



Mathematics teacher educators' documents, praxeologies, and beliefs: a holistic model

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Abstract

The research presented in this paper aims to shed light on the complex role of mathematics teacher educators who led courses for prospective mathematics teachers. Theoretically, the research combines the documentational approach to didactics and the meta-didactical transposition frameworks, complementing the study with the analysis of the teacher educators' beliefs about the prospective teachers' needs. The methodology entails a comparative case study related to two courses, taking place in quite different contexts, led by the two authors independently, and a collaborative self-study of teacher education approach, based on eight reciprocal semi-structured interviews between the authors. The result is a holistic model of mathematics teacher educators' work that takes into account their practices, including the design of teaching materials based on a variety of resources, justifying discourses and personal beliefs. This can deepen our insight into the expertise of mathematics teacher educators and, prospectively, can be fruitful for their professional development.

Keywords Beliefs · Documentation work · Meta-didactical praxeologies · Teacher educators · Self-study

Introduction

In the last decades, considerable attention has been given in research to the knowledge required of the mathematics teachers to be effective in their work (e.g., Ball et al., 2008; Rowland, 2013; Shulman, 1987). Shulman's (1987) theoretical framework enlists seven categories for teacher knowledge, which soon became the foundation for describing the Mathematical Knowledge for Teaching (MKT), defined as the "mathematical knowledge needed to perform the recurrent tasks of teaching mathematics to students" (Ball et al., 2008, p. 399). While MKT has been of great interest to researchers in mathematics education, the knowledge required of mathematics teacher educators (MTEs) has

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received far less attention (Chapman, 2008). Some authors (e.g., Beswick & Chapman, 2013) wonder whether the same models of mathematics teachers' knowledge can be also applied to MTEs, but they acknowledge inevitable differences. Zopf (2010) defined MKTT (Mathematical Knowledge for Teaching Teachers) as "the mathematical knowledge used by mathematics teacher educators in the work of teaching mathematics to teachers" (p. 11), and claimed that it is different in some ways from MKT, because MTEs have to help prospective and practicing teachers develop their MKT, so they need knowledge at a different level (Masingila et al., 2018). Furthermore, the need for nuanced approaches that capture the flexibility inherent in this field without rigid compartmentalization of knowledge is recognized in the literature (Scheiner & Bowers, 2023).

An important prompt for researchers in mathematics teacher education can be identified in Chapman's (2021) challenge to move beyond category-based descriptions of MTE knowledge in order to capture its complex nature. A major element of difficulty in addressing this complexity is that often MTEs learn by researching their own practices (Chapman, 2008; Tzur, 2001) and share their experience with other researchers in the field (e.g., Chauvot, 2009; Jaworski & Wood, 2008; Tzur, 2001). For example, Tzur (2001), building on his personal experiences of becoming an MTE, testifies to the importance of promoting MTEs' appreciation for reflecting about their own practice. However, with self-based methodologies, it is particularly difficult for MTEs to consider factors from their own affective domain, even though they constitute a key element for understanding their practice. Indeed, MTEs' beliefs and personal vision of what mathematics teachers should learn have an impact on the content and general orientation of their education programs (Escudero-Ávila et al., 2021). Probably also due to these difficulties, a research gap still exists in relation to practitioner-research conducted by MTEs, which utilizes self-based methodologies.

In this paper, we aim to take up Chapman's (2021) challenge to move beyond category-based descriptions of MTEs' knowledge, providing a characterization which enables us to capture the complex nature of their work and expertise with a holistic model incorporating MTEs' practices, included the design of teaching materials, justifying discourses, and beliefs. The reason of this aim is the theoretical necessity of considering this complexity of MTE's professional life, overcoming dichotomies (Sinclair et al., 2022) and including their practices, knowledge, and beliefs in a single interpretive model, which draws on all components. This aim is addressed through a comparative case study approach, entailing the analysis and the comparison of two cases, in which two MTEs (the authors) are researchers in mathematics education who each led a method course for the teaching/learning of mathematics devoted to prospective teachers (PTs). Since they work in two quite different contexts, the two MTEs adopt different practices and corresponding justifying discourses (praxeologies). The Meta-Didactical Transposition (MDT) framework (Arzarello et al., 2014; Cusi et al., 2023) enables us to study their praxeologies, that are more specifically called *meta-didactical praxeologies*. Meta-didactical praxeologies are referred to the professional development of teachers (whether PTs or in-service teachers), and regard the building of didactic praxeologies (Chevallard, 1985). As part of their meta-didactical praxeologies, the two MTEs rely on different kinds of *resources* to design teaching materials with specific utilization schemes for the work with the PTs. Therefore, to study MTEs' design work and their interaction with resources, we adopt the Documentational Approach to Didactics (DAD) framework (Gueudet & Trouche, 2009, 2012).

The potentialities of studying the interactions between teachers and resources from different theoretical perspectives have been explored by Trouche et al. (2019), who combined the DAD, the Anthropological Theory of Didactics (ATD) (Chevallard, 1985;

2019; Chevallard & Bosch, 2020), and the Cultural–Historical Activity Theory (CHAT) (Engeström, 2014; Leont’ev 1981) frameworks, to “develop a more nuanced view of the processes at stake” (Trouche et al., 2019, p. 54). We also seek to combine the DAD framework with the study of praxeologies (Chevallard, 2019), which, in our case, are meta-didactical ones, because we aim to study MTEs’ interaction with resources in the context of mathematics teacher education.

A combination (in the sense of networking, as detailed in Prediger et al., 2008) of the MDT and the DAD frameworks has been already adopted by Pocalana and Robutti (2023) for analyzing the intertwined evolutions of two MTEs’ meta-didactical praxeologies and documentation work, according to the evolving goal of the teacher professional development course that they were leading. In that case, however, the course under scrutiny was only one, led by both MTEs’ together, and it was addressed to in-service teachers. In this paper, we extend the research reported by Pocalana and Robutti (2023) with the comparison of two different courses for PTs, carried out in different contexts, in order to test and possibly adapting the model introduced in that study (Fig. 1), complementing it with MTEs’ beliefs about the needs of the PTs attending their respective courses.

In light of what has been said, the aim of this paper can be specified in studying the work and the expertise of MTEs from three different and intertwined points of view: their documentation work, their meta-didactical praxeologies, and their beliefs. Studying all these aspects in a holistic way, shedding light on the relationships among them, could provide a deeper insight into MTEs’ expertise and, in turn, is likely to have implications for their learning and development. Therefore, our research question is formulated as follows:

RQ) What relationships can be identified between MTEs’ documentation work, meta-didactical praxeologies, and personal beliefs about the PTs attending their courses?

