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ORIGINAL PAPER

Time-frequency representations on Lorentz spaces

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Abstract

In this paper we study boundedness properties of the short-time Fourier transform (STFT) on the Lorentz spaces $L^{p,q}(\mathbb{R}^d)$. The same results apply to the cases of the Wigner and τ-Wigner transforms. Reinterpreting these results in terms of operators we obtain boundedness properties for Weyl and τ-Weyl operators. We conclude with an application to the uncertainty principle of Donoho-Stark in the context of Lorentz spaces.

Keywords Lorentz spaces · Wigner transform · Weyl operators

Mathematics Subject Classification 46E30 · 47G30 · 42B10

1 Introduction

Time-frequency analysis, born to formulate mathematical models for the study of acoustic signals, has actually developed into a vast area of harmonic analysis whose ideas have deeply influenced various areas of applied research such as quantization, acoustics, geophysics, and biomedicine, whereas, at a more theoretic level, its constructions have shown connections with group representations, operator theory and C^*-algebras.

The starting point is the observation that the Fourier transform, although defining an isomorphism on $L^2(\mathbb{R})$, furnishes, in explicit way, only the distribution of the frequencies of a signal $f(t)$, whereas the information about their location in time remains

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“hidden” in the complex phase of $\widehat{f}(\omega)$. In the effort to overcome this drawback various other types of representations of signals have been defined. All of them share the feature of expressing the energy of the signal by a function, or distribution, which depends not only on the frequency but also on the time variable. From here the name *time-frequency representations* (or *transforms*), of which the *short-time Fourier transform* (STFT), and the *Wigner transform* are maybe the two most important examples. The first one, denoted by $V_g f(x,\omega)$, where for generality we suppose $x,\omega \in \mathbb{R}^d$, is obtained by localizing a signal f by multiplication with a *window* function g, which is translated in time, before taking the Fourier transform (formula (2.1)).

Considered as a map $(f,g) \longrightarrow V_g f$, the STFT defines a sesquilinear transform, which is connected with the well-known *(cross-)Wigner* transform $W(f,g)(x,\omega)$ (formula (2.2)), by the equality (see [12], Lemma 4.3.1)

$$W(f,g)(x,\omega) = 2^d e^{4\pi i x\omega} V_{\tilde{g}} f(2x, 2\omega). \tag{1.1}$$

where $\tilde{g}(x) = g(-x)$.

Equation (1.1) shows that L^p-boundedness properties of the two transforms are equivalent (with suitable adaptation of the bounding constants). Boundedness of the two transforms in the L^p setting has been widely studied, see e.g. [3, 4, 6], and the references therein. Actually a wide spectrum of possible functional settings is available in literature for both representations, in particular we recall the scale of modulations spaces, which are a refinement of the L^p spaces suitably designed to measure the time-frequency content of signals, see [7, 12, 13].

Parallel to modulation spaces, a different remarkable refinement of the L^p spaces, which presents interest both theoretically and in applications, are the Lorentz spaces, see e.g. [9–11] and the references therein. In this functional setting the behavior of time-frequency representations has been studied in [19–21], but the literature is not so vast. Following these lines, we present in this paper a further study of the boundedness properties of the STFT in Lorentz spaces.

The paper is organized as follows. In Sect. 2 we introduce the necessary notations and recall the definition and the main properties of Lorentz spaces.

Section 3 is focused on the use of the technique of *restricted weak type operators* and the corresponding Caldéron operators. We show that the STFT is of restricted weak type in two main situations (Theorems 3.6 and 3.7) each of them implying the boundedness of the STFT $V : (f,g) \longrightarrow V_g f$ as sesquilinear map on suitable Lorentz spaces. We remark however that the technique we use does not cover the case $V : L^{2,1}(\mathbb{R}^d) \times L^{2,1}(\mathbb{R}^d) \longrightarrow L^{2,1}(\mathbb{R}^{2d})$. Due to the lack of boundedness of the (partial) Fourier transform on $L^{2,1}$, we conjecture that boundedness does not hold for the STFT either, but at present this remains an open question. We complete then our results with counter-examples which disprove boundedness of the STFT in the remaining cases.

Section 4 is dedicated to the τ-calculus. More precisely we consider the τ-Wigner representation, $\tau \in [0,1]$, (formula (2.3)), which is a one-parameter modification of the classical Wigner tranform (recaptured for $\tau = 1/2$). We refer to [3, 5] for its setting in L^p space and its basic properties. Whereas Sandıkçı proved in [20] the boundedness of that τ-Wigner distribution on Lorentz spaces $L^{p,q}$ with indices p and q satisfying

$1 < p < 2$ and $q \geq 1$, in this paper, we complete these results deducing general boundedness properties from those of the STFT. Classically boundedness properties of time-frequency representations are strictly connected with corresponding properties of the associated linear operators, see e.g. [1, 8, 16], we conclude therefore the section reinterpreting these results in terms of the associated τ-Weyl operators (formula (4.2)). Finally in Sect. 5 we give an application to the uncertainty principle. More precisely, using the boundedness of the STFT previously obtained, we prove a qualitative version of Lieb's uncertainty principle, classically formulated in $L^2(\mathbb{R}^d)$, for signals in the Lorentz spaces.

The results presented are a contribution to fill the gap in the literature concerning the use of Lorentz spaces in time-frequency analysis. Besides the interest in itself, we hope they could find relevance in stimulating further developments in at least the following directions. First of all, although we conjecture that the boundedness fails in the cases not covered by our results, clearly thorough further investigation producing explicit counterexamples could be desirable. Secondly, the qualitative application to the Donoho-Stark uncertainty principle suggests that further applications regarding other uncertainty inequalities could possibly be derived. Thirdly, in the lines of some recent papers devoted to find optimizers of uncertainty inequalities on Euclidean spaces, Riemannian manifolds, and locally compact Abelian groups (see [14, 15, 17]), a recasting of the whole subject in some of these more general settings could be an interesting topic for a further research.

2 Basic definitions and results

In this section we collect some basic definitions and results on Lorentz spaces that we need in the following. We denote as usual by $\mathcal{S}(\mathbb{R}^d)$ and $L^p(\mathbb{R}^d)$, $1 \leq p \leq \infty$, the classical Schwartz space of rapidly decreasing functions and the Lebesgue space, respectively. The operators $T_x f(t) = f(t-x)$ and $M_\omega f(t) = e^{2\pi i \omega t} f(t)$ are called translation and modulation operators for $x, \omega \in \mathbb{R}^d$, respectively. The compositions

$$T_x M_\omega f(t) = e^{2\pi i \omega (t-x)} f(t-x) \qquad \text{or} \qquad M_\omega T_x f(t) = e^{2\pi i \omega t} f(t-x)$$

are called time frequency shifts.

For $f \in L^1(\mathbb{R}^d)$ the Fourier transform $\widehat{f}$ (or $\mathcal{F} f$) is defined as

$$\widehat{f}(t) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x t} dx,$$

where $xt = \sum_{i=1}^d x_i t_i$ is the usual inner product on $\mathbb{R}^d$.

Fix a function $g \neq 0$ (called the window function). The short-time Fourier transform (STFT) of a function f with respect to g is given by

$$V_g f(x, \omega) = \int_{\mathbb{R}^d} f(t) \overline{g(t-x)} e^{-2\pi i t \omega} dt, \tag{2.1}$$

for all $x, \omega \in \mathbb{R}^d$. In principle $V_g f$ is defined whenever the integral is convergent; for instance, it is known that if $f, g \in L^2(\mathbb{R}^d)$ then $V_g f \in L^2(\mathbb{R}^d \times \mathbb{R}^d)$ and $V_g f$ is uniformly continuous (see [12]).
We recall the following result:

Proposition 2.1 ([5], Proposition 6.3) *The short-time Fourier transform*

$$V : (f, g) \in L^{p'}(\mathbb{R}^d) \times L^p(\mathbb{R}^d) \longrightarrow V_g f \in L^q(\mathbb{R}^{2d})$$

is bounded if and only if $q \geq 2$, $q' \leq p \leq q$, *where* $\frac{1}{q} + \frac{1}{q'} = 1$. *In particular,*

$$\|V_g f\|_{L^q(\mathbb{R}^{2d})} \leq \|f\|_{L^{p'}(\mathbb{R}^d)} \|g\|_{L^p(\mathbb{R}^d)}.$$

The cross-Wigner distribution of $f, g \in L^2(\mathbb{R}^d)$ is defined to be

$$W(f, g)(x, \omega) = \int_{\mathbb{R}^d} f\left(x + \frac{t}{2}\right) \overline{g\left(x - \frac{t}{2}\right)} e^{-2\pi i t\omega} dt. \tag{2.2}$$

If $f = g$, then $W(f, f) = Wf$ is called the Wigner distribution of $f \in L^2(\mathbb{R}^d)$.

For $\tau \in [0, 1]$ and $f, g \in \mathcal{S}(\mathbb{R}^d)$, the τ-Wigner transform is defined as

$$W_\tau(f, g)(x, \omega) = \int_{\mathbb{R}^d} f(x + \tau t) \overline{g(x - (1 - \tau)t)} e^{-2\pi i t\omega} dt. \tag{2.3}$$

If $\tau = \frac{1}{2}$, then the τ-Wigner transform is the cross-Wigner distribution. Moreover, for $\tau = 0$, W_0 is the Rihaczek transform,

$$W_0(f, g)(x, \omega) = R(f, g)(x, \omega) = e^{-2\pi i x\omega} f(x) \overline{\widehat{g}(\omega)},$$

and for $\tau = 1$, W_1 is the conjugate Rihaczek transform,

$$W_1(f, g)(x, \omega) = \overline{R(g, f)}(x, \omega) = e^{2\pi i x\omega} \overline{g(x)} \widehat{f}(\omega).$$

For $\tau \in (0, 1)$, define $V_g^\tau f$ as

$$V_g^\tau f(x, \omega) = V_g f\left(\frac{1}{1 - \tau} x, \frac{1}{\tau}\omega\right);$$

then the τ-Wigner transform can be rewritten as

$$W_\tau(f, g)(x, \omega) = \frac{1}{|\tau|^d} e^{2\pi i \frac{1}{\tau} x w} V_{A_\tau g}^\tau f(x, \omega), \tag{2.4}$$

where the operator A_τ is defined by

$$A_\tau : h(t) \to h\left(\frac{\tau - 1}{\tau} t\right).$$

We now recall some definitions and theorems about Lorentz spaces. They all can be found in [2].

