Supersymmetric Localization and Nonconformal $\mathcal{N}=2$ Supersymmetric Yang-Mills Theories in the Perturbative Regime

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We examine the relation between supersymmetric localization on $\mathbb{S}^4$ and standard QFT results for nonconformal theories in flat space. Specifically, we consider 1/2 BPS circular Wilson loops in four-dimensional $\mathrm{SU}(N)$ $\mathcal{N}=2$ supersonic Yang-Mills theories with massless hypermultiplets in an arbitrary representation $\mathcal{R}$ such that the β function is nonvanishing. On $\mathbb{S}^4$, localization maps this observable into an interacting matrix model. Despite broken conformal symmetry at the quantum level, we show that within a specific regime of validity the matrix model predictions are consistent with perturbation theory in flat space up to order g^6. At this order, the reorganization of Feynman diagrams based on the matrix model potential, which has been widely tested in conformal models, also applies in nonconformal setups and is realized, in perturbative field theory, through highly nontrivial interference mechanisms.

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Introduction—Localization techniques have represented a major breakthrough in the study of supersymmetric gauge theories on compact manifolds at the nonperturbative level [1]. Several exact results have been obtained for partition functions [2–6], Wilson loops [7–13], line defects [12,14–16], and other supersymmetric observables [17–22], enabling nontrivial checks of the AdS/CFT duality [23–26] also in nonmaximally supersymmetric models.

Technically, the finite volume of spacetime plays an essential role in the localization procedure and serves as a (natural) gauge-invariant regulator for IR divergences. Conversely, the UV structure of the theory is left unchanged by the compactification and generally requires a renormalization. In superconformal models, the localization predictions naturally extend to the infinite flat space, and it is possible to compare them with standard field theory approaches. This program has been actively conducted in four dimensions, where the matrix model generated by supersymmetric localization on $\mathbb{S}^4$ was successfully tested at weak coupling against perturbative approaches for BPS Wilson loops [27–29] and special local correlators [30–32]. These analyses reveal that perturbative computations in flat space are captured by a one-loop effective action on $\mathbb{S}^4$ [1], which provides an elegant reorganization of the different Feynman diagrams [29].

However, when the gauge theory contains dimensionful parameters, such as a mass term in $\mathcal{N}=2^*$ theories or a scale generated by dimensional transmutation, conformal symmetry in flat space is broken. As a result, the short and long distance properties of the model are different and the calculation in $\mathbb{R}^4$ and on $\mathbb{S}^4$ are no longer expected to match. In particular, when the theory contains a mass scale, observables calculated on $\mathbb{S}^4$ acquire a dependence on the dimensionless parameter constructed by the product of the mass scale and of the radius of the sphere, leading to different results with respect to the flat space. This scenario was examined in [33] studying the vacuum expectation value of the half-BPS circular Wilson loop in $\mathcal{N}=2^*$ supersymmetric Yang-Mills (SYM) and finding that the perturbative two-loop computation of the observable on $\mathbb{S}^4$ coincides with the localization result, while the analogous flat-space computation differs.

While a mass deformation violates the conformal invariance at the classical level and affects both the structure of the propagators and of the action [34,35], the presence of a nonvanishing β function yields a breaking of the map between $\mathbb{S}^4$ and $\mathbb{R}^4$ at the *quantum* level. However, supersymmetric localization still provides explicit expressions in terms of a one-loop exact running coupling constant [1], in analogy to the flat-space computations. Therefore, it is important to investigate whether the conventional perturbative series in flat Euclidean space, when

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expressed in terms of the running coupling, is encoded in the localization effective action or to understand which part of it (if any) is univocally determined by the localization approach.

In this Letter, we consider SU(N) $\mathcal{N}=2$ SYM theories defined in flat space with massless hypermultiplets in a generic representation $\mathcal{R}$. The $\mathcal{N}=2$ vector multiplet contains the gauge field A_μ, a complex scalar ϕ, and two Weyl fermions, while the hypermultiplets are described by two complex scalars along with their fermionic superpartners. In these theories, the β function is one-loop exact, i.e.,

$$\beta(g)=\beta_0 g^3, \quad \text{where}\, \beta_0=\frac{i_\mathcal{R}-N}{8\pi^2}. \tag{1}$$