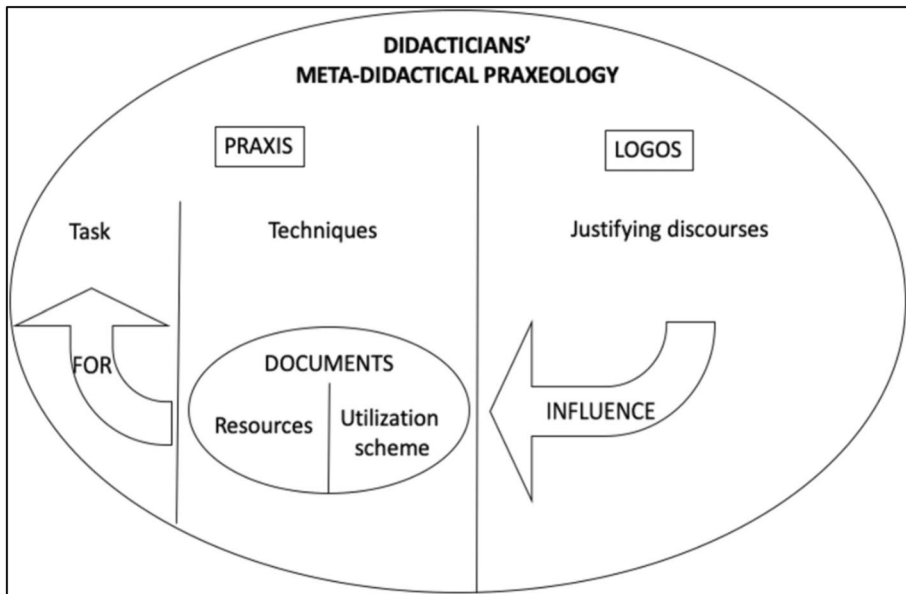


Fig. 1 Model for MTEs’ work, based on the MDT and the DAD frameworks (Pocalana & Robutti, 2023) (*Didacticians* was the word used to identify the MTEs)

Documentational approach to didactics

The DAD framework (Gueudet & Trouche, 2009, 2010, 2012; Gueudet et al., 2012; Trouche et al., 2019) has been introduced to study the work and the professional development of teachers, using different kinds of resources to prepare their lessons. Its emergence is concomitant with the “conceptual developments grounded in didactics (Vergnaud, 2009) and ergonomics (Vérillon & Rabardel, 1995)” (Trouche et al., 2023, p. 498) coming from France. In the DAD framework, resources are intended in a holistic way as anything, material or symbolic, digital or non-digital, capable of re-sourcing mathematics teachers’ work (Adler, 2000).

The fundamental concept on which DAD is based is that of *document*: a set of *resources* and related *utilization schemes*. The notion of *scheme* comes from Vergnaud (2009) and encompasses the invariant organization of activity for facing a class of situations with the same aim. The utilization schemes for a set of resources are constituted by: classes of situations in which the resources are used, rules of action, which are visible elements in the way the resources are used, and operational invariants, which constitute the invisible cognitive structure guiding the teacher’s action and are both driving forces and outcomes of the teacher’s activity. Therefore, documents comprehend both a material and a psychological component. The process by which documents are generated on the basis of a set of resources is named *documentational genesis* (Gueudet & Trouche, 2009). A documentational genesis entails the intertwined evolution of resources and teachers who use, organize, transform and design resources (Trouche et al., 2023).

During the years, the DAD has been applied also to contexts different from the study of teachers’ work and growth. For example, Gueudet et al. (2012) observed the documentational genesis of trainee teacher educators generating documents for each other’s training; while, Kieran et al. (2013) studied the work of researchers designing documents for students. Furthermore, Psycharis and Kalogeria (2018) focused on trainee teacher educators designing documents for teachers, in the context of a teacher professional development course. Pocalana and Robutti (2022, 2023) studied the documentational genesis of researchers with the role of MTEs for in-service mathematics teachers. What is still missing are examples of applications of the DAD framework to the study of the work of researchers having the role of MTEs in a method course for the teaching/learning of mathematics devoted to PTs.

Meta-didactical transposition

The MDT framework (Arzarello et al., 2014) is based on Chevallard’s ATD (Chevallard & Bosch, 2020; Chevallard, 1985, 2019), whose main underlying concept is that of *praxeology*. A praxeology is constituted by two components: the *praxis* (or “know how”) component, that is the combination of a type of task to be solved and a technique to solve it, and the *logos* (or “know why”) component that is a combination of a technology (justification of the technique) and a theory. Praxeologies can be mathematics, if they are referred to mathematical tasks, techniques to solve them, and a *logos* based on mathematical theories (Chevallard, 1985; 2019). They can be didactic praxeologies (Chevallard, 1985, 2019), if they are referred to teaching mathematics, so their *praxis* and *logos* components are related to teach some specific mathematical praxeology. Miyakawa and Wilson (2019) adopt the concept of *paradidactic praxeologies* to study teachers’ activities connected

to teaching, which are not directly teaching in their classrooms. Indeed, the paradidactic system, according to Winsløw (2012), regards teachers' activities outside the classroom, such as preparing lessons, reflecting on the lessons, meeting with colleagues, etc. Although these activities can occur under the supervision of a teacher educator (Otaki & Asami-Johansson, 2021), in the paradidactic system, the focus is on teachers' praxeologies.

In the MDT framework (Aldon et al., 2013; Robutti et al., 2019), the praxeologies under scrutiny are meta-didactical ones, in the sense that their *praxis* and *logos* components are related to the collaborative building by researchers (with the role of teacher educators) and teachers of didactic praxeologies, guided by meta-reflections about them developed during a teacher professional development course. Thus, meta-didactical praxeologies are specifically conceptualized to address the collaborative work of researchers and teachers, emphasizing the equalitarian position of the members of the two communities and their resulting intertwined evolution toward shared *praxis* and *logos* components.

What is still missing is a development of the concept of meta-didactical praxeology enabling us to study the work of researchers in mathematics education (who are MTEs) leading courses for PTs, in which the type of task consists into building and reflecting upon didactic praxeologies during courses devoted to students needing to be prepared to become teachers. Another research gap that remains to be filled regards the connection between this kind of meta-didactical praxeologies and MTEs' personal beliefs about the needs of the PTs attending their course. Indeed, Cusi et al. (2023), in their description of the evolution of the MDT framework named MDT.2, identified personal beliefs as one of the motivational agents for the evolution of teachers' meta-didactical praxeologies, in the context of a professional development course. Pocalana and Robutti (2024) addressed the double direction relationship between the evolution of teachers' meta-didactical praxeologies and the evolution of their beliefs, considered both as agent and consequence of each other. The present study is aimed at investigating if and how a connection between MTEs' meta-didactical praxeologies and beliefs can be detected, taking in consideration their documentation work as part of their praxeologies. Therefore, we address this aim focusing on a specific moment of the evolution process of MTEs' meta-didactical praxeologies.

