Alethic Pluralism and Kripkean Truth

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According to alethic pluralism, there is more than one way of being true: truth is not unique, in that there is a plurality of truth properties each of which pertains to a specific domain of discourse. This paper shows how such a plurality can be represented in a coherent formal framework by means of a Kripke-style construction that yields intuitively correct extensions for distinct truth predicates. The theory of truth it develops can handle at least three crucial problems that have been raised in connection with alethic pluralism: mixed compounds, mixed inferences, and semantic paradoxes.

1 OVERVIEW

Alethic pluralism defines itself in opposition to alethic monism, the view that some unique truth property belongs to all true sentences, and draws its appeal from the apparently puzzling consequences of the latter. As long as a theory of truth assumes that there is just one way of being true, its plausibility may vary extensively accross domains of discourse. Some theories of truth — such as correspondence theories — are quite intuitive when applied to empirical truths, but less convincing when applied to mathematical truths. Conversely, other theories of truth — such as coherence theories — seem reasonable when applied to mathematical truths, but harder to defend when applied to empirical truths. The pluralist diagnoses this variance as the effect of an overgeneralization: although ascriptions of truth in a given domain can be explained in terms of a given property, no single property is necessary and sufficient to explain ascriptions of truth in every domain.[1]

According to alethic pluralism, the predicate 'true' as used in ordinary language is associated with different truth properties. For example, consider the following sentences:

(1) Snow is white

(2) 2+2=4

1. Lynch 2011, Edwards 2018, Ferrari 2021, among others, spell out some crucial epistemological, metaphysical, and normative implications of this point.

On the plausible assumption that (1) and (2) are both true, the predicate 'true' applies to both of them, as in the following sentences:

(3) 'Snow is white' is true

(4) '2+2=4' is true

Intuitively, any sentence is equivalent to its truth predication, as the *Disquotation Schema* implies:

(DS) 'φ' is true if and only if φ

However, since (1) expresses an empirical statement while (2) expresses a mathematical statement, it may be argued that the property one ascribes to (1) when one utters (3) is not the same as the property one ascribes to (2) when one utters (4). That is, it may be argued that in the first case the property ascribed is correspondence, while in the second it is coherence. More generally, the idea that underlies alethic pluralism is that distinct domains of discourse determine distinct truth properties.[2]

A basic question that arises in connection with this idea is whether the truth properties so individuated are mutually exclusive. According to a version of alethic pluralism known as *strong pluralism*, the answer is affirmative: there is no single truth property that belongs to all true sentences. According to a weaker version known as *moderate pluralism*, instead, the answer is negative: some truth property is generic, in that it belongs to all true sentences. This paper focuses on strong pluralism, which we regard as the most neat and straightforward version of alethic pluralism, even though our formal framework can easily be adjusted to accomodate moderate pluralism as well. While moderate pluralism has to explain how the generic truth property differs from the domain-specific truth properties without collapsing into monism, strong pluralism can coherently avoid generic truth talk, as it contemplates only domain-specific truth properties that are all ontologically on a par. From now on, 'alethic pluralism' will be used univocally to refer to strong pluralism.

Although the linguistic and metaphysical implications of alethic pluralism have been widely debated, to our knowledge there is no fully developed formal theory in which distinct truth properties are represented in a consistent and rigorous way. Yet such a theory would undoubtedly serve the pluralist cause, for it would provide a unified account of truth predications that shows what the distinct truth properties postulated have in common from a logical point of view.

2. The two properties just mentioned are merely illustrative. For example, coherence might be replaced by superassertibility, as suggested by Wright 1992, p. 48. Or alternatively, one might endorse some form of correspondence pluralism, as in Sher 2023a, and think that different kinds of correspondence are involved in the two cases, for correspondence takes different forms in different fields, such as empirical science and mathematics. Our framework can seamlessly accommodate such variations.

Some attempts in this direction have already been made. In particular, Yu provided a pluralist formal semantics for a language without truth predicates, while Cotnoir suggested how distinct truth predicates can be defined in a framework that adopts type distinctions.[3] But none of these attempts has produced a full-fledged pluralist theory of truth. In a language without truth predicates it is impossible to model sentences in which truth is explicitly ascribed, such as (3) and (4), and the equivalence conveyed by (DS) is not expressible. On the other hand, in a framework with a typed truth predicate, (DS) is only partially vindicated, and a crucial problem remains: there is no way to account for other important uses of the truth predicate. Here are two examples:

(5) Every consequence of Euclid's axioms is true

(6) What Aristotle said is not true

As Kripke famously pointed out in his landmark work on truth, sentences such as (5) and (6), which involve explicit truth predications, cannot be properly accounted for unless 'true' is interpreted in a type-free way.[4] Moreover, while one might think that the use of 'true' in (3) and (4) is dispensable — on the ground that (3) and (4) are replaceable with (1) and (2) in virtue of (DS) — (5) and (6) are paradigmatic cases of *ineliminable* uses of 'true': in order to adequately express what (5) and (6) say without a truth predicate, one would need infinitely long sentences.[5]

Our aim is to develop a pluralist version of Kripke's theory of truth. This theory will make it possible to account for domain variations involved in truth predications — including contexts where truth is ineliminable — with no type restrictions. More importantly, it will make it possible to define a pluralist-friendly notion of logical consequence that can incorporate distinct truth predications. In order to do so, we define a formal model for two truth predicates T_1 and T_2, which are intended as object-linguistic representations of distinct truth properties. We work with just two truth predicates for the sake of simplicity, as it is the minimum number required in order for the theory to qualify as pluralist. But the model can be generalized to n truth predicates in a straightforward way. On the assumption that 'true' can express two distinct properties, as in (1) and (2), T_1 and T_2 represent those properties. More precisely, we take T_1 and T_2 to characterize distinct sets of sentences that have those properties. For example, we may suppose that T_1 stands for empirical truth and T_2

3. Yu 2017, Cotnoir 2013a. The suggestion in the latter work has been developed in different directions by Lubrano 2014 and Gamester 2023.

4. Kripke 1975.

5. This point, which is well established in the literature on truth, was made in Quine 1986, p. 12. For subsequent discussion see, e.g., Halbach 1999, and Picollo and Schindler 2018b.

stands for mathematical truth. So, the idea is that formalization in a language that contains T_1 and T_2 enables us to represent truth predications in a logically perspicuous way. In such a language, (3) and (4) can be formalized by using T_1 and T_2 respectively.[6]

Moreover, since natural language includes combinations of sentences that belong to different domains, such as the following, it is reasonable to expect that our formal language can represent mixed truth predications.

(7) Snow is white and 2+2=4

(8) Snow is white or 2+2=4

In particular, it is reasonable to expect that, for any two predicates T_i and T_j in the language, there is an operation $\circ$ such that $T_{i\circ j}$ is also a truth predicate. The operation $\circ$ should intuitively obey the following principles:

DEFINITION 1

1. $i \circ i = i$
2. $i \circ j = j \circ i$
3. $(i \circ j) \circ k = i \circ (j \circ k)$

Clauses 1-3 say that $\circ$ is idempotent, commutative, and associative. Note that these three properties entail that $T_{(i\circ j)\circ i} = T_{i\circ j}$, that is, mixing a mixed truth predicate $T_{i\circ j}$ with a truth predicate T_i that already appears as a constituent in the former yields no distinct truth predicate. The rationale behind this requirement is that if one takes a combination of empirical and mathematical truth, further combining that combination with, say, empirical truth does not yield a distinct kind of combination.[7]

The structure of the paper is as follows. Section 2 provides some definitions and preliminary clarifications. Sections 3-7 develop a model-theoretic construction that yields the desired interpretation of T_1 and T_2 and of a mixed predicate $T_{1\circ 2}$. Sections 8-10 explain how the theory of truth outlined can handle three crucial problems that have been raised in connection with alethic pluralism: mixed compounds, mixed inferences, and semantic paradoxes. Finally, section 11 draws attention to some crucial philosophical implications.

6. We remain neutral on the metaphysics of truth properties, as our talk of properties is to be understood in a purely semantic sense. As explained above, we restrict consideration to the extensions of truth properties. For example, for a sentence to be true in the empirical sense is for the sentence to belong to the extension of T_1.

7. As Yu 2017 shows, an operation with the properties of $\circ$ can be defined on a set of domains in order to obtain a pluralist interpretation of a formal language — propositional or first-order — that does not contain truth predicates. Instead, we apply $\circ$ directly to the truth predicates, without relying on preliminary assumptions about mixed domains.

Let L^- be a first-order language whose logical constants are $\neg, \wedge, \vee, \forall$. L^- is our *base language*, and we call L^-_A the set of its atomic formulas. A key semantic assumption about L^- is that its structures are rich enough to encode its syntax. More specifically, we require that L^- has at least one countable acceptable model such that, for every object a in the domain, there is a constant c_a in L^-. We call *base models* the structures of L^-.[8]

Let L be a language obtained by adding to L^- the truth predicates $T_1, T_2, T_{1\circ 2}$. Since the formulas of L can be coded by assigning to them elements of the domains of base models, truth predications can be represented in L by applying $T_1, T_2, T_{1\circ 2}$ to names denoting codes of formulas. Thus, for example, (3) and (4) can be formalized in L as $T_1{\ulcorner}\varphi{\urcorner}$ and $T_2{\ulcorner}\psi{\urcorner}$ respectively, where ${\ulcorner}\varphi{\urcorner}$ and ${\ulcorner}\psi{\urcorner}$ are closed terms that name φ and ψ. When a formula of L does not contain any of the predicates $T_1, T_2, T_{1\circ 2}$, we call it *truth-free*.

Note that the truth predicates of L are exactly three because, given the properties of $\circ$, there can be no mixed truth predicate other than $T_{1\circ 2}$.[9] As will become clear, the extensions assigned to $T_1, T_2, T_{1\circ 2}$ in our construction guarantee that the symbol $\circ$ appearing in $T_{1\circ 2}$ satisfies Definition 1.

The first element we need for our semantics is a pluralist-friendly kind of model:

DEFINITION 2 An *MD-frame* for L is a pair $\langle M, D\rangle$, where M is a base model and D is a function that maps L^-_A into the set $\{1, 2\}$.

Let $D_1 = \{\varphi \in \mathsf{L}^-_A : D(\varphi) = 1\}$. Let $D_2 = \{\varphi \in \mathsf{L}^-_A : D(\varphi) = 2\}$. D_1 and D_2 are understood as base domains, that is, sets of atomic sentences individuated by their subject matter: D_1 is a set of atomic empirical sentences, while D_2 is a set of atomic mathematical sentences. The underlying assumption here is that the subject matter of a sentence determines the kind of truth property that can be ascribed to the sentence itself. That is, empirical sentences, when true, are empirically true, while mathematical sentences, when true, are mathematically

8. The requirement of acceptability is needed for technical purposes, i.e. ensuring a smooth functioning of the coding machinery that makes it possible to formulate truth predications, including self-referential sentences, and to carry out Kripke-style constructions. For a definition of acceptable models see Moschovakis 1974, Chapter 5. The requirement that L^- has an individual constant for every element of the domain makes the interpretation of quantifiers much easier.