Definition 2.2 Let f be a measurable function on $\mathbb{R}^d$. The decreasing rearrangement of f is the function

$$f^* : [0, \infty) \to [0, \infty]$$

defined as

$$f^*(t) = \inf\{\lambda : \mu_f(\lambda) \leq t\}, \quad t \geq 0, \tag{2.5}$$

where

$$\mu_f(\lambda) = \mu\{x \in \mathbb{R}^d : |f(x)| > \lambda\}, \quad \lambda \geq 0, \tag{2.6}$$

and μ is the Lebesgue measure. We denote by f^{**} the maximal function of f^*, defined by

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)ds, \quad t > 0. \tag{2.7}$$

The most important property of f^* is that it has the same distribution function as f. It follows that

$$\left(\int_{\mathbb{R}^d} |f(x)|^p d\mu(x)\right)^{\frac{1}{p}} = \left(\int_0^\infty [f^*(t)]^p dt\right)^{\frac{1}{p}}, \tag{2.8}$$

when $p \in (0, \infty)$.

Theorem 2.3 (Hardy-Littlewood inequality) *Let f, g be two measurable functions on $\mathbb{R}^d$. Then*

$$\int_{\mathbb{R}^d} |f(x)g(x)|dx \leq \int_0^\infty f^*(s)g^*(s)ds. \tag{2.9}$$

For $p, q \in (0, \infty]$, the Lorentz space $L^{p,q}(\mathbb{R}^d)$ is the space of all measurable functions f on $\mathbb{R}^d$ for which $\|f\|_{p,q}$ is finite, where

$$\|f\|_{p,q} = \begin{cases} \left(\int_0^\infty \left[t^{\frac{1}{p}} f^*(t)\right]^q \frac{dt}{t}\right)^{\frac{1}{q}}, & 0 < q < \infty \\ \sup\limits_{0<t<\infty} \{t^{\frac{1}{p}} f^*(t)\}, & q = \infty. \end{cases} \tag{2.10}$$

By (2.8), it follows that $\|f\|_{p,p} = \|f\|_p$ and so $L^{p,p} = L^p$ for every $p \in (0, \infty]$. Also, $L^{\infty,q} = \{0\}$ for every $0 < q < \infty$. We remark that (2.10) yields in general only quasi-norms, however the following proposition holds:

Proposition 2.4 *For $p \in (1, \infty]$ and $q \in [1, \infty]$, we can substitute in (2.10) the function f^* with f^{**}, getting a norm for the Lorentz space $L^{p,q}(\mathbb{R}^d)$ equivalent to the quasi-norm in (2.10).*

Proposition 2.5 *For $p \in (0, \infty]$ and $q, r \in (0, \infty]$ with $q \leq r$ we have that there exists $C > 0$, depending on p, q, r, such that*

$$\|f\|_{p,r} \leq C\|f\|_{p,q}$$

In particular, we have the embedding $L^{p,q} \hookrightarrow L^{p,r}$.

Theorem 2.6 (G.H. Hardy) *Let $\psi : (0, \infty) \to \mathbb{R}$ be a measurable nonnegative function, $-\infty < \lambda < 1$, and $1 \leq q \leq \infty$. Then*

$$\left\{ \int_0^\infty \left(t^{\lambda-1} \int_0^t \psi(s) ds \right)^q \frac{dt}{t} \right\}^{1/q} \leq \frac{1}{1-\lambda} \left\{ \int_0^\infty \left(t^\lambda \psi(t) \right)^q \frac{dt}{t} \right\}^{1/q} \tag{2.11}$$

and

$$\left\{ \int_0^\infty \left(t^{1-\lambda} \int_t^\infty \psi(s) \frac{ds}{s} \right)^q \frac{dt}{t} \right\}^{1/q} \leq \frac{1}{1-\lambda} \left\{ \int_0^\infty \left(t^{1-\lambda} \psi(t) \right)^q \frac{dt}{t} \right\}^{1/q}. \tag{2.12}$$

Definition 2.7 Let $1 \leq p, q, r \leq \infty$ and let T be a bilinear operator. We say that T is of restricted weak type $(p, q; r)$, if there exists a constant $M \geq 0$ such that

$$\sup_{0<t<\infty} \{t^{1/r} [T(\chi_E, \chi_F)]^{**}(t)\} \leq M\mu(E)^{1/p} \mu(F)^{1/q}$$

for every meausurable sets E and F with finite measure where,

- T is supposed to be definite on each pair of simple functions (f, g), where a simple function is a finite linear combination of characteristic functions of sets of finite measure.
- $\mu(E)$ and $\mu(F)$ are the (Lebesgue) measures of E and F.

Proposition 2.8 *Let $1 \leq p, q < \infty$ and $1 < r \leq \infty$. Then T is of restricted weak type $(p, q; r)$ if and only if T extends uniquely to a bounded bilinear operator*

$$T : L^{p,1} \times L^{q,1} \to L^{r,\infty}.$$

Definition 2.9 Let us consider $1 \leq p_j, q_j, r_j \leq \infty$, $j = 1, \ldots, m$ and define

$$\sigma = \left\{ \left(\frac{1}{p_j}, \frac{1}{q_j}, \frac{1}{r_j} \right) \right\}_{j=1}^m. \tag{2.13}$$

The corresponding Calderón operator is defined as

$$S_\sigma(f, g)(t) = \int_0^\infty \int_0^\infty f(u)g(v) \min_{j=1,\ldots,m} \left\{ \frac{u^{1/p_j} v^{1/q_j}}{t^{1/r_j}} \right\} \frac{du}{u} \frac{dv}{v}.$$

Theorem 2.10 *Let* $1 \leq p_j, q_j, r_j \leq \infty$, $j = 1, \ldots, m$ *and define* σ *as in (2.13). Let* T *be a bilinear operator and suppose that* T *is of restricted weak type* $(p_j, q_j; r_j)$ *for every* $j = 1, \ldots, m$. *Then there exists a constant* $M \geq 0$ *such that for every* $t > 0$,

$$T(f, g)^*(t) \leq M S_\sigma(f^*, g^*)(t),$$

for all simple functions f, g.

Moreover, the tensor-product operator $T(f, g)$ which is defined by

$$(f, g) \to f(u)g(v) \equiv (f \otimes g)(u, v)$$

is of restricted weak types $(1, 1; 1)$ and $(\infty, \infty; \infty)$. Hence by Theorem 2.10, it satisfies

$$\begin{aligned} T(f, g)^*(t) &\leq S_\sigma(f^*, g^*)(t) \\ &= \int_0^\infty \int_0^\infty f^*(u)g^*(v) \min\left\{\frac{uv}{t}, 1\right\} \frac{du}{u}\frac{dv}{v}, \end{aligned}$$

for all simple functions f, g, where σ is the set

$$\sigma = \{(0, 0, 0), (1, 1, 1)\}.$$

Theorem 2.11 *Suppose* $1 < p < \infty$ *and* $1 \leq a, b, c \leq \infty$ *with* $\frac{1}{a} + \frac{1}{b} = \frac{1}{c} + 1$. *Then the tensor product operator* T *satisfies*

$$\|T(f, g)\|_{p,c} \leq c\|f\|_{p,a}\|g\|_{p,b}.$$

The next result can be found for instance in [9].

Proposition 2.12 (Young inequality for convolution on the half-line) *Let* $f, g : (0, +\infty) \to \mathbb{R}$, *and* $p, q, r \in [1, \infty]$ *such that*

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}.$$

Suppose that $f \in L^p(\mathbb{R}_+, \frac{dx}{x})$ *and* $g \in L^q(\mathbb{R}_+, \frac{dx}{x})$. *Then the convolution*

$$(f *_{\mathbb{R}_+} g)(x) = \int_0^\infty f(y)g\left(\frac{x}{y}\right)\frac{dy}{y} \tag{2.14}$$

belongs to $L^r(\mathbb{R}_+, \frac{dx}{x})$ *and satisfies*

$$\|f *_{\mathbb{R}_+} g\|_{L^r(\mathbb{R}_+, \frac{dx}{x})} \leq C\|f\|_{L^p(\mathbb{R}_+, \frac{dx}{x})}\|g\|_{L^q(\mathbb{R}_+, \frac{dx}{x})}$$

for a constant $C > 0$.

Proposition 2.13 (Hardy-Littlewood-Stein inequality) *Suppose that* $1 < p < 2$, $\frac{1}{p} + \frac{1}{p'} = 1$, *and* $0 < q \leq \infty$. *Then the following inequality holds*

$$\|\widehat{f}\|_{p',q} \leq K\|f\|_{p,q}.$$

Remark 2.14 In the following we use the results of this section in the case when T is the short-time Fourier transform (2.1). Observe that $V_g f$ is linear in f but conjugate linear in g; however, the previous results on bilinear operators T hold also in this case, since $(\overline{g})^* = g^*$ for every measurable function g.

3 Main results

In this section we analyze boundedness properties of the short-time Fourier transform on Lorentz spaces. To this aim we start by proving the following result.