Beliefs and self-study

Philipp (2007) defined beliefs as “lenses that affect one’s view of some aspect of the world or as dispositions toward action” (p. 259). Teacher’s beliefs are often conceived as tacit, often unconsciously held assumptions about students, classrooms, and the curriculum materials to be taught (Kagan, 1992). We have known for decades that mathematics teachers’ beliefs about mathematics and its teaching have a primary role in influencing how they teach (e.g., Beswick, 2005; Ernest, 1989; Philipp, 2007) and, in turn, in influencing their students’ beliefs about mathematics (e.g., Grootenboer, 2008; Mosvold & Fauskanger, 2014; Pajares, 1992). We also know that PTs’ beliefs are influenced by their previous experiences as school students, their MTEs, the curriculum documents that they study, and their practicum experiences (Marshman, 2021).

If mathematics teachers’ beliefs have received a lot of attention, especially in the last decades, conversely, MTEs’ beliefs have been less studied (Goos, 2009). Probably, this is also due to the fact that often MTEs study their own work and report about their actions, tensions and doubts in conducting their courses, and it is particularly difficult for them to

study their own beliefs. Furthermore, according to Leatham (2006), it is not guaranteed that an individual is able to express their beliefs or even be aware of them.

Despite these difficulties, Chapman (2008) addresses the relationship between beliefs and self-study in the case of mathematics PTs reflecting on their own beliefs on both content and pedagogical knowledge. According to Chapman (2008), indeed, self-study in PTs education is related to the conceptualization of beliefs as a basis of what teachers do and the reasons why they do it. In her studies, beliefs are considered at different levels:

- beliefs as a basis of knowledge. Beliefs are associated with teachers' knowledge about their roles, their students, mathematics and the context of their teaching and provide a basis of the knowledge useful for mathematics teaching;
- beliefs as a basis of disposition. Dispositions are intended as habits of mind, guided by teachers' beliefs related to values such as caring, fairness, social justice, etc.;
- beliefs as a basis of action. Beliefs are considered to have a strong relationship with what teachers do in their classrooms.

It seems natural to think that, also in the case of MTEs, beliefs play an important role in influencing their knowledge, disposition and action. However, much more than in PTs' case, MTEs' beliefs are likely to be influenced by theoretical studies and research (Bergsten & Grevholm, 2008), especially if they are researchers in mathematics education. For this reason, when studying MTEs' beliefs and practices, we cannot ignore their theoretical background and the *logos* permeating the discourses about PTs' professional development (broadly) shared in the mathematics education community.

Therefore, in this study, we consider MTEs both in their institutional position as part of their professional community, taking into account their praxeologies, and as individuals holding personal beliefs, thus providing a holistic and comprehensive insight into their complex work.

Methodology

The context of the study

The two courses for PTs analyzed in this paper are led independently by two researchers in mathematics education (the authors), in contexts which present several differences:

- As far as the course led by the first author (henceforth, course A) is concerned, the PTs have already taken a bachelor's degree in natural sciences, and are attending a master's degree in natural science, which includes courses both in science education and mathematics education. In the case of the course led by the second author (henceforth, course B), instead, the PTs have a bachelor's degree in mathematics and are attending a master's degree in mathematics with many courses in mathematics education and history of mathematics.
- The PTs attending course A have taken less university courses in mathematics (only one) with respect to the PTs attending course B.
- Coherently with their academic studies, the PTs attending course A can become mathematics and science teachers in lower secondary schools (grades 6–8); while, the PTs

attending course B can become mathematics teachers both in lower and upper (grades 9–13) secondary schools.

The MTEs are also different, especially in terms of experience as teacher educators. The first author (henceforth, MTE A), indeed, is leading a course for PTs for the first time, after having completed her doctoral degree in mathematics education. In the previous five years, she has led professional development courses for in-service mathematics teachers (Pocalana & Robutti, 2023; Pocalana & Robutti, 2022; Pocalana et al., 2023). On the contrary, the second author (henceforth, MTE B) is a full professor of mathematics education with more than thirty years of experience teaching PTs.

On the other side, the MTEs presents some similarities, for example they both have a degree in mathematics (MTE B has a degree in Physics too) and they both have taught in secondary schools (MTE A in lower secondary and MTE B in upper secondary schools) for several years before dedicating to research. Due also to the fact that MTE B has been MTE A's doctoral supervisor, the two MTEs share their background, both from the point of view of the institutional references (MIUR, 2010, 2012) and of the research in mathematics education. More specifically, their theoretical references encompass contextualized didactical approaches to the teaching/learning of mathematics: student-centered (Noyes, 2012), laboratory (Arzarello & Robutti, 2008) and inquiry-based (Laursen & Rassmussen, 2019).

Data collection and data analysis

For this study, we collected data from the transcripts of eight second-person semi-structured interviews (Bissel et al., 2021) of the two MTEs, conducted online and video-recorded. In the first four interviews, MTE A interviewed MTE B and in the last four interviews, vice versa.

The main questions, constituting the basic structure of the interviews, are: “Can you describe how your course is structured?”, “What resources did you rely on to design your teaching materials?”, “How did you use these resources with the PTs?”; “Can you explain the reasons why you decided to organize the lessons in this way?”. In addition to the main questions, each MTE's description of her own choices and practices was always followed by questions of the type “Why this choice?”, “Why do you think this is useful?”. Besides the interviews, we collected data from all the teaching materials designed by the MTEs for the PTs, collected in the online platforms of their respective courses.

As a general approach for our study, we adopt a *self-study of teacher education* (S-STEP) methodology (Kitchen et al., 2020). More precisely, taking into account the interactions among the MTEs and their collaboration as a critical part of their inquiry into self-study, the research methodology can be described as a collaborative self-study (Bullock, 2020). Self-study has been defined as a “post-qualitative methodology” (Hamilton et al., 2020, p. 305): researchers studying self to understand practice and ways of thinking about practice.