In the previous expression, we denoted with $i_\mathcal{R}$ the Dynkin index of the representation $\mathcal{R}$ (the Dynkin index is defined by $\mathrm{Tr}_\mathcal{R} T^a T^b = i_\mathcal{R}\delta^{ab}$, with the normalization that $i_F=1/2$ for the fundamental representation). Throughout this Letter, we will consider asymptotically free theories, where $\beta_0<0$. We will focus on the $1/2$ BPS circular Wilson loop operator in the fundamental representation

$$\hat{W}=\frac{1}{N}\mathrm{tr}\mathcal{P}\exp\left\{g_B\int_C \mathrm{d}\tau\left[iA_\mu\dot{x}^\mu+\frac{R}{\sqrt{2}}(\phi+\phi^\dagger)\right]\right\}, \tag{2}$$

where g_B is the bare coupling constant, $\mathcal{P}$ is the path-ordering operator, and the integral is over a circle C of radius R parametrized as $x_\mu(\tau)=R(\cos\tau,\sin\tau,\mathbf{0})$.

In the following, we will show that the matrix model predictions match standard perturbation theory up to three-loop accuracy [36] *within* a specific range of validity [see Eq. (8)]. (The full technical details of the Feynman diagram computations will be given in an upcoming paper.) Specifically, supersymmetric localization predicts two corrections proportional to $\zeta(3)$ which, in perturbative field theory, possess a different origin: one of them is present also in superconformal cases [28,29,31] and arises from a Feynman integral which has the same form on $\mathbb{R}^4$ and $\mathbb{S}^4$, while the second one emerges by interference effects between evanescent terms and the UV divergence of the bare coupling constant.

BPS Wilson loop on $\mathbb{S}^4$ via localization—Compactifying the theory on a four sphere of radius R and for which C is an equator, supersymmetric localization [1] enables us to compute the expectation value of the half-BPS Wilson loop defined in Eq. (2) by an integral over a traceless Hermitian $N\times N$ matrix a:

$$\mathcal{W}=\frac{1}{\mathcal{Z}}\int \mathrm{d}a\, \mathrm{e}^{-\mathrm{tr}a^2-S_{\mathrm{int}}(a,g)}\hat{\mathcal{W}}(a), \tag{3}$$

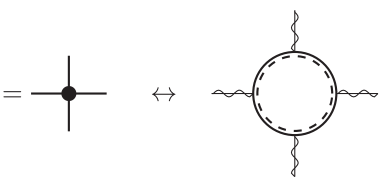
where the matrix operator $\hat{\mathcal{W}}$ reads

$$\hat{\mathcal{W}}=\frac{1}{N}\mathrm{tr}\exp\left(\frac{ga}{\sqrt{2}}\right)=1+\frac{g^2}{4N}\mathrm{tr}\, a^2+\mathcal{O}(g^4) \tag{4}$$

and the partition function $\mathcal{Z}$ is given by the same integral without the insertion of $\hat{\mathcal{W}}$. The integration measure is such that $\int \mathrm{d}a\,\mathrm{e}^{-\mathrm{tr}a^2}=1$. If we neglect the instanton contributions, the interaction potential $S_{\mathrm{int}}(a,g)$ arises from the one-loop determinants around the fixed point of the localizing action and is given by [37]

$$S_{\mathrm{int}}(a,g)=-\sum_{m=2}^{\infty}\left(-\frac{g^2}{8\pi^2}\right)^m\frac{\zeta(2m-1)}{m}\mathrm{Tr}'_\mathcal{R}a^{2m}, \tag{5}$$

where $\mathrm{Tr}'_\mathcal{R}\bullet=(\mathrm{Tr}_\mathcal{R}\bullet-\mathrm{Tr}_{\mathrm{Adj}}\bullet)$. This combination of traces describes the matter content of the difference between the $\mathcal{N}=2$ models under consideration and $\mathcal{N}=4$ SYM. From the perspective of perturbative field theory, Eq. (5) suggests constructing the interaction contributions by considering diagrams with internal lines in representation $\mathcal{R}$, and then subtracting identical contributions in which $\mathcal{R}=\mathrm{Adj}$. Diagrammatically, we will depict the corrections characterized by matter in the so-called *difference theory* [28,29] by a double solid/dashed line. For instance, the expected correspondence between a contribution in the matrix model involving the quartic vertex $\mathrm{Tr}'_\mathcal{R}a^4$ and usual Feynman diagrams is

$\mathrm{Tr}'_\mathcal{R}\, a^4 =$  , (6)

where the superposition of a wiggly/straight line denotes the vector-multiplet field propagation.

