Università degli Studi di Torino
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Classical and Quantum aspects of Bosonic String Field Theory

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"*Φύσις κρύπτεσθαι φιλεῖ.*"
"*Nature loves to hide.*"

***H**eraclitus, (Fragment)*.

Acknowledgements

English Version

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Abstract

This thesis presents several developments in bosonic string field theory (SFT), addressing both formal and physical aspects of the theory.

We construct a gauge-invariant kinetic term for SFT on target spaces with boundaries by introducing boundary terms and new degrees of freedom localized on the boundary, ensuring a well-defined variational principle and restoring gauge invariance. The construction is illustrated through explicit examples in open and closed string sectors, including the linear dilaton background.

Open–closed SFT is reformulated in the coalgebraic language, allowing a systematic derivation of the nilpotent structure that governs quantum consistency within the Batalin-Vilkovisky (BV) framework. This formalism is applied to describe D-brane backreaction in the large-N limit, allowing us to derive the classical closed SFT action corresponding to the new background without branes.

A systematic procedure for adding stubs to generic classical and quantum SFTs is developed, accompanied by a proof of invariance of tree- and loop-level amplitudes under such deformations.

Finally, SFT methods are used to study short bulk RG flows in two-dimensional CFTs with D-branes. By computing the on-shell disk action for slightly relevant bulk perturbations, we determine the shift in the g-function and the string coupling constant at next-to-leading order. This provides an alternative approach to computations traditionally requiring higher-loop conformal perturbation theory.

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CHAPTER 1

Introduction

1.1 Introduction

String Theory (ST) began as an attempt in the late 1960s to describe the strong nuclear force [1]. Yet, it soon became clear that strings could provide a deeper insight: a consistent framework in which gravity, matter and gauge interactions admit a consistent unified quantum description. In the 1970s, the first-quantized formulation of strings was developed, treating them as one-dimensional objects propagating in spacetime, with dynamics encoded in a two-dimensional conformal field theory (CFT) on the worldsheet (see for instance [2–5] for review).

Despite its successes, this perturbative framework has clear limitations, and the fundamental picture remains elusive. It is background dependent: each formulation assumes a fixed spacetime, whereas in principle the universe's geometry should emerge dynamically. It is also fundamentally perturbative, making it difficult to capture truly non-perturbative phenomena, such as D-brane decay or transitions between distinct string vacua. Furthermore, the formalism is intrinsically on-shell, designed to compute S-matrix elements rather than off-shell quantities. This limits its connection to conventional quantum field theory (QFT) and complicates the study of infrared (IR) effects or low-energy effective actions.

To overcome these challenges, the 1980s saw the birth of String Field Theory (SFT). Early efforts by Siegel laid the foundation [6–8], culminating in Witten's covariant cubic action for open bosonic strings [9]. SFT represents a second-quantized description of strings, namely a QFT describing the infinite tower of string modes. It provides a genuine off-shell and potentially background-independent framework, in which classical solutions describe different string backgrounds and quantum fluctuations capture dynamics around them.

The 1990s brought a series of major breakthroughs. Zwiebach extended SFT to closed strings [10] and to the open-closed string scenario [11], while Ashoke Sen and collaborators investigated tachyon condensation in open SFT [12–14] and made crucial contributions to background independence conjecture [15–20].

In the mid-2000s, a decisive breakthrough was achieved by Martin Schnabl, who constructed the first analytic solution for the tachyon vacuum in open bosonic SFT [21]. This solution provided a rigorous and elegant description of the decay of unstable D-brane systems due to their tachyonic instabilities, offering analytic confirmation of Sen's conjectures on tachyon condensation. In the years that followed, further exact solutions were constructed [22], including those for marginal deforma-

tions triggered by boundary operators [23–26]. A major milestone was later reached with the work of Erler and Maccaferri, who developed a systematic procedure to construct analytic solutions describing arbitrary bosonic D-brane systems [27, 28], thereby proving the background independence conjecture for open SFT.

The work presented in this thesis builds on this long line of development. It contributes along four interconnected directions. First, it develops boundary terms in SFT actions, leading to gauge-invariant kinetic terms on target spaces with boundaries [29, 30]. Second, it reformulates open-closed SFT using the coalgebraic language, enabling a systematic treatment of D-brane backreaction [31, 32]. Third, it introduces a procedure for adding stubs in generic quantum SFTs [33]. Finally, it applies SFT techniques to study short RG flows triggered by slightly relevant bulk operators in CFTs with D-branes [34].

For readers interested in a broader perspective, several reviews provide detailed historical and technical accounts, starting from the classic foundational surveys [35, 36] to more recent comprehensive treatments of open, closed, and supersymmetric SFTs [37–43].

The present dissertation is divided into seven chapters, organized as follows: Chapter 2 reviews the fundamental aspects of bosonic SFT. We begin in Section 2.1 with the basic definitions and conventions of the worldsheet CFT, focusing on the ingredients needed to set the stage for constructing the SFT action. Section 2.2 then develops the kinetic term, following the standard construction aimed at obtaining an action that dynamically restricts the string field to the space of physical states. In this framework, the kernel of the BRST charge is interpreted as the space of solutions to the equations of motion, while cohomological equivalence classes are understood as gauge orbits.

This section also includes a review of key aspects of the Batalin–Vilkovisky (BV) quantization procedure, which plays a central role in the second quantization of ST. In particular, we provide a pedagogical proof of the classical and quantum consistency of the free SFT BV master action and discuss the gauge-fixing procedure within this framework.

Finally, in Section 2.3, we introduce interactions in SFT. Guided by the principle that on-shell amplitudes must reproduce the well-known string scattering amplitudes through Feynman diagram expansions, we describe their off-shell generalization and the crucial splitting of the moduli space into fundamental and Feynman regions, which underlies the construction of interaction vertices. We also emphasize the

geometrical perspective, deriving the BV geometrical master equation and exploring its connection with the conditions for quantum consistency established in Section 2.2.

Chapter 3, based on our works [29, 30], is devoted to the construction of a gauge-invariant kinetic term for SFT defined on target spaces with boundaries. We begin in Section 3.1 with an introductory discussion of the general consequences and challenges of formulating QFTs on manifolds with boundaries. Through a series of instructive examples — a free scalar theory, abelian Chern–Simons theory, and General Relativity — we identify the key issues related to the variational principle and gauge invariance, and extract useful hints for how to modify the standard SFT kinetic term.

Section 3.2 shows that, when SFT is formulated on a manifold with boundaries without imposing appropriate fall-off conditions, the variational principle becomes ill-defined and gauge invariance is lost. We demonstrate that an infinite family of boundary terms can in principle be added to restore a well-defined variational principle and avoid overconstraining the space of solutions. However, from the viewpoint of gauge invariance the situation is more subtle: the introduction of an additional degree of freedom localized on the boundary is required. This construction is carried out in Section 3.3, where we present the final expression for the gauge-invariant kinetic term.

The chapter concludes in Section 3.4 with explicit examples. We first analyze the open string sector at the massless and first massive level, and the massless closed string sector, where we show the matching (up to second order) between our action and the low-energy effective action for gravity coupled to the dilaton and the Kalb–Ramond field, including the Gibbons–Hawking–York term in half–Euclidean space with fluctuating boundary. As a final example, we extend the construction to SFT defined on the linear dilaton background, repeating the computation for the massless open and closed string sectors in this setting.

Chapter 4 is another review chapter, devoted to the mathematical and algebraic structures underlying SFT, which will be crucial for the remainder of this thesis. Section 4.1 begins with a brief review of multi-string products and the coalgebraic formalism, illustrated with the concrete examples of classical OSFT and quantum CSFT. The goal is to rewrite the corresponding SFT actions in the so-called Wess–Zumino–Witten form.

We then examine the conditions for classical and quantum consistency within the BV framework, pedagogically deriving the nilpotency conditions that enforce

the homotopy relations governing the multi-string products of these theories. As is well known in the literature, these structures turn out to be an A_∞-algebra and a quantum L_∞-algebra, respectively. We also devote a dedicated subsection to homotopy algebras, with the aim of clarifying their physical interpretation and geometrical meaning.

Finally, Section 4.2.1 highlights the main advantage of this algebraic approach by studying the construction of the Wilsonian effective action in SFT. In particular, we explore its relation to the homotopy transfer, a set of mathematical tools and theorems that systematically allow, at least formally, the construction of an effective action obtained by integrating out a subsector of the degrees of freedom in the path integral approach.

After these preliminaries, Chapter 5, based on our works [31, 32], addresses the fundamental problem of open–closed duality through SFT methods, focusing in particular on the description of D-brane backreaction in the large-N limit.

Section 5.1 introduces the problem and outlines our strategy. We then rewrite the open–closed SFT action in terms of open–closed multi-string vertices (Section 5.2), and in Section 5.3 elevate the formalism to the coalgebraic level, obtaining a WZW-like action that compactly encodes the dynamics of open and closed strings. From this formulation we can directly derive the nilpotency condition imposed by the BV quantum master equation, which guarantee the consistency of the theory (Section 5.4). At the end of this section we further decompose the nilpotent operator to read off the homotopy relations order by order in the topological expansion, thereby clarifying their geometrical interpretation.

In Section 5.5, we specialize the construction to a background with a stack of N identical branes. By making the N-dependence explicit and performing the 't Hooft limit, we obtain a quantum but planar open–closed SFT action. The description of closed-string backreaction then requires two steps, carried out in Section 5.6. First, we integrate out the open-string sector using the homotopy transfer techniques developed in Chapter 4, obtaining a classical closed SFT action with a tadpole term, whose leading contribution is given by the boundary state. This signals that the perturbative vacuum is no longer a solution of the classical equations of motion, reflecting the fact that the background has shifted. As a second step, we perform the corresponding vacuum shift, thus arriving at a classical CSFT defined on a new background without D-branes.

Chapter 6 is an adapted reprint of our work [33], where we developed a system-

atic procedure for adding stubs to a generic classical or quantum SFT. Section 6.1 recalls the notion of stub deformations, while Section 6.2 presents our construction. The idea is to first introduce a completely decoupled auxiliary string field, then perform a field redefinition given by a rotation depending on the stub parameter, and finally integrate out the stub-rotated auxiliary field using homotopy transfer. In this way we obtain the stubbed theory, which consistently reduces to the original one when the stub parameter is set to zero.

As a first illustration, we apply this procedure to a simple cubic toy model, which captures the essential features of the construction. We then extend the analysis to explicit solutions of the BV quantum master equation, including closed SFT and open–closed SFT. In Section 6.3, we provide a general proof that tree-level and loop amplitudes remain invariant under stub deformation. Finally, as a non-trivial application, we discuss in detail how the stub deformation modifies the decomposition of moduli space in the one-loop open-string tadpole of open–closed SFT.

Chapter 7, a reprint of [34], investigates bulk RG flows in two-dimensional (B)CFTs triggered by slightly relevant operators of weight $(1-y, 1-y)$, with y small and positive. Our goal is to compute the change in the g-function and in the string coupling constant up to second order in y.

After describing in Section 7.1 how the background independence of SFT provides a powerful framework for approaching Conformal Perturbation Theory (CPT) problems, section 7.2 reviews the on-shell disk action in open–closed SFT and explains how it can be used to compute the shifts in the g-function and in the string coupling constant. We then specialize to bulk marginal deformations, focusing on deformations of a Narain lattice of d free bosons. By evaluating the disk action for a solution describing a deformation of the Narain modulus in the presence of generic boundary states, we identify the change in the string coupling constant induced by the closed-string sector.

Section 7.3 applies this framework to short bulk RG flows, setting up a bulk – boundary perturbative scheme that associates an open-string vacuum-shift solution to a given closed SFT solution. We analyze the conditions under which the open-string vacuum shift remains perturbative and write the explicit closed- and open-string solutions needed to compute the disk action up to subleading order, corresponding to a two-loop calculation in CPT.

Section 7.4 presents the main technical results, computing the disk action for small RG flows in the limit of large closed-string stubs. As a non-trivial consistency

check, the disk action is independent of the specific SFT data used to define the interaction vertices. Finally, Section 7.5 tests the general result against the expected change in boundary entropy for Cardy boundary states in large-m minimal models undergoing Zamolodchikov short RG flows triggered by $\phi_{(1,3)}$. Matching the g-function part of the disk action, we also identify a second-order change in the string coupling constant, which parallels the result for the marginally deformed Narain lattice.

Finally, Chapter 8 concludes the thesis by summarizing the main results presented in the preceding chapters. We also discuss possible future directions and outline perspectives for further research.

CHAPTER 2

Bosonic String Field Theory

2.1 World-sheet CFT

In this section, we review the basic aspects of a world-sheet CFT, taking the opportunity to fix our conventions and to set the stage for the construction of SFT, which will be developed in the following sections.

We define a world-sheet CFT as a critical CFT given by the tensor product of the bc-ghost system with central charge $c_{\rm gh} = -26$, and a matter CFT with central charge $c_{\rm m} = 26$

$$CFT = CFT_{\rm m} \otimes CFT_{\rm gh}\,, \tag{2.1}$$

with

$$c_{\rm tot} = c_{\rm m} + c_{\rm gh} = 0\,, \tag{2.2}$$

where we are using the subscripts $\rm m$ and $\rm gh$ to denote the matter and ghost sectors, respectively.

In general, it is possible to consider a matter component with $c_{\rm m} \neq 26$ and this yields to the so-called non-critical string models. In this framework, an auxiliary CFT is introduced

$$CFT = CFT_{\rm m} \otimes CFT_{\rm aux} \otimes CFT_{\rm gh}\,, \tag{2.3}$$

with an appropriate central charge $c_{\rm aux}$ to restore criticality

$$c_{\rm tot} = c_{\rm m} + c_{\rm aux} + c_{\rm gh} = 0\,. \tag{2.4}$$

Examples of such auxiliary sectors include the Liouville model (see [44, 45] for review), which compensates for the matter central charge by construction, having $c_{\rm liuv} = 26 - c_{\rm m}$. Another possibility, first introduced in the context of SFT in [46, 47], and more recently reconsidered in [48], consists of adding a linear dilaton sector, namely a Coulomb gas CFT with an appropriate value of background charge β. Indeed, as we will explain in more detail in Section 3.4.4, this non-unitary CFT has central charge $c_\beta = 1 - 12\beta^2$, and therefore, we can adjust the background charge to compensate for the matter central charge. This last construction will be further analyzed in Chapter 7, in the context of non-critical Minimal ST [49].

Notice that all these constructions can be simply adapted to the open string sector by considering a D-brane system characterized by a boundary state $\|B\rangle\!\rangle$ encoding the boundary conditions, and then studying the corresponding world-sheet BCFT (see [50] for review).

2.1.1 (b, c)-ghost CFT

The ghost sector of the world-sheet CFT is the same for all backgrounds, and it is characterized by two Grassmann-odd holomorphic fields $b(z)$ and $c(z)$, and two Grassmann-odd anti-holomorphic fields $\bar{b}(\bar{z})$ and $\bar{c}(\bar{z})$. The dynamics of these objects is determined by the following world-sheet (WS) action

$$S_{\rm gh}(b, c, \bar{b}, \bar{c}) = \frac{1}{2\pi} \int_{\rm WS} d^2z \left(b\partial c + \bar{b}\bar{\partial}\bar{c} \right) , \tag{2.5}$$

where ∂ and $\bar{\partial}$ are the derivatives with respect to the world-sheet coordinates z and $\bar{z}$. In what follows, we will mostly concentrate on the holomorphic sector, as the analysis can be straightforwardly extended to the antiholomorphic one.

Following the usual CFT prescription, we expand the fields as

$$c(z) = \sum_{n\in\mathbb{Z}} c_n z^{-n+1} , \qquad c_n = \oint_0 \frac{dz}{2\pi i}\, z^{n-2} c(z) , \tag{2.6a}$$

$$b(z) = \sum_{n\in\mathbb{Z}} b_n z^{-n-2} , \qquad b_n = \oint_0 \frac{dz}{2\pi i}\, z^{n+1} b(z) , \tag{2.6b}$$

so that the n-th Laurent mode has weight $-n$.

Following the canonical quantization scheme, the modes are upgraded to operators satisfying the following algebra

$$[b_n, c_m] = \delta_{n+m,0} , \tag{2.7}$$

where, here and in the following, $[\cdot, \cdot]$ must be understood as the graded commutator

$$[A, B] = AB - (-)^{\epsilon(A)\epsilon(B)} BA , \tag{2.8}$$

where $\epsilon(\cdot)$ denotes the Grassmann parity. The remaining commutators not reported

in (2.7) vanish.

Furthermore, by using (2.6), (2.7) translates into the following OPE

$$b(z)c(w) = \frac{1}{z-w} + (\text{regular terms})\,. \tag{2.9}$$

After that, we can introduce the $\mathrm{SL}(2,\mathbb{C})$-invariant vacuum $|0\rangle$ such that it satisfies

$$c_n\,|0\rangle = 0\,, \qquad \langle 0|\,c_{-n} = 0\,, \qquad \forall n \geq 2\,. \tag{2.10a}$$

$$b_n\,|0\rangle = 0\,, \qquad \langle 0|\,b_{-n} = 0\,, \qquad \forall n \geq -1\,, \tag{2.10b}$$

with the reality conditions

$$c_n^\dagger = c_{-n}\,, \qquad\qquad b_n^\dagger = b_{-n}\,, \tag{2.11}$$

and normalized such that

$$\langle 0|\,c_{-1}\bar{c}_{-1}c_0\bar{c}_0c_1\bar{c}_1\,|0\rangle = -1\,. \tag{2.12}$$

As far as the open string scenario, we deal with the $\mathrm{SL}(2,\mathbb{R})$-invariant vacuum which can be, consistently, normalized as

$$\langle 0|\,c_{-1}c_0c_1\,|0\rangle = -1\,. \tag{2.13}$$

Notice that, due to (2.7) and (2.10), every non-vanishing overlap can be reduced to the expressions above.

We shall now define the ghost Hilbert space $\mathcal{H}_{\mathrm{gh}}$ which is spanned by states obtained by applying the creation operators, according to (2.10), on the $\mathrm{SL}(2,\mathbb{C})$ vacuum

$$\mathcal{H}_{\mathrm{gh}} \coloneqq \mathrm{Span}(b_{-n_1}\dots b_{-n_N}\bar{b}_{-\bar{n}_1}\dots\bar{b}_{-\bar{n}_{\bar{N}}}c_{-m_1}\dots c_{-m_M}\bar{c}_{-\bar{m}_1}\dots\bar{c}_{-\bar{m}_{\bar{M}}}\,|0\rangle)\,, \tag{2.14}$$

for all $N,\bar{N},M,\bar{M} \geq 0$, $n_i,\bar{n}_i \geq 2$ and $m_j,\bar{m}_j \geq -1$.

Let us now discus the ghost Virasoro algebra, the ghost energy-momentum reads

$$T_{\rm gh}(z) = -2 : b\partial c : (z) + : c\partial b : (z), \tag{2.15}$$

where $: \cdots :$ identifies the normal ordering on $|0\rangle$ according to (2.10), namely the annihilation operators are placed to the right and the creation operators to the left, taking into account the signs coming from the anti-commutators.

Starting from (2.15) and using the OPE (2.9), we can prove that $b(z)$ and $c(z)$ are primary holomorphic fields of weights $(-1, 0)$ and $(2, 0)$, respectively, and that $c_{\rm gh} = -26$. In formulae

$$T_{\rm gh}(z)b(w) = \frac{\partial b(w)}{z-w} + \frac{2b(w)}{(z-w)^2} + (\text{regular terms})\,, \tag{2.16a}$$

$$T_{\rm gh}(z)c(w) = \frac{\partial c(w)}{z-w} + \frac{-c(w)}{(z-w)^2} + (\text{regular terms})\,, \tag{2.16b}$$

$$T_{\rm gh}(z)T_{\rm gh}(w) = \frac{\partial T_{\rm gh}(w)}{z-w} + \frac{2T_{\rm gh}(w)}{(z-w)^2} + \frac{-26}{2(z-w)^4} + (\text{regular terms})\,. \tag{2.16c}$$

Moreover, we can expand (2.15) in Virasoro modes

$$T_{\rm gh}(z) = \sum_{n\in\mathbb{Z}} L_n^{(\rm gh)} z^{-n-2}\,, \qquad\qquad L_n^{(\rm gh)} = \oint_0 \frac{dz}{2\pi i}\, z^{n+1} T_{\rm gh}(z)\,, \tag{2.17}$$

which can be explicitly written as

$$L_n^{(\rm gh)} = \sum_{k\in\mathbb{Z}} (n-k) : b_{k+n} c_{-k} :\,, \tag{2.18}$$

and satisfy the following commutator rules

$$[L_n^{(\rm gh)}, b_m] = (n-m) b_{m+n}\,, \tag{2.19a}$$

$$[L_n^{(\rm gh)}, c_m] = -(2n+m) c_{m+n}\,, \tag{2.19b}$$

$$[L_n^{(\rm gh)}, L_m^{(\rm gh)}] = (n-m) L_{n+m}^{(\rm gh)} - \frac{26}{12} n(n^2-1)\delta_{n+m,0}\,, \tag{2.19c}$$

where in the last equation we can recognize the Virasoro algebra with $c_{\rm gh} = -26$. Furthermore, we can verify that: first, the reality condition on the ghost modes (2.11)

implies the Hermiticty of the Hamiltonian, namely

$$L_n^{(\mathrm{gh})\dagger} = L_{-n}^{(\mathrm{gh})}, \tag{2.20}$$

and second, $|0\rangle$ is indeed the $\mathrm{SL}(2,\mathbb{C})$-invariant vacuum as it is annihilated by the $\mathrm{SL}(2,\mathbb{C})$ generators L_{-1}, L_0 and L_1.