Lemma 3.1 *The short-time Fourier transform*

$$V : (f, g) \to V_g f$$

is of:

(a) restricted weak type $(2, 2; 2)$,
(b) restricted weak type $(1, \infty; \infty)$,
(c) restricted weak type $(\infty, 1; \infty)$.

Proof (a) By Proposition 2.1 with $p = q = 2$ and Proposition 2.5 we have

$$\|V_g f\|_{2,\infty} \leq C_1\|V_g f\|_{2,2} = C_1\|V_g f\|_2 = C_1\|f\|_2\|g\|_2 \leq C\|f\|_{2,1}\|g\|_{2,1};$$

then the result follows from Proposition 2.8.

(b) Let $E, F \subset \mathbb{R}^d$ be measurable sets with finite measure. By Proposition 2.1 we have

$$\begin{aligned}
\sup_{0<t<\infty}\left\{t^{1/\infty}[V_{\chi_F}\chi_E]^{**}(t)\right\} &= \sup_{0<t<\infty}\left\{\frac{1}{t}\int_0^t (V_{\chi_F}\chi_E)^*(s)ds\right\}\\
&\leq \sup_{0<t<\infty}\left\{\frac{1}{t}\|(V_{\chi_F}\chi_E)^*\|_{L^\infty(0,t)}\int_0^t ds\right\}\\
&\leq \|(V_{\chi_F}\chi_E)^*\|_{L^\infty(0,+\infty)}\\
&= \|V_{\chi_F}\chi_E\|_{L^\infty(\mathbb{R}^{2d})}\\
&\leq \|\chi_E\|_{L^1(\mathbb{R}^d)}\|\chi_F\|_{L^\infty(\mathbb{R}^d)} = \mu(E),
\end{aligned}$$

so the short-time Fourier transform is of restricted weak type $(1, \infty; \infty)$.

(c) Point (c) can be proved in the same way as point (b).

□

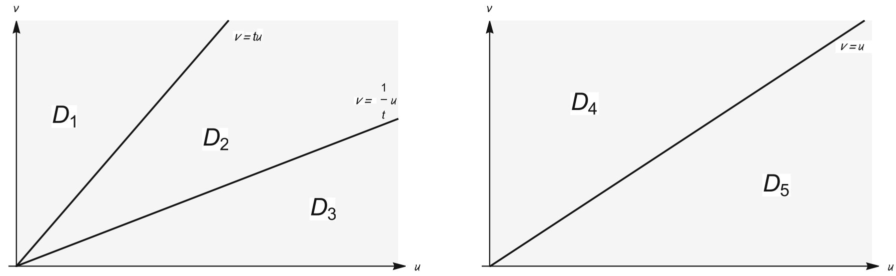

Fig. 1 Definition of the subsets D_1, D_2, D_3, D_4, D_5.

Now, following Definition 2.9, we consider

$$\sigma = \left\{ \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right), (1, 0, 0), (0, 1, 0) \right\} \tag{3.1}$$

and the corresponding Calderón operator

$$S_\sigma(f, g)(t) = \int_0^\infty \int_0^\infty f(u)g(v) \min\left\{\sqrt{\frac{uv}{t}}, u, v\right\} \frac{du}{u}\frac{dv}{v}. \tag{3.2}$$

Remark 3.2 Let us consider the subsets of $\{(u, v) \in \mathbb{R}^2 : u \geq 0,\ v \geq 0\}$ indicated in Fig. 1.

We have that:

- For $t > 1$, $\min\left\{\sqrt{\frac{uv}{t}}, u, v\right\} = \begin{cases} u, & \text{if } (u, v) \in D_1 \\ \sqrt{\frac{uv}{t}}, & \text{if } (u, v) \in D_2 \\ v, & \text{if } (u, v) \in D_3. \end{cases}$
- For $t \in (0, 1]$, $\min\left\{\sqrt{\frac{uv}{t}}, u, v\right\} = \begin{cases} u, & \text{if } (u, v) \in D_4 \\ v, & \text{if } (u, v) \in D_5. \end{cases}$

Remark 3.3 We observe that, by Lemma 3.1, Theorem 2.10 and Remark 2.14, for all simple functions f, g on $\mathbb{R}^d$, and $r \in [1, \infty]$, $c \in [1, \infty)$, there exists $M > 0$ such that

$$\|V_g f\|_{L^{r,c}(\mathbb{R}^{2d})} \leq M \left\{ \int_0^\infty \left[t^{1/r} S_\sigma(f^*, g^*)(t)\right]^c \frac{dt}{t} \right\}^{1/c}, \tag{3.3}$$

where σ is given by (3.1) and S_σ is the corresponding Calderón operator (3.2); for $c = \infty$ we have

$$\|V_g f\|_{L^{r,\infty}(\mathbb{R}^{2d})} \le M \sup_{0<t<\infty} \Big\{ t^{1/r} S_\sigma(f^*, g^*) \Big\}, \tag{3.4}$$

where S_σ is the same operator as before.

In order to find estimation for $\|V_g f\|_{r,c}$, we shall estimate the right-hand side of (3.3) and (3.4).

Proposition 3.4 *Let $c \in [1, \infty]$, $r \in (2, \infty]$, $p \in [r', r]$ with $p \neq 2$, and $a, b \geq 1$ such that $\frac{1}{a} + \frac{1}{b} = 1 + \frac{1}{c}$. Then there exists a constant $C > 0$ such that*

$$\left\{ \int_0^\infty \Big[t^{1/r} S_\sigma(f^*, g^*)(t) \Big]^c \frac{dt}{t} \right\}^{1/c} \le C \|f\|_{p',a} \|g\|_{p,b} \tag{3.5}$$

for every $f \in L^{p',a}(\mathbb{R}^d)$, $g \in L^{p,b}(\mathbb{R}^d)$ where $\frac{1}{p} + \frac{1}{p'} = 1$ and σ and S_σ are given by (3.1) and (3.2). In (3.5) we mean that, for $c = \infty$, the left-hand side is $\sup_{0<t<\infty} \big\{ t^{1/r} S_\sigma(f^*, g^*) \big\}$ as in (3.4).

Proof We start by considering the case $c < \infty$. Observe that for $a_j \in \mathbb{R}$, $a_j \geq 0$, $j = 1, \ldots, m$, and $\alpha > 0$ we have

$$(a_1 + \cdots + a_m)^\alpha \le m^\alpha (a_1^\alpha + \cdots + a_m^\alpha). \tag{3.6}$$

Then

$$\begin{aligned} \left\{ \int_0^\infty \Big[t^{1/r} S_\sigma(f^*, g^*)(t) \Big]^c \frac{dt}{t} \right\}^{1/c} &\le 2^{1/c} \Bigg\{ \left\{ \int_1^\infty \Big[t^{1/r} S_\sigma(f^*, g^*)(t) \Big]^c \frac{dt}{t} \right\}^{1/c} \\ &\quad + \left\{ \int_0^1 \Big[t^{1/r} S_\sigma(f^*, g^*)(t) \Big]^c \frac{dt}{t} \right\}^{1/c} \Bigg\}. \end{aligned} \tag{3.7}$$

Let us consider the first term in (3.7). By definition of S_σ and using Remark 3.2 we have

$$\begin{aligned} &\left\{ \int_1^\infty \Big[t^{1/r} S_\sigma(f^*, g^*)(t) \Big]^c \frac{dt}{t} \right\}^{1/c} \\ &= \left\{ \int_1^\infty \left[t^{1/r} \int_0^\infty \int_0^\infty f^*(u) g^*(v) \min\left\{ \sqrt{\frac{uv}{t}}, u, v \right\} \frac{du}{u} \frac{dv}{v} \right]^c \frac{dt}{t} \right\}^{1/c} \\ &= \left\{ \int_1^\infty \left[t^{1/r} \iint_{D_1} f^*(u) g^*(v) u \frac{du}{u} \frac{dv}{v} + t^{1/r} \iint_{D_2} f^*(u) g^*(v) \sqrt{\frac{uv}{t}} \frac{du}{u} \frac{dv}{v} \right.\right. \\ &\quad \left.\left. + t^{1/r} \iint_{D_3} f^*(u) g^*(v) v \frac{du}{u} \frac{dv}{v} \right]^c \frac{dt}{t} \right\}^{1/c}; \end{aligned}$$

by using (3.6) first on $[\cdots]^c$ and then on $\{\cdots\}^{1/c}$ we can estimate the previous expression as follows:

$$\begin{aligned}
&\left\{\int_1^\infty \left[t^{1/r} S_\sigma(f^*, g^*)(t)\right]^c \frac{dt}{t}\right\}^{1/c} \\
&\quad \le C\left\{\left\{\int_1^\infty \left[t^{1/r} \iint_{D_1} f^*(u) g^*(v) du \frac{dv}{v}\right]^c \frac{dt}{t}\right\}^{1/c}\right. \\
&\qquad + \left\{\int_1^\infty \left[t^{1/r} \iint_{D_2} f^*(u) g^*(v) \sqrt{\frac{uv}{t}} \frac{du}{u} \frac{dv}{v}\right]^c \frac{dt}{t}\right\}^{1/c} \\
&\qquad \left. + \left\{\int_1^\infty \left[t^{1/r} \iint_{D_3} f^*(u) g^*(v) \frac{du}{u} dv\right]^c \frac{dt}{t}\right\}^{1/c}\right\},
\end{aligned} \tag{3.8}$$

where $C = 3^{1+1/c}$. Reasoning in the same way in the second term of (3.7), we finally get that there exists a constant D, depending on the index c, such that

$$\begin{aligned}
&\left\{\int_0^\infty \left[t^{1/r} S_\sigma(f^*, g^*)(t)\right]^c \frac{dt}{t}\right\}^{1/c} \\
&\quad \le D\left\{\left\{\int_1^\infty \left[t^{1/r} \iint_{D_1} f^*(u) g^*(v) du \frac{dv}{v}\right]^c \frac{dt}{t}\right\}^{1/c}\right. \\
&\qquad + \left\{\int_1^\infty \left[t^{1/r} \iint_{D_2} f^*(u) g^*(v) \sqrt{\frac{uv}{t}} \frac{du}{u} \frac{dv}{v}\right]^c \frac{dt}{t}\right\}^{1/c} \\
&\qquad + \left\{\int_1^\infty \left[t^{1/r} \iint_{D_3} f^*(u) g^*(v) \frac{du}{u} dv\right]^c \frac{dt}{t}\right\}^{1/c}\Bigg\} \\
&\qquad + \left\{\int_0^1 \left[t^{1/r} \iint_{D_4} f^*(u) g^*(v) du \frac{dv}{v}\right]^c \frac{dt}{t}\right\}^{1/c} \\
&\qquad \left. + \left\{\int_0^1 \left[t^{1/r} \iint_{D_5} f^*(u) g^*(v) \frac{du}{u} dv\right]^c \frac{dt}{t}\right\}^{1/c}\right\}.
\end{aligned} \tag{3.9}$$

In order to prove the result, we then have to estimate the five terms appearing in (3.9).

