The materiality of mathematical imagination

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In this paper we draw attention to the material dimension of mathematical imagination with the aim to emphasize the creative role of imagination in mathematics teaching and learning. Drawing on previous studies of imagination in mathematical activity and post-humanist visions of affect and the body, we suggest framing imagination as a bodily and affective engagement with mathematics, a way of thinking with (not of) concepts, rather than a cognitive simulation of mental creations. Our account of the materiality of mathematical imagination is exemplified through the analysis of brief episodes from a teaching experiment that engaged grade 5 students in working with a spirograph.

Keywords: Mathematical imagination, body-tool interaction, materiality, affect, diagrams.

On imagination

Imagination is often treated and understood as a faculty of the mind, a cognitive ability activated by the individual through the creation of mental images. In this view, imagination does not pertain to the body and is mainly seen as a type of representational processing of something occurring mentally. For example, in the neuroscientific theory of Gallese and Lakoff (2005), imagination is a form of simulation that shares a neural substrate with action observation and execution. According to these researchers, imagination is correlated with the sensory-motor system through multimodal neurons.

With the purpose of broadening the scope of embodied cognition toward encompassing the imaginary, Nemirovsky and Ferrara (2009) have introduced *mathematical imagination* as a particular type of imagination. Mathematics learning is viewed in terms of the development of mathematical imagination and the expression perceptuo-motor-imaginary activity is used to capture the fact that the activity of imagining is fully part of any perceptual and motor activity. Mathematical imagination is a state of readiness for the enactment of possible actions, a vision that helps “illuminate the roles of tools and materials—not as “embodiments” of mathematical ideas, but as means to productively extend the horizon of possibilities that students come to entertain” (Nemirovsky & Ferrara, 2009, p. 173). The “what could be” character of these (empirical or pure) possibilities resonates with the virtuality that the philosopher Gilles Châtelet (2000) attributes to mathematical concepts, which we can envision as the horizon of possibilities that *concepts* come to entertain, as they are open to future alterations. This pushes us to move beyond the cognitive and shift attention to the relations between human and non-human bodies as the learning assemblage in mathematical activity, and to focus on the *materiality* of mathematical activity.

Other studies have characterized imagination as materially and physically situated, as depending on attention to features and resources of the local environment (e.g., Hutchins, 2010; Jornet & Steier, 2015). Accordingly, a learner’s ability to explore possibilities and to bring new ideas into the world depends to a great extent on the ways that aspects of the setting can be productively suitable to depict these ideas. Steier and colleagues (2019) talk of imagination as a social process focusing on how a group of people may explore (imagined) possibilities through communicative processes.

In this paper, we want to emphasise not the simulative, but the *creative* role of imagination in mathematics teaching and learning. To this aim, we suggest framing imagination more as a type of bodily and affective engagement with mathematics, a way of thinking *with* (not of) mathematical concepts. Drawing on post-humanist perspectives that value the material dimension of mathematical activity, we propose that attention to the materiality of mathematical imagination may enrich our understanding of mathematical embodiment. We will use brief episodes from the mathematics classroom to illustrate our account of the materiality of imagination.

On bodily and affective relations

In their investigation of the material aspects of mathematical practice, Sinclair and Jablonka (2022) discuss mathematical embodiment from a post-humanist perspective. These authors introduce three examples to focus readers' attention on the specific ways in which both the body and the environment are at play both in thinking and in learning mathematics. While they point to the tendency to pursue separate strands of research in the literature, they criticize perspectives that insist on isolating the human from the environment or the mind from the body, because they diminish the perceived relevance of both. Instead, "combined attention to tools and the body may provide insight into issues such as the role of affect and memory in mathematics learning, as well as production of mathematical inscriptions" (Sinclair & Jablonka, 2022, p. 424). On the one hand, there could be a shift to make explicit the ways in which the body takes up tools; on the other, to focus more on the movements of the body "*with* or *on* things" rather than alone. As these researchers stress, since mathematical activity does not occur at a distance from the material world and the body (considered in the wide sense of the senses and various body parts) body-tool interactions should be the focus of exploration.

A similar perspective has also been taken in educational research centred on the idea of *assemblage* to talk about the complex interactions of body, tools, and mathematical concepts in the classroom. For example, de Freitas et al. (2017) use the term *learning assemblage* to describe the provisional arrangements that involve multiple bodies (including students and technology) moving together and learning together. Ferrara & Ferrari (2022b) look at how the learning assemblage is sustained by coordinated movements around the surfaces at play in an activity with kindergarten children (the floor and the screen of a tablet). In both examples, the assemblage of learners, concepts and tools is taken up as the unit of study and helps account for the materiality of the expressive (such as discourse and sign) and the ways that agency is distributed across the human-tool interaction in the learning situation (for a discussion of distributed agency, see Rotman, 2008; de Freitas & Sinclair, 2012). Instead of focusing on tools as used by learners and on their affordances for learners, we can therefore focus on how learners mobilise tools to think *with* and perform creative and unscripted ways of thinking that were not present or available before.

