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Global well-posedness of a class of weakly hyperbolic Cauchy problems with variable multiplicities on $\mathbb{R}^d$

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A B S T R A C T

We study a class of weakly hyperbolic Cauchy problems on $\mathbb{R}^d$, involving linear operators with characteristics of variable multiplicities, whose coefficients are unbounded in the space variable. The behavior in the time variable is governed by a suitable "shape function". We develop a parameter-dependent symbolic calculus, corresponding to an appropriate subdivision of the phase space. By means of such calculus, a parametrix can be constructed, in terms of (generalized) Fourier integral operators naturally associated with the employed symbol class. Further, employing the parametrix, we prove $\mathscr{S}(\mathbb{R}^d)$-well-posedness and give results about the global regularity of the solution, within a scale of weighted Sobolev space, encoding both smoothness and decay at infinity of temperate distributions. In particular, loss of

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decay appears, together with the well-known phenomenon of loss of smoothness.

1. Introduction

In this paper we deal with the global decay and regularity properties of the solution of certain weakly hyperbolic Cauchy problems, associated with linear partial differential operators with smooth coefficients, polynomially growing in the space variable x. Setting

$$L = -D_t^2 u + \sum_{j=1}^{d} \left(a_j(t,x)D_j^2 + b_j(t,x)D_j\right) + c(t,x), \tag{1.1}$$

our model Cauchy problem is

$$\begin{cases} Lu(t,x) = g(t,x), \ (t,x) \in [0,T] \times \mathbb{R}^d, \\ u(0,x) = \varphi(x), u_t(0,x) = \psi(x), \ x \in \mathbb{R}^d, \end{cases} \tag{1.2}$$

where $D_j = -i\partial/\partial x_j$, $D_t = -i\partial/\partial t$ and $a_j(t,x), b_j(t,x), c(t,x)$, $j = 1,\dots,d$, are functions in $C([0,T] \times \mathbb{R}^d) \cap C^\infty((0,T] \times \mathbb{R}^d)$, for (a suitably small) $T > 0$. We define

$$a(t,x,\xi) = \sum_{j=1}^{d} \left(a_j(t,x)\xi_j^2 + b_j(t,x)\xi_j\right) + c(t,x), \tag{1.3}$$

and assume that $a(t,\cdot,D) = \mathrm{Op}(a(t))$ is a differential operator with coefficients that are smooth on $(0,T] \times \mathbb{R}^d$, but, possibly, with characteristic roots of variable multiplicity and *fast oscillations* (see [42]) at $t = 0$. Compared with the hyperbolic operators with variable multiplicities studied in [1], we here consider different conditions on the characteristic roots. Namely, the behavior with respect to the time variable t is governed by a shape function $\lambda(t)$, as explained in Section 2.1 below. The function a is assumed to be a family of symbols belonging to one of the Weyl-Hörmader classes $S(m,g)$, namely, where the metric on the phase space $\mathbb{R}^{2d}$ is

$$g_{(x,\xi)}(y,\eta) = \langle x\rangle^{-2}|y|^2 + \langle \xi\rangle^{-2}|\eta|^2, \tag{1.4}$$

where $\langle\eta\rangle$ stands for[1] $(e+|\eta|^2)^{1/2}, \eta\in\mathbb{R}^d$. Most often, we will employ weights of the form $m(x,\xi)=\langle x\rangle^m\langle\xi\rangle^\mu$, that is, the so-called SG-symbols, in their most standard form $S^{m,\mu}(\mathbb{R}^{2d})$, will be involved (cf. Cordes [12], Parenti [38]). However, we will also need more general symbols of SG-type, that is, those associated with the metric

$$g_{(x,\xi)}(y,\eta)=\langle x\rangle^{-2r_1}\langle\xi\rangle^{2\rho_1}|y|^2+\langle x\rangle^{2r_2}\langle\xi\rangle^{-2\rho_2}|\eta|^2.$$

In this case, with the same choice of weight above, we will denote $S(m,g)=S^{m,\mu}_{(r_j,\rho_j)}(\mathbb{R}^{2d})=S^{m,\mu}_{r_1,r_2,\rho_1,\rho_2}(\mathbb{R}^{2d})$. If

$$0\le r_2\le r_1\le 1 \quad\text{and}\quad 0\le\rho_1\le\rho_2\le 1,$$

then g is feasible and m is g-continuous in this case. If, in addition, $r_2,\rho_1<1$, then g is strongly feasible (see Coriasco and Toft [22], Hörmander [29], Nicola and Rodino [36]). Such conditions will actually be satisfied in our analysis below.

We will also assume that the family of symbols a given in (1.3) admits a lower bound, namely,

$$|a(t,x,\xi)|\ge C\lambda(t)^2\langle x\rangle^2\langle\xi\rangle^2, \tag{1.5}$$

for a positive constant C. Notice that (1.5) does not imply ellipticity of a in the SG-classes for $t=0$, see Section 2.4 below for details. Then, the operator in (1.2) will turn out to be weakly hyperbolic with variable multiplicities, since it will hold $\lambda(0)=0$, $\lambda(t)>0$, $t\in(0,T]$.

Unlike similar approaches, involving different symbol classes, we allow unbounded coefficients with respect to space variable $x\in\mathbb{R}^d$. Indeed, we will assume the following polynomial upper bounds on the family of symbols a and their derivatives: for all $\alpha,\beta\in\mathbb{N}^d$ and $k\in\mathbb{N}$,

$$|D_t^kD_x^\alpha D_\xi^\beta a(t,x,\xi)|\le C_{k\alpha\beta}\,\lambda(t)^2\langle x\rangle^{2-|\alpha|}\langle\xi\rangle^{2-|\beta|}\Sigma(t)^k, \tag{1.6}$$

for some $C_{k\alpha\beta}>0$ and every $(t,x,\xi)\in(0,T]\times\mathbb{R}^d\times\mathbb{R}^d$. In (1.6), the non-negative function $\Sigma(t)$ is defined by $\Sigma(t):=-\lambda(t)\ln(\Lambda(t))/\Lambda(t)$, where $\Lambda(t):=\int_0^t\lambda(\tau)\,d\tau$, and it captures the 'local' weakening of 'generalized' Lipschitz conditions, as described in Section 2.2 below. Inequality (1.6) means that, for all $t\in[0,T]$, the complete symbol $a(t,\cdot,\cdot)$ belongs to the so-called SG-classes of order $(2,2)$.

The SG-calculus appears in various other environments. An invariant definition of the above problems can be given on a class of noncompact manifolds, the so-called SG-manifolds, cf. Schrohe [45], which includes the manifolds with ends, see, e.g. [15,19,24,34].

[1] The modification in the definition of the weight $\langle y\rangle$, usually given by $\sqrt{1+|y|^2}$, is due to the presence, in the sequel, of logarithms of products of the type $\langle x\rangle\langle\xi\rangle$, which need not to vanish anywhere on $\mathbb{R}^{2d}$. All the usual symbol estimates and definitions are completely equivalent to those involving the standard definition of the weight.

SG-operators are also the local representation of the so-called *scattering operators* on asymptotically Euclidean manifolds, see, e.g., Melrose [35].

Indeed, operators of the form (1.1), whose symbols satisfy the estimates (1.6), arise in such geometric contexts, as local representations, in admissible coordinate charts, of natural operators, as we now show (see, e.g., [17, Section 5.3] for more details). Consider a class of modified wave operators of the form $L = \Box_{g(t)} - V(t)$, with the D'Alembert operator $\Box_{g(t)}$ and a t-dependent potential $V(t)$, on manifolds of the form $\mathbb{R}_t \times M_x$, equipped with a family of hyperbolic metrics $g(t) = \mathrm{diag}(-1, h(t))$, where $h(t)$ is a suitable family of t-dependent Riemannian metrics on the manifold with ends M. In this way,

$$L = \Box_{g(t)} - V(t) = -\partial_t^2 + \Delta_{h(t)} - V(t) = -\partial_t^2 + P(t), \tag{1.7}$$

where $\Delta_{h(t)}$ is the Laplace-Beltrami operator on M associated with the metric $h(t)$ and we have set $P(t) = \Delta_{h(t)} - V(t)$.

Assume now $\dim M = 2$ and consider the cylinder in $\mathbb{R}^3$ given by

$$\{(u, v, z) \in \mathbb{R}^3 \colon u^2 + v^2 = 1, z > 1\},$$

that is, the manifold $\mathcal{C} = S^1 \times (1, +\infty)$, as the local model of one of the "ends" of M. The first step consists in equipping $\mathcal{C}$ with a $\mathscr{S}$-structure, namely, a SG-admissible atlas (see [12,45]), as described in [17, Example 5.1]. As a second step, for any $m > 0$, define a t-dependent family of metrics $\mathfrak{h}(t)$ on $\mathbb{R}^3_{u,v,z}$ by

$$\mathfrak{h}(t) = \frac{1}{\Phi(t)} \cdot \mathrm{diag}\left(\frac{z^2}{4\langle z\rangle^m}, \frac{z^2}{4\langle z\rangle^m}, \frac{1}{\langle z\rangle^m}\right),$$

and denote by $h(t)$ the metrics on $\mathcal{C}$ obtained by pulling-back there the metrics $\mathfrak{h}(t)$. By computations similar to those in [17, Example 5.1], here taking into account the conformal factor $(\Phi(t))^{-1}$, we find that, in the local coordinates $x = (x_1, x_2)$ of the admissible atlas above,

$$\Delta_{h(t)} = \Phi(t)\langle x\rangle^m \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right).$$

Choosing then $V(t, x) = \Phi(t)\langle x\rangle^m$, we finally obtain

$$P(t) = -\Phi(t)\langle x\rangle^m (1 - \Delta),$$

with Δ denoting the Euclidean, flat Laplacian. Then, the operator L in (1.7) is of the form (1.1) when $m = 2$. With an appropriate choice of the conformal factor $(\Phi(t))^{-1}$, the associated symbol family $a(t, x, \xi)$ satisfies (1.6). Choosing, for instance, $\Phi(t) = t^\gamma$, $\gamma > 0$, we would be considering a contracting space $(M, h(t))$, singular at $t = 0$, with a scaling factor $\sqrt{\Phi(t)}$, see, e.g., [37, Sec. 9.1], [40] and [47]. The operators (1.1) involve, in general, also lower order terms, that is, additional effects, not necessarily related to the (varying) geometry of the underlying space.

The theory of strictly hyperbolic Cauchy problems, for first order systems and PDEs of arbitrary order in the SG-setting, can be found in Cordes [12]. In particular, Cordes showed that Cauchy problems of the type (1.2), with smooth coefficients in time within the SG-environment, are well-posed in the Schwartz spaces $\mathscr{S}(\mathbb{R}^d), \mathscr{S}'(\mathbb{R}^d)$, and in the weighted Sobolev spaces $H^{s,\sigma}(\mathbb{R}^d)$. The latter, also known as Sobolev-Kato spaces, are the scale of L^2-modelled spaces naturally associated with the SG-calculus, and are defined as

$$H^{s,\sigma}(\mathbb{R}^d) = \{u \in \mathscr{S}'(\mathbb{R}^d) : \mathrm{Op}(\omega_{s,\sigma})u \in L^2(\mathbb{R}^d)\},$$

where $\omega_{s,\sigma}(x,\xi) = \langle x\rangle^s \langle \xi\rangle^\sigma$.

Weakly hyperbolic first order systems and m-th order PDEs with constant multiplicities in this same setting and their well-posedness have been originally studied by Coriasco [14], together with the associated theory of Fourier integral operators [13], and subsequently by Coriasco and Rodino [20] and Ascanelli, Abdeljawad and Coriasco [1].

Classes of Fourier integral operators globally defined on $\mathbb{R}^d$, involving amplitudes and/or phase functions of SG-type have been considered, e.g., by Cappiello [3], Cordero, Nicola and Rodino [11], Coriasco and Ruzhansky [21], Ruzhansky and Sugimoto [44], and others (see the reference lists of the quoted papers). One novelty of our approach here is that we refine the calculus of standard SG-Fourier integral operators, taking into account the subdivision into zones of the phase space $\mathbb{R}^{2d}$ implied by the symbol class we need to use, in view of the degeneracy of the characteristic roots at $t = 0$.

A prototype of the Cauchy problems fitting in the environment studied in this paper is the following one (see Example 6.5 below for additional details):

$$\begin{cases} \partial_{tt}u(t,x) + t^{2r}\left[2 + \cos\left(\ln\left(\dfrac{1}{t^{r+1}}\right)\right)^{\ell}\right](1+|x|^2)(1-\Delta_x)u(t,x) = g(t,x), \\ u(0,x) = \varphi(x), \quad u_t(0,x) = \psi(x), \end{cases} \tag{1.8}$$

for some $r \geq 1$ and $0 \leq \ell \leq 2$ (see also Example 2.13 below). The results obtained in the sequel allow to conclude that (1.8) is well-posed in $\mathscr{S}(\mathbb{R}^d)$ (see Theorem 6.4). Moreover, assuming $\varphi \in H^{s,\sigma}(\mathbb{R}^d), \psi \in H^{s-1,\sigma-1}(\mathbb{R}^d)$ and $g \in C\Big([0,T], H^{s,\sigma}(\mathbb{R}^d)\Big)$, for suitably large s and σ, there exists $s_a > 0$ such that the unique solution $u(t,\cdot)$ belongs to $H^{s-s_a,\sigma-s_a}(\mathbb{R}^d)$ for all $t \in [0,T_0]$, with $T_0 > 0$ sufficiently small (see Theorem 6.3). Namely, the solution reveals both a finite loss of regularity and a finite loss of decay with respect to the initial data.

Similar results hold true for the general problem (1.2), provided that the real coefficients a_j and the coefficients b_j and c of the lower order terms satisfy

$$
\begin{aligned}
|D_t^k D_x^\alpha a_j(t,x)| &\le C_{k\alpha}\lambda(t)^2\langle x\rangle^{2-|\alpha|}\Sigma(t)^k,\\
|D_t^k D_x^\alpha b_j(t,x)| &\le C_{k\alpha}\lambda(t)^2\langle x\rangle^{1-|\alpha|}\frac{|\ln\lambda(t)|}{\Lambda(t)}\Sigma(t)^k,\\
|D_t^k D_x^\alpha c(t,x)| &\le C_{k\alpha}\lambda(t)^2\langle x\rangle^{-|\alpha|}\left(\frac{\ln\lambda(t)}{\Lambda(t)}\right)^2\Sigma(t)^k,\\
|D_t^k D_x^\alpha \mathrm{Im}\, b_j(t,x)| &\le C_{k\alpha}\lambda(t)\langle x\rangle^{1-|\alpha|}\Sigma(t)^{k+1},
\end{aligned}
\tag{1.9}
$$

for some $C_{k\alpha} > 0$, independent of $t \in (0,T]$ and all $x \in \mathbb{R}^d$. Additionally, the roots $\lambda_1(t,x,\xi)$ and $\lambda_2(t,x,\xi)$ of the principal symbol

$$
\mathfrak{L}_p(t,\tau,x,\xi) := -\tau^2 + \sum_{j=1}^{d} a_j(t,x)\xi_j^2 \tag{1.10}
$$

of the operator L, are assumed to be real-valued functions satisfying the inequalities

$$
|\lambda_j(t,x,\xi)| \le c\lambda(t)|\xi|\langle x\rangle, \qquad j=1,2, \tag{1.11}
$$

$$
|\lambda_1(t,x,\xi)-\lambda_2(t,x,\xi)| \ge \delta_1\lambda(t)|\xi|\langle x\rangle, \tag{1.12}
$$

for some positive constant c and δ_1, independent of $t \in [0,T]$, $(x,\xi) \in \mathbb{R}^{2d}$. In Proposition 2.3 we will prove that these assumptions are equivalent to an hyperbolicity condition on the characteristic roots.

The main novelty of the present paper is then the possibility to prove well-posedness results in $\mathscr{S}(\mathbb{R}^d)$, and show phenomena of loss of regularity and loss of decay, for weakly-hyperbolic models in the form (1.2) even if the coefficients have fast oscillations as $t \to 0^+$ (see [33] for a classification of oscillating behavior) and are unbounded with respect to the space variable (of at most quadratic growth for the principal part, and linear growth for the first order terms).

Comparing again with the results about variable multiplicities hyperbolic SG operators studied in [1], under the hypotheses of involutive characteric roots and coefficients smooth with respect to the t variable, another difference is that here we can express hyperbolicity also in terms of properties of the coefficients, not just through the behavior of the characteristic roots. Moreover, in view of the specific control of degeneracy at $t = 0$ of the coefficients or, equivalently, of the characteristic roots, a further difference is that, instead of a loss of smoothness and decay equal to the order of the operator, here we express such loss by means of the quantity s_a mentioned above. It is beyond the scope of the analysis performed in this paper to obtain a sharp estimate of the quantity s_a, which actually depends not only on the coefficients of the given equation, but also on other quantities introduced in the definition of different zones of the phase space.

Comparing with results available in cases of bounded coefficients with oscillating behavior, on one hand, it is well known that the presence of oscillations can break-down the well-posedness of weakly hyperbolic models, even in $C^\infty(\mathbb{R}^d)$. Indeed, in [9] the authors constructed a second order equation $\partial_t^2 u - a(t)\Delta u = f(t,x)$, with smooth speed of propagation $a(t) \in C^\infty([0,\infty])$ satisfying $a(t) > 0$ for $t < T$, oscillating for $t \to T^-$ and being identically zero for $t \geq T$, which is ill-posed in $C^\infty(\mathbb{R}^d)$ (see also [10]). On the other hand, the presence of coefficients unbounded with respect to the space variable can destroy the well-posedness in $\mathscr{S}(\mathbb{R}^d)$, as the next simple counterexample from [20] already illustrates. Consider the operator

$$L = D_t^2 + \langle x \rangle^{2\varepsilon}, \quad \varepsilon > 0.$$

The solution of $Lu(t,x) = 0$ is given by

$$u(t,x) = h_1(x) e^{t\langle x\rangle^\varepsilon} + h_2(x) e^{-t\langle x\rangle^\varepsilon},$$

with h_1, h_2 depending on the initial conditions. While the Cauchy problem associated with L is locally in x well-posed, it is immediate to observe that the same problem is not well-posed in $\mathscr{S}(\mathbb{R}^d)$, $\mathscr{S}'(\mathbb{R}^d)$, or any $H^{s,\sigma}(\mathbb{R}^d)$, $s, \sigma \in \mathbb{R}$ (see [20] for additional details). This shows that studying well-posedness in these genuinly global environments on $\mathbb{R}^d$ differs from studying the analogous problem locally in the space variables.

Papers [9] and [10] paved the way to several works in which the influence of oscillations on C^∞ well-posedness of problem (1.2) has been investigated under different assumptions on the coefficients. In particular, the results obtained in [30,46] allow to prove that Cauchy problem (1.2) is C^∞ well-posed and the solution shows a finite loss of regularity if the coefficients satisfy

$$\begin{aligned}
|D_t^k D_x^\alpha a_j(t,x)| &\leq C_{k\alpha} \lambda(t)^2 \Sigma(t)^k, \\
|D_t^k D_x^\alpha b_j(t,x)| &\leq C_{k\alpha} \lambda(t)^2 \frac{|\ln \lambda(t)|}{\Lambda(t)} \Sigma(t)^k, \\
|D_t^k D_x^\alpha c(t,x)| &\leq C_{k\alpha} \lambda(t)^2 \left(\frac{\ln \lambda(t)}{\Lambda(t)}\right)^2 \Sigma(t)^k, \\
|D_t^k D_x^\alpha \mathrm{Im} b_j(t,x)| &\leq C_{k\alpha} \lambda(t) \Sigma(t)^{k+1},
\end{aligned}$$

for some $C_{k\alpha} > 0$, independent of $t \in (0,T]$ and $x \in \mathbb{R}^d$, and the roots $\lambda_1(t,x,\xi)$ and $\lambda_2(t,x,\xi)$ of the principal symbol $\mathfrak{L}_p$ are real-valued functions satisfying the inequalities

$$\begin{aligned}
|\lambda_j(t,x,\xi)| &\leq c\lambda(t)|\xi|, \qquad j = 1,2, \\
|\lambda_1(t,x,\xi) - \lambda_2(t,x,\xi)| &\geq \delta_1 \lambda(t)|\xi|,
\end{aligned}$$

for some positive constant c and δ_1, independent of $t \in [0,T]$, $(x,\xi) \in \mathbb{R}^{2d}$. In [46, Chapter 5] the sharpness of the conditions required on the lower order terms is proved. These assumptions are known as Levi conditions and they turn out to be necessary to get the C^∞ well-posedness. In particular, the latter is not guaranteed if very fast oscillations appear, namely, the logaritmic term in $\Sigma(t)$ is replaced by $(-\ln \Lambda(t))^\gamma$ for some $\gamma > 1$, as thoroughly discussed in [28] in the case of time-dependent coefficients. In view of these facts, we conjecture that our conditions about the oscillating behavior of the coefficients are optimal for the $\mathscr{S}(\mathbb{R}^d)$ well-posedness of (1.2) as well. The analysis of the sharpness of the assumptions for the well-posedness of problem (1.2) in $\mathscr{S}(\mathbb{R}^d)$ will be the subject of future studies.

We recall that many authors have also considered strictly hyperbolic equations with non-smooth coefficients. In this direction, it has been observed that Lipschitz conditions play a crucial role in the Sobolev regularity results. Colombini and Lerner [8] have shown that Log-Lipschitz conditions on the coefficient are optimal for Sobolev regularity. Whenever dealing with Log-Lipschitz coefficients, a finite loss of derivatives occurs for the solution of the Cauchy problem as shown by means of examples in [4–6,8]. Other forms of weakening of the Lipschitz condition have been studied extensively in the works of Kubo and Reissig [31], Colombini, Del Santo and Reissig [7], see also Ghisi and Gobbino [26] for cases of finite and infinite loss of derivatives. In [2], Ascanelli and Cappiello considered SG-hyperbolic models with coefficients satisfying Log-Lipschitz conditions. Pattar and Kiran have studied strictly hyperbolic Cauchy problems on $\mathbb{R}^n$ with unbounded and singular coefficients in [39]. Extensions of our analysis in similar directions will be the subject of forthcoming papers.

Finally, we also mention that many authors have considered differential operators (1.1) with coefficients which only depend on the time variable t. In this case one can study the global in time well-posedness (that is, for $t \in [0, \infty)$) of problem (1.2) and investigate long time decay estimates (that is, for $t \to \infty$) for the norm of the solution in suitable functional spaces (see, for instance, [25,27,43] and the references therein).

In this paper, we study in detail a second order weakly SG-hyperbolic operator, in the case where the symbols satisfy a weakened form of Lipschitz condition, encoded through the function $\Sigma(t)$ appearing in (1.6) above. In the procedure for determining the parametrix, we first reformulate the original Cauchy problem into a corresponding Cauchy problem for a first order 2×2 system, modulo smoothing elements. We then perform a diagonalization of the 2×2 system, in the spirit of the procedures by Yagdjian [46], and Kubo and Reissig [31] (see also Kumano-go [32]). The latter requires suitable extensions of some of the results coming from the local symbolic calculi, similarly to the procedures considered, for instance, in [14,20], which need to be carefully modified. Efficient tools to achieve such needed extensions come, in particular, by some of the results by Coriasco and Toft [22]. A special feature of the SG-setting is that one can expect a finite loss of decay along with a loss of derivatives, as observed, e.g., in [2,14, 20]. This happens indeed, and we show that within our main results. Notice that the analysis can be carried through for SG-hyperbolic operators of arbitrary order, fully combining the mentioned theories in [13,14,20,22] with the approaches in [31,46]. To keep this exposition within a reasonable length, here we prefer to focus on second order operators, which anyway allows to highlight the main new features apparing in the SG-environment.

The paper is organized as follows. In Section 2 we recall known facts about the SG-calculus, and describe our symbolic setting and the assumptions about the coefficients of the operator in (1.2). One first main result, achieved in this section, is the proof of an analog, in our setting, of an equivalency between hyperbolicity conditions, in term of properties of the characteristic roots and of the coefficients of the operator. This also indicates the SG-type symbol classes we then need to consider in our analysis, as well as the subdivision into zones of the phase space. We here state the properties of such symbol classes. In the subsequent Section 3 we focus on the associated classes of pseudodifferential and Fourier integral operators, and establish their calculus. Equipped with these two tools, we then follow the classical approach to solve (1.2): in Section 4 we switch to a Cauchy problem for a 2×2 first order system, and diagonalize it, determine the parametrix of the diagonalized system in Section 5, and finally state and prove our main results about the properties of the solutions to (1.2) in the concluding Section 6. The parametrix construction relies on the properties of the solutions of the Hamiltonian systems associated with the characteristic roots of the operators. In this respect, a careful analysis is needed, in view of the explicit presence of the x variable in the definition of the zones, as well as the unboundedness of the characteristic roots with respect to it and their behavior with respect to the time variable. In the sequel we will sometimes write $A \lesssim B$ when $A \leq cB$ for a suitable constant $c > 0$, and we set $A \asymp B$ when $A \lesssim B \lesssim A$ (in particular, we will adopt this notation when the value of the specific constants c, c' is not crucial).