Finally, we introduce the ghost number operator

$$N_{\mathrm{gh}} \coloneqq \oint_0 \frac{dz}{2\pi i}\, j_{\mathrm{gh}}(z) + \oint_0 \frac{d\bar{z}}{2\pi i}\, \bar{j}_{\mathrm{gh}}(\bar{z})\,, \tag{2.21}$$

where the non-primary weight $(1,0)$ and $(0,1)$ ghost currents are defined as

$$j_{\mathrm{gh}}(z) = -:bc:(z)\,, \qquad \bar{j}_{\mathrm{gh}}(\bar{z}) = -:\bar{b}\bar{c}:(\bar{z})\,, \tag{2.22}$$

which assign 1 to the c-ghost oscillators and -1 to the b-ghost oscillators

$$\mathrm{gh}(c_n) \coloneqq [N_{\mathrm{gh}}, c_n] = 1\,, \qquad \mathrm{gh}(b_n) \coloneqq [N_{\mathrm{gh}}, b_n] = -1\,, \tag{2.23a}$$

$$\mathrm{gh}(\bar{c}_n) \coloneqq [N_{\mathrm{gh}}\,, \bar{c}_n] = 1\,, \qquad \mathrm{gh}(\bar{b}_n) \coloneqq [N_{\mathrm{gh}}, \bar{b}_n] = -1\,. \tag{2.23b}$$

2.1.2 The BRST charge and the string spectrum

Given a matter CFT carrying central charge $c_{\mathrm{m}} = 26$ with energy-momentum tensor $T_{\mathrm{m}}(z)$

$$T_{\mathrm{m}}(z) = \sum_{n\in\mathbb{Z}} L_n^{(\mathrm{m})} z^{-n-2}\,, \qquad L_n^{(\mathrm{m})} = \oint_0 \frac{dz}{2\pi i}\, z^{n+1} T_{\mathrm{m}}(z)\,, \tag{2.24}$$

satisfying the OPE

$$T_{\mathrm{m}}(z)T_{\mathrm{m}}(w) = \frac{\partial T_{\mathrm{m}}(w)}{z-w} + \frac{2T_{\mathrm{m}}(w)}{(z-w)^2} + \frac{c_{\mathrm{m}}}{2(z-w)^4} + (\text{regular terms})\,, \tag{2.25}$$

or equivalently the Virasoro algebra

$$[L_n^{(\mathrm{m})}, L_m^{(\mathrm{m})}] = (n-m)L_{n+m}^{(\mathrm{m})} + \frac{c_{\mathrm{m}}}{12} n(n^2-1)\delta_{n+m,0}\,, \tag{2.26}$$

we can introduce the BRST operator following the definition of the BRST quantization procedure with (2.26) as the constraint algebra

$$Q = \sum_{n\in\mathbb{Z}} c_{-n} L_n^{(\mathrm{m})} + \frac{1}{2} : c_{-n} L_n^{(\mathrm{gh})} : + \bar{c}_{-n} \bar{L}_n^{(\mathrm{m})} + \frac{1}{2} : \bar{c}_{-n} \bar{L}_n^{(\mathrm{gh})} : \,. \tag{2.27}$$

This operator can be expressed as a contour integral

$$Q = \oint_0 \frac{dz}{2\pi i}\, j_B(z) + \oint_0 \frac{d\bar{z}}{2\pi i}\, \bar{j}_B(\bar{z})\,, \tag{2.28}$$

by introducing the weight $(1,0)$ and $(0,1)$ primary BRST currents

$$j_B(z) = cT_{\mathrm{m}}(z) + : bc\partial c : (z) + \frac{3}{2}\partial^2 c(z)\,, \tag{2.29}$$

$$\bar{j}_B(\bar{z}) = \bar{c}\bar{T}_{\mathrm{m}}(\bar{z}) + : \bar{b}\bar{c}\bar{\partial}\bar{c} : (\bar{z}) + \frac{3}{2}\bar{\partial}^2 \bar{c}(\bar{z})\,. \tag{2.30}$$

characterized by the following self OPE

$$\begin{aligned} j_B(z) j_B(w) = {} & \frac{26 - c_{\mathrm{m}}}{12} \frac{c\partial^3 c(w)}{z-w} + \frac{18 - c_{\mathrm{m}}}{4} \frac{c\partial^2 c(w)}{(z-w)^2} + \frac{18 - c_{\mathrm{m}}}{2} \frac{c\partial c(w)}{(z-w)^3} \\ & + (\text{regular terms})\,. \end{aligned} \tag{2.31}$$

From the definitions above, and using the various OPE discussed so far, we can compute the commutators

$$[Q, b(z)] = T(z)\,, \tag{2.32a}$$

$$[Q, c(z)] = c\partial c(z)\,, \tag{2.32b}$$

$$[Q, Q] = \frac{c_{\mathrm{m}} - 26}{12} \sum_{n\in\mathbb{Z}} n(n^2-1)(c_{-n}c_n + \bar{c}_{-n}\bar{c}_n) = 0\,, \tag{2.32c}$$

$$[N_{\mathrm{gh}}, Q,] = Q\,, \tag{2.32d}$$

where $T(z)$ is the total energy-momentum tensor

$$T(z) \coloneqq T_{\mathrm{m}}(z) + T_{\mathrm{gh}}(z)\,. \tag{2.33}$$

Let us focus on the last two commutation relations. Equation (2.32c) shows that, as expected, BRST nilpotency is unobstructed (i.e., no anomalies arise in the quantization process) when the total central charge sums to zero, whereas equation (2.32d) confirms that Q increases the ghost number by one.

All these operators are understood to act on the total space

$$\mathcal{H} \coloneqq \mathcal{H}_{\mathrm{m}} \otimes \mathcal{H}_{\mathrm{gh}} \,, \tag{2.34}$$

where $\mathcal{H}_{\mathrm{m}}$ is the matter Hilbert space. In particular, we denote by $\mathcal{H}_{\mathrm{o}}$ the Hilbert space associated with the total BCFT for the open string sector, and by $\mathcal{H}'_{\mathrm{c}}$ the total Hilbert space for the closed string sector. The rationale behind the prime in the notation for the closed string sector is that the relevant space of states in this framework is not the full Hilbert space, but rather, as we will motivate below, a subspace satisfying the so-called level-matching conditions

$$\mathcal{H}_{\mathrm{c}} \coloneqq \{|s\rangle \in \mathcal{H}'_{\mathrm{c}} : b_0^- \, |s\rangle = L_0^- \, |s\rangle = 0\} \,, \tag{2.35}$$

where we defined the combinations of zero modes

$$b_0^{\pm} \coloneqq b_0 \pm \bar{b}_0 \,, \qquad L_0^{\pm} \coloneqq L_0 \pm \bar{L}_0 \,, \qquad c_0^{\pm} \coloneqq \frac{1}{2}(c_0 \pm \bar{c}_0) \,, \tag{2.36}$$

which obey the following commutation relations

$$[b_0^{\pm}, c_0^{\pm}] = 1 \,, \tag{2.37a}$$

$$[b_0^{\pm}, c_0^{\mp}] = 0 \,, \tag{2.37b}$$

$$[Q, b_0^{\pm}] = L_0^{\pm} \,, \tag{2.37c}$$

$$[Q, c_0^{\pm}] = -\frac{1}{2} \sum_{n \in \mathbb{Z}} n(c_{-n} c_n \pm \bar{c}_{-n} \bar{c}_n) \,, \tag{2.37d}$$

the first two follow directly from (2.7), while the last two are consequences of (2.32a) and (2.32b). Furthermore, using Jacobi identity and (2.37c) we obtain

$$[Q, L_0^{\pm}] = 0 \,. \tag{2.38}$$

Note that, from the viewpoint of conformal transformations, L_0^- generates rotations of the complex plane; therefore, level-matched states remain invariant under such phase transformations

$$e^{i\theta} \circ \Phi(0,\bar{0}) = \Phi(0,\bar{0})\,, \tag{2.39}$$

for $\Phi \in \mathcal{H}_{\rm c}$. It is then natural to introduce an equivalence relation among conformal maps f acting on level-matched states

$$f \sim f \circ e^{i\theta}\,, \tag{2.40}$$

this equivalence will be crucial in defining the off-shell amplitudes in the closed string sector, as we will describe in Section 2.3.1.

Moreover, due to the presence of the zero modes, a vector $\Phi \in \mathcal{H}'_{\rm c}$ can always be decomposed by applying the projectors

$$P_{\rm s}^- \coloneqq b_0^- c_0^-\,, \qquad \bar{P}_{\rm s}^- \coloneqq 1 - P_{\rm s}^- = c_0^- b_0^-\,, \tag{2.41a}$$

$$P_{\rm s}^+ \coloneqq b_0^+ c_0^+\,, \qquad \bar{P}_{\rm s}^+ \coloneqq 1 - P_{\rm s}^+ = c_0^+ b_0^+\,, \tag{2.41b}$$

obtaining

$$\Phi = \Phi_{\downarrow\downarrow} + c_0^+\tilde{\Phi}_{\uparrow\downarrow} + c_0^-\tilde{\Phi}_{\downarrow\uparrow} + c_0^- c_0^+\tilde{\Phi}_{\uparrow\uparrow}\,, \tag{2.42}$$

where $\Phi_{\downarrow\downarrow}, \tilde{\Phi}_{\uparrow\downarrow}, \tilde{\Phi}_{\downarrow\uparrow}, \tilde{\Phi}_{\uparrow\uparrow} \in \mathrm{Ker}(b_0^-) \cap \mathrm{Ker}(b_0^+)$. Therefore, the level matching condition set to zero $\tilde{\Phi}_{\downarrow\uparrow}$ and $\tilde{\Phi}_{\uparrow\uparrow}$ and restrict $\Phi_{\downarrow\downarrow}, \tilde{\Phi}_{\uparrow\downarrow} \in \mathrm{Ker}(L_0^-)$, ending up on

$$\Phi = \Phi_{\downarrow\downarrow} + c_0^+\tilde{\Phi}_{\uparrow\downarrow} \in \mathcal{H}_{\rm c}\,. \tag{2.43}$$

Similarly, in the open string sector we can use the holomorphic zero modes b_0 and c_0 to define

$$P_{\rm s} \coloneqq b_0 c_0\,, \qquad \bar{P}_{\rm s} \coloneqq 1 - P_{\rm s} = c_0 b_0\,, \tag{2.44}$$

and then decompose a generic state as

$$\Psi = \Psi_{\downarrow} + c_0 \tilde{\Psi}_{\downarrow} \in \mathcal{H}_{\mathrm{o}} \,, \tag{2.45}$$

where, $\Psi_{\downarrow}, \tilde{\Psi}_{\downarrow} \in \mathrm{Ker}(b_0)$.

The nilpotency of Q allows us to define the absolute BRST cohomology as

$$H_{\mathrm{abs}}(Q) \coloneqq \frac{\mathrm{Ker}(Q)}{\mathrm{Im(Q)}} \,, \tag{2.46}$$

which is the space of BRST-closed states $Q\,|s\rangle = 0$, modulo BRST-exact states of the form $|s\rangle = Q\,|\lambda\rangle$ for some $|\lambda\rangle \in \mathcal{H}$. In other words, two states $|s\rangle$ and $|s'\rangle$ are considered equivalent if they differ by a BRST-exact state: $|s\rangle \sim |s'\rangle$ whenever $|s\rangle - |s'\rangle = Q\,|\lambda\rangle$. The term "absolute" indicates that this cohomology is taken over the entire Hilbert space $\mathcal{H}$, without imposing additional constraints.

In general, one may restrict the BRST cohomology to a subspace, such as the level-matched space $\mathcal{H}_{\mathrm{c}}$ in the closed string sector. In this context, it is useful to define the relative cohomology with respect to b_0^- [1]

$$H_{\mathrm{rel}}(Q; b_0^-) \coloneqq \{\Psi \in \mathrm{Ker}(Q) \cap \mathrm{Ker}(b_0^-),\, \nexists \Lambda \in \mathrm{Ker}(b_0^-) : \Psi = Q\Lambda\} \,. \tag{2.47}$$

This definition differs from the absolute one in two key respects: first, all states must also be annihilated by b_0^-; second, the equivalence relation now requires $|s\rangle - |s'\rangle = Q\,|\lambda\rangle$ for some $|\lambda\rangle \in \mathcal{H}_{\mathrm{c}}$, so that states equivalent in the absolute cohomology may be distinct in the relative cohomology.

A concrete example of a state that is trivial in the absolute cohomology but non-trivial in the relative cohomology is provided by the ghost dilaton

$$D = (c_1 c_{-1} - \bar{c}_1 \bar{c}_{-1})\,|0\rangle = Q c_0^-\,|0\rangle \,, \tag{2.48}$$

[1] We formulated the level-matched cohomological problem by imposing only $b_0^- = 0$, since, in this context, the condition $L_0^- = 0$ is redundant. Indeed, (i) if we impose $b_0^- = 0$, then $\mathrm{Ker}(Q) \subset \mathrm{Ker}(L_0^-)$, as can be seen from (2.37c), and (ii) there are no Q-exact states annihilated by L_0^- whose trivializer is not annihilated by L_0^-, because L_0^- and Q commute. Therefore, $H_{\mathrm{rel}}(Q; b_0^-) = H_{\mathrm{rel}}(Q; b_0^-, L_0^-)$. Thus, it might seem unnecessary to impose $L_0^- = 0$ as a subsidiary condition, as it has no cohomological consequences. However, we require this condition because only under this restriction we can define well-posed interaction vertices, as we will describe in Section 2.3.1.

where the last equality follows from (2.37d) and the fact that Q annihilates the vacuum. Thus, D is BRST-exact and trivial in the cohomology of $\mathcal{H}'_{\rm c}$. However, since $c_0^- |0\rangle \notin \mathcal{H}_{\rm c}$ (indeed $b_0^- c_0^- |0\rangle \neq 0$), D defines a nontrivial class in the relative cohomology of $\mathcal{H}_{\rm c}$.

Absolute and relative BRST cohomology are fundamental to understanding the physical spectrum of ST. In the open bosonic string, physical states correspond to elements of the absolute cohomology at ghost number 1 in $\mathcal{H}_{\rm o}$. In the closed string, the physical spectrum is in one-to-one correspondence with the ghost number 2 sector of the relative cohomology in $\mathcal{H}_{\rm c}$.

In the following, we will refer to these as on-shell states, anticipating the SFT formulation in which elements of $\mathcal{H}_{\rm c}$ and $\mathcal{H}_{\rm o}$ are promoted to dynamical fields whose equations of motion restrict them to lie within the BRST kernel and the cohomological classes of equivalence are understood as different gauge orbits of the theory.

2.1.3 Inner products

In this section we will equip the Hilbert spaces $\mathcal{H}_{\rm o}$ and $\mathcal{H}_{\rm c}$ of appropriate inner products, which will be one of the key ingredients in the construction of the SFT action. Given a generic (B)CFT, we naturally define the so-called BPZ inner product in the corresponding Hilbert space. This inner product implements the notion of overlap between 'in' and 'out' states within the framework of radial quantization. Specifically, given two states $\Psi_1, \Psi_2 \in \mathcal{H}$ and using the state operator correspondence, we define

$$\langle \Psi_1 | \Psi_2 \rangle = \langle 0 | \, I \circ \Psi_1(0) \Psi_2(0) \, |0\rangle \ , \tag{2.49}$$

where $I(z)$ is the inversion map, defined differently in the open and closed string sectors

$$I(z) = \begin{cases} I_{\rm o}(z) = -\frac{1}{z} & \text{open string}\,, \\ I_{\rm c}(z) = +\frac{1}{z} & \text{closed string}\,. \end{cases} \tag{2.50}$$

In the open string case, $I_{\rm o}$ maps the upper half-plane (UHP) into itself, exchanging the interior and exterior of the unit half-disk. In the closed string case, $I_{\rm c}$ maps the complex plane into itself exchanging the interior of the unit disk with the exterior of the unit disk. In this case the choice of sign is conventional, closed string states are insensitive to phase transformations (2.39).

Consistently, we observe that (2.49) is graded symmetric

$$\langle \Psi_1 | \Psi_2 \rangle = (-)^{\epsilon(\Psi_1)\epsilon(\Psi_2)} \langle \Psi_2 | \Psi_1 \rangle \ , \tag{2.51}$$

because the vacuum is invariant under $\mathrm{SL}(2,\mathbb{C})$ (or $\mathrm{SL}(2,\mathbb{R})$) transformations, and the square of the inversion map is the identity.

We can now introduce the notion of BPZ dual. Given an operator O acting on $\mathcal{H}$ we define the BPZ dual O^* through the relation

$$\langle \Psi_1 | O \Psi_2 \rangle = (-)^{\epsilon(O)\epsilon(\Psi_1)} \langle O^* \Psi_1 | \Psi_2 \rangle \ . \tag{2.52}$$

An operator is said to be BPZ even or BPZ odd if it satisfies

$$O^* = \pm O \, , \tag{2.53}$$

respectively. From these definitions we have the following properties

$$O^{**} = O \, , \tag{2.54a}$$

$$(O_1 + O_2)^* = O_1^* + O_2^* \, , \tag{2.54b}$$

$$(O_1 O_2)^* = (-)^{\epsilon(O_1)\epsilon(O_2)} O_2^* O_1^* \, . \tag{2.54c}$$

To avoid confusion, we emphasize that BPZ conjugation is distinct from complex conjugation. For instance, for a complex number $a \in \mathbb{C}$, we have $a^* = a$ under BPZ conjugation.

The BPZ conjugate of the Laurent modes of a conformal field $\phi(z)$ with weight h

can be expressed as

$$\phi_n^* = \oint_0 \frac{dz}{2\pi i} \left(I \circ \phi(z)\right) z^{h+n-1} \,, \tag{2.55}$$

therefore, if $\phi(z)$ is primary we obtain

$$\phi_n^* = \begin{cases} (-)^{n+h}\phi_{-n} & \text{open string}\,, \\ (-)^{h}\phi_{-n} & \text{closed string}\,. \end{cases} \tag{2.56}$$

In particular, the BRST charge, being the zero mode of a weight-1 primary field (2.29), is BPZ odd

$$Q^* = -Q\,. \tag{2.57}$$

Grading and symplectic forms

Motivated by the fact that the cohomology is in one to one correspondence with the ghost number 1 sector of $\mathcal{H}_{\rm o}$ and the ghost number 2 sector of $\mathcal{H}_{\rm c}$, it is natural to introduce a new grading defined as

$$d(\Psi) = \begin{cases} \mathrm{gh}(\Psi) - 1 & \text{open string}\,, \\ \mathrm{gh}(\Psi) - 2 & \text{closed string}\,, \end{cases} \tag{2.58}$$

so as the cohomology can be always searched in the degree-0 sector. For brevity, we denote $(-)^{d(\Psi)}$ as $(-)^{\Psi}$. Operators O acting on the Hilbert space are assigned to have degrees equal to their ghost numbers, satisfying

$$d(O\Psi) = d(O) + d(\Psi)\,. \tag{2.59}$$

For instance, the BRST charge has degree 1, while the zero mode b_0 has degree -1. In $\mathcal{H}_{\rm o}$, we define the open string symplectic form

$$\omega_{\rm o}(\Psi_1, \Psi_2) \coloneqq (-)^{\Psi_1} \langle \Psi_1 | \Psi_2 \rangle \,, \tag{2.60}$$

which is graded antisymmetric

$$\omega_{\rm o}(\Psi_1,\Psi_2) = -(-)^{\Psi_1\Psi_2}\omega_{\rm o}(\Psi_2,\Psi_1)\,, \tag{2.61}$$

as a direct consequence of (2.51), and non-vanishing only when

$$d(\Psi_1)+d(\Psi_2)=1\,, \tag{2.62}$$

reflecting the requirement to saturate the total ghost number to 3 for a non-zero overlap. Furthermore, the BPZ conjugation property can be equivalently rewritten as

$$\omega_{\rm o}(\Psi_1,O\Psi_2) = (-)^{O\Psi_1}\omega_{\rm o}(O^*\Psi_1,\Psi_2)\,. \tag{2.63}$$

In this context, a BPZ-odd operator, such as Q, is said to be cyclic with respect to the symplectic form, since it satisfies

$$\omega_{\rm o}(\Psi_1,Q\Psi_2) = (-)^{\Psi_1\Psi_2}\omega_{\rm o}(\Psi_2,Q\Psi_1)\,. \tag{2.64}$$

In the closed string sector we define the symplectic form as

$$\omega_{\rm c}(\Phi_1,\Phi_2) \coloneqq (-)^{\Phi_1}\,\langle\Phi_1|c_0^-|\Phi_2\rangle\,, \tag{2.65}$$

where the insertion of c_0^- is needed to have a non-vanishing overlap through the BPZ product of two level matched states as $(P_{\rm s}^-)^* = \bar{P}_{\rm s}^-$. Furthermore, (2.65) is degenerate in $\mathcal{H}_{\rm c}'$ but not in $\mathcal{H}_{\rm c}$ as can be easily seen by using the decomposition (2.42). Indeed, the degeneracy in $\mathcal{H}_{\rm c}'$ relies in the fact that $\omega_{\rm c}$ is non-sensitive to the components $\tilde{\Psi}_{\downarrow\uparrow}$ and $\tilde{\Psi}_{\uparrow\uparrow}$, which are those fixed to zero by the level matching condition. Moreover, also in this case the symplectic form is, consistently, graded antisymmetric

$$\omega_{\rm c}(\Phi_1,\Phi_2) = -(-)^{\Phi_1\Phi_2}\omega_{\rm c}(\Phi_2,\Phi_1)\,, \tag{2.66}$$

thanks to the fact that $(c_0^-)^* = -c_0^-$.

According the grading definition in $\mathcal{H}_{\mathrm{c}}$, the ghost number saturation rule in the closed string sector implies that (2.65) is non-vanishing only if $d(\Phi_1) + d(\Phi_2) = 1$, as for ω_{o}.

