^{r+1} \leq \dim E_{p,q}^2$.

Proof. 1. For each pair of fixed indices p, q , at each page r there are differentials

$$E_{p-r, q+r-1}^r \xleftarrow{d_{p,q}^r} E_{p,q}^r \xleftarrow{d_{p+r, q-r+1}^r} E_{p+r, q-r+1}^r,$$

so the dimension of $E_{p,q}^{r+1} \cong \ker d_{p,q}^r / \operatorname{im} d_{p+r, q-r+1}^r$ is

$$\dim E_{p,q}^{r+1} = \dim \ker d_{p,q}^r - \dim \operatorname{im} d_{p+r, q-r+1}^r = \dim E_{p,q}^r - \dim \operatorname{im} d_{p,q}^r - \dim \operatorname{im} d_{p+r, q-r+1}^r.$$

This argument can be applied recursively to $\dim E_{p,q}^i$ for all $r \geq i > 2$.

2. The inequality $\dim E_{p,q}^{r+1} \leq \dim E_{p,q}^2$ follows from the fact that $E_{p,q}^{r+1}$ is a subquotient of $E_{p,q}^r$, for each r . To prove the other inequality, let us observe that, since the dimension of the image of a linear map is upper bounded by the dimension of both the domain and the codomain, for each differential $d_{p,q}^i$ we have

$$\dim \operatorname{im} d_{p,q}^i \leq \dim E_{p-i, q+i-1}^i$$

and for each differential $d_{p+i, q-i+1}^i$ we have

$$\dim \operatorname{im} d_{p+i, q-i+1}^i \leq \dim E_{p+i, q-i+1}^i.$$

We can now apply to the right hand side of the equation of (1.) the inequalities

$$\begin{aligned} \dim \operatorname{im} d_{p,q}^i &\leq \dim E_{p-i, q+i-1}^i \leq \dim E_{p-i, q+i-1}^2 \\ \dim \operatorname{im} d_{p+i, q-i+1}^i &\leq \dim E_{p+i, q-i+1}^i \leq \dim E_{p+i, q-i+1}^2, \end{aligned}$$

for any $2 \leq i \leq r$. $\square$

It is worth observing that, depending on the indices p, q, r , several differentials and terms in Proposition 7.1 can be trivial in our situation. For example, in the Mayer-Vietoris spectral sequence associated with an n -parameter filtration we know that $\operatorname{im} d_{p,q}^i$ and $E_{p-i, q+i-1}^2$ are zero whenever $p < i$, while $\operatorname{im} d_{p+i, q-i+1}^i$ and $E_{p+i, q-i+1}^2$ are zero whenever $p+i \geq n$.

Moving toward the proof of the lower bound inequality (Theorem 7.3), the following simple fact will be useful.

Lemma 7.2. Consider the map $i_q^u : H_q(\cup_j X^{u-e_j}) \rightarrow H_q(X^u)$ and the commutative triangle $\operatorname{mer}_q^u = i_q^u \bar{\varepsilon}_q^u$ as in (3.3). It holds that

$$\dim \operatorname{im} i_q^u \leq \dim \operatorname{im} \operatorname{mer}_q^u + \dim H_q(\cup_j X^{u-e_j}) - \dim \operatorname{im} \bar{\varepsilon}_q^u.$$

Proof. Let us preliminarily observe that, given a composition h of linear maps $f : U \rightarrow V$ followed by $g : V \rightarrow W$ between finite dimensional vector spaces, we have

$$\dim \operatorname{im} g = \dim \operatorname{im} h + \dim V - \dim(\operatorname{im} f + \ker g). \quad (7.1)$$

Indeed, from

$$h(U) = g(f(U)) \cong f(U) / \{x \in f(U) \mid g(x) = 0\}$$

we obtain $\dim \operatorname{im} h = \dim \operatorname{im} f - \dim(\operatorname{im} f \cap \ker g)$. Now we can sum $\dim \ker g$ to both sides of the equation, use the rank-nullity formula $\dim \ker g = \dim V - \dim \operatorname{im} g$ on the left-hand side, use Mayer-Vietoris' formula to express the right-hand side as $\dim(\operatorname{im} f + \ker g)$, and rearrange to obtain (7.1).