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2. Formulation of the hyperbolic Cauchy problem in $\mathbb{R}^d$

In this section we discuss an appropriate parameter-dependent global symbol class of SG type, associated with the Cauchy problem (1.2). In particular, we will state the requirements on the coefficients that guarantee the SG-hyperbolicity of (1.2) (that is, hyperbolicity in $\mathbb{R}^d$ with respect to the calculus associated with the metric (1.4)), and necessary conditions on the lower order terms that ensure its well-posedness. These conditions also dictate the definition of the SG-classes which must be used in the construction of the fundamental solution.

2.1. Shape function

Following [46], we introduce a real-valued, positive function $\lambda(t)$, which allows to describe the speed at which characteristics collide at $t = 0$ and the qualitative behavior with respect to t of the coefficients in (1.2). In particular, $\lambda(t)$ belongs to $C^\infty([0,T])$ and satisfies $\lambda(0) = \lambda'(0) = 0$, $\lambda'(t) \neq 0$ for $t > 0$. The hypotheses on λ of course imply $\lambda'(t) > 0$ for $t > 0$. Moreover, the following inequality holds

$$|\lambda^{(k)}(t)| \leq c \left(\frac{\lambda'(t)}{\lambda(t)}\right)^{k-1} |\lambda'(t)|,$$

for non-negative integers k. The choice of $\lambda(t)$ with $\lambda(0) = 0$ makes (1.2) a multiple characteristic partial differential equation at $t = 0$, and the whole operator one with characteristic roots of variable multiplicities. This function quantifies the vanishing order of the principal part of (1.2) with respect to t, and thus the behavior of the characteristic roots. Furthermore, the integral of $\lambda(t)$,

$$\Lambda(t) = \int_0^t \lambda(s) ds,$$

plays a crucial role in the determination of the phase function in the Fourier integral operators we will deal with. We assume that $\lambda^2/\Lambda \in C^\infty([0,T])$ and there exist $c_1 > 1/2$ and $0 < C_1 < 1$ constants such that

$$c_1 \frac{\lambda(t)}{\Lambda(t)} \leq \frac{\lambda'(t)}{\lambda(t)} \leq C_1 \frac{\lambda(t)}{\Lambda(t)}, \tag{2.1}$$

for $t \in (0,T]$. The estimates (2.1) define a bound on the growth of the coefficients encoded by the symbols $a(t,x,\xi)$, given in (1.3), with respect to t. Typical examples of shape functions that satisfy the required assumptions are given by

$$\lambda(t) = t^r \qquad \text{or} \qquad \lambda(t) = \underbrace{\exp(-\exp(\ldots\exp(|t|^{-r})))}_{k \text{ exponents}}$$

for r, k integers numbers, $k \geq 0$ and $r \geq 2$.

2.2. Zones of the phase space

Employing the function $\Lambda(t)$ we define a partition of $[0,T] \times \mathbb{R}^{2d}$: for all $(x,\xi) \in \mathbb{R}^{2d}$ we fix $t_{x,\xi}$ the unique solution to equation

$$\Lambda(t_{x,\xi})\, \langle x\rangle\langle\xi\rangle = N \ln\langle x\rangle\langle\xi\rangle, \quad |x| + |\xi| \geq M, \tag{2.2}$$

where $N > 0$ is a sufficiently large parameter and $M > 0$. Then, we define the *hyperbolic zone* as

$$Z_{\text{hyp}}(N) = \{(t, x, \xi) \in [0, T] \times \mathbb{R}^{2d} : t \geq t_{x,\xi}\},$$

and the *pseudodifferential zone* as

$$Z_{\text{pd}}(N) = \{(t, x, \xi) \in [0, T] \times \mathbb{R}^{2d} : t \leq t_{x,\xi}\}.$$

For future aims, we further partition the hyperbolic zone into as

$$Z_{\text{hyp}}(N) = Z_{\text{osc}}(N) \cup Z_{\text{reg}}(N),$$

defining the *regular zone* as

$$Z_{\text{reg}}(N) = \{(t, x, \xi) \in [0, T] \times \mathbb{R}^{2d} : t'_{x,\xi} \leq t\},$$

and the *oscillation zone* as

$$Z_{\text{osc}}(N) = \{(t, x, \xi) \in [0, T] \times \mathbb{R}^{2d} : t_{x,\xi} \leq t \leq t'_{x,\xi}\}.$$

Here, for all $(x, \xi) \in \mathbb{R}^{2d}$, the time function $t = t'_{x,\xi}$ satisfies the equation

$$\Lambda(t'_{x,\xi})\, \langle x \rangle \langle \xi \rangle = 2N(\ln(\langle x \rangle \langle \xi \rangle))^2 \text{ for } |x| + |\xi| \geq M. \tag{2.3}$$

The next Lemma 2.1 holds true.

Lemma 2.1. *For all M and N sufficiently large, there exist $d_1, d_2 > 0$ such that it holds*

$$-d_1 \ln(\langle x \rangle \langle \xi \rangle) \leq \ln \lambda(t_{x,\xi}) \leq -d_2 \ln(\langle x \rangle \langle \xi \rangle),$$

for every $(t, x, \xi) \in Z_{\text{hyp}}(N)$ with $|x| + |\xi| \geq M$. As a consequence, for all $t \in [0, T]$ each couple (x_t, ξ_t) that solves the equation

$$\Lambda(t)\, \langle x_t \rangle \langle \xi_t \rangle = N \ln(\langle x_t \rangle \langle \xi_t \rangle), \tag{2.4}$$

and $|x_t| + |\xi_t| \geq M$ satisfies the inequalities

$$-d_1 \ln(\langle x_t \rangle \langle \xi_t \rangle) \leq \ln \lambda(t) \leq -d_2 \ln(\langle x_t \rangle \langle \xi_t \rangle).$$

Proof. We of course have

$$-\ln \lambda(t_{x,\xi}) + \ln \lambda(T) = \int_{t_{x,\xi}}^{T} \frac{\lambda'(t)}{\lambda(t)}\, dt.$$

By assumption (2.1), there exists $c_1, C_1 > 0$ such that

$$c_1 \int_{t_{x,\xi}}^{T} \frac{\lambda(t)}{\Lambda(t)}\,dt \leq -\ln\lambda(t_{x,\xi}) + \ln\lambda(T) \leq C_1 \int_{t_{x,\xi}}^{T} \frac{\lambda(t)}{\Lambda(t)}\,dt.$$

Employing (2.2), it follows

$$\begin{aligned} &c_1 \ln[N\ln(\langle x\rangle\langle\xi\rangle)] - c_1\ln(\langle x\rangle\langle\xi\rangle) - c_1\ln\Lambda(T) + \ln\lambda(T)\\ &\qquad \geq \ln\lambda(t_{x,\xi}) \geq \\ &\qquad - C_1\ln(\langle x\rangle\langle\xi\rangle) + C_1\ln[N\ln(\langle x\rangle\langle\xi\rangle)] - C_1\ln\Lambda(T) + \ln\lambda(T). \end{aligned}$$

For any N (arbitrarily large), and $|x|+|\xi| \geq M$, M sufficiently large, we may guarantee

$$-d_1\ln(\langle x\rangle\langle\xi\rangle) \leq \ln\lambda(t_{x,\xi}) \leq -d_2\ln(\langle x\rangle\langle\xi\rangle).$$

The other part of the statement follows by similar considerations, recalling the definition of $Z_{\mathrm{hyp}}(N)$. □

Remark 2.2.

(1) Lemma 2.1 implies that, for every $(t,x,\xi) \in Z_{\mathrm{hyp}}(N)$ with $|x|+|\xi| \geq M$ and $t \in [0,T]$

$$d_2\ln(\langle x\rangle\langle\xi\rangle) \leq |\ln\lambda(t_{x,\xi})| \leq d_1\ln(\langle x\rangle\langle\xi\rangle)$$

and, for all $t \in [0,T]$ each couple (x_t,ξ_t) which solves (2.4) satisfies

$$d_2\ln(\langle x_t\rangle\langle\xi_t\rangle) \leq |\ln\lambda(t)| \leq d_1\ln(\langle x_t\rangle\langle\xi_t\rangle).$$

(2) As a consequence of Lemma 2.1, there exists $C > 0$ such that

$$\langle x_t\rangle\langle\xi_t\rangle = \frac{N\ln(\langle x_t\rangle\langle\xi_t\rangle)}{\Lambda(t)} \leq C\frac{|\ln\lambda(t)|}{\Lambda(t)}.$$

Moreover, for M sufficiently large, in the hyperbolic zone we may estimate

$$\frac{|\ln\lambda(t)|}{\langle x\rangle\langle\xi\rangle\Lambda(t)} \lesssim \frac{1}{N}.$$

Indeed, we first prove that for all $t \in [0,T]$ and $(x,\xi) \in \mathbb{R}^{2d}$, if $(t,x,\xi) \in Z_{\mathrm{hyp}}(N)$, then $\langle x\rangle\langle\xi\rangle \geq \langle x_t\rangle\langle\xi_t\rangle$. In fact, assume that $\langle x\rangle\langle\xi\rangle < \langle x_t\rangle\langle\xi_t\rangle$. Then, since the function $\zeta(s) = \frac{s}{\ln(s)}$ is increasing on $[e,+\infty)$, we have

$$\frac{\Lambda(t)\langle x\rangle\langle\xi\rangle}{\ln(\langle x\rangle\langle\xi\rangle)} < \frac{\Lambda(t)\langle x_t\rangle\langle\xi_t\rangle}{\ln(\langle x_t\rangle\langle\xi_t\rangle)} = N,$$

that is, $(t,x,\xi) \in Z_{\mathrm{pd}}(N)$, which is a contradiction. It follows, by point (1) and the previous considerations,

$$\frac{1}{\Lambda(t)} = \frac{\langle x_t\rangle\langle\xi_t\rangle}{N\ln(\langle x_t\rangle\langle\xi_t\rangle)} \le \frac{d_1\langle x_t\rangle\langle\xi_t\rangle}{N|\ln(\lambda(t))|} \le \frac{d_1\langle x\rangle\langle\xi\rangle}{N|\ln(\lambda(t))|},$$

which gives the claim.

2.3. Assumptions on the coefficients

We suppose that there exists a positive constant N such that the zeros $\tau_j(t,x,\xi)$, $j = 1,2$, of the complete symbol of the operator L,

$$\mathfrak{L}(t,\tau,x,\xi) = -\tau^2 + a(t,x,\xi) = 0, \tag{2.5}$$

are smooth functions on $Z_{\mathrm{hyp}}(N)$ and, for any $k \in \mathbb{N}$, $\alpha,\beta \in \mathbb{N}^d$, the following inequalities hold:

$$|D_t^k D_x^\alpha D_\xi^\beta \tau_j(t,x,\xi)| \le C_{k\alpha\beta}\lambda(t)\langle x\rangle^{1-|\alpha|}\langle\xi\rangle^{1-|\beta|}\left(\frac{\lambda(t)}{\Lambda(t)}\ln\left(\frac{1}{\Lambda(t)}\right)\right)^k, \tag{2.6}$$

$$|\tau_1(t,x,\xi) - \tau_2(t,x,\xi)| \ge \delta\lambda(t)\langle x\rangle\langle\xi\rangle, \tag{2.7}$$

$$|D_t^k D_x^\alpha D_\xi^\beta \mathrm{Im}\tau_j(t,x,\xi)| \le C_{k\alpha\beta}\langle x\rangle^{-|\alpha|}\langle\xi\rangle^{-|\beta|}\left(\frac{\lambda(t)}{\Lambda(t)}\ln\left(\frac{1}{\Lambda(t)}\right)\right)^{k+1}, \tag{2.8}$$

for some positive constants δ, $C_{k\alpha\beta}$, and $(t,x,\xi) \in Z_{\mathrm{hyp}}(N)$. Proposition 2.3 below allows to relate the conditions on the roots $\tau_j(t,x,\xi)$, $j = 1,2$, of the complete symbol with suitable assumptions on the roots $\lambda_j(t,x,\xi)$ of the principal symbol of the operator L, defined by (1.10), and on the coefficients $a_j(t,x)$, $b_j(t,x)$, $j = 1,\dots,d$ and $c(t,x)$. Such equivalent formulation of SG-hyperbolicity of L is proved by adapting the argument of the analogous statement in [46]. We sketch the argument below, focusing on the specific aspect involving here the unboundedness of the coefficients, characteristic roots, and symbols, with respect to the space variable $x \in \mathbb{R}^d$.

Proposition 2.3. *The following conditions **(A)** and **(H)** are equivalent:*

(A) *For all $t \in [0,T]$, $x,\xi \in \mathbb{R}^d$, the zeros $\lambda_1(t,x,\xi)$ and $\lambda_2(t,x,\xi)$ of the principal symbol (1.10) of the operator L are real-valued functions that satisfy inequalities (1.11) and (1.12), for some positive constant c and δ, independent of $t \in [0,T]$, $(x,\xi) \in \mathbb{R}^{2d}$. Furthermore, the coefficients $a_j(t,x)$, $b_j(t,x)$, $j = 1,\dots,d$, and $c(t,x)$ satisfy the inequalities in (1.9).*

(H) *There exists a positive constant N_0 such that, for all $N \geq N_0$, the zeros $\tau_1(t,x,\xi)$ and $\tau_2(t,x,\xi)$ of the complete symbol (2.5) of the operator L are smooth functions on $Z_{\rm hyp}(N)$ and the inequalities (2.6)-(2.8) hold for all $(t,x,\xi) \in Z_{\rm hyp}(N)$.*

Remark 2.4. The equivalence result given in Proposition 2.3 is inspired by Proposition 2.1.21 in [46]. However, the conditions assumed here, on the coefficients of the differential operator L and the roots of the principal symbol (1.10), are deeply different with respect to the ones provided in [46]. These differences underline the novelty of this work: we investigate the well-posedness of problem (1.2) in the global framework of $\mathbb{R}^d$, considering coefficients that are not necessarily bounded with respect to x, but possibly of polynomial growth, provided that the roots of the principal symbol satisfy the global (in space) separation condition (1.12).

Proof. We prove the implication **(H)** $\Longrightarrow$ **(A)**. Since it holds

$$a_j(t,x) = \frac{1}{2} D^2_{\xi_j}(\tau_1(t,x,\xi)\tau_2(t,x,\xi)),$$

for all $(t,x,\xi) \in Z_{\rm hyp}(N)$ we may estimate

$$|D_t^k D_x^\alpha a_j(t,x)| \leq C_{k\alpha} \lambda(t)^2 \langle x\rangle^{2-|\alpha|} \left(\frac{\lambda(t)}{\Lambda(t)} \ln\left(\frac{1}{\Lambda(t)}\right)\right)^k. \tag{2.9}$$

By induction, we may prove the desired estimates also for the coefficients b_j, $j = 1,\dots,d$ and c. Indeed, it holds

$$D_t^k D_x^\alpha b_j(t,x) = i D_t^k D_x^\alpha (D_{\xi_j}(\tau_1\tau_2) - 2a_j(t,x)\xi_j),$$

and

$$D_t^k D_x^\alpha c(t,x) = D_t^k D_x^\alpha \Big(\tau_1\tau_2 - \sum_{j=1}^d (a_j(t,x)\xi_j^2 + b_j(t,x)\xi_j)\Big);$$

In particular, by (2.6) and (2.9) we have

$$\begin{aligned}|D_t^k D_x^\alpha b_j(t,x)| \leq & C_{k\alpha}\lambda(t)^2\langle x\rangle^{2-|\alpha|}\langle\xi\rangle \left(\frac{\lambda(t)}{\Lambda(t)} \ln\left(\frac{1}{\Lambda(t)}\right)\right)^k \\ \leq & C_{k\alpha}\lambda(t)^2\langle x\rangle^{1-|\alpha|}\frac{|\ln\lambda(t)|}{N\Lambda(t)} \left(\frac{\lambda(t)}{\Lambda(t)} \ln\left(\frac{1}{\Lambda(t)}\right)\right)^k,\end{aligned}$$

and then,

$$
\begin{aligned}
|D_t^k D_x^\alpha c(t,x)| \leq & C_{k\alpha}\lambda(t)^2\langle x\rangle^{2-|\alpha|}\langle \xi\rangle^2\left(\frac{\lambda(t)}{\Lambda(t)}\ln\left(\frac{1}{\Lambda(t)}\right)\right)^k \\
\leq & C_{k\alpha}\lambda(t)^2\langle x\rangle^{-|\alpha|}\left(\frac{|\ln\lambda(t)|}{N\Lambda(t)}\right)^2\left(\frac{\lambda(t)}{\Lambda(t)}\ln\left(\frac{1}{\Lambda(t)}\right)\right)^k,
\end{aligned}
$$

for all $(t,x)\in[0,T]\times\mathbb{R}^d$. Indeed, for any fixed $t\in[0,T]$ and $x\in\mathbb{R}^d$, it is possible to choose $\xi\in\mathbb{R}^d$ (depending on t and x) equal to the unique solution to the equation

$$
\Lambda(t)\langle x\rangle\langle\xi\rangle = N\ln(\langle x\rangle\langle\xi\rangle).
$$

With such choice of ξ, we may estimate

$$
\langle x\rangle\langle\xi\rangle \leq C\frac{|\ln(\lambda(t))|}{N\Lambda(t)},
$$

as a consequence of Lemma 2.1 (see Remark 2.2).

Let us now prove that $\mathrm{Im} a_j = 0$. Applying estimate (2.8) we find

$$
\begin{aligned}
|\mathrm{Im}\, a_j(t,x)| &= \left|\mathrm{Im}\,\partial^2_{\xi_j}(\tau_1(t,x,\xi)\tau_2(t,x,\xi))\right| \\
&\lesssim \langle x\rangle\langle\xi\rangle^{-1}\frac{\lambda(t)^2}{\Lambda(t)}\ln\left(\frac{1}{\Lambda(t)}\right),
\end{aligned}
$$

for all $(t,x,\xi)\in Z_{\mathrm{hyp}}(N)$. In particular, for any fixed $t>0$ and $x\in\mathbb{R}^n$ there exists M sufficiently large such that $(t,x,\xi)\in Z_{\mathrm{hyp}}(N)$ for all $|\xi|>M$. Taking $|\xi|\to+\infty$ the right-hand side tends to 0, whereas the left-hand side does not depend on ξ. This allows to conclude that $\mathrm{Im} a_j = 0$. As a consequence, the following identity holds:

$$
\begin{aligned}
|\mathrm{Im}\, b_j(t,x)| &= \left|\mathrm{Im}\,\partial_{\xi_j}(\tau_1(t,x,\xi)\tau_2(t,x,\xi))\right| \\
&\lesssim \langle x\rangle\frac{\lambda(t)^2}{\Lambda(t)}\ln\left(\frac{1}{\Lambda(t)}\right).
\end{aligned}
$$

In order to prove (1.11) we define, for $i=1,2$, $\mu_i(t,x,\xi)$ and $\gamma_i(t,x,\xi)$ such that

$$
\lambda_i(t,x,\xi) = \lambda(t)\langle x\rangle|\xi|\mu_i(t,x,\xi) \text{ and } \tau_i(t,x,\xi) = \lambda(t)\langle x\rangle|\xi|\gamma_i(t,x,\xi).
$$

Then, μ_i and γ_i satisfy

$$
-\mu_i^2 + (\lambda(t)\langle x\rangle|\xi|)^{-2}\sum_{j=1}^d a_j(t,x)\xi_j^2 = 0; \tag{2.10}
$$

$$
-\gamma_i^2 + (\lambda(t)\langle x\rangle|\xi|)^{-2}\sum_{j=1}^d a_j(t,x)\xi_j^2 + B_1 = 0, \tag{2.11}
$$

$$B_1 := (\lambda(t)\langle x\rangle|\xi|)^{-2}\Big(\sum_{j=1}^{d} b_j(t,x)\xi_j + c(t,x)\Big).$$

In view of conditions (2.6)-(2.8), the functions γ_i satisfy:

$$|D_t^j D_x^\alpha D_\xi^\beta \gamma_i(t,x,\xi)| \le C_{k\alpha\beta}\langle x\rangle^{-|\alpha|}\langle\xi\rangle^{-\beta}\left(\frac{\lambda(t)}{\Lambda(t)}\ln\left(\frac{1}{\Lambda(t)}\right)\right)^j, \tag{2.12}$$

$$|\gamma_1(t,x,\xi) - \gamma_2(t,x,\xi)| \ge \delta, \tag{2.13}$$

$$|D_t^j D_x^\alpha D_\xi^\beta \mathrm{Im}\gamma_i(t,x,\xi)| \le C_{k\alpha\beta}\frac{1}{\Lambda(t)}\langle x\rangle^{-1-|\alpha|}\langle\xi\rangle^{-1-|\beta|}\left(\frac{\lambda(t)}{\Lambda(t)}\ln\left(\frac{1}{\Lambda(t)}\right)\right)^j,$$

for all $(t,x,\xi) \in Z_{\text{hyp}}(N)$. Furthermore, by using the obtained estimates for the coefficients b_j, $j = 1,\dots,d$, and c we may estimate the perturbation B_1 by

$$|D_t^k D_x^\alpha D_\xi^\beta B_1| \lesssim \frac{1}{N}\langle x\rangle^{-|\alpha|}\langle\xi\rangle^{-|\beta|}\left(\frac{\lambda(t)}{\Lambda(t)}\ln\left(\frac{1}{\Lambda(t)}\right)\right)^k. \tag{2.14}$$

Let us consider the polynomial in the variable μ defined by $P(t,x,\xi;\mu) := (\mu-\gamma_1)(\mu-\gamma_2)$. By (2.10) and (2.11) we derive that μ_1 and μ_2 are the two solutions to equation

$$P(t,x,\xi;\mu) + B_1 = 0.$$

In particular, condition (2.13) guarantees that $P(t,x,\xi,\cdot)$ has two simple roots. Moreover, the roots γ_i of (2.11) analytically depend on the perturbation B_1 in some neighborhood of the origin. Then, if B_1 has modulus sufficiently small we may write the following series expansion of $\mu_i(t,x,\xi)$,

$$\mu_i(t,x,\xi) = \gamma_i(t,x,\xi) + \sum_{n=1}^{\infty} c_n^{(i)}(t,x,\xi)B_1^n(t,x,\xi), \quad i = 1,2,$$

where

$$\begin{aligned} c_n^{(i)}(t,x,\xi) &= \frac{1}{2\pi i}\oint\limits_{|\omega-\gamma_i|=\rho} \frac{(\omega-\omega_i)P'_\omega(t,x,\xi;\omega)}{P(t,x,\xi;\omega)^{n+1}} \\ &= \frac{1}{(n-1)!}\left[\frac{d^{n-1}}{d\omega^{n-1}}\left\{\left[\frac{\omega-\gamma_i}{P(t,x,\xi;\omega)}\right]^{n+1} P'_\omega(t,x,\xi;\omega)\right\}\right]_{\omega=\gamma_i}, \end{aligned}$$

where $0 < 2\rho < \delta$. In particular, we have the inequality

$$c_n^{(i)}(t,x,\xi) \le c\delta^{-1}(2c/\rho\delta)^n,$$

for all $(t, x, \xi) \in Z_{\text{hyp}}(N)$, with a constant c independent of t, x, ξ. As a consequence, the radius of convergence given by the Cauchy-Hadamard formula $R_i = 1/\limsup_{n\to+\infty}(c_n^{(i)})^{\frac{1}{n}}$ is independent of (t, x, ξ). Moreover, due to estimate (2.14), the series $\sum_n c_n^{(i)} B_i^n$ can be made arbitrarily small taking N sufficiently large. Then, by estimate (2.14), for N sufficiently large we derive

$$|\mu_1(t, x, \xi) - \mu_2(t, x, \xi)| \geq \delta \tag{2.15}$$

for all $(t, x, \xi) \in Z_{\text{hyp}}(N)$. Let us prove the following estimate:

$$|D_t^k D_x^\alpha D_\xi^\beta \mu_i(t, x, \xi)| \leq C_{k\alpha\beta} \langle x \rangle^{-|\alpha|} \langle \xi \rangle^{-|\beta|} \left(\frac{\lambda(t)}{\Lambda(t)} \ln \left(\frac{1}{\Lambda(t)} \right) \right)^k, \tag{2.16}$$

for all $(t, x, \xi) \in Z_{\text{hyp}}(N)$. To this aim, let us denote $y = (t, x, \xi) \in (0, T] \times \mathbb{R}^{2d}$. By applying the formula for the derivative of implicit functions to $P(t, x, \xi; \mu)$, for any multi-index $r \in \mathbb{N}^{2d+1}$ and $i = 1, 2$, we obtain the identity

$$\partial_y^r \mu_i(y) = (P'_\omega(y; \mu_i(y)))^{-1}(-B_1(y) + \mathcal{E}_i^r(y)),$$

where

$$\begin{aligned} \mathcal{E}_i^r(y) &= \partial_y^r \gamma_1(y)(\mu_i(y) - \gamma_2(y)) + \partial_y^r \gamma_2(y)(\mu_i(y) - \gamma_1(y)) \\ &\quad - \sum_{\substack{r_1+r_2=r \\ r_1, r_2 \neq r}} \frac{r!}{r_1! r_2!} \partial_y^{r_1}(\mu_i(y) - \gamma_1(y)) \partial_y^{r_2}(\mu_i(y) - \gamma_2(y)). \end{aligned}$$

By (2.14) and (2.15) we may estimate, for N sufficiently large,

$$|P'_\omega(y, \mu_i(y))| = |2\mu_i - (\gamma_1 + \gamma_2)| > \delta$$

for all $y \in Z_{\text{hyp}}(N)$. Then, (2.16) follows from (2.12) and our previous considerations on $\sum_n c_n^{(i)} B_i^n$. □

*2.4. Generalized parameter-dependent **SG** symbol classes*

Following [22], we consider, for any real numbers m, μ, and $r_j, \rho_j \geq 0$, $j = 1, 2$, the general class of SG-type symbols $S^{m,\mu}_{r_1,r_2,\rho_1,\rho_2}$, which consists of all $a \in C^\infty(\mathbb{R}^{2d})$ such that, for all multi-indices $\alpha, \beta \in \mathbb{N}^d$, there is a constant $C_{\alpha\beta} > 0$ that satisfies

$$|D_x^\alpha D_\xi^\beta a(x, \xi)| \leq C_{\alpha\beta} \langle x \rangle^{m - r_1|\alpha| + r_2|\beta|} \langle \xi \rangle^{\mu + \rho_1|\alpha| - \rho_2|\beta|},$$

for all $(x, \xi) \in \mathbb{R}^{2d}$. In particular, we denote the special classes $S^{m,\mu}_{1,0,1,0}$ and $S^{m,\mu}_{1-\varepsilon,\varepsilon,1-\varepsilon,\varepsilon}$, $\varepsilon \in (0, 1)$, by $S^{m,\mu}$ and $S^{m,\mu}_{(\varepsilon)}$, respectively.