Additionally, the presence of c_0^- in (2.65) implies that the notion of BPZ conjugation differs from the dual defined through the symplectic form ω_{c}. This contrasts with the open string sector, where both notions coincide, as shown in equation (2.63). We have already encountered the fundamental example P_{s}^- whose BPZ dual is its complementary $\bar{P}_{\mathrm{s}}^-$, whereas within the context of the symplectic form it is self-dual, meaning that it can be freely moved between the left and right entries of the symplectic pairing. In general we have that

$$\omega_{\mathrm{c}}(\Phi_1, O\Psi_2) = (-)^{O\Phi_1}\omega_{\mathrm{c}}(O^*\Phi_1, \Phi_2) + \omega_{\mathrm{c}}(\Phi_1, [b_0^-, [c_0^-, O]]\Phi_2)\,. \tag{2.67}$$

Therefore, the BPZ conjugate and the dual with respect to ω_{c} of an operator acting on $\mathcal{H}_{\mathrm{c}}$ coincide if the second term of the above equation vanishes. This means that, ultimately, the operators for which the two notions of conjugation differ are those containing the projector P_{s}^-. For instance, $[b_0^-, [c_0^-, Q]] = 0$, and thus, even in the closed string scenario, the BRST charge is cyclic with respect to the corresponding symplectic form

$$\omega_{\mathrm{c}}(\Phi_1, Q\Phi_2) = (-)^{\Phi_1\Phi_2}\omega_{\mathrm{c}}(\Phi_2, Q\Phi_1)\,. \tag{2.68}$$

We will see in the following that this property is crucial for constructing a gauge-invariant kinetic term for SFT.

Basis Vectors and Poisson bi-vector

Additionally, it is useful to write the symplectic form in components using the notation

$$(\omega_{\mathrm{o}})_{ab} \coloneqq \omega_{\mathrm{o}}(o_a, o_b)\,, \qquad (\omega_{\mathrm{o}})^{ab} \coloneqq \omega_{\mathrm{c}}(o^a, o^b)\,, \tag{2.69}$$

$$(\omega_{\mathrm{c}})_{ab} \coloneqq \omega_{\mathrm{c}}(c_a, c_b)\,, \qquad (\omega_{\mathrm{c}})^{ab} \coloneqq \omega_{\mathrm{c}}(c^a, c^b)\,, \tag{2.70}$$

where o_a and c_a are the basis vectors of $\mathcal{H}_{\rm o}$ and $\mathcal{H}_{\rm c}$ which are dual to o^a and c^a in the sense that

$$\omega_{\rm c}(c^a, c_b) = -\omega_{\rm c}(c_b, c^a) = \delta^a{}_b\,, \tag{2.71}$$

$$\omega_{\rm o}(o^a, o_b) = -\omega_{\rm c}(o_b, o^a) = \delta^a{}_b\,. \tag{2.72}$$

Therefore, the basis vectors and their duals have degrees such that they sum up to 1. Notice also the completeness relations

$$\mathbf{1}_{\mathcal{H}_{\rm c}} = c_a\omega_{\rm c}\left(c^a, \cdot\right) = -c^a\omega_{\rm c}\left(c_a, \cdot\right)\,, \tag{2.73}$$

$$\mathbf{1}_{\mathcal{H}_{\rm o}} = o_a\omega_{\rm o}\left(o^a, \cdot\right) = -o^a\omega_{\rm o}\left(o_a, \cdot\right)\,. \tag{2.74}$$

We can now define the closed and open Poisson bi-vectors as

$$U_{\rm c} \coloneqq \frac{(-1)^{c^a}}{2} c_a \wedge c^a = \frac{(-1)^{c^a}}{2}\left(c_a \otimes c^a + c^a \otimes c_a\right) \in \mathcal{H}_{\rm c}^{\wedge 2} \subset \mathcal{H}_{\rm c}^{\otimes 2}\,, \tag{2.75}$$

$$U_{\rm o} \coloneqq \frac{(-1)^{o^a}}{2} o_a \wedge o^a = \frac{(-1)^{o^a}}{2}\left(o_a \otimes o^a + o^a \otimes o_a\right) \in \mathcal{H}_{\rm o}^{\wedge 2} \subset \mathcal{H}_{\rm o}^{\otimes 2}\,, \tag{2.76}$$

where $\wedge$ is the graded symmetrized tensor product. They are the inverse of the corresponding symplectic forms in the sense that

$$(\omega_{\rm c} \otimes \mathbf{1})(1 \otimes U_{\rm c}) = \mathbf{1}_{\mathcal{H}_{\rm c}}\,, \tag{2.77}$$

$$(\omega_{\rm o} \otimes \mathbf{1})(1 \otimes U_{\rm o}) = \mathbf{1}_{\mathcal{H}_{\rm o}}\,, \tag{2.78}$$

which, after applying to a (closed or open) state (of even degree, for simplicity) read

$$(\omega_{\rm c} \otimes \mathbf{1})(\Phi \otimes U_{\rm c}) = \Phi\,, \tag{2.79}$$

$$(\omega_{\rm o} \otimes \mathbf{1})(\Psi \otimes U_{\rm o}) = \Psi\,, \tag{2.80}$$

as can be easily checked using the completeness relations (2.73) and (2.74) .

2.1.4 Free boson CFT

Until now, we have left complete freedom in the choice of the matter CFT, and for most of this thesis, we will not need to specify it. However, when discussing concrete examples, we will mostly consider the direct sum of d free bosons $X^\mu(z,\bar{z})$ for $\mu = 0, ..., d-1$, which, as usual, are interpreted as the world-sheet embedding maps describing the d flat, non-compact directions of the target space. Throughout the thesis, we will always have in mind the euclidean signature $\eta_{\mu\nu} = \mathrm{diag}(+,\dots,+)$. The WS action on the complex plane reads

$$S_{\mathrm{m}}(X^\mu, \bar{X}^\mu) = \frac{1}{2\pi\alpha'}\int_{\mathrm{WS}} d^2z\, \eta_{\mu\nu}\partial X^\mu(z)\partial\bar{X}^\nu(\bar{z})\,, \tag{2.81}$$

where, as usual, we split the scalars $X^\mu(z,\bar{z})$ as the sum of their holomorphic and anti-holomorphic part, $X^\mu(z)$ and $\bar{X}^\mu(\bar{z})$, which can be expanded in modes as

$$X^\mu(z) = +c^\mu + \frac{1}{2}x^\mu - i\frac{\alpha'}{2}p^\mu\log(z) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu z^{-n}\,, \tag{2.82a}$$

$$\bar{X}^\mu(\bar{z}) = -c^\mu + \frac{1}{2}x^\mu - i\frac{\alpha'}{2}p^\mu\log(\bar{z}) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\bar{\alpha}_n^\mu \bar{z}^{-n}\,, \tag{2.82b}$$

where c^μ is a constant that cancels out in the total embedding map, x^μ is the position zero mode, and p^μ is the momentum. Following the canonical quantization procedure the above Laurent modes are upgraded to operators obeying the algebra

$$[\hat{x}^\mu, \hat{p}^\nu] = i\eta^{\mu\nu}\,, \tag{2.83a}$$

$$[\alpha_n^\mu, \alpha_m^\nu] = n\eta^{\mu\nu}\delta_{n+m,0}\,, \tag{2.83b}$$

$$[\bar{\alpha}_n^\mu, \bar{\alpha}_m^\nu] = n\eta^{\mu\nu}\delta_{n+m,0}\,. \tag{2.83c}$$

It is useful to define the weight 1 current

$$j^\mu(z) = i\sqrt{\frac{2}{\alpha'}}\partial X^\mu(z)\,, \tag{2.84}$$

which in modes can be written as

$$j^\mu(z) = \sum_{n\in\mathbb{Z}} \alpha_n^\mu z^{-n-1}\,, \qquad\qquad \alpha_n^\mu = \oint_0 \frac{dz}{2\pi i} j^\mu(z) z^n\,, \tag{2.85}$$

where we defined

$$\alpha_0^\mu = \begin{cases} \sqrt{\frac{\alpha'}{2}}\hat{p}^\mu & \text{closed string}\,, \\ \sqrt{2\alpha'}\hat{p}^\mu & \text{open string}\,, \end{cases} \tag{2.86}$$

similarly, for the anti-holomorphic sector.

The commutation relations (2.83) translate into the following short-distance behavior for the matter currents

$$j^\mu(z) j^\nu(w) = \frac{1}{(z-w)^2} + (\text{regular terms})\,. \tag{2.87}$$

Exactly as for the ghost sector, we introduce the matter component of the $\mathrm{SL}(2,\mathbb{C})$-invariant vacuum $|0\rangle$ (or $\mathrm{SL}(2,\mathbb{R})$-invariant vacuum for the open string scenario) such that

$$\alpha_n^\mu |0\rangle = 0\,, \qquad \langle 0| \alpha_{-n}^\mu = 0\,, \qquad \forall n \geq 1\,, \tag{2.88}$$

with the reality condition

$$(\alpha_n^\mu)^\dagger = \alpha_{-n}^\mu\,. \tag{2.89}$$

We can then define a generic momentum eigenstate as

$$|p\rangle = e^{ip\cdot\hat{x}} |0\rangle\,, \tag{2.90}$$

whose BPZ dual is

$$\langle p| = \langle 0| e^{ip\hat{x}}\,, \tag{2.91}$$

using the fact that the position zero mode is BPZ even, as can be explicitly seen by applying the inversion map (2.50) to (2.82a). Therefore, the matter Hilbert space

$\mathcal{H}_{\rm m}$ is naturally defined as the space spanned by vectors constructed by applying matter creation oscillator modes, according to (2.88), on the vacuum at generic momentum.

$$\mathcal{H}_{\rm m} = \mathrm{Span}(\alpha^{\mu_1}_{-n_1} \dots \alpha^{\mu_N}_{-n_N} \bar{\alpha}^{\nu_1}_{-m_1} \dots \bar{\alpha}^{\nu_M}_{-m_M} |p\rangle)\,, \tag{2.92}$$

for all $N, M \geq 0$, $n_i, m_j > 0$ and $p \in \mathbb{R}$. Moreover, we normalize the vacuum such that

$$\langle p_1 | p_2 \rangle = (2\pi)^d \delta^{(d)}(p_2 + p_1)\,, \tag{2.93}$$

which is consistent to the fact that $\hat{p}$ is a BPZ odd operator since it is proportional to the zero mode of a weight 1 primary field. In the open string sector, when considering a Dp-brane, the space-time momentum p^μ lies along the brane. Consequently, the d-dimensional Dirac delta function in (2.93) is replaced by $\delta^{(p+1)}(p_2 + p_1)$. Therefore, in the full world-sheet (B)CFT the basic overlaps are

$$\langle p_1 |\, c_{-1}\bar{c}_{-1} c_0^- c_0^+ c_1 \bar{c}_1\, | p_2 \rangle = -\frac{1}{2}(2\pi)^d \delta^{(d)}(p_1 + p_2)\,, \tag{2.94}$$

for the closed string sector and

$$\langle p_1 |\, c_{-1} c_0 c_1\, | p_2 \rangle = -(2\pi)^{p+1} \delta^{(p+1)}(p_1 + p_2)\,, \tag{2.95}$$

for the open string defined on a D-branes system with $p+1$ Newman directions.

Consider now the energy-momentum tensor, which follows directly from the action (2.81) and takes the form

$$T_m(z) = \frac{1}{2}\eta_{\mu\nu} : j^\mu j^\nu : (z)\,, \tag{2.96}$$

where $: \cdots :$ denotes normal ordering in the matter sector, defined with respect to (2.88). Using this expression and the short distance behaviors (2.87) we obtain

$$T_{\rm m}(z) j^\mu(w) = \frac{\partial j^\mu(w)}{(z-w)} + \frac{j^\mu(w)}{(z-w)^2} + (\text{regular terms})\,, \tag{2.97a}$$

which tell us that $j^\mu(z)$ is a weight $(1,0)$ primary field.

Furthermore, $T_{\rm m}(z)$ obeys the following self-OPE

$$T_{\rm m}(z)T_{\rm m}(w) = \frac{\partial T_{\rm m}(w)}{z-w} + \frac{2T_{\rm m}(w)}{(z-w)^2} + \frac{d}{2(z-w)^4} + (\text{regular terms})\,. \tag{2.98}$$

Equivalently, using (2.24), its Laurent modes,

$$L_n^{(\rm m)} = \frac{1}{2}\eta_{\mu\nu}\sum_{k\in\mathbb{Z}} :\alpha_{n-k}^\mu \alpha_k^\nu :\,, \tag{2.99}$$

generate the Virasoro algebra with central charge $c_{\rm m} = d$. Therefore, as it is well known, if we aim to construct our worldsheet CFT without additional auxiliary subsectors, we must require $d = 26$ to achieve criticality. Furthermore, by focusing on the matter Virasoro zero modes, we observe that they can be expressed as

$$L_n^{(\rm m)} = \begin{cases} \frac{\alpha'}{2}p^2 + N_{\rm m} & \text{closed string}\,, \\ 2\alpha' p^2 + N_{\rm m} & \text{open string}\,, \end{cases} \tag{2.100}$$

where we defined the matter level operator

$$N_{\rm m} \coloneqq \sum_{k>0} \eta_{\mu\nu}\alpha_{-k}^\mu \alpha_k^\nu\,, \tag{2.101}$$

which counts the number of matter oscillators weighted by minus the mode number

$$[N_{\rm m}, \alpha_{-n}^\mu] = n\alpha_{-n}^\mu\,. \tag{2.102}$$

In general, we refer to the concept of level using the total level operator, which considers both matter and ghost CFTs

$$N \coloneqq \begin{cases} L_0 - \frac{\alpha'}{2}p^2 & \text{closed string}\,, \\ L_0 - 2\alpha' p^2 & \text{open string}\,. \end{cases} \tag{2.103}$$

This operator yields the zero-momentum conformal weight of a state. In the closed string sector, by imposing the weight-zero condition on physical states arising from

the BRST quantization scheme, we obtain the following conditions in terms of the zero-mode combinations

$$L_0^- = 0 \longrightarrow N = \bar{N}\,, \tag{2.104a}$$

$$L_0^+ = 0 \longrightarrow \alpha' m^2 = 4N\,, \tag{2.104b}$$

where the first equation represents level matching, and the second is the mass-shell condition. For instance, in this convention, the tachyon arises at level -1, the massless fields at level 0, and so on. Notably, as a consequence of (2.38), the BRST charge preserves the level

$$[Q, N] = 0\,. \tag{2.105}$$

2.2 Free String Field Theory

In this section, we describe the construction of the SFT kinetic terms in both the open- and closed-string sectors. We begin with the classical formulation and then prepare it for quantization using the BV formalism by constructing the BV master action. Subsequently, we focus on the gauge-fixing procedure: first, we introduce the commonly used Siegel gauge and its corresponding propagator, and then generalize this to a more abstract approach based on the introduction of a contracting homotopy operator.

2.2.1 Classical kinetic term

In SFT, the fundamental dynamical variable is the classical open- or closed-string field Ψ_0, defined as a generic degree-0 element of $\mathcal{H}_{\mathrm{o}}$ or $\mathcal{H}_{\mathrm{c}}$

$$\Psi_0 = \psi_0^\alpha e_\alpha^{(0)}\,, \tag{2.106}$$

where $e_\alpha^{(0)}$ are the $gh = 1$ and $gh = 2$ basis vectors of $\mathcal{H}_{\mathrm{o}}$ and $\mathcal{H}_{\mathrm{c}}$, respectively. The coefficients ψ_0^α represent Grassmann-even classical target-space fields, and the index α runs over both discrete and continuous labels of the basis. The qualifier “classical”

for the string field will be clarified when we will discuss the BV quantization scheme.

For instance, in bosonic closed ST, the classical string field takes the form

$$\Psi_0 = \int \frac{d^d k}{(2\pi)^d} \Big(\tilde{t}(k) c_1 \bar{c}_1 + \Big[-\frac{1}{2} \tilde{e}_{\mu\nu}(k) \alpha^\mu_{-1} \bar{\alpha}^\nu_{-1} c_1 \bar{c}_1 + \tilde{e}(k) c_1 c_{-1} + \tilde{\tilde{e}}(k) \bar{c}_1 \bar{c}_{-1} + \\ + i \sqrt{\frac{\alpha'}{2}} c_0^+ \left(\tilde{e}_\mu(k) c_1 \alpha^\mu_{-1} + \tilde{\tilde{e}}_\mu(k) \bar{c}_1 \bar{\alpha}^\mu_{-1} \right) \Big] + \dots \Big) |k\rangle \,, \tag{2.107}$$

whereas, in the open string sector on a space-filling D-brane, the string field reads

$$\Psi_0 = \int \frac{d^d k}{(2\pi)^d} \Big(\tilde{T}(k) c_1 + \Big[\tilde{A}_\mu(k)\, c_1 \alpha^\mu_{-1} - i \sqrt{\frac{\alpha'}{2}} \tilde{B}(k)\, c_0 \Big] + \dots \Big) |k\rangle \,, \tag{2.108}$$

with the states organized by increasing level. Tildes denote Fourier transforms

$$T(x) = \int \frac{d^d k}{(2\pi)^d} \tilde{T}(k) e^{ikx} \,. \tag{2.109}$$

The target-space fields arising from the level -1 component are the tachyons $t(x)$ and $T(x)$. At level 0 in the open-string sector, we obtain the Maxwell gauge field $A_\mu(x)$ and the Nakanishi–Lautrup field $B(x)$. We postpone the discussion of the interpretation of closed-string target-space fields to Chapter 3.

We can then define the SFT classical free action

$$S_{\text{kin}}(\Psi_0) = \frac{1}{2} \omega(\Psi_0, Q\Psi_0) \,, \tag{2.110}$$

where ω is understood as ω_{o} or ω_{c}, depending on the case. Notice that in this expression, the ghost number is correctly saturated. In the closed-string case, the insertion of c_0^- is crucial. As anticipated, the action is constructed so that on-shell states lie in the BRST kernel, and cohomologically equivalent states are identified as gauge orbits. Indeed, by varying the action and using the facts that the symplectic form ω is non-degenerate and Q is cyclic, we obtain

$$Q\Psi_0 = 0 \,, \tag{2.111}$$

as the equation of motion, and the action is invariant under the gauge transformation

$$\delta_g \Psi_0 = Q\Lambda_{-1}, \tag{2.112}$$

thanks to the nilpotency of Q. In (2.112) Λ_{-1} is the classical string field gauge parameter, defined as the most generic ghost number 1 and 0 state in $\mathcal{H}_{\rm c}$ and $\mathcal{H}_{\rm o}$, respectively. For instance, in the examples of bosonic closed- and open- ST, it can be written as

$$\Lambda_{-1} = \int \frac{d^d k}{(2\pi)^d} \Big(\Big[\frac{i}{\sqrt{2\alpha'}} (\tilde{\xi}_\mu(k) c_1 \alpha^\mu_{-1} - \tilde{\bar{\xi}}_\mu(k) \bar{c}_1 \bar{\alpha}^\mu_{-1}) - \tilde{\mu}(k) c_0^+ \Big] + \dots \Big) |k\rangle \ , \tag{2.113}$$

and

$$\Lambda_{-1} = \int \frac{d^d k}{(2\pi)^d} \Big(\frac{i}{2\alpha'} \tilde{\lambda}(k) + \dots \Big) |k\rangle \ . \tag{2.114}$$

We can immediately notice that, as expected, the tachyons are gauge invariant, since it is not possible to build level -1 closed string states at $\mathrm{gh} = 1$ and level -1 open string states at $\mathrm{gh} = 0$.

In particular, by replacing the level -1 open string field component into the action, using equation (2.56) to find the BPZ conjugate, and truncating the BRST charge (2.27) to the non-vanishing terms in this sector, we end up with

$$\begin{aligned} S_{\rm kin}(T) &= \frac{1}{2} \int \frac{d^d k_1}{(2\pi)^d} \frac{d^d k_2}{(2\pi)^d} \tilde{T}(k_1) \tilde{T}(k_2) \, \langle k_1 | c_{-1} c_0 (\alpha' \hat{p}^2 - 1) c_1 | k_2 \rangle \\ &= \frac{\alpha'}{2} \int d^d x \, T(x) \Big(\partial^2 + \frac{1}{\alpha'} \Big) T(x) \, , \end{aligned} \tag{2.115}$$

where we used the delta function from the basic overlap (2.95) in its integral representation to perform the inverse Fourier transform, yielding the action in position space. As expected, this describes a free scalar field with a negative mass squared, $m^2 = -1/\alpha'$, corresponding to the open-string tachyon.

We will provide the explicit expression of the free action for the classical string fields at higher levels in Chapter 3, where we will consider the presence of boundary term. Until now, we have assumed that the target space has no boundaries and that

the fields satisfy suitable fall-off conditions at infinity. However, it is also interesting to consider situations in which the target space does have boundaries, as we will discuss in detail in Chapter 3.

Furthermore, notice that the possibility of computing the kinetic term level by level relies on two key facts: Q does not change the level (2.105) and the symplectic structure couples only states at the same level.

Let us now examine the gauge transformation (2.112). Because the BRST charge is nilpotent, not every string-field gauge parameter is independent: if two parameters differ by a Q-exact state, they generate the same gauge transformation. Consequently, some gauge transformations are themselves redundant, leading to a hierarchy where each level of gauge parameter has its own gauge invariance, called gauge-for-gauge transformation

$$\delta_g \Lambda_{-n} = Q\Lambda_{-n-1}\,, \tag{2.116}$$

with string field gauge parameters at lower ghost numbers, $n > 0$. Since the ghost number is not bounded from below, this process continues ad infinitum. Such theories are called infinitely[2] reducible gauge theories and necessitate an infinite tower of ghosts-for-ghosts in the quantization procedure. This is one of the reasons why we adopt the BV formalism for quantization, as it provides a general framework for quantizing gauge theories with complex gauge symmetry structures, including those that exhibit infinite reducibility.

2.2.2 The BV master action for SFT

As explained in the previous section, it is natural to apply the BV quantization scheme in the context of SFT. The BV formalism was introduced by Batalin and Vilkovisky in the early 1980s in [51, 52]. For comprehensive introductions and reviews, see [53,

[2]Examples of finitely reducible gauge theories include Abelian p-form theories, characterized by the action $S = -\frac{1}{2}\int F \wedge *F$, where $*$ denotes the Hodge dual and F is the field strength obtained by applying the exterior derivative d to the p-form A. It is evident that the theory is invariant under the gauge transformation $\delta A = d\lambda_{p-1}$, where λ_{p-1} is a $(p-1)$-form. This gauge transformation itself possesses a gauge symmetry: $\delta\lambda_{p-1} = d\lambda_{p-2}$, and so on, continuing until $\delta\lambda_1 = d\lambda_0$. Unlike ghost number, which is unbounded from below, the form degree cannot decrease below zero. Consequently, this hierarchy terminates at the $(p-1)$-th stage

54], whereas for applications to SFT, see [55, 56]. Here, we briefly review the main points of the BV formalism, aiming to construct the BV master action for the free SFT and demonstrate its quantum consistency.