9. More generally, it can be shown that, for any number of indices, the number of resulting truth predicates respecting Definition 1 is given by the recursive function f defined as: $\begin{cases} f(0) = 0 \\ f(n+1) = 2f(n) + 1 \end{cases}$ (o)

true. It may also be assumed that, at the atomic level, empirical truth amounts to correspondence, and mathematical truth amounts to coherence.[10] Definition 2 implies that each truth-free atomic formula belongs to one and only one base domain, so that the resulting truth properties are mutually exclusive.[11]

In order to appreciate the generality of Definition 2 it suffices to consider the limit case in which D assigns 1 to every element of L_A^-. In this case the value 1 is the only value and the frame yields only one base domain, as $D_1 = \mathsf{L}_A^-$, so we have the situation theorized by alethic monism. A MD-frame of this kind may be called *monist*, while a MD-frame in which D assigns distinct values to some elements of L_A^- may be called *pluralist*. In what follows we focus on pluralist MD-frames where the elements of D_1 represent simple empirical sentences and the elements of D_2 represent simple mathematical sentences. But it is important to see that, for every pluralist MD-frame $\langle M, D\rangle$, there is a corresponding monist MD-frame $\langle M, D'\rangle$, where D' assigns 1 to every element of L_A^-.

Definition 2 provides the basis for our model-theoretic construction. As is well-known, Kripke suggests a way of defining an extension and an anti-extension for a truth predicate that satisfies the intuitive condition expressed by (DS), that is, such that any sentence is equivalent to its truth predication. He constructs the two sets in stages, starting from the truth-free atomic formulas that are satisfied (not satisfied) in the base model, and then defining each stage as the result of applying a chosen evaluation schema to the formulas in the previous stage. The process of building more and more stages eventually stabilizes because it reaches a *fixed point*, that is, a point where any further application of the evaluation schema leaves the set of formulas unchanged. In our construction, this method is applied to distinct truth predicates simultaneously.

As a first step, we define four sets Ei_1, Ai_1, Ei_2, Ai_2, where the first two are associated with T_1, and the second two are associated with T_2. The i stands for 'impure', because the definition of Ei_1, Ai_2, Ei_2, Ai_2 takes into account formulas of L that contain both elements of D_1 and elements of D_2. The sentences in Ei_1 may be called *broadly empirically true*, as some of them include constituents that are true in the empirical sense and constituents that are true in the mathematical sense. Similarly, the sentences in Ei_2 may be called *broadly mathematically*

10. This way of individuating truth properties in terms of subject matter is adopted, among other works, in Lynch 2013, Wyatt 2013, Yu 2017, Kim and Pedersen 2018. However, subject matter could be replaced by more substantive metaphysical notions. In any case, the distinction induced by D on M is meant to be just the kind of distinction that explains the existence of different truth properties according to alethic pluralism.

11. This exclusiveness condition is emphasized in Cotnoir 2013a, Yu 2017, Kim and Pedersen 2018. Note that if exclusiveness were not required, that is, if D could assign both 1 and 2 to the same atomic formula, the frame could accommodate moderate pluralism.

true. For example, (7) and (8) qualify as both broadly empirically true and broadly mathematically true. The sentences in Ai_1 and Ai_2 may be called *broadly empirically false* and *broadly mathematically false* for a similar reason.

As a second step we define two further sets $E_{1\circ 2}$ and $A_{1\circ 2}$ such that $E_{1\circ 2}$ intersects Ei_1 and Ei_2, and $A_{1\circ 2}$ intersects Ai_1 and Ai_2. $E_{1\circ 2}$ and $A_{1\circ 2}$ are respectively the extension and anti-extension of $T_{1\circ 2}$. Again, (7) and (8) illustrate the part of $E_{1\circ 2}$ that overlaps with Ei_1 and Ei_2. The case of $A_{1\circ 2}$ is entirely similar.

Finally, we define four pure sets E_1, A_1, E_2, A_2 by "subtracting" $E_{1\circ 2}$ from Ei_1 and Ei_2, and $A_{1\circ 2}$ from Ai_1 and Ai_2. E_1 and A_1 are respectively the extension and anti-extension of T_1. In other words, *strictly empirically true sentences*, such as (1), are a proper subset of broadly empirically true sentences. Similarly, E_2 and A_2 are respectively the extension and anti-extension of T_2. In other words, *strictly mathematically true* sentences, such as (2), are a proper subset of broadly mathematically true sentences. The same hold for empirical falsity and mathematical falsity.

3 CONSTRUCTION OF Ei_1, Ei_2, Ai_1, Ai_2

This section presents the construction of Ei_1, Ei_2, Ai_1, Ai_2 for any MD-frame $\langle M, D\rangle$. We employ a simultaneous induction to define a jump $\mathcal{J}$ on four arbitrary subsets of M, call them U_1, U_2, V_1, V_2. The elements of these sets are objects that encode formulas of L, so for the sake of simplicity we will refer to such formulas directly. $\mathcal{J}$ applies to U_1 and U_2 Kripke-style clauses for preservation of truth based on D_1 and D_2 respectively, and applies to V_1 and V_2 Kripke-style clauses for preservation of falsity based on D_1 and D_2 respectively. This means that the definition of $\mathcal{J}$ combines four definitions — that is, four jumps $\mathcal{H}_1, \mathcal{H}_2, \mathcal{I}_1, \mathcal{I}_2$ — one for each of the preservation properties described above. Recall that $\mathcal{P}(M)^4 = \mathcal{P}(M) \times \mathcal{P}(M) \times \mathcal{P}(M) \times \mathcal{P}(M)$.[12]

DEFINITION 3 With $S = \langle U_1, U_2, V_1, V_2\rangle \in \mathcal{P}(M)^4$, let $\mathcal{J}(S) = \langle \mathcal{H}_1(U_1), \mathcal{H}_2(U_2), \mathcal{I}_1(V_1), \mathcal{I}_2(V_2)\rangle$, where:

1. $\varphi \in \mathcal{H}_1(U_1)$ if
   - 1.1. $\varphi \in D_1$ and $M \models \varphi$, or
   - 1.2 φ is $\neg\psi$ and $\psi \in V_1$, or
   - 1.3 φ is $\psi \wedge \chi$ and $\psi, \chi \in U_1 \cup U_2$, and $\psi \in U_1$ or $\chi \in U_1$, or
   - 1.4 φ is $\psi \vee \chi$ and $\psi \in U_1$ or $\chi \in U_1$, or
   - 1.5 φ is $\forall x\psi$ and, for some closed term t, $(\psi)^t_x \in U_1$, and, for every closed term t, $(\psi)^t_x \in U_1 \cup U_2$, or

12. For the Kripke-style construction presented in this section, we follow the template of Nicolai and Rossi, forthcoming, to which we refer the reader for further details. For a more comprehensive presentation, see Cantini 1989.

1.6 φ is $T_1{}^{\ulcorner}\psi^{\urcorner}$ and $\psi \in U_1$.

2. $\varphi \in \mathcal{H}_2(U_2)$ if

   2.1 $\varphi \in D_2$ and $M \models \varphi$, or

   2.2 φ is $\neg\psi$ and $\psi \in V_2$, or

   2.3 φ is $\psi \wedge \chi$ and $\psi, \chi \in U_1 \cup U_2$, and $\psi \in U_2$ or $\chi \in U_2$, or

   2.4 φ is $\psi \vee \chi$ and $\psi \in U_2$ or $\chi \in U_2$;

   2.5 φ is $\forall x\psi$ and, for some closed term t, $(\psi)^t_x \in U_2$, and, for every closed term t, $(\psi)^t_x \in U_1 \cup U_2$, or

   2.6 φ is $T_2{}^{\ulcorner}\psi^{\urcorner}$ and $\psi \in U_2$.

3. $\varphi \in \mathcal{I}_1(V_1)$ if

   3.1 $\varphi \in D_1$ and $M \not\models \varphi$, or

   3.2 φ is $\neg\psi$ and $\psi \in U_1$, or

   3.3 φ is $\psi \wedge \chi$ and $\psi \in V_1$ or $\chi \in V_1$, or

   3.4 φ is $\psi \vee \chi$ and $\psi, \chi \in V_1 \cup V_2$ and $\psi \in V_1$ or $\chi \in V_1$, or

   3.5 φ is $\forall x\psi$ and for some closed term t, $(\psi)^t_x \in V_1$, or

   3.6 φ is $T_1{}^{\ulcorner}\psi^{\urcorner}$ and $\psi \in V_1$.

4. $\varphi \in \mathcal{I}_2(V_2)$ if

   4.1 $\varphi \in D_2$ and $M \not\models \varphi$, or

   4.2 φ is $\neg\psi$ and $\psi \in U_2$, or

   4.3 φ is $\psi \wedge \chi$ and $\psi \in V_2$ or $\chi \in V_2$, or

   4.4 φ is $\psi \vee \chi$ and $\psi, \chi \in V_1 \cup V_2$ and $\psi \in V_2$ or $\chi \in V_2$, or

   4.5 φ is $\forall x\psi$ and for some closed term t, $(\psi)^t_x \in V_2$, or

   4.6 φ is $T_2{}^{\ulcorner}\psi^{\urcorner}$ and $\psi \in V_2$

Item 1 provides a characterization of $\mathcal{H}_1$ based on D_1. Condition 1.1 is satisfied when φ is an atomic formula of D_1 that is made true by M. This is the simplest kind of broadly empirically true sentence. Condition 1.2 is satisfied when φ is the negation of a formula that belongs to V_1: the negation of a broadly empirically false sentence is broadly empirically true.[13] Condition 1.3 is satisfied when φ is a conjunction such that both its conjuncts belong to U_1 or U_2, and at least one of them belongs to U_1: a conjunction is broadly empirically true when each of its conjuncts is true in some sense and at least one of them is broadly empirically true. Condition 1.4 is satisfied when φ is a disjunction such that at least one of its disjuncts belongs to U_1: a disjunction is broadly empirically true when at least one of its

13. This claim is not entirely uncontroversial. For example, it is rejected by Edwards 2008, 2009, Gamester 2019.

disjuncts is broadly empirically true. Condition 1.5 is satisfied when φ is a quantified formula such that at least some of its instances is in U_1 and all its instances are either in U_1 or in U_2: a quantified sentence is broadly empirically true when at least one of its instances is broadly empirically true and all its instances are true in some sense. Finally, condition 1.6 is satisfied when φ is a formula in which T_1 is applied to a formula that is in U_1: a predication of empirical truth to a broadly empirically true sentence is broadly empirically true.

Item 2 is similar to item 1, as it provides a characterization of $\mathcal{H}_2$ based on D_2. Conditions 2.1-2.6 are like conditions 1.1-1.6 but switch 1 and 2 in the subscript. While 1 concerns broad empirical truth, 2 concerns broad mathematical truth. So the two domains D_1 and D_2 determine distinct truth properties in virtue of their subject matter. Note that understanding the distinction between D_1 and D_2 along the lines just suggested is consistent with assuming a unique formal relation between models and formulas. The symbol $\models$, which occurs in conditions 1.1 and 2.1, is the same in both definitions.