1. Estimation of the first term of (3.9): Since $D_1 = \{(u, v) : 0 < u < \frac{v}{t},\ v > 0\}$ (see Fig. 1), for $p \le r$ we have

$$\left\{\int_1^\infty \left[t^{1/r} \iint_{D_1} f^*(u) g^*(v) du \frac{dv}{v}\right]^c \frac{dt}{t}\right\}^{1/c}$$

$$\leq \left\{ \int_1^\infty \left[t^{1/p} \int_0^\infty g^*(v) \Big(\int_0^{v/t} f^*(u)du \Big) \frac{dv}{v} \right]^c \frac{dt}{t} \right\}^{1/c}$$
$$\leq \left\{ \int_0^\infty \left[\int_0^\infty v^{1/p} g^*(v) \Big(\frac{t}{v} \Big)^{1/p} \Big(\int_0^{v/t} f^*(u)du \Big) \frac{dv}{v} \right]^c \frac{dt}{t} \right\}^{1/c},$$

where we can estimate with the integral between 0 and ∞ in the t-variable since all the quantities appearing inside are non-negative. Writing

$$G(y) = y^{1/p} g^*(y), \quad F(y) = y^{1/p} \int_0^{1/y} f^*(u)du, \tag{3.10}$$

recalling the definition of convolution in $\mathbb{R}_+$, see (2.14), and by Proposition 2.12 for $a, b \geq 1$ such that $\frac{1}{a} + \frac{1}{b} = 1 + \frac{1}{c}$, we get

$$\left\{ \int_1^\infty \left[t^{1/r} \iint_{D_1} f^*(u) g^*(v) du \frac{dv}{v} \right]^c \frac{dt}{t} \right\}^{1/c}$$
$$\leq \left\{ \int_0^\infty \left[(F *_{\mathbb{R}_+} G)(t) \right]^c \frac{dt}{t} \right\}^{1/c} \leq \|F\|_{L^a(\mathbb{R}_+, \frac{dt}{t})} \|G\|_{L^b(\mathbb{R}_+, \frac{dt}{t})}$$
$$= \left\{ \int_0^\infty \Big(t^{1/p} \int_0^{1/t} f^*(u)du \Big)^a \frac{dt}{t} \right\}^{1/a} \left\{ \int_0^\infty \Big(t^{1/p} g^*(t) \Big)^b \frac{dt}{t} \right\}^{1/b};$$

by the change of variable $t = \frac{1}{s}$ in the first factor and by Hardy inequality (2.11) with $\lambda = 1 - \frac{1}{p}$,

$$\left\{ \int_1^\infty \left[t^{1/r} \iint_{D_1} f^*(u) g^*(v) du \frac{dv}{v} \right]^c \frac{dt}{t} \right\}^{1/c}$$
$$\leq \|g\|_{p,b} \left\{ \int_0^\infty \Big((s)^{-1/p} \int_0^s f^*(u)du \Big)^a \frac{ds}{s} \right\}^{1/a}$$
$$\leq \|g\|_{p,b} \frac{1}{\frac{1}{p}} \left\{ \int_0^\infty \Big((s^{1-\frac{1}{p}}) f^*(s) \Big)^a \frac{ds}{s} \right\}^{1/a}$$
$$= p \|g\|_{p,b} \left\{ \int_0^\infty \Big(s^{1/p'} f^*(s) \Big)^a \frac{ds}{s} \right\}^{1/a}$$
$$= p \|f\|_{p',a} \|g\|_{p,b}.$$

Summarizing, we have estimated the first term of (3.9) as

$$\left\{ \int_1^\infty \left[t^{1/r} \iint_{D_1} f^*(u) g^*(v) du \frac{dv}{v} \right]^c \frac{dt}{t} \right\}^{1/c} \leq p \|f\|_{p',a} \|g\|_{p,b} \tag{3.11}$$

for $1 \le p \le r,\ a, b \ge 1$, such that $\frac{1}{a} + \frac{1}{b} = 1 + \frac{1}{c}$.

2. Estimation of the second term of (3.9): Since $D_2 = \{(u, v) : \frac{u}{t} < v < tu,\ u > 0\}$, for every $1 \le p_1 \le r$, taking into account that all the functions involved in the integrals are non-negative, we have

$$\left\{ \int_1^\infty \left[t^{1/r} \iint_{D_2} f^*(u) g^*(v) \sqrt{\frac{uv}{t}} \frac{du}{u} \frac{dv}{v} \right]^c \frac{dt}{t} \right\}^{1/c}$$
$$\le \left\{ \int_1^\infty \left[t^{1/p_1 - 1/2} \int_0^\infty f^*(u) \left(\int_{u/t}^{tu} g^*(v) \frac{dv}{\sqrt{v}} \right) \frac{du}{\sqrt{u}} \right]^c \frac{dt}{t} \right\}^{1/c}$$
$$\le \left\{ \int_0^\infty \left[t^{1/p_1 - 1/2} \int_0^\infty f^*(u) \left(\int_{u/t}^{\infty} g^*(v) \frac{dv}{\sqrt{v}} \right) \frac{du}{\sqrt{u}} \right]^c \frac{dt}{t} \right\}^{1/c}$$
$$= \left\{ \int_0^\infty \left[\int_0^\infty u^{1/p_1} f^*(u) \left(\frac{t}{u}\right)^{1/p_1 - 1/2} \left(\int_{u/t}^{\infty} g^*(v) \frac{dv}{\sqrt{v}} \right) \frac{du}{u} \right]^c \frac{dt}{t} \right\}^{1/c}.$$

Writing

$$F_1(y) = y^{1/p_1} f^*(y), \quad G_1(y) = y^{1/p_1 - 1/2} \int_{1/y}^\infty g^*(v) \frac{dv}{\sqrt{v}} \tag{3.12}$$

and recalling the definition (2.14) of convolution in $\mathbb{R}_+$, we get

$$\left\{ \int_1^\infty \left[t^{1/r} \iint_{D_2} f^*(u) g^*(v) \sqrt{\frac{uv}{t}} \frac{du}{u} \frac{dv}{v} \right]^c \frac{dt}{t} \right\}^{1/c} \le$$
$$\le \left\{ \int_0^\infty \left[F_1 *_{\mathbb{R}_+} G_1(t) \right]^c \frac{dt}{t} \right\}^{1/c} = \| F_1 *_{\mathbb{R}_+} G_1 \|_{L^c(\mathbb{R}_+, \frac{dt}{t})}.$$

By Young inequality (see Proposition 2.12)

$$\left\{ \int_1^\infty \left[t^{1/r} \iint_{D_2} f^*(u) g^*(v) \sqrt{\frac{uv}{t}} \frac{du}{u} \frac{dv}{v} \right]^c \frac{dt}{t} \right\}^{1/c}$$
$$\le \|F_1\|_{L^a(\mathbb{R}_+, \frac{dt}{t})} \|G_1\|_{L^b(\mathbb{R}_+, \frac{dt}{t})}$$
$$= \left\{ \int_0^\infty \left(t^{1/p_1} f^*(t) \right)^a \frac{dt}{t} \right\}^{1/a} \left\{ \int_0^\infty \left[t^{1/p_1 - 1/2} \left(\int_{1/t}^\infty g^*(v) \frac{dv}{\sqrt{v}} \right) \right]^b \frac{dt}{t} \right\}^{1/b}$$
$$= \|f\|_{p_1, a} \left\{ \int_0^\infty \left[s^{1/2 - 1/p_1} \left(\int_s^\infty \sqrt{v} g^*(v) \frac{dv}{v} \right) \right]^b \frac{ds}{s} \right\}^{1/b},$$

for $a, b \ge 1$ such that $\frac{1}{a} + \frac{1}{b} = 1 + \frac{1}{c}$, where in the last line we put $t = \frac{1}{s}$ in the integral. By Hardy inequality (2.12) with $\lambda = \frac{1}{2} + \frac{1}{p_1}$ (assuming that $p_1 > 2$ in such a way that $\lambda < 1$), $q = b$ and $\psi(v) = \sqrt{v} g^*(v)$, we obtain

$$\left\{ \int_1^\infty \left[t^{1/r} \iint_{D_2} f^*(u) g^*(v) \sqrt{\frac{uv}{t}} \frac{du}{u} \frac{dv}{v} \right]^c \frac{dt}{t} \right\}^{1/c}$$

$$\leq \|f\|_{p_1,a} \frac{1}{1 - \frac{1}{2} - \frac{1}{p_1}} \left\{ \int_0^\infty \left((t^{1-\frac{1}{2}-\frac{1}{p_1}}) \sqrt{t} g^*(t) \right)^b \frac{dt}{t} \right\}^{1/b}$$

$$= \frac{2p_1}{p_1 - 2} \|f\|_{p_1,a} \|g\|_{p_1',b}.$$

Summarizing, we have estimated the second term of (3.9) as

$$\left\{ \int_1^\infty \left[t^{1/r} \iint_{D_2} f^*(u) g^*(v) \sqrt{\frac{uv}{t}} \frac{du}{u} \frac{dv}{v} \right]^c \frac{dt}{t} \right\}^{1/c} \leq \frac{2p_1}{p_1 - 2} \|f\|_{p_1,a} \|g\|_{p_1',b} \tag{3.13}$$

for $1 \leq p_1 \leq r,\ p_1 > 2,\ a, b \geq 1$, such that $\frac{1}{a} + \frac{1}{b} = 1 + \frac{1}{c}$.