Some research strives to explore the *affective* relations that arise from body-tool interactions in mathematical activities, thinking of affect not in terms of individual states (Hannula, 2012) but rather in terms of relational engagement with and experience of mathematical concepts (e.g., de Freitas et al., 2019; Sinclair & Ferrara, 2021). Massumi (2015) already argued that it is the situated, relational entanglement of bodies that is primary and not the feeling and thinking individual, when looking at the embodied mode of human existence. Material ecologies, in which affects are dispersed, are created by tools because of their interactional potential, and experiences with different affective tonalities can be associated with different concepts and the use of different tools. Learners' actions and gestures are treated as affective states that mobilize questions and concepts, and the affective modulation of the mathematical encounters becomes a way of better understanding collaborative achievements in the classroom and significant aspects of mathematics learning. In this vein, the mathematical work is creative and inventive: potentially, it always implicates new possibilities for the bodies to assemble, to move and feel together, to coordinate with each other, new possibilities for learners to *think with concepts*.

Drawing on these studies, we see the new possibilities above—the *what could be* that is yet to come—as a fundamental aspect of the material practice of doing mathematics, especially with tools. By focusing on them, in this paper we aim to study the embodied and material dimensions of imagining in the mathematics classroom, and we propose to rethink mathematical imagination as a bodily and affective engagement with concepts that bring forth the new. We will exemplify our view discussing brief episodes in which grade 5 children are (imaginatively) engaged with a spirograph, without using it to draw. The next section presents the method of our study and the participants.

Method, tool, and participants

Our data comes from a teaching experiment that lasted a total of 4 meetings of 3 hours each, in which we carried out a series of activities related to the use of a spirograph with a class of grade 5 students. A spirograph is a tool made up of a set of wheels and rings that can be combined thanks to their toothed edges. When a wheel is moved inside a ring through the tip of a pen inserted in one of its holes, the wheel rolls inside the ring and the pen draws a closed curve. We are interested in the use of the spirograph because, through the combined motions of parts, it allows for drawing (beautiful) curves having specific mathematical properties that students can explore through diagrams. We also studied the use of the spirograph in a previous work presented at CERME (Ferrara & Ferrari, 2022a).

For this paper, we focus on the very first activity with the tool, which was guided by two requests: 1) to manipulate the tool and imagine the kind of things one can do with it; 2) to draw what is imagined the tool would draw when in use. This piece of the teaching experiment is relevant to our discussion of mathematical imagination as it engages children in imagining what the tool would produce and how before they use it. While we agree that mathematical imagination is at play even when not "explicitly requested", we believe that this type of task has the power to elicit new forms of reasoning and move thinking towards the unscripted. In fact, the task of imagining making/doing something with a tool is already a way of attending to the material dimension of the activity and the involvement of the tool in interaction with the body. Instructing to *imagine* what the tool does without using it, uniquely situates the imagination as a tool itself to be used, instead of the spirograph.

We present data from group work and a collective discussion led by the first author and the teacher (who acted as an active observer within the classroom). Data consists of video recordings of a focus group and the discussion, as well as written protocols and the diagrams created on large paper sheets. Methodologically speaking, the analysis focuses attention on the ways that movement partakes in the activity, looking at the assembling of children's bodies (actions, gestures, postural and facial expression, eye movement), diagrams and tools, and at how new relations arise from it.

Mathematical imagination(s)

The first day the class is divided into groups. The researcher has just presented the spirograph by mentioning its parts (wheels and rings). D, a child who had already seen (perhaps used) the tool, goes in front of the class to simulate how it can be used. As soon as D points to a hole to insert a pen and shows the wheel rotating inside the ring, some children exclaim "It's a circle!". D shakes his head expressing dissensus, while others seem to agree that the drawing will have a circular shape. We focus here on the next few minutes, using three brief episodes from the first day of the activity.

Imagining movement(s)

While the researcher (R) collects impressions from the class (holding the spirograph in her hands in front of the children), A suddenly exclaims [in the following event, students P and Al are A's group mates; S is part of another group]:

A: Hm, no, no, because … the little wheel moves (*repeats three times a right-hand rotation, with thumb and index finger closed to create a little circle*, Figure 1a)
P: Hm, but the pen doesn't move
A: It moves and makes the pen shift (*produces a wide then smaller right-hand rotation*, Figures 1b and 1c)
P: No, the distance is always the same
A: Well, for me it's not a circle (S: It depends …)
R: Which distance?
A: That [the wheel] (*stands pointing to the spirograph in R's hands*), when it turns (*rotates repetitively her right hand clockwise*), it turns around the thing (*rotates her right hand counterclockwise*), so the pen also turns, and so it does all (*shifting her rotating gesture to the desk while it becomes narrower*) … circlets
Al: It depends on where you put it (*moves close to the researcher*) … If you put the pencil or pen here and you turn, sometimes it's more internal, sometimes more external (*points to a hole on the wheel, then rotates the wheel following the position of the hole with his gaze*, Figure 2)
A: [to P while Al speaks] See? It changes, the tip also shifts, so it doesn't make a circle!