Definition 2.5. An element $a \in S^{m,\mu}$ is called (md- or SG-)elliptic if it also satisfies the lower bound

$$|a(x,\xi)| \geq C\langle x\rangle^m \langle \xi\rangle^\mu, \quad x,\xi \in \mathbb{R}^d, |x|+|\xi| \geq M.$$

More generally, $a \in S^{m,\mu}$ is called (SG-)hypoelliptic if it satisfies the lower bound

$$|a(x,\xi)| \geq C\langle x\rangle^{m'} \langle \xi\rangle^{\mu'}, \quad x,\xi \in \mathbb{R}^d, |x|+|\xi| \geq M,$$

for some $m' \leq m$, $\mu' \leq \mu$, and for any $\alpha,\beta \in \mathbb{N}^d$ there exists $C_{\alpha\beta} > 0$ such that

$$|D_x^\alpha D_\xi^\beta a(x,\xi)| \leq C|a(x,\xi)|\langle x\rangle^{-|\alpha|}\langle \xi\rangle^{-|\beta|}.$$

Note that an extended family of SG-symbols is defined in [16], by employing more general weights $\omega(x,\xi)$ in place of $\langle x\rangle^m\langle\xi\rangle^\mu$ in (5.22).

Definition 2.6. By S_{2N}^{pd} we denote the class of all symbols $p \in L_\infty\Big([0,T],\ C^\infty(\mathbb{R}^{2d})\Big)$ satisfying, for $(t,x,\xi) \in Z_{\text{pd}}(2N)$ and for all multi-indices $\alpha,\beta \in \mathbb{N}^d$, the estimates

$$\operatorname*{ess\,sup}_{t\in[0,t_{x,\xi}]} |\partial_\xi^\beta \partial_x^\alpha p(t,x,\xi)| \leq C_{\alpha\beta}\, \langle x\rangle^{1-|\alpha|}\langle\xi\rangle^{1-|\beta|}.$$

Definition 2.7. By $S^{m,\mu}\{\kappa,\ell\}_N^{\text{hyp}}$ we denote the class of all symbol families $a \in C([0,T],S^{m,\mu}) \cap C^\infty((0,T]\times\mathbb{R}^{2d})$ such that, for all $k \in \mathbb{N}$ and $\alpha,\beta \in \mathbb{N}^d$, we have, for suitable constants $C_{k\alpha\beta} > 0$,

$$|D_t^k D_x^\alpha D_\xi^\beta a(t,x,\xi)| \leq C_{k\alpha\beta}\langle x\rangle^{m-|\beta|}\langle\xi\rangle^{\mu-|\alpha|}\lambda(t)^\kappa \left(\frac{\lambda(t)}{\Lambda(t)}\left(\ln\frac{1}{\Lambda(t)}\right)\right)^{\ell+k} \tag{2.17}$$

for every $(t,x,\xi) \in Z_{\text{hyp}}(N)$. Similarly, we denote by $S^{m,\mu}\{\kappa,\ell\}_N^{\text{reg}}$ the class of all symbol families in $C([0,T],S^{m,\mu})\cap C^\infty((0,T]\times\mathbb{R}^{2d})$ that satisfy (2.17) for all $(t,x,\xi) \in Z_{\text{reg}}(N)$.

Remark 2.8. The class $S^{m,\mu}\{\kappa,\ell\}_N^{\text{reg}}$ has the same properties as that of $S^{m,\mu}\{\kappa,\ell\}_N^{\text{hyp}}$ except for regularity behavior. In particular, the regularity in $S^{m-l,\mu-l}\{\kappa,\ell+l\}_N^{\text{reg}}$ is $(\ln(\langle x\rangle\langle\xi\rangle))^{-1}$ better than those of symbols from $S^{m-l+1,\mu-\ell+1}\{\kappa,\ell+l-1\}_N^{\text{reg}}$ for $l \geq 0$. This deviation of the regularity across hierarchy classes will play a role in the diagonalization procedure below.

Some hierarchical properties of the symbol class introduced in Definition 2.7 are listed below. They follow by adapting arguments to prove similar properties of the classes employed in [46], so we omit their proofs.

Proposition 2.9.

(1) $S^{m,\mu}\{\kappa,\ell\}^{\text{hyp}}_{N_2} \subset S^{m,\mu}\{\kappa,\ell\}^{\text{hyp}}_{N_1}$ *for* $N_1 \geq N_2$.
(2) $S^{m,\mu}\{\kappa,\ell\}^{\text{hyp}}_{N} \subset S^{m+l,\mu+l}\{\kappa,\ell-l\}^{\text{hyp}}_{N}$ *for* $l \geq 0$. *By this property there is no difference in regularity of the symbols from* $S^{m,\mu}\{\kappa,\ell\}^{\text{hyp}}_{N}$ *and* $\cap_{l\geq 0} S^{m-l,\mu-l}\{\kappa,\ell+l\}^{\text{hyp}}_{N}$ *in* $Z_{\text{hyp}}(N)$.
(3) *If* $a(t,x,\xi) \in S^{m_1,\mu_1}\{\kappa_1,\ell_1\}^{\text{hyp}}_{N}$ *and* $b(t,x,\xi) \in S^{m_2,\mu_2}\{\kappa_2,\ell_2\}^{\text{hyp}}_{N}$, *then* $a(t,x,\xi)b(t,x,\xi) \in S^{m_1+m_2,\mu_1+\mu_2}\{\kappa_1+\kappa_2,\ell_1+\ell_2\}^{\text{hyp}}_{N}$.
(4) *If* $a(t,x,\xi) \in S^{m,\mu}\{\kappa,\ell\}^{\text{hyp}}_{N}$ *and* $\kappa > 0$, *then* $D_t a(t,x,\xi) \in S^{m,\mu}\{\kappa-1,\ell+1\}^{\text{hyp}}_{N}$
(5) *If* $a \in S^{m,\mu}\{\kappa,\ell\}^{\text{hyp}}_{N}$ *is constant in* $Z_{\text{pd}}(N)$ *and* $\ell \geq 0$, *then*

$$\partial_t^l a \in L_\infty\Big([0,T],\ S^{\tilde m+l,\tilde\mu+l}\Big)$$

for all $l \geq 0$ *and where* $\tilde m = \max\{0, m+\ell\}$ *and* $\tilde\mu = \max\{0,\mu+\ell\}$. *Indeed, for* $(t,x,\xi) \in Z_{\text{hyp}}(N)$ *we have*

$$\begin{aligned}|\partial_t^l a(t,x,\xi)| &\leq \langle x\rangle^m \langle\xi\rangle^\mu \lambda(t)^\kappa \left(\frac{\lambda(t)}{\Lambda(t)}\ln\left(\frac{1}{\Lambda(t)}\right)\right)^{\ell+l}\\ &\leq C_l \langle x\rangle^{m+\ell+l}\langle\xi\rangle^{\mu+\ell+l},\end{aligned}$$

being $\lambda(t)$ *uniformly bounded in* $[0,T]$ *and*

$$\frac{\lambda(t)}{\Lambda(t)}\ln\left(\frac{1}{\Lambda(t)}\right) \leq \langle x\rangle\langle\xi\rangle.$$

A straightforward modification of the procedures of asymptotic summation, using the hierarchy of symbol classes $S^{m,\mu}\{\kappa,\ell\}^{\text{hyp}}_{N}$, yield the next Lemma 2.10.

Lemma 2.10. *Assume that the symbols* $a_l \in S^{m-l,\mu-l}\{\kappa,\ell\}^{\text{hyp}}_{N}$, $\kappa \geq 0$, *vanish in* $Z_{\text{pd}}(N)$. *Then, there is a symbol* $a \in S^{m,\mu}\{\kappa,\ell\}^{\text{hyp}}_{N}$ *with support in* $Z_{\text{hyp}}(N)$ *such that*

$$a - \sum_{l=0}^{k-1} a_l \in S^{m-k,\mu-k}\{\kappa,\ell\}^{\text{hyp}}_{N},\ \text{for all } k \geq 1.$$

The symbol a *is uniquely determined modulo* $C^\infty([0,T], S^{-\infty,-\infty})$.

Remark 2.11. We note that if (2.6) holds for $j = 1,2$, for all (t,x,ξ) in $Z_{\text{hyp}}(N)$, then the function a defined in (1.3) belongs to $S^{2,2}\{2,0\}^{\text{hyp}}_{N}$. Moreover, for all $(t,x,\xi) \in Z_{\text{pd}}(N)$ it satisfies

$$|\partial_x^\alpha \partial_\xi^\beta a(t,x,\xi)| \lesssim \langle x\rangle^{2-|\alpha|}\langle\xi\rangle^{2-|\beta|}.$$

Further, inequality (2.7) implies that $a(t, x, \xi)$ satisfies an ellipticity condition depending on the parameter t, which degenerates in $t = 0$, that is,

$$|a(t, x, \xi)| \gtrsim \lambda(t)^2 \langle x \rangle^2 \langle \xi \rangle^2,$$

for all $(t, x, \xi) \in Z_{\text{hyp}}(N)$. In particular, $a(0, x, \xi)$ need not to be SG-elliptic.

Remark 2.12. Notice that the results in Proposition 2.9 and Lemma 2.10 hold true, with trivial modifications, also for hierarchies associated with the subdivision into zones, and based on the more general classes $S^{m,\mu}_{r_1,r_2,\rho_1,\rho_2}$, $r_j, \rho_j \geq 0$, $j = 1, 2$, $r_2, \rho_1 < 1$, and, in particular, on the classes $S^{m,\mu}_{(\varepsilon)}$, $\varepsilon \in (0, 1)$, mentioned above. We will tacitly use these results as well in the sequel, wherever needed.

Now we provide an example of operator belonging to the class we are studying. The Cauchy problem can, in this case, be solved explicitely, so that we can, in particular, study the decay properties of the solution in relation to those of the initial data, which is one of the new, interesting features on which we are focusing.

Example 2.13. Choose a shape function λ, satisfying the hypotheses described in Section 2.1, and consider the partial differential equation

$$Lu(t, x) = (\partial_t - \lambda(t)x\partial_x)(\partial_t + \lambda(t)x\partial_x)u(t, x) = 0, \quad (t, x) \in [0, T] \times \mathbb{R}, \tag{2.18}$$

with the unknown u satisfying

$$u(0, x) = f(x) \text{ and } u_t(0, x) = g(x).$$

The solution is given by

$$u(t, x) = f(xe^{-\Lambda(t)}) + \int_0^t g\left(xe^{2\Lambda(s)-\Lambda(t)}\right) ds, \tag{2.19}$$

as it can be checked directly.

By straightforward computations, with the notation in (1.3), dropping the index $j \equiv 1$, we here have

$$a(t, x) = \lambda(t)^2 x^2, \; b(t, x) = i(\lambda'(t) - \lambda(t)^2)x, \text{ and } c(t, x) = 0.$$

The roots of the principal symbol (1.10), $\lambda_{1,2}(t, x, \xi) = \pm\lambda(t)|x\xi|$, are real-valued, satisfy (1.11), but violate (1.12). It is also immediate to notice that

$$a(t, x, \xi) = \lambda(t)^2 x^2 \xi^2 + i(\lambda'(t) - \lambda(t)^2)x\xi$$

does not fulfill (1.5). However, the coefficients satisfy the estimates (1.9). Indeed, the estimates for c are trivial, while the estimates involving order 0 derivatives and any x-derivatives of a and b are immediate, given their product form. So, only the behavior of the t-derivatives needs to be checked. To this aim, we employ the properties of $\lambda(t)$, which imply, in particular,

$$|\lambda'(t)| = \left|\frac{\lambda'(t)}{\lambda(t)}\lambda(t)\right| \le C_1\frac{\lambda^2(t)}{\Lambda(t)}.$$

Then,

$$\begin{aligned}
|D_t\lambda(t)^2| &= 2|\lambda(t)\,\lambda'(t)| \le 2\lambda(t)^2\frac{\lambda(t)}{\Lambda(t)} \lesssim \lambda(t)^2\left(\frac{\lambda(t)}{\Lambda(t)}\left(\ln\frac{1}{\Lambda(t)}\right)\right),\\
|D_t^2\lambda(t)^2| &= |2(\lambda'(t))^2 + 2\lambda(t)\,\lambda''(t)|\\
&\lesssim \lambda(t)^2\left(\frac{\lambda(t)}{\Lambda(t)}\left(\ln\frac{1}{\Lambda(t)}\right)\right)^2 + \lambda(t)\left|\frac{\lambda'(t)}{\lambda(t)}\right|\,|\lambda'(t)|\\
&\lesssim \lambda(t)^2\left(\frac{\lambda(t)}{\Lambda(t)}\left(\ln\frac{1}{\Lambda(t)}\right)\right)^2 + \lambda(t)\frac{\lambda(t)}{\Lambda(t)}\lambda(t)\frac{\lambda(t)}{\Lambda(t)}\\
&\lesssim \lambda(t)^2\left(\frac{\lambda(t)}{\Lambda(t)}\left(\ln\frac{1}{\Lambda(t)}\right)\right)^2,
\end{aligned}$$

and the estimates for the higher order derivatives follow by induction. This proves the estimates for the coefficient a. The estimates for the coefficient b can be obtained in a completely similar fashion, taking into account that $\left|\dfrac{\ln\lambda(t)}{\Lambda(t)}\right| \gtrsim 1$.

We conclude the analysis of this example observing that, in this case, the solution (2.19) has the same decay of the initial data (as $|x| \to +\infty$), in spite of the fact that the operator has characteristics with variable multiplicites (distinct for $t \in (0, T]$, both collapsing to zero at $t = 0$). This shows that the decay loss phenomenon, which occurs for equation (1.2) under Assumption **(A)** (or, equivalently, Assumption **(H)**) in Proposition 2.3, is strongly related to the behavior of the Hamiltonian flows generated by the characteristic roots, which *transports the smoothness and decay singularities*, encoded by suitable *global wave-front sets*, see [16–18] and Section 5.1 below. For the operator L in (2.18), the flow generated by λ_1 is given, for $x, \xi > 0$, by $(x, \xi) \mapsto (x, \xi)\exp(\Lambda(t) - \Lambda(s))$ (and similar expressions in the other quadrants and for λ_2), which *preserves directions* ∞x_0 (see [17,18]), and then *decay singularities* as well. Results about the *propagation of global singularities* for the Cauchy problems studied in this paper, further extending those in [1,17,18], will appear elsewhere.

3. Generalized pseudodifferential operators and Fourier integral operators of SG-type

Let $a \in S^{m,\mu}_{r_1,r_2,\rho_1,\rho_2}$ be such that $r_j, \rho_j \geq 0$, $j = 1, 2$, $\rho_1 \leq \rho_2$ and $r_1 \geq r_2$. Then, the pseudodifferential operator associated with a, denoted by $\mathrm{Op}(a)$, is a linear and continuous operator on $\mathcal{S}(\mathbb{R}^d)$ defined by the formula

$$(\mathrm{Op}(a)f)(x) = (2\pi)^{-d} \iint e^{i(x-y)\cdot\xi} a(x,\xi) f(y) dy d\xi.$$

Given a SG-pseudodifferential operator A, we will, as customary, denote by $\sigma(A)$ its symbol. We will also denote by $\sigma_p(A)$ a principal part of the symbol of A (often, this will be the leading term of an asymptotic expansion). Explicitly, if $A \in \mathrm{Op}(S^{m,\mu}_{r_1,r_2,\rho_1,\rho_2})$, then $\sigma(A) - \sigma_p(A) \in S^{m-(r_1-r_2),\mu-(\rho_2-\rho_1)}_{r_1,r_2,\rho_1,\rho_2}$. Since in the sequel we will generally have $r_1 > r_2$ and $\rho_2 > \rho_1$, we recover an analog of the usual notion of principal part of a symbol in the (generalized) SG-setting.

The next Lemma 3.1 is the main composition result for the generalized parameter-dependent SG symbols introduced in the previous section. This result follows from the properties of the symbol calculus. Again, we omit the proof, and refer to [12] and [46].

Lemma 3.1. *Let $a \in S^{m_1,\mu_1}\{\kappa_1, \ell_1\}^{\mathrm{hyp}}_N$ and $b \in S^{m_2,\mu_2}\{\kappa_2, \ell_2\}^{\mathrm{hyp}}_N$ be two symbols that are constant in $Z_{\mathrm{pd}}(N)$. Then, the composed operator $\mathrm{Op}(c) = \mathrm{Op}(a)\,\mathrm{Op}(b)$ admits a symbol $c \in S^{m_1+m_2,\mu_1+\mu_2}\{\kappa_1+\kappa_2, \ell_1+\ell_2\}^{\mathrm{hyp}}_N$ which satisfies*

$$c(t,x,\xi) \sim \sum_{\alpha} \frac{i^{|\alpha|}}{\alpha!} D^\alpha_\xi a(t,x,\xi) D^\alpha_x b(t,x,\xi),$$

modulo a regularizing symbol from $C^\infty\Big([0,T], S^{-\infty,-\infty}(\mathbb{R}^d \times \mathbb{R}^d)\Big)$.

Given the composition rule we can now discuss the parametrix of an operator $\mathrm{Op}(a)$. We will denote the parametrix of $\mathrm{Op}(a)$ by $\mathrm{Op}(a)^\sharp$. This means that the following equations hold true, with I the identity operator, modulo $C^\infty\Big([0,T], \mathrm{Op}(S^{-\infty,-\infty})\Big)$:

$$\mathrm{Op}(a)\,\mathrm{Op}(a)^\sharp - I = 0 \text{ and } \mathrm{Op}(a)^\sharp\,\mathrm{Op}(a) - I = 0.$$

It is well known that, when a is an elliptic SG-symbol of order (m,μ), $\mathrm{Op}(a)$ admits a parametrix $\mathrm{Op}(a)^\sharp$ of order $(-m,-\mu)$. A similar statement holds true for operators associated with hypoelliptic symbols, but in this case the parametrix has a different order (see, for instance, Theorem 1.3.6 in [36]). The result extends to matrix-valued symbols, as we show explicitly in the next Lemma 3.2.

Lemma 3.2. *Assume that a is a matrix-valued symbol with entries in $S^{0,0}\{0,0\}^{\mathrm{hyp}}_N$ and that there exists $a' \in S^{0,0}\{0,0\}^{\mathrm{hyp}}_N$, a' constant in $Z_{\mathrm{pd}}(N)$, such that $a - a' \in$*

$L_\infty\Big([0,T],\ S^{-\varepsilon,-\varepsilon}(\mathbb{R}^{2d})\Big)$ *for some* $\varepsilon > 0$. *If* $\mathrm{Op}(a)$ *is elliptic, that is,* $|\det(a(t,x,\xi))| \geq C > 0$ *for all* $(t,x,\xi) \in [0,T] \times \mathbb{R}^{2d}$*, then, there exists a parametrix* $\mathrm{Op}(a)^\sharp = \mathrm{Op}(a^\sharp)$*, such that* $a^\sharp \in S^{0,0}\{0,0\}_N^{\mathrm{hyp}}$*. Moreover,* $a^\sharp - (a')^{-1} \in L_\infty\Big([0,T],\ S^{-\varepsilon,-\varepsilon}(\mathbb{R}^{2d})\Big)$*.*

Proof. We set $a_0^\sharp(t,x,\xi) := (a'(t,x,\xi))^{-1}$. By a standard argument, it turns out that the matrix-valued symbol $a_0^\sharp$ belongs to $S^{0,0}\{0,0\}_N^{\mathrm{hyp}}$. Using Lemma 3.1, we can then define symbols $a_k^\sharp$ by means of the recursive scheme

$$\sum_{|\alpha|=1}^{k} \tfrac{1}{\alpha!}\Big(D_\xi^\alpha a(t,x,\xi)\Big)\Big(\partial_x^\alpha a^\sharp_{k-|\alpha|}(t,x,\xi)\Big) =: -a(t,x,\xi)a_k^\sharp(t,x,\xi).$$

By the hypotheses, $a_k^\sharp(t,x,\xi) \equiv 0$ in $Z_{\mathrm{pd}}(N)$, while the calculus implies $a_k^\sharp \in S^{-k,-k}\{0,0\}_N^{\mathrm{hyp}}$, $k \geq 1$. Employing Lemma 2.10, we obtain a symbol $a_R^\sharp \in S^{0,0}\{0,0\}_N^{\mathrm{hyp}}$ and a right parametrix $\mathrm{Op}(a_R^\sharp)$ such that

$$a_R^\sharp - \sum_{l=0}^{k-1} a_l^\sharp \in S^{-k,-k}\{0,0\}_N^{\mathrm{hyp}}\ ,\ a_R^\sharp(t,x,\xi) = a_0^\sharp(t,x,\xi) \text{ in } Z_{\mathrm{pd}}(N),$$

and

$$\mathrm{Op}(a)\,\mathrm{Op}(a_R^\sharp) - I \in C^\infty([0,T], \mathrm{Op}(S^{-\infty,-\infty})),$$

where I denotes the identity operator. In a completely similar fashion, the existence of a left parametrix $\mathrm{Op}(a_L^\sharp)$ such that $\mathrm{Op}(a_L^\sharp)\,\mathrm{Op}(a) - I \in C^\infty([0,T], \mathrm{Op}(S^{-\infty,-\infty}))$ can be shown. By a standard argument, it follows that $\mathrm{Op}(a_L^\sharp)$ and $\mathrm{Op}(a_R^\sharp)$ coincide modulo $C^\infty([0,T], \mathrm{Op}(S^{-\infty,-\infty}))$. Then, we have proven the existence of the parametrix, defined by

$$\mathrm{Op}(a)^\sharp = \mathrm{Op}(a_L^\sharp),$$

which is uniquely determined modulo $C^\infty([0,T], \mathrm{Op}(S^{-\infty,-\infty}))$. □

3.1. Fourier integral operators of SG*-type*

We review and extend notions about the class of SG-Fourier integral operators in the global setting on $\mathbb{R}^d$, by first defining a parameter-dependent admissible class of phase functions in the SG context.

Definition 3.3. A real-valued function $\varphi \in C^\infty([0,T]; S^{1,1}(\mathbb{R}^{2d}))$ is called a (smooth family of) *simple phase* if

$$\langle \nabla_\xi \varphi(t,x,\xi)\rangle \asymp \langle x \rangle \quad \text{and} \quad \langle \nabla_x \varphi(t,x,\xi)\rangle \asymp \langle \xi \rangle$$

are fulfilled, uniformly with respect to x, $\xi \in \mathbb{R}^{2d}$ and $t \in [0, T]$. Moreover, the (smooth family of) simple phase function is called *regular* if

$$|\det(\nabla_\xi \nabla_x \varphi(t, x, \xi))| \geq c,$$

for some $c > 0$ independent of t, x and ξ.

In [23], it was discussed that the regular phase function φ defines two globally invertible (families of) mappings, namely, $\xi \to \varphi'_x(t, x, \xi)$ and $x \to \varphi'_\xi(t, x, \xi)$. Then, the mappings generated by the first derivatives of the admissible regular phase functions give rise to SG-diffeomorphism with $S^{0,0}$ parameter-dependence.

Definition 3.4. The generalized Fourier integral operator $\mathrm{Op}_{\varphi(t)}(a(t))$ of SG type I, with phase φ and amplitude a, is a linear operator given by

$$[\mathrm{Op}_{\varphi(t)}(a(t))u](t, x) = (2\pi)^{-d} \iint e^{i(\varphi(t,x,\xi) - y\cdot\xi)} a(t, x, \xi) u(y) dy d\xi,$$

and the generalized FIO $\mathrm{Op}^*_{\varphi(t)}(b(t))$ of SG type II, with phase φ and amplitude b, is a linear operator given by

$$[\mathrm{Op}^*_{\varphi(t)}(b(t))u](t, x) = (2\pi)^{-d} \iint e^{i(x\cdot\xi - \varphi(t,y,\xi))} \overline{b(t, y, \xi)} u(y) dy d\xi.$$

Suppose a, b and φ are given as in Definition 3.4. Then, the parameter-dependent operators $\mathrm{Op}_{\varphi(t)}(a(t))$ and $\mathrm{Op}^*_{\varphi(t)}(b(t)$ are linear and continuous on $\mathcal{S}(\mathbb{R}^d)$, and uniquely extendable to linear and continuous operators on $\mathcal{S}'(\mathbb{R}^d)$.

