Construction of 2D Explicit Cubic Quasi-Interpolating Splines in Bernstein-Bézier Form

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Construction of 2D explicit cubic quasi-interpolating splines in Bernstein-Bézier form

D. Barrera, S. Eddargani, M.J. Ibáñez and S. Remogna

Abstract In this paper, the construction of C^1 cubic quasi-interpolants on a three-direction mesh of $\mathbb{R}^2$ is addressed. The quasi-interpolating splines are defined by directly setting their Bernstein-Bézier coefficients relative to each triangle from point and gradient values in order to reproduce the polynomials of the highest possible degree. Moreover, additional global properties are required. Finally, we provide some numerical tests confirming the approximation properties.

1 Introduction

In many scientific applications and mathematical problems the approximation of functions from their values or some derivatives at given points is present, and quasi-interpolation is a simple and useful procedure in this context thanks to its particular properties (see e.g. the book [1] for a general overview on this topic). Indeed, the construction of classical approximants, e.g. interpolants, often requires the resolution of linear systems, instead quasi-interpolants are local approximants avoiding this problem.

Here we focus on spline quasi-interpolation and we recall there are several schemes that allow to represent them (see e.g. the book [2] and the reference therein), for example using compactly supported spanning functions, like B-splines or box splines, or using local and stable minimal determining sets. Starting from [3, 4] and

Domingo Barrera and María José Ibáñez
Department of Applied Mathematics, University of Granada, Campus de Fuentenueva s/n, 18071-Granada, Spain, e-mail: {dbarrera,mibanez}@ugr.es

Salah Eddargani
Department of Mathematics, University of Rome Tor Vergata, Rome, Italy, e-mail: eddargani@mat.uniroma2.it

Sara Remogna
University of Torino, Via Carlo Alberto 10, 10123, Torino, Italy, e-mail: sara.remogna@unito.it

going on with [5, 6, 7, 8, 9, 10], another local approach has been adopted in the literature and it is based on the Bernstein-Bézier (BB-) representation of polynomials, by setting the spline BB-coefficients to appropriate combinations of the given data values, by using local portions of the data in such a way that the C^1 smoothness conditions are satisfied as well as the required polynomial reproduction. In particular, in [5, 6, 7] C^1 quartic and cubic quasi-interpolants on type-1 triangulations, exact on the space of cubic and quadratic polynomials, respectively, are constructed. In [8] such a method has been applied for the construction of C^1 quadratic quasi-interpolants exact on quadratic polynomials, defined on a uniform triangulation of type-1 endowed with a Powell–Sabin refinement. In [9] the method has been modified by combining a quasi-interpolating spline with one step of the so called Modified Buttertly Interpolatory Subdivision Scheme, to construct C^1 quartic interpolating splines on regular type-1 triangulations. Moreover, in [10], quasi-interpolating schemes constructed by using this method have been applied to digital elevation models.

We remark that in the above papers, the BB-coefficients are determined using only the values of the function to be approximated. In this context, in the present paper we propose the construction of C^1-cubic Hermite splines on a uniform three-direction triangulation, whose BB-coefficients are determined by the values of the function and its gradient at the vertices of the triangulation and the associated quasi-interpolation operator is exact on quadratic polynomials. The resulting spline (obtained by imposing C^1 smoothness and quadratic polynomial reproduction) depends on five parameters that we fix imposing additional properties.

In particular, in Section 2 we give notations and preliminaries used in the paper. In Section 3 we define the problem and we prove the existence of a 5-parametric family of spline quasi-interpolants. In Section 4 we present some strategies to fix the free parameters and in Section 5 we provide some numerical tests confirming the approximation properties.

2 Notations and preliminaries

Given a triangulation Δ of the real plane, a polynomial p_d of degree less than or equal to d can be represented on each triangle T induced by Δ with vertices $v_1 = \left(v_{1,1}, v_{1,2}\right)$, $v_2 = \left(v_{2,1}, v_{2,2}\right)$ and $v_3 = \left(v_{3,1}, v_{3,2}\right)$ in terms of its Bernstein basis. If $\tau := (\tau_1, \tau_2, \tau_3)$ are the barycentric coordinates with respect to T, defined by the equalities

$$(x, y) = \tau_1 \left(v_{1,1}, v_{1,2}\right) + \tau_2 \left(v_{2,1}, v_{2,2}\right) + \tau_3 \left(v_{3,1}, v_{3,2}\right) \quad \text{and} \quad \tau_1 + \tau_2 + \tau_3 = 1$$

for $(x, y) \in T$, then

$$p_d\,(x, y) = \sum_{|\alpha|=d} b_\alpha^d B_\alpha^d\,(\tau)\,, \tag{1}$$

where $|\alpha| := \alpha_1 + \alpha_2 + \alpha_3$ stands for the length of the multi-index $\alpha := (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}_0^2$, and

$$B_\alpha^d(\tau) := \frac{d!}{\alpha!}\tau^\alpha = \frac{d!}{\alpha_1!\alpha_2!\alpha_2!}\tau_1^{\alpha_1}\tau_2^{\alpha_2}\tau_3^{\alpha_3}$$

for the Bernstein polynomials of degree d on T. The real numbers b_α^d are said to be the Bernstein-Bézier (BB-) coefficients of p_d on T. They are related to the called domain points relative to T, which are defined as $\xi_\alpha^d := \frac{\alpha_1}{d}v_1 + \frac{\alpha_2}{d}v_2 + \frac{\alpha_3}{d}v_3$. It is well-known that the graph of the surface $z = p(x, y)$ on T lies in the convex hull of the set $\left\{\left(\xi_\alpha^d, b_\alpha^d\right), |\alpha| = d\right\}$ of control points.