Now consider items 3 and 4, which characterize $\mathcal{I}_1$ and $\mathcal{I}_2$. Condition 3.1 is satisfied when φ is an atomic formula of D_1 that is made false by M. This is the simplest kind of broadly empirically false sentence. Conditions 3.2-3.6 provide falsity conditions for complex formulas containing $\neg$, $\wedge$, $\vee$, $\forall$, T_1, so they are dual counterparts of conditions 1.2-1.6. The clauses for $\mathcal{I}_2$ are like those for $\mathcal{I}_1$, switching 1 and 2 in the subscripts. The definitions of $\mathcal{I}_1$ and $\mathcal{I}_2$ are understood as characterizations of broad empirical falsity and broad mathematical falsity.[14]

In order to construct Ei_1, Ai_1, Ei_2, Ai_2, we need to iterate $\mathcal{J}$ until we reach a point in which any further application of $\mathcal{J}$ does not change the quadruple to which it is applied. The iterations of $\mathcal{J}$, starting from the empty set $\varnothing$, along the ordinals are defined by transfinite induction as follows, where δ is a limit ordinal and $\emptyset$ abbreviates $\langle \varnothing, \varnothing, \varnothing, \varnothing \rangle$.

DEFINITION 4 $\mathcal{J}^0(\emptyset) := \emptyset$
$\mathcal{J}^{\alpha+1}(\emptyset) := \mathcal{J}(\mathcal{J}^{\alpha}(\emptyset))$
$\mathcal{J}^{\delta}(\emptyset) := \bigcup_{\beta<\delta} \mathcal{J}^{\beta}(\emptyset)$

14. Note that in all the items 1-4 the universal quantifier is defined substitutionally. This is techically the simplest way to go, although it is not the only one. As is well known, the Kripkean method is equally compatible with objectual quantification. So it would be feasible to adopt, say, objectual clauses in 1.5 and 3.5 but substitutional clauses in 2.5 and 4.5. For example, it might be argued that objectual quantification suits the empirical domain, as the semantics of correspondence is objectual, while substitutional quantification suits the mathematical domain, as the semantics of coherence is substitutional. But since our choice of the empirical/mathematical distinction is merely illustrative, as explained above, and it is at least conceivable that two distinct truth properties are semantically analogous, the simplification adopted will do no harm.

It is provable that there is a least ordinal α, the so-called *closure ordinal*, such that $\mathcal{J}^{\alpha}(\emptyset) = \mathcal{J}(\mathcal{J}^{\alpha}(\emptyset))$.[15] For such α, we call $\mathcal{J}^{\alpha}(\emptyset)$ the *fixed point* of $\mathcal{J}$ generated by $\emptyset$.

A key feature of $\mathcal{J}^{\alpha}(\emptyset)$ is that it is *consistent* in the sense that neither of the first two sets in the quadruple overlaps with some of the other two sets.

PROPOSITION 1 $\mathcal{J}^{\alpha}(\emptyset)$ is consistent.

Proof. The proof is by transfinite induction on the construction of $\mathcal{J}^{\alpha}(\emptyset)$. Clearly, $\mathcal{J}^{0}(\emptyset) = \emptyset$ is consistent. Moreover, for δ limit, $\mathcal{J}^{\delta}(\emptyset)$ is consistent just in case, for every $\gamma < \delta$, $\mathcal{J}^{\gamma}(\emptyset)$ is consistent. So the consistency of limit stages of the construction depend on the consistency of successor stages. In order to prove that every successor stage is consistent, suppose the contrary, i.e. that some successor stage is inconsistent. Therefore, there is a least ordinal γ such that $\mathcal{J}^{\gamma}(\emptyset)$ is consistent but $\mathcal{J}^{\gamma+1}(\emptyset)$ is not. This means that $\mathrm{H}_1^{\gamma}(\varnothing) \cap (\mathcal{I}_1^{\gamma}(\varnothing) \cup \mathcal{I}_2^{\gamma}(\varnothing)) = \varnothing = \mathcal{H}_2^{\gamma}(\varnothing) \cap (\mathcal{I}_1^{\gamma}(\varnothing) \cup \mathcal{I}_2^{\gamma}(\varnothing))$,

but there is at least one formula φ such that either $\varphi \in \mathcal{H}_1^{\gamma+1}(\varnothing) \cap (\mathcal{I}_1^{\gamma+1}(\varnothing) \cup \mathcal{I}_2^{\gamma+1}(\varnothing))$ or $\varphi \in \mathcal{H}_2^{\gamma+1}(\varnothing) \cap (\mathcal{I}_1^{\gamma+1}(\varnothing) \cup \mathcal{I}_2^{\gamma+1}(\varnothing))$. Suppose the first disjunct holds (if the second holds, the reasoning is entirely similar). Then it is provable that none of the clauses of Definition 3 can yield an inconsistency. Here we consider three cases.
Case 1: φ is $\forall x \psi$. By γ's minimality and (3), both the following obtain:

- for every closed term s, $(\psi)_x^s \in \mathcal{H}_1^{\gamma}(\varnothing)$ (by clause 1.5 of Definition 3),
- for some closed term t, $(\psi)_x^t \in \mathcal{I}_1^{\gamma}(\varnothing)$ (clause 3.5), or $(\psi)_x^t \in \mathcal{I}_2^{\gamma}(\varnothing)$ (clause 4.5)

But then $(\psi)_x^t \in \mathcal{H}_1^{\gamma}(\varnothing) \cap (\mathcal{I}_1^{\gamma}(\varnothing) \cup \mathcal{I}_2^{\gamma}(\varnothing))$, against (3).
Case 2: φ is $T_1{}^{\ulcorner}\psi^{\urcorner}$. By γ's minimality and (3), both the following obtain:

- $\psi \in \mathcal{H}_1^{\gamma}(\varnothing)$ (clause 1.6),
- $\psi \in \mathcal{I}_1^{\gamma}(\varnothing)$ (clause 3.6), or $\psi \in \mathcal{I}_2^{\gamma}(\varnothing)$ (clause 4.6)

But then $\psi \in \mathcal{H}_1^{\gamma}(\varnothing) \cap (\mathcal{I}_1^{\gamma}(\varnothing) \cup \mathcal{I}_2^{\gamma}(\varnothing))$, against (3).
Case 3: φ is $T_2{}^{\ulcorner}\psi^{\urcorner}$. Since there is no $\mathcal{H}_1$-clause to include any formula of the form $T_2(t)$, there must be a least ordinal $\beta \leq \gamma$ such that $T_2{}^{\ulcorner}\psi^{\urcorner} \in \mathcal{H}_1^{\beta}(\varnothing)$. However, such an ordinal cannot be any successor ordinal (because, as said, there is no $\mathcal{H}_1$-clause to include any formula of the form $T_2(t)$), nor any limit ordinal, because the latter would require a least successor ordinal stage where $T_2{}^{\ulcorner}\psi^{\urcorner}$ is included. Therefore β must be 0. But this is impossible since $\mathcal{H}_1^{\beta}(\varnothing) = \varnothing$. □

15. Moschovakis 1974, Theorem 1A.1.

Once the consistency of our fixed point is established, we can define Ei_1, Ai_1, Ei_2, Ai_2 as follows, letting iterations be defined as above:

DEFINITION 5 For α the closure ordinal of $\mathcal{J}$,

$$Ei_1 = \mathcal{H}_1^\alpha(\varnothing),$$

$$Ei_2 = \mathcal{H}_2^\alpha(\varnothing),$$

$$Ai_1 = \mathcal{I}_1^\alpha(\varnothing),$$

$$Ai_2 = \mathcal{I}_2^\alpha(\varnothing).$$

Definitions 3 and 5 together imply that, as long as the initial MD-frame is pluralist, the intersection of Ei_1 and Ei_2 is not empty, and the same goes for the intersection of Ai_1 and Ai_2. As explained above, Ei_1, Ai_1, Ei_2, Ai_2 are impure. Ei_1 and Ei_2 do not provide disjoint extensions for T_1 and T_2, as some sentences are both broadly empirically true and broadly mathematically true. Similarly, Ai_1 and Ai_2 do not provide disjoint anti-extensions, as some sentences are both broadly empirically false and broadly mathematically false.

4 MIXED EXTENSION AND ANTI-EXTENSION

This section provides the definition of $E_{1\circ2}$ and $A_{1\circ2}$, the extension and the anti-extension of the mixed truth predicate $T_{1\circ2}$. The construction of $E_{1\circ2}$ starts from Ei_1 and Ei_2. Since Ei_1 and Ei_2 contain sentences that are both broadly empirically and broadly mathematically true, we can use them to "carve out" $E_{1\circ2}$, as it were: their intersection constitutes the starting point of the construction. The same holds for $A_{1\circ2}$, whose construction starts from the intersection of Ai_1 and Ai_2. In the following definition, clauses 1.1 and 2.1 set the starting point of the construction respectively. The rest of the construction is essentially as in Definition 3.

DEFINITION 6 With $S = \langle P, Q\rangle \in \mathcal{P}(M)^2$, let $\mathcal{K}(S) = \langle \mathcal{F}(P), \mathcal{G}(Q)\rangle$, where:

1 $\varphi \in \mathcal{F}(P)$ if

- 1.1 $\varphi \in Ei_1$ and $\varphi \in Ei_2$, or
- 1.2 φ is $\neg\psi$ and $\psi \in Q$, or
- 1.3 φ is $\psi \wedge \chi$ and $\psi \in P$ and $\chi \in P$, or
- 1.4 φ is $\psi \vee \chi$ and $\psi \in P$ or $\chi \in P$, or
- 1.5 φ is $\forall x\psi$ and for every closed term t, $(\psi)^t_x \in P$, or
- 1.6 φ is $T_{1\circ2}{}^{\ulcorner}\psi^{\urcorner}$ and $\psi \in P$.

2 $\varphi \in \mathcal{G}(Q)$ if

- 2.1 $\varphi \in Ai_1$ and $\varphi \in Ai_2$, or

2.2 φ is $\neg\psi$ and $\psi \in P$, or

2.3 φ is $\psi \wedge \chi$ and $\psi \in Q$ or $\chi \in Q$, or

2.4 φ is $\psi \vee \chi$ and $\psi \in Q$ and $\chi \in Q$, or

2.5 φ is $\forall x\psi$ and for some closed term t, $(\psi)^t_x \in Q$, or

2.6 φ is $T_{1\circ 2}{}^{\ulcorner}\psi^{\urcorner}$ and $\psi \in Q$.

As in the case of Ei_1, Ai_1, Ei_2, Ai_2, in order to obtain $E_{1\circ 2}$ and $A_{1\circ 2}$ we need to iterate $\mathcal{K}$ until we reach a fixed point. So we define its iterations by transfinite induction on the ordinals, as before. These iterations will have a closure ordinal, i.e. an ordinal α such that $\mathcal{K}^\alpha(\emptyset) = \mathcal{K}(\mathcal{K}^\alpha(\emptyset)$, where $\emptyset$ abbreviates $\langle\emptyset,\emptyset\rangle$. For such α, $\mathcal{K}^\alpha(\emptyset)$ is the fixed point of $\mathcal{K}$ generated by $\emptyset$. This provides the desired mixed extension and anti-extension, that is, $E_{1\circ 2}$ and $A_{1\circ 2}$.