We state the next Theorem 3.5 about composition of a Fourier integral operator with a pseudodifferential operator; for the sake of brevity we omit the proof that follows the approaches of [23,44].

Theorem 3.5. *Let $\varphi \in C^\infty([0, T]; S^{1,1})$ be a smooth family of simple and regular phase functions, and let $b \in S^{m_1,\mu_1}\{\kappa_1, \ell_1\}^{\mathrm{hyp}}_N$ with $\mathrm{supp}(b) \subset Z_{\mathrm{hyp}}(N)$. Let $p \in S^{m_2,\mu_2}\{\kappa_2, \ell_2\}^{\mathrm{hyp}}_N$ with $\mathrm{supp}(p) \subset Z_{\mathrm{hyp}}(N)$. Then, the composition*

$$\mathrm{Op}_{\varphi(t)}(c(t)) = \mathrm{Op}(p(t))\, \mathrm{Op}_{\varphi(t)}(b(t))$$

is a smooth family of Fourier integral operators with amplitude

$$c(t, x, \xi) \in S^{m_1+m_2,\mu_1+\mu_2}\{\kappa_1 + \kappa_2, \ell_1 + \ell_2\}^{\mathrm{hyp}}_N,$$

supported in $Z_{\mathrm{hyp}}(N)$ with a suitable choice of N. Moreover, we have the asymptotic expansion

$$c(t,x,\xi) \sim \sum_\alpha \frac{i^{|\alpha|}}{\alpha!}(D_\xi^\alpha p)(t,x,\nabla_x\varphi(t,x,\xi))D_y^\alpha[e^{i\Phi(t,x,y,\xi)}b(t,y,\xi)]_{y=x},$$

where $\Phi(t,x,y,\xi) = \varphi(t,y,\xi) - \varphi(t,x,\xi) + (x-y)\cdot\nabla_x\varphi(t,x,\xi)$.

We conclude the section by recalling some basic facts concerning the Sobolev-Kato spaces, as well as the boundedness properties of the operators we treated above on such scale of spaces. We refer to [12–14,20] for the basic tools employed in the proofs and leave the details for the reader.

Proposition 3.6. *The following properties of the weighted Sobolev spaces* $H^{s,\sigma}(\mathbb{R}^d)$ *hold true.*

(1) $L^2(\mathbb{R}^d) = H^{0,0}(\mathbb{R}^d)$.
(2) *If* $s_1 \le s_2$ *and* $\sigma_1 \le \sigma_2$, *then* $H^{s_2,\sigma_2}(\mathbb{R}^d) \hookrightarrow H^{s_1,\sigma_1}(\mathbb{R}^d)$. *Moreover, if* $s_1 < s_2$ *and* $\sigma_1 < \sigma_2$, *then the embedding* $H^{s_2,\sigma_2}(\mathbb{R}^d) \hookrightarrow H^{s_1,\sigma_1}(\mathbb{R}^d)$ *is compact.*
(3) $\cap_{s,\sigma\in\mathbb{R}} H^{s,\sigma}(\mathbb{R}^d) = \mathscr{S}(\mathbb{R}^d)$ and $\cup_{s,\sigma\in\mathbb{R}} H^{s,\sigma}(\mathbb{R}^d) = \mathscr{S}'(\mathbb{R}^d)$.
(4) *The operator* $\mathrm{Op}(\omega_{t,\tau})$, $\omega_{t,\tau}(x,\xi) = \langle x\rangle^t\langle\xi\rangle^\tau$, *is a continuous, invertible, linear operator from* $H^{s,\sigma}(\mathbb{R}^d)$ *to the space* $H^{s-t,\sigma-\tau}(\mathbb{R}^d)$. *In particular,* $u \in H^{s,\sigma}(\mathbb{R}^d)$ *if and only if* $\mathrm{Op}(\omega_{s,\sigma})u \in L^2(\mathbb{R}^d)$.

Theorem 3.7. *Let the real-valued phase functions* $\varphi(t)$ *be simple and regular. Let* $a(t) \in S^{0,0}\{0,0\}_N^{\mathrm{hyp}}$. *Then, the parameter-dependent operator* $\mathrm{Op}_{\varphi(t))}(a(t))$ *is* L^2*-bounded, and satisfies, for some constant* $C > 0$,

$$||\mathrm{Op}_{\varphi(t)}(a(t))||_{L^2\to L^2} \le C \sup_{|\alpha|,|\beta|\le 2d+1} ||\partial_y^\alpha\partial_\xi^\beta a(t,y,\xi)||_{L^\infty}.$$

Moreover, $\mathrm{Op}_{\varphi(t)}(a(t))$ *is a bounded linear operator from* $H^{s,\sigma}(\mathbb{R}^d)$ *to* $H^{s,\sigma}(\mathbb{R}^d)$ *for all* $s,\sigma \in \mathbb{R}$.

The proof of Theorem 3.7 follows with a minor modification of the proof of [44, Theorem 3.3], see also [13].

Remark 3.8. As above, we remark that the results in this section extend to operators defined by means of amplitude hierarchies associated with the subdivision into zones, and based on the more general classes $S^{m,\mu}_{r_1,r_2,\rho_1,\rho_2}$, $r_j,\rho_j \ge 0$, $j = 1,2$, $r_2,\rho_1 < 1$, and, in particular, on the classes $S^{m,\mu}_{(\varepsilon)}$, $\varepsilon \in (0,1)$. We will tacitly make use of such variant results whenever they will be needed.

4. Diagonalization procedure

The so-called perfect diagonalization [31,46] is a procedure to switch from (1.2) to a 2×2 system, and to suitably "decouple the equations" of the system, so that each of them can be solved by means of Fourier integral operators separately. It consists of various steps, taking into account the subdivision into zones of the phase space. Similar to [31], the first step is carried out in all the zones, the second step is carried out in the hyperbolic zone $Z_{\text{hyp}}(N)$ (that is, away from $t=0$), and the third step allows a refinement in the regular zone $Z_{\text{reg}}(N)$. This can be achieved by means of the smoothness properties of the symbol class $S^{m,\mu}\{\kappa,\ell\}_N^{\text{reg}}$. The diagonalization produces then equations which are equivalent to (1.2), modulo rapidly decreasing/smoothing elements. Since we are interested in the smoothness and decay properties of the solutions, such terms do not affect the claims, so we will often ignore/avoid writing them, and consider equalities modulo such remainders.

Let us denote by $\rho(t,x,\xi)$ the positive root of the equation

$$\rho(t,x,\xi)^2 = 1 + \frac{\lambda(t)^2}{\Lambda(t)}\langle x\rangle\langle\xi\rangle \ln(\langle x\rangle\langle\xi\rangle).$$

The function $\rho(t,x,\xi)$ is monotonic with respect to $t\in[0,T]$, since $c_1 > 1/2$ in (2.1). In the next Lemma 4.1 further properties of $\rho(t,x,\xi)$ are proved.

Lemma 4.1. *The function $\rho(t,x,\xi)$ has the following properties:*

- $\rho \in S^{1,1}\{1,0\}_N^{\text{hyp}}$, *for some $N>0$;*
- $\rho \in C\big([0,T], S^{\frac12+\varepsilon,\frac12+\varepsilon}\big)$, *for every $\varepsilon>0$;*
- $\partial_t^j\rho \in C\big((0,T], S^{j+\varepsilon,j+\varepsilon}\big)$, *for every $\varepsilon>0$ and $j\ge 1$.*

In particular, for any $(t,x,\xi)\in Z_{\text{pd}}(N)$, it holds

$$|\partial_t^j D_x^\alpha D_\xi^\beta \rho(t,x,\xi)| \lesssim \frac{\lambda(t)}{\sqrt{\Lambda(t)}}\left(\frac{1}{\sqrt{\Lambda(t)}}\right)^j \langle x\rangle^{\frac12+\varepsilon-|\alpha|}\langle\xi\rangle^{\frac12+\varepsilon-|\beta|},$$

for any $\alpha,\beta\in\mathbb{N}^d$ such that $|\alpha|+|\beta|\ge 1$.

Proof. The claims follow by a modification of the argument in [46, Lemma 2.1.27]. In particular, it is straightforward to prove the inequality

$$|\partial_x^\alpha\partial_\xi^\beta\partial_t^j\rho(t,x,\xi)| \le C_{\alpha\beta}\langle x\rangle^{-|\alpha|}\langle\xi\rangle^{-|\beta|}|\partial_t^j\rho(t,x,\xi)|, \quad j=0,1. \tag{4.1}$$

Indeed, for small $t\in[0,T]$, we recall that $\frac{\lambda^2(t)}{\Lambda(t)}\in C^\infty([0,T])$. On the other hand, for $t\in(0,T]$ we can apply condition (2.1). The remaining details are left for the reader. □

For a given $N > 0$, we define the symbol

$$\begin{aligned} h(t,x,\xi) &= \rho(t,x,\xi)\,\chi\left(\frac{\Lambda(t)\langle x\rangle\langle\xi\rangle}{N\ln(\langle x\rangle\langle\xi\rangle)}\right) \\ &\quad + \lambda(t)\langle x\rangle\langle\xi\rangle\left[1-\chi\left(\frac{\Lambda(t)\langle x\rangle\langle\xi\rangle}{N\ln(\langle x\rangle\langle\xi\rangle)}\right)\right], \end{aligned} \tag{4.2}$$

where $\chi \in C_0^\infty(\mathbb{R})$ is a cutoff function such that $\chi(\eta) \equiv 1$ for $|\eta| \le 1$, $\chi(\eta) \equiv 0$ for $|\eta| \ge 2$, and $0 \le \chi(\eta) \le 1$.

Lemma 4.2. *The parameter-dependent symbol h in (4.2) has the following properties:*

- $\partial_t^j h \in C([0,T], S^{1+j,1+j})$;
- $h(t,x,\xi) \in S^{1,1}\{1,0\}_N^{\mathrm{hyp}}$, *for some* $N > 0$.

Moreover, there exist constants $c, C > 0$ such that, for all $(t,x,\xi) \in [0,T] \times \mathbb{R}^{2d}$, it holds

$$\max\{c, \lambda(t)\langle x\rangle\langle\xi\rangle\} \le\; h(t,x,\xi) \le C\langle x\rangle\langle\xi\rangle, \tag{4.3}$$

and, for all α and $\beta \in \mathbb{N}^d$,

$$|\partial_x^\alpha \partial_\xi^\beta h(t,x,\xi)| \le C_\alpha \langle x\rangle^{-|\alpha|}\langle\xi\rangle^{-|\beta|} h(t,x,\xi).$$

Finally, for all $(t,x,\xi) \in Z_{\mathrm{pd}}(2N)$ and $\alpha, \beta \in \mathbb{N}^d$, we have

$$|\partial_t \partial_x^\alpha \partial_\xi^\beta h(t,x,\xi)| \le C_\alpha \langle x\rangle^{-|\alpha|}\langle\xi\rangle^{-|\beta|}|\partial_t \rho(t,x,\xi)|.$$

Remark 4.3. Observe that, in view of Definition 2.5, taking into account (4.3), it turns out that the parameter-dependent symbol h is globally hypoelliptic. In particular, in the hyperbolic zone it is actually an elliptic symbol.

Setting $U(t) = (U_1(t), U_2(t))^T = (\mathrm{Op}(h(t))u(t), D_t u(t))^T$, the Cauchy problem (1.2) is equivalent, modulo $C^1([0,T], \mathscr{S}(\mathbb{R}^d))$, to the Cauchy problem

$$\begin{cases} D_t U(t) - K(t)U(t) = G(t) \\ U(0) = U_0, \end{cases} \tag{4.4}$$

where

$$G(t,x) = (0, g(t,x))^T, \quad U_0(x) = ([\mathrm{Op}(h(0))\varphi](x), -i\psi(x))^T,$$

$K(t) = A(t) - (D_t H)(t)\, H(t)^\sharp$, and the matrix-valued operators $H(t)$ and $A(t)$ are

$$A(t) := \begin{pmatrix} 0 & \mathrm{Op}(h(t)) \\ \mathrm{Op}(a(t))\,\mathrm{Op}(h(t))^\sharp & 0 \end{pmatrix} \text{ and } H(t) := \begin{pmatrix} \mathrm{Op}(h(t)) & 0 \\ 0 & 1 \end{pmatrix}.$$

As above, $P^\sharp$ denotes the parametrix of the operator P. Indeed $H(t)^\sharp$ exists, since $\det \sigma(H(t)) = h(t)$, and $h(t)$ is hypoelliptic. In the next Lemma 4.4 we deduce the properties of the matrix $K(t)$ of the coefficients of the system in (4.4).

Lemma 4.4. *The matrix-valued parameter-dependent symbol $\sigma(K)$ belongs to $S^{\mathrm{pd}}_{2N} \cap S^{1,1}\{1,0\}^{\mathrm{hyp}}_N$.*

Proof. On the one hand, as a consequence of Lemma 4.1 and 4.2, it follows that $D_t H \in S^{1,1}\{1,0\}^{\mathrm{hyp}}_N$. Then, by the calculus, $(D_t H)H^\sharp \in S^{1,1}\{1,0\}^{\mathrm{hyp}}_N$, since $H^\sharp \in S^{0,0}\{0,0\}^{\mathrm{hyp}}_N$. On the other hand, the assumptions (1.9) on the coefficients a_j, b_j, $j = 1,\dots,d$, and c, guarantee that $\mathrm{Op}(a)\,\mathrm{Op}(h)^\sharp \in S^{1,1}\{1,0\}^{\mathrm{hyp}}_N \cap S^{\mathrm{pd}}_{2N}$, as a consequence of (4.3) (see also Remark 2.11). Since $H^\sharp \in S^{-1,-1}\{-1,0\}^{\mathrm{hyp}}_N$ and it is uniformly bounded in $Z_{\mathrm{pd}}(2N)$, we conclude that $\mathrm{Op}(a)\,\mathrm{Op}(h)^\sharp \in S^{1,1}\{1,0\}^{\mathrm{hyp}}_N \cap S^{\mathrm{pd}}_{2N}$. □

Let us now define

$$\mathfrak{t}_j(t,x,\xi) = d_j \rho(t,x,\xi)\chi\left(\frac{\Lambda(t)\langle x\rangle\langle\xi\rangle}{N\ln(\langle x\rangle\langle\xi\rangle)}\right) + \tau_j(t,x,\xi)\left(1-\chi\left(\frac{\Lambda(t)\langle x\rangle\langle\xi\rangle}{N\ln(\langle x\rangle\langle\xi\rangle)}\right)\right),$$

where $\tau_j(t,x,\xi) = d_j\sqrt{a(t,x,\xi)}$, $j = 1,2$, with $d_2 = -d_1 = 1$, are the roots of the complete symbol (2.5) of L. The symbol $\mathfrak{t}_j(t,x,\xi)$, $j = 1,2$, are introduced to allow the construction of the fundamental solution close to $t = 0$, due to jump in the multiplicity of the characteristic roots there.

Lemma 4.5. *The functions $\mathfrak{t}_1$ and $\mathfrak{t}_2$ satisfy the following properties:*

a) $\mathfrak{t}_k \in S^{\mathrm{pd}}_{2N} \cap S^{1,1}\{1,0\}^{\mathrm{hyp}}_N$, $k = 1,2$, *with* $\mathfrak{t}_2 - \mathfrak{t}_1 = 2\mathfrak{t}_2$. *In particular, for any* $(t,x,\xi) \in Z_{\mathrm{pd}}(N)$ *it holds*

$$|D^\alpha_x D^\beta_\xi \mathfrak{t}_k(t,x,\xi)| \lesssim \frac{\lambda(t)}{\sqrt{\Lambda(t)}}\langle x\rangle^{\frac12+\varepsilon-|\alpha|}\langle\xi\rangle^{\frac12+\varepsilon-|\beta|}, \quad k = 1,2,$$

for any $\alpha,\beta \in \mathbb{N}^n$ *such that* $|\alpha|+|\beta| \ge 1$;

b) $\partial^j_t \mathfrak{t}_k \in L_\infty([0,T], S^{1+j,1+j})$, $k = 1,2$, *for all* $j = 0,1,\dots$;

c) $\partial^j_t (\mathfrak{t}_k/h) \in L_\infty([0,T], S^{1+j,1+j})$, $k = 1,2$, *for all* $j = 0,1,\dots$;

d) $|D^\beta_x D^\alpha_\xi D_t \frac{\mathfrak{t}_k(t,x,\xi)}{\rho(t,x,\xi)}| \le C_{\alpha\beta}\langle\xi\rangle^{-|\alpha|}\langle x\rangle^{-|\beta|}\frac{|\partial_t\rho(t,x,\xi)|}{\rho(t,x,\xi)}$, $k = 1,2$, for $(t,x,\xi) \in Z_{\mathrm{pd}}(2N)$.

To begin the diagonalization of the system in (4.4), we introduce the matrix-valued parameter-dependent operator

$$M = \begin{pmatrix} I & I \\ \operatorname{Op}(\mathfrak{t}_1)\operatorname{Op}(h)^\sharp & \operatorname{Op}(\mathfrak{t}_2)\operatorname{Op}(h)^\sharp \end{pmatrix}, \tag{4.5}$$

whose principal symbol is

$$m_0 = \sigma_p(M) = \begin{pmatrix} 1 & 1 \\ \frac{\mathfrak{t}_1}{h} & \frac{\mathfrak{t}_2}{h} \end{pmatrix}.$$

Since $\det(m_0) = \frac{\mathfrak{t}_2 - \mathfrak{t}_1}{h} = \frac{2\mathfrak{t}_2}{h} \gtrsim 1$, for $(t, x, \xi) \in [0, T] \times \mathbb{R}^{2d}$, the operator M is elliptic. By Lemma 4.4, M then admits a parametrix $M^\sharp$. Note that, by their definition, the symbols of $M, M^\sharp$ are constant in $Z_{\mathrm{pd}}(N)$, while their entries belong to $S^{0,0}\{0,0\}^{\mathrm{hyp}}_N$.

We look for the fundamental solution $E = E(t, s)$ to (4.4), that is, an operator family $E(t, s)$ satisfying

$$D_t E(t, s) - K(t)E(t, s) = 0\ ,\ E(s, s) = I. \tag{4.6}$$

In view of the properties of the calculus of the generalized parameter-dependent SG-operators, established in the previous sections, we can adapt the arguments in [31,46]. We first set $E_0(t, s) = M^\sharp(t)E(t, s)$, which then satisfies

$$\begin{aligned} D_t E_0 &= M^\sharp(A + (D_t H)H^\sharp)E + (D_t M^\sharp)E \\ &= (M^\sharp AM)E_0 + (D_t M^\sharp + M^\sharp(D_t H)H^\sharp)ME_0 + \hat{R}_1 E \\ &= \mathcal{D}E_0 + \mathcal{B}_1 E_0 + \hat{R}_1 E, \end{aligned}$$

where

$$\mathcal{D} = \operatorname{Op}(\sigma_p(M^\sharp AM)),\ \mathcal{B}_1 = (M^\sharp AM - \mathcal{D}) + (D_t M^\sharp + M^\sharp(D_t H)H^\sharp)M,$$

and $\hat{R}_1 \in C^\infty\Big([0, T], \operatorname{Op}(S^{-\infty,-\infty})\Big)$ is a smooth family of regularizing operators.

We first examine $\mathcal{D}$. By Lemmas 3.1 and 3.2, looking at the top order terms of the asymptotic expansions of the involved compositions and parametrix, we see that

$$\sigma(\mathcal{D}) = \sigma_p(M^\sharp AM) = \begin{cases} \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix} & \text{in } \ Z_{\mathrm{hyp}}(2N), \\ \begin{pmatrix} \frac{\mathfrak{t}_1^2 + \tau_1^2}{2\mathfrak{t}_1} & \frac{\mathfrak{t}_2^2 - \tau_2^2}{2\mathfrak{t}_2} \\ \frac{\mathfrak{t}_1^2 - \tau_1^2}{2\mathfrak{t}_1} & \frac{\mathfrak{t}_2^2 + \tau_2^2}{2\mathfrak{t}_2} \end{pmatrix} & \text{in } Z_{\mathrm{pd}}(2N). \end{cases} \tag{4.7}$$

Then,

$$\sigma(\mathcal{D}) = \begin{pmatrix} \mathfrak{t}_1 & 0 \\ 0 & \mathfrak{t}_2 \end{pmatrix} + \sigma(Q),$$

where $\sigma(Q) \in S^{\rm pd}_{2N}$ and $\sigma(Q) \equiv 0$ in $Z_{\rm hyp}(2N)$. Again by the definition of M and the properties of the calculus, we can write

$$\sigma(M^\sharp AM) = \sigma(\mathcal{D}) + q_0 + r_0,$$

where $q_0 \equiv 0$ in $Z_{\rm pd}(2N)$, $q_0 \in S^{\rm pd}_N \cap S^{0,0}\{0,0\}^{\rm hyp}_N$ and $r_0 \in C^\infty([0,T],\, S^{-\infty,-\infty})$.

We now consider $\mathcal{B}_1$. Using the equalities

$$M^\sharp M = I \Rightarrow (D_t M^\sharp)M = -M^\sharp(D_t M),$$

which hold modulo smooth families of regularizing operators, we find

$$\begin{aligned}\mathcal{B}_1 &= (M^\sharp AM - \mathcal{D}) + (D_t M^\sharp + M^\sharp(D_t H)H^\sharp)M \\ &= (M^\sharp AM - \mathcal{D}) + (-M^\sharp D_t M + M^\sharp(D_t H)H^\sharp M) + \tilde{R}_1\end{aligned}$$

with $\tilde{R}_1 \in C^\infty([0,T]; \mathrm{Op}(S^{-\infty-\infty}))$. Again by the properties of the calculus, we can write

$$\begin{aligned}&\sigma(-M^\sharp D_t M + M^\sharp(D_t H)H^\sharp M) \\ &\qquad = \sigma_p(-M^\sharp D_t M) + \sigma_p(M^\sharp(D_t H)H^\sharp M) + q_1 + r_1,\end{aligned}$$

where $q_1 \equiv 0$ in $Z_{\rm pd}(N)$, $q_1 \in S^{\rm pd}_N \cap S^{-1,-1}\{0,1\}^{\rm hyp}_N$, and $r_1 \in C^\infty([0,T],\, S^{-\infty,-\infty})$. Looking at the top terms of the asymptotic expansions, by direct calculation we have

$$\sigma_p(M^\sharp(D_t H)H^\sharp M) = \begin{pmatrix} \frac{D_t h}{2h} & \frac{D_t h}{2h} \\ \frac{D_t h}{2h} & \frac{D_t h}{2h} \end{pmatrix}$$

and

$$\sigma_p(M^\sharp D_t M) = \frac{h}{2\mathfrak{t}_2}\begin{pmatrix} -D_t(\frac{\mathfrak{t}_1}{h}) & -D_t(\frac{\mathfrak{t}_2}{h}) \\ D_t(\frac{\mathfrak{t}_1}{h}) & D_t(\frac{\mathfrak{t}_2}{h}) \end{pmatrix}.$$

Summing up, it follows that there exist $q \in S^{\rm pd}_N \cap S^{0,0}\{0,0\}^{\rm hyp}_N$ and $r \in C^\infty([0,T],$ $S^{-\infty,-\infty})$ such that

$$\sigma(\mathcal{B}_1) = b_1 = \begin{pmatrix} \frac{D_t \mathfrak{t}_2}{2\mathfrak{t}_2} & -\frac{D_t \mathfrak{t}_2}{2\mathfrak{t}_2} + \frac{D_t h}{h} \\ \frac{D_t \mathfrak{t}_2}{2\mathfrak{t}_2} + \frac{D_t h}{h} & \frac{D_t \mathfrak{t}_2}{2\mathfrak{t}_2} \end{pmatrix} + q + r.$$

This shows $b_1 \in S^{0,0}\{0,1\}^{\rm hyp}_N$. Moreover, we can estimate

$$|D^\beta_x D^\alpha_\xi b_1(t,x,\xi)| \le C_{\alpha\beta}\langle x\rangle^{-|\beta|}\langle \xi\rangle^{-|\alpha|}\left(\rho(t,x,\xi) + \frac{\partial_t \rho(t,x,\xi)}{\rho(t,x,\xi)}\right), \tag{4.8}$$

for all $(t, x, \xi) \in Z_{\mathrm{pd}}(2N)$. Indeed, (4.8) follows by Lemma 4.1 and (4.1), noticing that

$$|D_x^\alpha D_\xi^\beta a(t, x, \xi)| \le C_{\alpha\,\beta} \langle x\rangle^{-|\alpha|} \langle \xi\rangle^{-|\beta|} \rho^2(t, x, \xi),$$

and

$$|D_x^\alpha D_\xi^\beta \mathfrak{t}_j(t, x, \xi)| \le C_{\alpha\,\beta} \langle x\rangle^{-|\alpha|} \langle \xi\rangle^{-|\beta|} \rho(t, x, \xi), \quad j = 1, 2,$$

for all $(t, x, \xi) \in Z_{\mathrm{pd}}(2N)$, as a consequence of (1.9).