We are interested in constructing spline functions on the triangulation Δ, so it is useful to recall the conditions on the BB-coefficients of its restrictions to the triangles that guarantee the C^r regularity.

Suppose a polynomial $\widetilde{p}_d$ of degree d is defined on the triangle $\widetilde{T}$ of vertices v_4, v_3 and v_2, thus sharing with T the edge defined by v_2 and v_3. Then,

$$\widetilde{p}_d(x, y) = \sum_{|\alpha|=d} \widetilde{b}_\alpha^d \widetilde{B}_\alpha^d(\widetilde{\tau}), \ (x, y) \in \widetilde{T}, \tag{2}$$

where $\left\{\widetilde{B}_\alpha^d(\widetilde{\tau}), |\alpha| = d\right\}$ is the basis of Bernstein polynomials of $\mathbb{P}_d\left(\widetilde{T}\right)$, which are expressed in terms of the corresponding barycentric coordinates $\widetilde{\tau} := (\widetilde{\tau}_1, \widetilde{\tau}_2, \widetilde{\tau}_3)$. Then, the following result holds [2, Lemma 2.29]:

Lemma 1 *The polynomials p and $\widetilde{p}$ given in (1) and (2), respectively, join with C^r smoothness across the edge defined by v_2 and v_3 if*

$$\widetilde{b}_{n,j,k} = \sum_{|\beta|=n} b_{\beta_1, k+\beta_2, j+\beta_3} B_\beta^n(\tau), \ j + k = d - n, \ n = 0, \dots, r,$$

where τ denotes the barycentric coordinates of v_4 with respect to T.

In particular, with $\tau = (\tau_1, \tau_2, \tau_3)$, for C^1 smoothness the equalities

$$\widetilde{b}_{0,j,k} = b_{0,k,j}, \ j + k = d, \tag{3}$$

$$\widetilde{b}_{0,j,k} = \tau_1 b_{1,k,j} + \tau_2 b_{0,k+1,j} + \tau_3 b_{0,k,j+1}, \ j + k = d - 1, \tag{4}$$

are required [2, eq (2.50) in Thm. 2.28].

In this work, we consider the uniform triangulation Δ_3 defined by the vectors $e_1 := (h, h)$, $e_2 := (h, -h)$ and $e_3 := e_1 + e_2$, with a given $h > 0$. It gives rise to vertices $v_{i,j} := ie_1 + je_2$, $i, j \in \mathbb{Z}$. It also produces two types of triangles. The first one is $T_{i,j} := \left[v_{i,j}, v_{i+1,j+1}, v_{i+1,j}\right]$ and the other one $\widetilde{T}_{i,j} := \left[v_{i,j}, v_{i+1,j+1}, v_{i,j+1}\right]$ (see Fig. 1).

The quasi-interpolating splines will be constructed in the space

$$S_3^1(\Delta_3) := \left\{s \in C^1\left(R^2\right) : s_{|T} \in P_3 \text{ for all } T \in \Delta_3\right\}.$$

According to (1), their BB-coefficients on each triangle of Δ_3 will be directly setting. Given $s \in S_3^1(\Delta_3)$, its restriction to a specific triangle T (equal to $T_{i,j}$ or $\widetilde{T}_{i,j}$) can be written as

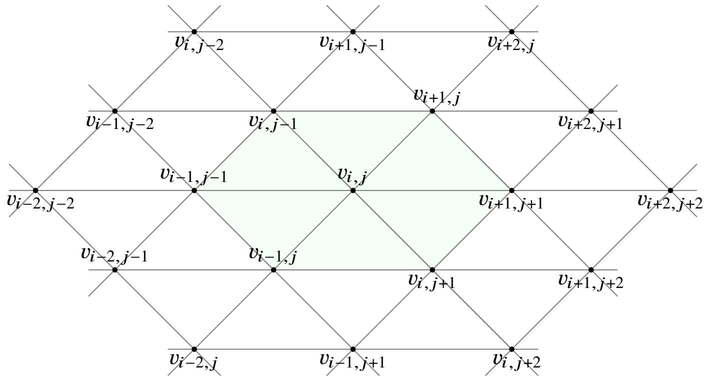

Fig. 1 The triangulation Δ_3 and the hexagon $H_{i,j}$ centered at $v_{i,j}$ defined by its six closest vertices.

$$s_{|T} = \sum_{|\alpha|=3} b_\alpha^T B_\alpha^T,$$

where a superscript is used to show that the Bernstein polynomials depend on the triangle considered and the BB-coefficients will be defined from the available information on the function to be approximated.