Collaborative self-study entails an interaction between colleagues, based on dialog and an inquiry attitude, aimed at extending themselves beyond a comfort zone (Bullock, 2020). The collaborative endeavor contributes to overcome the MTEs' difficulties in studying themselves underlined by Leatham (2006). Indeed, self-study is self-focused but, collaborating with each other, researchers can elaborate reflections on their practice that might resonate with others' experiences (Loughran, 2005). The commitment to dialog, entailing the examination of practices for both understanding and improvement, is a key element for establishing trustworthiness in this kind of research (Hamilton et al., 2020). Due to the fact

that MTEs, in their complex contexts, are constantly moving and becoming, quality and trustworthiness in their research rely on their “nuanced accounts of practice as lived and understood by the person conducting the study” (ibid., p. 310).

Data analysis has been conducted in three rounds with a qualitative approach. In each round, the analysis has been conducted at first separately by both the researchers and, in a second time, discussed until reaching a complete agreement.

In the first round, we adopted a methodology inspired by the reflective investigation (Gueudet & Trouche, 2012) typical of the DAD framework, in which the researchers try to access teachers’ (in our case, MTEs’) thinking through reflective analysis (Trouche et al., 2019). Reflective analysis refers to a process in which an individual critically reflects on an experience with the aim of gaining a better understanding of significant aspects of the experience. We relied on MTEs’ descriptions of their own interaction with resources, and we collaboratively provided an account of the different kinds of resources on which the two MTEs based their design work, and the utilization schemes that they adopted in the generation of their documents for their respective courses. The focus of this analysis is “on activity, on what actors in the educational environment use and/or design for their work. [...] This approach is more directly linked to activity [...] rather than to characteristics or features of the actors (their knowledge or beliefs, for example)” (Sinclair et al., 2022, p. 400). This conceptualization of the DAD approach is coherent with the model proposed in Pocalana & Robutti (2023), in which MTEs’ generation of documents is conceived as part of the *praxis* component of their meta-didactical praxeologies.

In particular, we selected three documents for each MTE that not only capture the essence of their interactions with PTs but are also sufficiently distinct from one another, ensuring a comprehensive account of the MTEs’ practices. For each document, we provided a description of:

- the resources on which the MTEs relied on, according to their respective answers to the question asked in the interview: “What resources did you rely on to design your teaching materials?”
- The rules of action of each document, according to MTEs’ answers to the question: “How did you use these resources with the PTs?”
- The operational invariants underlying each document, according to MTEs’ answers to the questions: “Can you explain the reasons why you decided to organize the lessons in this way?”

All the information retrieved in the first round has been schematically organized in tables structured as Table 1.

In the second round, we analyzed again the data coming from the transcripts of the answers to all the questions of the interviews, to identify the general meta-didactical praxeologies of the two MTEs for co-building and reflecting on didactic praxeologies with PTs in their respective courses. In particular, we identified the techniques adopted to address their specific type of task (*praxis* component), and the technologies and theories, that is their theoretical discourses justifying their choices (*logos* component). Every human activity, and their outputs, can be described in terms of praxeologies, in view of the fact that the type of task can be considered “in a very general sense, irrespective of its volume or pettiness” (Chevallard et al., 2015, p. 2615). According to Chevallard (1999), technologies are rational discourses justifying the techniques. The technological discourses encompass assertions, some more implicit than others, that can be subject to scrutiny. This scrutiny leads to a higher level of justification and explanation, that of

Table 1 Scheme of analysis for MTEs' documents

Document n generated by MTE x	
Resources	Utilization schemes
All the resources on which the MTE declared to have relied on for the design of her teaching materials	<p><i>Class of situations</i>: Type of situation in which the resources have been employed</p> <p><i>Rules of action</i>: Specific ways in which the MTE utilized the resources with the PTs</p> <p><i>Operational invariants</i>: Guiding principles that underlay the rules of action</p>

theory. In the context of praxeologies, theory plays a role in relation to technology analogous to what technology does in relation to technique. One can conceive of an ongoing justificatory regression ad infinitum, suggesting the possibility of a theory of the theory, and so forth. However, the three-level description presented here (technique/technology/theory) typically suffices to comprehend the activity under analysis.

Due to the characteristics of the justifying discourses in the meta-didactical praxeologies under scrutiny (which are not mathematical praxeologies), we chose to analyze technologies and theories as a unique component, referring to them as *logos*, because the difference between these two components is more nuanced in respect to the case of mathematical praxeologies. In our case, indeed, the *logos* component is intended as a discourse shared to some extent among the community of researchers in mathematics education who led courses for PTs (or in-service teachers) about what kind of didactic praxeologies are important to be built and how it is appropriate to do it. While, in mathematical praxeologies, the *logos* component is constituted by mathematical technologies and theories, for meta-didactical praxeologies like those under scrutiny in this paper, there is a vast reference literature (e. g., Jaworski & Wood, 2008; Beswick, & Chapman, 2020) which constitutes the basis of the *logos*. Indeed, it is a growing corpus of shared thoughts, suggestions, and reflections that could inspire and guide the work of MTEs. This is coherent with what is claimed in Chevallard et al. (2015):

“In ATD a theory is thus a hypothetical reality that assumes the form of a (necessarily fuzzy) set of explicit and implicit statements about the object of the theory. A theory is in truth the current state of a dialectic process of theorization of which it offers an instantaneous and partial view that may prove delusive” (p. 2619).

Accordingly, we can say that MTEs' *logos* is permeated by the broader literature in mathematics education which they consider as theoretical reference for their work with the PTs, generally referred to the teaching and learning of mathematics, to the students' needs, students' cognitive and affective aspects, etc., and the institutional references based on the curricular requirements of their national school system (MIUR, 2010, 2012). To account for the general meta-didactical praxeology of each MTE, we produced a table for each of them, structured like Table 2.

A third round of qualitative analysis has been conducted on the transcripts of the MTEs' interviews, to identify their personal beliefs about the needs of the PTs attending their courses.