Some remarks are in order. First, note that $E_{1\circ 2}$ and $A_{1\circ 2}$ do not contain truth-free atomic formulas. The reason is that a formula can get in the first stage of the construction only if it belongs to both Ei_1 and Ei_2, or to both Ai_1 and Ai_2, and no truth-free atomic formula satisfies such condition. This rules out the possibility of *mixed atomics*, that is, truth-free atomic sentences that belong to more than one domain. We are aware that the issue of mixed atomics is controversial, and that any serious attempt to settle it would go beyond the scope of this paper. Nonetheless, we regard the exclusion of mixed atomics as a reasonable simplification, which is in line with the idea that empirical and mathematical truths are fundamental in a way in which mixed truths are not: if mixed truth existed at the atomic level, it would no longer be derivative in the sense of resulting from the combination of empirical truth and mathematical truth.[16]

Second, the reader might wonder why we employed a further fixed point construction to define the extension and the anti-extension of $T_{1\circ 2}$ rather than simply taking the intersections of the two pairs of sets defined above, that is, by stipulating that $E_{1\circ 2} = Ei_1 \cap Ei_2$ and $A_{1\circ 2} = Ai_1 \cap Ai_2$. The reason is that if we did that, $T_{1\circ 2}$ would not iterate. Let χ stand for (7). Clearly, $\chi \in Ei_1 \cap Ei_2$. By the stipulation above we would thus get that $\chi \in E_{1\circ 2}$, so $T_{1\circ 2}{}^{\ulcorner}\chi^{\urcorner}$ should hold. However, $T_{1\circ 2}{}^{\ulcorner}T_{1\circ 2}{}^{\ulcorner}\chi^{\urcorner\urcorner}$ would not hold, because $T_{1\circ 2}{}^{\ulcorner}\chi^{\urcorner} \notin Ei_1 \cap Ei_2$. Our construction avoids this outcome, as it yields that $\chi \in E_{1\circ 2}$, $T_{1\circ 2}{}^{\ulcorner}\chi^{\urcorner} \in E_{1\circ 2}$, $T_{1\circ 2}{}^{\ulcorner}T_{1\circ 2}{}^{\ulcorner}\chi^{\urcorner\urcorner} \in E_{1\circ 2}$, and so on.

Finally, one might ask why we first define Ei_1, Ai_1, Ei_2, Ai_2 and then carve $E_{1\circ 2}$ and $A_{1\circ 2}$ out of them instead of directly defining pure estensions and anti-extensions for T_1 and T_2. The answer is that the latter option, although technically feasible, has different logical implications. Consider the case of disjunction. If one wants to define strict empirical truth rather than broad empirical truth, one cannot

16. For further discussion, see Sher 2005 and Edwards 2018. We thank an anonymous reviewer for helpful comments on this point.

simply stipulate that a disjunction is strictly true in a given sense if and only if at least one of its disjuncts is strictly true in that sense. On such a stipulation, (8) would qualify as both strictly empirically true and strictly mathematically true, with the effect of trivializing the very notion of strict truth. The clause for disjunction should rather be along the following lines: a disjunction is strictly true in a given sense if and only if one of its disjuncts is strictly true in that sense, and the other is either true or false in the same sense, or strictly false in the other sense. This way (8) counts as neither strictly empirically true nor strictly mathematically true. However, as a consequence of the clause just stated, disjunction would have a "weak Kleene" behaviour: the inference from γ to $\gamma \vee \delta$ would fail according to any notion of logical consequence based on some truth-preservation property, because it could be the case that γ is strictly true in a given sense but δ is neither strictly true or false in that sense nor false in the other sense.[17] As the next section shows, in our construction disjunction has a "strong Kleene" behaviour, in that $\gamma \vee \delta$ follows from γ.

5 PURE EXTENSIONS AND ANTI-EXTENSIONS

The interpretation of T_1 and T_2 can now be properly specified. The extensions of T_1 and T_2 — E_1 and E_2 — are defined as follows:

DEFINITION 7 $E_1 = Ei_1 \setminus E_{1\circ 2}$

DEFINITION 8 $E_2 = Ei_2 \setminus E_{1\circ 2}$

In other words, the set of strictly empirically true sentences is obtained by subtracting the mixed true sentences from the broadly empirically true sentences, and the set of strictly mathematically true sentences is obtained by subtracting the mixed true sentences from the broadly mathematically true sentences. From now on we will omit the qualification 'strictly', given that in our account empirical truth may be identified with strict empirical truth, and mathematical truth may be identified with strict mathematical truth.

The anti-extensions of T_1 and T_2 — A_1 and A_2 — are defined as follows:

DEFINITION 9 $A_1 = Ai_1 \setminus A_{1\circ 2}$

DEFINITION 10 $A_2 = Ai_2 \setminus A_{1\circ 2}$

A_1 is the set of strictly empirically — or simply empirically — false sentences, and A_2 is the set of strictly mathematically — or simply mathematically — false sentences. These two sets are obtained

17. Granted, the inference would hold if one adopted a notion of consequence based on preservation of non-falsity. Recall that fixed-point constructions such as the ones employed here are not compatible with essentially negative clauses, e.g. we cannot require a disjunction to be strictly mathematically true when at least one disjunct is and the other is *not* strictly empirically true.

by subtracting all the mixed false sentences from the set of broadly empirically false sentences and from the set of broadly mathematically false sentences.

Definitions 7-10 provide an interpretation for T_1 and T_2. The mixed truth predicate $T_{1\circ 2}$ is interpreted by taking $E_{1\circ 2}$ as its extension and $A_{1\circ 2}$ as its anti-extension. So we end up with three predicates, $T_1, T_2, T_{1\circ 2}$, each with its own extension and anti-extension. One crucial property of the sets so defined is that they are pairwise disjoint. By Definitions 7 and 8, $E_1 \cap E_2 = \emptyset$, $E_1 \cap E_{1\circ 2} = \emptyset$, and $E_2 \cap E_{1\circ 2} = \emptyset$. By Definitions 9 and 10, $A_1 \cap A_2 = \emptyset$, $A_1 \cap A_{1\circ 2} = \emptyset$, and $A_2 \cap A_{1\circ 2} = \emptyset$. Moreover, none of the sets $E_1, E_2, E_{1\circ 2}$ intersects any of the sets $A_1, A_2, A_{1\circ 2}$. Note that, since $E_{1\circ 2}$ is disjoint from E_1 and E_2, being true in the mixed sense does not amount to being true in some generic sense, and the same goes for mixed falsity.

The constructions of $E_1, E_2, E_{1\circ 2}, A_1, A_2, A_{1\circ 2}$ provide truth and falsity conditions for empirical, mathematical, and mixed sentences. We state them in the following proposition, where E_i (A_i) is any of $E_1, E_2, E_{1\circ 2}(A_1, A_2, A_{1\circ 2})$, and $\varphi, \psi \in \mathsf{L}$.

PROPOSITION 2

1 $\neg\varphi \in E_i$ iff $\varphi \in A_i$;

2 $\neg\varphi \in A_i$ iff $\varphi \in E_i$;

3 $\varphi \wedge \psi \in E_i$ iff $\varphi \in E_i$ and $\psi \in E_i$;

4 (a) $\varphi \wedge \psi \in A_1$ iff $\varphi, \psi \notin A_2 \cup A_{1\circ 2}$, and $\varphi \in A_1$ or $\psi \in A_1$;
(b) $\varphi \wedge \psi \in A_2$ iff $\varphi, \psi \notin A_1 \cup A_{1\circ 2}$, and $\varphi \in A_2$ or $\psi \in A_2$;
(c) $\varphi \wedge \psi \in A_{1\circ 2}$ iff $\varphi \in A_{1\circ 2}$ or $\psi \in A_{1\circ 2}$;

5 (a) $\varphi \vee \psi \in E_1$ iff $\varphi, \psi \notin E_2 \cup E_{1\circ 2}$, and $\varphi \in E_1$ or $\psi \in E_1$;
(b) $\varphi \vee \psi \in E_2$ iff $\varphi, \psi \notin E_1 \cup E_{1\circ 2}$, and $\varphi \in E_2$ or $\psi \in E_2$;
(c) $\varphi \vee \psi \in E_{1\circ 2}$ iff $\varphi \in E_{1\circ 2}$ or $\psi \in E_{1\circ 2}$;

6 $\varphi \vee \psi \in A_i$ iff $\varphi \in A_i$ and $\psi \in A_i$;

7 $\forall x\varphi(x) \in E_i$ iff for every closed term t, $\varphi(t) \in E_i$;

8 (a) $\forall x\varphi(x) \in A_1$ iff for all closed terms s, $\varphi(s) \notin A_2 \cup A_{1\circ 2}$, and for some closed term t, $\varphi(t) \in A_1$;
(b) $\forall x\varphi(x) \in A_2$ iff for every closed term s, $\varphi(s) \notin A_1 \cup A_{1\circ 2}$, and for some closed term t, $\varphi(t) \in A_2$;
(c) $\forall x\varphi(x) \in A_{1\circ 2}$ iff for some closed term t, $\varphi(t) \in A_{1\circ 2}$;

9 $T_i\ulcorner\varphi\urcorner \in E_i$ iff $\varphi \in E_i$;

10 $T_i\ulcorner\varphi\urcorner \in A_i$ iff $\varphi \in A_i$.

Proof. 1 and 2 follow from clauses 1.2, 2.2, 3.2, 4.2 of Definition 3 and from clauses 1.2 and 2.2 of Definition 6. 3 follows from clauses 1.3 and 2.3 of Definition 3 and from clause 1.3 of Definition 6. 4 follows from clauses 3.3 and 4.3 of Definition 3 and from clause 2.3 of Definition 6. 5 follows from clauses 1.4 and 2.4 of Definition 3 and from clause 1.4 of Definition 6. 6 follows from clauses 3.4 and 4.4 of Definition 3 and from clause 2.4 of Definition 6. 7 follows from clause 1.5 and 2.5 of Definition 3 and from clause 1.5 of Definition 6. 8 follows from clauses 3.5 and 4.5 of Definition 3 and from clause 2.5 of Definition 6. 9 follows from clauses 1.6 and 2.6 of Definition 3 and from clause 1.6 of Definition 6. Finally, 10 follows from clauses 3.6 and 4.6 of Definition 3 and from clause 2.6 of Definition 6. □

1 and 2 imply that the truth and falsity conditions for negated sentences are the same for empirical, mathematical, and mixed sentences. 3, 6, and 7 show that the truth conditions for conjunctions, the falsity conditions for disjunctions, and the truth conditions for quantified sentences are also the same for empirical, mathematical, and mixed sentences. 4, 5, and 8, instead, involve separate treatments for the falsity conditions of conjunctions, the truth conditions of disjunctions, and the falsity conditions of quantified sentences. Finally, 9 and 10 show that the principle known as *naïveté* holds for each of the predicates $T_1, T_2, T_{1\circ 2}$. The pluralist version of naïveté stated in 9 and 10 is a generalization of the usual monist version, since the latter results from 9 and 10 when E_1 is the only extension and A_1 is the only anti-extension, that is, when the construction starts from a monist MD-frame.

It is important to note that naïveté is not quite the same thing as *intersubstitutivity*, the principle stated as follows: for any formula φ and any substitution φ^t that results from replacing at least one occurrence of a subformula ψ of φ with $T_i{}^{\ulcorner}\psi^{\urcorner}$, or *vice versa*, $\varphi \in E_i$ if and only if $\varphi^t \in E_i$, and $\varphi \in A_i$ if and only if $\varphi^t \in A_i$. Although both naïveté and intersubstitutivity accord with the intuition conveyed by (DS), the latter is stronger than the former, as it implies that the equivalence between a sentence and its truth predication holds within any sentence.