3. Estimation of the third term of (3.9): We observe that $D_3 = \{(u, v) : 0 < v < \frac{u}{t},\ u > 0\}$ (see Fig. 1); moreover, $D_1 = \{(u, v) : 0 < u < \frac{v}{t},\ v > 0\}$. Then interchanging u and v we obtain

$$\left\{ \int_1^\infty \left[t^{1/r} \iint_{D_3} f^*(u) g^*(v) \frac{du}{u} dv \right]^c \frac{dt}{t} \right\}^{1/c}$$

$$= \left\{ \int_1^\infty \left[t^{1/r} \iint_{D_1} f^*(v) g^*(u) \frac{dv}{v} du \right]^c \frac{dt}{t} \right\}^{1/c}.$$

So the third term of (3.9) corresponds to the first term of (3.9) with f and g interchanged. By (3.11) we then get

$$\left\{ \int_1^\infty \left[t^{1/r} \iint_{D_3} f^*(u) g^*(v) \frac{du}{u} dv \right]^c \frac{dt}{t} \right\}^{1/c} \leq p \|f\|_{p_1,b_1} \|g\|_{p_1',a_1} \tag{3.14}$$

for $1 \leq p_1 \leq r, a_1, b_1 \geq 1$ such that $\frac{1}{a_1} + \frac{1}{b_1} = 1 + \frac{1}{c}$.

4. Estimation of the fourth term of (3.9): Since $D_4 = \{(u, v) : 0 < u < v,\ v > 0\}$ we have

$$\left\{ \int_0^1 \left[t^{1/r} \iint_{D_4} f^*(u) g^*(v) du \frac{dv}{v} \right]^c \frac{dt}{t} \right\}^{1/c}$$

$$= \left\{ \int_0^1 \left[t^{1/r} \int_0^\infty g^*(v) \Big(\frac{1}{v} \int_0^v f^*(u) du \Big) dv \right]^c \frac{dt}{t} \right\}^{1/c}$$

$$= \left\{ \int_0^1 \left[t^{1/r} \int_0^\infty g^*(v) f^{**}(v) dv \right]^c \frac{dt}{t} \right\}^{1/c}$$

$$= \int_0^\infty f^{**}(v) g^*(v) dv \Big(\int_0^1 t^{\frac{c}{r}-1} dt \Big)^{1/c}$$

$$= \Big(\frac{r}{c}\Big)^{1/c} \int_0^\infty \Big(v^{1/p_2} f^{**}(v) \Big) \Big(v^{1/p_2'} g^*(v) \Big) \frac{dv}{v}$$

for every $p_2 \in [1, \infty]$, since $\frac{1}{p_2} + \frac{1}{p_2'} = 1$. Now by applying Hölder inequality for $\mathbb{R}_+$ with the measure $\frac{dv}{v}$, we have that for every $q \in [1, \infty]$

$$\left\{ \int_0^1 \left[t^{1/r} \iint_{D_4} f^*(u) g^*(v) du \frac{dv}{v} \right]^c \frac{dt}{t} \right\}^{1/c}$$

$$\leq \Big(\frac{r}{c}\Big)^{1/c} \| v^{1/p_2} f^{**}(v) \|_{L^q(\mathbb{R}_+, \frac{dv}{v})} \| v^{1/p_2'} g^*(v) \|_{L^{q'}(\mathbb{R}_+, \frac{dv}{v})}$$

$$= \Big(\frac{r}{c}\Big)^{1/c} \Big(\int_0^\infty \Big(v^{1/p_2} f^{**}(v) \Big)^q \frac{dv}{v} \Big)^{1/q} \Big(\int_0^\infty \Big(v^{1/p_2'} g^*(v) \Big)^{q'} \frac{dv}{v} \Big)^{1/q'}$$

$$\leq D \| f \|_{p_2, q} \| g \|_{p_2', q'} \tag{3.15}$$

by Proposition 2.4, where (3.15) holds for every $p_2, q \in [1, \infty]$. We observe that, if we fix $a, b \geq 1$, such that $\frac{1}{a} + \frac{1}{b} = 1 + \frac{1}{c}$ and we fix in (3.15) $q = a$, we get that $b < q'$ and then by Proposition 2.5, we get $\| g \|_{p_2', q'} \leq C \| g \|_{p_2', b}$. Then from (3.15) we deduce that

$$\left\{ \int_0^1 \left[t^{1/r} \iint_{D_4} f^*(u) g^*(v) du \frac{dv}{v} \right]^c \frac{dt}{t} \right\}^{1/c} \leq D_1 \| f \|_{p_2, a} \| g \|_{p_2', b} \tag{3.16}$$

for every $p_2 \in [1, \infty]$ and $a, b \geq 1$ such that $\frac{1}{a} + \frac{1}{b} = 1 + \frac{1}{c}$.

5. Estimation of the fifth term of (3.9): We observe that $D_5 = \{(u, v) : 0 < v < u,\ u > 0\}$ (see Fig. 1) and $D_4 = \{(u, v) : 0 < u < v,\ v > 0\}$. Then we can proceed as in (3) and, interchanging u and v, we obtain

$$\left\{ \int_0^1 \left[t^{1/r} \iint_{D_5} f^*(u) g^*(v) \frac{du}{u} dv \right]^c \frac{dt}{t} \right\}^{1/c}$$

$$= \left\{ \int_0^1 \left[t^{1/r} \iint_{D_4} f^*(v) g^*(u) \frac{dv}{v} du \right]^c \frac{dt}{t} \right\}^{1/c}.$$

So the fifth term of (3.9) corresponds to the fourth one with f and g interchanged. By (3.16) we then get

$$\left\{ \int_0^1 \left[t^{1/r} \iint\limits_{D_5} f^*(u) g^*(v) \frac{du}{u} dv \right]^c \frac{dt}{t} \right\}^{1/c} \leq D_1 \| f \|_{p_3', b_1} \| g \|_{p_3, a_1} \tag{3.17}$$

for every $p_3 \in [1, \infty]$ and $a_1, b_1 \geq 1$ such that $\frac{1}{b_1} + \frac{1}{a_1} = 1 + \frac{1}{c}$.

Now, to conclude the proof, we observe that by (3.9), (3.11), (3.13), (3.14), (3.16), (3.17), we obtain the estimate of Proposition 3.4 for $1 \leq p,\ p' \leq r,\ p' > 2$ and $a, b \geq 1$ such that $\frac{1}{a} + \frac{1}{b} = 1 + \frac{1}{c}$. On the other hand, since

$$S_\sigma(f, g)(t) = S_\sigma(g, f)(t)$$

(cf. (3.2)), it is enough that one between p and p' is greater than 2, that is true $\Leftrightarrow$ $p \neq 2$. Then we need that:

(i) $r > 2$
(ii) $p \leq r$
(iii) $p' \leq r$
(iv) $a, b \geq 1$ such that $\frac{1}{a} + \frac{1}{b} = 1 + \frac{1}{c}$.

These requirements can be written as $r > 2$, $p \in [r', r]$, $p \neq 2$, and $a, b \geq 1$ such that $\frac{1}{a} + \frac{1}{b} = 1 + \frac{1}{c}$. This corresponds to the hypotheses of Proposition 3.4, that is then is proved for $c < \infty$.

The case $c = \infty$ can be treated in the same way. Indeed, estimate (3.9) becomes

$$\begin{aligned} \sup_{0<t<\infty} \left[t^{1/r} S_\sigma(f^*, g^*)(t) \right] \leq D \Bigg\{ & \sup_{1<t<\infty} \left[t^{1/r} \iint\limits_{D_1} f^*(u) g^*(v) du \frac{dv}{v} \right] \\ & + \sup_{1<t<\infty} \left[t^{1/r} \iint\limits_{D_2} f^*(u) g^*(v) \sqrt{\frac{uv}{t}} \frac{du}{u} \frac{dv}{v} \right] \\ & + \sup_{1<t<\infty} \left[t^{1/r} \iint\limits_{D_3} f^*(u) g^*(v) \frac{du}{u} dv \right] \\ & + \sup_{0<t<1} \left[t^{1/r} \iint\limits_{D_4} f^*(u) g^*(v) du \frac{dv}{v} \right] \\ & + \sup_{0<t<1} \left[t^{1/r} \iint\limits_{D_5} f^*(u) g^*(v) \frac{du}{u} dv \right] \Bigg\}. \end{aligned} \tag{3.18}$$

Then the computations are exactly the same as before; there are just two points in the estimates of the first and second terms where something new appears.