Due to the fact that people often are not able to clearly express their beliefs in a discursive way, and sometimes they are not even aware of them (Leatham, 2006), we did not ask direct questions about MTEs' beliefs in the interviews. We tried to extrapolate MTEs'

Table 2 Scheme of analysis of MTEs' praxeologies

Meta-didactical praxeology adopted by MTE x	
Praxis	Logos
<p><i>Type of task:</i> To co-build with PTs didactical praxeologies for teaching mathematics at the school level of reference, focusing on student-centered didactic approaches coherent with the institutional reference of the national curriculum (MIUR, 2010, 2012)</p> <p><i>Technique:</i> Practices adopted by the MTE to address the type of task, including the generation of specific documents useful for the co-building of didactical praxeologies with PTs</p>	<p>Theoretical discourse justifying the <i>praxis</i>, mainly based on the literature in mathematics education and on the institutional reference of the national curriculum (MIUR, 2010, 2012)</p>

beliefs on their students' needs from the transcripts of their answers provided during the whole interviews, and particularly to questions like: "Why did you act in this way?" and "Why do you think this is useful?", with an interpretative approach (Cohen et al., 2018). Identifying the subjective meaning attributed to the practices is the aim of an interpretive approach to research. This approach also involves understanding the social context within which an actor's intentions make sense. In educational research, the importance given to the subjective interpretations of educational phenomena by educational practitioners, "challenges the positivist assumption of an objective reality that can be interpreted by causal explanations and universal laws" (Carr & Kemmis, 1986, p. 98).

In the end, we tried to highlight the relationships among the analyzed elements, taking into account the fact that we needed to intersect different levels: the personal level of MTEs' beliefs and the institutional level of the *logos* shared (in the sense stated above) among the community of MTEs', thus common to the two MTEs' under scrutiny in our study. The personal and the institutional level are, indeed, intertwined, because MTEs, besides having an institutional position and institutional references to follow, also have a personal relationship with the theory constituting the *logos* shared in their community (Croset & Chaachoua, 2016, for example, discussed the notion of "personal praxeologies" to take into account these two levels). Both the institutional and the personal level could, according to our working hypothesis, influence the MTE's design of teaching materials based on different kinds of resources. For our analysis, we relied on an adaptation of the model (Fig. 1) proposed by Pocalana and Robutti (2023) and applied in Pocalana et al. (2024), complemented with MTE's personal beliefs (Fig. 2), with the aim of testing its validity. In this new model, the utilization schemes of the documents generated by the MTEs intersect the *logos* component of their meta-didactical praxeologies. This adaptation aims to take into account the fact that the operational invariants of the utilization schemes are indeed part of the MTEs' justifying discourses for their practices.

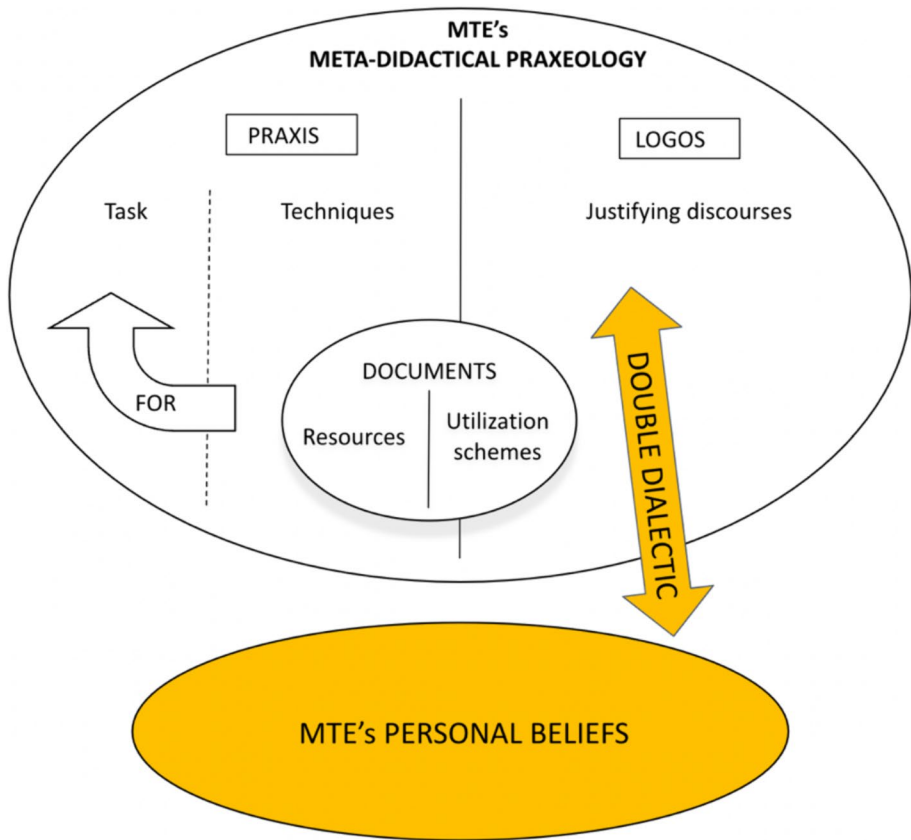


Fig. 2 Model for MTEs' work, based on the MDT and the DAD frameworks, with the integration of MTEs' personal beliefs

Results

In this section, we report the results related to the cases of the two MTEs', presented at first separately, and subsequently compared to delineate common features and to derive common conclusions. For both cases, we first describe three documents, enlisting the resources on which they are based and the relative utilization schemes. For each document, we report the MTE's excerpts testifying for her beliefs influencing the generation of that specific document. In the end, for each MTE, we delineate her praxeology for the design and implementation of her course, highlighting the relationships with her beliefs about PTs' needs and her documentation work.

Table 3 Resources and utilization schemes of Document 1, generated by MTE A

Document 1 generated by MTE A	
Resources	Utilization schemes
<ul style="list-style-type: none"> • Italian school institutional reference: the national curriculum for lower secondary schools (MIUR, 2012) • Italian school institutional reference: the national assessment framework for mathematics (INVALSI, 2018) 	<p><i>Class of situations:</i> Presenting the Italian institutional references for lower secondary schools</p> <p><i>Rules of action:</i> Guiding collective critical readings and discussions of the most relevant parts of the Italian school institutional references presented to the PTs, with specific focus on the mathematics teaching/learning approaches proposed for lower secondary schools</p> <p><i>Operational invariants:</i></p> <ul style="list-style-type: none"> • It is important that PTs know the Italian institutional references and realize what is required for the teaching/learning of mathematics • It is also important that they understand the close relationship between the two Italian school institutional references: national curriculum and national assessment framework

The case of MTE A

Documentation work and beliefs

In this section, we present three examples of documents generated by MTE A for her course for PTs, describing the main resources on which they are based and their utilization schemes (Tables 3, 4 and 5). Besides that, we trace connections between the MTE A documentation work and her beliefs about the needs of the PTs attending her course.

The generation of this document is influenced by MTE A's belief that the PTs attending her course are not yet familiar with the Italian national curriculum and that they are at risk of referring to dated curricula and to what is present in textbooks—not always coherent with the current curriculum¹—for understanding what must be taught at school.