Exactly as in the Faddeev-Popov approach, the first step consists of introducing ghost fields: starting with the ghost associated with the gauge transformation, which has opposite Grassmann parity relative to the gauge parameter, and then introducing further ghost fields for each gauge-for-gauge transformation, with alternating Grassmann parity. In our case, we introduce an infinite tower of target space ghost fields ψ_n^α for $n < 0$, with $\epsilon(\psi_n^\alpha) = |n| \pmod 2$. Collecting these fields together with the classical fields ψ_0^α, we obtain what is generically called the set of fields.

The second step consists of introducing a target space antifield for each field, denoted as $\psi_n^{*\alpha}$, with opposite Grassmann parity, $\epsilon(\psi_n^*) = \epsilon(\psi_n)+1$, which we rename as $\psi_n^{*\alpha} = \psi_{-n+1}^\alpha$ for $n \leq 0$, so that we can collect both the fields and the antifields in a unique set $\{\psi_n^\alpha\}_{n\in\mathbb{Z}}$, with alternating Grassmann parity $\epsilon(\psi_n^\alpha) = |n| \pmod 2$. In the context of SFT, it is natural to see the target-space fields as coefficients of vectors in the open or closed string Hilbert spaces. In particular we organize them as follows

$$\Psi = \sum_{n\in\mathbb{Z}} \psi_n^\alpha e_\alpha^{(n)} = \underbrace{\underbrace{\cdots + \psi_{-1}^\alpha e_\alpha^{(-1)}}_{\text{ghost fields}} + \underbrace{\psi_0^\alpha e_\alpha^{(0)}}_{\text{classical fields}}}_{\text{fields}} + \underbrace{\psi_1^\alpha e_\alpha^{(1)} + \dots}_{\text{antifields}}, \tag{2.117}$$

where $e_\alpha^{(n)}$ are the $\mathrm{gh} = n+1$ and $\mathrm{gh} = n+2$ basis vectors of $\mathcal{H}_\mathrm{o}$ and $\mathcal{H}_\mathrm{c}$, respectively. This decomposition of the string field in fields and antifields is due to Thorn in [57]. The rationale behind this choice is that, by doing so, fields and antifields are paired through the preexisting symplectic structure defined in Section 2.1.3.

Clearly, there is an infinite number of possible choices of collecting fields and antifields in the string field respecting the symplectic paring. For instance, looking at the decompositions (2.43) and (2.45) we can collect the antifields to be the coefficients of the states annihilated by c_0 or c_0^+ and the fields as the coefficients of the states annihilated by b_0 or b_0^+. This observation will be crucial in the gauge fixing procedure, indeed our strategy will be to collect all the antifields as coefficients of states which do not satisfy the gauge fixing equations and then setting them to zero.

The decomposition just described, based on (2.43) and (2.45), yields to the Siegel gauge as we will explain better in the next section.

Therefore, in full generality we can define the total BV closed string field as a generic Grassmann-even element of the G-module[3] $\mathcal{H}_{\rm c}^{\rm G}$ and the total BV open string field as a generic Grassmann-odd element of the G-module $\mathcal{H}_{\rm o}^{\rm G}$. These two spaces are spanned by the same basis vectors as $\mathcal{H}_{\rm c}$ and $\mathcal{H}_{\rm o}$, respectively, but admit coefficients with both Grassmann parities (see [58] for more details). In other words, we define the BV open and closed string fields as

$$\Psi = \psi^a o_a \,, \qquad\qquad \Phi = \phi^a c_a \,, \tag{2.118}$$

introducing a target space grading defined as

$$d(\psi^a) = -d(o_a) \,, \qquad\qquad d(\phi^a) = -d(c_a) \,, \tag{2.119}$$

and related to the Grassmann parity through $\epsilon(\psi^a) = d(\psi^a) \pmod 2$. Notice that in these conventions the BV string field has always degree 0. In the following, for the sake of simplicity, we will refer to the G-modules $\mathcal{H}_{\rm c}^{\rm G}$ and $\mathcal{H}_{\rm o}^{\rm G}$ simply as $\mathcal{H}_{\rm c}$ and $\mathcal{H}_{\rm o}$.

The third step in the BV quantization scheme is to construct the so-called master action, which depends on both fields and antifields and reduces to the original action when the ghost fields and antifields are set to zero. This action is referred to as the classical master action if it satisfies the classical BV master equation

$$(S, S) = 0 \,, \tag{2.120}$$

where $(\cdot,\cdot)$ denotes the BV bracket, defined as follows

$$(\cdot,\cdot)_c = \frac{\overleftarrow{\partial}}{\partial\phi^a}(\omega_{\rm c})^{ab}\frac{\overrightarrow{\partial}}{\partial\phi^b} \,, \tag{2.121a}$$

$$(\cdot,\cdot)_o = \frac{\overleftarrow{\partial}}{\partial\psi^a}(\omega_{\rm o})^{ab}\frac{\overrightarrow{\partial}}{\partial\psi^b} \,, \tag{2.121b}$$

where the arrows above the derivatives indicate the direction in which they act. Two

[3] G stands for Grassmann

derivatives with opposite arrows are related through the following expression

$$\frac{\overrightarrow{\partial}}{\partial\phi^b}F = (-)^{\phi^b(F+1)}F\frac{\overleftarrow{\partial}}{\partial\phi^b}\,. \tag{2.122}$$

A BV master action is said to be quantum if it satisfies the quantum BV master equation

$$\frac{1}{2}(S,S) + \Delta S = 0\,, \tag{2.123}$$

or equivalently

$$\Delta\left(e^S\right) = 0\,, \tag{2.124}$$

where we introduced the BV Laplacian

$$\Delta_c = \frac{1}{2}(-)^{\phi^a}(\omega_{\rm c})^{ab}\frac{\overrightarrow{\partial}}{\partial\phi^a}\frac{\overrightarrow{\partial}}{\partial\phi^b}\,, \tag{2.125a}$$

$$\Delta_o = \frac{1}{2}(-)^{\psi^a}(\omega_{\rm o})^{ab}\frac{\overrightarrow{\partial}}{\partial\psi^a}\frac{\overrightarrow{\partial}}{\partial\psi^b}\,. \tag{2.125b}$$

Equation (2.123) ensures the quantum consistency of the action, implying that the path integral is independent of the gauge-fixing choice. In the literature, (2.123) can be found in different forms, depending on conventions. Here, we are setting $\hbar = 1$ and considering the Euclidean action $S_{\rm E} = -S$, the minus is related to our conventions (2.94) and (2.95) which yields to a definite negative action. Indeed, as can be explicitly seen in (2.115), the momentum appears in the kinetic term as $-p^2$ and not as p^2 as expected in the euclidean action.

For the free SFT, we can consistently define the BV master action as

$$S_{\rm kin}(\Psi) = \frac{1}{2}\omega(\Psi, Q\Psi)\,, \tag{2.126}$$

where the only difference compared to (2.110) is that now we are considering the full string field. In the following, we will prove that this action satisfies both (2.120) and (2.123).

To do so, we first compute the derivative

$$\frac{\overrightarrow{\partial}}{\partial\psi^b}S_{\text{kin}}(\Psi) = (-)^{\psi^b}\omega_{\text{o}}(o_b, Q\Psi)\,, \tag{2.127}$$

where we used cyclicity to combine the two contributions given by the Leibniz rule. Furthermore, notice that bringing the derivative inside the symplectic form introduces a sign due to the relation between the symplectic form and the BPZ product

$$\frac{\overrightarrow{\partial}}{\partial\psi_1^b}\omega(\Psi_1,\Psi_2) = \frac{\overrightarrow{\partial}}{\partial\psi_1^b}(-)^{\Psi_1}\langle\psi_1^a o_a|\Psi_2\rangle = (-)^{\Psi_1}\langle o_b|\Psi_2\rangle = (-)^{\psi_1^b}\omega(o_b,\Psi_2)\,, \tag{2.128}$$

as $d(\psi_1^b) + d(o_b) = d(\Psi_1)$. To keep track of the correct sign factors, we can use the mnemonic that the symbol ω is a Grassmann-odd object. By applying (2.122) in (2.120) and using (2.127), we obtain

$$(S_{\text{kin}}, S_{\text{kin}})_{\text{o}} = (-)^{\psi^b}(\omega_{\text{o}})^{ab}\omega_{\text{o}}(o_a, Q\Psi)\omega_{\text{o}}(o_b, Q\Psi) = -\omega_{\text{o}}(Q\Psi, Q\Psi) = 0\,, \tag{2.129}$$

where we reconstructed the identity twice (see (2.74)) and used cyclicity and nilpotency of Q.

Finally, also the BV Laplacian annihilates (2.126)

$$\Delta_{\text{o}}S_{\text{kin}} = -\frac{1}{2}\omega_{\text{o}}(o_a, Qo^a) = 0\,, \tag{2.130}$$

because, in the above equation, the degrees cannot sum up to 1. This would require the basis vectors and their duals to have opposite degree, which is not possible by definition.

2.2.3 Siegel gauge

The most common gauge fixing choice is the so-called Siegel gauge

$$b_0\Psi = 0\,, \tag{2.131}$$

in the open string sector, and

$$b_0^+ \Phi = 0\,, \tag{2.132}$$

in the closed string sector. The advantages of this gauge fixing are that the cohomology can be searched in $\mathrm{Ker}(L_0)$ or $\mathrm{Ker}(L_0^+)$, as can be immediately seen from equation (2.32a), and it has a clear geometrical interpretation, as we will discuss in Section 2.3.3. In the following, we will focus on the open string scenario, but everything can be straightforwardly adapted to the closed string case. From the viewpoint of the decomposition (2.45), fixing the Siegel gauge means setting $\tilde{\Psi}_\downarrow = 0$. This can be achieved from any field satisfying $\tilde{\Psi}_\downarrow \notin \mathrm{Ker}(L_0)$, and thus $\Psi \notin \mathrm{Ker}(L_0)$, by choosing

$$\Lambda_\mathrm{s} = -\frac{1}{L_0}\bar{P}_0\tilde{\Psi}_\downarrow = -\frac{b_0}{L_0}\bar{P}_0\Psi\,, \tag{2.133}$$

as gauge parameter. Where, we introduced the projector P_0, which projects onto $\mathrm{Ker}(L_0)$, and its complementary $\bar{P}_0 = 1 - P_0$. Let us now split the total string field using the projectors P_0 and then the component outside the $\mathrm{Ker}(L_0)$ using P_s, defined in (2.44),

$$\Psi = P_0\Psi + P_s\bar{P}_0\Psi + \bar{P}_s\bar{P}_0\Psi := \psi + R + \tilde{R}\,, \tag{2.134}$$

notice that $[P_0, P_s] = 0$, as P_s contains only weight-zero modes. Therefore, in this formalism, fixing the Siegel gauge means setting $\tilde{R} = 0$, and then (2.81) becomes

$$S_\mathrm{g.f.}(\psi, R) = \frac{1}{2}\omega_\mathrm{o}(\psi, Q\psi) + \frac{1}{2}\omega_\mathrm{o}(R, c_0L_0R)\,, \tag{2.135}$$

with the subsidiary condition $(Q - c_0L_0)R = 0$. This is called the out-of-Siegel-gauge constraint, which is needed since the starting equations of motion for $\tilde{R}$ contain R

and vice versa

$$\mathrm{EOM}_R(\psi, R, \tilde{R}) = \bar{P}_s Q(R + \tilde{R}) \longrightarrow \mathrm{EOM}_R(\psi, R, 0) = c_0 L_0 R\,, \tag{2.136a}$$

$$\mathrm{EOM}_{\tilde{R}}(\psi, R, \tilde{R}) = P_s Q(R + \tilde{R}) \longrightarrow \mathrm{EOM}_{\tilde{R}}(\psi, R, 0) = (Q - c_0 L_0) R\,, \tag{2.136b}$$

to write the above equations, it is crucial to note that $P_s^* = \bar{P}_s$ and then varying the action with respect to R or $\tilde{R}$ return the equation of motion with the complementary projector. Therefore, when one fixes the Siegel gauge, (2.136a) gives the equation of motion which comes form (2.135) whereas the relation(2.136b) is lost and must be imposed on the side of the action.

Looking at (2.133) it is natural to define the Siegel propagator

$$h_0 = \frac{b_0}{L_0}(1 - P_0)\,, \tag{2.137}$$

as it inverts the kinetic operator outside of $\mathrm{Ker}(L_0)$

$$[h_0, Q] = \bar{P}_0 \longrightarrow h_0 Q R = R\,. \tag{2.138}$$

The above expression is the so-called Hodge-Kodaira decomposition. We will return to the general interpretation of this formula later; for now, let us conclude the description of the Siegel propagator. It is particularly useful to rewrite (2.137) in the Schwinger parametrization

$$\frac{1}{L_0} = \int_0^1 \frac{dt}{t}\, t^{L_0} = \int_0^\infty ds\, e^{-sL_0}\,, \tag{2.139}$$

where the two Schwinger parameters are related through $t = e^{-s}$. This integral representation is well defined for $L_0 > 0$. The divergences for $L_0 < 0$ are artificial meaning that this parametrization is not appropriate. However, we can still use this parametrization by regularizing these divergences due to the propagation of states with total negative weight using the following prescription

$$\frac{1}{L_0} \to \frac{1}{L_0 + \epsilon}\bigg|_{\epsilon \to 0} = \int_0^1 \frac{dt}{t}\, t^{L_0 + \epsilon}\bigg|_{\epsilon \to 0}\,, \tag{2.140}$$

assuming that the computation is carried out for values of ϵ where the integral converges, with the effect of returning what we would have obtained by directly applying $\frac{1}{L_0}$. This regularization has been introduced in [59] and is equivalent to the prescription discussed in [60]. The divergence for $L_0 = 0$ is genuine but we are protected by the $\bar{P}_0$ in (2.137). Therefore, (2.137) can be rewritten as

$$h_0 = \int_0^1 \frac{dt}{t}\, t^{L_0+\epsilon} b_0 \bar{P}_0 \Big|_{\epsilon\to 0} . \tag{2.141}$$

Similarly, in the closed string scenario

$$h_0^+ = \frac{b_0^+}{L_0^+}\bar{P}_0^+ = \int_0^1 \frac{dt}{t} t^{L_0^+ + \epsilon} b_0^+ \bar{P}_0^+ \Big|_{\epsilon\to 0} , \tag{2.142}$$

where P_0^+ projects onto $\mathrm{Ker}(L_0^+)$.

This construction of the propagator regulates the IR divergences in the string amplitudes. Indeed, as we will discuss in detail in Section 2.3.3, the propagator can be seen as a strip or a tube identifying the propagating open or closed string, respectively, and the Schwinger parameter s quantifies their length. Thus, the IR divergences at the boundary of moduli space correspond to the limit $s \to \infty$, which are regularized as explained above. Of course, in general, different gauge-fixing conditions can be chosen, which might not have a clear geometrical interpretation but can still be consistent[4].

2.2.4 Abstract gauge fixing

In a more abstract way, we can fix the gauge by imposing

$$h\Psi = 0\,, \tag{2.143}$$

[4]An interesting generalization of the Siegel gauge is provided by the so-called linear b-gauges, explored in [61] where the gauge fixed states are annihilated by combination of antighost oscillators. Among these there is the Schnabl gauge [21] used to find the analytic solution for tachyon condensation in open SFT

for h generic BPZ-even, degree -1 operator obeying the Hodge-Kodaira decomposition

$$[h, Q] = 1 - P\,, \tag{2.144}$$

where P is a BPZ even, degree 0 projector commuting with the BRST charge $[P, Q] = 0$, and satisfying the annihilation relations

$$h^2 = Ph = hP = 0\,. \tag{2.145}$$

Clearly, this gauge fixing is reachable in $\bar{P}$ by choosing $\Lambda_h = -h\Psi$ and h can be interpreted as the propagator as it inverts Q in $\bar{P}$. The operators Q, h and P with the properties just described form a strong deformation retract (SDR) usually schematized as

$$\circlearrowleft (-h)\,(\mathcal{H}, Q) \underset{I}{\overset{\Pi}{\rightleftarrows}} (\Pi\mathcal{H}, \Pi Q I)\,, \tag{2.146}$$

where in this context $-h$ is called contracting homotopy operator and we defined the canonical inclusion

$$I : \Pi\mathcal{H} \to P\mathcal{H} \subset \mathcal{H}\,, \tag{2.147}$$

and the canonical projector

$$\Pi : \mathcal{H} \to \Pi\mathcal{H}\,, \tag{2.148}$$

such that

$$P = I\Pi\,, \tag{2.149}$$

and

$$\Pi I = 1_{\Pi\mathcal{H}}\,. \tag{2.150}$$

Furthermore, (2.144) together with (2.145) tell us that (P, hQ, Qh) are orthogonal projectors. This partitions the total space $\mathcal{H}$ into

$$\mathcal{H} = A \oplus B \oplus C\,, \tag{2.151}$$

where we have defined

$$A \coloneqq P\mathcal{H} = \mathrm{Span}\{a_i\}\,, \tag{2.152}$$

$$B \coloneqq Qh\mathcal{H} = \mathrm{Span}\{b_i\}\,, \tag{2.153}$$

$$C \coloneqq hQ\mathcal{H} = \mathrm{Span}\{c_i\}\,. \tag{2.154}$$

We also define dual basis vectors by

$$\omega(a^i, a_j) = \omega(b^i, c_j) = \omega(c^i, b_j) = \delta^i_j\,, \tag{2.155}$$

from which we can write down the completeness relation

$$1_{\mathcal{H}} = a_i\omega(a^i, \cdot) + b_i\omega(c^i, \cdot) + c_i\omega(b^i, \cdot)\,. \tag{2.156}$$

Under this decomposition the symplectic form is given by

$$\omega_{ij} = \begin{pmatrix} \omega^A_{ij} & 0 & 0 \\ 0 & 0 & \omega^{BC}_{ij} \\ 0 & -\omega^{BC}_{ij} & 0 \end{pmatrix}\,, \tag{2.157}$$

where we defined

$$\omega^A_{ij} \coloneqq \omega(a_i, a_j)\,, \tag{2.158}$$

$$\omega^{BC}_{ij} \coloneqq \omega(b_i, c_j)\,. \tag{2.159}$$

Therefore, the string field can be written as

$$\Psi = \psi + \tilde{R} + R = \alpha^i a_i + \beta^i b_i + \gamma^i c_i. \tag{2.160}$$

The distinction between fields and antifields in $\bar{P}\mathcal{H}_{\rm o}$ depends on a choice of lagrangian submanifold (a maximal subspace of $\bar{P}\mathcal{H}_{\rm o}$ where ω vanishes). In particular, as anticipated in the previous section, it is possible to arrange all antifields of $\bar{P}\mathcal{H}_{\rm o}$ into B. Consequently, the Lagrangian submanifold that defines the gauge fixing will be given by the condition $\beta^i = 0$ or equivalently $h\Psi = 0$. Thus, the gauge fixed path integral becomes

$$Z_{\rm g.f.} = \int_L \mathcal{D}\Psi\, e^{S(\Psi)} = \int \prod_{ijk} d\alpha^i d\beta^k d\gamma^j\, \delta(\beta^k) e^{S(\Psi)} = \int \prod_{ij} d\alpha^i d\gamma^j\, e^{S_{\rm g.f.}(\psi,R)}\,, \tag{2.161}$$

and it is possible to prove that it is independent on the choice of L if S is a solution of the quantum BV master action (2.123). Again in the above equation the action appears in the path integral with a plus due to our normalization, because we want a definite negative object at the exponent in the path integral[5].

Finally, if there exits an operator Δ which satisfies (2.144) with $\bar{P} = 1$, namely

$$[Q, \Delta] = 1 \tag{2.162}$$

implies that Q has no cohomology due to the fact that a generic Q-closed state can always be written as $\Psi = Q\Delta\Psi$. Finding Δ is the typical strategy to prove that a BRST charge has no cohomology.[6]

2.3 Interacting String Field Theory

In this section, we define the full interacting SFT action. While the kinetic term is constructed to ensure that solutions to the equations of motion belong to the cohomology of the BRST charge, the guiding principle for defining the interaction terms is that, through the Feynman diagram expansion, they must reproduce the expected string amplitudes when the states are taken to be on-shell. Here, "on-shell" refers to

[5] Note that there is a subtlety regarding the tachyon at zero momentum. In this scenario, it is not possible to compensate for the negative mass, leading to an ill-defined path integral. As explained in [62], this problem can be approached by Wick rotating the tachyon mode.

[6] For instance, in [63], I. Ellwood and M. Schnabl demonstrate that there is no cohomology around the tachyon vacuum by constructing the corresponding homotopy operator.

solutions of the linearized equations of motion. Therefore, the starting point will be to define an appropriate off-shell continuation of string amplitudes. Then, contrary to the usual approach in QFT, we aim to understand how these amplitudes can be consistently decomposed into contributions arising from fundamental interaction vertices, which will constitute the interacting action, and those that can be interpreted as resulting from the Feynman diagram expansion, namely, lower-order fundamental vertices connected by propagators. This leads us to impose recursive constraints on the fundamental vertices, which can be summarized in the so-called geometrical BV master equation. As we will see explicitly, this equation implies (2.123).

2.3.1 Off-shell amplitudes

As anticipated, the starting point is to construct the off-shell generalization of string amplitudes. For concreteness, we will describe the closed SFT scenario, which can be readily generalized to open and open-closed SFT frameworks (see [37, 64]). In full generality, the off-shell genus g amplitudes with k bulk punctures are defined as a symmetric multilinear maps

$$\mathcal{A}_k^g : \mathcal{H}_{\rm c}^{\wedge k} \to \mathbb{C}\,, \tag{2.163}$$

such that:

1. When acting on k physical states, they reproduce the expected on-shell amplitudes derived from ST.

2. They vanish when one of the insertions is BRST-exact, and the others are BRST-closed; that is, Q-exact states decouple from physical interactions.