In each triangle a cubic spline is uniquely determined by ten BB-coefficients, linked to ten domain points. The subset D consisting of the domain points of all the triangles can be written as $D = \bigcup_{i,j} D_{i,j}$, where

$$D_{i,j} := \left\{v_{i,j}, c_{i,j}, \widetilde{c}_{i,j}\right\} \cup \left\{u_{i,j}^{k,m}, k, m \in \{-1, 0, 1\}, k + m \neq 0\right\},$$

$c_{i,j}$ and $\widetilde{c}_{i,j}$ being the barycenters of $T_{i,j}$ and $\widetilde{T}_{i,j}$, respectively, and

$$u_{i,j}^{k,m} := \frac{1}{3}\left(2v_{i,j} + v_{i+k,j+m}\right).$$

This partition is essential for the construction to be proposed. As the triangulation is uniform, it will be sufficient to define the BB-coefficients associated with the points in $D_{i,j}$. Fig. 2 shows the domain points linked to the BB-coefficients that determine a cubic spline in the triangles $T_{i,j}$ and $\widetilde{T}_{i,j}$.

3 C^1 cubic Hermite quasi-interpolation

In this section, we define quasi-interpolating splines $Qf \in S_3^1(\Delta_3)$ to a given function $f \in C^1(\mathbb{R}^2)$ by assuming that the values of f and its gradient at the vertices are known. The BB-coefficients of Qf are set on each triangle T as follows:

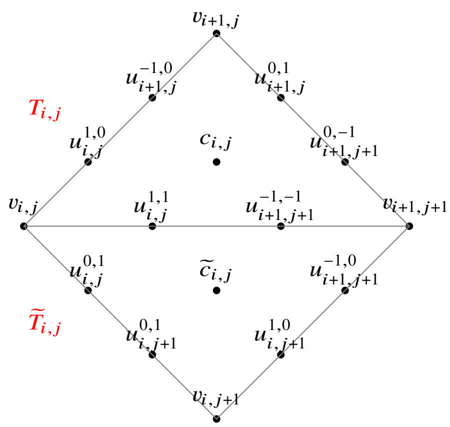

Fig. 2 The domain points in $T_{i,j}$ and $\widetilde{T}_{i,j}$.

$$\begin{aligned} Qf_{|T_{i,j}} &= V_{i,j}B_{3,0,0}^{T_{i,j}} + U_{i,j}^{1,1}B_{2,1,0}^{T_{i,j}} + U_{i,j}^{1,0}B_{2,0,1}^{T_{i,j}} + U_{i+1,j+1}^{-1,-1}B_{1,2,0}^{T_{i,j}} \\ &+ C_{i,j}B_{1,1,1}^{T_{i,j}} + U_{i+1,j}^{-1,0}B_{1,0,2}^{T_{i,j}} + V_{i+1,j+1}B_{0,3,0}^{T_{i,j}} + U_{i+1,j+1}^{0,-1}B_{0,2,1}^{T_{i,j}} \\ &+ U_{i+1,j}^{0,1}B_{0,1,2}^{T_{i,j}} + V_{i+1,j}B_{0,0,3}^{T_{i,j}}, \\ Qf_{|\widetilde{T}_{i,j}} &= V_{i,j+1}B_{3,0,0}^{\widetilde{T}_{i,j}} + U_{i,j+1}^{1,0}B_{2,1,0}^{\widetilde{T}_{i,j}} + U_{i,j+1}^{0,1}B_{2,0,1}^{\widetilde{T}_{i,j}} + U_{i+1,j+1}^{-1,0}B_{1,2,0}^{\widetilde{T}_{i,j}} \\ &+ \widetilde{C}_{i,j}B_{1,1,1}^{\widetilde{T}_{i,j}} + U_{i,j}^{0,1}B_{1,0,2}^{\widetilde{T}_{i,j}} + V_{i+1,j+1}B_{0,3,0}^{\widetilde{T}_{i,j}} + U_{i+1,j+1}^{-1,-1}B_{0,2,1}^{\widetilde{T}_{i,j}} \\ &+ U_{i,j}^{1,1}B_{0,1,2}^{\widetilde{T}_{i,j}} + V_{i,j}B_{0,0,3}^{\widetilde{T}_{i,j}}. \end{aligned} \tag{5}$$

Note that the BB-coefficients have been named as their corresponding domain points using capital letters.

Expressions (5) involve the BB-coefficients associated with four vertices, ten domain points of type u and those of the two barycentres. Taking into account that Δ_3 is a uniform partition, it will be sufficient to define the BB-coefficients of the domain points appearing in $D_{i,j}$. They will be linear combinations of the values of f and its first order partial derivatives $\partial_{1,0}f$ and $\partial_{0,1}f$ at the seven vertices in the hexagon $H_{i,j}$ (see Fig. 1). For example, the BB-coefficient associated with the domain point $v_{i,j}$ has the following form:

$$\begin{aligned} V_{i,j} &= \alpha_{0,0,0}\ f\left(v_{i,j}\right) + \alpha_{0,0,1}\ f\left(v_{i+1,j+1}\right) + \alpha_{0,0,2}\ f\left(v_{i+1,j}\right) + \alpha_{0,0,3}\ f\left(v_{i,j-1}\right) \\ &+ \alpha_{0,0,4}\ f\left(v_{i-1,j-1}\right) + \alpha_{0,0,5}\ f\left(v_{i-1,j}\right) + \alpha_{0,0,6}\ f\left(v_{i,j+1}\right) \\ &+ \alpha_{1,0,0}\ \partial_{1,0} f\left(v_{i,j}\right) h + \alpha_{1,0,1}\ \partial_{1,0} f\left(v_{i+1,j+1}\right) h + \alpha_{1,0,2}\ \partial_{1,0} f\left(v_{i+1,j}\right) h \\ &+ \alpha_{1,0,3}\ \partial_{1,0} f\left(v_{i,j-1}\right) h + \alpha_{1,0,4}\ \partial_{1,0} f\left(v_{i-1,j-1}\right) h + \alpha_{1,0,5}\ \partial_{1,0} f\left(v_{i-1,j}\right) h \\ &+ \alpha_{1,0,6}\ \partial_{1,0} f\left(v_{i,j+1}\right) h + \alpha_{0,1,0}\ \partial_{0,1} f\left(v_{i,j}\right) h + \alpha_{0,1,1}\ \partial_{0,1} f\left(v_{i+1,j+1}\right) h \\ &+ \alpha_{0,1,2}\ \partial_{0,1} f\left(v_{i+1,j}\right) h + \alpha_{0,1,3}\ \partial_{0,1} f\left(v_{i,j-1}\right) h + \alpha_{0,1,4}\ \partial_{0,1} f\left(v_{i-1,j-1}\right) h \\ &+ \alpha_{0,1,5}\ \partial_{0,1} f\left(v_{i-1,j}\right) h + \alpha_{0,1,6}\ \partial_{0,1} f\left(v_{i,j+1}\right) h. \end{aligned} \tag{6}$$

This expression can be simplified if three *masks* $\alpha_{0,0} := \left(\alpha_{0,0,\ell}\right)_{0\le\ell\le6}$, $\alpha_{1,0} := \left(\alpha_{1,0,\ell}\right)_{0\le\ell\le6}$ and $\alpha_{0,1} := \left(\alpha_{0,1,\ell}\right)_{0\le\ell\le6}$ are introduced, as well the notation $g_{i,j} := \left(g\left(v_{i,j}\right), g\left(v_{i+1,j+1}\right), g\left(v_{i+1,j}\right), g\left(v_{i,j-1}\right), g\left(v_{i-1,j-1}\right), g\left(v_{i-1,j}\right), g\left(v_{i,j+1}\right)\right)$ is introduced for a given function g. Thus, equality (6) can be written as

$$V_{i,j} = \alpha_{0,0}\ f_{i,j} + \alpha_{1,0}\ h\ \partial_{1,0} f_{i,j} + \alpha_{0,1}\ h\ \partial_{0,1} f_{i,j}. \tag{7}$$

Similarly, for $k, m \in \{-1, 0, 1\}$ such that $k + m \neq 0$, we write

$$U_{i,j}^{k,m} = \beta_{0,0}^{k,m}\ f_{i,j} + \beta_{1,0}^{k,m}\ h\ \partial_{1,0} f_{i,j} + \beta_{0,1}^{k,m}\ h\ \partial_{0,1} f_{i,j}. \tag{8}$$

Finally, for the BB-coefficients $C_{i,j}$ and $\widetilde{C}_{i,j}$ relative to the barycenters, we write

$$C_{i,j} = \gamma_{0,0}\ f_{i,j} + \gamma_{1,0}\ h\ \partial_{1,0} f_{i,j} + \gamma_{0,1}\ h\ \partial_{0,1} f_{i,j} \tag{9}$$

and

$$\widetilde{C}_{i,j} = \widetilde{\gamma}_{0,0}\ f_{i,j} + \widetilde{\gamma}_{1,0}\ h\ \partial_{1,0} f_{i,j} + \widetilde{\gamma}_{0,1}\ h\ \partial_{0,1}\ f_{i,j}. \tag{10}$$

All masks $\alpha_{0,0}$, $\alpha_{1,0}$, $\alpha_{0,1}$, $\beta_{0,0}^{k,m}$, $\beta_{1,0}^{k,m}$, $\beta_{0,1}^{k,m}$, $\gamma_{0,0}$, $\gamma_{1,0}$, $\gamma_{0,1}$, $\widetilde{\gamma}_{0,0}$, $\widetilde{\gamma}_{1,0}$ and $\widetilde{\gamma}_{0,1}$ must be computed in order to produce a C^1 cubic quasi-interpolant Qf. The previous local and linear construction results in the quasi-interpolation operator $Q : C^1\left(\mathbb{R}^2\right) \longrightarrow S_3^1\left(\Delta_3\right)$ defined by $Q[f] := Qf$. It can only reproduce $\mathbb{P}_2$ since the order of approximation of $S_3^1\left(\Delta_3\right)$ is only three [11]. This will be the exactness required of the operator.

As far as the regularity of Qf is concerned, since the triangulation is uniform, it is sufficient to impose it on the three edges emanating from the vertex $v_{0,0}$. Therefore, conditions (3) and (4) must be satisfied. The former are automatically satisfied by construction, so the class C^1 must be imposed by requiring the fulfillment of (4). In the case of the edge $\left[v_{0,0}, v_{1,1}\right]$, the values τ_1, τ_2 and τ_3 to be used correspond to the barycentric coordinates of the vertex $v_{0,1}$ with respect to the triangle $T_{0,0}$, which are $(1, 1, -1)$. For the edge $\left[v_{0,0}, v_{1,0}\right]$, the coordinates of $v_{0,-1}$ with respect to $T_{0,0}$ are also equal to $(1, 1, -1)$. The same result holds for the barycentric coordinates of $v_{-1,0}$ with respect to $T_{0,0}$, needed by the C^1 continuity across the edge $\left[v_{0,0}, v_{-1,0}\right]$.