One way to see the difference between naïveté and intersubstitutivity is to realize that intersubstitutivity holds for T_1 and T_2 but not for $T_{1\circ 2}$. While φ and φ^t are always intersubstitutable when $i = 1$ or $i = 2$, they are not intersubstitutable in some cases in which $i = 1 \circ 2$, and this is exactly as it should be. To illustrate, suppose that γ and δ stand for (1) and (2) respectively, so that $\gamma \wedge \delta$ stands for (7). Then $\gamma \wedge \delta \in E_{1\circ 2}$. But $T_{1\circ 2}{}^{\ulcorner}\gamma^{\urcorner} \wedge T_{1\circ 2}{}^{\ulcorner}\delta^{\urcorner} \notin E_{1\circ 2}$, otherwise, by condition 3 of proposition 2, $T_{1\circ 2}{}^{\ulcorner}\gamma^{\urcorner}$ and $T_{1\circ 2}{}^{\ulcorner}\delta^{\urcorner}$ would be in $E_{1\circ 2}$. Note that, by condition 9 of proposition 2 we also get that $T_{1\circ 2}{}^{\ulcorner}\gamma \wedge \delta^{\urcorner} \in E_{1\circ 2}$, as is plausible to assume. This means that $T_{1\circ 2}$ should not distribute over

subsentences of sentences that are in $E_{1\circ 2}$, lest it trivializes the very notion of mixed truth.

6 PK-MODELS

The definitions set out in sections 3-5 specify a construction method that applies to MD-frames. For any MD-frame $\langle M, D\rangle$, the method provides an interpretation of L that yields extensions for $T_1, T_2, T_{1\circ 2}$. We call such an interpretation a *Pluralist Kripkean model*, or PK-*model* for short.

In a PK-model, each of the predicates $T_1, T_2, T_{1\circ 2}$ is associated with an extension and an anti-extension. So we can group the formulas of L in three sets: $E_1 \cup A_1, E_2 \cup A_2, E_{1\circ 2} \cup A_{1\circ 2}$. The first two sets may be identified with domains properly understood, that is, pure domains that extend D_1 and D_2 respectively. Instead, $E_{1\circ 2} \cup A_{1\circ 2}$ is a domain only in a spurious sense. From now on we will say that, in a PK-model, a formula φ is of *type* 1 when $\varphi \in E_1 \cup A_1$, is of *type* 2 when $\varphi \in E_2 \cup A_2$, and is of *type* $1 \circ 2$ when $\varphi \in E_{1\circ 2} \cup A_{1\circ 2}$. A formula that does not belong to any of these three categories is a formula with no type.

To illustrate the notion of formula with no type, consider a formula $T_1{}^{\ulcorner}\varphi^{\urcorner}$ such that $\varphi \in D_2$ and $M \models \varphi$. This formula can represent the statement that (2) is empirically true, which involves a category mistake. Since $\varphi \notin E_1$, we get that $T_1{}^{\ulcorner}\varphi^{\urcorner} \notin E_1$. Since $\varphi \notin A_1$, we get that $T_1{}^{\ulcorner}\varphi^{\urcorner} \notin A_1$. So $T_1{}^{\ulcorner}\varphi^{\urcorner}$ is not of type 1. Moreover, $T_1{}^{\ulcorner}\varphi^{\urcorner}$ cannot be of type 2 or $1 \circ 2$ because it contains T_1. More generally, we call *improper predications* the formulas in which a truth predicate is combined with a sentence of the wrong type. The fact that these formulas have no type accords with the intuition that a sentence that involves a category mistake is not evaluable as true or false.

PK-models inherit from MD-frames their neutrality as to the distinction between alethic monism and alethic pluralism. Just as an MD-frame can be monist or pluralist, depending on whether or not D assigns 1 to every element of L^{-}_{A}, a PK-model can be monist or pluralist, depending on the MD-frame on which it is based. More precisely, every pluralist PK-model has a corresponding monist version in the following sense:

DEFINITION 11 For any pluralist PK-model $\mathcal{M}$ based on a MD-frame $\langle M, D\rangle$, a *monist reduction* of $\mathcal{M}$ is a PK-model $\mathcal{M}'$ based on $\langle M, D'\rangle$, where D' assigns the index 1 to every element of L^{-}_{A}.

In other words, a monist reduction of $\mathcal{M}$ is a PK-model that makes true, or false, exactly the same sentences as $\mathcal{M}$, but it makes them all true, or false, in the same sense. Definition 11 provides one way to describe the difference between monism and pluralism. It may be the case that, for a set of sentences E, the monist and the pluralist agree

that E is the set of truths. But while the monist thinks that all the sentences in E are true in the same sense — the unique, unqualified sense indicated by the index 1 — the pluralist thinks that a further distinction can be drawn within E in terms of distinct senses of being true. This is not to say that the monist and the pluralist generally agree on the set of truths: the set of sentences that qualify as true for the monist — say, correspondence truths — is typically a proper subset of the set of sentences that the pluralist regards as truth-apt, which may include sentences pertaining to the aesthetical domain. Yet an extensional agreement of the kind considered is possible at least in principle.

The notion of monist reduction yields the following equivalence, which holds for every PK-model:

PROPOSITION 3 If $\mathcal{M}$ is a pluralist PK-model and $\mathcal{M}'$ is a monist reduction of $\mathcal{M}$, for every φ such that φ is of type i in $\mathcal{M}$, $\varphi \in E_i$ in $\mathcal{M}$ iff $\varphi \in E_1$ in $\mathcal{M}'$, and $\varphi \in A_i$ in $\mathcal{M}$ iff $\varphi \in A_1$ in $\mathcal{M}'$.

Proof. Let $\mathcal{M}$ be a pluralist PK-model based on a MD-frame $\langle M, D\rangle$. Let $\mathcal{M}'$ be a monist reduction of $\mathcal{M}$, where $\langle M, D'\rangle$ is the frame of $\mathcal{M}'$. Assume that, in $\mathcal{M}$, φ is a formula of type i. Then it can be shown by induction that $\varphi \in E_i$ in $\mathcal{M}$ iff $\varphi \in E_1$ in $\mathcal{M}'$, and $\varphi \in A_i$ in $\mathcal{M}$ iff $\varphi \in A_1$ in $\mathcal{M}'$.
Base. Let $\varphi \in \mathsf{L}^-_A$. Suppose that $\varphi \in E_i$ in $\mathcal{M}$, that is, $M \models \varphi$ and $\varphi \in D_i$ in $\mathcal{M}$. Since D' assigns 1 to φ, it follows that $\varphi \in D_1$ in $\mathcal{M}'$, hence $\varphi \in E_1$ by clause 1.1 of Definition 3. Conversely, from the supposition that $\varphi \in E_1$ in $\mathcal{M}'$ we get that $\varphi \in E_i$ in $\mathcal{M}$. Now suppose that $\varphi \in A_i$ in $\mathcal{M}$, that is, $M \not\models \varphi$ and $\varphi \in D_i$ in $\mathcal{M}$. Since D' assigns 1 to φ, it follows that $\varphi \in D_1$ in $\mathcal{M}'$, hence $\varphi \in A_1$ by clause 3.1 of Definition 3. Conversely, from the supposition that $\varphi \in A_1$ in $\mathcal{M}'$ we get that $\varphi \in A_i$ in $\mathcal{M}$.
Inductive step. Suppose that the conjunction of biconditionals to be proved holds up to complexity n, and let φ be a formula of complexity $n+1$. By considering six cases for the first conjunct and six cases for the second, and relying respectively on clauses 1.2-1.6 and 3.2-3.6 of Definition 3, it can be shown that the same conjunction holds for φ as well. □

Note that if φ has no type, the biconditional stated in proposition 3 holds vacuously, that is, it is still the case that φ has type i in $\mathcal{M}$ if and only if it has type 1 in $\mathcal{M}'$.

7 PK-CONSEQUENCE

To complete our picture, we define a relation of logical consequence in terms of PK-models. Let us start from the most basic notion of validity, according to which an argument from a set of premises $\varphi_1, \ldots, \varphi_n$ to

a conclusion ψ is valid when, necessarily, if $\varphi_1, \ldots, \varphi_n$ are true, ψ is true. This notion can be made precise in different ways, and by itself does not imply preservation of a single truth property. As Gamester pointed out, the criterion of validity, when phrased in its most general form, simply requires a relation of structural dependence along the following lines:

DEFINITION 12 An argument from a set of premises $\varphi_1, \ldots, \varphi_n$ to a conclusion ψ is valid just in case, if the truth of φ_1 consists in F_1, ..., the truth of φ_n in F_n, and the truth of ψ in G, then, necessarily, if φ_1 is $F_1, \ldots$, and φ_n is F_n, then ψ is G.[18]

In other words, an argument is valid when the truth of its conclusion depends on the truth of its premises in the right way, so that ψ cannot fail to be true when $\varphi_1, \ldots, \varphi_n$ are true. The definition of logical consequence we suggest rests precisely on this idea.

Let it be granted that, for every PK-model and every formula of type i, E_i is the extension that the model assigns to T_i. The relation of PK-consequence between a set of formulas $\varphi_1, \ldots, \varphi_n$ and a formula ψ, in symbols $\models_{\mathsf{PK}}$, is defined as follows:

DEFINITION 13 $\varphi_1, \ldots, \varphi_n \models_{\mathsf{PK}} \psi$ iff for every PK-model, if $\varphi_1 \in E_i, \ldots, \varphi_n \in E_k$, and $i, \ldots, k$ are the types of $\varphi_1, \ldots, \varphi_n$ respectively, then $\psi \in E_j$, where j is the type of ψ.

An immediate consequence of this definition is that, when the formulas involved are all of the same type (i.e. when $i = \cdots = k = j$), PK-consequence boils down to preservation of truth relative to that type. Suppose that $\varphi_1, \ldots, \varphi_n \models_{\mathsf{PK}} \psi$, and consider an MD-frame in which $\varphi_1, \ldots, \varphi_n$ and ψ are all formulas of type i. For any PK-model based on that frame we get that, if $\varphi_1 \in E_i, \ldots, \varphi_n \in E_i$, then $\psi \in E_i$. Note, further, that PK-inferences can also obtain between sentences that have no type. For example, if φ has no type, $\varphi \models_{\mathsf{PK}} \varphi$ still holds, because Definition 13 is vacuously satisfied.

Our pluralist notion of logical consequence generalizes the Kripkean notion of logical consequence defined for a single truth predicate. If $\models_{\mathsf{K}}$ stands for logical consequence in a monist Kripkean language, the following holds:

PROPOSITION 4 If $\varphi_1, \ldots, \varphi_n \models_{\mathsf{K}} \psi$, then $\varphi_1, \ldots, \varphi_n \models_{\mathsf{PK}} \psi$.

Proof. Assume that $\alpha_1, \ldots, \alpha_n \models_{\mathsf{K}} \beta$. Consider an arbitrary PK-model $\mathcal{M}$, and suppose that $\alpha_1 \in E_i, \ldots, \alpha_n \in E_k$ in $\mathcal{M}$, where $i, \ldots, k$ are the types of $\alpha_1, \ldots, \alpha_n$ respectively. Now let $\mathcal{M}'$ be a monist reduction of $\mathcal{M}$. By proposition 3, $\alpha_1 \in E_1, \ldots, \alpha_n \in E_1$. Given the initial assumption, it follows that $\beta \in E_1$. By proposition 3 again, it follows that $\beta \in E_j$, where j is the type of β. □

18. Gamester 2019, p. 38, notation adapted. The same point has been made by Lynch and Pedersen 2018. For a critical discussion, see Strollo 2022a, 2024.