Importantly, in (3) and (5), $g=g(R)$ is the running coupling constant

$$\frac{1}{g^2}=\frac{1}{g_*^2}+\beta_0\log M^2R^2, \tag{7}$$

where $g_*=g_*(M)$ is the renormalized coupling at the UV cut-off M. This scale enters the matrix model because the representation $\mathcal{R}$ is associated with a nonvanishing β function. This requires a regularization for the one-loop fluctuation determinants which involves additional hypermultiplets of mass M, see [1,37]. For perturbation theory to be applicable, asymptotic freedom requires that

$$\Lambda\ll\frac{1}{R}\ll M, \quad \text{where}\ \Lambda=M\mathrm{e}^{\frac{1}{2g_*^2\beta_0}} \tag{8}$$

is the *infrared* strong coupling scale generated by dimensional transmutation. In flat space, R is simply the radius of the circular Wilson loop mapped to the equator of $\mathbb{S}^4$ under a stereographic projection. This means that to connect the results on $\mathbb{S}^4$ with those in flat space, it is not necessary to take $R \to \infty$.

Let us also stress that when $1/R$ approaches Λ, the running coupling g is of order $\mathcal{O}(1)$, requiring a resummation of the perturbative series, and the observable also receives nonperturbative powerlike corrections $C_n(R\Lambda)^n$. (In certain multicolor models, such as $\mathcal{N} = 2^*$ or the massive deformation of $\mathcal{N} = 2$ SQCD, the coefficients C_n were determined on $\mathbb{S}^4$ by localization techniques [38]. Instantons, which we neglected in the matrix model, would also contribute to the observable with terms of this type.) We expect that such *infrared* contributions differ between the sphere and flat space due to the conformal anomaly. Conversely, when $1/R$ approaches M, the (massive) regulating degrees of freedom become relevant and the theory itself changes.

The matrix model (3) is *formally* analogous to that employed in the conformal case in [29], so that we can apply the same techniques for the perturbative calculation. Up to order g^6, the prediction is

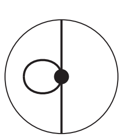
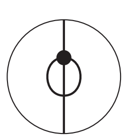
$$\mathcal{W} = \mathcal{W}_0 + \frac{g^6\zeta(3)}{2^9\pi^4 N}\langle \mathrm{tr}a^2 \mathrm{Tr}'_{\mathcal{R}} a^4\rangle_{0,c} + \mathcal{O}(g^8)$$
$$= \mathcal{W}_0 + \frac{3g^6\zeta(3)}{2^8\pi^4 N}\mathcal{K}'_4 + \frac{g^6\zeta(3)}{16\pi^2}C_F N\beta_0 + \mathcal{O}(g^8), \qquad (9)$$

where the subscript 0, c denotes the connected correlator in the Gaussian matrix model, while $C_F = (N^2-1)/2N$ is the fundamental Casimir. Moreover, $\mathcal{W}_0$ captures the Wilson loop expectation value in the free matrix model and, in $\mathcal{N} = 4$ SYM, it resums all the *ladderlike* diagrams. Its explicit expression reads [7,8]

$$\mathcal{W}_0 = \frac{1}{N}L^1_{N-1}(-g^2/4)\mathrm{e}^{\frac{g^2}{8}(1-1/N)} = 1 + \frac{g^2}{4}C_F + \frac{g^4 C_F(2N^2-3)}{192N} + \frac{g^6 C_F(N^4-3N^2+3)}{4608N^2} + \mathcal{O}(g^8), \qquad (10)$$

where L^1_{N-1} is a Laguerre polynomial.

To evaluate the connected correlator for arbitrary $\mathcal{R}$, we introduced the free contraction $\langle a^a a^b\rangle_0 = \delta^{ab}$ and employed the usual Wick theorem. The two interaction terms, characterized by the color factors $C_F N\beta_0$ and $\mathcal{K}'_4 = \mathrm{Tr}'_{\mathcal{R}} T^a T^e T_a T_e = 2NC_F[C_{\mathcal{R}} - (Ni_{\mathcal{R}}/2) - (N^2/2)]$, are associated with the two contractions of the matrix model quartic vertex:

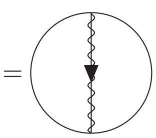
, . (11)

The correspondence between matrix vertices and field theory matter loops (6) suggests that these interactions should arise from single-exchange field theory diagrams. This connection has been checked long ago in [28,29] for generic superconformal setups, where only the $\mathcal{K}'_4$ structure is present. (In [29], the test has been extended to four-loop order in generic superconformal setups.) However, in nonconformal models it is not obvious if this correspondence persists.