Applying Equation (7.1) to the commutative triangle (3.3) yields

$$\dim \operatorname{im} i_q^u = \dim \operatorname{im} \operatorname{mer}_q^u + \dim H_q(\cup_j X^{u-e_j}) - \dim(\operatorname{im} \bar{\varepsilon}_q^u + \ker i_q^u).$$

We obtain the stated inequality by observing that $\dim(\operatorname{im} \bar{\varepsilon}_q^u + \ker i_q^u) \geq \dim \operatorname{im} \bar{\varepsilon}_q^u$. $\square$

Theorem 7.3. For an n -parameter filtration $\{X^u\}_{u \in \mathbb{Z}^n}$, for each grade $u \in \mathbb{Z}^n$, and for each $q \geq 0$, we have

$$\mu_q(u) \geq \xi_0^q(u) + \xi_1^{q-1}(u) - \sum_{i=1}^{n-1} \xi_{i+1}^{q-i}(u) + R,$$

where

$$R = \sum_{r=2}^{n-1} \left(\sum_{i=1}^{r-1} \dim \operatorname{im} d_{i,q-i}^r + \sum_{i=r+1}^{n-1} \dim \operatorname{im} d_{i,q-i}^r + \sum_{i=1}^{n-1} \dim \operatorname{im} d_{i+r,q-i-r+1}^r \right)$$

is a non-negative integer.

Proof. By standard application of the rank-nullity formula to the long exact sequence of the pair $(X^u, \cup_j X^{u-e_j})$, we know that $\dim H_q(X^u, \cup_j X^{u-e_j}) = \dim \operatorname{coker} i_q^u + \dim \ker i_{q-1}^u$. Hence, we get

$$\mu_q(u) = (\dim H_q(X^u) - \dim \operatorname{im} i_q^u) + (\dim H_{q-1}(\cup_j X^{u-e_j}) - \dim \operatorname{im} i_{q-1}^u). \quad (7.2)$$

On the right hand side we can apply Lemma 7.2 to both $\dim \operatorname{im} i_q^u$ and $\dim \operatorname{im} i_{q-1}^u$. We obtain

$$\begin{aligned} \mu_q(u) &\geq \xi_0^q(u) - \sum_{i=0}^{n-1} \dim E_{i,q-i}^n + \dim E_{0,q}^n - \dim \operatorname{im} \operatorname{mer}_{q-1}^u + \dim E_{0,q-1}^n \\ &= \xi_0^q(u) - \sum_{i=1}^{n-1} \dim E_{i,q-i}^n - \dim \operatorname{im} \operatorname{mer}_{q-1}^u + \dim E_{0,q-1}^n \end{aligned} \quad (7.3)$$

by recalling that $\operatorname{im} \bar{\varepsilon}_q^u \cong E_{0,q}^\infty \cong E_{0,q}^n$ and $H_q(\cup_j X^{u-e_j}) \cong E_{0,q}^n \oplus E_{1,q-1}^n \oplus \cdots \oplus E_{n-1,q-n+1}^n$ (Theorem 4.5 and Proposition 4.3), together with the fact that $\xi_0^q(u) = \dim H_q(X^u) - \dim \operatorname{im} \operatorname{mer}_q^u$ (Section 3). We can now observe that, by Proposition 7.1,

$$\dim E_{p,q}^n = \dim E_{p,q}^2 - \sum_{r=2}^{n-1} \dim \operatorname{im} d_{p,q}^r - \sum_{r=2}^{n-1} \dim \operatorname{im} d_{p+r,q-r+1}^r,$$

to express the last member of the inequality (7.3) as

$$\begin{aligned} \xi_0^q(u) - \sum_{i=1}^{n-1} \left(\dim E_{i,q-i}^2 - \sum_{r=2}^{n-1} \dim \operatorname{im} d_{i,q-i}^r - \sum_{r=2}^{n-1} \dim \operatorname{im} d_{i+r,q-i-r+1}^r \right) \\ - \dim \operatorname{im} \operatorname{mer}_{q-1}^u + \left(\dim E_{0,q-1}^2 - \sum_{r=2}^{n-1} \dim \operatorname{im} d_{0,q-1}^r - \sum_{r=2}^{n-1} \dim \operatorname{im} d_{r,q-r}^r \right). \end{aligned}$$