Summarizing, we proved that the fundamental solution $E(t, s)$ satisfying (4.6) can be represented, modulo smooth families of regularizing operators, in the form $E(t, s) = M(t)E_0(t, s)M^\sharp(s)$, where M is the matrix-valued elliptic operator (4.5), with $\sigma(M) \in S^{0,0}\{0, 0\}_N^{\mathrm{hyp}}$, and $E_0 = E_0(t, s)$ solves

$$D_t E_0 - \mathcal{D} E_0 + \mathcal{B}_1 E_0 + R_1 E_0 = 0, E_0(s, s) = I,$$

with $\mathcal{D}$ the matrix-valued diagonal pseudodifferential operator with symbol given in (4.7), $\mathcal{B}_1$ a matrix-valued pseudodifferential operator with symbol $b_1 \in S_N^{\mathrm{pd}} \cap S^{0,0}\{0, 1\}_N^{\mathrm{hyp}}$, satisfying (4.8) for all $(t, x, \xi) \in Z_{\mathrm{pd}}(2N)$, and R_1 a smooth family of regularizing operator with matrix-valued symbol in $C^\infty([0, T], S^{-\infty,-\infty})$.

Our next goal is to diagonalize $\mathcal{B}_1$ modulo $S^{-1,-1}\{-1, 2\}_N^{\mathrm{hyp}}$. Since we are looking for the fundamental solution modulo regularizing terms, from now on we will no longer indicate them explicitly in the computations.

Proposition 4.6. *There exist an elliptic, matrix-valued pseudodifferential operator N_1 with $\sigma(N_1) \in S^{0,0}\{0, 0\}_N^{\mathrm{hyp}}$, $\sigma(N_1) \equiv I$ in $Z_{\mathrm{pd}}(N)$, a diagonal matrix-valued pseudodifferential operator $\mathcal{D}_1$ with $\sigma(\mathcal{D}_1) \in S_{2N}^{\mathrm{pd}} \cap S^{0,0}\{0, 1\}_{2N}^{\mathrm{hyp}}$, and a matrix-valued pseudodifferential operator $\mathcal{B}_2$ with $\sigma(\mathcal{B}_2) \in S_{2N}^{\mathrm{pd}} \cap S^{-1,-1}\{-1, 2\}_{2N}^{\mathrm{hyp}}$, such that*

$$(D_t - \mathcal{D} + \mathcal{B}_1)N_1 = N_1(D_t - \mathcal{D} + \mathcal{D}_1 + \mathcal{B}_2)$$

holds, modulo a smooth family of regularizing operators with symbol in $C^\infty([0, T], S^{-\infty,-\infty})$; in particular, $\sigma(\mathcal{B}_2)$ satisfies estimate (4.8) in $Z_{\mathrm{pd}}(2N)$.

Proof. The proof is achieved by an argument similar to those in [14,31,46], relying on the properties of the parameter-dependent calculus we are using. For the sake of completeness, we here provide some details. Let $\mathcal{D}_1$ and $N^{(1)}$ be the pseudodifferential operators with matrix-valued symbols

$$\sigma(\mathcal{D}_1)(t, x, \xi) = \left(1 - \chi\left(\frac{\Lambda(t)\langle x\rangle\langle \xi\rangle}{N \ln(\langle x\rangle\langle \xi\rangle)}\right)\right) \begin{pmatrix} b_{1,11}(t, x, \xi) & 0 \\ 0 & b_{1,22}(t, x, \xi) \end{pmatrix},$$

and, respectively,

$$\sigma(N^{(1)})(t,x,\xi) = \left(1 - \chi\left(\frac{\Lambda(t)\langle x\rangle\langle\xi\rangle}{N\ln(\langle x\rangle\langle\xi\rangle)}\right)\right)\begin{pmatrix} 0 & \frac{b_{1,12}(t,x,\xi)}{\mathrm{t}_1-\mathrm{t}_2} \\ \frac{b_{1,21}(t,x,\xi)}{\mathrm{t}_2-\mathrm{t}_1} & 0 \end{pmatrix},$$

where there appear the entries of the matrix-valued symbol

$$b_1 = \sigma(\mathcal{B}_1) = \begin{pmatrix} b_{1,11} & b_{1,12} \\ b_{1,21} & b_{1,22} \end{pmatrix}.$$

Our considerations above show that $\sigma(\mathcal{D}_1) \in S^{0,0}\{0,1\}^{\mathrm{hyp}}_N$, and also $\sigma(N^{(1)}) \in S^{-1,-1}\{-1,1\}^{\mathrm{hyp}}_N$. Moreover, by their definition,

$$\sigma(\mathcal{D}_1)(t,x,\xi) = \sigma(N^{(1)})(t,x,\xi) = 0 \quad \text{for } (t,x,\xi) \in Z_{\mathrm{pd}}(N).$$

Let $N_1 = I + N^{(1)}$. Then, setting

$$\begin{aligned}\mathcal{B}_2 &= (D_t - \mathcal{D} + \mathcal{B}_1)(I + N^{(1)}) - (I + N^{(1)})(D_t - \mathcal{D} + \mathcal{D}_1) \\ &= D_t N^{(1)} + [N^{(1)}, \mathcal{D}] + \mathcal{B}_1 N^{(1)} - N^{(1)}\mathcal{D}_1 + \mathcal{B}_1 - \mathcal{D}_1,\end{aligned}$$

it follows

$$(D_t - \mathcal{D} + \mathcal{B}_1)N_1 = N_1(D_t - \mathcal{D} + \mathcal{D}_1 + \mathcal{B}_2).$$

By a direct calculation, using Lemma 3.1, we find

$$\sigma_p(\mathcal{B}_1(1-\chi) - \mathcal{D}_1 + [N^{(1)}, \mathcal{D}]) \equiv 0;$$

which implies

$$\sigma_p(\mathcal{B}_2) = \sigma_p(\tilde{\mathcal{B}}_2 + \chi\mathcal{B}_1 N_1),$$

where

$$\tilde{\mathcal{B}}_2 := D_t N^{(1)} + (1-\chi)\mathcal{B}_1 N^{(1)} - N^{(1)}\mathcal{D}_1.$$

In particular, $\sigma(\tilde{\mathcal{B}}_2) \equiv 0$ in $Z_{\mathrm{pd}}(N)$ and both the symbols

$$\sigma(D_t N^{(1)}) \text{ and } \sigma(\mathcal{B}N^{(1)} - N^{(1)}\mathcal{D}_1)$$

belong to $S^{\mathrm{pd}}_{2N} \cap S^{-1,-1}\{-1,2\}^{\mathrm{hyp}}_N$, as a consequence of (4.8).

On the other hand, $\sigma_p(\chi\mathcal{B}_1 N_1) \equiv 0$ in $Z_{\mathrm{hyp}}(N)$ and it belongs to S^{pd}_{2N} too. That is, $\mathcal{B}_2$ satisfies the desired properties.

Now, let us show that, for a sufficiently large N, the pseudodifferential operator N_1 is a elliptic, with symbol belonging to $S^{0,0}\{0,0\}^{\mathrm{hyp}}_N$. By its definition, $\sigma(N_1) \equiv 1$ in $Z_{\mathrm{pd}}(N)$ and $n_1(t,x,\xi) = \sigma(N_1)(t,x,\xi) \in S^{-1,-1}\{-1,1\}^{\mathrm{hyp}}_N$. Moreover,

$$
\begin{aligned}
|\sigma_p(N_1)(t,x,\xi)| &\leq \frac{C}{\langle x\rangle\langle\xi\rangle\lambda(t)}\left(\frac{\lambda(t)}{\Lambda(t)}\left(\frac{1}{\ln\Lambda(t)}\right)\right)\\
&\leq \frac{C(\ln(\langle x\rangle\langle\xi\rangle))}{\langle x\rangle\langle\xi\rangle\Lambda(t)}\\
&\leq \frac{C}{N} \quad \text{in } Z_{\text{hyp}}(N).
\end{aligned}
$$

Consequently, a large N yields $|\sigma_p(N_1)(t,x,\xi)| \geq \frac{1}{2}$ in $[0,T]\times\mathbb{R}^{2d}$, using $\sigma(N_1)\equiv I$ in $Z_{\text{pd}}(N)$. This gives, together with Lemma 3.2, the existence of a parametrix $N_1^\sharp$ with $\sigma(N_1^\sharp)\in S^{0,0}\{0,0\}_N^{\text{hyp}}$. $\square$

We conclude the section with a further steps of the diagonalization, localized in $Z_{\text{reg}}(N)\subset Z_{\text{hyp}}(N)$. Observe that, for a symbol $p\in S^{-p,-p}\{-p,p+1\}_N^{\text{hyp}}$, we can estimate

$$
\begin{aligned}
\int_{t_{x,\xi}}^{t}\Big|p(\tau,x,\xi)\Big|d\tau &\leq \int_{t_{x,\xi}}^{t}\tfrac{C_p}{\langle x\rangle^p\langle\xi\rangle^p\lambda(\tau)^p}\left(\tfrac{\lambda(\tau)}{\Lambda(\tau)}\ln\tfrac{1}{\Lambda(\tau)}\right)^{p+1}d\tau\\
&\leq \frac{C_p(\ln(\langle x\rangle\langle\xi\rangle))^{(p+1)}}{(\langle x\rangle\langle\xi\rangle\Lambda(t'_{x,\xi}))^p}\\
&= \frac{C_p}{(2N)^p}(\ln(\langle x\rangle\langle\xi\rangle))^{(p+1)-2p}
\end{aligned}
$$

where $t_{x,\xi}$ is defined as in (2.2). In the oscillations subzone, corresponding to $\mathcal{D}+\mathcal{B}_1$, we achieved remainder of the type $S^{0,0}\{0,0\}_{2N}^{\text{hyp}}+S^{-1,-1}\{-1,2\}_{2N}^{\text{hyp}}$. We can actually improve the diagonalization scheme modulo operators with symbols from

$$
\begin{aligned}
\mathcal{HG}_N = {} & S_{2N}^{\text{pd}}\\
&\cap\Big(S^{0,0}\{0,0\}_{2N}^{\text{hyp}}+S^{-1,-1}\{-1,2\}_{2N}^{\text{hyp}}\Big)\\
&\cap\Big(\bigcap_{p\geq 0}S^{-p,-p}\{-p,p+1\}_N^{\text{reg}}\Big).
\end{aligned}
\tag{4.9}
$$

Theorem 4.7. *There exist matrix-valued pseudodifferential operators N_2, $\mathcal{D}_2$, $\mathcal{B}_\infty$ such that*

$$
(D_t-\mathcal{D}+\mathcal{D}_1+\mathcal{B}_2)N_2 = N_2(D_t-\mathcal{D}+\mathcal{D}_2+\mathcal{B}_\infty)
$$

holds modulo $C^\infty([0,T],\text{Op}(S^{-\infty,-\infty}))$, with the elliptic symbol $\sigma(N_2)\in S^{0,0}\{0,0\}_N^{\text{hyp}}$ satisfying $\sigma(N_2)=1$ in $Z_{\text{pd}}(N)\cup Z_{\text{osc}}(N)$, the diagonal matrix $\sigma(\mathcal{D}_2)\in S^{0,0}\{0,0\}_N^{\text{reg}}+S^{-1,-1}\{-1,2\}_N^{\text{reg}}$ vanishing in $Z_{\text{pd}}(N)\cup Z_{\text{osc}}(N)$ and $\sigma(\mathcal{B}_\infty)(t,x,\xi)\in\mathcal{HG}_N$; in particular, $\sigma(\mathcal{B}_\infty)$ satisfies estimate (4.8) in $Z_{\text{pd}}(2N)$.

We omit the details of the proof, since it follows similar lines to that of Theorem 4.6.

5. The parametrix of the diagonalized problem

As a consequence of Proposition 4.6 and Theorem 4.7, in order to construct a fundamental solution $E = E(t,s)$ solving (4.6) it is sufficient to study the system

$$D_tE - \mathcal{D}E + \mathcal{D}_2E + \mathcal{B}_\infty E + R_\infty E = 0, \quad E(s,s) = I\,.$$

Since it is enough, for our aims, to obtain E modulo smooth families of regularizing operators, we will ignore the term $R_\infty E$. As customary in this approach, we construct the parametrix in two steps, corresponding to the diagonal terms $(D_t - \mathcal{D} + \mathcal{D}_2)$ and to the non-diagonal term $\mathcal{B}_\infty$.

Firstly, we construct the parametrix E_1 of

$$D_tE_1 - \mathcal{D}E_1 + \mathcal{D}_2E_1 \sim 0,\ E_1(s,s) \sim I.$$

As it turns out, the parametrix E_1 is a diagonal Fourier Integral Operator. The main difficulty here is in determining the inhomogeneous phase function of SG type. Further, to determine the amplitude of E_1, we recognize that the terms of the expression $-\mathcal{D}+\mathcal{D}_2$ have symbols in different classes, i.e.,

$$\sigma(\mathcal{D}) \in S^{\mathrm{pd}}_N \cap S^{1,1}\{1,0\}^{\mathrm{hyp}}_N$$

and

$$\sigma(\mathcal{D}_2) \in S^{\mathrm{pd}}_N \cap \left(S^{0,0}\{0,0\}^{\mathrm{reg}}_N + S^{-1,-1}\{-1,2\}^{\mathrm{reg}}_N\right).$$

In view of this, we determine the phase function based on the top-order term $-\mathcal{D}$ and determine the amplitude using the whole term $-\mathcal{D} + \mathcal{D}_2$.

Secondly, we construct the parametrix $Q(t,s)$ corresponding to the non-diagonal part such that $E = E_1Q$. As we do not have a diagonal structure in this case, we cannot obtain the parametrix as a diagonal Fourier integral operator. Nevertheless, we can take advantage of the fact that $\sigma(\mathcal{B}_\infty) \in \mathcal{HG}_N$ and observe that the parametrix is indeed a pseudodifferential operator.

5.1. Construction of the phase functions

Let us denote by $\vartheta = \vartheta(t,x,\xi)$ the real part of one of the functions $\mathsf{t}_k,\ k = 1,2$. We consider the Hamiltonian flow $(q,p) = (q,p)(t,s,y,\eta) = \psi_{s,t}(y,\eta)$, defined as the solution to

$$\begin{aligned} \frac{dq}{dt} &= \nabla_\xi\vartheta(t,q,p), \quad q(s,s,y,\eta) = y, \\ \frac{dp}{dt} &= -\nabla_x\vartheta(t,q,p), \quad p(s,s,y,\eta) = \eta. \end{aligned} \tag{5.1}$$

For convenience, in the sequel we will sometimes denote $q(t, s, y, \eta)$ by $q(t, s)$ and, similarly, $p(t, s, y, \eta)$ by $p(t, s)$. Following the approach used in [46], we prove the next result.

Lemma 5.1. *There exists $T_0 \in (0, T]$ such that the solution $(q(t, s, y, \eta),\ p(t, s, y, \eta))$ to (5.1) exists uniquely on $[0, T_0]^2 \times \mathbb{R}^d_y \times \mathbb{R}^d_\eta$. Moreover,*

$$\partial_t^k \partial_s^l \partial_y^\alpha \partial_\eta^\beta (q(t, s, y, \eta), p(t, s, y, \eta)) \in C([0, T_0]^2 \times \mathbb{R}^d_y \times \mathbb{R}^d_\eta).$$

In particular, it holds

$$\begin{aligned} p(t, s) &= \eta + \int_s^t \nabla_x \vartheta(\tau, q(\tau, s), p(\tau, s)) d\tau, \\ q(t, s) &= y - \int_s^t \nabla_\xi \vartheta(\tau, q(\tau, s), p(\tau, s)) d\tau, \end{aligned} \tag{5.2}$$

for all $(s, t) \in [0, T_0]^2$ and $(y, \eta) \in \mathbb{R}^{2d}$.

Proof. Let us define

$$f(t, q, p) = (\nabla_\xi \vartheta(t, q, p), -\nabla_x \vartheta(t, q, p)).$$

Since $\vartheta \in C([0, T], S^{1,1}) \subset C([0, T], C^\infty(\mathbb{R}^{2d}))$, we get that f is continuous together with its partial derivatives $\partial_{p_i} f$ and $\partial_{q_i} f$ in $[0, T] \times \mathbb{R}^{2d}$. As a consequence (see [41, Theorem 15]), for all $s \geq 0$ and $(y, \eta) \in \mathbb{R}^{2d}$ there exists $r_s > 0$ and $\sigma_s > 0$, depending on s, y and η, such that the solution $(p(t, \tau, \tilde{y}, \tilde{\eta}), q(t, \tau, \tilde{y}, \tilde{\eta}))$ to (5.1) with initial condition $\tilde{y}$ and $\tilde{\eta}$ is defined and continuous with respect to the variables $t, \tau, \tilde{y}, \tilde{\eta}$ for

$$|t - \tau| < r_s, \quad |\tau - s| < \sigma_s, \quad \|(\tilde{y}, \tilde{\eta}) - (y, \eta)\| < \sigma.$$

The existence of a common interval of definition $[0, T_0]$, independent of y and η, is not trivial.

Let $(y, \eta) \in \mathbb{R}^{2d}$, with $|y| > M$ and $|\eta| > L$. We denote

$$\begin{aligned} z &:= (z_1, z_2) := (p/|y|, q/|\eta|); \\ f &:= (f_1(t, z_1, z_2), f_2(t, z_1, z_2)) \\ &\ := (|y|^{-1} \nabla_\xi \vartheta(t, |y| z_1, |\eta| z_2), -|\eta|^{-1} \nabla_x \vartheta(t, |y| z_1, |\eta| z_2)). \end{aligned}$$

Then, z satisfies the equation

$$\frac{dz}{dt} = f(t, z), \quad z|_{t=s} = (\omega, \zeta) \in \mathbb{R}^{2d},\ |\omega| = |\zeta| = 1. \tag{5.3}$$

It remains to prove that for all $|y| > M$ and $|\eta| > L$ there exists a common domain of existence for $z(t, s, \omega, \zeta, y, \eta)$. We consider

$$\Pi := \{(t, z) \in [0, T] \times \mathbb{R}^{2d} : \ |z_1 - \omega| + |z_2 - \zeta| \leq a^2 \leq 1/4\},$$
$$\Pi_r := \{(s, t, z) \in [0, T]^2 \times \mathbb{R}^{2d} : \ t + s \leq r, (t, z) \in \Pi\}.$$

It is obvious that there exist M_1 and K positive constants such that

$$|f| \leq M_1, \quad \left|\frac{\partial f_i}{\partial z_j}\right| \leq K, \quad i, j = 1, 2,$$

uniformly in Π with respect to $|y|$ and $|\eta|$. As a consequence (see [41], Theorem 15 and formulae (19), (22) in Section 21), the solution to (5.3) exists and is continuous in Π_r if $(s, t) \in [0, T_0]^2$ with

$$T_0 \leq r \leq \frac{a}{M_1}, \quad r \leq \frac{k}{4d^2 K}, \text{ for some } k < 1.$$

In particular, this solution depends smoothly on the parameters y and η. □

Let us describe the behavior of the solution $(q(t, s, y, \eta), p(t, s, y, \eta))$. To this end we introduce an auxiliary point $\tilde{t}_{x,\xi}$ such that

$$\Lambda(\tilde{t}_{x,\xi})\langle x\rangle\langle\xi\rangle = N_1 \ln(\langle x\rangle\langle\xi\rangle),$$

where $N_1 < N$. Clearly, it holds $\tilde{t}_{x,\xi} < t_{x,\xi}$.

Lemma 5.2. *For $(s, y, \eta) \in Z_{\text{hyp}}(N)$ there exists $N_1 < N$ such that $(t, q(t, s), p(t, s)) \in Z_{\text{hyp}}(N_1)$ for all $t \in [s, T_0]$, taking $T_0 > s$ sufficiently small.*

Proof. To describe the behavior of the solution $(q(t, s, y, \eta), p(t, s, y, \eta))$ with respect to the zones we consider the properties of the function $\vartheta(t, x, \xi)$: for a sufficiently small T_0,

$$|\nabla_x \vartheta(t, x, \xi)| \leq c\lambda(t)\langle\xi\rangle \text{ and } |\nabla_\xi \vartheta(t, x, \xi)| \leq c\lambda(t)\langle x\rangle,$$

for all $(t, x, \xi) \in Z_{\text{hyp}}(N)$. So, we obtain, for $0 \leq s \leq t \leq T_0$,

$$|\frac{d}{dt}\langle p(t)\rangle^2| \leq c\langle p(t)\rangle^2\lambda(t) \text{ and } |\frac{d}{dt}\langle q(t)\rangle^2| \leq c\langle q(t)\rangle^2\lambda(t).$$

By applying Gronwall's lemma we get

$$\langle p(s)\rangle e^{2c(\Lambda(s)-\Lambda(t))} \leq \langle p(t)\rangle \leq \langle p(s)\rangle e^{2c(\Lambda(t)-\Lambda(s))}$$

and

$$\langle q(s)\rangle e^{2c(\Lambda(s)-\Lambda(t))} \le \langle q(t)\rangle \le \langle q(s)\rangle e^{2c(\Lambda(t)-\Lambda(s))}.$$

Note that the function $\Lambda(t)$ is a positive, strictly increasing, and continuous function. So, we can choose the interval $[0, T_0]$ sufficiently small, $t > s$ and $N_1 < N$ appropriately so that we have

$$\begin{aligned}\frac{\Lambda(t)\langle q(t)\rangle\langle p(t)\rangle}{N_1 \ln(\langle q(t)\rangle\langle p(t)\rangle)} &\ge \frac{\Lambda(s)\langle q(s)\rangle\langle p(s)\rangle}{N \ln(\langle q(s)\rangle\langle p(s)\rangle)} \frac{N e^{c[\Lambda(s)-\Lambda(t)]}}{N_1(1+\frac{c[\Lambda(t)-\Lambda(s)]}{\ln(\langle q(s)\rangle\langle p(s)\rangle)})} \\ &\ge \frac{\Lambda(s)\langle q(s)\rangle\langle p(s)\rangle}{N \ln(\langle q(s)\rangle\langle p(s)\rangle)} \frac{N}{N_1} \frac{e^{-c[\Lambda(t)-\Lambda(s)]}}{(1+c[\Lambda(t)-\Lambda(s)])}.\end{aligned}$$

Choosing $N_1 < N$, there exists $\varepsilon > 0$ such that $N e^{-\omega} > N_1(1+\omega)$ for all $\omega \in [0, \varepsilon)$. Then, there exists T_0 sufficiently small such that, for all $0 \le s \le t \le T_0$, it holds

$$\frac{\Lambda(t)\langle q(t)\rangle\langle p(t)\rangle}{N_1 \ln(\langle q(t)\rangle\langle p(t)\rangle)} \ge \frac{\Lambda(s)\langle q(s)\rangle\langle p(s)\rangle}{N \ln(\langle q(s)\rangle\langle p(s)\rangle)}.$$

The desired result follows by the definition of $Z_{\text{hyp}}(N)$. □

Lemma 5.3. *Let $\alpha, \beta \in \mathbb{N}^d$ and $j, k \in \mathbb{N}$ with $j + k \in \{0, 1\}$. For $T > 0$ sufficiently small, there exist constants $C_{jk\alpha\beta}$, such that, for all $(y, \eta) \in \mathbb{R}^{2d}$, if $0 \le s, t \le t_{y,\eta} \le T$ it holds*

$$\begin{aligned}|D_t^j D_s^k D_y^\alpha D_\eta^\beta (q(t,s,y,\eta) - y)| &\le C_{jk\alpha\beta} \langle y\rangle^{\frac{1}{2}+\varepsilon-|\alpha|} \langle\eta\rangle^{-\frac{1}{2}+\varepsilon-|\beta|} \\ &\times |\sqrt{\Lambda(t)} - \sqrt{\Lambda(s)}|^{1-j-k} \left(\frac{\lambda(t)}{\sqrt{\Lambda(t)}}\right)^j \left|\ln\left(\frac{\Lambda(t)}{\Lambda(s)}\right)\right|^k, \\ |D_t^j D_s^k D_y^\alpha D_\eta^\beta (p(t,s,y,\eta) - \eta) &\le C_{jk\alpha\beta} \langle y\rangle^{-\frac{1}{2}+\varepsilon-|\alpha|} \langle\eta\rangle^{\frac{1}{2}+\varepsilon-|\beta|} \\ &\times |\sqrt{\Lambda(t)} - \sqrt{\Lambda(s)}|^{1-j-k} \left(\frac{\lambda(t)}{\sqrt{\Lambda(t)}}\right)^j \left|\ln\left(\frac{\Lambda(t)}{\Lambda(s)}\right)\right|^k.\end{aligned} \tag{5.4}$$

Further, there exist constants T_0, M_1, $T_0 \in (0, T]$, $M_1 > M$ such that for any j, k positive integers and α, β multi-indices, there exist constants $C_{jk\alpha\beta}$, such that for all $(y, \eta) \in \mathbb{R}^{2d}$, $|y| + |\eta| \ge M$, we have the following estimates:

- *for $\tilde{t}_{y,\eta} \le s \le t \le T_0$ or $0 \le s \le \tilde{t}_{y,\eta} \le t_{y,\eta} \le t \le T_0$ it holds*

$$\begin{aligned}&|D_t^j D_s^k D_y^\alpha D_\eta^\beta (q(t,s,y,\eta) - y)| \\ &\quad \le C_{jk\alpha\beta} \Lambda(t) \langle y\rangle^{1-|\alpha|} \langle\eta\rangle^{-|\beta|} \left(\frac{\lambda(t)}{\Lambda(t)} \ln\left(\frac{1}{\Lambda(t)}\right)\right)^j \left(\frac{\lambda(s)}{\Lambda(s)} \ln\left(\frac{1}{\Lambda(s)}\right)\right)^k, \\ &|D_t^j D_s^k D_y^\alpha D_\eta^\beta (p(t,s,y,\eta) - \eta)| \\ &\quad \le C_{jk\alpha\beta} \Lambda(t) \langle y\rangle^{-|\alpha|} \langle\eta\rangle^{1-|\beta|} \left(\frac{\lambda(t)}{\Lambda(t)} \ln\left(\frac{1}{\Lambda(t)}\right)\right)^j \left(\frac{\lambda(s)}{\Lambda(s)} \ln\left(\frac{1}{\Lambda(s)}\right)\right)^k.\end{aligned} \tag{5.5}$$

Proof. We first note that for $(\tau, x, \xi) \in Z_{\mathrm{pd}}(N)$ it holds $\vartheta(\tau, x, \xi) = \rho(\tau, x, \xi) \in C([0,T], S^{\frac{1}{2}+\varepsilon, \frac{1}{2}+\varepsilon})$. Thus, it holds

$$|\nabla_x \vartheta(\tau, x, \xi)| \lesssim \frac{\lambda(t)}{\sqrt{\Lambda(t)}} \langle x \rangle^{-\frac{1}{2}+\varepsilon} \langle \xi \rangle^{\frac{1}{2}+\varepsilon},$$

and

$$|\nabla_\xi \vartheta(\tau, x, \xi)| \lesssim \frac{\lambda(t)}{\sqrt{\Lambda(t)}} \langle x \rangle^{\frac{1}{2}+\varepsilon} \langle \xi \rangle^{-\frac{1}{2}+\varepsilon}.$$

Since (p, q) satisfies (5.1) and $0 \le s, t, \le t_{x,\xi}$, we find

$$\begin{aligned}
\left| \frac{d}{dt} \langle p(t,s) \rangle^2 \right| &= 2 \langle p(t,s) \rangle |\nabla_x \vartheta(t, q, p)| \\
&\le 2 \frac{\lambda(t)}{\sqrt{\Lambda(t)}} \langle p(t,s) \rangle \langle q(t,s) \rangle^{-\frac{1}{2}+\varepsilon} \langle p(t,s) \rangle^{\frac{1}{2}+\varepsilon} \\
&\le C \frac{\lambda(t)}{\sqrt{\Lambda(t)}} \langle p(t,s) \rangle^2.
\end{aligned}$$

Thus, by Gronwall' inequality we may estimate for all $s, t \in [0, T_0]$,

$$\begin{aligned}
\langle \eta \rangle \lesssim \langle \eta \rangle e^{C(\sqrt{\Lambda(s)} - \sqrt{\Lambda(t)})} &\lesssim \langle p(t, s, y, \eta) \rangle \\
&\lesssim \langle \eta \rangle e^{C(\sqrt{\Lambda(t)} - \sqrt{\Lambda(s)})} \lesssim \langle \eta \rangle,
\end{aligned}$$

that is, $\langle p \rangle \sim \langle \eta \rangle$. Similarly, we find

$$\langle q(t, s, y, \eta) \rangle \sim \langle y \rangle.$$

By using representation (5.2), we get also

$$\begin{aligned}
|p(t, s, y, \eta) - \eta| &\lesssim \langle y \rangle^{-\frac{1}{2}+\varepsilon} \langle \eta \rangle^{\frac{1}{2}+\varepsilon} |\sqrt{\Lambda(t)} - \sqrt{\Lambda(s)}|, \\
|q(t, s, y, \eta) - y| &\lesssim \langle y \rangle^{\frac{1}{2}+\varepsilon} \langle \eta \rangle^{-\frac{1}{2}+\varepsilon} |\sqrt{\Lambda(t)} - \sqrt{\Lambda(s)}|.
\end{aligned}$$

In order to give the estimate for $j = k = 0$ and $|\alpha| = |\beta| = 1$, we define

$$\begin{aligned}
Q_1 = \nabla_y q(t, s, y, \eta), \quad Q_2 = \nabla_\eta q(t, s, y, \eta), \\
P_1 = \nabla_y p(t, s, y, \eta), \quad P_2 = \nabla_\eta p(t, s, y, \eta),
\end{aligned}$$

which satisfy

$$\begin{cases} \frac{d}{dt}\begin{pmatrix} Q_1 & Q_2 \\ P_1 & P_2 \end{pmatrix} = \begin{pmatrix} -\nabla_\eta\nabla_y\vartheta & -\nabla_\eta\nabla_\eta\vartheta \\ \nabla_y\nabla_y\vartheta & \nabla_\eta\nabla_y\vartheta \end{pmatrix}\begin{pmatrix} Q_1 & Q_2 \\ P_1 & P_2 \end{pmatrix} \\ \begin{pmatrix} Q_1 & Q_2 \\ P_1 & P_2 \end{pmatrix}_{t=s} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \end{cases}$$

Let

$$\begin{aligned} E(t) = \|Q_1(t) - I\|^2 + \|\langle y\rangle^{-1}\langle\eta\rangle Q_2(t)\|^2 \\ + \|\langle y\rangle\langle\eta\rangle^{-1}P_1(t)\|^2 + \|P_2(t) - I\|^2. \end{aligned} \tag{5.6}$$

It is easy to check that the following estimates hold:

$$\frac{d}{dt}E(t) \lesssim \frac{\lambda(t)}{\sqrt{\Lambda(t)}}\langle y\rangle^{-\frac{1}{2}+\varepsilon}\langle\eta\rangle^{-\frac{1}{2}+\varepsilon}(E(t) + \sqrt{E(t)}).$$

Then, taking account that $E(s) = 0$, we conclude that

$$E(t) \lesssim (\Lambda(t) - \Lambda(s))\langle y\rangle^{-1+2\varepsilon}\langle\eta\rangle^{-1+2\varepsilon}.$$

This allows to conclude the desired estimates

$$\begin{aligned} |\nabla_y(q(t,s,y,\eta) - y)| &\lesssim \langle y\rangle^{-\frac{1}{2}+\varepsilon}\langle\eta\rangle^{-\frac{1}{2}+\varepsilon}(\sqrt{\Lambda(t)} - \sqrt{\Lambda(s)}), \\ |\nabla_\eta(q(t,s,y,\eta) - y)| &\lesssim \langle y\rangle^{\frac{1}{2}+\varepsilon}\langle\eta\rangle^{-\frac{3}{2}+\varepsilon}(\sqrt{\Lambda(t)} - \sqrt{\Lambda(s)}), \\ |\nabla_y(p(t,s,y,\eta) - \eta)| &\lesssim \langle y\rangle^{-\frac{3}{2}+\varepsilon}\langle\eta\rangle^{\frac{1}{2}+\varepsilon}(\sqrt{\Lambda(t)} - \sqrt{\Lambda(s)}), \\ |\nabla_\eta(p(t,s,y,\eta) - \eta)| &\lesssim \langle y\rangle^{-\frac{1}{2}+\varepsilon}\langle\eta\rangle^{-\frac{1}{2}+\varepsilon}(\sqrt{\Lambda(t)} - \sqrt{\Lambda(s)}). \end{aligned}$$

By induction on $|\alpha + \beta|$, (5.4) can be proved for all α and β when $j = k = 0$.

In order to give the estimate for the derivatives with respect to s, we consider the auxiliary system

$$\frac{dQ}{dt} = -\nabla_\xi\vartheta(t+s, Q, P), \qquad \frac{dP}{dt} = \nabla_x\vartheta(t+s, Q, P), \tag{5.7}$$

with initial conditions

$$Q(0,s,y,\eta) = y, \quad P(0,s,y,\eta) = \eta.$$

Then,

$$q(t,s,y,\eta) = Q(t-s,s,y,\eta), \qquad p(t,s,y,\eta) = P(t-s,s,y,\eta).$$

Differentiating (5.7) with respect to s we obtain

$$\begin{aligned}\frac{d}{dt}\left|\left(\frac{dQ}{ds}\right)(t-s)\right|^2 &= 2\left|\left(\frac{dQ}{ds}\right)(t-s)\right|\left(\frac{d}{ds}(-\nabla_\xi\vartheta(t+s,Q,P))\right)(t-s)\\ &= -2\left|\left(\frac{dQ}{ds}\right)(t-s)\right|\left(\frac{d}{dt}\nabla_\xi\vartheta\right)(t,Q(t-s),P(t-s))\\ &\lesssim \left|\left(\frac{dQ}{ds}\right)(t-s)\right|\frac{\lambda(t)}{\Lambda(t)}\langle q\rangle^{\frac{1}{2}+\varepsilon}\langle p\rangle^{-\frac{1}{2}+\varepsilon},\end{aligned}$$

since $\partial_t\rho \in C([0,T],S^{1+\varepsilon,1+\varepsilon})$. Applying Gronwall' inequality, we obtain the desired estimate

$$|\partial_s q(t,s,y,\eta)| \lesssim \langle y\rangle^{\frac{1}{2}+\varepsilon}\langle\eta\rangle^{-\frac{1}{2}+\varepsilon}\left|\ln\left(\frac{\Lambda(t)}{\Lambda(s)}\right)\right|.$$

Similarly, we may estimate

$$|\partial_s p(t,s,y,\eta)| \lesssim \langle y\rangle^{-\frac{1}{2}+\varepsilon}\langle\eta\rangle^{\frac{1}{2}+\varepsilon}\left|\ln\left(\frac{\Lambda(t)}{\Lambda(s)}\right)\right|.$$

Finally, for $k=0$ and $j=1$, the desired estimates can be derived directly by equation (5.1). Indeed, we get

$$|\partial_t q(t,s,y,\eta)| = |\nabla_\xi\vartheta(t,q,p)| \lesssim \frac{\lambda(t)}{\sqrt{\Lambda(t)}}\langle y\rangle^{\frac{1}{2}+\varepsilon}\langle\eta\rangle^{-\frac{1}{2}+\varepsilon}.$$

Similarly, we get the estimate for the derivatives of p with respect to t.

By using the same approach, we can prove estimate (5.5) for all α, $\beta \in \mathbb{N}^d$ and $j,k \in \mathbb{N}$ such that $j+k \in \{0,1\}$. In particular, we note that, if $(s,y,\eta) \in Z_{\mathrm{hyp}}(N_1)$, then there exists $N_0 < N_1$ such that $(t,q(t,s,y,\eta),p(t,s,y,\eta))$ belongs to $Z_{\mathrm{hyp}}(N_0)$, as a consequence of Lemma 5.2. Moreover, it holds $\langle p(t,s,y,\eta)\rangle \sim \langle\eta\rangle$ and $\langle q(t,s,y,\eta)\rangle \sim \langle y\rangle$. In particular, it is easy to prove that the energy $E(t)$ defined by (5.6) satisfies the estimate

$$\frac{d}{dt}E(t) \lesssim \lambda(t)(E(t)+\sqrt{E(t)}).$$

Finally, the proof of (5.5) for $k+j=1$ exploits the non-increasing monotonicity of the function $\lambda(t)/\Lambda(t)$, that is guaranteed by assumption (2.1), being $C_1 < 1$.

In order to prove estimate (5.5) for $0 \le s \le \tilde{t}_{y,\eta} \le t_{y,\eta} \le t \le T_0$ we combine the estimates obtained in the hyperbolic and pseudodifferential zones. In particular, we note that for all $\tau \in [s,\tilde{t}_{y,\eta}]$ it holds $(\tau,p(\tau,s,y,\eta),q(\tau,s,y,\eta)) \in Z_{\mathrm{pd}}(N_1)$ and $\vartheta(\tau,q,p) = \rho(\tau,q,p)$. Moreover, we may estimate

$$|(D_y^\alpha D_\eta^\beta\rho)(t,p,q)| \lesssim \frac{\lambda(t)}{\sqrt{\Lambda(t)}}\langle q\rangle^{\frac{1}{2}-|\alpha|}\langle p\rangle^{\frac{1}{2}-|\beta|}\sqrt{\ln(\langle q\rangle\langle p\rangle)},$$

and then

$$\int_s^{\tilde t_{y,\eta}} |D_y^\alpha D_\eta^\beta \vartheta(\tau,q,p)|\,d\tau \lesssim \sqrt{\Lambda(\tilde t_{y,\eta})}\langle q\rangle^{\frac12-|\alpha|}\langle p\rangle^{\frac12-|\beta|}\sqrt{\ln(\langle q\rangle\langle p\rangle)}$$
$$= \frac{1}{N}\Lambda(t)\langle q\rangle^{1-|\alpha|}\langle p\rangle^{1-|\beta|},$$

being

$$\ln(\langle q\rangle\langle p\rangle) \lesssim \Lambda(t)\langle q\rangle\langle p\rangle,$$

for all $(t,q,p)\in Z_{\mathrm{hyp}}(N)$. □

Remark 5.4. As a consequence of Lemma 5.3 we may conclude that $q(t,s)$ and $p(t,s)$ satisfy

$$\tfrac{q(t,s)-y}{\Lambda(t)-\Lambda(s)},\ \partial_t q(t,s),\ \partial_s q(t,s) \in L_\infty\Big([0,T_0]^2, S^{1,0}(\mathbb{R}^n_y\times\mathbb{R}^n_\eta)\Big),$$
$$\tfrac{p(t,s)-\eta}{\Lambda(t)-\Lambda(s)},\ \partial_t p(t,s),\ \partial_s p(t,s) \in L_\infty\Big([0,T_0]^2, S^{0,1}(\mathbb{R}^n_y\times\mathbb{R}^n_\eta)\Big).$$

The obtained information about the behavior of the Hamiltonian flow allows to prove the following result.

Lemma 5.5. *There exists a constant T_1 with $0<T_1\le T_0$ such that the mappings*

$$x = q(t,s,\cdot,\eta) : y\in\mathbb{R}^d \to x\in\mathbb{R}^d,$$
$$\xi = p(t,s,y,\cdot) : \eta\in\mathbb{R}^d \to \xi\in\mathbb{R}^d,$$

with parameters (t,s,η), $s,t\in[0,T_1]$, both admit an inverse mapping $y(t,s,x,\eta)$ and $\eta(t,s,x,\xi)$, respectively, satisfying the following estimates, for $j=0,1$ and $\alpha,\beta\in\mathbb{N}^d$, with suitable positive constants $C_{j\alpha\beta}$:

- *for $0\le s,t\le t_{x,\xi}$*

$$|D_t^j D_x^\alpha D_\xi^\beta(y(t,s,x,\xi)-x)| \le C_{j\alpha\beta}\langle x\rangle^{\frac12+\varepsilon-|\alpha|}\langle\xi\rangle^{-\frac12+\varepsilon-|\beta|}$$
$$\times|\sqrt{\Lambda(t)}-\sqrt{\Lambda(s)}|^{1-j}\left(\frac{\lambda(t)}{\sqrt{\Lambda(t)}}\right)^j$$

and

$$|D_t^j D_x^\alpha D_\xi^\beta(\eta(t,s,x,\xi)-\xi)| \le C_{j\alpha\beta}\langle x\rangle^{-\frac12+\varepsilon-|\alpha|}\langle\xi\rangle^{\frac12+\varepsilon-|\beta|}$$
$$|\sqrt{\Lambda(t)}-\sqrt{\Lambda(s)}|^{1-j}\left(\frac{\lambda(t)}{\sqrt{\Lambda(t)}}\right)^j;$$

- *for* $\tilde{t}_{x,\xi} \le s \le t \le T_0$ *or* $0 \le s \le \tilde{t}_{x,\xi} \le t_{x,\xi} \le t \le T_0$

$$|D_t^j D_x^\alpha D_\xi^\beta (y(t,s,x,\xi) - x)| \le C_{j\alpha\beta} \Lambda(t) \langle x\rangle^{1-|\alpha|} \langle \xi\rangle^{-|\beta|} \left(\frac{\lambda(t)}{\Lambda(t)} \ln\left(\frac{1}{\Lambda(t)}\right)\right)^j$$

and

$$|D_t^j D_x^\alpha D_\xi^\beta (\eta(t,s,x,\xi) - \xi)| \le C_{j\alpha\beta} \Lambda(t) \langle x\rangle^{-|\alpha|} \langle \xi\rangle^{1-|\beta|} \left(\frac{\lambda(t)}{\Lambda(t)} \ln\left(\frac{1}{\Lambda(t)}\right)\right)^j.$$

Proof. Applying Lemma 5.2, it is easy to get that there exists $T_1 > 0$ and $\varepsilon > 0$ with $0 < T_1 \le T_0$ and $0 < \varepsilon < 1$ such that

$$\left| \frac{\partial}{\partial_y} q(t,s,y,\xi) - I \right| \le 1 - \varepsilon, \qquad \text{for } s,t \in [0,T_1],\ y,\eta \in \mathbb{R}^d,\ |y| + |\xi| \ge M.$$

As a consequence, the invertibility of $q(t,s,\cdot,\xi)$ follows. The desired estimates can be derived after noticing that

$$y(t,s,x,\xi) - x = y(t,s,x,\xi) - q(t,s,y(t,s,x,\xi),\xi),$$

following the same approach of [46], employing the estimates in Lemma 5.3. The same idea can be used to prove the existence of the inverse function $\eta = \eta(t,s,x,\xi)$ and the corresponding estimates. □

Now, we deal with the construction of the phase function $\varphi = \varphi(t,s,x,\xi)$ solving the eikonal equation

$$\begin{cases} \partial_t \varphi(t,s,x,\xi) - \vartheta(t,x,\nabla_x \varphi(t,s,x,\xi)) = 0, \\ \varphi(s,s,x,\xi) = x \cdot \xi\,. \end{cases} \tag{5.8}$$

Lemma 5.6. *Let* $\varphi = \varphi(t,s,x,\xi)$ *be defined as*

$$\varphi(t,s,x,\xi) = v(t,s,y(t,s,x,\xi),\xi) \tag{5.9}$$

where

$$v(t,s,y,\eta) = y \cdot \eta - \int_s^t \Big(p \cdot \nabla_\xi \vartheta - \vartheta\Big)\Big(\tau, q(\tau,s,y,\eta), p(\tau,s,y,\eta)\Big) d\tau\,.$$

Then, φ solves the Cauchy problem (5.8). Moreover, for all $\alpha, \beta \in \mathbb{N}^d$, there exist positive constants $C_{\alpha\beta}$ such that, for all $(x, \xi) \in \mathbb{R}^{2d}$, it holds

$$\left| D_x^\alpha D_\xi^\beta \Big(\varphi(t, s, x, \xi) - x \cdot \xi \Big) \right| \leq C_{\alpha\beta} \langle x \rangle^{\varepsilon - |\alpha|} \langle \xi \rangle^{\varepsilon - |\beta|}, \tag{5.10}$$

if $0 \leq s, t \leq t_{x,\xi}$, with $\varepsilon > 0$ arbitrarily small, and

$$\left| D_x^\alpha D_\xi^\beta \Big(\varphi(t, s, x, \xi) - x \cdot \xi \Big) \right| \leq C_{\alpha\beta} \langle x \rangle^{1 - |\alpha|} \langle \xi \rangle^{1 - |\beta|} |\Lambda(t) - \Lambda(s)|, \tag{5.11}$$

if $\max(s, t) \geq \tilde{t}_{x,\xi}$.

Proof. The fact that the function φ defined in (5.9) solves the Cauchy problem (5.8) follows by classical results (see, for instance, [32]). The desired estimates follow by (5.9), employing Faa' di Bruno formula to evaluate the derivatives of composite functions, together with the estimates obtained in Lemma 5.3. In particular, in estimate (5.10) we use that

$$\langle x \rangle^{\frac{1}{2}} \langle \xi \rangle^{\frac{1}{2}} \max\{\sqrt{\Lambda(t)}, \sqrt{\Lambda(s)}\} \leq \sqrt{N \ln(\langle x \rangle \langle \xi \rangle)},$$

being (s, x, ξ) and (t, x, ξ) in $Z_{\mathrm{pd}}(N)$. □

From now on, we denote by φ^- and φ^+ the solutions to (5.8), with ϑ equal to the real part of $\mathfrak{t}_1$ and, respectively, the real part of $\mathfrak{t}_2$.

Lemma 5.7. *Suppose $\psi_{s,t}(x, \xi) = (y, \eta) = (\psi^1_{s,t}(x, \xi), \psi^2_{s,t}(x, \xi))$ be the canonical relation associated with the Hamiltonian $\vartheta = \vartheta(t, x, \xi)$ and let*

$$b(t, x, \xi) = e^{i \int_t^T a(\tau, x, \xi) d\tau},$$

with $a(t, x, \xi) \in S^{0,0}\{1, 0\}^{\mathrm{hyp}}_N$ supported in $Z_{\mathrm{hyp}}(N)$. Then,

$$c(t, x, \xi) = b(t, \psi_{s,t}(x, \xi)) \in S^{0,0}\{0, 0\}^{\mathrm{hyp}}_{N_1}$$

for $0 \leq s < t \leq T$ and sufficiently large $N_1 \geq N$.

Proof. We assume T sufficiently small in $Z_{\mathrm{hyp}}(N)$ to ensure that the canonical transformation $\psi_{s,t}$ is a SG-diffeomorphism. This implies, in particular, $\psi^1_{s,t} \in S^{1,0}$, $\psi^2_{s,t} \in S^{0,1}$, $\langle \psi^1_{s,t}(x, \xi) \rangle \asymp \langle x \rangle$ and $\langle \psi^2_{s,t}(x, \xi) \rangle \asymp \langle \xi \rangle$, uniformly with respect to s, t, $0 \leq s \leq t \leq T$.