Indeed, in the estimate of the first term of (3.18), we get now $\|F *_{\mathbb{R}_+} G\|_{L^\infty(\mathbb{R}_+)}$, where F and G are as in (3.10); we then estimate

$$\|F *_{\mathbb{R}_+} G\|_{L^\infty(\mathbb{R}_+)} \le \|F\|_{L^a(\mathbb{R}_+,\frac{dt}{t})}\|G\|_{L^b(\mathbb{R}_+,\frac{dt}{t})}$$

for $\frac{1}{a}+\frac{1}{b}=1$, and here one between a and b can be ∞. If $b=\infty$,

$$\|G\|_{L^b(\mathbb{R}_+,\frac{dt}{t})} = \|G\|_{L^\infty(\mathbb{R}_+)} = \sup_{0<t<\infty}\{t^{1/p}g^*(t)\} = \|g\|_{p,\infty} = \|g\|_{p,b};$$

moreover if $a=\infty$,

$$\begin{aligned}\|F\|_{L^a(\mathbb{R}_+,\frac{dt}{t})} = \|F\|_{L^\infty(\mathbb{R}_+)} &= \sup_{0<t<\infty}\left\{t^{\frac{1}{p}}\int_0^{1/t} f^*(u)du\right\}\\ &= \sup_{0<s<\infty}\left\{s^{-\frac{1}{p}}\int_0^{s} f^*(u)du\right\}\\ &= \sup_{0<s<\infty}\left\{s^{1-\frac{1}{p}}\frac{1}{s}\int_0^{s} f^*(u)du\right\};\end{aligned}$$

then for $a=\infty$, by (2.7) and Proposition 2.4 we get

$$\|F\|_{L^a(\mathbb{R}_+,\frac{dt}{t})} = \sup_{0<s<\infty}\left\{s^{\frac{1}{p'}}f^{**}(s)\right\} \le C\|f\|_{p',\infty} = \|f\|_{p',a}.$$

Then for the first term of (3.18) we obtain the same estimate (3.11).
Concerning the second term we find now $\|F_1 *_{\mathbb{R}_+} G_1\|_{L^\infty(\mathbb{R}_+)}$, where F_1 and G_1 are as in (3.12); we then estimate

$$\|F_1 *_{\mathbb{R}_+} G_1\|_{L^\infty(\mathbb{R}_+)} \le \|F_1\|_{L^a(\mathbb{R}_+,\frac{dt}{t})}\|G_1\|_{L^b(\mathbb{R}_+,\frac{dt}{t})}$$

for $\frac{1}{a}+\frac{1}{b}=1$, and again both a and b could now be ∞. As before, if $a=\infty$,

$$\|F_1\|_{L^a(\mathbb{R}_+,\frac{dt}{t})} = \|F_1\|_{L^\infty(\mathbb{R}_+)} = \sup_{0<t<\infty}\{t^{\frac{1}{p_1}}f^*(t)\} = \|f\|_{p_1,\infty} = \|f\|_{p_1,a}.$$

If $b=\infty$, we have

$$\begin{aligned}\|G_1\|_{L^b(\mathbb{R}_+,\frac{dt}{t})} = \|G_1\|_{L^\infty(\mathbb{R}_+)} &= \sup_{0<t<\infty}\left\{t^{\frac{1}{p_1}-\frac{1}{2}}\int_{1/t}^{\infty} g^*(v)\frac{dv}{\sqrt{v}}\right\}\\ &= \sup_{0<t<\infty}\left\{t^{\frac{1}{p_1}-\frac{1}{2}}\int_{1/t}^{\infty} v^{\frac{1}{p_1'}}g^*(v)\frac{dv}{v^{\frac{1}{p_1'}+\frac{1}{2}}}\right\}\\ &\le \sup_{0<t<\infty}\left\{t^{\frac{1}{p_1}-\frac{1}{2}}\sup_{0<v<\infty}\left\{v^{\frac{1}{p_1'}}g^*(v)\right\}\int_{1/t}^{\infty}\frac{dv}{v^{\frac{1}{p_1'}+\frac{1}{2}}}\right\}\end{aligned}$$

$$= \frac{2 - p_1'}{2p_1'} \|g\|_{p_1',b}.$$

So also for the second term of (3.18) we find again the same estimate (3.13). The estimates of the other terms of (3.18) do not present substantial differences with respect to the case $c < \infty$; the proof of Proposition 3.4 is then complete. □

Theorem 3.5 ([9], Theorem 1.4.13) *The simple functions are dense in $L^{p,q}$ when $0 < q < \infty$.*

We have now the following result.

Theorem 3.6 *Let $c \in [1, \infty)$, $r \in (2, \infty]$, $p \in [r', r]$, $p \neq 2$, and $a, b \geq 1$ such that $\frac{1}{a} + \frac{1}{b} = 1 + \frac{1}{c}$. Then the short-time Fourier transform is bounded as an operator*

$$V : L^{p',a}(\mathbb{R}^d) \times L^{p,b}(\mathbb{R}^d) \longrightarrow L^{r,c}(\mathbb{R}^{2d}). \tag{3.19}$$

Proof For all simple functions, we have by Remark 3.3 (considering the case $c < \infty$)

$$\|V_g f\|_{L^{r,c}(\mathbb{R}^{2d})} \leq M \left\{ \int_0^\infty \left[t^{1/r} S_\sigma(f^*, g^*)(t) \right]^c \frac{dt}{t} \right\}^{1/c}$$

where σ and S_σ are given by (3.1) and (3.2). Then, by Proposition 3.4 we have

$$\|V_g f\|_{L^{r,c}(\mathbb{R}^{2d})} \leq MC \|f\|_{p',a} \|g\|_{p,b},$$

and so (3.19) is proved for f and g simple. By Theorem 3.5 the conclusion follows by density arguments, since the condition $c < \infty$ implies also $a, b < \infty$. □

In Theorem 3.6 we require $p \neq 2$; in the case $p = 2$ we have the following result.

Theorem 3.7 *Let $r \in (2, \infty)$. Then the short-time Fourier transform is bounded as an operator*

$$V : L^{2,1} \times L^{2,1} \longrightarrow L^{r,1}.$$

Proof We know (cf. Proposition 2.1) that V is a continuous map between:

$$V : L^{p'} \times L^p \longrightarrow L^r,$$

for every (p, r) in the shadowed zone in Fig. 2:

In particular we have

$$\begin{cases} V : L^2 \times L^2 \longrightarrow L^2 \\ V : L^2 \times L^2 \longrightarrow L^\infty \end{cases}$$

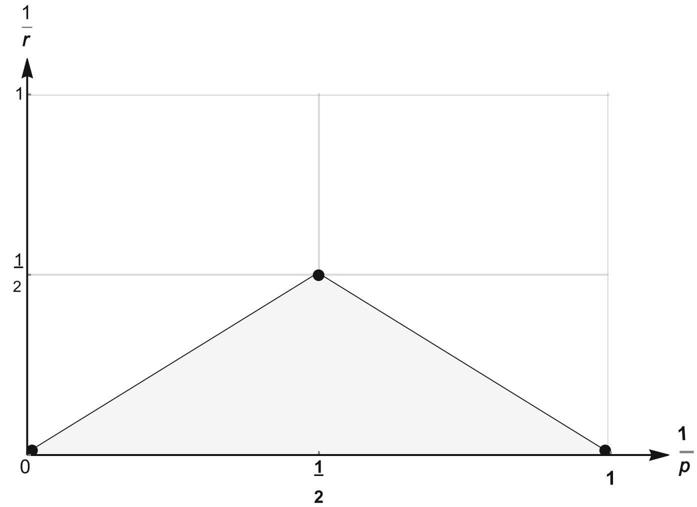

Fig. 2 Boundedness zone for the short-time Fourier transform.

and taking into account the continuous inclusion $L^{p,a} \subset L^{p,b}$ for $a \leq b$ we obtain that V defines two bounded maps

$$\begin{cases} V : L^{2,1} \times L^{2,1} \longrightarrow L^{2,\infty} \\ V : L^{2,1} \times L^{2,1} \longrightarrow L^{\infty,\infty}. \end{cases}$$

So V is of restricted weak type $\{(2,2;2),(2,2;\infty)\}$. This implies (by Theorem 2.10) that $\exists M \geq 0, \forall t > 0$:

$$V(f,g)^*(t) \leq M S_\sigma(f^*, g^*) \tag{3.20}$$

with $\sigma = \left\{\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right),\left(\frac{1}{2},\frac{1}{2},0\right)\right\}$, for all simple functions f, g, where S_σ is given by

$$S_\sigma(f,g)(t) = \int_0^\infty \int_0^\infty f(u)g(v) \min\left\{\sqrt{\frac{uv}{t}}, \sqrt{uv}\right\} \frac{du}{u}\frac{dv}{v}.$$

Then

$$S_\sigma(f,g)(t) = \begin{cases} \int_0^\infty \int_0^\infty f(u)g(v)\sqrt{\frac{uv}{t}}\frac{du}{u}\frac{dv}{v}, & \text{for} \quad t > 1 \\ \int_0^\infty \int_0^\infty f(u)g(v)\sqrt{uv}\frac{du}{u}\frac{dv}{v}, & \text{for} \quad 0 < t \leq 1. \end{cases}$$

Consider now $r \in (2,\infty)$, $c \in [1,\infty]$ and

$$\left\{\int_0^\infty \left[t^{1/r} S_\sigma(f^*,g^*)(t)\right]^c \frac{dt}{t}\right\}^{1/c} \leq 2^{1/c}\left\{\left\{\int_0^1 \left[t^{1/r} S_\sigma(f^*,g^*)(t)\right]^c \frac{dt}{t}\right\}^{1/c}\right.$$

$$
\begin{aligned}
&+\left\{\int_1^\infty \left[t^{1/r}S_\sigma(f^*,g^*)(t)\right]^c\frac{dt}{t}\right\}^{1/c}\Bigg\}\\
&=2^{1/c}\Bigg\{\left\{\int_0^1\left[t^{1/r}\int_0^\infty\int_0^\infty f^*(u)g^*(v)\sqrt{uv}\frac{du}{u}\frac{dv}{v}\right]^c\frac{dt}{t}\right\}^{1/c} \qquad (3.21)\\
&+\left\{\int_1^\infty\left[t^{1/r}\int_0^\infty\int_0^\infty f^*(u)g^*(v)\sqrt{\frac{uv}{t}}\frac{du}{u}\frac{dv}{v}\right]^c\frac{dt}{t}\right\}^{1/c}\Bigg\}.
\end{aligned}
$$