The $\wedge$ in (2.163) denotes the symmetric tensor product which symmetrize over bulk punctures

$$\Phi_1 \wedge \cdots \wedge \Phi_n \coloneqq \sum_{\sigma\in\mathbf{S}_n} (-1)^{\epsilon_\sigma} \Phi_{\sigma(1)} \otimes \cdots \otimes \Phi_{\sigma(n)}, \tag{2.164}$$

where $\mathbf{S}_n$ is the group of permutations.

From the first-quantized ST perspective, the g loop k-point amplitude with local operators $\Phi_1, \ldots, \Phi_k$ is given by an appropriate correlator integrated over the moduli space $\mathcal{M}_k^g$, with real dimensionality

$$d_k^g := \dim(\mathcal{M}_k^g) = 6g - 6 + 2k \,, \tag{2.165}$$

describing the punctured (or marked) Riemann surface Σ_k^g, with Euler characteristic

$$\chi(\Sigma_k^g) = 2 - 2g - k \,. \tag{2.166}$$

A key consequence of going off-shell is that the insertions are no longer primary fields, and thus the correlators lose their invariance under conformal transformations. To address this, we introduce a local coordinate map $f_a(w_a)$ for each puncture, inserting the closed string at $f_a(0)$, as reported in Figure 2.1. These maps are defined as generic analytic functions mapping the unit disk in the local coordinate frame $|w_a| \leq 1$, where the bulk field is inserted at the origin, to a disk domain surrounding the puncture at $f_a(0)$, referred to as local coordinate patches. We require that these local coordinate maps be non-overlapping, for reasons that will become clear later. By specifying local coordinate maps at each puncture, we introduce

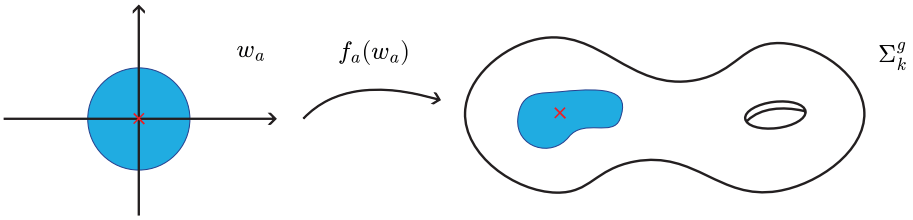


Figure 2.1: Each closed string puncture, located at the origin of the local coordinate frame with coordinates w_a, is inserted on the Riemann surface Σ_k^g through a local coordinate map $f_a(w_a)$.

off-shell data necessary for defining off-shell amplitudes. However, this choice is not unique; different selections of local coordinate maps lead to different off-shell extensions of the same on-shell amplitude. These ambiguities will manifest as different formulations of the SFT action, all of which share the same S-matrix elements and then are related by field redefinitions.

To formalize this concept, we consider the fiber bundle $\mathcal{P}_k^g$, defined for $(k \geq$

$3, g = 0)$ or $(k \geq 0, g > 0)$, whose base is the moduli space and whose fibers correspond to all possible local coordinate maps around each puncture. A point $u \in \mathcal{P}_k^g$ corresponds to a decorated Riemann surface, namely a genus g surface with k insertions at fixed value of moduli and specified local coordinate maps around each puncture. Furthermore, the level-matching condition (specifically $L_0^- = 0$) implies that closed string states are insensitive to phase transformations. Thus, in closed SFT, it is useful to define the quotient space $\hat{\mathcal{P}}_k^g$ by modding out local coordinate maps that differ by a phase. Therefore, choosing specific local coordinate maps corresponds to selecting a section $\mathcal{S}_k^g$ of $\hat{\mathcal{P}}_k^g$, Figure 2.2. Note that a global section of the bundle of

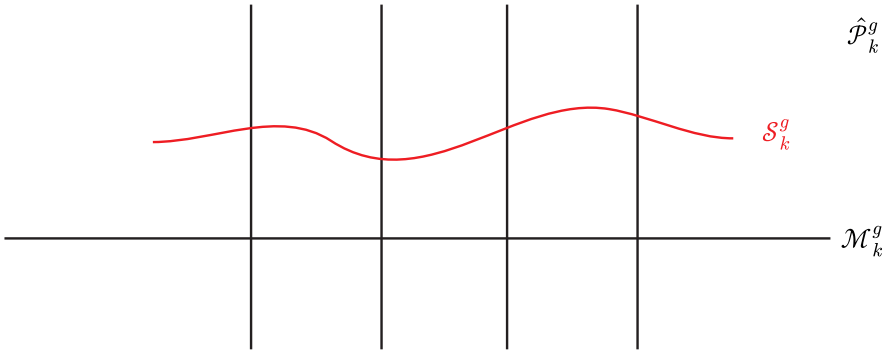


Figure 2.2: $\hat{\mathcal{P}}_k^g$ is the fibre bundle whose base is the moduli space $\mathcal{M}_k^g$, and whose fibre (depicted as the vertical line in the drawing) is given by the possible local coordinate maps for each puncture, modulo those that differ by a phase. Making a choice of local coordinate maps at each point in the moduli space corresponds to selecting a section $\mathcal{S}_k^g$ of the fibre bundle.

decorated Riemann surfaces cannot exist if the fibers contain information about the phase of the local coordinates. This is because it is impossible to fix the phase of the local coordinate continuously over the entire Riemann surface, as explained in [65]. This is the reason why we imposed $L_0^- = 0$ as a subsidiary constraint in constructing the closed string Hilbert space.

Therefore, the off-shell amplitude can be expressed as

$$\mathcal{A}_k^g(\Phi_1, \ldots, \Phi_k) = \frac{\kappa^{-\chi_k^g}}{k!} \sum_{\sigma \in \mathbf{S}_k} (-1)^{\epsilon_\sigma} \int_{\mathcal{S}_k^g} \Omega_p^{(g,k)}(\Phi_{\sigma(1)}, \ldots, \Phi_{\sigma(k)}) \,, \tag{2.167}$$

where κ is the string coupling constant appearing with a power given by the usual topological expansion and $\Omega_p^{(g,k)}$ is an appropriate $p = d_k^g$-form. By "appropriate," we mean that it must satisfy conditions (1) and (2). In the following we will use the

notation

$$\mathcal{A}_k^g(\Phi_1,\dots,\Phi_k) \coloneqq \frac{\kappa^{-\chi_k^g}}{k!} A_k^g(\Phi_1 \wedge \cdots \wedge \Phi_k)\,, \tag{2.168}$$

to make the combinatorial factor and the string coupling constant explicit.

To define $\Omega_p^{(g,k)}$, we must introduce a parametrization of $\hat{\mathcal{P}}_k^g$. Given a marked Riemann surface Σ_k^g, we can decompose it into a collection of k disjoint disks D_a surrounding the k insertions, and $2g-2+k$ spheres S_i with three holes, as depicted in Figure 2.3. We refer to the boundaries of these disks and spheres collectively as

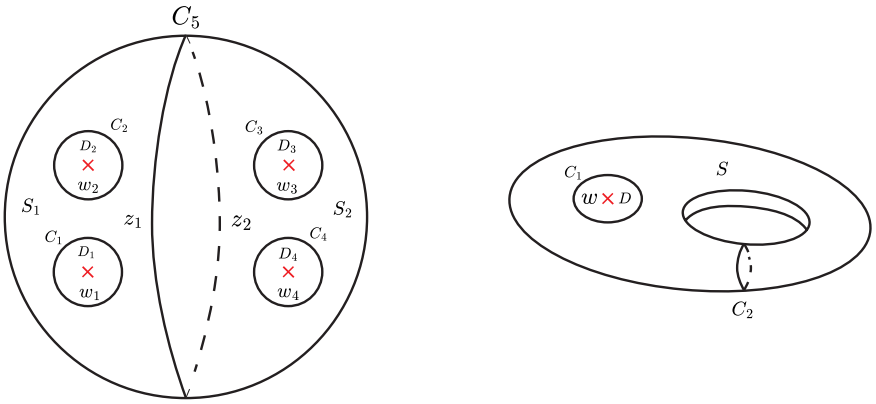


Figure 2.3: On the left, the sphere–disk decomposition of a sphere with three closed string insertions, and on the right, the same decomposition for a torus with one closed string insertion.

circles. These circles can be categorized into two types: (i) boundaries shared by a disk and a sphere, $C_{ai} \coloneqq D_a \cap S_i$, and (ii) boundaries shared between two spheres, $C_{ij} \coloneqq S_i \cap S_j$. We will treat all these circles collectively, denoting them as C_s with a single index $s = 1,\dots,3g-3+2k$.

Let us introduce complex coordinate systems for each sphere, denoted by z_i , and for each disk, denoted by w_a. The latter are the local coordinate systems without necessarily imposing the condition that the circle lies at $|w_a| = 1$. In general, we denote by σ_s and τ_s the complex coordinate systems on the two sides of the circle: σ_s on the left and τ_s on the right, with respect to a given orientation of the circle Figure 2.4.

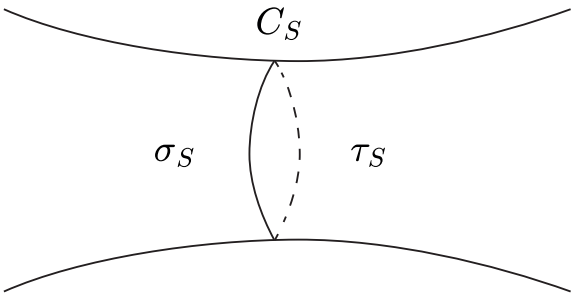


Figure 2.4: Convention for the naming of local coordinates on the two sides of a circle.

We introduce the transition functions as

$$\sigma_s = F_s(\tau_s; u_\alpha), \tag{2.169}$$

where the u_α are parameters which define these functions (they can be seen as the Taylor/Laurent expanding modes of F_s around the insertion point). Observe that, through this decomposition of Σ_k^g , the set of transition functions uniquely determines the Riemann surface, as well as other spaces of interest, by defining appropriate equivalence classes:

1. $\mathcal{M}^g$, identifying transition functions differing by reparametrization of z_i and w_a;
2. $\mathcal{M}_k^g$, identifying transition functions differing by reparametrization of z_i and w_a, keeping fix the inserting points at $w_a = 0$;
3. $\mathcal{P}_k^g$, identifying transition functions differing by reparametrization of z_i;
4. $\hat{\mathcal{P}}_k^g$, identifying transition functions differing by reparametrization of z_i and and phase rotations of w_a.

Therefore, by considering the fourth case we can use u_α as coordinates of the infinite-dimensional space $\hat{\mathcal{P}}_k^g$ and then define the form $\Omega_p^{(g,k)}$ as

$$\Omega_p^{(g,k)}(\Phi_1, \ldots, \Phi_k) = \left(\frac{i}{2\pi}\right)^{\frac{d_n^g}{2}} \langle \bigwedge_{i=1}^{d_n^g} B^{\alpha_i}(u) du_{\alpha_i} \prod_{j=0}^{k} f_j \circ \Phi_j(0, \bar{0}) \rangle_{\Sigma_k^g}, \tag{2.170}$$

where as expected we have the correlator on the punctured Riemann surface $\langle \cdot \rangle_{\Sigma_k^g}$

with the local operator inserted through the local coordinate maps f_j. The normalizing factors will be motivated in the next section (2.215) and in the integration measure, summation over repeated indices α_i is understood. Here, $B^{\alpha_i}(u)$ represents the Beltrami differential, which is defined as a sum over the circles C_s of combinations of b-ghost modes, obtained by computing the contour integral of the b-ghost fields

$$B^{\alpha}(u) \coloneqq \sum_s \oint_{C_s} \frac{d\sigma_s}{2\pi i} v_s^{\alpha}(\sigma_s;u) b(\sigma_s) + \oint_{C_s} \frac{d\bar{\sigma}_s}{2\pi i} \bar{v}_s^{\alpha}(\sigma_s;u) \bar{b}(\bar{\sigma}_s)\,, \tag{2.171}$$

where $v_s^{\alpha}(\sigma_s;u)$ is called Shiffer vector and it is given by

$$v_s^{\alpha}(\sigma_s;u) \coloneqq \frac{\partial}{\partial u_{\alpha}} F_s(\tau_s;u)|_{\tau_s=F_s^{-1}(\sigma_s;u)}\,, \tag{2.172}$$

similarly for the anti-holomorphic sector.

Note that the contour integral is performed according to the conventions where σ_s is on the left and τ_s on the right. For instance, if τ_s is the coordinate on a disk, then the integral is taken to run clockwise around the puncture located at $\tau_s = 0$.

In brief, (2.171) introduces the b-ghost insertions required to saturate the ghost number in the correlator, by transforming local vertex operators into integrated ones, as we will see below.

In (2.167) the form (2.170) is integrated over the d_k^g dimensional section $\mathcal{S}_k^g$. Introducing d_k^g real coordinates y_λ for $\lambda = 1,\dots,d_k^g$, which parametrize $\mathcal{S}_k^g$ by imposing $u_\alpha = u_\alpha(y)$, we can use the chain rule to rewrite (2.170) on the section as

$$\Omega_p^{(g,k)}(\Phi_1,\dots,\Phi_k)|_{\mathcal{S}_k^g} = \left(\frac{i}{2\pi}\right)^{\frac{d_n^g}{2}} \langle \bigwedge_{i=1}^{d_n^g} B^{\lambda_i}(y) dy_{\lambda_i} \prod_{j=0}^{k} f_j \circ \Phi_i(0,\bar{0})\rangle_{\Sigma_k^g}\,, \tag{2.173}$$

where we defined

$$B^{\lambda}(y) \coloneqq B^{\alpha}(u(y)) \frac{\partial u_\alpha(y)}{\partial y_\lambda}\,, \tag{2.174}$$

which can be written as (2.171) by introducing the Shiffer vectors on the section

$$v_s^\lambda(\sigma_s; y) \coloneqq \frac{\partial}{\partial y_\lambda} F_s(\tau_s; u(y))|_{\tau_s = F_s^{-1}(\sigma_s; u(y))} \,. \tag{2.175}$$

Notice that the measure is now expressed as a finite sum of terms $B^\lambda(y)dy_\lambda$, in contrast to equation (2.170), which involves an infinite sum $B^\alpha(u)dy_\alpha$.

To illustrate this construction, let us consider two fundamental examples: the three- and four-point amplitudes on the sphere involving closed string vertex operators $c\bar{c}V$, where V is a weight $(1,1)$ primary field, showing that (2.167) gives the expected result on-shell. The three-point amplitude on the sphere is straightforward. Indeed, the moduli space is zero-dimensional ($d_3^0 = 0$), and fixing a section simply means choosing three local coordinate maps for the three insertions. However, since the tree fields are on-shell, we lose all information about the local coordinate maps, ending up with

$$A_3^0(c\bar{c}V, \dots, c\bar{c}V) = \langle \prod_{i=1}^{3} c\bar{c}V(\hat{z}_i, \bar{\hat{z}}_i) \rangle_{\Sigma_3^0} \,, \tag{2.176}$$

which is independent on the $\hat{z}_i$.

Let us now consider the more instructive example of the four-point amplitude on the sphere. In this case, the moduli space has real dimension $d_4^0 = 2$, and the space $\hat{\mathcal{P}}_4^0$ is parametrized by five transition maps according to Figure 2.3. Analogous to the previous example, since we are working with vertex operators, we can specify the local coordinate maps by fixing the locations of the insertions. Therefore, a possible two-dimensional section is constructed by first imposing $z_1 = F_5(z_2) = z_2$ to glue the two spheres, and then choosing local coordinate maps such that $F_1(0) = \hat{z}_1$, $F_2(0) = \hat{z}_2$, $F_3(0) = \hat{z}_3$ for $\hat{z}_1$, $\hat{z}_2$ and $\hat{z}_3$ fixed and

$$F_4(w_4; y) = w_4 + y \,, \qquad\qquad \bar{F}_4(\bar{w}_4; \bar{y}) = \bar{w}_4 + \bar{y} \,, \tag{2.177}$$

Therefore, by doing so, we are parametrizing the two-dimensional section through the location of the fourth insertion, and thus $\mathcal{S}_4^0$ corresponds to the complex plane

$(y, \bar{y})$. In this setup, the components of the Schiffer vector can be written as

$$v_s^y = \delta_{s,5}, \qquad v_s^{\bar{y}} = 0, \tag{2.178a}$$

$$\bar{v}_s^y = 0, \qquad \bar{v}_s^{\bar{y}} = \delta_{s,5}. \tag{2.178b}$$

Therefore, the two components of resulting Beltrami form are given by

$$B^y(y) = -\oint_y \frac{dz_1}{2\pi i} b(z_2)\,, \qquad B^{\bar{y}}(\bar{y}) = -\oint_{\bar{y}} \frac{dz_1}{2\pi i} \bar{b}(\bar{z}_2)\,, \tag{2.179}$$

where the minus signs arise from the convention used in defining the integral over circles. Specifically, by construction, the circle C_5 surrounds the puncture running clockwise, whereas the above integrals are taken to be counter-clockwise. Hence the measure appearing in (2.173) becomes

$$\bigwedge_{i=1}^{2} B^{\lambda_i}(y,\bar{y}) dy_{\lambda_i} = (dy \wedge d\bar{y}) \oint_y \frac{dz_2}{2\pi i} \oint_{\bar{y}} \frac{d\bar{z}_2}{2\pi i} b(z_2)\bar{b}(\bar{z}_2)\,, \tag{2.180}$$

and finally by plugging it back into (2.167) we obtain

$$A_4^0(c\bar{c}V, \ldots, c\bar{c}V) = \frac{1}{2\pi i} \int dy \wedge d\bar{y}\, \langle \prod_{i=1}^{3} c\bar{c}V(\hat{z}_i, \bar{\hat{z}}_i) V(y,\bar{y}) \rangle_{\Sigma_4^0}\,, \tag{2.181}$$

which is the well-known expression for the amplitude of four on-shell closed string vertex operators on the sphere.

Furthermore, it is possible to show[7] that in this context the BRST charge behaves as an exterior derivative

$$\begin{aligned} \Omega_p^{(g,k)}(Q\Phi_1, \ldots, \Phi_k) + \cdots + (-)^{\epsilon(\Phi_1)+\cdots+\epsilon(\Phi_k)} \Omega_p^{(g,k)}(\Phi_1, \ldots, Q\Phi_k) \\ = (-)^p d\Omega_{p-1}^{(g,k)}(\Phi_1, \ldots, \Phi_k)\,, \end{aligned} \tag{2.182}$$

which has two main consequences that guarantee the amplitude is well-defined:

- On-shell amplitudes involving one or more Q-exact states vanish. Specifically, consider equation (2.182) with $k-1$ BRST-invariant insertions, $\phi_1, \ldots, \phi_{k-1}$

[7] The detailed proof can be found in [10]

and $Q\Lambda$. This results in

$$\Omega_p^{(g,k)}(Q\Lambda, \phi_2, \ldots, \phi_{k-1}) = (-)^p d\Omega_{p-1}^{(g,k)}(\Lambda, \phi_1, \ldots, \phi_{-1})\,, \tag{2.183}$$

which gives zero when integrated over $\mathcal{S}_k^g$ then proving that

$$A_k^g(Q\Lambda, \phi_1, \ldots, \phi_{k-1}) = 0\,. \tag{2.184}$$

- On-shell amplitudes are independent form the choice of the section. Let $\mathcal{S}$ and $\mathcal{S}'$ denote two sections, and $\mathcal{R}$ the region in $\hat{\mathcal{P}}_k^g$ interpolating between them. The difference between on-shell amplitudes evaluated on these two sections is given by

$$\left(\int_{\mathcal{S}} - \int_{\mathcal{S}'}\right)\Omega_p = \int_{\partial\mathcal{R}} \Omega_p = \int_{\mathcal{R}} d\Omega_p\,, \tag{2.185}$$

and then we can use (2.182) to bring the exterior derivative into Ω_p by acting with BRST charge on the states $\phi_1, \ldots, \phi_k$ which are by assumption Q-closed. Hence,

$$\left(\int_{\mathcal{S}} - \int_{\mathcal{S}'}\right)\Omega_p^{(g,k)}(\phi_1, \ldots, \phi_k) = 0\,. \tag{2.186}$$

2.3.2 Interacting SFT action

Having defined the off-shell amplitudes in (2.167), our goal is to construct the vertices of the SFT action such that the Feynman diagram expansion reproduces these amplitudes. This approach may seem counter-intuitive compared to conventional QFT, where one starts with an action and computes Feynman diagrams to obtain amplitudes. Here, we reverse the logic: starting from the off-shell amplitudes, we aim to determine the corresponding action.

A crucial observation is that off-shell amplitudes suffer from IR divergences arising from the boundaries of moduli space. For the action to be well-defined, its vertices must be finite. Therefore, we introduce a strategy: given a section $\mathcal{S}_k^g$, we split it into two regions, the fundamental domain $\mathcal{V}_k^g$, and the Feynman region

$\mathcal{F}_k^g$. The latter contains the parts of the section that contribute to divergences in the amplitude. For instance, in the example of the four-point amplitude on the sphere discussed in the previous section, divergences arise from the collision of insertions. In our parametrization, these correspond to the limits $y \to \hat{z}_1, \hat{z}_2, \hat{z}_3$. In this context, we interpret the regions surrounding these points as the Feynman region, and the remainder as the fundamental domain, as illustrated in Figure 2.5. Where we denoted the three disconnected component of $\mathcal{F}_4^0$ as $\mathcal{F}_s$, $\mathcal{F}_t$, and $\mathcal{F}_u$, corresponding to the three channels in which two cubic fundamental vertices are connected by a propagator, analogous to the $s-$, $t-$, and $u-$channels in QFT.