Field theory approach in flat space—We regularize Feynman diagrams by *dimensionally reducing* the theory to $d = 4 - 2\epsilon$ dimensions with $\epsilon > 0$ [7]. This scheme preserves the extended supersymmetry of the model but breaks classical conformal symmetry since g_B is dimensionful. Consequently, the v.e.v. of the half-BPS Wilson loop operator (2) can only depend on the dimensionless combination $\hat{g}_B = g_B R^\epsilon$:

$$\langle \hat{W}\rangle \equiv W = 1 + \hat{g}_B^2 W_2 + \hat{g}_B^4 W_4 + \hat{g}_B^6 W_6 + \mathcal{O}(\hat{g}_B^8). \qquad (12)$$

One-loop corrections: The one-loop correction $\hat{g}_B^2 W_2$ arises from the following single-exchange diagram

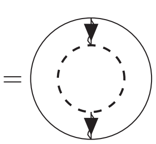
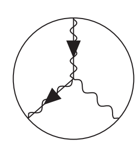
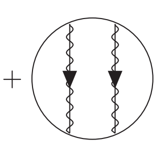
$W_2 =$  $= C_F A_1(\epsilon)$, (13)

where we used the graphical notation of Eq. (6) to denote the gauge field or adjoint scalar propagation inside the Wilson loop and we defined the functions

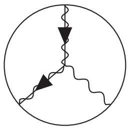
$$A_n(\epsilon) = \frac{1}{8}\pi^{n\epsilon}\Gamma^n(1-\epsilon)\frac{\sec(n\pi\epsilon)\Gamma(-n\epsilon)}{\Gamma(-2n\epsilon)\Gamma(1+n\epsilon)} = \frac{1}{4} + \mathcal{O}(\epsilon). \qquad (14)$$

Note that we do not include the factors $\hat{g}_B$ when we give the explicit result of a diagram.

Two-loop corrections: The Feynman diagrams which contribute at order $\hat{g}_B^4$ are organized in three different classes [37]:

$W_4 =$  $+$  $+$ . (15)

The internal bubble in the first diagram denotes the one-loop correction to the adjoint scalar and gauge field propagator. Specifically, the dashed line is associated with the matter fields in the representation $\mathcal{R}$ which run in the virtual loop. This correction, as well as the diagrams with internal vertices, exhibit a (UV) singular behavior when $\epsilon \to 0$. The singularity in the *Mercedes-like* diagrams arises when two points on the contour collide and is such that [7,37]



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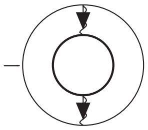
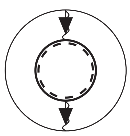


$$- \epsilon \frac{\zeta(3) C_F N}{8\pi^2} + \mathcal{O}(\epsilon^2) \;, \quad (16)$$

where the internal bubble on the right-hand side denotes the one-loop correction to the adjoint scalar and gauge field propagator in $\mathcal{N} = 4$ SYM, where the hypermultiplets are in the adjoint representation. The previous expression reveals in $\mathcal{N} = 4$ SYM, all the interaction diagrams cancel each other out, and the observable receives contributions only from the ladderlike corrections. The evanescent $\zeta(3)$ term results from a triple *path-ordered* integration and possesses the same color factor, proportional to C_F, of the single exchange diagrams (13). Upon renormalization, the UV poles of the bare coupling g_B interfere with the evanescent factor, leading to a finite three-loop contribution.

Substituting Eq. (16) in Eq. (15), we find that

$W_4 =$



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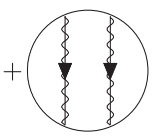
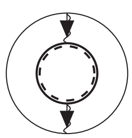


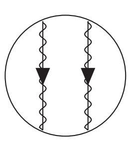
$$- \epsilon \frac{\zeta(3) C_F N}{8\pi^2} + \dots \;, \quad (17)$$

where in the first diagram we employed the double dashed/line of (6) to describe the one-loop propagators in the *difference theory*. Finally, we find that



$$= C_F \frac{\beta_0}{\epsilon(2\epsilon - 1)} A_2(\epsilon) \;. \quad (18)$$

Surprisingly, also the ladderlike diagrams provide an evanescent factor proportional to $\zeta(3)$. Using the well-known properties of the non-Abelian exponentiation of the Wilson loop operator, we obtain