Proposition 1 *Qf is C^1 continuous if and only if*

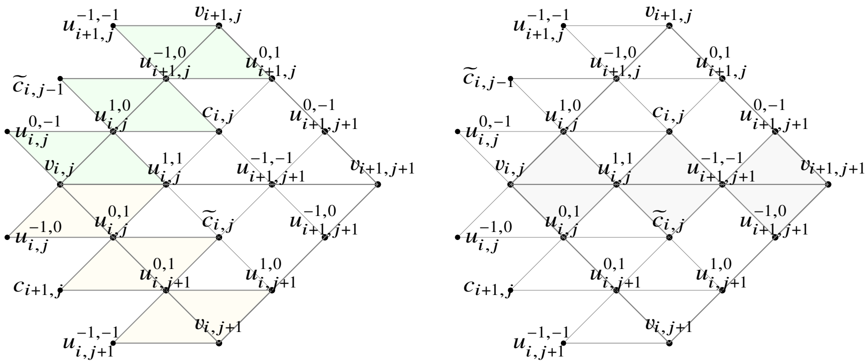

Fig. 3 The conditions equivalent to the C^1 smoothness of Qf .

$$\begin{aligned}
U^{0,-1}_{0,0} + U^{1,1}_{0,0} &= V_{0,0} + U^{1,1}_{0,0}, & \widetilde{C}_{0,-1} + C_{0,0} &= U^{1,0}_{0,0} + U^{-1,0}_{1,0}, & U^{-1,1}_{1,0} + U^{0,1}_{1,0} &= U^{-1,0}_{1,0} + V_{1,0},\\
V_{0,0} + U^{1,1}_{0,0} &= U^{1,0}_{0,0} + U^{0,1}_{0,0}, & C_{0,0} + \widetilde{C}_{0,0} &= U^{1,1}_{0,0} + U^{-1,-1}_{1,1}, & U^{-1,-1}_{1,1} + V_{1,1} &= U^{0,-1}_{1,1} + U^{-1,0}_{1,1},\\
U^{1,1}_{0,0} + U^{-1,0}_{0,0} &= V_{0,0} + U^{0,1}_{0,0}, & C_{-1,0} + \widetilde{C}_{0,0} &= U^{0,1}_{0,0} + U^{0,1}_{0,1}, & U^{0,1}_{0,1} + V_{0,1} &= U^{-1,-1}_{0,1} + U^{1,0}_{0,1}.
\end{aligned} \tag{11}$$

Proof Equations (11) are the result of applying the equalities (3) and (4) taking into account the domain points involved (see Fig. 3). □

Each of the above nine equalities corresponds to a linear functional whose action on an arbitrary function f must be zero. The image by such a linear functional is a linear combination of values of f, $\partial_{1,0}f$ and $\partial_{0,1}f$ at the vertices of the set $S := D_{0,0} \cup D_{1,1} \cup D_{1,0} \cup D_{0,-1} \cup D_{-1,-1} \cup D_{-1,0} \cup D_{0,1}$. The coefficients of such a linear combination must be zero, giving rise to linear equations that must be satisfied.

Proposition 2 *The problem of finding a quasi-interpolation operator Q exact on $\mathbb{P}_2$ and such that $Qf \in S^1_3(\Delta_3)$ is defined by means of BB-coefficients given in (7)-(10) has a 5-parametric family of solutions.*

Proof The solution of the problem is found by solving the system of equations provided by the C^1 smoothness and those resulting from imposing equality between each of the BB-coefficients of Qm_μ and the corresponding one of m_μ, $|\mu| \le 2$, at each of the triangles $T_{0,0}$ and $\widetilde{T}_{0,0}$, where $m_\mu(x, y) := x^{\mu_1} y^{\mu_2}$. This solution is calculated by means of a Computer Algebra System. The parameters are $\alpha_{0,0,2}$, $\alpha_{1,0,2}$, $\alpha_{0,1,2}$, $\alpha_{0,0,3}$ and $\alpha_{1,0,3}$. □

4 Choice of parameters

The existence of degrees of freedom makes it possible to construct quasi-interpolants with additional properties. The exactness on $\mathbb{P}_2$ of the operator Q^* provided by the above proposition implies that for each triangle T induced by the triangulation Δ_3 the quasi-interpolation error $\|f - Q^*[f]\|_{C^1,T}$ relative to T is of order $O(h^3)$, where $\|f\|_{C^1,T} := \|f\|_{\infty,T} + h\left\|\partial_{1,0}f\right\|_{\infty,T} + h\left\|\partial_{0,1}f\right\|_{\infty,T}$ (see, e.g. [12]). More precisely,

$$\|f - Q^*[f]\|_{C^1,T} \leq (1 + \|Q^*\|_{C^1}) \operatorname{dist}_{C^1,T}(f, \mathbb{P}_2),$$

with $\operatorname{dist}_{C^1,T}(f,\mathbb{P}_2) = \inf_{p\in\mathbb{P}_2}\|f-p\|_{C^1,T}$. It is straightforward to prove that

$$\begin{aligned}\|Q^*\|_{C^1} \leq \max\Big\{&\left\|\alpha_{0,0}\right\|_1 + \left\|\alpha_{1,0}\right\|_1 + \left\|\alpha_{0,1}\right\|_1,\\ &\left\|\gamma_{0,0}\right\|_1 + \left\|\gamma_{1,0}\right\|_1 + \left\|\gamma_{0,1}\right\|_1, \left\|\widetilde{\gamma}_{0,0}\right\|_1 + \left\|\widetilde{\gamma}_{1,0}\right\|_1 + \left\|\widetilde{\gamma}_{0,1}\right\|_1;\\ &\left\|\beta_{0,0}^{k,m}\right\|_1 + \left\|\beta_{1,0}^{k,m}\right\|_1 + \left\|\beta_{0,1}^{k,m}\right\|_1, \ k,m \in \{-1,0,1\}, \ k+m \neq 0\Big\},\end{aligned}$$

where $\|v\|_1 := \sum_{\ell=1}^{N}|v_\ell|$ for $v \in \mathbb{R}^N$.