Here we assume, as usual, that $\models_{\mathsf{K}}$ is defined as preservation of truth in standard Kripkean models. But the kind of generalization suggested could equally be applied to other definitions. For example, if $\models_{\mathsf{K}}$ were defined as preservation of non-falsity, we could provide a definition of $\models_{\mathsf{PK}}$ in terms of non-falsity, which extends $\models_{\mathsf{K}}$ exactly in the same way. The same goes for definitions that give rise to substructural consequence relations, such as 'strict-tolerant' and 'tolerant-strict' consequence.[19]

The account of logical consequence just provided completes the presentation of our pluralist theory of truth, which we will call PKT, for *Pluralist Kripkean Truth*. More explicitly:

DEFINITION 14 $\mathsf{PKT} := \{\langle \Gamma, \varphi \rangle \mid \Gamma \models_{\mathsf{PK}} \varphi\}$

In the next three sections we will show how PKT can handle three crucial problems that have been widely discussed in connection with alethic pluralism.

8 MIXED COMPOUNDS

The first problem concerns *mixed compounds*, that is, complex sentences whose constituents belong to different domains. For example, (7) is a mixed compound. Since (7) is intuitively true, the question naturally arises of what kind of truth property should be ascribed to it. That is, how should one read the word 'true' as it occurs in the following sentence?

(9) 'Snow is white and 2+2=4' is true

Several attempts have been made to provide a satisfactory answer to this question, which poses a serious challenge to alethic pluralism.[20]

In particular, one key issue to be addressed — which has not yet received a clear answer — is whether or how mixed compounds differ from *pure compounds*, that is, complex sentences whose constituents belong to the same domain. Compare (7) with the following sentence:

(10) Snow is white and grass is green

On the one hand, it might be argued that (10) is not like (1): the truth of (10), like the truth of (7), cannot be understood as correspondence, for there is no such thing as the conjunctive fact that snow is white and grass is green. On the other hand, however, (7) and (10) seem to differ in one crucial respect: the truth of (10), unlike the truth of (7),

19. Cobreros et al. 2012.

20. The problem is addressed in Tappolet 2000, Edwards 2008, Cotnoir 2009, Edwards 2009, Cook 2011, Yu 2017, Lynch and Pedersen 2018, Gamester 2019, among other works.

seems entirely reducible to the truth of sentences which correspond to reality.[21]

In this section we focus on some basic constraints on mixed compounds, without providing a detailed discussion of the extant proposals. More precisely, we consider three general conditions which may plausibly be regarded as desiderata for a pluralist account of mixed compounds, in order to show that PKT satisfies all of them. In what follows, domains are understood as pure domains, as explained in section 6.[22]

The first condition concerns pure compounds, such as (10):

(C1) A compound of sentences associated with the same domain is also associated with the same domain.

Intuitively, if the semantic value of a complex sentence depends on the semantic value of its constituents, and its constituents are true in the same sense, the complex sentence should itself be true in that sense. This is to say that (10) should be true in the same sense in which its constituents are true.

The second condition concerns mixed compounds, such as (7):

(C2) A compound of sentences belonging to distinct domains should be associated with a domain distinct from those associated with any of its constituents.

Intuitively, if the semantic value of a complex sentence depends on the semantic value of its constituents, and its constituents are true in distinct senses, the complex sentence should be true in a further sense other than that of its constituents. For example, since (1) and (2) are true in two distinct ways, (7) must be true in a third way. The idea behind of C2 is that the parts of the compound should contribute equally to the truth value of the whole compound.[23] If (7) were true in the sense in which (1) is true, but not in the sense in which (2) is true, (1) and (2) would not contribute equally to the truth value of (7).[24]

The third condition concerns the connection between truth and falsity:

(C3) The negation of a domain-specifically true sentence is domain-specifically false, and vice versa.

21. This difference is not generally recognized. For instance, Edwards 2009, Kim and Pedersen 2018, and Gamester 2019 treat the two cases alike when addressing the issue of mixed compounds.

22. Yu 2017 considers similar conditions, although his use of 'domain' is not exactly the same.

23. This is a generalization of an idea expressed in Lynch and Pedersen 2018, p. 550. See also Kim and Pedersen 2018, p. 110.

24. Here we are ruling out the possibility that (7) is both true in the same sense in which (1) is true and true in the same sense in which (2) is true, for that would trivialize the notion of strict truth, as noted in section 4.

Intuitively, if a sentence is true in a given sense, its negation should be false in that sense, and the other way round. So, for example, the negation of (1) should be false in the same sense in which (1) is true.

PKT satisfies C1-C3. C1 is satisfied because, for any $\varphi, \psi \in \mathsf{L}$, if φ and ψ belong to the same set — E_1 or E_2 — the same goes for $\varphi \wedge \psi$ and $\varphi \vee \psi$. In the case of (10), this means that there is a truth property, empirical truth, that the complex sentence shares with its constituents.

C2 is satisfied because, for any φ, ψ, if φ and ψ belong to distinct sets — E_1 and E_2 — then $\varphi \wedge \psi$ and $\varphi \vee \psi$ belong to a third set, namely $E_{1\circ 2}$. In the case of (7), this means that the truth property of the complex sentence, mixed truth, is distinct from the truth properties of its constituents, empirical truth and mathematical truth.

C3 is satisfied because, for any $\varphi \in \mathsf{L}$, if φ belongs to one of the two extensions, E_1 or E_2, then $\neg\varphi$ belongs to the corresponding anti-extension, and vice versa. In the case of (1), which is empirically true, this means that its negation is empirically false.

The principles about $\wedge$ stated above can be extended to $\forall$. More specifically, if for every closed term t, $(\varphi)^t_x \in E_1$, then $\forall x\varphi \in E_1$. Similarly, if for every closed term t, $(\varphi)^t_x \in E_1$ or $(\varphi)^t_x \in E_2$ and it is not the case that every $(\varphi)^t_x$ belongs to the same set E_1 or E_2, then $\forall x\varphi \in E_{1\circ 2}$. Although the debate on mixed compounds mostly focuses on sentential truth-functional connectives, quantified sentences are also compounds. Consider the following sentence:

(11) Everything Aristotle said is true

Since Aristotle made statements about different subjects, if (11) is true, it is true in virtue of different kinds of truths. PKT predicts exactly this, as it treats quantified sentences as generalized conjunctions. Suppose that Aristotle asserted two kinds of truths, empirical truths and mathematical truths. By clause 1.5 of Definition 3 we get that (11) is true in the mixed sense.

Two important points emerge from the foregoing remarks. First, being a true compound is not the same thing as being true in the mixed sense. This is illustrated by (10), which differs from (7): a complex sentence can be true without being true in the mixed sense. Second, empirical truth is not the same thing as correspondence, and mathematical truth is not the same thing as coherence. In our frames, the distinction between correspondence and coherence emerges at the level of the base domains. Let us assume that, for any atomic sentence in D_1, being made true by M means corresponding to reality, and for any atomic sentence in D_2, being made true by M means being coherent. From Definition 7 we then get that empirical truth depends on correspondence to reality in the sense that, for every formula in E_1, all its atomic constituents belong to D_1. Similarly, from Definition 8 we get that mathematical truth depends on coherence in the same sense. However, this does not mean that correspondence

and coherence transfer to complex sentences. For example, it may coherently be claimed that (10) is empirically true but there is no conjunctive fact to which it corresponds.

9 MIXED INFERENCES

The second problem, which is directly related to the first, concerns *mixed inferences*, that is, inferences involving sentences that belong to different domains. Here is an example:

(1) Snow is white
(2) 2+2=4
---
(7) Snow is white and 2+2=4

The inference from (1) and (2) to (7) is apparently valid. But since (1) and (2) belong to distinct domains, this cannot be explained by saying that a single truth property is necessarily preserved in the step from (1) and (2) to (7). So the question naturally arises of how can one make sense of its validity. More generally, the pluralist has to explain the apparent validity of some mixed inferences without appealing to necessary truth preservation as understood by the monist.[25]

As in the case of mixed compounds, we will not provide a detailed discussion of the extant proposals. The aim of this section is twofold. First, we show that PKT provides a coherent account of mixed inferences. Then, we draw attention to a general hypothesis about mixed inferences that seems independently justified, and explain how PKT accords with this hypothesis.

Let us start with the account of mixed inferences. As explained in section 7, the relation of logical consequence defined in PKT is based on the idea of structural dependence expressed by Definition 12, namely, that if the premises have certain truth properties, the conclusion must have some truth property determined by those properties. In the case of non-mixed inferences, the property of the conclusion is the same as the property of the premises, while in the case of mixed inferences it can differ, which is precisely what happens in the inference from (1) and (2) to (7).

Note that our account of validity cannot be understood in terms of preservation of some generic truth property. In the case of valid non-mixed inferences this is easily seen: if the premises are true in one domain-specific sense, the conclusion will be true in the same domain-specific sense. In the case of valid mixed inferences, if the premises are true in n different senses, the conclusion will be true in a sense that depends on the n aforementioned senses. This truth property, again, is not a generic one, as explained in section 8.

25. This issue is addressed in Tappolet 1997, 2000, Beall 2000, Pedersen 2006, Cotnoir 2013b, Yu 2017, Strollo 2018, 2022b, Gamester 2019, among other works.

Let us now consider the following relation between arguments, which we call *projection*:

DEFINITION 15 For any two arguments A and A', A' is a projection of A if and only if there is one-to-one correspondence between the sentences in A and the sentences in A' such that, for any sentence φ in A, the corresponding sentence φ' in A' has the same truth conditions as φ but makes explicit at the syntactic level the truth property of φ.

Normally, when one assertively utters a sentence φ, one conveys the thought that φ is true. In fact it is usually regarded as part of the normative constraints that define assertion as a speech act that one correctly asserts φ only if φ is true. So, asserting φ implies taking φ to be true. On the other hand, it seems that one cannot assert that φ is true without thereby asserting φ. This is why it is usually taken for granted that φ is equivalent to its truth predication, as implied by (DS). As long as such equivalence is assumed, for any argument A there is an argument A' that is a projection of A, where each sentence of A is replaced by the corresponding truth predication. In particular, the following argument is a projection of the argument above:

(3) 'Snow is white' is true
(4) '2+2=4' is true
---
(9) 'Snow is white and 2+2=4' is true

The hypothesis that we will consider — call it the *Projection Hypothesis* — is the following:

(PH) For any two arguments A and A', if A' is a projection of A, then A is valid if and only if A' is valid.

Clearly, (PH) owes its plausibility to the intuitive equivalence between a sentence and its truth predication. In the example considered, (1), (2), and (7) are intuitively equivalent to (3), (4), and (9) respectively. The plausibility of this equivalence is neutral with respect to the distinction between alethic monism and alethic pluralism. The same goes for (PH) itself.

The key difference between alethic monism and alethic pluralism emerges when Definition 15 is applied to the case of mixed inferences. Let A be mixed. According to alethic monism, A stands in the relation defined with A' when the sentences in A are replaced in A' by truth predications that are all of the same kind. In our example, this is to say that the word 'true' in (3), (4), and (9) is understood in the same way. According to alethic pluralism, instead, each sentence in A must be replaced in A' by a truth predication of the appropriate kind. In our example, this is to say that the word 'true' in (3), (4), and (9) is understood in different ways.