Proposition 4.1 states that $\dim E_{p,q}^2 = \xi_{p+1}^q(u)$ when $p > 0$, and $\dim E_{0,q}^2 = \dim \operatorname{im} \operatorname{mer}_q^u + \xi_1^q(u)$. Upon substitution of these terms in the previous expression we obtain

$$\begin{aligned} \xi_0^q(u) - \sum_{i=1}^{n-1} \left(\xi_{i+1}^{q-i}(u) - \sum_{r=2}^{n-1} \dim \operatorname{im} d_{i,q-i}^r - \sum_{r=2}^{n-1} \dim \operatorname{im} d_{i+r,q-i-r+1}^r \right) \\ - \dim \operatorname{im} \operatorname{mer}_{q-1}^u + \dim \operatorname{im} \operatorname{mer}_q^u + \xi_1^{q-1}(u) - \sum_{r=2}^{n-1} \dim \operatorname{im} d_{0,q-1}^r - \sum_{r=2}^{n-1} \dim \operatorname{im} d_{r,q-r}^r, \end{aligned}$$

and rearranging:

$$\begin{aligned} \xi_0^q(u) + \xi_1^{q-1}(u) - \sum_{i=1}^{n-1} \xi_{i+1}^{q-i}(u) + \sum_{i=1}^{n-1} \sum_{r=2}^{n-1} \dim \operatorname{im} d_{i,q-i}^r + \sum_{i=1}^{n-1} \sum_{r=2}^{n-1} \dim \operatorname{im} d_{i+r,q-i-r+1}^r \\ - \sum_{r=2}^{n-1} \dim \operatorname{im} d_{0,q-1}^r - \sum_{r=2}^{n-1} \dim \operatorname{im} d_{r,q-r}^r. \end{aligned}$$

We can now observe that $\sum_{r=2}^{n-1} \dim \operatorname{im} d_{0,q-1}^r = 0$, since all the involved differentials target zero spaces, and that all the summands of $\sum_{r=2}^{n-1} \dim \operatorname{im} d_{r,q-r}^r$ cancel out with some summands of $\sum_{i=1}^{n-1} \sum_{r=2}^{n-1} \dim \operatorname{im} d_{i,q-i}^r$, namely those for which $i = r$. $\square$

We refer to the inequality $\mu_q(u) \geq \xi_0^q(u) + \xi_1^{q-1}(u) - \sum_{i=1}^{n-1} \xi_{i+1}^{q-i}(u)$ of Theorem 7.3 as lower bound for $\mu_q(u)$ in terms of the Betti tables.

7.2. An upper bound for homological Morse numbers

We prove an upper bound in terms of the Betti tables for the homological Morse numbers $\mu_q(u)$ of an n -parameter filtration, with $n \geq 2$.

Proposition 7.4. *For an n -parameter filtration $\{X^u\}_{u \in \mathbb{Z}^n}$, for each grade $u \in \mathbb{Z}^n$, and for each $q \geq 0$, we have*

$$\dim H_q(\cup_j X^{u-e_j}) \leq \dim \operatorname{im} \operatorname{mer}_q^u + \sum_{i=1}^n \xi_i^{q-i+1}(u).$$

Proof. By Proposition 4.3, $H_q(\cup_j X^{u-e_j}) \cong E_{0,q}^n \oplus E_{1,q-1}^n \oplus \cdots \oplus E_{n-1,q-n+1}^n$. For all p, q we have $\dim E_{p,q}^n \leq \dim E_{p,q}^2$ (Proposition 7.1). We obtain

$$\dim H_q(\cup_j X^{u-e_j}) \leq \dim E_{0,q}^2 + \dim E_{1,q-1}^2 + \cdots + \dim E_{n-1,q-n+1}^2,$$

hence the claim is a consequence of Proposition 4.1. $\square$

Keeping in mind the equality $\dim \operatorname{im} \operatorname{mer}_q^u = \dim H_q(X^u) - \xi_0^q(u)$, we note that, in the case of $n = 2$ parameters, we have $E_{p,q}^2 = E_{p,q}^\infty$ for all p, q (Proposition 4.3), hence Proposition 7.4 can be stated as an equality:

$$\dim H_q(X^{u-e_1} \cup X^{u-e_2}) = \dim H_q(X^u) - \xi_0^q(u) + \xi_1^q(u) + \xi_2^{q-1}(u).$$

We can now prove the following upper bound:

Theorem 7.5. *For an n -parameter filtration $\{X^u\}_{u \in \mathbb{Z}^n}$, for each grade $u \in \mathbb{Z}^n$, and for each $q \geq 0$, we have*

$$\mu_q(u) \leq \sum_{i=0}^n \xi_i^{q-i}(u).$$

Proof. Since $\text{mer}_q^u = i_q^u \bar{\epsilon}_q^u$, as in the commutative diagram (3.3), we have

$$\dim \text{im mer}_q^u \leq \dim \text{im } i_q^u, \quad (7.4)$$

for all $q \in \mathbb{Z}$ and $u \in \mathbb{Z}^n$. For a fixed grade $u \in \mathbb{Z}^n$, as a consequence of a simple argument on the long exact sequence of relative homology of the pair $(X^u, \cup_j X^{u-e_j})$, we can write, as we did before in (7.2),

$$\mu_q(u) = (\dim H_q(X^u) - \dim \text{im } i_q^u) + (\dim H_{q-1}(\cup_j X^{u-e_j}) - \dim \text{im } i_{q-1}^u).$$

By (7.4), the first parenthesis in the right-hand term is upper bounded by $\dim H_q(X^u) - \dim \text{im mer}_q^u = \xi_0^q(u)$. For the second parenthesis we use Proposition 7.4 to see that

$$\dim H_{q-1}(\cup_j X^{u-e_j}) \leq \dim \text{im mer}_{q-1}^u + \sum_{i=1}^n \xi_i^{q-i}(u),$$

and (7.4) to conclude that

$$\dim H_{q-1}(\cup_j X^{u-e_j}) - \dim \text{im } i_{q-1}^u \leq \sum_{i=1}^n \xi_i^{q-i}(u).$$

Putting together the inequalities for the two parentheses we obtain the stated upper bound for $\mu_q(u)$. $\square$

7.3. Sharpness of lower and upper bounds

In this subsection we show that, for a fixed $u \in \mathbb{Z}^n$ and $q \geq 0$, the lower bound

$$\mu_q(u) \geq \xi_0^q(u) + \xi_1^{q-1}(u) - \sum_{i=1}^{n-1} \xi_{i+1}^{q-i}(u)$$

of Theorem 7.3 and the upper bound

$$\mu_q(u) \leq \sum_{i=0}^n \xi_i^{q-i}(u)$$

of Theorem 7.5 for $\mu_q(u)$ in terms of the Betti tables are sharp. As the previous inequalities are trivially seen to be equalities in the case when X consists of only one 0-cell, we aim at showing that equalities can be attained in situations in which any of the involved $\xi_i^k(u)$ is non-zero. The examples we provide focusing on the case of filtrations with $n = 3$ parameters are general enough to be easily generalized to any number of parameters, as shown for instance in Fig. 1 where the same construction is repeated for $n = 1$ and $q = 0$, $n = 2$ and $q = 1$, $n = 3$ and $q = 2$, and can be easily inferred for $n > 3$ and $q = n - 1$.

Lower bound For $n = 3$ and $q = 2$ we show examples of filtrations $\{X^u\}_{u \in \mathbb{Z}^n}$ in which, for a fixed $u \in \mathbb{Z}^n$, $\mu_2(u) = \xi_0^2(u) + \xi_1^1(u) - \xi_2^1(u) - \xi_3^0(u)$ holds and all $\xi_i^k(u)$ of the right-hand side are non-zero. First, let us notice that taking the disjoint union of two filtered cell complexes results in adding both their homological Morse numbers μ_q and their Betti tables. It is therefore enough to provide examples of the following cases:

- (i) $\mu_2(u) = \xi_0^2(u) - \xi_3^0(u)$, with $\xi_0^2(u), \xi_3^0(u) > 0$,
- (ii) $\mu_2(u) = \xi_0^2(u) - \xi_2^1(u)$, with $\xi_0^2(u), \xi_2^1(u) > 0$,
- (iii) $\mu_2(u) = \xi_0^2(u) + \xi_1^1(u)$, with $\xi_0^2(u), \xi_1^1(u) > 0$.