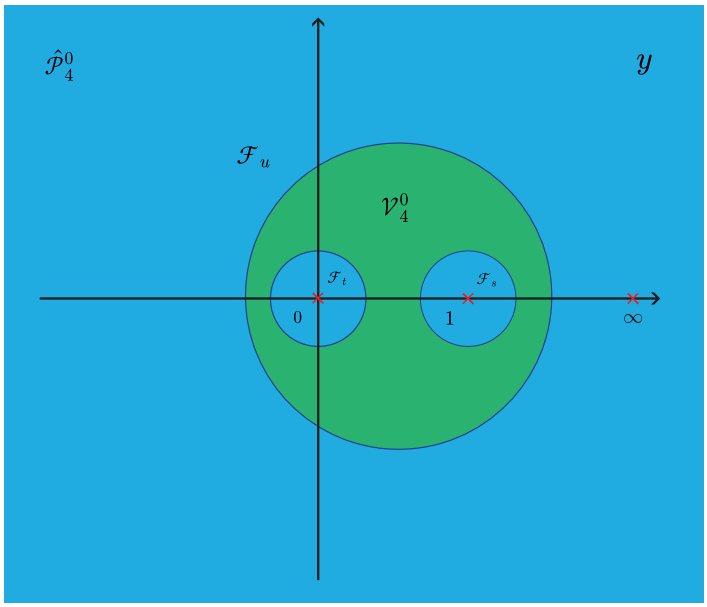


Figure 2.5: Decomposition of the moduli space for a sphere with four closed string insertions, $\mathcal{M}_4^0$, into the fundamental region (in green) $\mathcal{V}_4^0$ and the Feynman region (in blue) $\mathcal{F}_4^0$, which is given by the union of three disconnected pieces, $\mathcal{F}_u$, $\mathcal{F}_v$, and $\mathcal{F}_w$, surrounding the three fixed punctures in 0, 1 and ∞.

Given $\Phi \in \mathcal{H}_c$, we define the interacting closed SFT action as the sum over all possible vertices

$$S_c(\Phi) = \sum_{g,k} \kappa^{-\chi_k^g} \frac{1}{k!} V_k^g(\Phi^{\wedge k}) \,, \tag{2.187}$$

weighted by the string coupling constant according to the topological expansion. The

terms in (2.187) are given by the kinetic term

$$V_2^0(\Phi^{\wedge 2}) \coloneqq \omega_{\rm c}(\Phi, Q_{\rm c}\Phi)\,, \tag{2.188}$$

and the interaction vertices

$$V_k^g(\Phi^{\wedge k}) = \int_{\mathcal{V}_k^g} \Omega_p^{(g,k)}(\Phi^{\wedge k})\,, \tag{2.189}$$

With the condition that the full off-shell amplitude is given by the Feynman rules

$$A_k^g(\Phi^{\wedge k}) = V_k^g(\Phi^{\wedge k}) + F_k^g(\Phi^{\wedge k})\,, \tag{2.190}$$

where $F_k^g(\Phi^{\wedge k})$ identifies the contribution of Feynman diagram. Therefore, (2.190) implies that

$$F_k^g(\Phi^{\wedge k}) = \int_{\mathcal{F}_k^g} \Omega_p^{(g,k)}(\Phi^{\wedge k})\,, \tag{2.191}$$

imposing that the section is fully covered

$$\mathcal{S}_k^g = \mathcal{V}_k^g \cup \mathcal{F}_k^g\,. \tag{2.192}$$

Notice that once we define the kinetic term (2.188) and the fundamental vertices (2.189), we can use the gauge-fixed path integral to compute the off-shell amplitudes. Throughout, we work in the Siegel gauge. At any fixed order in the topological expansion, the amplitude consists of the corresponding fundamental vertex and contributions from diagrams built by connecting lower-order vertices via Siegel propagators. Thus, the first step is to verify that the Feynman diagrams constructed using Siegel propagators reproduce the expression in (2.191). Next, we must determine how the Feynman regions $\mathcal{F}_k^g$ are specified in terms of the lower-order fundamental domains. This enables us to reformulate equation (2.192) as a recursive relation involving only the fundamental vertices. This is the so-called BV geometric master equation, which encodes the constraints that the fundamental domains must satisfy in order to define a consistent SFT action, as we will discus

better in Section 2.3.4.

2.3.3 Plumbing fixture and propagators

In this section, we will make the previous discussion concrete by introducing the so-called plumbing fixture procedure, namely, a prescription for gluing either two punctured Riemann surfaces (separating plumbing fixture) or a Riemann surface to itself (non-separating plumbing fixture). As we will see, these correspond to connecting the respective vertices via the Siegel propagator.

For the separating plumbing fixture: consider two fundamental domains, $\mathcal{V}_{k_1}^{g_1}$ and $\mathcal{V}_{k_2}^{g_1}$, i.e., two marked Riemann surfaces $\Sigma_{k_1}^{g_1}$ and $\Sigma_{k_2}^{g_2}$ equipped with local-coordinate maps $f_a^{(1)}(w_a^{(1)})$ and $f_a^{(2)}(w_a^{(2)})$ at their punctures. Choose one puncture on $\Sigma_{k_1}^{g_1}$ with local coordinate $w_1^{(1)}$ and one on $\Sigma_{k_2}^{g_2}$ with local coordinate $w_1^{(2)}$ then glue the two surfaces using the twisted gluing rule

$$w_1^{(1)} w_1^{(2)} = t e^{i\theta} \coloneqq q\,, \tag{2.193}$$

for $0 \leq \theta \leq 2\pi$ and $0 \leq t \leq 1$. From the view point of the coordinates on the spheres, which share the gluing circle $C_{\hat{s}}$, (2.193) can be written as

$$z^{(1)} = f_1^{(1)} \circ I_q \circ \left(f_1^{(2)}\right)^{-1} (z^{(2)})\,, \tag{2.194}$$

where we defined $I_q(w) = \frac{q}{w}$. Looking at (2.194), we can explicitly see the steps of this gluing procedure, which consist of

1. Mapping the exterior of the local coordinate patch surrounding the puncture in $\Sigma_{k_2}^{g_2}$ into the exterior of the unit disk in the local coordinate frame $w_1^{(2)}$, via $\left(f_1^{(2)}\right)^{-1}$.

2. Mapping the exterior of the unit disk in the $w_1^{(2)}$ frame to the interior of the unit disk in the $w_1^{(1)}$ frame, applying a dilation and a twist parametrized by t and θ, respectively.

3. Mapping the interior of the unit disk in the $w_1^{(1)}$ frame into the interior of the local coordinate patch surrounding the puncture in $\Sigma_{k_1}^{g_1}$, via $f_1^{(1)}$.

These steps are illustrated graphically in Figure 2.6, and yield a Riemann surface $\Sigma^{g_1+g_2}_{k_1+k_2-2}$ characterized by

$$F_{\hat{s}}(z^{(2)}) = f_1^{(1)} \circ I_q \circ \left(f_1^{(2)}\right)^{-1}(z^{(2)})\,, \tag{2.195}$$

as transition function on the gluing circle and local coordinate maps inherited from $\hat{\mathcal{P}}^{g_1}_{k_1}$ and $\hat{\mathcal{P}}^{g_2}_{k_2}$. Consequently, this construction identifies a subregion of the section $\mathcal{S}^{g_1+g_2}_{k_1+k_2-2}$, parametrized by t and θ which contains the IR divergence as $t \to 0$ as this corresponds to two Riemann surfaces connected by a very long tube.

Similarly, in the non-separating scenario, consider a Riemann surface Σ^g_k with two punctures, characterized by local coordinates w_1 and w_2. We then glue them together using the twisted gluing rule in equation (2.193). This procedure yields a subregion of the moduli space $\mathcal{S}^{g+1}_{k-2}$, parametrized by t and θ, which contains the region responsible for the IR divergence, namely, a Riemann surface with a very long handle as $t \to 0$.

In the following, we will prove that equation (2.191), with $\mathcal{F}^g_k$ obtained through the plumbing fixture procedure, yields the same result as the Feynman diagram expansion in the Siegel gauge. Therefore, focusing on the separating plumbing fixture case, the object we aim to compute is

$$F_{11}(\{\Phi^{(1)}\},\{\Phi^{(2)}\}) = \int_{\mathcal{F}_{11}} \Omega(\{\Phi^{(1)}\},\{\Phi^{(2)}\})\,, \tag{2.196}$$

where $\mathcal{F}_{11}$ denotes the subregion of $\mathcal{S}^{g_1+g_2}_{k_1+k_2-2}$ obtained via twist-gluing; $\{\Phi^{(1)}\} \coloneqq \{\Phi^{(1)}_2, \dots \Phi^{(1)}_{k_1}\}$ and $\{\Phi^{(2)}\} \coloneqq \{\Phi^{(2)}_2, \dots \Phi^{(2)}_{k_2}\}$ are the sets of local operators inserted on the first and second surfaces, and we denote $\Omega^{g_1+g_2,\,k_1+k_2-2}_p$ simply as Ω .

Given the transition function in (2.195), the holomorphic component of the Schiffer vectors are

$$v^q_s(z^{(1)};q) = \delta_{s,\hat{s}}\dot{f}^{(1)}_1(w^{(1)}_1(z^{(1)}))\frac{w^{(1)}_1(z^{(1)})}{q}\,, \qquad\qquad v^{\bar{q}}_s = 0\,, \tag{2.197}$$

where $\dot{f}$ denotes the total derivative and $w^{(1)}_1(z^{(1)})$ is a shorthand for $\left(f^{(1)}_1\right)^{-1}(z^{(1)})$.

Substituting this into (2.171) and changing variables as $z^{(1)} = f_1^{(1)}(w)$, we obtain

$$B^q(q) = \frac{1}{q}\oint_0 \frac{dw}{2\pi i}\, w\, \Big(\dot{f}_1^{(1)}(w)\Big)^2\, b\Big(f_1^{(1)}(w)\Big) = f_1^{(1)} \circ \Big[\frac{1}{q} b_0\Big]\,. \tag{2.198}$$

Similarly for the anti-holomorphic sector. Therefore the differential form resulting from the gluing operation becomes

$$B^q(q)dq \wedge B^{\bar{q}}(\bar{q})d\bar{q} = f_1^{(1)} \circ \Big[-\frac{1}{2}(dq \wedge d\bar{q})\frac{b_0^+ b_0^-}{q\bar{q}}\Big]\,. \tag{2.199}$$

By adapting (2.173) to our example, we find

$$\begin{aligned}\Omega(\{\Phi^{(1)}\},\{\Phi^{(2)}\}) = \Big(\frac{i}{2\pi}\Big)^{\frac{1}{2}\left(d_{k_1}^{g_1}+d_{k_2}^{g_2}+2\right)} \\ \Big\langle\Big[\bigwedge_{i=1}^{d_{k_1}^{g_1}} B^{\lambda_i}(y^{(1)})dy^{(1)}_{\lambda_i} f_2^{(1)} \circ \Phi_2^{(1)}(0,\bar{0}) \dots f_{k_1}^{(1)} \circ \Phi_{k_1}^{(1)}(0,\bar{0})\Big] \\ \times f_1^{(1)} \circ \Big[-\frac{1}{2}(dq \wedge d\bar{q})\frac{b_0^+ b_0^-}{q\bar{q}}\Big] \\ \times F_{\hat{s}} \circ \Big[\bigwedge_{j=1}^{d_{k_2}^{g_2}} B^{\lambda_j}(y^{(2)})dy^{(2)}_{\lambda_j} f_2^{(2)} \circ \Phi_2^{(2)}(0,\bar{0}) \dots f_{k_2}^{(2)} \circ \Phi_{k_2}^{(2)}(0,\bar{0})\Big]\Big\rangle_{\hat{\Sigma}_q}\,,\end{aligned} \tag{2.200}$$

where we decomposed the total dimension of the section in terms of the dimensionalities of the moduli spaces of the two glued Riemann surfaces

$$d_{k_1+k_2-2}^{g_1+g_2} = d_{k_1}^{g_1} + d_{k_2}^{g_2} + 2\,. \tag{2.201}$$

We denote by $y^{(1)}_{\lambda_i}$ and $y^{(2)}_{\lambda_i}$ the coordinates of $\mathcal{S}_{k_1}^{g_1}$ and $\mathcal{S}_{k_2}^{g_2}$, respectively, and we call $\hat{\Sigma}_q$ the Riemann surface (depending on q and $\bar{q}$), obtained by the plumbing fixture procedure. Furthermore we use $z^{(1)}$ as global coordinate, this motivates the presence of $F_{\hat{s}}$ in front of the operators defined in the $z^{(2)}$ frame. Now let us apply the conformal transformation $\Big(f_1^{(1)}\Big)^{-1}$ and observe that the map I_q corresponds to the composition of a dilatation (generated by the adjoint action of the Virasoro

modes L_0 and $\bar{L}_0$) and an inversion. Therefore, (2.200) becomes

$$\begin{aligned}\Omega(\{\Phi^{(1)}\},\{\Phi^{(2)}\}) = \left(\frac{i}{2\pi}\right)^{\frac{1}{2}\left(d^{g_1}_{k_1}+d^{g_2}_{k_2}+2\right)} \\ \left\langle \left(f^{(1)}_1\right)^{-1} \circ \Big[\bigwedge_{i=1}^{d^{g_1}_{k_1}} B^{\lambda_i}(y^{(1)}) dy^{(1)}_{\lambda_i} f^{(1)}_2 \circ \Phi^{(1)}_2(0,\bar{0}) \dots f^{(1)}_{k_1} \circ \Phi^{(1)}_{k_1}(0,\bar{0}) \Big] \right. \\ \times \Big[-\frac{1}{2}(dq \wedge d\bar{q}) \frac{b^+_0 b^-_0}{q\bar{q}} q^{L_0} \bar{q}^{\bar{L}_0} \Big] \\ \left. \times I_{\rm c} \circ \left(f^{(2)}_1\right)^{-1} \circ \Big[\bigwedge_{j=1}^{d^{g_2}_{k_2}} B^{\lambda_j}(y^{(2)}) dy^{(2)}_{\lambda_j} f^{(2)}_2 \circ \Phi^{(2)}_2(0,\bar{0}) \dots f^{(2)}_{k_2} \circ \Phi^{(2)}_{k_2}(0,\bar{0}) \Big] \right\rangle_{\hat{\Sigma}'} ,\end{aligned} \tag{2.202}$$

where the correlator is now computed on the Riemann surface $\hat{\Sigma}'$, obtained via the conformal transformation $\frac{1}{q}\left(f^{(1)}_1\right)^{-1}$. The $\frac{1}{q}$ rescaling makes $\hat{\Sigma}'$ independent on q and it is related to the $q^{-L_0}q^{-\bar{L}_0}$ on the right, which together with $q^{L_0}q^{\bar{L}_0}$ on the left implement the plumbing fixture dilatation, we will clarify this point below. Let us now introduce the multi-linear products

$$\begin{aligned}&|\Sigma^g_{k-1}(\Phi_2,\dots\Phi_k)\rangle \coloneqq b^-_0 \delta_{L^-_0,0} \\ &I_{\rm c} \circ f_1^{-1} \circ \Big[\bigwedge_{j=1}^{d^g_k} B^{\lambda_j} dy_{\lambda_j} f_2 \circ \Phi_2(0,\bar{0}) \dots f_k \circ \Phi_k(0,\bar{0}) \Big] \Big| f_1^{-1} \circ \Sigma^g_k \Big\rangle ,\end{aligned} \tag{2.203}$$

where $\left|f_1^{-1} \circ \Sigma^g_k\right\rangle$ is a surface state defined such that

$$\left\langle 0 \middle| (\dots) \middle| f_1^{-1} \circ \Sigma^g_k \right\rangle = \langle (\cdots) \rangle_{f_1^{-1} \circ \Sigma^g_k} , \tag{2.204}$$

where the dots identify generic k closed string insertions. Notice that from the above definition we can deduce that

$$\left|\frac{1}{q} \circ \Sigma^g_k\right\rangle = q^{-L_0} q^{-\bar{L}_0} \left|\Sigma^g_k\right\rangle , \tag{2.205}$$

which motivates our previous statement about the rescaling of the surface.[8] Further-

[8]This result can be generalized for generic conformal transformations. Let us express the adjoint action of the transformation f on the field Φ as $f \circ \Phi = U_f \Phi U_f^{-1}$, where $U_f = \exp\left(\sum_n v_n L_n\right)$

more, we defined the multilinear product by inserting $b_0^- \delta_{L_0^-,0}$ to ensure that they are level matched, $|\Sigma^g_{k-1}(\Phi_2, \dots \Phi_k)\rangle \in \mathcal{H}_{\rm c}$.

Hence, (2.173) can be rewritten as

$$\Omega^{(g,k)}_p(\Phi_1, \dots, \Phi_k) = \left(\frac{i}{2\pi}\right)^{d^g_k} \omega_{\rm c}(\Phi_1, \Sigma^g_{k-1}(\Phi_2, \dots \Phi_k))\,, \tag{2.206}$$

and (2.202) becomes

$$\begin{aligned}\Omega(\{\Phi^{(1)}\}, \{\Phi^{(2)}\}) = &\left(\frac{i}{2\pi}\right)^{\frac{1}{2}\left(d^{g_1}_{k_1}+d^{g_2}_{k_2}+2\right)} \\ &\omega_{\rm c}(\Sigma^{g_1}_{k_1-1}(\{\Phi^{(1)}\}), \Big[-\frac{1}{2}(dq\wedge d\bar{q})\frac{b_0^+}{q\bar{q}} q^{L_0}\bar{q}^{\bar{L}_0}\Big]\Sigma^{g_2}_{k_2-1}(\{\Phi^{(2)}\})) + \Omega'(\{\Phi^{(1)}\}, \{\Phi^{(2)}\})\,.\end{aligned} \tag{2.207}$$

There are two subtleties in deriving the above equation. First, in (2.202) there is no $\delta_{L_0^-,0}$; therefore, to obtain the multi-string product, we must add and subtract it. This introduces an additional term, denoted by $\Omega'(\{\Phi^{(1)}\}, \{\Phi^{(2)}\})$, characterized by the operator in the square brackets and by $1 - \delta_{L_0^-,0}$. Schematically

$$\Omega'(\{\Phi^{(1)}\}, \{\Phi^{(2)}\}) = \omega_{\rm c}((\dots), \Big[-\frac{1}{2}(dq\wedge d\bar{q})\frac{b_0^+}{q\bar{q}} q^{L_0}\bar{q}^{\bar{L}_0}\Big](1-\delta_{L_0^-,0})(\dots))\,. \tag{2.208}$$

Below, we will show that this term vanishes when integrating over the plumbing-fixture parameter θ. Second, we must demonstrate that the resulting overlap corresponds to a correlator on $\hat{\Sigma}'$. Thanks to the state–operator correspondence, we can write

$$\Sigma^{g_2}_{k_2-1}(\Phi^{(2)}_2, \dots, \Phi^{(2)}_{k_2}) = \Sigma^{g_2}_{k_2-1}(\Phi^{(2)}_2, \dots, \Phi^{(2)}_{k_2})(0,\bar{0})\,|0\rangle\ , \tag{2.209}$$

similarly for $\Sigma^{g_1}_{k_1-1}(\Phi^{(1)}_2, \dots, \Phi^{(1)}_{k_1-1})$, and then by definition the BPZ product (2.207) gives a correlator on the sphere with one field inserted at the origin and the other at infinity. This corresponds to map the exterior of the local coordinate patch around $z^{(1)}$ in $\Sigma^{g_1}_{k_1}$ to the exterior of the unit disk surrounding the origin of the sphere, while the exterior of the patch around $z^{(2)}$ in $\Sigma^{g_2}_{k_2}$ is mapped to the interior of the unit disk.

and the coefficients v_n determine the transformation. Therefore, by imposing that $\langle \Phi_1 \dots \Phi_k\rangle_{\Sigma^g_k} = \langle f\circ\Phi_1 \dots f\circ\Phi_k\rangle_{f\circ\Sigma^g_k}$ we obtain $|f\circ\Sigma^g_k\rangle = U_f\,|\Sigma^g_k\rangle$ as transformation rule for surface states.

Hence, this is precisely the setup produced by the plumbing fixture construction in the $w_1^{(1)}$ frame, namely $\hat{\Sigma}'$, as you can see in Figure 2.6 for $q = 1$.

Let us then insert the completeness relation twice into (2.207) in order to isolate the operator in the middle

$$\Omega - \Omega' = \frac{i}{2\pi}\omega_{\rm c}(c^i, \Big[-\frac{1}{2}(dq \wedge d\bar{q})\frac{b_0^+}{q\bar{q}} q^{L_0}\bar{q}^{\bar{L}_0}\Big]c^j) \\ \times \Omega_{p_1}^{(g_1,k_1)}(c_i, \Sigma_{k_1-1}^{g_1}(\{\Phi^{(1)}\}))\Omega_{p_2}^{(g_2,k_2)}(c_j, \Sigma_{k_2-1}^{g_2}(\{\Phi^{(2)}\}))\,, \quad (2.210)$$

where we reconstructed the differential forms using (2.206). We are now ready to compute (2.196), which, thanks to the factorization in (2.210), becomes

$$F_{11}(\{\Phi^{(1)}\}, \{\Phi^{(2)}\}) = \frac{i}{2\pi}\int_{|q|\leq 1} \omega_{\rm c}(c^i, \Big[-\frac{1}{2}(dq \wedge d\bar{q})\frac{b_0^+}{q\bar{q}} q^{L_0}\bar{q}^{\bar{L}_0}\Big]c^j) \\ \times V_{k_1}^{g_1}(c_i, \Sigma_{k_1-1}^{g_1}(\{\Phi^{(1)}\}))V_{k_2}^{g_2}(c_j, \Sigma_{k_2-1}^{g_2}(\{\Phi^{(2)}\}))\,, \quad (2.211)$$

where the last two terms are the fundamental vertices associated with the Riemann surfaces $\Sigma_{k_1}^{g_1}$ and $\Sigma_{k_2}^{g_2}$, respectively. As anticipated, there are no contributions from $\Omega'(\{\Phi^{(1)}\}, \{\Phi^{(2)}\})$, as we will now clarify by explicitly computing the integral over the plumbing fixture moduli. By changing variables according to (2.193), we obtain

$$\int_{|q|\leq 1} -\frac{1}{2}(dq \wedge d\bar{q})\frac{b_0^+}{q\bar{q}} q^{L_0}\bar{q}^{\bar{L}_0} = -\left(\frac{i}{2\pi}\right)^{-1}\int_0^1 \frac{dt}{t}\, t^{L_0^+} b_0^+ \delta_{L_0^-,0}\,, \quad (2.212)$$

where we used the integral representation of the Kronecker delta

$$\int_0^{2\pi} \frac{d\theta}{2\pi} e^{i\theta L_0^-} = \delta_{L_0^-,0}\,, \quad (2.213)$$

which, as expected, annihilates $\Omega'(\{\Phi^{(1)}\}, \{\Phi^{(2)}\})$. Looking at (2.212), one recognizes, up to the numerical prefactor, the unregularized Siegel propagator (2.142). Therefore, by adopting the natural regularization scheme discussed in 2.2.3, we obtain

$$F_{11}(\{\Phi^{(1)}\}, \{\Phi^{(2)}\}) = -\omega_{\rm c}(c^i, h_s^+ c^j)\, V_{k_1}^{g_1}(c_i, \Sigma_{k_1-1}^{g_1}(\{\Phi^{(1)}\}))V_{k_2}^{g_2}(c_j, \Sigma_{k_2-1}^{g_2}(\{\Phi^{(2)}\}))\,, \quad (2.214)$$

which precisely matches the structure of the result from the path-integral approach. Finally, note that the numerical factors from the normalizations and the propagators cancel out. This cancellation motivates our initial normalization choice, which guarantees that

$$\left(\frac{i}{2\pi}\right)^{\frac{1}{2}d^{g_1+g_2}_{k_1+k_2-2}} = \left(\frac{i}{2\pi}\right)^{\frac{1}{2}d^{g_1}_{k_1}} \left(\frac{i}{2\pi}\right)^{\frac{1}{2}d^{g_2}_{k_2}} \left(\frac{i}{2\pi}\right) , \tag{2.215}$$

where the last factor is exactly what is needed to reconstruct the Siegel propagator.