It is not possible to achieve a higher order of global convergence than above, but it is feasible to attain higher orders of convergence at specific points, obtaining quasi-interpolants that are often called super-convergent. Specifically, we will ask that the quasi-interpolation error

$$\varepsilon[f](q) := f(q) - Q^*[f](q)$$

be of order greater than or equal to four at the midpoints of the edges of the triangulation.

Proposition 3 *It is satisfied that the quasi-interpolation error $\varepsilon(f)$ is of order four at the midpoints of the sides of Δ_3 if and only if*

$$\alpha_{0,0,2} = \lambda, \alpha_{0,0,3} = \frac{1}{6}(-5+12\lambda), \alpha_{1,0,2} = \frac{1}{36}(1-18\lambda), \alpha_{1,0,3} = \frac{1}{12}(5-18\lambda),$$
$$\alpha_{0,1,2} = -\frac{h}{9},$$

λ being an arbitrary value.

Proof The midpoints of the triangulation edges are the midpoints $e_{i,j}^{k,\ell}$ of the edges $\left[v_{i,j}, v_{i+k,j+\ell}\right]$, $k,\ell \in \{0,1\}$, $k+\ell \neq 0$, $i,j \in \mathbb{Z}$. Since the triangulation is uniform, it is sufficient to prove the claim for the midpoints of the triangle $T_{0,0}$. The exactness of Q^* on $\mathbb{P}_2$ implies that the linear functional ε is null on this space. The value of $Q^*[f]$ on the triangle $T_{0,0}$ is determined from the BB-coefficients

$$\left\{V_{0,0}, U_{0,0}^{1,1}, U_{0,0}^{1,0}, U_{1,1}^{-1,-1}, T_{0,0}, U_{1,0}^{-1,0}, V_{1,1}, U_{1,1}^{0,-1}, U_{1,0}^{0,1}, V_{1,0}\right\},$$

which are defined from the masks and the values of f, $\partial_{1,0}f$ and $\partial_{0,1}f$ at the vertices in S. The de Casteljau algorithm allows to easily calculate the value of the quasi-interpolant of each of the m_μ cubic monomials, $|\mu| = 3$, at the midpoints $e_{0,0}^{1,1}$, $e_{0,0}^{0,1}$ and $e_{1,0}^{0,1}$ lying in the edges $\left[v_{0,0}, v_{1,1}\right]$, $\left[v_{0,0}, v_{1,0}\right]$ and $\left[v_{1,0}, v_{1,1}\right]$, whose barycentric coordinates are $\left(\frac{1}{2}, \frac{1}{2}, 0\right)$, $\left(\frac{1}{2}, 0, \frac{1}{2}\right)$ and $\left(0, \frac{1}{2}, \frac{1}{2}\right)$, respectively. Regarding

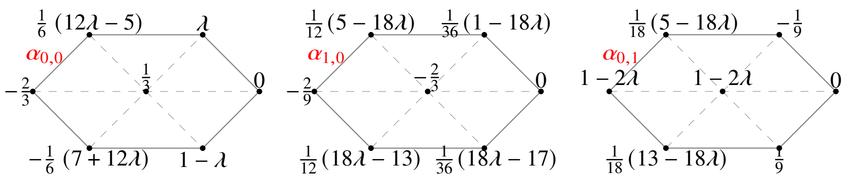

Fig. 4 Masks associated with vertices for achieving C^1 smoothness and superconvergence at midpoints of edges (and also at vertices), and ensuring exactness on $\mathbb{P}_2$.

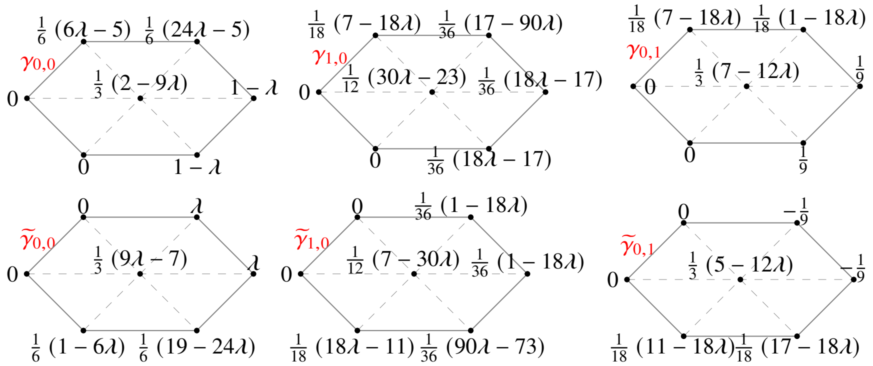

Fig. 5 Masks associated with barycenters.