MTE A: The Indicazioni Nazionali [the Italian national curriculum] are not well known, although they have been out for more than 10 years. If they [the PTs] don't know them, by the time they get to school, they'll end up ignoring them and having traditional textbooks as their only reference.

Besides that, according to MTE A, standardized forms of national assessment, like that carried out by the INVALSI institute, are probably seen by them (as well as by many in-service teachers) as an outside interference, a way of evaluating teachers' performance designed with criteria far from usual classroom practices, and as an unnecessary source of stress for students and teachers.

¹ In Italy, teachers are free to choose textbooks, without ministerial impositions. Usually there is the same textbook in the school. The textbooks are published by private editors and their scientific/educational content is not controlled by the Ministry.

Table 4 Resources and utilization scheme of Document 2, generated by MTE A

Document 2 generated by MTE A	
Resources	Utilization schemes
<ul style="list-style-type: none"> • Excerpts from the book Euclides' Elements on mathematical definitions of quadrilaterals (Frajese & Maccioni, 1970) • A Journal paper on mathematical definitions, described as based on freedom of choice and coherence (Ferrari, 2000) • GeoGebra Graphic View, for geometrical constructions of quadrilaterals, based on mathematical definitions • A journal paper on the role of dragging in construction of geometrical figures and exploration problems (Arzarello et al., 2002), in which the dragging test is presented as the bridge between the definition and the correct construction 	<p><i>Class of situations:</i> Working on the concept of mathematical definition</p> <p><i>Rules of action:</i></p> <ul style="list-style-type: none"> • Presenting PTs with different definitions of the main quadrilaterals (square, rhombus, rectangle, parallelogram, trapezoid), taken from Euclid's Elements and from texts currently in use in Italian lower secondary schools • Requesting PTs to construct the aforementioned quadrilaterals in the GeoGebra Graphic View, by imposing all the features present in the definitions. Requiring the dragging test (Arzarello et al., 2002) to see whether the drawings keeps these features <p><i>Operational invariants:</i></p> <ul style="list-style-type: none"> • It is important that PTs realize that the same mathematical object can be defined in different ways, imposing different properties. The essential element to emphasize is the need for coherence throughout the discussion • Through a dynamic geometry software, it is possible to verify, by means of the dragging test, if the construction of a geometric figure is correct, that is, if all the features enlisted in the definition are respected

MTE A: Teachers often see INVALSI tests as a means to evaluate their work, as a useless external interference and as a waste of time for them and for the students. Probably, my students [the PTs] inherited the same vision.

The generation of this document is influenced by MTE A's belief that the PTs attending her course have no clear understanding of the concept of mathematical definition. Probably they have never thought about which properties are necessary and/or sufficient to define a geometrical figure. Due to the fact that they do not have a degree in mathematics and many of them have attended schools other than scientific high schools, perhaps they have never had the opportunity to reflect on the epistemology and foundations of mathematics. So, they are likely to have no clear awareness of the nature of mathematics, in particular of its being based on freedom of choice about definitions, as well as rigor in drawing the consequences of choosing a certain definition. For example, MTE A said:

MTE A: They [the PTs attending a master's degree in natural sciences] only take a mathematics course before mine and many of them have not even attended a scientific high school. So, I cannot assume that they have reflected on the foundations of mathematics, for example on what a definition is, or on what a set of necessary and sufficient properties to define a geometric figure is.

The generation of this document is influenced by MTE A's belief that the PTs are at risk of continuing to reproduce the predominantly teacher-centered, lecture-based teaching practices they experienced as students. Indeed, MTE A said:

Table 5 Resources and utilization schemes of Document 3, generated by MTE A

Document 3 generated by MTE A	
Resources	Utilization schemes
<ul style="list-style-type: none"> • Excerpts from the book “Building Thinking Classrooms in Mathematics” (Liljedahl, 2021) • Material resources useful for experimenting with Liljedahl’s (2021) suggested teaching/learning approach: cellophane, whiteboard markers, eraser, paper tape 	<p><i>Class of situations:</i> Presenting and experiencing an inclusive teaching/learning approach to involve all the students in problem solving activities</p> <p><i>Rules of action:</i></p> <ul style="list-style-type: none"> • Presenting to the PTs excerpts from the book “Building Thinking Classrooms in Mathematics”, by Peter Liljedahl, with the main principles of the teaching/learning approach proposed by him • Requiring PTs to experiment in first person with this teaching/learning approach, by working in random groups of three to solve some of the mathematical problems proposed in the book, standing up and writing on vertical erasable surfaces (cellophane attached to the walls) <p><i>Operational invariants:</i></p> <ul style="list-style-type: none"> • PTs should be presented with inclusive teaching/learning approaches, considered as effective according to the mathematics education literature (Liljedahl, 2021) • PTs should experiment in first person the teaching/learning approaches that they are supposed to propose to their students in their future classes, in order to better understand their potentialities

MTE A: If they [the PTs] are not involved in firsthand experiences with alternative methods to the classic frontal teaching, when they are in the classroom they will not know what else to do but reproduce those old methods they know well. [...]

According to MTE A, to avoid this risk, PTs should be presented with alternative models, not very complex to implement and not requiring expensive equipment, that they experience as effective and engaging.

MTE A: It is necessary to propose inclusive, effective, not too complex strategies to be implemented even in classes with little technological equipment and in schools with scarce budgets.

MTE A’s meta-didactical praxeologies and beliefs

The examples of documents presented above were generated by MTE A as part of her general meta-didactical praxeology (Table 6) adopted to address the type of task of co-building with PTs didactical praxeologies for teaching mathematics at the lower secondary school, focusing on student-centered didactic approaches coherent with the institutional reference of the national curriculum (MIUR, 2012). The *logos* component of this praxeology includes ideas which are part of the theoretical discourse shared among the community of researchers in mathematics education in regard to (pre-service and in-service) teacher education. Indeed, the MTEs under scrutiny in this paper are researchers

Table 6 Meta-didactical praxeology adopted by MTE A

Meta-didactical praxeology adopted by MTE A

<i>Praxis</i>	<i>Logos</i>
<p><i>Type of task:</i> To co-build with PTs didactical praxeologies for teaching mathematics at the lower secondary school, focusing on student-centered didactic approaches coherent with the institutional reference of the national curriculum (MIUR, 2012)</p> <p><i>Technique:</i></p> <ul style="list-style-type: none"> • Presenting Italian school institutional references for lower secondary school teachers • Deepening students' insight into the basic concepts (such as that of definition), on the foundations of mathematics, and on the nature of mathematics • Presenting examples of teaching/ learning approaches considered effective in the literature 	<p>It is important to prepare Pts to teach mathematics:</p> <ul style="list-style-type: none"> • with a student-centered (Noyes, 2012), laboratory (Arzarello & Robutti, 2008) and inquiry-based (Laursen & Rassmussen, 2019) approach; • having awareness of the crucial role of the mathematics teachers in curriculum development (e. g., Clarke et al., 1996); • mastering mathematics basic concepts and having deep understanding of the nature of mathematics (e. g., Ponte & Chapman, 2008); • with an inclusive and engaging approach (e. g., Liljedahl, 2021)

in mathematics education, so the *logos* component of their praxeologies is connected with their institutional position.