Now, for the first term in (3.21), we have

$$
\left\{\int_0^1\left[t^{1/r}\int_0^\infty\int_0^\infty f^*(u)g^*(v)\sqrt{uv}\frac{du}{u}\frac{dv}{v}\right]^c\frac{dt}{t}\right\}^{1/c}\le K_1\|f\|_{2,1}\|g\|_{2,1},
$$

where $K_1=\left(\int_0^1 t^{\frac{c}{r}-1}dt\right)^{1/c}<\infty$ since $r<\infty$. For the second term in (3.21), we get

$$
\begin{aligned}
&\left\{\int_1^\infty\left[t^{1/r}\int_0^\infty\int_0^\infty f^*(u)g^*(v)\sqrt{\frac{uv}{t}}\frac{du}{u}\frac{dv}{v}\right]^c\frac{dt}{t}\right\}^{1/c}\\
&\quad=\left\{\int_1^\infty\left[t^{1/r}\int_0^\infty\int_0^\infty f^*(u)g^*(v)\frac{1}{\sqrt{t}}\frac{du}{\sqrt{u}}\frac{dv}{\sqrt{v}}\right]^c\frac{dt}{t}\right\}^{1/c}\\
&\quad=\int_0^\infty f^*(u)\frac{du}{\sqrt{u}}\cdot\int_0^\infty g^*(v)\frac{dv}{\sqrt{v}}\cdot\left(\int_1^\infty\left(t^{\frac{1}{r}-\frac{1}{2}}\right)^c\frac{dt}{t}\right)^{1/c}\\
&\quad=K_2\|f\|_{2,1}\|g\|_{2,1},
\end{aligned}
$$

where $K_2<\infty$ since $r>2$. So, we have

$$
\left\{\int_0^\infty\left[t^{1/r}S_\sigma(f^*,g^*)(t)\right]^c\frac{dt}{t}\right\}^{1/c}\le K\|f\|_{2,1}\|g\|_{2,1}
$$

for $K>0$ suitable constant, depending on r and c.

Now, for all simple functions f, g on $\mathbb{R}^d$ we have by (3.20)

$$
\|V_g f\|_{L^{r,c}(\mathbb{R}^{2d})}\le M\left\{\int_0^\infty\left[t^{1/r}S_\sigma(f^*,g^*)(t)\right]^c\frac{dt}{t}\right\}^{1/c}\le K\|f\|_{2,1}\|g\|_{2,1},
$$

where $r\in(2,\infty)$, $c\in[1,\infty]$. By density arguments we then have a bounded map

$$
V: L^{2,1}\times L^{2,1}\longrightarrow L^{r,c},\quad r\in(2,\infty),\quad c\in[1,\infty].
$$

As the continuous inclusion $L^{r,c_0} \hookrightarrow L^{r,c_1}$, $c_0 \leq c_1$, holds, the best result is obtained for the smallest c, that is $c = 1$:

$$V : L^{2,1} \times L^{2,1} \longrightarrow L^{r,1}.$$

□

Now we observe that, when comparing Theorems 3.6 and 3.7 with the boundedness of the short-time Fourier transform in Lebesgue spaces, our results do not cover all the cases of Proposition 2.1. In particular, it is well known that $V : L^2 \times L^2 \to L^2$; this would correspond to take $p = r = a = b = c = 2$ in Theorem 3.6, that is not allowed. In Theorem 3.7 we allow p to be equal to 2, but we still need $r > 2$, so the classical L^2 boundedness of the short-time Fourier transform is not obtained as a particular case of our results. This is related to the fact that Lorentz spaces are a finer scale of spaces than the Lebesgue ones, and this causes much more difficulties when trying to characterize all the situations where boundedness holds. On the other hand, we can show that in many of the cases when the hypotheses of Theorems 3.6 and 3.7 do not hold, the short-time Fourier transform fails to be bounded, as stated in the following result.

Theorem 3.8 *Let p and r be such that* $\max\{p, p'\} > r$, *and let* $a, b, c \in [1, \infty]$. *Then the short-time Fourier transform is not bounded as an operator*

$$V : L^{p',a}(\mathbb{R}^d) \times L^{p,b}(\mathbb{R}^d) \to L^{r,c}(\mathbb{R}^{2d}). \tag{3.22}$$

Proof We start by some observations concerning the decreasing rearrangement of a Gaussian. Let $\Lambda = (\lambda_1, \ldots, \lambda_d)$ with $\lambda_j > 0$ for every $j = 1, \ldots, d$, and consider the Gaussian

$$g_\Lambda(x) = e^{-\pi(\lambda_1 x_1^2 + \cdots + \lambda_d x_d^2)}, \quad x \in \mathbb{R}^d.$$

It is not difficult to prove that

$$g_\Lambda^*(t) = e^{-\Gamma(\frac{d}{2}+1)^{2/d}(\lambda_1 \cdots \lambda_d)^{1/d} t^{2/d}}, \quad t > 0, \tag{3.23}$$

where Γ is the Euler function. We prove the result in the case $d = 1$; the same proof holds in higher dimension. For $\lambda > 0$ we write $g_\lambda(t) = e^{-\pi\lambda t^2}$; the short-time Fourier transform on Gaussian functions can be explicitly computed, obtaining that for $\lambda, \mu > 0$

$$V_{g_\lambda} g_\mu(x, \omega) = \frac{1}{\sqrt{\lambda + \mu}} e^{-\pi i \frac{2\mu}{\lambda+\mu} x\omega} g_\Lambda(x, \omega), \tag{3.24}$$

where $\Lambda = (\frac{\lambda\mu}{\lambda+\mu}, \frac{1}{\lambda+\mu})$. By (3.23) and a simple change of variables we have

$$\|g_\lambda\|_{p,q}^q = \int_0^\infty \left[t^{1/p} g_\lambda^*(t)\right]^q \frac{dt}{t} = \int_0^\infty t^{q/p} e^{-q\Gamma(\frac{3}{2})^2 \lambda t^2} \frac{dt}{t}$$

$$= \frac{1}{\sqrt{\lambda}^{q/p}} \int_0^\infty s^{q/p} e^{-ks^2} \frac{ds}{s},$$

where k is a positive constant depending on q but not on λ. Then there exists a constant $C_{p,q} > 0$ such that

$$\|g_\lambda\|_{p,q} = C_{p,q}(\sqrt{\lambda})^{-1/p},$$

for every $\lambda > 0$; observe that the same result holds also in the case $p < \infty$, $q = \infty$. A similar computation gives

$$\|V_{g_\lambda} g_\mu\|_{p,q} = C'_{p,q} \frac{(\lambda + \mu)^{\frac{1}{p} - \frac{1}{2}}}{(\sqrt{\lambda\mu})^{\frac{1}{p}}},$$

for a positive constant $C'_{p,q}$ depending only on p and q. Suppose now that (3.22) is bounded; then for every $\lambda, \mu > 0$ it should happen that $\|V_{g_\lambda} g_\mu\|_{r,c} \le C\|g_\lambda\|_{p',a}\|g_\mu\|_{p,b}$, that means

$$\frac{(\lambda + \mu)^{\frac{1}{r} - \frac{1}{2}}}{(\sqrt{\lambda\mu})^{\frac{1}{r}}} \le C \frac{1}{(\sqrt{\lambda})^{1/p'}(\sqrt{\mu})^{1/p}};$$

taking $\mu = 1$ and letting $\lambda \to 0$ we see that the last inequality cannot be satisfied (and then (3.22) is not bounded) if $p' > r$, and interchanging the roles of λ and μ we have that it cannot be satisfied if $p > r$. The proof is then complete. □

4 τ-Wigner distribution and τ-Weyl Operators

The boundedness results that we have proved for the short-time Fourier transform can be transferred to the τ-Wigner transform (2.3) by means of the relation (2.4).

Theorem 4.1 *(i) Let $c \in [1, \infty)$, $r \in (2, \infty]$, $p \in [r', r]$, $p \neq 2$, and $a, b \ge 1$ such that $\frac{1}{a} + \frac{1}{b} = 1 + \frac{1}{c}$. Then for $\tau \in (0, 1)$*

$$W_\tau : L^{p',a}(\mathbb{R}^d) \times L^{p,b}(\mathbb{R}^d) \longrightarrow L^{r,c}(\mathbb{R}^{2d})$$

is bounded.

(ii) Let $r \in (2, \infty)$, $\frac{1}{r} + \frac{1}{r'} = 1$, and $1 \le a, b, c \le \infty$ with $\frac{1}{a} + \frac{1}{b} = \frac{1}{c} + 1$. Then for $\tau = 0$,

$$W_0 : L^{r,a}(\mathbb{R}^d) \times L^{r',b}(\mathbb{R}^d) \longrightarrow L^{r,c}(\mathbb{R}^{2d})$$

is bounded.

(iii) Let $r \in (2, \infty)$, $\frac{1}{r} + \frac{1}{r'} = 1$, *and* $1 \le a, b, c \le \infty$ *with* $\frac{1}{a} + \frac{1}{b} = \frac{1}{c} + 1$. *Then for* $\tau = 1$,

$$W_1 : L^{r',a}(\mathbb{R}^d) \times L^{r,b}(\mathbb{R}^d) \longrightarrow L^{r,c}(\mathbb{R}^{2d})$$

is bounded.