the midpoint $e_{0,0}^{1,1}$, the following results hold:

$$\begin{aligned}
\varepsilon\left[m_{3,0}\right]\left(e_{0,0}^{1,1}\right) &= \frac{h^3}{4}\left(12\alpha_{0,0,2} - 6\alpha_{0,0,3} - 5\right),\\
\varepsilon\left[m_{2,1}\right]\left(e_{0,0}^{1,1}\right) &= -\frac{h^2}{4}\left(4h\alpha_{0,0,2} + 24\alpha_{1,0,2} - 2h\alpha_{0,0,3} - 8\alpha_{1,0,3} + h\right),\\
\varepsilon\left[m_{1,2}\right]\left(e_{0,0}^{1,1}\right) &= \frac{h^2}{108}\left(540h\alpha_{0,0,2} + 288\alpha_{1,0,2} + 864\alpha_{0,1,2} + 18h\alpha_{0,0,3}\right.\\
&\quad \left.+288\alpha_{1,0,3} - 17h\right),\\
\varepsilon\left[m_{0,3}\right]\left(e_{0,0}^{1,1}\right) &= \frac{h^2}{36}\left(108h\alpha_{0,0,2} + 72\alpha_{1,0,2} + 18h\alpha_{0,0,3} + 72\alpha_{1,0,3} - 17h\right).
\end{aligned}$$

One will have superconvergence at $e_{0,0}^{1,1}$ if and only if the above expressions are equal to zero, which is equivalent to the claim in the statement. It is straightforward to check that these same conditions guarantee superconvergence at $e_{0,0}^{0,1}$ and $e_{1,0}^{0,1}$. □

Figures 4 and 5 show the masks provided by Proposition 3 for vertices and barycenters, respectively.

Next, the maks for $U_{i,j}^{k,m}$. Firstly, those associated with the values of f at the vertices:

$$\beta_{0,0}^{1,1} = \left(-\frac{1}{3}, 0, 2\lambda, \frac{1}{6}(6\lambda-5), 0, \frac{1}{6}(1-6\lambda), 2(1-\lambda)\right),$$
$$\beta_{0,0}^{1,0} = \left(\frac{3}{2}(1-2\lambda), 0, 2\lambda, \frac{1}{3}(9\lambda-5), \frac{1}{6}(1-6\lambda), 0, 1-\lambda\right)$$
$$\beta_{0,0}^{0,-1} = \left(\frac{1}{6}(13-18\lambda), 0, \lambda, \frac{1}{3}(12\lambda-5), -\frac{1}{2}(2\lambda+1), 1-\lambda, 0\right),$$
$$\beta_{0,0}^{-1,-1} = \left(1, 0, 0, \frac{1}{6}(18\lambda-5), -\frac{4}{3}, \frac{1}{6}(13-18\lambda), 0\right),$$
$$\beta_{0,0}^{-1,0} = \left(\frac{1}{6}(18\lambda-5), 0, 0, \lambda, \frac{1}{2}(2\lambda-3), \frac{1}{3}(7-12\lambda), 1-\lambda\right)$$
$$\beta_{0,0}^{0,1} = \left(\frac{3}{2}(2\lambda-1), 0, \lambda, 0, \frac{1}{6}(6\lambda-5), \frac{1}{3}(4-9\lambda), 2(1-\lambda)\right).$$

Regarding the masks linked to $\partial_{1,0} f$, the following masks were obtained:

$$\beta_{1,0}^{1,1} = \left(-\frac{8}{9}, 0, \frac{1-18\lambda}{18}, \frac{7-18\lambda}{18}, 0, \frac{18\lambda-11}{18}, \frac{18\lambda-17}{18}\right),$$
$$\beta_{1,0}^{1,0} = \left(\frac{2(9\lambda-8)}{9}, 0, \frac{1-18\lambda}{18}, \frac{29-90\lambda}{36}, \frac{18\lambda-1}{18}, 0, \frac{18\lambda-17}{36}\right),$$
$$\beta_{1,0}^{0,-1} = \left(\frac{2(9\lambda-7)}{9}, 0, \frac{1-18\lambda}{36}, \frac{5-18\lambda}{6}, \frac{6\lambda-5}{6}, \frac{18\lambda-17}{36}, 0\right),$$
$$\beta_{1,0}^{-1,-1} = \left(-\frac{4}{9}, 0, 0, \frac{2(2-9\lambda)}{9}, -\frac{4}{9}, \frac{2(9\lambda-7)}{9}, 0\right),$$
$$\beta_{1,0}^{-1,0} = \left(\frac{2(2-9\lambda)}{9}, 0, 0, \frac{1-18\lambda}{36}, \frac{1-6\lambda}{6}, \frac{18\lambda-13}{6}, \frac{18\lambda-17}{36}\right),$$
$$\beta_{1,0}^{0,1} = \left(\frac{2(1-9\lambda)}{9}, 0, \frac{1-18\lambda}{36}, 0, \frac{7-18\lambda}{18}, \frac{90\lambda-61}{36}, \frac{18\lambda-17}{18}\right).$$