As the symbol is a constant in the pseudodifferential zone, we can assume, without loss of generality, that we are working in the hyperbolic zone, i.e., $t \geq t_{x,\xi}$. Of course, since a is uniformly bounded, $|c(t, x, \xi)| \lesssim 1$. Let us now estimate the derivatives of first order. By the chain rule and the hypotheses, for $j = 1, \ldots, d$,

$$
\begin{aligned}
\left|\partial_{x_j} c(t,x,\xi)\right| &= |c(t,x,\xi)| \cdot \left| \sum_{k=1}^{d} \int_t^T \left[(\partial_{x_k} a)(\tau, \psi_{s,t}(x,\xi))(\partial_{x_j} \psi^{1k}_{s,t})(x,\xi) \right.\right. \\
&\qquad \left.\left. + (\partial_{\xi_k} a)(\tau, \psi_{s,t}(x,\xi))(\partial_{x_j} \psi^{2k}_{s,t})(x,\xi) \right] d\tau \right| \\
&\lesssim \sum_{k=1}^{d} \int_t^T \left[\lambda(\tau) \langle \psi^1_{s,t}(x,\xi)\rangle^{-1} \langle \psi^2_{s,t}(x,\xi)\rangle^{0} \langle x\rangle^0 \langle \xi\rangle^0 \right. \\
&\qquad \left. + \lambda(\tau) \langle \psi^1_{s,t}(x,\xi)\rangle^{0} \langle \psi^2_{s,t}(x,\xi)\rangle^{-1} \langle x\rangle^{-1} \langle \xi\rangle \right] d\tau \\
&\lesssim \left(\langle x\rangle^{-1} + \langle \xi\rangle^{-1} \langle x\rangle^{-1} \langle \xi\rangle \right) \int_t^T \lambda(\tau) d\tau \\
&\le \left[\Lambda(T) - \Lambda(t_{x,\xi}) \right] \langle x\rangle^{-1} \lesssim \langle x\rangle^{-1}.
\end{aligned}
$$

In a completely similar fashion, we obtain, for $j = 1, \dots, d$, the estimates

$$
\left|\partial_{\xi_j} c(t,x,\xi)\right| \lesssim \langle \xi\rangle^{-1}.
$$

Finally, by the estimates for the components of the Hamiltonian flow, proved above, with $N_1 \ge N$ sufficiently large, we find

$$
\begin{aligned}
|\partial_t c(t,x,\xi)| &= |c(t,x,\xi)| \cdot \left| -a(t, \psi_{s,t}(x,\xi)) \right. \\
&\qquad + \sum_{k=1}^{d} \int_t^T \left[(\partial_{x_k} a)(\tau, \psi_{s,t}(x,\xi))(\partial_t \psi^{1k}_{s,t})(x,\xi) \right. \\
&\qquad \left.\left. + (\partial_{\xi_k} a)(\tau, \psi_{s,t}(x,\xi))(\partial_t \psi^{2k}_{s,t})(x,\xi) \right] d\tau \right| \\
&\lesssim \lambda(t) + \sum_{k=1}^{d} \int_t^T \left[\lambda(\tau) \langle \psi^1_{s,t}(x,\xi)\rangle^{-1} \langle \psi^2_{s,t}(x,\xi)\rangle^{0} \lambda(t) \langle x\rangle \langle \xi\rangle^0 \right. \\
&\qquad \left. + \lambda(\tau) \langle \psi^1_{s,t}(x,\xi)\rangle^{0} \langle \psi^2_{s,t}(x,\xi)\rangle^{-1} \lambda(t) \langle x\rangle^0 \langle \xi\rangle \right] d\tau \\
&\lesssim \lambda(t) \left[1 + \left(\langle x\rangle^{-1} \langle x\rangle + \langle \xi\rangle^{-1} \langle \xi\rangle \right) \int_t^T \lambda(\tau) d\tau \right]
\end{aligned}
$$

$$\lesssim \lambda(t) \lesssim \frac{\lambda(t)}{\Lambda(t)} \ln \frac{1}{\Lambda(t)}.$$

Notice that the last inequality is equivalent to $1 \lesssim -\frac{\ln \Lambda(t)}{\Lambda(t)}$, which holds true for $t \geq t_{x,\xi}$. The estimates for the derivatives of higher arbitrary order follow inductively, by means of the Fáa di Bruno formula. □

5.2. Construction of the amplitudes

As usual, we look for operator families $E_2^{\mp}(t,s)$ of the form

$$E_2^{\mp}(t,s)w(x) = \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} e^{i(\varphi^{\mp}(t,s,x,\xi)-y.\xi)} e_2^{\mp}(t,s,x,\xi)w(y)dyd\xi, \quad w \in \mathscr{S}(\mathbb{R}^d),$$

with

$$\varphi^{\mp}(s,s,x,\xi) = x\cdot\xi\ ,\ e_2^{\mp}(s,s,x,\xi) = 1.$$

The asymptotic representation

$$e_2^{\mp}(t,s,x,\xi) \sim \sum_{j=0}^{\infty} e_{2,j}^{\mp}(t,s,x,\xi) \quad \text{modulo } C\Big([0,T_0]^2, S^{-\infty,-\infty}\Big),$$
$$e_{2,0}^{\mp}(s,s,x,\xi) = 1\ ,\ e_{2,j}^{\mp}(s,s,x,\xi) = 0 \quad \text{for } j \geq 1,$$

allows us to derive the transport equation. Namely, we need to study the action of $D_t - \mathrm{Op}(\mathfrak{t}_1(t))$ and $D_t - \mathrm{Op}(\mathfrak{t}_2(t))$, respectively, on E_2^- , E_2^+. We confine ourselves to the fundamental solution $E_2^-(t,s)$, corresponding to the amplitude function $e_2^-(t,s,x,\xi)$: the computations for the case $E_2^+(t,s)$ are similar. Formally, we can of course write

$$D_t E_2^-(t,s)w(x) = \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} e^{i(\varphi^-(t,s,x,\xi)-y\cdot\xi)} \Big(\partial_t\varphi^- \sum_{j=0}^{\infty} e_{2,j}^- + \tfrac{1}{i}\partial_t \sum_{j=0}^{\infty} e_{2,j}^-\Big) w(y)dyd\xi.$$

In order to find the expansion terms $e_{2,j}^-$, $j = 0,1,2,\ldots$, we introduce the abbreviations

$$g_\alpha(t,s,x,\xi) = \partial_\xi^\alpha \mathfrak{t}_1(t,x,\nabla_x\varphi^-(t,s,x,\xi)), \quad \text{for } |\alpha| \geq 1,$$
$$g_0(t,s,x,\xi) = -i \sum_{|\alpha|=2} \frac{1}{\alpha!} g_\alpha(t,s,x,\xi)\partial_x^\alpha \varphi^-(t,s,x,\xi),$$
$$Z(t,s) = D_t - \sum_{|\alpha|=1} g_\alpha(t,s,x,\xi)\partial_x^\alpha + g_0(t,s,x,\xi).$$

Then, according to the asymptotic expansion for the compositions, we obtain the equations

$$Z(t,s)e^-_{2,0} = 0 \text{ and } Z(t,s)e^-_{2,j} + r_{j-1} = 0,$$

for $0 \le t < T$, with the initial conditions

$$e^-_{2,0}(s,s) = 1 \text{ and } e^-_{2,j}(s,s) = 0,\ j = 1, 2, \dots,$$

where

$$\begin{aligned} r_j(t,s,x,\xi) = &\sum_{|\alpha|>2} g_\alpha(t,s,x,\xi) \sum_{\substack{|\alpha_1|+|\alpha_2|=|\alpha| \\ |\alpha_1|\neq 1}} D_x^{\alpha_1}\varphi^-(t,x,y) D_x^{\alpha_2} e^-_{2,j}(t,s,x,\xi) \\ &+ \sum_{|\alpha|=2} g_\alpha(t,s,x,\xi) D_x^\alpha e^-_{2,j}(t,s,x,\xi). \end{aligned}$$

The solutions $e^-_{2,j}(t,s,x,\xi)$, $j = 0, 1, 2, \dots$, to the above initial value problems are

$$\begin{aligned} e^-_{2,0}(t,s,x,\xi) &= \exp\left[-i\int_s^t g_0(\sigma,s,q(\sigma,s,y(t,s,x,\xi)),\xi)d\sigma\right], \\ e^-_{2,j}(t,s,x,\xi) &= -i\int_s^t r_{j-1}(\sigma,s,q(\sigma,s,y(t,s,x,\xi)),\xi) \\ &\qquad \times\exp\left[-i\int_s^t g_0(\sigma',s,q(\sigma',s,y(t,s,x,\xi)),\xi)d\sigma'\right] d\sigma. \end{aligned}$$

Lemma 5.8. *The following useful estimates hold:*

- *If* $0 \le s, t \le t_{x,\xi}$ *then*

$$|D_x^\alpha D_\xi^\beta g_\alpha(t,s,x,\xi)| \lesssim \langle x\rangle^{\frac{1}{2}+\varepsilon-|\alpha|}\langle\xi\rangle^{-\frac{1}{2}+\varepsilon-|\beta|}, \quad \text{for } |\alpha| = 1,$$

and

$$\begin{aligned} |D_x^\alpha D_\xi^\beta g_0(t,s,x,\xi)| &\lesssim \langle x\rangle^{-1+\varepsilon-|\alpha|}\langle\xi\rangle^{-1+\varepsilon-|\beta|}|t-s|, \\ |D_x^\alpha D_\xi^\beta e_0^\mp(t,s,x,\xi)| &\lesssim \langle x\rangle^{-1+\varepsilon-|\alpha|}\langle\xi\rangle^{-1+\varepsilon-|\beta|}|t-s|^2, \\ |D_x^\alpha D_\xi^\beta r_0(t,s,x,\xi)| &\lesssim \langle x\rangle^{-2+\varepsilon-|\alpha|}\langle\xi\rangle^{-2+\varepsilon-|\beta|}|t-s|^2; \end{aligned}$$

- *if* $\max\{s,t\} \ge \tilde{t}_{x,\xi}$ *then*

$$|D_x^\alpha D_\xi^\beta g_\alpha(t,s,x,\xi)| \lesssim \lambda(t)\langle x\rangle^{1-|\alpha|}\langle\xi\rangle^{-|\beta|}, \quad \text{for } |\alpha| = 1,$$

and

$$|D_x^\alpha D_\xi^\beta g_0(t,s,x,\xi)| \lesssim \langle x\rangle^{-|\alpha|}\langle\xi\rangle^{-|\beta|}\lambda(t)|\Lambda(t)-\Lambda(s)|,$$
$$|D_x^\alpha D_\xi^\beta e_0^{\mp}(t,s,x,\xi)| \lesssim \langle x\rangle^{-|\alpha|}\langle\xi\rangle^{-|\beta|}|\Lambda(t)-\Lambda(s)|^2,$$
$$|D_x^\alpha D_\xi^\beta r_0(t,s,x,\xi)| \lesssim \langle x\rangle^{-1-|\alpha|}\langle\xi\rangle^{-1-|\beta|}\lambda(t)|\Lambda(t)-\Lambda(s)|^2.$$

As a consequence, by using induction on $j \geq 0$ we get the following statement.

Proposition 5.9. *The parametrix* $E_2(t,s) = \operatorname{diag}\left(E_2^-(t,s), E_2^+(t,s)\right)$ *is a diagonal matrix of Fourier integral operators with*

$$E_2^{\mp}(t,s)w(x) = \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} e^{i(\varphi^{\mp}(t,s,x,\xi)-y\cdot\xi)} e_2^{\mp}(t,s,x,\xi)w(y)dyd\xi,$$
$$\varphi^{\mp}(s,s,x,\xi) = x\cdot\xi\ ,\ e_2^{\mp}(s,s,x,\xi) = 1\ .$$

The phase functions $\varphi^{\mp}$ *satisfy estimate* (5.10) *and* (5.11)*. Further, for* $(t,x,\xi) \in Z_{\text{pd}}(N)$*, the amplitude functions* $e_2^{\mp}$ *satisfy*

$$|D_x^\alpha D_\xi^\beta e_{2,j}^{\mp}(t,s,x,\xi)| \lesssim \langle x\rangle^{-1-j+\varepsilon-|\alpha|}\langle\xi\rangle^{-1-j+\varepsilon-|\beta|}|t-s|.$$

Moreover, there exists a positive number N_0 *such that, for* $(t,x,\xi) \in Z_{\text{hyp}}(N_0)$*, it holds*

$$|D_x^\alpha D_\xi^\beta e_{2,j}^{\mp}(t,s,x,\xi)| \lesssim \langle x\rangle^{-|\alpha|}\langle\xi\rangle^{-|\beta|}\Lambda(t)^j|\Lambda(t)-\Lambda(s)|$$

for $j = 0,1,2,\ldots$

Remark 5.10. We may write

$$E_2^{\mp}(t,s) = \operatorname{Op}(a^{\mp}(t,s)) + \operatorname{Op}_{\varphi^{\mp}(t,s)}(b^{\mp}(t,s)),$$

where $\operatorname{Op}(a^{\mp}(t,s))$ is a pseudodifferential operator with amplitude

$$a^{\mp}(t,s,x,\xi) := \chi\left(\frac{\Lambda(t)\langle x\rangle\langle\xi\rangle}{2N\ln(\langle x\rangle\langle\xi\rangle)}\right)e^{i(\varphi^{\mp}(t,s,x,\xi)-x\cdot\xi)}e_2^{\mp}(t,s,x,\xi),$$

and $\operatorname{Op}_{\varphi^{\mp}(t,s)}(b^{\mp}(t,s))$ is the SG Fourier integral operator of type I (see Definition 3.4) with phase $\varphi^{\mp}$ and amplitude

$$b^{\mp}(t,s,x,\xi) = \left(1-\chi\left(\frac{\Lambda(t)\langle x\rangle\langle\xi\rangle}{2N\ln(\langle x\rangle\langle\xi\rangle)}\right)\right)e_2^{\mp}(t,s,x,\xi).$$

We note that, as a consequence of (5.10) and Lemma 5.8, the symbol $a^{\mp}$ belongs to $L_\infty([0,T_0]^2, S_{(\varepsilon)}^{-1+\varepsilon,-1+\varepsilon})$, according to the definition given in (5.22).

Lemma 5.11. *Let $a(t,x,\xi) \in S^{-p,-p}\{-p,p+k\}_N^{\mathrm{reg}}$ be supported in the regular zone with $t'_{x,\xi}$ defined in (2.3). Then, the symbol*

$$b(t,x,\xi) = e^{i\int_t^T a(\tau,x,\xi)d\tau}$$

satisfies

$$|\partial_x^\alpha \partial_\xi^\beta b(t,x,\xi)| \leq C_{\alpha\beta}\langle x\rangle^{-|\alpha|}\langle \xi\rangle^{-|\beta|}$$

for any $p \geq k$.

Proof. We need to prove the estimate on in $Z_{\mathrm{reg}}(N)$ as $b(t,x,\xi)$ is constant outside $Z_{\mathrm{reg}}(N)$. For $|\alpha| = |\beta| = 0$, the result clearly holds. We can then proceed inductively, by observing that, for some positive constants C_j, C'_j, we have

$$\left|\partial_{x_j} b(\tau,x,\xi)\right| \leq C_j \int_t^T |\partial_{x_j} a(\tau,x,\xi)|d\tau.$$

By the estimate on the symbols in $S^{-p,-p}\{-p,p+k\}_N^{\mathrm{reg}}$ we have

$$\begin{aligned}\left|\partial_{x_j} b(\tau,x,\xi)\right| &\leq C_j\langle x\rangle^{-1}\int_t^T \tfrac{1}{\langle x\rangle^p\langle\xi\rangle^p\lambda(\tau)^p}\left(\tfrac{\lambda(\tau)}{\Lambda(\tau)}\ln\tfrac{1}{\Lambda(\tau)}\right)^{p+k}d\tau \\ &\leq C_j\langle x\rangle^{-1}\frac{(\ln(\langle x\rangle\langle\xi\rangle))^{(p+k)}}{(\langle x\rangle\langle\xi\rangle\Lambda(t'_{x,\xi}))^p}.\end{aligned}$$

In $Z_{\mathrm{reg}}(N)$ we can simplify the above expression to

$$\left|\partial_{x_j} b(\tau,x,\xi)\right| \leq \langle x\rangle^{-1}\frac{C_j}{(2N)^p}(\ln(\langle x\rangle\langle\xi\rangle))^{(p+k)-2p} \leq C'_j\langle x\rangle^{-1} \text{ as } p \geq k.$$

Similarly, we have $|\partial_{\xi_j} b(\tau,x,\xi)| \leq C'_j\langle\xi\rangle^{-1}$. The estimates for the higher order derivatives are proved inductively, by means of the hypotheses and the Fáa di Bruno formula. □

Theorem 5.12 *(Egorov's Theorem). Let $E_2^{\mp}(t,s)$ be the fundamental solutions given in Proposition 5.9 with $0 \leq s \leq t$. Assume that $p \in S^{0,0}\{0,0\}_N^{\mathrm{hyp}}$ and it is supported in $Z_{\mathrm{hyp}}(N)$. Then, for a sufficiently large N_1, the operator*

$$\mathrm{Op}(p_1(t,s)) = E_2^{\mp}(s,t)\,\mathrm{Op}(p(t))E_2^{\mp}(t,s)$$

is a pseudodifferential operator with symbol $p_1(\cdot,s,\cdot,\cdot) \in S^{0,0}\{0,0\}_{N_1}^{\mathrm{hyp}}$ defined for all $s \in [0,t)$ for $t \in [0,T_1]$ and vanishing in the pseudodifferential zone.

The proof follows in similar lines to that of the standard Egorov's theorem, cf. [46]. For the sake of completeness, we illustrate the argument.

Proof. Let s be fixed while t runs over $[0, s]$, then we denote

$$P_1(t) := P_1(s,t) = E_2^-(t,s)\,\mathrm{Op}(p(s))E_2^-(s,t), \quad 0 \le t \le s.$$

Clearly, $P_1(t)$ solves, modulo smoothing operators, the initial value problem

$$D_t P_1(t) = \mathrm{Op}(\mathfrak{t}_1(t))P_1(t) - P_1(t)\,\mathrm{Op}(\mathfrak{t}_1(t)), \quad P_1(s) = \mathrm{Op}(p(s)). \tag{5.12}$$

We are going to construct an approximating solution, $Q(t)$ to (5.12) in the form of a pseudodifferential operator $Q(t) := Q(s,t) = \mathrm{Op}(q(t,s))$ with a parameter dependent symbol $q(t,s)$ belonging to $S^{0,0}$. So we determine $Q(t)$ solving the Cauchy problem

$$\frac{\partial}{\partial t}Q(t) = i\,\mathrm{Op}(\mathfrak{t}_1(t))Q(t) - iQ(t)\,\mathrm{Op}(\mathfrak{t}_1(t)) + R(t), \quad Q(s) = \mathrm{Op}(p(s)),$$

where $R(t)$ is a smooth family of regularizing operators on $\mathbb{R}^d$.

In order to solve for $q(t,s,x,\xi)$, we assume an asymptotic expansion in the form

$$q(t,s,x,\xi) \sim q_0(t,s,x,\xi) + q_1(t,s,x,\xi) + \cdots,$$

where $q_j(t,s,x,\xi) \in C^\infty([0,T]; S^{-j,-j}(\mathbb{R}^d \times \mathbb{R}^d))$. Therefore, we first solve the transport equation

$$\frac{\partial q_0}{\partial t} - \sum_{j=1}^{d} \frac{\partial \varphi}{\partial \xi_j}\frac{\partial q_0}{\partial x_j} + \sum_{j=0}^{d} \frac{\partial \varphi}{\partial x_j}\frac{\partial q_0}{\partial \xi_j} = 0,$$

or, written equivalently by means of the Hamiltonian flow,

$$\left(\frac{\partial}{\partial t} - \mathcal{H}_{\mathfrak{t}_1}\right) q_0 = 0,$$

with the initial condition $q_0(s,s,x,\xi) = p(s,x,\xi)$.

As $p(t,s,x,\xi)$ vanishes on the pseudodifferential zone, evidently, $q_0(t,s,x,\xi) = 0$ for all $(s,x,\xi) \in Z_{\mathrm{pd}}(N), t \in [0,s]$. To solve for $q_0(t,s,x,\xi)$ in the hyperbolic zone, we observe that, by choosing k and T_1 sufficiently small, we have a SG-diffeomorphism, that is, $(y,\eta) \to (x,\xi)$ with

$$(x,\xi) = (x(t,s,y,\eta), \xi(t,s,y,\eta)) : \mathbb{R}^d_y \times \mathbb{R}^d_\eta \to \mathbb{R}^d_x \times \mathbb{R}^d_\xi.$$

For the parameters t, s satisfying $t \le s < T_1$ there exists an inverse mapping $(x,\xi) \to (y,\eta)$ with

$$(y,\eta) = (y(t,s,x,\xi), \eta(t,s,x,\xi)) : \mathbb{R}^d_x \times \mathbb{R}^d_\xi \to \mathbb{R}^d_y \times \mathbb{R}^d_\eta,$$

for all $(t,x,\xi) \in Z_{\mathrm{pd}}(N), (s,x,\xi) \in Z_{\mathrm{hyp}}(N), t,s \leq T_1$. Moreover, the function $q_0(t,s,x,\xi)$ is constant on the integral curves of the Hamiltonian flow corresponding to $\mathfrak{t}_1$. Thus, by the initial condition $q_0(s,s,x,\xi) = p(s,x,\xi)$ we have

$$q_0(t,s,x,\xi) = p(s, y(t,s,x,\xi), \eta(t,s,x,\xi)).$$

Similarly, we can determine the lower order terms $q_l(t,s,x,\xi)$ for $l = 1,2,\ldots$ by the equation

$$\frac{\partial q_l}{\partial t} - \sum_{j=1}^{d} \frac{\partial \varphi}{\partial \xi_j}\frac{\partial q_l}{\partial x_j} + \sum_{j=0}^{d} \frac{\partial \varphi}{\partial x_j}\frac{\partial q_l}{\partial \xi_j} = \zeta_l(t,s,x,\xi)$$

with

$$\zeta_l(t,s,x,\xi) = \sum_{i=0}^{l-1} \sum_{|\alpha|=l-i+1} \frac{1}{\alpha!}\left(D^\alpha_\xi \varphi \partial^\alpha_x q_i - D^\alpha_\xi q_i \partial^\alpha_x \varphi\right), \quad l = 1,2,\ldots$$

and the initial condition $q_k(s,s,x,\xi) = 0$.

By a variant of the usual asymptotic expansion argument, we can show that there exists a symbol $q(t,s,x,\xi) \sim \sum_{l=0}^{\infty} q_l(t,s,x,\xi)$. Furthermore, by Lemma 5.1 and Lemma 5.7, we have

$$q(t,s,x,\xi) \in S^{0,0}\{0,N_1\}^{\mathrm{hyp}}.$$

The uniqueness of the solution modulo a smooth family of regularizing operators $R(t)$ is also a consequence of our approach. The proof is complete. □

5.3. Parametrix for the diagonal terms

In this section we will construct the parametrix to

$$D_t E_1 - \mathcal{D} E_1 + \mathcal{D}_2 E_1 = 0, \quad E_1(s,s) = I. \tag{5.13}$$

The equalities of course hold modulo smoothing terms. Using $E_2 = E_2(t,s)$ from the previous section, we define

$$E_1(t,s) = E_2(t,s) Q_1(t,s), \;\; Q_1(s,s) = I\,.$$

Substituting into (5.13) gives the Cauchy problem

$$D_t Q_1 + E_2(s,t)\mathcal{D}_2(t)E_2(t,s)Q_1 = 0\,, \;\; Q_1(s,s) = I\,. \tag{5.14}$$

By the Egorov's Theorem 5.12, we have that the matrix-valued operator $R_1(t,s) = E_2(s,t)\mathcal{D}_2(t)E_2(t,s)$ consists of pseudodifferential operators with principal symbol

$$r_1 = r_1(t,s,x,\xi) = \sigma_p(\mathcal{D}_2)\Big(t, \psi_{s,t}(x,\xi)\Big).$$

Since $\sigma(\mathcal{D}_2)$ belongs to $S^{0,0}\{0,0\}_N^{\text{hyp}} + S^{-1,-1}\{-1,2\}_N^{\text{reg}}$ and vanishes in $Z_{\text{pd}}(N) \cup Z_{\text{osc}}(N)$, we may write

$$\mathcal{D}_2 = \mathcal{D}_{2,0} + \mathcal{D}_{2,1},$$

where

$$d_{2,0} := \sigma(\mathcal{D}_{2,0}) \in S^{0,0}\{0,0\}_N^{\text{hyp}}, \quad d_{2,0} \equiv 0 \text{ in } Z_{\text{pd}}(N);$$

and

$$d_{2,1} := \sigma(\mathcal{D}_{2,1}) \in S^{-1,-1}\{-1,2\}_N^{\text{reg}}, \quad \sigma(\mathcal{D}_{2,1}) \equiv 0 \text{ in } Z_{\text{pd}}(N) \cup Z_{\text{osc}}(N).$$

As a consequence of Lemma 5.5, we see that the compositions $d_{2,0}(t,\psi_{s,t})$ and $d_{2,0}(t,\psi_{s,t})$ satisfy

$$\begin{aligned}
&|D_x^\alpha D_\xi^\beta d_{2,0}(t,\psi_{s,t}(x,\xi))| \lesssim \Lambda(t)\langle x\rangle^{-|\alpha|}\langle\xi\rangle^{-|\beta|},\\
&|D_x^\alpha D_\xi^\beta d_{2,1}(t,\psi_{s,t}(x,\xi))| \lesssim \langle x\rangle^{-1-|\alpha|}\langle\xi\rangle^{-1-|\beta|}\frac{\lambda(t)}{\Lambda(t)}\left(\ln\left(\frac{1}{\Lambda(t)}\right)\right)^2.
\end{aligned}$$

In particular, since $\Lambda(t) \leq T\lambda(t)$ and, for all $(t,x,\xi) \in Z_{\text{hyp}}(N)$, it holds

$$\langle x\rangle^{-1}\langle\xi\rangle^{-1}\frac{\lambda(t)}{\Lambda(t)}\left(\ln\left(\frac{1}{\Lambda(t)}\right)\right)^2 \lesssim \lambda(t),$$

we may estimate

$$|D_x^\alpha D_\xi^\beta r_1(t,s,x,\xi)| \lesssim \lambda(t)\langle x\rangle^{-|\alpha|}\langle\xi\rangle^{-|\beta|} \tag{5.15}$$

for all $(t,x,\xi) \in Z_{\text{hyp}}(N)$. The obtained estimates allow to prove that the fundamental solution to (5.14) is given by a parameter-dependent pseudodifferential operator

$$Q_1(t,s)w(x) = \int_{\mathbb{R}^n} e^{i\,x\cdot\xi} q_1(t,s,x,\xi)\hat{w}(\xi)d\xi, \quad q_1(s,s,x,\xi) = 1. \tag{5.16}$$

Let us determine the matrix amplitude q_1 by means of an asymptotic expansion

$$q_1(t,s,x,\xi) \sim \sum_{j=0}^{\infty} q_{1,j}(t,s,x,\xi).$$

Inserting (5.16) into (5.14) we get the following system of ordinary differential equations for $q_{1,j}$:

$$D_t q_{1,0} + r_1(t,s,x,\xi) q_{1,0} = 0, \quad q_{1,0}(s,s,x,\xi) = 1,$$

and, for all $j > 0$,

$$D_t q_{1,j} + r_1(t,s,x,\xi) q_{1,j} = -\sum_{|\alpha|=1}^{j} \tfrac{1}{\alpha!} D_\xi^\alpha r_1(t,s,x,\xi) \partial_x^\alpha q_{1,j-|\alpha|}(t,s,x,\xi),$$

with the initial condition $q_{1,j}(s,s,x,\xi) = 0$. The solution of this system is given by

$$q_{1,0}(t,s,x,\xi) = \exp\Big(-i\int_s^t r_1(\tau,s,x,\xi)d\tau\Big),$$

$$q_{1,j}(t,s,x,\xi) = -i\int_s^t \exp\Big(-i\int_\sigma^t r_1(\tau,s,x,\xi)d\tau\Big)$$
$$\times \sum_{|\alpha|=1}^{j} \tfrac{1}{\alpha!} D_\xi^\alpha r_1(\sigma,s,x,\xi) \partial_x^\alpha q_{1,j-|\alpha|}(\sigma,s,x,\xi)d\sigma.$$

Employing the estimate (5.15), we can show that the function $q_{1,j} = q_{1,j}(t,s,x,\xi)$ satisfies

$$\Big|\partial_x^\beta \partial_\xi^\alpha q_{1,j}(t,s,x,\xi)\Big| \le C_{\alpha\beta} \langle x\rangle^{-j-|\beta|} \langle\xi\rangle^{-j-|\alpha|} |\Lambda(t) - \Lambda(s)|,$$

for each $j = 0, 1, \ldots$

Collecting all the results of this section, we have proved the next Proposition 5.13.