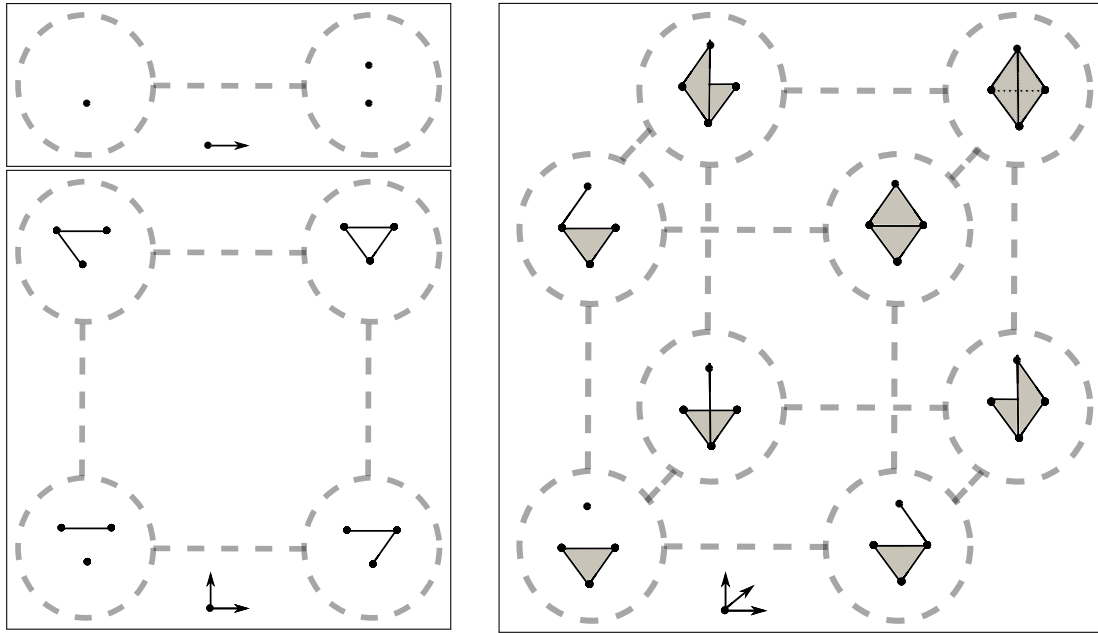


Fig. 1. Filtrations with n -parameters ($1 \leq n \leq 3$) of an $(n-1)$ -sphere for which inequalities of Theorem 7.3 are sharp.

Cases (i) and (ii) in particular illustrate the interesting situation of no critical cells entering at u (which implies $\mu_2(u) = 0$), with $\xi_0^2(u) = 1$ being compensated by $\xi_3^0(u) = 1$ and $\xi_2^1(u) = 1$, respectively.

For (i), consider Fig. 1 (right), where u is the maximum grade shown in the filtration, and only the grades $u - e_\sigma$ with $\sigma \subseteq \{1, 2, 3\}$ are shown. In this filtration, X^u is homeomorphic to a 2-sphere, triangulated as the boundary $\partial\Delta^3$ of a 3-simplex. At the minimum displayed grade $u - e_1 - e_2 - e_3$, only the union of the 0-skeleton of X^u and one of its 2-faces has entered the filtration.

For (ii), we can consider a similar filtration with at grade $u - e_1 - e_2 - e_3$ the union of the 1-skeleton of $X^u = \partial\Delta^3$ and one of its 2-faces. In this case, then, $X^{u-e_1-e_2} = X^{u-e_1-e_3} = X^{u-e_2-e_3} = X^{u-e_1-e_2-e_3}$.

Finally, for (iii), we can set $X^w = \emptyset$ at all grades w , except for $X^{u-e_1} \subseteq X^u$ which is the inclusion of $X^{u-e_1} \simeq S^1$ into $X^u \simeq S^2$ as its equator, with the entrance of $\mu_2(u) = 2$ critical 2-cells at u , and with $\xi_0^2(u) = \xi_1^1(u) = 1$.

Upper bound For $n = 3$ and $q = 3$ we show examples of filtrations $\{X^u\}_{u \in \mathbb{Z}^n}$ in which, for a fixed $u \in \mathbb{Z}^n$, $\mu_3(u) = \xi_0^3(u) + \xi_1^2(u) + \xi_2^1(u) + \xi_3^0(u)$ holds and all $\xi_i^k(u)$ are non-zero.

Consider the cases (i) and (ii) we illustrated above. Adding a 3-cell at grade u so that $X^u = \Delta^3$ we have $\mu_3(u) = 1$ and, respectively, $\xi_3^0(u) = 1$ or $\xi_2^1(u) = 1$, with the other Betti tables being zero.

Mimicking (iii) described above, if $X^w = \emptyset$ at all grades, except for $X^{u-e_1} \subseteq X^u$ being the inclusion of $X^{u-e_1} \simeq S^2$ into $X^u \simeq S^3$, then we have $\mu_3(u) = 2$ and $\xi_0^3(u) = \xi_1^2(u) = 1$.

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