Finally, the masks associated with $\partial_{0,1} f$ are

$$\beta_{0,1}^{1,1} = \left(2\left(1-2\lambda\right), 0, -\frac{2}{9}, \frac{7-18\lambda}{18}, 0, \frac{11-18\lambda}{18}, \frac{2}{9}\right),$$
$$\beta_{0,1}^{1,0} = \left(\frac{11-18\lambda}{6}, 0, -\frac{2}{9}, \frac{2\left(1-3\lambda\right)}{3}, \frac{11-18\lambda}{18}, 0, \frac{1}{9}\right),$$
$$\beta_{0,1}^{0,-1} = \left(\frac{5-6\lambda}{6}, 0, -\frac{1}{9}, \frac{5-18\lambda}{9}, \frac{29-54\lambda}{18}, \frac{1}{9}, 0\right),$$
$$\beta_{0,1}^{-1,-1} = \left(0, 0, 0, \frac{1-6\lambda}{6}, 2\left(1-2\lambda\right), \frac{5-6\lambda}{6}, 0\right),$$
$$\beta_{0,1}^{-1,0} = \left(\frac{1-6\lambda}{6}, 0, 0, -\frac{1}{9}, \frac{25-54\lambda}{18}, \frac{13-18\lambda}{9}, \frac{1}{9}\right),$$
$$\beta_{0,1}^{0,1} = \left(\frac{7-18\lambda}{6}, 0, -\frac{1}{9}, 0, \frac{7-18\lambda}{18}, \frac{2\left(2-2\lambda\right)}{3}, \frac{2}{9}\right).$$

Corollary 1 *The masks in Proposition 3 produce quasi-interpolants that are also superconvergent at the vertices.*

Proof It is sufficient to check that $\varepsilon\left[m_\mu\right]\left(v_{0,0}\right) = 0$, for $|\mu| = 3$. □

After selecting four parameters by imposing superconvergence on the midpoints of the sides, only one parameter remains, which is susceptible to be selected. One possibility is to check the behaviour of the quasi-interpolation error at the vertices of the corresponding quasi-interpolant $Q_\lambda^*[f]$. It is easy to prove that

$$\varepsilon\left[m_{4,0}\right]\left(v_{0,0}\right) = \varepsilon\left[m_{0,4}\right]\left(v_{0,0}\right) = -\frac{4}{3}h^4, \varepsilon\left[m_{2,2}\right]\left(v_{0,0}\right) = \frac{4}{9}h^4, \varepsilon\left[m_{1,3}\right]\left(v_{0,0}\right) = 0$$

and

$$\varepsilon\left[m_{3,1}\right]\left(v_{0,0}\right) = 2h^4\left(2\lambda-1\right).$$

Therefore, the choice $\lambda = 1/2$ produces quasi-interpolants $Q_{1/2}^*[f]$ with symmetric behaviour with respect to the errors at the vertices for the quartic monomials.

5 Numerical tests

We test the performance of the operator $Q_{1/2}^*$ by considering the classical Franke and Nielson test functions. They are [13, 14]

$$f\left(x,y\right) = \frac{1}{2}\exp\left(-\left(\left(9x-7\right)^2+\frac{1}{4}\left(9y-3\right)^2\right)\right) + \frac{3}{4}\exp\left(-\frac{1}{49}\left(9x+1\right)^2 - \frac{1}{10}\left(9y+1\right)\right)$$
$$-\frac{1}{5}\exp\left(-\left(9x-4\right)^2-\left(9y-7\right)^2\right) + \frac{3}{4}\exp\left(-\left(\left(9x-2\right)^2+\left(9y-2\right)^2\right)\right),$$
$$g\left(x,y\right) = \frac{1}{2}y\cos^4\left(4\left(x^2+y-1\right)\right).$$

	f		g	
n	error	NCO	error	NCO
8	3.624×10^{-1}	–	5.258×10^{-1}	–
16	8.836×10^{-2}	2.036	1.062×10^{-1}	2.307
32	8.742×10^{-3}	3.337	9.658×10^{-3}	3.459
64	7.303×10^{-4}	3.581	7.426×10^{-4}	3.701
128	7.550×10^{-5}	3.274	6.381×10^{-5}	3.541

Table 1 Estimations of the quasi-interpolation errors for Franke and Nielson functions provided by the operator $Q^*_{1/2}$ for $h = 1/n$.

Their quasi-interpolants are computed on $\Omega := [0,1] \times [0,1]$ for which evaluations of f and g at points outside Ω but close to its boundary are necessary.

Table 1 shows approximate values of the $\left\| f - Q^*_{1/2}[f] \right\|_{\infty,\Omega}$ and $\left\| g - Q^*_{1/2}[g] \right\|_{\infty,\Omega}$ for $h = 1/n$, with $n = 8, 16, 32, 46, 128$. They are estimated from the values of the test function and its quasi-interpolant at 28 points in each triangle. It also includes the numerical approximation orders, computed as the rate

$$\text{NCO} := \log\left(\frac{\mathtt{E}(h_2)}{\mathtt{E}(h_1)}\right) / \log\left(\frac{h_2}{h_1}\right),$$

where $\mathtt{E}(h)$ stands for the estimated error associated with the step-length h.

The results confirm the theoretical results regarding the performance of $Q^*_{1/2}$.

6 Conclusions

In this paper, we have proposed the construction of C^1 cubic quasi-interpolants on a three-direction mesh of $\mathbb{R}^2$. The quasi-interpolating splines have been defined by directly setting their BB-coefficients from point and gradient values in order to reproduce quadratic polynomials, the highest possible degree. The resulting spline depends on five parameters that we have fixed imposing additional properties. Finally, we have provided some numerical tests confirming the approximation properties.

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