The choice of the reference literature for the design and the implementation of the course appears to be influenced by MTE A's beliefs about the needs of the PTs attending her course. Indeed, among all the relevant literature in the field of (pre-service or in-service) mathematics teacher education, MTE A chose a subset of references according to what she deemed most useful in her case.

The case of MTE B

Documentation work and beliefs

In this section, we present three examples of documents generated by MTE B for her course for PTs (Tables 7, 8 and 9), describing the main resources on which they are based and their utilization schemes, tracing connections between the MTE B's documentation work and her beliefs about the needs of the PTs attending her course.

The generation of this document is influenced by MTE B's belief that the PTs' attending her course, having followed many mathematics courses during their bachelor's degree but none in mathematics education, need to understand what mathematics education deals with.

MTE B: They [the PTs] have taken many mathematics courses up to now, but mathematics education is a total novelty for them.

Table 7 Resources and utilization schemes of Document 1, generated by MTE B

Document 1 generated by MTE B	
Resources	Utilization schemes
<ul style="list-style-type: none"> • Prompt for PTs' essay "What math is for me", that is an adaptation of the prompt "Me and math" adopted by Di Martino and Zan (2010) • Audio-recording of Emma Castelnuovo's <i>lectio magistralis</i> "Teaching math", held in Rome in 2007, in occasion of the tribute to her career as teacher-researcher (Castelnuovo, 2007) • Discussion forum on a Moodle platform with the title "Discussion about Emma Castelnuovo's <i>lectio magistralis</i>" 	<p><i>Class of situations:</i> Introductory "lecture/non-lecture"</p> <p><i>Rules of action:</i></p> <ul style="list-style-type: none"> • Asking PTS to write an anonymous essay as a response to the open prompt "What math is for me" during the first lesson of the course • After that, inviting them to listen to the audio-recording of Emma Castelnuovo's <i>lectio magistralis</i> "Teaching math" (Castelnuovo, 2007). For example, drawing the PTs attention on the pieces in which Castelnuovo spoke about the changes she made to the traditional teaching/learning approach to geometry in lower secondary schools • In the end, opening a forum discussion on this <i>lectio magistralis</i>, in which the PTs can open threads based on what aroused their interest and discuss among them over the following week <p><i>Operational invariants:</i> It is important to understand how PTs view mathematics. For example,</p> <ul style="list-style-type: none"> • from the essay based on the prompt "What math is for me", their experience and their relationship with mathematics emerge; • discussing about Emma Castelnuovo's <i>lectio magistralis</i> the PTs can express their ideas about teaching mathematics

Furthermore, MTE B thinks that she needs to understand the PTs' view of mathematics and their relationship with the subject, in order to be able to address possible issues in this respect.

MTE B: The first two hours with them, it's not a lecture given by myself but it is a way to enter in a world that is the world of mathematics education [...]. It is a lecture/non-lecture because I want to recover from them their ideas, the sense they attribute to mathematics.

The generation of this document is influenced by MTE B's belief that in our culture, in which the PTs are immersed, we privilege the transmission of knowledge through written words in textbooks, which is not the best way to favor students' understanding and learning. For this reason, PTs should come to know that, already many centuries ago, someone had proposed a more effective teaching/learning model based on students' firsthand experiences.

For example, MTE B said:

MTE B: We, in Western culture, from the invention of printing to today, have leaned heavily on the book, thinking that the book was enough to teach the masses, but this is not the case. Therefore, it has been spreading more and more a symbolic-reconstructive learning, which is not the most stable one. The most stable one, indeed, is the motor-perceptive one.

Table 8 Resources and utilization schemes of Document 2, generated by MTE B

Document 2 generated by MTE B	
Resources	Utilization schemes
<ul style="list-style-type: none"> ● The Socratic dialog Meno by Plato (Plato, 2016) ● Comenius' book Didactica Magna (Comenius, 2011) ● Antinucci's book "La scuola si è rotta" ("The school broke down") (Antinucci, 2001) ● Discussion forum on a Moodle platform about the recurrent concepts in the previous 3 resources (Antinucci, 2001; Comenius, 2011; Keatinge, 1907; Plato, 2016) 	<p data-bbox="597 284 1057 384"><i>Class of situations:</i> "Setting the scene" in the first lessons by presenting and discussing excerpts from books relevant for mathematics education, dating back to different historical periods</p> <p data-bbox="597 389 730 412"><i>Rules of action:</i></p> <ul style="list-style-type: none"> ● Asking three PTs to read the piece of the Socratic dialog Meno (Plato, 2016) referred to the duplication of the square, respectively, impersonating Socrates, Meno and Meno's slave. Particularly, ask them to focus on the pieces in which Socrates explains his maieutic strategy to Meno ● Requiring PTs to read in class some excerpts from the books Didactica Magna by Comenius (2011) and "La scuola si è rotta" by Antinucci (2001) to show <i>a fil rouge</i> on the importance of direct experience in the teaching/learning process ● Then, opening a discussion on the online forum regarding the recurring concepts in all the resources covered <p data-bbox="597 763 797 786"><i>Operational invariants:</i></p> <ul style="list-style-type: none"> ● The piece of the dialog (Plato, 2016) between Socrates, Meno and the slave, referred to the duplication of the square, is a particularly significant piece because Socrates does not explain to the slave how to get the side of a square with twice the area of the starting square, but keep questioning him until the slave comes up with the solution on his own ● The book by Comenius (2011) is a milestone for mathematics education and it contains very modern ideas. For example, the ideal school is described as a workshop, a laboratory where teachers show examples, offer suggestions starting from the analysis of problems, encourage work in heterogeneous groups ● Antinucci's (2001) book, which is much more recent, connects very well to the previous resources because it sustains the importance of the perceptual-motor learning, which has proven to be more stable than the symbolic one which occurs through the theoretical exposition of rules in a book ● The forum is aimed to discuss recurring aspects in all the resources presented, such as the concept of school as a workshop in which the students are active protagonists