Proposition 5.13. *The parametrix $E_1 = E_1(t,s)$ to the operator $D_t - \mathcal{D}_1 + \mathcal{D}_2$ can be written as $E_1(t,s) = E_2(t,s)Q_1(t,s)$, where $E_2(t,s)$ is the diagonal matrix of Fourier integral operator from Proposition 5.9 and $Q_1 = Q_1(t,s)$ is a diagonal pseudodifferential operator with symbol belonging to $W^1_\infty\Big([0,T_0]^2, S^{0,0}\Big)$.*

5.4. Parametrix for the full system

Finally, we devote ourself to find $E = E(t,s)$ such that

$$D_t E - \mathcal{D}E + \mathcal{D}_2 E + \mathcal{B}_\infty E = 0, \quad E(s,s) = I,$$

modulo smoothing terms. Setting, as above, $E(t,s) = E_1(t,s)Q(t,s)$ we have to study

$$D_t Q + E_1(s,t)\mathcal{B}_\infty(t)E_1(t,s)Q = 0, \quad Q_1(s,s) = I. \tag{5.17}$$

The matrix operator $\mathcal{B}_\infty$ does not have diagonal form, so Egorov's theorem cannot be applied in this case. However, employing the property $\sigma(\mathcal{B}_\infty) \in \mathcal{HG}_N$ (see (4.9)), we will be able to prove that the composition $E_1(s,t)\mathcal{B}_\infty(t)E_1(t,s)$ is still a parameter-dependent pseudodifferential operator for all $0 \le s \le t \in [0,T]$.

Let us first consider the composition $\mathcal{B}_\infty(t)E_1(t,s)$ of the matrix $\mathcal{B}_\infty = (\mathcal{B}^{jk}_\infty)$ with the diagonal matrix $E_1(t,s) =$ diag $(E_1^-(t,s), E_1^+(t,s))$. Notice that, as a consequence of Proposition 5.13, there exists $e_1^\mp \in L_\infty\Big([0,T_0]^2, S^{0,0}\Big)$ such that

$$E_1(t,s)^\mp w(x) = \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} e^{i(\varphi^\mp(t,s,x,\xi)-y\cdot\xi)} e_1^\mp(t,s,x,\xi) w(y)dyd\xi.$$

Applying Theorem 3.5, we may conclude that $\mathcal{B}_\infty(t)E_1(t,s)$ is a matrix-valued Fourier integral operator

$$\mathcal{B}_\infty(t)E_1(t,s) = \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} e^{i(\varphi^\mp(t,s,x,\xi)-y\cdot\xi)} r_\mp(t,s,x,\xi) w(y)dyd\xi$$

with symbols having entries admitting asymptotic expansions of the form

$$\begin{aligned} r^{jk}_\mp(t,&s,x,\xi) \\ &\sim \sum_\alpha \frac{i^{|\alpha|}}{\alpha!}(D^\alpha_\xi \sigma(\mathcal{B}^{jk}_\infty))(t,s,x,\nabla_x\varphi^\mp(t,x,\xi)) D^\alpha_y[e^{i\Phi^\mp(t,x,y,\xi)} e_1^\mp(t,s,y,\xi)]_{y=x}, \end{aligned}$$

where $\Phi^\mp(t,x,y,\xi) = \varphi^\mp(t,y,\xi) - \varphi^\mp(t,x,\xi) + (x-y)\cdot\nabla_x\varphi^\mp(t,x,\xi)$.

By using Lemma 5.2, Lemma 5.11, Proposition 5.9, Theorem 5.12 and Proposition 5.13, we can obtain for all $\alpha,\beta \in \mathbb{N}^d$, the estimate

$$\begin{aligned} |D^\alpha_x D^\beta_\xi r_\mp(t,s,x,\xi)| &\lesssim C_{\alpha\beta}\langle x\rangle^{-|\alpha|}\langle\xi\rangle^{-|\beta|} \\ &\times \begin{cases} \lambda(t)\langle x\rangle^{-p}\langle\xi\rangle^{-p}\left(\frac{1}{\Lambda(t)}\ln\left(\frac{1}{\Lambda(t)}\right)\right)^{p+1} & \text{in } Z_{\text{reg}}(2N), \\ 1+\lambda(t)\langle x\rangle^{-1}\langle\xi\rangle^{-1}\left(\frac{1}{\Lambda(t)}\ln\left(\frac{1}{\Lambda(t)}\right)\right)^2 & \text{in } Z_{\text{osc}}(2N), \\ \langle x\rangle\langle\xi\rangle & \text{in } Z_{\text{pd}}(2N), \end{cases} \end{aligned}$$

where $p \in \mathbb{N}$ can assume an arbitrary large value.

Since we want to prove that $\mathcal{B}_\infty(t)E_1(t,s)$ is a pseudodifferential operator, we write it as

$$\mathcal{B}_\infty(t)E_1(t,s) = \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} e^{i\varphi^\mp(t,s,x,\xi)-ix\cdot\xi} r_\mp(t,s,x,\xi) w(y)dyd\xi,$$

and we prove that

$$\tilde{r}_{\mp}(t,s,x,\xi)) := e^{i\varphi^{\mp}(t,s,x,\xi)-ix\cdot\xi} r^{\mp}(t,s,x,\xi)$$

is the symbol of a pseudodifferential operator.

We note that the exponential term satisfies:

$$|D_x^\alpha D_\xi^\beta e^{i\varphi^{\mp}(t,s,x,\xi)-ix\cdot\xi}| \lesssim C_{\alpha\beta}\langle x\rangle^{-(1-\varepsilon)|\alpha|+\varepsilon|\beta|}\langle\xi\rangle^{\varepsilon|\alpha|-(1-\varepsilon)|\beta|} \tag{5.18}$$

if $\max\{s,t\} \le t_{x,\xi}$ and

$$|D_x^\alpha D_\xi^\beta e^{i\varphi^{\mp}(t,s,x,\xi)-ix\cdot\xi}| \lesssim C_{\alpha\beta}\langle x\rangle^{|\beta|}\langle\xi\rangle^{|\alpha|}\Lambda(t)^{|\alpha|+|\beta|},$$

if $t \ge s \ge t_{x,\xi}$. In particular, if (t,x,ξ) belongs to $Z_{\mathrm{osc}}(2N)$ we may estimate

$$\langle x\rangle^{\beta}\Lambda(t)^{|\beta|} \le 2N\langle\xi\rangle^{-|\beta|}(\ln(\langle x\rangle\langle\xi\rangle))^{2|\beta|} \lesssim \langle x\rangle^{\varepsilon|\beta|}\langle\xi\rangle^{-(1-\varepsilon)|\beta|},$$

and, similarly,

$$\langle\xi\rangle^{|\alpha|}\Lambda(t)^{|\alpha|} \le 2N\langle x\rangle^{-|\alpha|}(\ln(\langle x\rangle\langle\xi\rangle))^{2|\alpha|} \lesssim \langle x\rangle^{-(1-\varepsilon)|\alpha|}\langle\xi\rangle^{\varepsilon|\alpha|},$$

for any $\varepsilon > 0$ arbitrarily small. Then, (5.18) holds also in $Z_{\mathrm{osc}}(2N)$. Moreover, if $t \ge \max\{s, t'_{x,\xi}\}$, we obtain

$$\begin{aligned}&|D_x^\alpha D_\xi^\beta \tilde{r}_{\mp}(t,s,x,\xi)|\\ &\quad\lesssim \sum_{\substack{|\alpha_1|+|\alpha_2|=|\alpha|\\ |\beta_1|+|\beta_2|=|\beta|}} \lambda(t)\frac{\langle x\rangle^{|\beta_2|-|\alpha_1|-p}\langle\xi\rangle^{|\alpha_2|-|\beta_1|-p}}{\Lambda(t)^{p+1-|\alpha_2|-|\beta_2|}}\left(\ln\left(\frac{1}{\Lambda(t)}\right)\right)^{p+1};\end{aligned}$$

taking $p > 1$ sufficiently large we can estimate

$$\Lambda(t)^{p-|\alpha_2|-|\beta_2|} > N(\ln(\langle x\rangle\langle\xi\rangle))^{2p-2|\alpha_2|-2|\beta_2|}(\langle x\rangle\langle\xi\rangle)^{-p+|\alpha_2|+|\beta_2|}.$$

Taking into account that

$$\ln\left(\frac{1}{\Lambda(t)}\right) \le \ln(\langle x\rangle\langle\xi\rangle),$$

we may conclude

$$\begin{aligned}&|D_x^\alpha D_\xi^\beta \tilde{r}_{\mp}(t,s,x,\xi)|\\ &\quad\lesssim \frac{\lambda(t)}{\Lambda(t)}\ln\left(\frac{1}{\Lambda(t)}\right)\langle x\rangle^{-(1-\varepsilon)|\alpha|+\varepsilon|\beta|}\langle\xi\rangle^{\varepsilon|\alpha|-(1-\varepsilon)|\beta|}(\ln(\langle x\rangle\langle\xi\rangle))^{-p}\end{aligned}$$

for any $\varepsilon > 0$ arbitrarily small.

Similarly, since $\sigma(\mathcal{B}_\infty)$ belongs to $S^{0,0}\{0,0\}^{\text{hyp}}_{2N} + S^{-1,-1}\{-1,2\}^{\text{hyp}}_{2N}$ we may estimate for all $(t,x,\xi) \in Z_{\text{osc}}(2N)$

$$|D_x^\alpha D_\xi^\beta \tilde{r}_\mp(t,s,x,\xi)| \lesssim \langle x\rangle^{-(1-\varepsilon)|\alpha|+\varepsilon|\beta|}\langle\xi\rangle^{\varepsilon|\alpha|-(1-\varepsilon)|\beta|} \times \left(1 + (\ln(\langle x\rangle\langle\xi\rangle))^2\langle x\rangle^{-1}\langle\xi\rangle^{-1}\frac{\lambda(t)}{\Lambda(t)^2}\right),$$

for any α and β multi-indices.

Finally, since $\sigma(\mathcal{B}_\infty)$ satisfies (4.8) if $(s,x,\xi),(t,x,\xi) \in Z_{\text{pd}}(2N)$, then we may estimate

$$|D_x^\alpha D_\xi^\beta \tilde{r}_\mp(t,s,x,\xi)| \lesssim \langle x\rangle^{-(1-\varepsilon)|\alpha|+\varepsilon|\beta|}\langle\xi\rangle^{\varepsilon|\alpha|-(1-\varepsilon)|\beta|} \times \left(\rho(t,x,\xi) + \frac{\partial_t\rho(t,x,\xi)}{\rho(t,x,\xi)}\right),$$

for any α and β multi-indices. Summarizing, we obtain that for all $(s,t) \in [0,T_0]^2$ the operator $\mathcal{B}_\infty(t)E_1(t,s)$ is a pseudodifferential operator with symbol $\tilde{r}_\mp$ satisfying

$$|D_x^\alpha D_\xi^\beta \tilde{r}_\mp(t,s,x,\xi)| \le C_{\alpha\beta}\langle x\rangle^{-(1-\varepsilon)|\alpha|+\varepsilon|\beta|}\langle\xi\rangle^{\varepsilon|\alpha|-(1-\varepsilon)|\beta|} g_p(t,x,\xi)$$

where

$$g_p(t,x,\xi) := \begin{cases} \frac{\lambda(t)}{\Lambda(t)}\ln\left(\frac{1}{\Lambda(t)}\right)(\ln(\langle x\rangle\langle\xi\rangle))^{-p} & \text{in } Z_{\text{reg}}(2N), \\ 1 + (\ln(\langle x\rangle\langle\xi\rangle))^2\langle x\rangle^{-1}\langle\xi\rangle^{-1}\frac{\lambda(t)}{\Lambda(t)^2} & \text{in } Z_{\text{osc}}(2N), \\ \rho(t,x,\xi) + \frac{\partial_t\rho(t,x,\xi)}{\rho(t,x,\xi)} & \text{in } Z_{\text{pd}}(2N). \end{cases} \tag{5.19}$$

In particular, there exist K_1, K_2, K_3, positive constants depending only on N, such that

$$\int_{t'_{x,\xi}}^{T} g_p(t,x,\xi)\,dt \le \ln(\Lambda(t'_{x,\xi}))^2(\ln(\langle x\rangle\langle\xi\rangle))^{-p} \le K_1,$$

$$\int_{t_{x,\xi}}^{t'_{x,\xi}} g_p(t,x,\xi)\,dt \le \frac{(\ln(\langle x\rangle\langle\xi\rangle))^2}{\Lambda(t_{x,\xi})\langle x\rangle\langle\xi\rangle} \le K_2\ln(\langle x\rangle\langle\xi\rangle),$$

and

$$\int_0^{t_{x,\xi}} g_p(t,x,\xi)\,dt \le \frac{K_3}{2}\left(1 + \sqrt{\langle x\rangle\langle\xi\rangle\ln(\langle x\rangle\langle\xi\rangle)}\int_0^{t_{x,\xi}}\frac{\lambda(t)}{\sqrt{\Lambda(t)}}\,dt\right) \le K_3\ln(\langle x\rangle\langle\xi\rangle).$$

In the same way we can treat the operator

$$R(t,s) := E_1(s,t)\Big(\mathcal{B}_\infty(t)E_1(t,s)\Big). \tag{5.20}$$

We have proved the next Proposition 5.14.

Proposition 5.14. *For all $s \le t \in [0,T_0]$, the matrix operator $R = R(t,s)$ defined by (5.20) is a pseudodifferential operator whose symbol $\sigma(R(t,s))$ belongs to $L_\infty\Big([0,T_0]^2, S^{1,1}_{(\varepsilon)}\Big)$ and satisfies*

$$|D_x^\alpha D_\xi^\beta \sigma(R)(t,s,x,\xi)| \lesssim C_{\alpha\beta}\langle x\rangle^{-(1-\varepsilon)|\alpha|+\varepsilon|\beta|}\langle\xi\rangle^{\varepsilon|\alpha|-(1-\varepsilon)|\beta|}g_p(t,x,\xi),$$

for every $p \ge 0$ and $\varepsilon > 0$ arbitrarily small, with the function g_p defined by (5.19).

Following the same approach in [46, Proposition 3.9.1] one can prove the following Lemma 5.15, that allows to estimate the loss of regularity and the loss of decay due to the *bad behavior* in the pseudodifferential and oscillation zones.

Lemma 5.15. *Let $Q(t,s)$ the solution, modulo smoothing operators, to the Cauchy problem*

$$D_tQ + R(t,s)Q = 0, \quad Q(t,s) = I, \tag{5.21}$$

where $\sigma(R) \in L_\infty([0,T_0]^2, S^{m,\mu}_{r_1,r_2,\rho_1,\rho_2})$. Assume that, for all $\alpha,\beta \in \mathbb{N}^d$, there exists $C_{\alpha\beta} > 0$ such that

$$|D_x^\alpha D_\xi^\beta \sigma(R)| \le C_{\alpha\beta}\langle x\rangle^{-r_1|\alpha|+r_2|\beta|}\langle\xi\rangle^{\rho_1|\alpha|-\rho_2|\beta|}g(t,x,\xi), \tag{5.22}$$

for some $g \in C([0,T_0]^2 \times \mathbb{R}^{2d})$. Suppose that $g(t,x,\xi) \lesssim \langle x\rangle^\ell\langle\xi\rangle^\omega$ for some $\ell,\omega \in \mathbb{R}$ and it holds

$$\int_0^{T_0} g(t,x,\xi)dt \le K\ln(\langle x\rangle\langle\xi\rangle).$$

Then, there exists a solution Q to (5.21) with matrix symbol $\sigma(Q)$ satisfying

$$|D_x^\alpha D_\xi^\beta \sigma(Q)| \le C_{\alpha\beta}\langle x\rangle^{K-r_1|\alpha|+r_2|\beta|}\langle\xi\rangle^{K+\rho_1|\alpha|-\rho_2|\beta|}(\ln(\langle x\rangle\langle\xi\rangle))^{|\alpha|+|\beta|+1}.$$

Such solution is unique modulo $C([0,T_0]^2, \mathrm{Op}(S^{-\infty,-\infty}))$.

As a consequence of the results proved above, employing Lemma 5.15, we finally obtain the concluding result of this section, the next Proposition 5.16.

Proposition 5.16. *The parametrix $E = E(t, s)$ to the operator $D_t - \mathcal{D} + \mathcal{D}_2 + \mathcal{B}_\infty$ can be written as $E(t, s) = E_1(t, s)Q(t, s)$, where $E_1 = E_1(t, s)$ is the matrix of Fourier integral operators given in (5.17) and Q is a matrix of parameter-dependent pseudodifferential operators with symbol belonging to*

$$L_\infty\Big([0, T_0]^2, S_\varepsilon^{K_0,K_0}\Big) \cap W^1_\infty\Big([0, T_0]^2, S_\varepsilon^{K_0+1+\varepsilon,K_0+1+\varepsilon}\Big),$$

for every small $\varepsilon > 0$. Here, the constant K_0 describes the loss of derivatives and decay coming from the pseudodifferential zone $Z_{\mathrm{pd}}(2N)$ and the oscillations subzone $Z_{\mathrm{osc}}(2N)$.

Remark 5.17. We notice that the function g_p in (5.19) is integrable in $[0, T]$. In particular, there exists a constant K_0 such that

$$\int_0^T g_p(t, x, \xi)\, dt \lesssim K_0 \ln(\langle x\rangle\langle \xi\rangle). \tag{5.23}$$

This can be shown observing that, since Λ is increasing in $[0, T]$, we may estimate

$$\begin{aligned}\int_{t'_{x,\xi}}^T \frac{\lambda(t)}{\Lambda(t)^{q+1}} \ln\left(\frac{1}{\Lambda(t)}\right)^{q+1} dt &\lesssim \frac{1}{\Lambda(t'_{x,\xi})^q} \ln\left(\frac{1}{\Lambda(t'_{x,\xi})}\right)^{q+1}\\ &\lesssim \langle x\rangle^q\langle \xi\rangle^q \ln(\langle x\rangle\langle \xi\rangle),\end{aligned}$$

for any exponent $q > 0$. The value of K_0 in (5.23) determines the loss of derivatives and the loss of decay obtained in Proposition 5.16.

6. Solution of the Cauchy problem

The analysis performed in the previous sections allows now us to prove our main results, namely, well-posedness, regularity and decay of the solutions to (1.2). The uniqueness of the solution of (1.2) in the weighted Sobolev spaces follows by standard arguments, see [12]. The same is true concerning the existence and uniqueness of the fundamental solution of the system in (4.4), which we have determined modulo smoothing operators. Let us state this result precisely, in the next Lemma 6.1.

Lemma 6.1. *The fundamental solution $F(t, s)$ of (4.4) has the representation $F(t, s) = E(t, s) + \mathcal{G}_\infty(t, s)$, where $E(t.s)$ comes from Proposition 4.6, Theorem 4.7, and Propositions 5.13 and 5.16. Moreover, $\mathcal{G}_\infty(t, s) = \mathrm{Op}(g_\infty(t, s))$ with $g \in W^1_\infty\Big([0, T_0]^2, SG^{-\infty,-\infty}\Big)$, for a suitably small $T_0 \in (0, T]$.*

The next Corollary 6.2 follows then immediately, by the Duhamel's principle.

Corollary 6.2. *The unique solution $U = U(t)$ of (4.4) can be written as*

$$U(t,x) = F(t,0)U_0(x) + \int_0^t F(t,s)G(s,x)ds.$$

Corollary 6.2 implies our first main result, Theorem 6.3, about solutions of the Cauchy problem (1.2) in the scale of Sobolev-Kato spaces.

Theorem 6.3 *(Loss of regularity and decay for solutions in weighted Sobolev spaces). Consider the Cauchy problem (1.2), where the coefficients a_j, b_j, $j = 1,\ldots,d$, and c satisfy the assumptions in Proposition 2.3 (A). Assume also that the initial data satisfy $\varphi \in H^{s,\sigma}(\mathbb{R}^d), \psi \in H^{s-1,\sigma-1}(\mathbb{R}^d)$ and that $g \in C\Big([0,T], H^{s,\sigma}(\mathbb{R}^d)\Big)$, with suitably large s,σ. Then, for a suitably small $T_0 \in (0,T]$, the Cauchy problem (1.2) admits a unique solution*

$$\begin{aligned} &u \in C\Big([0,T_0], H^{s-s_a,\sigma-s_a}(\mathbb{R}^d)\Big) \\ &\quad \cap C^1\Big([0,T_0], H^{s-s_a-1,\sigma-s_a-1}(\mathbb{R}^d)\Big) \cap C^2\Big([0,T_0], H^{s-s_a-2,\sigma-s_a-2}(\mathbb{R}^d)\Big). \end{aligned}$$

The loss of derivatives and decay s_a depends on the constant K_0 from Proposition 5.16.

Proof. The claim follows immediately by the decomposition $F = E + \mathcal{G}_\infty$ of the fundamental solution of the system (4.4) from Lemma 6.1, the relation between its solution U and the solution u of (1.2), the mapping properties of SG pseudodifferential and Fourier integral operators on the Sobolev-Kato spaces, and the properties of E, proved in Section 5. □

Since $\mathscr{S}(\mathbb{R}^d) = H^{\infty,\infty}(\mathbb{R}^d) = \cap_{s,\sigma\in\mathbb{R}} H^{s,\sigma}(\mathbb{R}^d)$, from Theorem 6.3 we deduce our second and final main result, the next Theorem 6.4.

Theorem 6.4 *($\mathscr{S}(\mathbb{R}^d)$-well-posedness). Consider the Cauchy problem (1.2), where the coefficients a_j, b_j, $j = 1,\ldots,d$, and c satisfy the assumptions in Proposition 2.3 (A). Assume also that the initial data satisfy $\varphi,\psi \in \mathscr{S}(\mathbb{R}^d)$ and that $g \in C\Big([0,T], \mathscr{S}(\mathbb{R}^d)\Big)$. Then, for a suitably small $T_0 \in (0,T]$, the Cauchy problem (1.2) admits a unique solution $u \in C^2\Big([0,T_0], \mathscr{S}(\mathbb{R}^d)\Big)$.*

Example 6.5. Choose a shape function λ, satisfying the hypotheses described in Section 2.1, and consider the Cauchy problem

$$\begin{cases} \partial_{tt}u(t,x) + \lambda(t)^2 \left[2 + \cos\ln\left(\left(\dfrac{1}{\Lambda(t)}\right)\right)^{\ell}\right](1+|x|^2)(1-\Delta_x)u(t,x) = 0 \\ u(0,x) = \varphi(x), \quad u_t(0,x) = \psi(x), \end{cases} \tag{6.1}$$

$t \in [0,T]$, $x \in \mathbb{R}^d$, for some $0 \le \ell \le 2$. For the operator in (6.1) we have

$$a_j(t,x) = \lambda(t)^2 \left[2 + \cos\left(\ln\left(\frac{1}{\Lambda(t)} \right) \right)^{\ell} \right] \langle x \rangle^2 = c(t,x),$$

$$b_j(t,x) = 0, \quad j = 1, \dots, d,$$

so that

$$a(t,x,\xi) = \lambda(t)^2 \left[2 + \cos\left(\ln\left(\frac{1}{\Lambda(t)} \right) \right)^{\ell} \right] \langle x \rangle^2 \langle \xi \rangle^2.$$

Evaluating explicitely the two roots τ_1 and τ_2 of the complete symbol $a(t,x,\xi)$ it is easy to derive that Assumption **(H)** in Proposition 2.3 holds true, and then our theory applies to the operator in (6.1). Notice that, being $0 \le \ell \le 2$, with the same approach used in the proof of Proposition 2.3 it is also possible to prove that the coefficient $c(t,x)$ satisfies

$$|D_t^k D_x^\alpha c(t,x)| \le C_{k\alpha} \lambda(t)^2 \langle x \rangle^{-|\alpha|} \left(\frac{\ln \lambda(t)}{\Lambda(t)} \right)^2 \left(\frac{\lambda(t)}{\Lambda(t)} \ln\left(\frac{1}{\Lambda(t)} \right) \right)^k,$$

for suitable positive constants $C_{k\alpha}$ depending on $k \in \mathbb{N}$ and $\alpha \in \mathbb{N}^d$. In particular, in view of the presence of the oscillating t-dependent factor, here logarithms indeed appear in the coefficients estimates.

Declaration of competing interest

None declared.

Data availability

No data was used for the research described in the article.

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