Figure 1. A gestures movement of the wheel (a) and the pen (b-c)

Figure 2. Al describes a hole's movement while turning the wheel inside the ring

In the first episode, we see that some children imagine a circle emerging from the rotation of the wheel. While we might interpret that the children "confuse" the shape of the curve with the circular movement of the inner gear, the drawing of a circle would be possible with any configuration of gears by choosing a hole in the exact middle of the wheel. While gestures, gazes and words actualize the distance from the rim, imagining the curve involves a complex body-tool interaction, an assemblage of tool, children and diagrams. No matter whether the tool is far or close (for A and Al respectively), different movements are brought forth: of the wheel inside the ring and of the hole-pen from rim, while the wheel turns. Movements, which are coordinated in the drawing, now are captured by bodily and affective engagements with the curve: repeated wide arm and hand gestures (A) or very precise actions with one gear (Al), which scan space through a movement that always occurs in the same way but implicates different shapes, while awakening the temporality of the event.

Imagining self-intersection(s)

A spirograph and the first task are given to each group: "Imagine what the spirograph does" (when it works). The children, seated all around a desk, can manipulate the spirograph but cannot use it to draw on paper. We focus on the group made of A, P, and Al. The children begin with moving a wheel inside a ring and observe that one of the holes is (almost) in the middle of the wheel, so when you roll the wheel, the hole stays "always in the same position", and this will result in a circle. Otherwise, if the chosen hole is next to the rim, "it shifts", so the children predict that it would not draw a circle:

A: For me, it comes out something like this (*draws self-intersecting pieces of the curve on the desk with her right-hand index finger*, Figure 3a)

This gesture propagates across the desk when Al replicates it (Figure 3b). On the other hand, P is more interested in comparing the different wheels: he superimposes them and looks for corresponding ("common") holes or tries to combine more than one wheel inside one ring (Figure 3c).

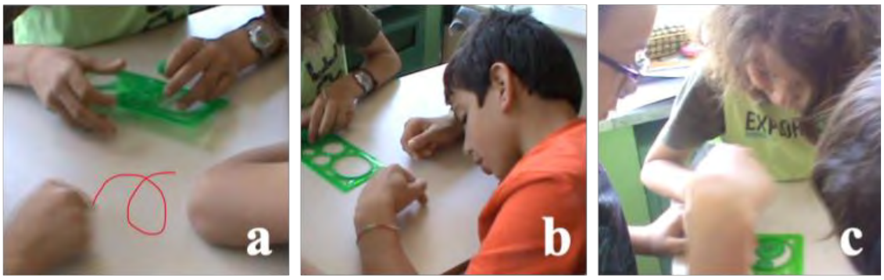

Figure 3. a) A draws a piece of the curve on the desk with her right-hand index finger; b) Al propagates the spiralling gesture on the desk, multiple times; c) P explores combinations of gears

During group work, A, P and Al recover the previous expectations about the curve in case the hole is in the middle of the wheel. Imagining the creation of the curve in a more general case, A gestures on the desk, repetitively drawing with her right-hand index finger little self-intersections connected one another and articulated along a circular movement. The gesture propagates across the desk and Al begins doing similar movements with his left-hand index finger, increasing the intersections and the speed (also note Al's posture). Affects are distributed across these rhythmic and percolating gestures, which bring forth both the dynamic and iterative nature of the curve and the fascination that pervades the activity. The tool captures the students' attention as P relates the different wheels to the rings in such a way that it becomes difficult to separate his imaginative process (and that of the group mates) from the body-tool interaction. We see that, when sketching first diagrams on the desk surface, the students' engagement changes, shifting to imagining (and gesturing and assembling with) self-intersections of the curve that recruit the dynamic relations among parts of the tool.