Table 9 Resources and utilization schemes of Document 3, generated by MTE B

Document 3 generated by MTE B	
Resources	Utilization schemes
<ul style="list-style-type: none"> Italian institutional references: the national curricula for lower and upper secondary schools from the early 1900s to nowadays 	<p><i>Class of situations:</i> Fostering the awareness of historical development of the Italian national curriculum for mathematics</p> <p><i>Rules of action:</i></p> <ul style="list-style-type: none"> Creating a folder in the Moodle platform, in which to insert the texts of the Italian national curricula for mathematics, starting from the first school laws Guiding the collective reading and discussing of some key passages of each curriculum Choosing some mathematical topics and follow their evolution over the decades, to understand if they are always present and if the way of dealing with them changes <p><i>Operational invariants:</i> It is important that PTs broadly understand the evolution of the Italian national curriculum in respect to mathematics, not to know in detail all the topics enlisted in them, but to be aware of the evolution of the conceptualization of the teaching/learning of mathematics as a response to social changes</p>

The generation of this document is influenced by MTE B's belief that the PTs need to understand the evolution of the mathematics curriculum in the Italian institutional references in order, for example, to recognize that some topics that were deemed very important in a specific historical period may not be so central in a successive period.

MTE B: It is essential to give them the institutional contextual dimension of their country to make them understand that things have not always been as we see them today [...]. If a textbook offers you a certain thing and you have this historical awareness of the curricular changes, you understand that this stuff maybe dates back to the fifties and that today it is totally anachronistic.

According to MTE B, this awareness is crucial to understand that mathematics teachers need to update their teaching planning according to the evolution of the educational needs of their students in the present society.

MTE B: If you enter a mathematics department of a school and they suggest you see the equations done in a certain way, and do the monomials, etc., you understand that this thing was contextualized perhaps in Italy in 1965, but today this is no good.

MTE B's meta-didactical praxeologies and beliefs

The examples of documents presented above were generated by MTE B as part of her general meta-didactical praxeology (Table 10) adopted to address the type of task of co-building with PTs didactical praxeologies for teaching mathematics at the upper secondary school, focusing on student-centered didactic approaches coherent with the national

Table 10 Meta-didactical praxeology adopted by MTE B

Meta-didactical praxeology adopted by MTE B	
Praxis	Logos
<p><i>Type of task:</i> To co-build with PTs didactical praxeologies for teaching mathematics at the upper secondary school, focusing on student-centered didactic approaches coherent with the institutional reference of the national curriculum (MIUR, 2010)</p> <p><i>Technique:</i></p> <ul style="list-style-type: none"> • At the beginning of the course, giving PTs an introduction to the field of mathematics education, from an historical point of view, following a <i>fil rouge</i> on the crucial role of embodied experience in the teaching/learning of mathematic • Giving PTs essays for making them express their view about mathematics and their relationship with the subject <p>Providing PTs with an historical overview of both the origin of mathematics education and the Italian national curriculum</p>	<p>It is important to prepare PTs to teach mathematics:</p> <ul style="list-style-type: none"> • with a student-centered (Noyes, 2012), laboratory (Arzarello & Robutti, 2008) and inquiry-based (Laursen & Rasmussen, 2019) approach; • having awareness of the historical evolution of mathematics and school programs (e.g., Gerofsky, 2023; Giacardi, 2015); • having awareness of the crucial role of embodied experience in the teaching/learning of mathematics (e.g., Nemirowsky et al., 2004); • understanding the view that PTs hold about mathematics and their relationship with the subject (e.g., Escudero-Ávila et al., 2021)

curriculum (MIUR, 2010). The *logos* component of her praxeology is constituted by general principles and broad ideas regarding (pre-service and in-service) teacher education.

The choice of the reference literature among the large variety of studies in the field is influenced by MTE B's beliefs about the needs of the PTs attending her course. Indeed, MTE B chose her theoretical references according to what she deemed most important for the PTs.

Discussion and conclusion

In this paper, we discussed the cases of two MTEs leading each a method course for the teaching/learning of mathematics devoted to PTs. From the comparison of the two cases, we derived that, despite the differences connected to the contexts and the relative differences in the design and implementation of the courses for PTs, a common pattern of relationships can be highlighted between the analyzed elements. Through the analysis of MTEs' documentation work, meta-didactical praxeologies and beliefs about the needs of the PTs attending their courses, we derived a complex intertwining of relationships among all these aspects, confirming the validity of the model adopted for data analysis (Fig. 2).

More specifically, we highlighted how MTEs' personal beliefs about the PTs influenced their documentation work, both in the choice of resources to rely on and in regard to their utilization schemes, even though beliefs about PTs are more general and are not only linked to specific documents based on specific resources. So, in the utilization scheme

of a document, we can find elements influenced by more general MTEs' beliefs. As far as the relationship between MTEs' meta-didactical praxeologies and documentation work is concerned, we confirmed the adaptation (Fig. 2) of the model proposed by Pocalana and Robutti(2023) (Fig. 1), positioning MTEs' documentation work as part of their meta-didactical praxeology for the design and the implementation of the course for PTs, with the utilization schemes at the intersection between the *praxis* and the *logos* components. Besides that, we traced connections between MTEs' personal beliefs about the needs of the PTs attending their respective courses and the *logos* of their meta-didactical praxeologies. Indeed, MTEs' personal beliefs influence the choice of which components of the general *logos*, referred to the professional development of PTs shared in the community of mathematics education, to consider for the design of their specific course, in its specific institutional context. The beliefs of the MTEs, who also are researchers in mathematics education, in turn, are influenced by the *logos* that permeate their studies and their own professional development (Bergsten & Grevholm, 2008) generating a double dialectic between these two elements, pertaining one—beliefs—to the personal level and the other—*logos*—to the institutional level. In particular, as regards to MTEs' generation of documents for their respective course for PTs, considered as part of their meta-didactical praxeologies, their beliefs influence the choice of resources and the relative utilization schemes.

In light of what has been said, we confirm the validity of the adaptation of the model introduced in the work of Pocalana and Robutti (2023) with the new positioning of the documentation work and the integration of MTEs' beliefs having a dialectic relationship with the *logos* component of their meta-didactical praxeologies, also influencing their documentation work (Fig. 2). This new model can provide a fresh insight into the complexity of MTEs' work, opening a venue for future research which takes into account their practices, justifying discourses, and beliefs in a holistic way. To deepen our insight, it could also be promising to study the work of a greater number of MTEs', not limited to the authors, in different contexts, in order to test the validity of the proposed model.

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Declarations

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