2.3.4 Geometrical BV master equation

In the preceding section, we clarified the geometric interpretation of the Siegel propagator, establishing a systematic procedure to determine $\mathcal{F}^g_k$ by gluing lower-order Riemann surfaces. This approach allows us to rewrite (2.192) in terms of fundamental domains only, yielding the BV geometrical master equation. By construction, it is clear that imposing (2.192) is equivalent to requiring that, at a given order in the topological expansion, the boundary of the corresponding fundamental domain $\partial\mathcal{V}^g_k$ must coincide with the boundary of the Feynman regions, namely the subregion of $\mathcal{S}^g_k$ obtained applying the plumbing fixture procedure with $t = 1$

$$w_1 w_2 = e^{i\theta} , \tag{2.216}$$

In the example of the four-point amplitude on the sphere, this region corresponds to the three circles surrounding the insertions (see Figure 2.5), generated by applying the separating plumbing-fixture procedure to two spheres with three insertions, glued in all possible ways, corresponding to the three possible channels. It follows that this construction leads to recursive constraints on the vertices. Concretely, we begin with the three-point function on the sphere, which has a zero-dimensional moduli space. The corresponding vertex is fixed once we specify the three local coordinate maps. This, completely determines the three circles mentioned above, thus constraining the construction of the quartic vertex. To make this reasoning more concrete, we define two operators acting on the space of decorated Riemann surfaces

1. Closed geometrical BV bracket $(\mathcal{V}^{g_1}_{k_1}, \mathcal{V}^{g_2}_{k_2})_{\rm c}$: Given two fundamental domains $\mathcal{V}^{g_1}_{k_1}$ and $\mathcal{V}^{g_2}_{k_2}$, this is the subregion of $\hat{\mathcal{P}}^{g_1+g_2}_{k_1+k_2-2}$ obtained by applying the separating plumbing fixture (2.216) to all inequivalent pairs of punctures, one on each surface.

2. Closed geometrical BV Laplacian $\Delta_{\rm c}\mathcal{V}^g_k$: Given a fundamental domain $\mathcal{V}^g_k$, this is the subregion of $\hat{\mathcal{P}}^{g+1}_{k-2}$ obtained by gluing together two punctures on Σ^g_k via (2.216), in all possible ways.

Therefore, the consistency condition (2.192) can be expressed as

$$\partial\mathcal{V}^g_k + \Delta_{\rm c}\mathcal{V}^{g-2}_{k+2} + \frac{1}{2}\sum_{\substack{g_1+g_2=g\\ k_1+k_2=k+2}} (\mathcal{V}^{g_2}_{k_1}, \mathcal{V}^{g_2}_{k_2})_{\rm c} = 0\,, \tag{2.217}$$

subject to the restrictions $(k \geq 3, g = 0)$ or $(k \geq 1, g = 1)$ or $(k \geq 0, g \geq 2)$. The case $(k \geq 3, g = 0)$ arises because for $k < 3$, the corresponding decorated Riemann-surface space does not exist. The case $(k \geq 1, g = 1)$ is subtler. Although $\hat{\mathcal{P}}^1_0$ can be consistently defined, the fundamental domain $\mathcal{V}^1_0$ does not exist. That is because the boundary region of the moduli space cannot be constructed via the plumbing-fixture procedure, in particular, we would need $\mathcal{V}^{09}_2$[9]. Furthermore, the BV geometrical master equation (2.217), can be rewritten in a more compact form as

$$\partial\mathcal{V} + \Delta_{\rm c}\mathcal{V} + \frac{1}{2}(\mathcal{V}, \mathcal{V})_{\rm c} = 0\,, \tag{2.218}$$

where we defined the total fundamental domains as

$$\mathcal{V} \coloneqq \sum_{g,k} \kappa^{-\chi^g_k}\mathcal{V}^g_k\,. \tag{2.219}$$

We use the same notation for these geometric operators acting on decorated Riemann surfaces and for the BV differential structure because they are closely related. To

[9]For more details on this and about how to impose constraints on the corresponding vertex V^2_0 see Section 6 of [20]

appreciate this connection, we define the linear functional

$$f(\mathcal{V}_k^g) \coloneqq \frac{1}{k!}\int_{\mathcal{V}_k^g} \Omega_p^{(g,k)}(\Phi^{\wedge k})\,. \tag{2.220}$$

One can show that the following relations hold

$$\Big(f(\mathcal{V}_{k_1}^{g_1}), f(\mathcal{V}_{k_2}^{g_2})\Big)_{\mathrm{c}} = -f\Big((\mathcal{V}_{k_1}^{g_1}, \mathcal{V}_{k_2}^{g_2})_{\mathrm{c}}\Big)\,, \tag{2.221a}$$

$$\Big(V_2^0, f(\mathcal{V}_{k_1}^{g_1})\Big)_{\mathrm{c}} = -2f(\partial\mathcal{V}_k^g)\,, \tag{2.221b}$$

$$\Delta_{\mathrm{c}} f(\mathcal{V}_k^g) = -f(\Delta_{\mathrm{c}}\mathcal{V}_k^g)\,. \tag{2.221c}$$

As explained in [20], these identities can be derived by explicitly applying the BV differential structure and using the surface-state formalism introduced earlier to rewrite the resulting expression as an integral over a new domain. Furthermore, to obtain (2.221b) it crucial to use (2.182). Therefore, noticing that the action is given by

$$S_{\mathrm{c}}(\Phi) = \frac{1}{2}V_2^0(\Phi^{\wedge 2}) + f(\mathcal{V}) + V_0^1\,. \tag{2.222}$$

and using (2.221), we obtain

$$\frac{1}{2}(S,S) + \Delta S = -f\Big(\partial\mathcal{V} + \Delta_{\mathrm{c}}\mathcal{V} + \frac{1}{2}(\mathcal{V},\mathcal{V})_{\mathrm{c}}\Big)\,, \tag{2.223}$$

which shows that the geometric BV master equation implies the BV master equation (2.123). Notice that (2.218) is a stricter condition: the quantum master equation alone (2.123) does not constrain the constant terms of the action. The vertices $V_0^{g\geq 2}$ are constrained only via (2.218). Indeed,

$$f(\partial\mathcal{V}_0^{g\geq 2}) = -\frac{1}{2}\Big(V_2^0, f(\mathcal{V}_0^{g\geq 2})\Big)_{\mathrm{c}} = 0\,. \tag{2.224}$$

This is expected, since adding a constant term to an action with a well-defined path integral does not spoil its quantum consistency, it merely affects the normalization.

In the following, we will focus only on the dynamical part, and thus we must only ensure that the BV master action satisfies the BV quantum master equation (2.123).

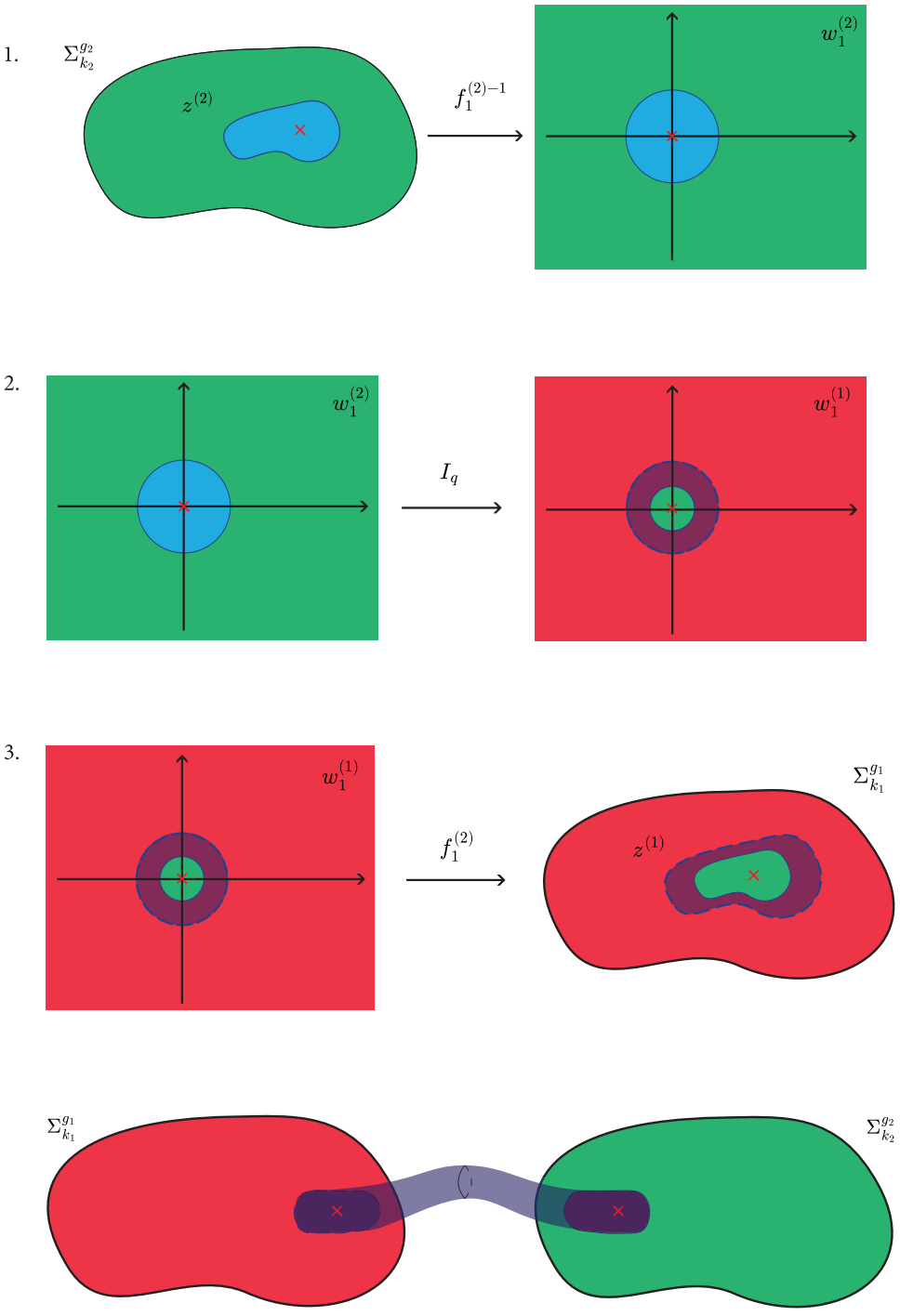


Figure 2.6: Step-by-step procedure for the separating plumbing fixture, as explained in Section 2.3.3.

CHAPTER 3

Boundary Modes in String Field Theory

3.1 Boundary terms in QFT and SFT

In this chapter, we review the main results of our works presented in [29, 30], where a complete gauge-invariant construction of the free action for open and closed SFT is proposed in a target space $M \subset \mathbb{R}^d$, having a boundary ∂M.

The reason we are interested in extending the SFT action (2.81) to this kind of framework is that there are indications that the usual formulation may be incomplete. For example, in [66], it was shown that classical solutions in closed SFT have a vanishing action. However, there exist interesting solutions that require boundary terms, as reported in [48], where the authors perturbatively constructed a closed SFT solution using a non-normalizable linear-dilaton profile. Naively plugging this solution back into the boundary-term-free action yields a non-vanishing result, whose physical interpretation is intriguingly explored from the CFT point of view in [67]. To fully address this issue, one likely needs to explicitly include boundary terms in the action.

Before focusing on the SFT scenario, let us first discuss how we can define a generic QFT on a manifold $M \subset \mathbb{R}^d$ with boundary ∂M. If we naively approach this problem by simply replacing the domain of integration

$$\int_{\mathbb{R}^d} \to \int_M , \tag{3.1}$$

then the resulting action can suffer from two serious deficiencies:

1. it does not give a well-defined variational principle,
2. it is no longer gauge-invariant.

To better understand the origin of these problems, and to obtain hints about how to resolve them in the context of SFT, in the following we will consider three instructive examples:

1. Free boson.
2. Abelian Chern-Simons theory.
3. General relativity (GR).

In particular, we will see that the first example is affected only by the first problem, the second only by the second, and GR by both.

Let us first consider a free boson in a d-dimensional Euclidean space. Assuming that the field is well behaved at infinity, the action can be written in two equivalent ways by integrating by parts

$$S(\phi) = \frac{1}{2}\int_{\mathbb{R}^d} d^d x\, \phi(-\partial_\mu \partial^\mu)\phi = \frac{1}{2}\int_{\mathbb{R}^d} d^d x\, \partial_\mu \phi \partial^\mu \phi\,. \tag{3.2}$$

When we define this theory on a manifold M with boundary ∂M, the two expressions above are no longer equivalent, and thus we obtain two different actions

$$S(\phi) = \frac{1}{2}\int_{M} d^d x\, \phi(-\partial_\mu \partial^\mu)\phi\,, \tag{3.3}$$

and

$$S'(\phi) = \frac{1}{2}\int_{M} d^d x\, \partial_\mu \phi \partial^\mu \phi\,, \tag{3.4}$$

which differ by a boundary term

$$S'(\phi) = S(\phi) + \frac{1}{2}\int_{\partial M} d^{d-1} y\, \sqrt{\gamma}\, n_\mu (\phi \partial^\mu \phi)\,, \tag{3.5}$$

where γ is the determinant of the induced metric and n^μ is the normal unit vector of ∂M, pointing outside M. This can be easily seen by integrating by parts in (3.4) using the divergence theorem

$$\int_M d^d x\, \partial_\mu f^\mu(x) = \int_{\partial M} d^{d-1} y\, \sqrt{\gamma}\, n_\mu(y) f^\mu(x(y))\,, \tag{3.6}$$

where $x(y)$ describes the embedding of ∂M into $M \subset \mathbb{R}^d$. The actions (3.3) and (3.4) look very similar, but when considering the variational principle we notice two fundamentally different behaviours. Let us begin by studying the variation of (3.4)

$$\delta S'(\phi) = \int_M d^d x\, \delta\phi\, (-\partial_\mu \partial^\mu \phi) + \int_{\partial M} d^{d-1} y\, \sqrt{\gamma}\, n_\mu\, \delta\phi\, \partial^\mu \phi\,. \tag{3.7}$$

The bulk part of the variation gives the standard wave equation $\partial^2\phi = 0$, and the boundary contribution can be set to zero by Dirichlet

$$\delta\phi\Big|_{\partial M} = 0\,,$$

or Neumann boundary conditions

$$n_\mu \partial^\mu\phi\Big|_{\partial M} = 0\,,$$

which are consistent boundary conditions for the wave equation. This implies that (3.4) has a well-defined variational principle. The situation is different for (3.3), whose variation yields

$$\begin{aligned}\delta S(\phi) =& -\frac{1}{2}\int_M d^dx\ (\delta\phi(-\partial_\mu\partial^\mu)\phi + \phi(-\partial_\mu\partial^\mu)\delta\phi)\\ =& \int_M d^dx\,\delta\phi\,(-\partial_\mu\partial^\mu\phi) +\\ &-\frac{1}{2}\int_{\partial M} d^{d-1}y\,\sqrt{\gamma}\,n_\mu\,\delta\phi\,\partial^\mu\phi + \frac{1}{2}\int_{\partial M} d^{d-1}y\,\sqrt{\gamma}\,n_\mu\phi\,\delta\partial^\mu\phi\,. \end{aligned} \tag{3.8}$$

Here we notice the presence of an additional boundary contribution that cannot be set to zero without over-constraining the space of solutions. Thus, in this case the variational principle is ill-defined. We can therefore understand a general property: the origin of this extra boundary term lies in the presence of a second derivative in the transverse direction in the action. To obtain a well-posed variational principle, we must add a boundary term implementing the integration by parts of one ∂_μ from the kinetic operator $\partial_\mu\partial^\mu$, thereby producing an action involving only first derivatives

$$S(\phi) \to S'(\phi) = S(\phi) + S_{\partial M}(\phi)\,. \tag{3.9}$$

In the case of the free boson, this boundary term is

$$S_{\partial M}(\phi) = \frac{1}{2}\int_{\partial M} d^{d-1}y\,\sqrt{\gamma}\,n_\mu(\phi\partial^\mu\phi)\,, \tag{3.10}$$

as can be directly seen in (3.5).

As we will discuss in the body of this chapter, an analogous boundary contribution can be added in various ways to the naive SFT action in order to render the variational principle well defined.

The story becomes more involved when we consider theories in which gauge invariance is spoiled. In such cases, a simple prescription like the one just described is generally not sufficient to restore it. Indeed, the loss of gauge invariance is not fundamentally related to the presence of second-order derivatives; rather, it is a direct consequence of the presence of a boundary. To obtain a first glimpse of this issue (and, at the same time, a crucial hint towards its resolution), let us consider $3d$ free abelian Chern–Simons theory

$$S_{\text{CS}}(A) = \frac{1}{2}\int_{\mathbb{R}^3} A \wedge dA\,. \tag{3.11}$$

Assuming that the gauge connection A goes to zero fast enough at infinity, (3.11) is trivially invariant under the gauge transformation

$$\delta_g A = d\xi\,, \tag{3.12}$$

but if we define this theory on a 3-manifold with boundary by simply replacing the domain of integration

$$S_{\text{CS}}(A) = \frac{1}{2}\int_{M} A \wedge dA\,, \tag{3.13}$$

then it is no longer invariant under the gauge transformation (3.12)

$$\delta_g S_{\text{CS}}(A) = \frac{1}{2}\int_M d\xi \wedge dA = \frac{1}{2}\int_M d\left(\xi \wedge dA\right) = \frac{1}{2}\int_{\partial M} \xi \wedge dA \neq 0\,. \tag{3.14}$$

Since gauge transformations are merely redundancies in our description, they cannot be broken by the introduction of a boundary. In other words, if we are to consistently add a boundary to a gauge theory, there must exist a way to do so while preserving gauge invariance. In this case, the natural solution is to introduce a new degree of freedom living on the boundary, equipped with a gauge transformation law that

compensates for the bulk non-invariance.[1]

In particular, we introduce a scalar boundary field m with action

$$S_{\partial M}(A,m) = -\frac{1}{2}\int_{\partial M} m \wedge dA\,, \tag{3.15}$$

and gauge transformation

$$\delta_g m = \xi\,, \tag{3.16}$$

so that a fully gauge-invariant action for abelian Chern–Simons theory with boundary can be written as

$$S'_{\mathrm{CS}}(A,m) = S_{\mathrm{CS}}(A) + S_{\partial M}(A,m) = \frac{1}{2}\int_M A\wedge dA - \frac{1}{2}\int_{\partial M} m\wedge dA\,. \tag{3.17}$$

The original action (3.13) can then be recovered from (3.17) by gauging the field m away. Doing so completely fixes the gauge parameter at the boundary (as follows from (3.16)), meaning that (3.13) can be regarded as a partially gauge-fixed action, which is invariant under (3.12) provided that ξ is restricted to vanish on ∂M.

The last interesting classical example that we want to discuss before moving to SFT is GR. It is known that to achieve a well-defined variational principle we must add the Gibbons–Hawking–York (GHY) boundary term to the usual Einstein–Hilbert action [71, 72]

$$S_{\mathrm{GR}} = S_{\mathrm{EH}} + S_{\mathrm{GHY}} = \int_M d^dx\,\sqrt{g}\,R + \int_{\partial M} d^{d-1}y\,\sqrt{\gamma}\,2K\,, \tag{3.18}$$

where K is the extrinsic scalar curvature of the boundary. As we will review in Appendix A, the variation of this action reads

$$\delta S_{\mathrm{GR}} = \int_M d^dx\,\sqrt{g}\,\left(\tfrac{1}{2}Rg^{\alpha\beta} - R^{\alpha\beta}\right)\delta g_{\alpha\beta} + \int_{\partial M} d^{d-1}y\,\sqrt{\gamma}\,\left(K\gamma^{\alpha\beta} - K^{\alpha\beta}\right)\delta g_{\alpha\beta}\,, \tag{3.19}$$

[1] The introduction of this additional boundary degree of freedom gives rise to physical excitations propagating on the boundary called edge modes [68, 69] (see [70] for a review).

where we can recognize the well-known Einstein field equation in the bulk and an additional contribution at the boundary, which can be set to zero by imposing Dirichlet

$$\delta g_{\alpha\beta}\Big|_{\partial M} = 0\,, \tag{3.20}$$

or generalized Neumann

$$(K\gamma^{\alpha\beta} - K^{\alpha\beta})\Big|_{\partial M} = 0 \tag{3.21}$$

boundary conditions, thereby making the variational principle well defined.