Imagining repeated turn(s)

The teacher assigns a new task to the groups: "On this sheet of paper, draw what you think will come out by using that tool, *without* actually using it". Student A first mimes a movement for the curve in front of her, then draws the curve shown in Figure 4a. Student P suddenly intervenes as follows:

P: Because, for me, this is a circle (*draws a circular shape*) first it goes close, then far, then close (*draws inside the circular shape, creating first the top part of the diagram,* Figure 4b top) …
A: No [while P draws], but it makes a turn, look (*points with the pen to P's drawing*)
P: Well, yes (*draws self-intersecting pieces like in A's diagram*, Figure 4b bottom)
A: Look, pretend, here there's the circle (*mimes a circle with joint thumbs and index fingers of her hands*), somebody makes a circle (*P begins moving his right-hand index finger inside A's hands, while making a feeble puff*) ... *Make a circle!* [to P] (P: Yes!, *miming a circle with his hands*) Here, I keep the point (*points to a position inside P's circle with her right-hand index finger*), this one turns, so this one turns, and [it] does like this (*gestures movement*, Figure 4c) [Al begins drawing on paper]
Al: (*slowly draws a new diagram*, Figure 4d)

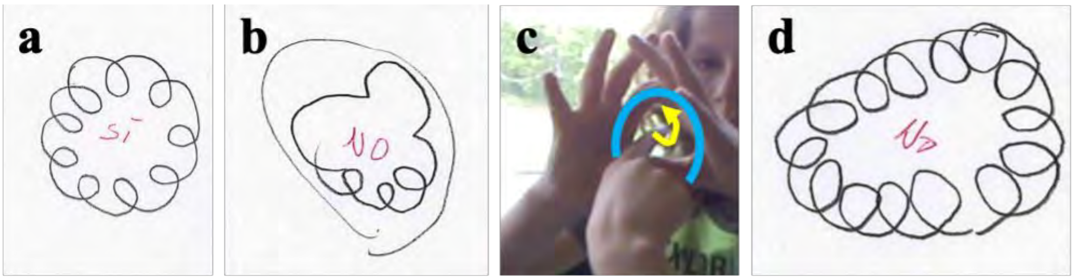

Figure 4. a) A's diagram; b) P's diagram; c) A and P's combining gestures; d) Al's diagram

While the tool is at disposal but not yet used to draw, first diagrams are created on paper by capturing space in ways that bring forth (the need for) intersections in relation to the changing distance already mentioned (P), as well as relations of symmetry (A) and regularity (Al), which capture the presence of circular gears combined in movement through repeated turns. We see how relevant aspects of the mathematical imagination of the students are also given by A and P's coordinated movement (Figure 4c) to assemble a gestural configuration of gears through which the two children can engage, and think, with the moving point that draws the curve, and by the perception of a closed curve (that we observe in the produced drawings). We see how imagination excavates new perceptions and resources through bodily engagement when students are actually prevented from using the tool. Not only do the hands describe the resulting coordinated movement, i.e., the curve, but they help us unfold imagination as fundamentally dealing with temporality, movement, constraints and possibilities.

For five more minutes the children continue to draw diagrams on the shared paper. Then, they classify their diagrams as those that they have imagined may or may not be produced, writing "Yes/No" in red inside their drawings. For example, A's diagram above is classified as an expected shape for the curve, whereas P's and Al's diagrams are discarded as possibilities for the spirograph, perhaps because of their asymmetries. After this task, the activity of the children shifted to drawing curves and exploring their properties by using the tool (we do not consider it here, for the sake of space).

Discussion

In this paper, we have used three episodes in order to illustrate how mathematical imagination is at play in the mathematical activity. The children-spirograph interaction, without using the tool to draw, gives rise to bodily explorations that bring forth new ways of moving with the tool (its parts) as well as new mathematical relations (new assemblages of bodies, tool and concepts): circles (same distance from one point, the middle), closed curves that are not circles (changing distance from a boundary, the rim), self-intersections (movements on the desk), regularity and symmetry of turns (diagrams on paper). The reasoning occurs not only visually or spatially through the tool, but temporally across the imagined event, as the gaze or the hand follows the wheel turning or moves on the desk or the paper.

We see how imagination is a bodily and affective engagement with the curve that brings forth the new, involving and evolving mathematical relations. Movement becomes more and more structured, and bodies more and more coordinated, while mathematically relevant structures emerge (circles, self-intersections, symmetry). In the process, the activity develops as a flow across past and future, abstract and concrete (Ferrara & Ferrari, 2022a), expanding the body-tool interaction. The tool occasions opportunities for learners to move and feel together, as gestures, propagating on the desk or the paper, and diagrams articulate new mathematical relations, which make the mathematical work inventive. It is in the possibility of new ideas, movements and affects that we see the creative role of mathematical imagination, in line with a conceptualisation of mathematical inventiveness as that which brings forth or makes visible what was not present before (Sinclair et al., 2013).

We think that this view of mathematical imagination helps enrich our understanding of mathematical embodiment beyond cognitivist theories and the tendency to treat body and tool as separated, especially as possessing separated agencies. Instead, we have offered an account of how agency is distributed across body-tool interactions in which imagination partakes. Studying mathematical imagination as a type of bodily and affective engagement with mathematics allows us to pay attention to the potentiality of the body (in the broadest sense), and to consider the relevance of imagination to challenging and creative mathematics teaching and learning, including in terms of task design.

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