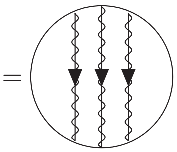


$$= \frac{C_F (2N^2 - 3)}{12N} A_1^2(\epsilon) - \epsilon \frac{C_F N \zeta(3)}{16\pi^2} \;. \quad (19)$$

The evanescent $\zeta(3)$-like term results from a nested quadruple integration over the Wilson loop contour associated with the *maximally non-Abelian* part of the diagram, namely, the contribution of the diagrams characterized by the Casimir eigenvalues with the color factor $C_F C_{\mathrm{adj}} = C_F N$. Note that this combination again coincides with the color factor associated with the single exchange diagrams (13).

Three-loop corrections: At order $\hat{g}_B^6$, we can use the fact that, up to evanescent corrections, all the interaction diagrams with internal line associated with vector-multiplet fields cancel the corrections resulting from hypermultiplets in the adjoint representation. This is the statement that in the $\mathcal{N} = 4$ theory the observable only receives *ladderlike* corrections, while in our case these cancellations reconstruct difference theory loop diagrams suggested by localization. We identify the following five classes of corrections:

$W_6 =$



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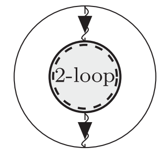

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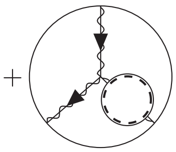

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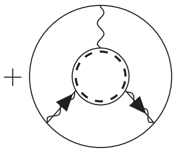

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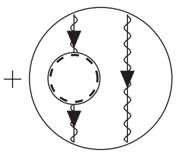

$$+ \mathcal{O}(\epsilon) \;. \quad (20)$$

The $\mathcal{O}(\epsilon)$ terms are analogous to those we encountered in Eq. (16) and, in an analogous way, they could yield finite four-loop corrections whose analysis is beyond our current goal. As we will see in the following section, the diagrams in (20) guarantee the correct renormalization properties of the Wilson loop [39–41]. The calculation of these contributions is extremely technical and will be examined in detail in an upcoming work [36]. We find

$$W_6 = C_F \frac{N^4 - 3N^2 + 3}{4608 N^2} + \frac{3\zeta(3)\mathcal{K}_4'}{2^8 \pi^4 N} + C_F \frac{(\beta_0)^2}{(2\epsilon^2 - \epsilon)^2} A_3 + \frac{C_F(2N^2 - 3)}{6N} \frac{\beta_0}{2\epsilon^2 - \epsilon} A_1 A_2 + \frac{7 C_F N \beta_0 \zeta(3)}{16\pi^2} + \mathcal{O}(\epsilon). \quad (21)$$

Note that at this perturbative order we generated two terms proportional to $\zeta(3)$. The first one, which involves the quantity $\mathcal{K}_4'$, arises from the single-exchange diagrams dressed with the two-loop corrections to the adjoint scalar and gauge field propagators in the difference method, i.e., the second class of diagrams in (20). This term was originally studied in [28] and arises from a well-known Feynman integral which is regular when $\epsilon \to 0$ and proportional to $\zeta(3)$. Being a finite and massless integral in four dimensions, it retains the same form and the same value on the sphere and in flat space.

The second contribution proportional to $\zeta(3)$ in Eq. (21) is characterized by the same color factor predicted by the matrix model in Eq. (9) and results from the last three classes of diagrams depicted in Eq. (20).

Renormalization and comparison with the localization approach—The dimensionally regularized Wilson loop v.e.v. W is ultraviolet divergent and must be renormalized in order to obtain a finite result. Since the operator is defined over a smooth contour, the divergences are removed just by the charge renormalization of the coupling

g_B [39–41] which, in terms of $\hat{g}_B$, amounts to

$$\hat{g}_B = (MR)^{\epsilon} g_* Z_{g_*}(\epsilon). \tag{22}$$

Here $g_*(M) \equiv g_*$ is the renormalized coupling at the renormalization scale M, while Z_{g_*} denotes the subtraction terms. The one-loop exactness of the β function (1) implies that in the MS scheme we find the following expression for the subtraction terms:

$$Z_{g_*}^2(\epsilon) = 1 + \frac{\beta_0 g_*^2}{\epsilon} + \frac{\beta_0^2 g_*^4}{\epsilon^2} + \mathcal{O}(g_*^6). \tag{23}$$