On the pluralist understanding of Definition 15, (PH) may be regarded as a crucial adequacy condition for a theory of truth that

aims to provide a coherent treatment of truth predications in the object language. For any mixed A, the following should hold: if A' is a projection of A where distinct kinds of truth predications of the sentences in A are made, then A is valid if and only if A' is valid. In our example, the pluralist has to explain why the inference from (1) and (2) to (7) is valid if and only if the inference from (3) and (4) to (9) is valid, where the latter is appropriately formalized by using distinct truth predicates.

PKT provides the required explanation. First consider the argument from (1) and (2) to (7), which are formalized in L respectively as $\varphi, \psi, \varphi \wedge \psi$. From Definition 13 we get that $\varphi, \psi \models_{\mathsf{PK}} \varphi \wedge \psi$. Note that $\varphi, \psi \models_{\mathsf{K}} \varphi \wedge \psi$, which entails $\varphi, \psi \models_{\mathsf{PK}} \varphi \wedge \psi$ by proposition 4. In particular, consider an MD-frame in which φ is of type 1 and ψ is of type 2. For any model based on such frame, we get that $\varphi \wedge \psi \in E_{1\circ 2}$, which is the outcome one would expect.

Now consider the argument from (3) and (4) to (9), which are formalized in L respectively as $T_1{}^{\ulcorner}\varphi^{\urcorner}, T_2{}^{\ulcorner}\psi^{\urcorner}, T_{1\circ 2}{}^{\ulcorner}\varphi \wedge \psi^{\urcorner}$. As in the previous case, Definition 13 entails that $T_1{}^{\ulcorner}\varphi^{\urcorner}, T_2{}^{\ulcorner}\psi^{\urcorner} \models_{\mathsf{PK}} T_{1\circ 2}{}^{\ulcorner}\varphi \wedge \psi^{\urcorner}$. In particular, consider an MD-frame in which φ is of type 1 and ψ is of type 2, hence the same goes for $T_1{}^{\ulcorner}\varphi^{\urcorner}$ and $T_2{}^{\ulcorner}\psi^{\urcorner}$ respectively. For any model based on such frame, we get that $\varphi \wedge \psi \in E_{1\circ 2}$, hence $T_{1\circ 2}{}^{\ulcorner}\varphi \wedge \psi^{\urcorner} \in E_{1\circ 2}$.

More generally, the following holds:

PROPOSITION 5 For any formula φ, let φ' be $T_i{}^{\ulcorner}\varphi_i{}^{\urcorner}$, where i is the type of φ_i, if φ has a type, and $\varphi' = \varphi$ otherwise. Then $\varphi_1, \ldots, \varphi_n \models_{\mathsf{PK}} \psi$ iff $\varphi'_1, \ldots, \varphi'_n \models_{\mathsf{PK}} \psi'$.

Proof. Let $\varphi'_1, \ldots, \varphi'_n$ and ψ' be obtained from $\varphi_1, \ldots, \varphi_n$ and ψ in the way required. By proposition 2 we get that, for every φ_i, if φ_i is a formula of type j, then $\varphi_i \in E_j$ iff $\varphi'_i \in E_j$, and similarly that $\psi \in E_j$ iff $\psi' \in E_j$. By Definition 13 it follows that $\varphi_1, \ldots, \varphi_n \models_{\mathsf{PK}} \psi$ iff $\varphi'_1, \ldots, \varphi'_n \models_{\mathsf{PK}} \psi'$. □

Proposition 5 shows how PKT validates (PH). Note that, if a formula φ has no type, then (PH) simply does not apply to it, and proposition 5 is vacuously satisfied.

10 SEMANTIC PARADOXES

Paradoxical sentences involving a single truth predicate are well-known. A paradigmatic example is the *Liar sentence*. In a language that contains a single truth predicate T, a term t_λ can be obtained by diagonalization from the open formula $\neg T(x)$ in such a way that $t_\lambda = {}^{\ulcorner}\neg T(t_\lambda/x)^{\urcorner}$ in the selected acceptable model.[26] Using λ to

26. Acceptable models in Moschovakis's sense guarantee the existence of sentences and terms such as the ones mentioned here to formalize a Liar sentence, in that they

abbreviate $\neg T(t_\lambda)$, we get that λ is equivalent to $\neg T^{\ulcorner}\lambda^{\urcorner}$, viz. λ "says of itself" that it is not true. In the original Kripkean construction, the inductive definition does not allow λ or $\neg\lambda$ to be part of the extension or the anti-extension of T.

In a language that contains distinct truth predicates, one can formulate distinct kinds of Liar sentences. Consider our language L. A sentence λ_1 can be defined following the above pattern as equivalent to $\neg T_1{}^{\ulcorner}\lambda_1{}^{\urcorner}$, so that it says of itself that it is not empirically true. Similarly, λ_2 can be defined as a sentence that says of itself that it is not mathematically true. The formal treatment of λ_1 and λ_2 in our semantics mirrors what happens in the original Kripke construction: neither they nor their negations are in any of the sets defined so far.

PROPOSITION 6 $\lambda_1, \lambda_2 \notin E_1 \cup E_2 \cup E_{1\circ 2} \cup A_1 \cup A_2 \cup A_{1\circ 2}$

Proof. First we establish that $\lambda_1, \lambda_2, \notin Ei_1 \cup Ei_2 \cup Ai_1 \cup Ai_2$. There are 8 cases. We just prove two of them in some detail — namely $\lambda_1 \notin Ei_1$ and $\lambda_1 \notin Ai_2$ — since the others are essentially analogous.
Case 1: $\lambda_1 \in Ei_1$. By definition of λ_1, this means that $\neg T_1{}^{\ulcorner}\lambda_1{}^{\urcorner} \in Ei_1$. Ei_1 is the least fixed point of $\mathcal{H}_1$. Therefore, if $\lambda_1 \in Ei_1$, there is a smallest successor ordinal $\delta + 1$ such that $\neg T_1{}^{\ulcorner}\lambda_1{}^{\urcorner} \in \mathcal{H}_1^{\delta+1}(\varnothing) \setminus \mathcal{H}_1^{\delta}(\varnothing)$. This means that (by clause 1.2 of Definition 3), $T_1{}^{\ulcorner}\lambda_1{}^{\urcorner} \in \mathcal{I}_1^{\delta}(\varnothing)$. But then, by construction, there is a least ordinal $\gamma + 1 \leq \delta$ such that $\mathrm{T}_1{}^{\ulcorner}\lambda_1{}^{\urcorner} \in \mathcal{I}_1^{\gamma+1}(\varnothing) \setminus \mathcal{I}_1^{\gamma}(\varnothing)$. This means that (clause 2.6), $\lambda_1 \in \mathcal{I}_1^{\gamma}(\varnothing)$, i.e. that $\neg T_1{}^{\ulcorner}\lambda_1{}^{\urcorner} \in \mathcal{I}_1^{\gamma}(\varnothing)$. As above, this means that there is a least ordinal $\beta + 1 \leq \gamma$ such that $\neg T_1{}^{\ulcorner}\lambda_1{}^{\urcorner} \in \mathcal{I}_1^{\beta+1}(\varnothing) \setminus \mathcal{I}_1^{\beta}(\varnothing)$, and thus (clause 2.2) that $T_1{}^{\ulcorner}\lambda_1{}^{\urcorner} \in \mathcal{H}_1^{\beta}(\varnothing)$. Finally, this means that, again by construction, there is an $\alpha + 1 \leq \beta$ such that $\mathrm{T}_1{}^{\ulcorner}\lambda_1{}^{\urcorner} \in \mathcal{H}_1^{\alpha+1}(\varnothing) \setminus \mathcal{H}_1^{\alpha}(\varnothing)$, and therefore (by clause 1.6), that $\lambda_1 \in \mathcal{H}_1^{\alpha}(\varnothing)$. But $\alpha < \delta + 1$, against the minimality of $\delta + 1$. A similar reasoning establishes that $\lambda_1 \notin Ai_1$, $\lambda_2 \notin Ei_2$, and $\lambda_2 \notin Ai_2$.
Case 2: $\lambda_1 \in Ai_2$. By definition of λ_1, this means that $\neg T_1{}^{\ulcorner}\lambda_1{}^{\urcorner} \in Ai_2$. Ai_2 is the least fixed point of $\mathcal{I}_2$. Therefore, if $\lambda_1 \in Ai_2$, there is a smallest successor ordinal $\beta + 1$ such that $\neg T_1{}^{\ulcorner}\lambda_1{}^{\urcorner} \in \mathcal{I}_2^{\beta+1}(\varnothing) \setminus \mathcal{I}_2^{\beta}(\varnothing)$. This means by clause 2.2 of Definition 3 that $T_1{}^{\ulcorner}\lambda_1{}^{\urcorner} \in \mathcal{H}_2^{\beta}(\varnothing)$. However, $\mathcal{H}_2$ has no clause to include any sentence of the form $T_1(t)$ in its construction. This entails that (i) β is a limit ordinal, and that (ii) there is no least successor $\alpha_0 + 1 \leq \beta$ such that $T_1{}^{\ulcorner}\lambda_1{}^{\urcorner} \in \mathcal{H}_2^{\alpha_0+1}(\varnothing)$. By (i) and (ii) and Definition 4, for every limit ordinal $\alpha \leq \beta$, $T_1{}^{\ulcorner}\lambda_1{}^{\urcorner} \in \mathcal{H}_2^{\alpha}(\varnothing)$. Therefore, $T_1{}^{\ulcorner}\lambda_1{}^{\urcorner} \in \mathcal{H}_2^{\omega}(\varnothing)$. But then, (ii) entails that there is no finite ordinal n such that $T_1{}^{\ulcorner}\lambda_1{}^{\urcorner} \in \mathcal{H}_2^{n+1}(\varnothing)$. Therefore $T_1{}^{\ulcorner}\lambda_1{}^{\urcorner} \in \mathcal{H}_2^{0}(\varnothing) = \varnothing$. An entirely similar reasoning establishes that $\lambda_1 \notin Ei_2$, $\lambda_2 \notin Ei_1$, and $\lambda_2 \notin Ai_1$.

can be easily taken to satisfy the so-called *strong* diagonal Lemma. See Heck 2007; Picollo 2020 for more details.

Having proved that $\lambda_1, \lambda_2, \notin Ei_1 \cup Ei_2 \cup Ai_1 \cup Ai_2$, one can prove the proposition. By Definitions 7-10 and the fact that $E_{1\circ 2} \neq \varnothing \neq A_{1\circ 2}$, we have that $E_1 \subsetneq Ei_1$, $E_2 \subsetneq Ei_2$, $A_1 \subsetneq Ai_1$, and $A_2 \subsetneq Ai_2$. Therefore, $\lambda_1, \lambda_2, \notin E_1 \cup E_2 \cup A_1 \cup A_2$. It remains to be shown that $\lambda_1, \lambda_2, \notin E_{1\circ 2} \cup A_{1\circ 2}$. But this is easily seen, given that the construction of $E_{1\circ 2}$ and $A_{1\circ 2}$ does not allow formulas which are not included in the construction of the previous impure sets. □

Proposition 6 shows that λ_1 and λ_2 are formulas with no type. Our theory thus provides a conceptually coherent and formally consistent account of Liar sentences, which generalizes Kripke's original theory to a pluralist framework. We conclude the section by addressing two possible objections.

The first objection concerns proposition 6. Consider λ_1. One might be tempted to think that it is intuitively wrong to say that λ_1 is neither true nor false. After all, λ_1 says that λ_1 is not empirically true and, well, it is not: whatever its semantic status — true, false, or otherwise — that status is not pertinent to the empirical domain. So, *a fortiori*, λ_1 is not empirically true. Similar considerations apply to λ_2.