Proof (i) Using Lemma 2 in [20] and Theorem 3.6, we have by (2.4)

$$\begin{aligned}\|W_\tau(f,g)\|_{r,c}^c &= \left\| \frac{1}{|\tau|^d} e^{2\pi i \frac{1}{\tau} xw} V_{A_\tau g}^\tau f \right\|_{r,c}^c \\ &= \frac{1}{|\tau|^{dc}} \|V_{A_\tau g}^\tau f\|_{r,c}^c \\ &= \frac{1}{|\tau|^{dc}} \big(|1-\tau| \cdot |\tau|\big)^{\frac{dc}{r}} \|V_{A_\tau g} f\|_{r,c}^c \\ &\le \frac{1}{|\tau|^{dc}} \big(|1-\tau| \cdot |\tau|\big)^{\frac{dc}{r}} MK \|f\|_{p',a}^c \|A_\tau g\|_{p,b}^c.\end{aligned}$$

Then, applying Lemma 1 in [20], we obtain

$$\begin{aligned}\|W_\tau(f,g)\|_{r,c}^c &\le \frac{1}{|\tau|^{dc}} \big(|1-\tau| \cdot |\tau|\big)^{\frac{dc}{r}} MK \|f\|_{p',a}^c \left| \frac{\tau}{1-\tau} \right|^{\frac{dc}{p}} \|g\|_{p,b}^c \\ &= |\tau|^{dc(\frac{1}{r}+\frac{1}{p}-1)} |1-\tau|^{dc(\frac{1}{r}-\frac{1}{p})} MK \|f\|_{p',a}^c \|g\|_{p,b}^c.\end{aligned}$$

This completes the proof.

(ii) Let $f \in L^{r,a}(\mathbb{R}^d)$ and $g \in L^{r',b}(\mathbb{R}^d)$. Then $\widehat{g} \in L^{r,b}(\mathbb{R}^d)$ and

$$\left\|\widehat{g}\right\|_{L^{r,b}} \le K \|g\|_{L^{r',b}} \tag{4.1}$$

by the Hardy-Littlewood-Stein inequality.

By using the equality $W_0(f,g)(x,\omega) = R(f,g)(x,\omega) = e^{-2\pi i x w} f(x)\overline{\widehat{g}(\omega)}$, inequality (4.1) and Theorem 2.11, we have

$$\begin{aligned}\|W_0(f,g)\|_{r,c} = \|R(f,g)\|_{r,c} &\le c\|f\|_{r,a}\|\overline{\widehat{g}}\|_{r,b} \\ &\le cK\|f\|_{r,a}\|g\|_{L^{r',b}}.\end{aligned}$$

This is the desired result.

(iii) The proof for $\tau = 1$ is similar to (ii).

□

The natural association between sesquilinear forms and operators yields, in the case of τ-Wigner transforms, the class of τ-Weyl operators, defined by

$$(\mathcal{W}_\tau^\varphi f, g)_{L^2(\mathbb{R}^d)} = (\varphi, W_\tau(g,f))_{L^2(\mathbb{R}^{2d})} \tag{4.2}$$

where $f, g \in L^2(\mathbb{R}^d), \varphi \in L^2(\mathbb{R}^{2d})$, with suitable extensions to other functional settings, and the correspondence $\varphi \longrightarrow \mathcal{W}_\tau^\varphi$ is a quantization, see for instance [5].

We give next sufficient conditions for the boundedness of the τ-Weyl quantization $\mathcal{W}_\tau^\varphi$ with symbol $\varphi \in L^{r',c'}(\mathbb{R}^{2d})$, where $\tau \in [0, 1]$.

Theorem 4.2 *(i) Let* $\tau \in (0, 1)$. *The quantization*

$$\varphi \in L^{r',c'}(\mathbb{R}^{2d}) \longrightarrow \mathcal{W}_\tau^\varphi \in B(L^{p,b}(\mathbb{R}^d), L^{p,a'}(\mathbb{R}^d))$$

is continuous for $c \in (1, \infty)$, $r \in (1, 2)$, $p \in [r, r']$, $p \neq 2$, *and* $a > 1$, $b \geq 1$ *such that* $\frac{1}{a} + \frac{1}{b} = 1 + \frac{1}{c}$.

(ii) Let $\tau = 0$. *The quantization*

$$\varphi \in L^{r',c'}(\mathbb{R}^{2d}) \longrightarrow \mathcal{W}_0^\varphi \in B(L^{r',b}(\mathbb{R}^d), L^{r',a'}(\mathbb{R}^d))$$

is continuous for $r \in (1, 2)$ *and* $a, c \in (1, \infty)$, $1 \leq b \leq \infty$ *such that* $\frac{1}{a} + \frac{1}{b} = 1 + \frac{1}{c}$.

(iii) Let $\tau = 1$. *The quantization*

$$\varphi \in L^{r',c'}(\mathbb{R}^{2d}) \longrightarrow \mathcal{W}_1^\varphi \in B(L^{r,b}(\mathbb{R}^d), L^{r,a'}(\mathbb{R}^d))$$

is continuous for $r \in (1, 2)$ *and* $a, c \in (1, \infty)$, $1 \leq b \leq \infty$ *such that* $\frac{1}{a} + \frac{1}{b} = 1 + \frac{1}{c}$.

Proof The result is a consequence of [5, Proposition 6.1] and Theorem 4.1. □

5 An application to the uncertainty principle

We conclude with an application of the results of the previous sections to Lieb's uncertainty principle giving an extension of this principle in the case of signals in suitable Lorentz spaces. As the STFT boundedness result that we shall use (Theorems 3.6 and 3.7) does not contain the explicit computation of the bounding constant, our result here will be of qualitative type, i.e. without exact determination of the involved constants, which could however be determined with considerably more cumbersome computations of the same arguments.

In order to prove our result we need the following extension of Hölder inequality to Lorentz spaces (see [18]).

Proposition 5.1 *(Hölder inequality in Lorentz spaces) For* $0 < p, p_1, p_2 < \infty$ *and* $0 < q, q_1, q_2 \leq \infty$ *satisfying* $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$, *we have*

$$\|fg\|_{L^{p,q}} \leq C\|f\|_{L^{p_1,q_1}}\|g\|_{L^{p_2,q_2}}, \tag{5.1}$$

whenever the right-hand side is finite, and with $C > 0$ *depending on the indices of the spaces.*

Proposition 5.2 *Suppose that $r \in (2, +\infty]$, $p \in [r', r]$, $p \neq 2$, $\frac{1}{a} + \frac{1}{b} \geq \frac{3}{2}$. If $f \in L^{p',a}(\mathbb{R}^d)$, $g \in L^{p,b}(\mathbb{R}^d)$ with $\|f\|_{p',a} = \|g\|_{p,b} = 1$ and*

$$\int_U |V_g f(x,\omega)|^2 \, dx \, d\omega \geq 1 - \varepsilon, \tag{5.2}$$

then

$$\mu(U) \geq C(1-\varepsilon)^{s/2} \tag{5.3}$$

where $\frac{1}{r} + \frac{1}{s} = \frac{1}{2}$ (with C depending on all indices but independent of f and g) and $\mu(U)$ is the Lebesgue measure of U.

Proof For simplicity let us indicate by C constants which can be different in different inequalities. Let χ_U be the characteristic function of U. As $\frac{1}{a} + \frac{1}{b} \geq \frac{3}{2}$ we can take $c \in [2, +\infty)$ such that $\frac{1}{a} + \frac{1}{b} = 1 + \frac{1}{c}$ and $q \in (0, +\infty]$ such that $\frac{1}{2} = \frac{1}{q} + \frac{1}{c}$; suppose moreover that $\frac{1}{2} = \frac{1}{r} + \frac{1}{s}$, then from the hypothesis (5.2) and Hölder inequality (5.1) we have

$$\sqrt{1-\varepsilon} \leq \|\chi_U V_g f\|_{2,2} \leq C \|V_g f\|_{r,c} \|\chi_U\|_{s,q}.$$

It can easily be checked that $\|\chi_U\|_{s,q} = \left(\frac{s}{q}\right)^{1/q} \mu(U)^{1/s}$, and, as $r > 2$, we can apply the continuity of the STFT (Theorem 3.6) which yields

$$\|V_g f\|_{r,c} \leq C \|f\|_{p',a} \|g\|_{p,b} = C.$$

as $\|f\|_{p',a} = \|g\|_{p,b} = 1$.

We obtain therefore

$$\sqrt{1-\varepsilon} \leq C \left(\frac{s}{q}\right)^{1/q} \mu(U)^{1/s},$$

i.e. the estimate

$$\mu(U) \geq C(1-\varepsilon)^{s/2}. \tag{5.4}$$

□

Remark 5.3 For any $p \neq 2$, we can always take $r = \infty$, and therefore $s = 2$, so that, for $\varepsilon \to 0$, the right-hand side of (5.3) goes to zero with order 1.

In the case $f \in L^{p',a}(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, $g \in L^{p,b}(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ we can compare this result with the classical *Lieb's uncertainty principle* which asserts that

$$\mu(U) \geq (1-\varepsilon)^{\frac{p}{p-2}} (p/2)^{\frac{2d}{p-2}}$$

for every $p > 2$, if $f, g \in L^2(\mathbb{R}^d)$, $\|f\|_2 = \|g\|_2 = 1$, and $\int_U |V_g f(x, \omega)|^2 \, dx \, d\omega \geq 1 - \varepsilon$.

As $\frac{p}{p-2} > 1$ for every $p > 2$, we see therefore that (5.3) represents a slight improvement in the order of the estimate.

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