Inserting Eq. (22) in the explicit expression (12) of W that follows from the previous results for $W_{2,4,6}$ all the (UV) divergences cancel and we can define the renormalized observable as

$$W_* = \lim_{\epsilon \to 0} W. \tag{24}$$

When $\epsilon \to 0$, the overall dependence on the scale M disappears and W_* satisfies the usual Callan-Symanzik equation [37]. This implies that W_* must actually depend on M, g_*, and R through the running coupling g defined in (7). This is in fact what happens; moreover, the explicit expression of $W_*(g)$ is quite simple and coincides perfectly with the localization result in (9):

$$W_*(g) = \mathcal{W}(g) + \mathcal{O}(g^8) \tag{25}$$

in the regime of validity specified in (8). In fact, only when $RM \gg 1$ the $\log RM$ terms, which arise when we replace the bare coupling with renormalized one (22), dominate over other scheme-dependent terms which we can then neglect to obtain the relation (25). Potentially, the presence of these large logarithmic contributions could make perturbation theory ill-defined. However, when $R\Lambda \ll 1$ we can resum these large logarithms in the effective coupling g which remains small. Beyond the range (8), we expect that the results on $\mathbb{S}^4$ differ from those in flat space by powerlike corrections proportional to $R\Lambda$ [see the discussion after Eq. (8)].

Some further comments are in order. First, we remark the crucial role of the evanescent factors in the two-loop corrections (16) and (19). Upon renormalization, these factors interfere with the UV poles of the bare coupling $\hat{g}_B$ (23) and provide finite (three-loop) corrections proportional to $\beta_0\zeta(3)$ which combine with the analogous ones in (21). However, it turns out that the $\zeta(3)$ terms resulting from the *Mercedes and lifesaver-like* diagrams, namely the correction depicted in (16) and the analogous ones in (20), do not contribute to the final result. Thus, in the perturbative field approach only the $\zeta(3)$ corrections resulting from the (two and three loop) double-exchange diagrams are relevant and reproduce the matrix model prediction. Importantly, this effect ties nicely in with the diagrammatic approach of the matrix model (11), where these $\zeta(3)$ corrections arise from single-exchange diagrams. In fact, the $\zeta(3)$-like part of the multiple-exchange corrections emerges from the *maximally non-Abelian* parts of the diagrams which, being proportional to C_F, behave as single-exchange diagrams in agreement with the matrix model prediction (11).

Let us also note that, once re-expressed in terms of the running coupling, the renormalized v.e.v. up to three-loop order can be described in terms of few diagrams. Beside the ladder corrections, there is the irreducible part of the three-loop single exchange—the second diagram in (20)—which is characterized by the color factor $\mathcal{K}_4'$ and is present also in the superconformal cases [28,29] and the term proportional to $\beta_0\zeta(3)$ that arises from a pinching limit of the maximally non-Abelian part of the double exchange ladder diagram.

Conclusions and future perspectives—In this Letter, we examined the relation between supersymmetric localization and standard perturbative techniques in flat space for generic $\mathcal{N} = 2$ SYM theories with nonvanishing β function.

We studied via localization the vacuum expectation value of the one-half BPS Wilson loop on $\mathbb{S}^4$. Within the regime described in (8), we showed that the matrix model predictions match standard perturbation theory based on Feynman diagrams techniques in flat space up to order g^6. We precisely related the matrix-model effective diagrams associated with the $\zeta(3)$ terms to the flat-space perturbative expansion. Our results not only provide a nontrivial test of the localization approach for nonconformal theories but also unveil the subtle reorganization of the conventional Feynman diagrams into the matrix-model average. It would be interesting to extend our analysis to the next perturbative order and try to generalize the understanding at all loops. It would also be interesting to take into account nonperturbative corrections. On $\mathbb{S}^4$, this is doable, at least at low instanton number, by including in the matrix model the Nekrasov partition function. Conversely, in flat space, the instanton computations are presumably complicated and beyond the scope of this work. Finally, let us underline that localization provides exact results, in the nonconformal case, also for classes of two-point functions of chiral operators that can be compared with flat-space perturbation theory [42]. It would be interesting to reanalyze these observables in light of the present computations. Exact all-orders expressions on $\mathbb{S}^4$ have been also used to study the large-order behavior of the perturbative series, in connection with resurgent techniques [43], for different $\mathcal{N} = 2$ SYM theories. Reconsidering the nonconformal case and its relation with a flat-space analysis could further improve our understanding of the perturbative results and their gauge-invariant resummation.

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