The reply to this objection is straightforward: given naïveté — that is, 9 and 10 of proposition 2 — there is no coherent way of accepting λ_1. For $\neg T_1{}^{\ulcorner}\lambda_1{}^{\urcorner}$ just *is* λ_1, and therefore accepting λ_1 together with naïveté entails accepting $T_1{}^{\ulcorner}\lambda_1{}^{\urcorner}$ as well. This contradicts the initial observation that λ_1 is not empirically true. Moreover, λ_1 is not a mathematical or a mixed truth, so T_2 and $T_{1\circ 2}$ do not apply either. Thus none of the available truth predicates is applicable. To forego the naïveté intuition seems a very high price for the sake of accepting $\neg T_1{}^{\ulcorner}\lambda_1{}^{\urcorner}$. Note that, since intersubstitutivity entails naïveté, rejecting the latter implies rejecting the former.[27]

The second objection concerns the analogy between Liar sentences and improper predications. Liar sentences and improper predications both end up as formulas of no type. So it might be contended that our account misses the difference between them, for category mistakes and semantic paradoxes are distinct phenomena.

To address this objection, one must consider fixed points other than $\mathcal{J}^{\alpha}(\varnothing)$. One key property of $\mathcal{J}$ is that it is *monotone*: for any two quadruples $S = \langle S_1, S_2, S_3, S_4\rangle$ and $S' = \langle S'_1, S'_2, S'_3, S'_4\rangle$, letting $S \preceq S'$ abbreviate $S_i \subseteq S'_i$ for every $1 \leq i \leq 4$, we get that if $S \preceq S'$, then $\mathcal{J}(S) \preceq \mathcal{J}(S')$. By the Knaster-Tarski Theorem, the fixed points of a monotone operator form a lattice.[28] In particular, $\mathcal{J}^{\alpha}(\varnothing)$ is the *least*

27. The same picture emerges in more complex cases. Let λ^* be a sentence equivalent to $\neg T_1{}^{\ulcorner}\lambda^*{}^{\urcorner} \wedge \neg T_2{}^{\ulcorner}\lambda^*{}^{\urcorner} \wedge \neg T_{1\circ 2}{}^{\ulcorner}\lambda^*{}^{\urcorner}$. Informally, λ^* "says of itself" that it is not true in any of the available senses of 'true'. As above, it seems that λ^*, whatever its semantic status, is neither empirically nor mathematically true, nor is it partly both. So, we might want to validate λ^* by placing it in some extension. But, whether such extension is E_1, E_2, or $E_{1\circ 2}$, it would follow that one of $T_1{}^{\ulcorner}\lambda^*{}^{\urcorner}, T_2{}^{\ulcorner}\lambda^*{}^{\urcorner}, T_{1\circ 2}{}^{\ulcorner}\lambda^*{}^{\urcorner}$ is to be accepted, which is absurd.

28. Tarski 1955.

fixed point of $\mathcal{J}$, in that for every fixed point S of $\mathcal{J}$ (i.e. every S such that $\mathcal{J}(S) = S$), we have that $\mathcal{J}^{\alpha}(\varnothing) \preceq S$.

Some sentences, while not in the extension or anti-extension of the least fixed point, belong to some consistent non-minimal fixed points. For example, let τ be a *truth-teller sentence* which "says of itself" that it is true. τ belongs to some consistent non-minimal fixed points, so it differs from λ. For this reason Kripke did not classify τ as paradoxical. The case of improper predications is similar. Although an improper predication does not belong to $\mathcal{J}^{\alpha}(\varnothing)$, and so behaves exactly like λ_1 and λ_2 as far as $\mathcal{J}^{\alpha}(\varnothing)$ is concerned, it belongs to some consistent non-minimal fixed point of $\mathcal{J}$. So it is not paradoxical according to Kripke's taxonomy, unlike λ_1 and λ_2. However, we should note that some Liar-like sentences pattern together with truth-tellers and improper truth predications. Let $\lambda_{1\circ 2}$ be a sentence equivalent to $\neg T_{1\circ 2}{}^{\ulcorner}\lambda_{1\circ 2}{}^{\urcorner}$. Although $\lambda_{1\circ 2}$ behaves like λ_1 and λ_2 in some crucial respects — in particular, $\lambda_{1\circ 2} \notin E_1 \cup E_2 \cup E_{1\circ 2} \cup A_1 \cup A_2 \cup A_{1\circ 2}$ — it differs from them in that it belongs to some consistent non-minimal fixed points, just like truth-teller sentences and improper truth predications.

11 PHILOSOPHICAL EPILOGUE

In the foregoing sections we have shown how a plurality of truth properties can be represented in a coherent formal framework by means of a suitable Kripke-style construction. Our semantics, which provides a generalization of Kripke's original semantics, yields intuitively correct extensions and anti-extensions for two basic predicates, T_1 and T_2, and one mixed truth predicate, $T_{1\circ 2}$. As emerges from sections 8-10, the resulting theory, PKT, can handle at least three crucial problems that have been raised in connection with alethic pluralism: mixed compounds, mixed inferences, and semantic paradoxes.

The relevance of our proposal to the debate on alethic pluralism, however, goes far beyond these three problems. The formal definitions and results presented above are intended to provide an intuitively adequate technical machinery that may prove useful to elucidate the very idea that there is a plurality of truth properties. In particular, the semantics outlined may help to get a deeper insight into three crucial questions that naturally arise in connection with alethic pluralism.

The first question concerns the relation between *pure truth* — empirical or mathematical — and *mixed truth*. As we have seen, in order to handle mixed compounds and mixed inferences it is crucial to treat mixed truth as one of the truth properties that a sentence can have. So, mixed truth is like pure truth in some sense. On the other hand, however, there is a sense in which mixed truth is *not* like pure truth, that is, mixed truth depends on pure truth in a way in which pure truth does not depend on mixed truth. How can the pluralist provide a coherent account of this apparent duplicity? Our theory

provides an answer to this question. Since $T_{1\circ 2}$ is exactly like T_1 and T_2 as far as its syntactic and semantic behaviour is concerned, $T_{1\circ 2}$ represents a truth property in the sense required. At the same time, however, one can account for the derivative status of mixed truth by employing a semantic notion of *grounding* defined along Kripkean lines. Consider the base domain D_1 in an arbitrary MD-frame $\langle M, D\rangle$. For any atomic formula φ in D_1 such that φ is made true by M, one can say that φ is grounded in M in the specific sense pertaining to D_1, that is, the empirical sense. The rest of the construction extends this grounding relation to complex formulas, so that any formula φ in E_1 turns out to be grounded in M in the same sense. Moreover, as long as grounding at the atomic level is understood as correspondence, it turns out that empirical truth is grounded in correspondence, which features as a fundamental property. A similar reasoning applies to mathematical truth, for in that case we get that mathematical truth is grounded in coherence. Since mixed truth results from the combination of empirical truth and mathematical truth, it is ultimately grounded in correspondence and in coherence. For example, the truth of (7) is grounded both in the empirical truth of (1), which in turn is grounded in correspondence, and in the mathematical truth of (2), which in turn is grounded in coherence. Thus, although pure truth and mixed truth are both grounded, the key difference between them is that in the case of mixed truth there is no such thing as a base mixed domain, which means that there is no such thing as a corresponding fundamental property. Mixed truth depends on pure truth in the sense that it is ultimately grounded in distinct fundamental properties that characterize distinct kinds of pure truth.[29]

The second question concerns *logical truth*: is logical truth one of the basic truth properties to be postulated, or is it rather a different kind of property that somehow overlaps with them? We think that the second disjunct is more plausible than the first. Logical truth is not to be regarded as a basic truth property along with empirical truth and mathematical truth, and is not derivative in the sense in which mixed truth is derivative. Rather, it is a different kind of property — truth in virtue of logical form — that belongs to sentences that have one of those properties. Definition 13 gives formal substance to this thought. Consider for example two formulas φ and ψ such that $\varphi \in D_1$ and $\psi \in D_2$. On our account, $\varphi \vee \neg\varphi \in E_1$ and $\psi \vee \neg\psi \in E_2$, that is, the first disjunction is empirically true while the second is mathematically true. But there is something that they have in common, namely, they are both true (each in its own sense) in virtue of their form. This

29. Formal explications and developments of the Kripkean notion of grounding can be found in Yablo 1982 and Leitgeb 2005. Note that the account of the derivative status of mixed truth just outlined can plausibly be phrased in terms of some metaphysical notion of grounding, that is, assuming that grounding is a form of constitutive determination or explanation. See Correia and Schnieder 2012 for a discussion of such notions.

emerges clearly from the fact that $\models_{\mathsf{PK}} \varphi \vee \neg\varphi$ and $\models_{\mathsf{PK}} \psi \vee \neg\psi$ hold no matter how one chooses the base model.

The third question concerns the *unity of truth*: if there is a plurality of truth properties, what is it that makes all of them *truth* properties? A plausible way to answer this question is to say, following Wright, that all the truth properties satisfy a set of basic principles which constitutes the concept of truth. As Wright puts it,

> the way to think the unity and the plurality of truth is this: there is a single *concept* of truth, and the *property* it presents can vary from area of discourse to area of discourse.[30]

Our theory provides a formal characterization of this idea insofar as it implies that the predicates $T_1, T_2, T_{1\circ 2}$ are similar in some crucial respects. In particular, one key principle that accounts for the unity of truth is naïveté. This principle expresses the idea conveyed by (DS), which Wright regards as constitutive of the concept of truth. Since $T_1, T_2, T_{1\circ 2}$ obey naïveté, the properties they represent are all truth properties: the basic structural feature they share — the fact that a sentence φ of type i is equivalent to its truth predication $T_i\ulcorner\varphi\urcorner$ — may be regarded as a distinctive mark of the concept of truth. What makes $T_1, T_2, T_{1\circ 2}$ represent distinct truth properties is that they have distinct extensions and anti-extensions. So, there is a sense in which empirical truth and mathematical truth share some basic formal features, while they differ in their content.

This is far from being a full-blooded account of the unity of truth. Although (DS) is essential to our understanding of truth, it is reasonable to expect that the set of principles that constitutes the concept of truth does not reduce to (DS). Note that alethic pluralism crucially differs from deflationism, as long as the latter is understood as the view that (DS) provides an exhaustive account of the use of 'true' in our talk and thought.[31] The pluralist may argue that it is part of the concept of truth that a certain normative function characterizes truth as distinct from assertibility. Moreover, and relatedly, the pluralist should explain the need of a variety of truth properties rather than a single truth property. This suggests that there are aspects of the concept of truth that are hardly reducible to (DS). However, a formal representation of such aspects would go far beyond the scope of our work, given the limited expressive resources of L. What we can provide here is at most a partial characterization of the concept of truth that is compatible with such an account.[32]

30. Wright 2013, p. 128. This line of thought is developed in Wright 1992, 1998, 2013.

31. See for example Horwich 1990 and Shapiro 1998. For other forms of deflationism, see Field 1994, and more recently Picollo and Schindler 2018a. Wright 1992 and Sher 2023b provide a criticism of deflationism from a pluralist perspective.

32. The assumption that (DS) characterizes the concept of truth may also be matter of controversy. As Price 1998 points out, the disquotational behaviour of 'true' may not be unique to the truth predicate.

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