Clearly, (3.18) is no longer invariant under infinitesimal diffeomorphism gauge transformations $\delta_g x^\mu = -\tau^\mu$ due to the presence of the boundary. For concreteness, let us assume flat space with flat boundary located at $z = 0$. This hypersurface can be defined through the implicit equation

$$F(x^\mu) = z = 0\,, \tag{3.22}$$

which is trivially not invariant under bulk diffeomorphisms transverse to the boundary. In this case, the strategy to restore gauge invariance is to consider a fluctuating boundary,

$$F(x^\mu) = z + \frac{1}{2}l(y^a)\,, \tag{3.23}$$

i.e. introducing a boundary degree of freedom, associated with the transverse displacement of the boundary, in addition to fluctuations of the bulk metric. This degree of freedom is equipped with a gauge transformation such that (3.23) is gauge invariant

$$\delta_g l(y^a) = 2\tau_z\,. \tag{3.24}$$

In Appendix A we explicitly show that, under these assumptions, the action obtained by expanding (3.18) around a flat boundary in flat space up to quadratic order is

gauge invariant.

From this perspective, fixing the boundary to be rigid corresponds to gauging away l, yielding a partially gauge-fixed action whose residual gauge symmetry is given by diffeomorphisms that preserve the boundary. In other words, full diffeomorphism invariance is realized only if the boundary is allowed to fluctuate, with the field l playing a role analogous to the field m in the Chern–Simons action (3.17).

Since gravity generally constitutes a subsector of ST, we should expect that analogous boundary degrees of freedom are necessary to consistently define SFT on a target space with a boundary.

To summarize, when defined on a target space with a boundary, we expect that the SFT action (2.126) yields an ill-defined variational principle and ceases to be gauge invariant. The goal of the remainder of this chapter is to describe in detail how to resolve these issues. In particular, we will address these two problems following the logic of the first two examples. Similarly to the free boson, we will add a boundary term with the aim of obtaining a well-defined variational principle, guided by the requirement that the kinetic term should involve only first derivatives in the transverse direction. We will then observe that the resulting action is no longer gauge invariant, and we will restore gauge invariance analogously to the Chern–Simons example by introducing a boundary degree of freedom living at the boundary, equipped with an appropriate action and gauge transformation. Consistently, we will see that the final gauge-invariant action reproduces GR plus the GHY term with a fluctuating boundary when evaluated on the closed-string massless sector.

3.2 Position-space approach to SFT

In this section, we introduce a general formalism applicable in the case of a free SFT on a flat non-compact d-dimensional target space. Adding a codimension-one boundary, we will demonstrate the failure of the naive SFT kinetic term to provide a well-defined gauge-invariant action for free target fields. We will always have in mind the euclidean signature $\eta_{\mu\nu} = \mathrm{diag}[+, \ldots, +]$ for all non-compact directions, to avoid subtleties associated with time-like and null boundaries,.

3.2.1 Notation and conventions

String fields in position space

A generic string field in the aforementioned setup can be expanded in the basis of the momentum eigenstates as

$$|\Psi\rangle = \sum_s \int_{\mathbb{R}^d} \frac{d^d k}{(2\pi)^d} \tilde{\Psi}_s(k) \alpha_s |k\rangle \,, \tag{3.25}$$

where the operators α_s collectively denote matter and ghost oscillators such that $\alpha_s \, |k\rangle \neq 0$. Hence, denoting by $\Psi_s(x)$ the Fourier transform

$$\Psi_s(x) = \int_{\mathbb{R}^d} \frac{d^d k}{(2\pi)^d} \, \tilde{\Psi}_s(k) \, e^{ik\cdot x} \,, \tag{3.26}$$

of the momentum-space expansion coefficients $\tilde{\Psi}_s(k)$, the string field (3.25) can be recast as a function of the zero modes $\hat{x}$ as

$$|\Psi\rangle = \Psi(\hat{x})|0\rangle \,, \tag{3.27}$$

where

$$\Psi(\hat{x}) = \sum_s \Psi_s(\hat{x}) \, \alpha_s \,. \tag{3.28}$$

BPZ inner product

On this space of string fields, let us consider the BPZ inner product

$$\langle \Phi | \Psi \rangle = \langle 0 | \Phi^*(\hat{x}) \Psi(\hat{x}) | 0 \rangle \,, \tag{3.29}$$

where * denotes the BPZ conjugation. This product can be evaluated in terms of the target fields $\Psi_s(x)$ by inserting the completeness relation

$$\int_{\mathbb{R}^d} d^d x \; |x\rangle \, \langle x| = \mathbb{I}$$

and noting that we can write

$$\langle x|\,0\rangle = \lim_{k\to 0}\langle x|\,k\rangle = \lim_{k\to 0} e^{ik\cdot x} = 1\,.$$

One then obtains

$$\langle\Phi|\Psi\rangle = \sum_{s,s'}\int_{\mathbb{R}^d} d^dx\, W_{ss'}\,\Phi_s(x)\Psi_{s'}(x)\,, \tag{3.30}$$

with the matrix elements $W_{ss'} \equiv \langle 0|\alpha_s^*\alpha_{s'}|0\rangle$.

Primed symplectic form

In the following, we will find it very convenient to make explicit the target integration over x by writing

$$\omega(\Phi,\Psi) = \int_{\mathbb{R}^d} d^dx\,\omega'(\Phi(x),\Psi(x))\,, \tag{3.31}$$

where ω' is the obvious reduced symplectic form for the states in which the position zero-mode operators $\hat{x}$ were replaced by their eigenvalues x

$$\Psi(\hat{x}) \longrightarrow \Psi(x)\,. \tag{3.32}$$

More explicitly, given that we decompose

$$\Psi(x) = \sum_s \Psi_s(x)\,\alpha_s\,,$$

(similarly as in (3.28) but now with all fields evaluated at x, not at the operator $\hat{x}$), we can define the primed symplectic form as

$$\omega'(\Phi(x),\Psi(x)) \equiv (-1)^{d_\Phi}\langle 0|\Phi^*(x)\#\Psi(x)|0\rangle = (-1)^{d_\Phi}\sum_{s,s'}\omega_{ss'}\,\Phi_s(x)\Psi_{s'}(x)\,.$$

with the matrix elements

$$\omega_{ss'} \equiv \begin{cases} \langle 0|\alpha_s^* c_0^- \alpha_{s'}|0\rangle & \text{closed string}\,, \\ \langle 0|\alpha_s^* \alpha_{s'}|0\rangle & \text{open string}\,. \end{cases} \tag{3.33}$$

3.2.2 Case with no boundaries

Let us start by writing down the free action in the flat space $\mathbb{R}^d$.

We will be assuming suitable fall-off conditions for the target fields $\Psi_s(x)$ so that one can integrate by parts without picking up any boundary contributions at infinity. In particular, for any field configurations $\Psi_s(x), \Psi_{s'}(x)$ (as well as for all their derivatives), we will assume that

$$\int_{\mathbb{R}^d} d^d x\, \Psi_s(x)\, \partial_\mu \Psi_{s'}(x) = -\int_{\mathbb{R}^d} d^d x\, (\partial_\mu \Psi_s(x)) \Psi_{s'}(x)\,. \tag{3.34}$$

Eventually, we will be interested in defining a consistent field theory on a domain $M \subset \mathbb{R}^d$, where we would like to allow the fields $\Psi_s(x)$ to take generic non-zero values on ∂M. To this end, we will assume that the fall-off conditions which guarantee the validity of (3.34) do not generally limit possible field configurations in the parts of M and its boundary ∂M which stay away from infinity.

The free action for $\Psi_s(x)$ then reads

$$S_0(\Psi) = \frac{1}{2}\omega(\Psi, Q\Psi) = \frac{1}{2}\int_{\mathbb{R}^d} d^d x\, \omega'(\Psi(x), Q\Psi(x))\,, \tag{3.35}$$

Inside the primed symplectic form, the BRST charge can be thought of as acting in the position representation and will be assumed to take the generic form

$$Q = -\alpha' c\, \partial_\mu \partial^\mu - \Omega^\mu \partial_\mu + Q'\,, \tag{3.36}$$

as can be seen by expanding (2.27). The decomposition (3.36) is defined so that Ω^μ and Q' contain neither x^μ nor the target derivatives ∂_μ. In particular, c denotes a c-ghost zero-mode[2], while Ω^μ and Q' are certain linear combinations of matter and

[2] c_0 for OSFT, $\frac{1}{2}c_0^+ = \frac{1}{4}(c_0 + \bar{c}_0)$ for CSFT

ghost oscillators with ghost-number one, for which the nilpotency of Q_{B} implies the algebra

$$c^2 = 0\,, \tag{3.37a}$$

$$[c, \Omega^\mu] = 0\,, \tag{3.37b}$$

$$-\alpha' \eta^{\mu\nu}[c, Q'] + \tfrac{1}{2}[\Omega^\mu, \Omega^\nu] = 0\,, \tag{3.37c}$$

$$[Q', \Omega^\mu] = 0\,, \tag{3.37d}$$

$$(Q')^2 = 0\,. \tag{3.37e}$$

Furthermore, we have the BPZ properties $c^* = -c$, $\Omega_\mu^* = +\Omega_\mu$, as well as $(Q')^* = -Q'$. As a consequence of these, the BRST charge (3.36) is a BPZ-odd operator, that is

$$Q^* = -Q\,. \tag{3.38}$$

However, since Q contains derivatives with respect to target coordinates, the cyclicity of Q with respect to ω crucially depends on the ability to integrate by parts without getting contributions at infinity. Indeed, it is only after making use of the fall-off conditions (3.34) of $\Psi_s(x)$ at infinity that the property (3.38) guarantees that Q is cyclic with respect to ω, namely

$$\omega(\Psi_1, Q\Psi_2) = -(-1)^{d(\Psi_1)}\omega(Q\Psi_1, \Psi_2)\,. \tag{3.39}$$

A generic variation of the action (3.35) is therefore given by

$$\delta S_0 = \omega(\delta\Psi, Q\Psi) = \int_{\mathbb{R}^d} d^d x\, \omega'(\delta\Psi(x), Q\Psi(x))\,, \tag{3.40}$$

with no extra boundary terms. This implies both the well-definedness of the variational principle given by the action (3.35), as well as its invariance under the gauge transformation $\delta_\Lambda \Psi = Q\Lambda$.

3.2.3 Introducing a boundary

We would now like to write down a free action for the target fields $\Psi_s(x)$ on a domain $M \subset \mathbb{R}^d$ with a codimension-one boundary ∂M where the fields $\Psi_s(x)$ are not constrained by any fall-off conditions at ∂M. In particular, we will no longer assume that integration by parts can be performed inside M without picking up contributions localized at ∂M. On the other hand, for the sake of simplicity, we will focus on configurations, where integrating by parts over ∂M produces no additional (corner) terms. This can be ensured either by assuming that $\partial(\partial M) = \emptyset$ or by adopting suitable fall-off conditions towards the boundaries of ∂M (corners of M).

Boundary operator

A naive candidate for such an action would be obtained by replacing

$$\omega \longrightarrow \omega_M \,,$$

in the action (3.35), where the symplectic form ω_M can be defined in terms of ω with an extra insertion of the indicator function Θ pointing on the inside of M, evaluated at $\hat{x}$. In particular, the distribution $\Theta(x)$ is defined to restrict integrals over $\mathbb{R}^d$ to those over M, namely

$$\int_{\mathbb{R}^d} d^d x\, \Theta(x)\, f(x) = \int_M d^d x\, f(x) \tag{3.41}$$

for any function f on $\mathbb{R}^d$. The symplectic form ω_M can then be expressed as

$$\omega_M(\Phi, \Psi) \equiv \omega(\Phi, \Theta(\hat{x})\Psi) = \int_M d^d x\, \omega'(\Phi(x), \Psi(x))\,. \tag{3.42}$$

The failure of the ability to integrate by parts on M without obtaining extra boundary terms is now reflected in Q being no-longer cyclic with respect to ω_M. Equivalently, one can say that the operator ΘQ is not BPZ-odd because $(\Theta Q)^* = -Q\Theta \neq -\Theta Q$ and therefore not cyclic with respect to ω. This failure of cyclicity can be parametrized by

the relation

$$\omega_M(\Psi_1, Q\Psi_2) = -(-1)^{d(\Psi_1)}\omega_M(Q\Psi_1, \Psi_2) + \omega(\Psi_1, B\Psi_2)\,, \tag{3.43}$$

where we have introduced the boundary operator[3]

$$B \equiv [\Theta, Q]\,. \tag{3.44}$$

Note that such B is BPZ even, namely $B^* = B$, and cyclic with respect to ω. Moreover, when acting on the oscillator Hilbert space inside the primed symplectic form, B has the effect of localizing at the boundary

$$B\,\psi = c\big[\delta_\mu(x)\partial^\mu\psi + \partial^\mu(\delta_\mu(x)\,\psi)\big] + \Omega^\mu\delta_\mu\,\psi\,, \tag{3.45}$$

where we have defined

$$\delta_\mu(x) \equiv \partial_\mu\Theta(x) = -n_\mu(y)\,\delta(x)\,, \tag{3.46}$$

with $n_\mu(y)$ being the globally defined normal unit vector of the boundary ∂M, pointing outside M. Where y^a denote coordinates on the manifold ∂M, which is now understood as being described by the embedding $x^\mu = x^\mu(y^a)$, and the $\delta = \delta(x)$ distribution-like operator is the Dirac delta localizing at the boundary ∂M

$$\omega_{\partial M}(\psi_1, \psi_2) = \omega(\psi_1, \delta\,\psi_2) \equiv \int_{\partial M} d^{D-1}y\,\sqrt{\gamma}\,\omega'(\psi_1(x(y)), \psi_2(x(y)))\,, \tag{3.47}$$

where we have introduced the induced metric on the boundary

$$\gamma_{ab} \equiv \partial_a x^\mu(y)\,\partial_b x^\nu(y)\,\eta_{\mu\nu}\,, \tag{3.48a}$$

$$\gamma \equiv \det(\gamma_{ab})\,. \tag{3.48b}$$

[3] A similar expression (replacing $\hat{x}$ with the full oscillator expansion of $X(z,\bar{z})$ inserted at the open-string midpoint) appeared in [73] and [74] in the context of the construction of a symplectic structure of OSFT. See also [75]. The construction of a phase–space symplectic structure in OSFT was recently revisited in [76], which allowed the authors to compute the energy of the rolling tachyon solution [77] and the lump solution [78] in Witten SFT.

Naive action

It follows that if we were to consider the action of the form

$$S_0(\Psi) = \frac{1}{2}\omega_M(\Psi, Q\Psi) = \frac{1}{2}\int_M d^d x\, \omega'(\Psi(x), Q\Psi(x))\,, \tag{3.49}$$

the variation of S_0 with respect to Ψ would have given

$$\begin{aligned}\delta S_0 &= \omega_M(\delta\Psi, Q\Psi) - \frac{1}{2}\omega(\delta\Psi, B\Psi) \\ &= \int_M d^d x\, \omega'(\delta\Psi(x), Q\Psi(x)) - \frac{1}{2}\int_{\partial M} d^{d-1}y\, \sqrt{\gamma}\, n_\mu(y)\, \omega'(\Psi(y), \alpha' c\partial^\mu \delta\Psi(y)) + \\ &\qquad + \frac{1}{2}\int_{\partial M} d^{d-1}y\, \sqrt{\gamma}\, n_\mu(y)\, \omega'(\delta\Psi(y), (\alpha' c\partial^\mu + \Omega^\mu)\Psi(y))\,,\end{aligned} \tag{3.50a}$$

where we have used the shortcut $\Psi(y^a) \equiv \Psi(x^\mu(y^a))$. The second term in the variation can be made vanishing by imposing Dirichlet boundary condition

$$\delta\Psi(x)\big|_{\partial M} = 0\,, \tag{3.51}$$

or the Neumann boundary condition

$$n_\mu(\alpha' c\partial^\mu + \Omega^\mu)\Psi(x)\big|_{\partial M} = 0\,. \tag{3.52}$$

However, forcing the first term to be zero (after having fixed Neumann or Dirichlet conditions) would generally over-constrain the system. The variational principle for the field theory on M given by the action (3.49) is therefore ill-defined. In addition to this deficiency, specializing to the gauge variation $\delta_\Lambda\Psi = Q\Lambda$, we find that

$$\delta_\Lambda S_0 = -\frac{1}{2}\omega(\Lambda, QB\Psi)\,. \tag{3.53}$$

Since the product QB is generally non-vanishing, we conclude that for general gauge parameter Λ, the r.h.s. of (3.53) will be non-zero. This leaves us with the conclusion that the action S_0 can, at the very best, be invariant under a restricted class of bulk gauge transformations which are given by such Λ for which the r.h.s. of (3.53)

vanishes for all Ψ.

It will be the subject of the remainder of this chapter to attempt to fix both the well-definedness of the variational principle given by S_0 as well as its gauge-invariance by adding suitable boundary terms. That is, we will seek a total action

$$S = S_0 + S_{\partial M}\,, \tag{3.54}$$

where the terms $S_{\partial M}$ localize on ∂M, such that the action S

1. gives a well-defined variational principle,
2. is invariant w.r.t. the gauge transformation $\delta_\Lambda \Psi = Q\Lambda$ of the target fields $\Psi_s(x)$,
3. reduces to S_0 upon assuming suitable fall-off conditions on $\Psi_s(x)$ towards ∂M.

3.3 Construction of a consistent gauge-invariant quadratic action

To construct a gauge-invariant action for a free string which constitutes a well-posed variational principle, we proceed in three steps.

3.3.1 Bulk action

To start with, we write an action where only first-order ∂_μ-derivatives are present, so that the variational principle is well-defined. This action is given by

$$S_{\rm cyc}(\Psi) = \frac{1}{2}\omega(\Psi, (\Theta Q - \delta\Gamma^*)\Psi)\,. \tag{3.55}$$

The first-order differential operator Γ^* is defined through the identification

$$B \equiv \delta\Gamma^* + \Gamma\delta\,, \tag{3.56}$$

which, according to (3.45) gives

$$\Gamma^* = -c\, n^\mu(y)\, \partial_\mu - \gamma^* = c\partial_\perp - \gamma^*, \tag{3.57}$$

where we have defined the normal derivative $\partial_\perp \equiv -n^\mu(y)\,\partial_\mu$, which points to the interior of M, and where γ^* is any operator which is free of normal derivatives (so that it commutes with δ) and satisfies

$$\gamma^* + \gamma = \Omega^\mu n_\mu(y)\,. \tag{3.58}$$

Among all possible choices, one option is $\gamma_{\rm simple}$, defined such that

$$\gamma^*_{\rm simple} = \gamma_{\rm simple}\,, \tag{3.59}$$

which effectively reproduces the results of [79] and [80].

Notice that the full kinetic operator is now cyclic and BPZ odd

$$(\Theta Q - \delta\Gamma^*)^* = Q^*\Theta - \Gamma\delta = -Q\Theta - \Gamma\delta = -\Theta Q + B - \Gamma\delta = -(\Theta Q - \delta\Gamma^*)\,. \tag{3.60}$$

Explicitly we find

$$S_{\rm cyc}(\Psi) = \frac{1}{2}\int_M d^dx\,\omega'(\partial_\mu\Psi(x), c\partial^\mu\Psi(x)) + \frac{1}{2}\int_M d^dx\,\omega'(\Psi(x), (Q' - \Omega^\mu\partial_\mu)\Psi(x)) + \\ + \frac{1}{2}\int_{\partial M} d^{d-1}y\,\sqrt{\gamma}\,\omega'(\Psi(y), \gamma^*\Psi(y))\,. \tag{3.61}$$

This action has now only first derivatives and so the variational principle is well-defined.

The action $S_{\rm cyc}$ gives a well-defined variational principle for any choice of Γ^*. However, its gauge variation is not zero and it is localized at the boundary. Namely, under the variation $\delta_g\Psi = Q\Lambda$, we obtain

$$\delta_g S_{\rm cyc} = -\omega(\Psi, \delta\Gamma^* Q\Lambda) = -\int_{\partial M} d^{d-1}y\,\sqrt{\gamma}\,\omega'(\Psi(x(y)), \Gamma^* Q\Lambda(x(y)))\,. \tag{3.62}$$

Notice that this total gauge variation is sensible to the gauge parameter Λ at the boundary up to its second normal derivative $\partial_\perp^2$. In particular the third derivative contribution potentially contained in $\Gamma^* Q$ is accompanied by $c^2 = 0$ and is therefore

not present.

3.3.2 Bulk-boundary action

In general it is not possible to choose γ in (3.57) such that $\delta\Gamma^* Q\Lambda = 0$.

To rescue the part of gauge invariance which has been broken by the boundary, we need to add boundary degrees of freedom whose gauge variation can compensate for (3.62). We draw inspiration from the analogous problem of abelian Chern–Simons theory with a boundary, discussed in Section 3.1. Looking at (3.62) we see that this can be canceled by promoting the gauge parameter $\Lambda(x)$ approaching the boundary with a 'boundary field' $\chi(x)$ naturally subject to the gauge transformation

$$\delta_g \chi = \Lambda \,. \tag{3.63}$$

With this preparation it is immediate to realize that (3.62) can be canceled by a bulk-boundary action

$$S_1(\Psi, \chi) = \omega(\Psi, \delta\Gamma^* Q\chi) = \int_{\partial M} d^{d-1}y \, \sqrt{\gamma}\, \omega'(\Psi(x(y)), \Gamma^* Q\chi(x(y)))\,, \tag{3.64}$$

when we vary with respect to χ. Notice that just as (3.62) only depends on the values of $\Lambda(x)$, $\partial_\perp \Lambda(x)$, $\partial_\perp^2 \Lambda(x)$ at the boundary $x \in \partial M$, the bulk-boundary action $S_1(\Psi, \chi)$ only depends on

$$\chi_{-1}(y) \equiv \chi(x)\Big|_{x\in\partial M}\,, \tag{3.65a}$$

$$\chi_0(y) \equiv \partial_\perp \chi(x)\Big|_{x\in\partial M}\,, \tag{3.65b}$$

$$\chi_1(y) \equiv \partial_\perp^2 \chi(x)\Big|_{x\in\partial M}\,. \tag{3.65c}$$

The indices $(-1, 0, 1)$ on χ are (mass) space-time dimensions which are defined so that the bulk field Ψ has conventionally dimension 1 and ∂_μ has also dimension 1. This means that the BRST charge Q has total mass-dimension two[4] and therefore the gauge parameter Λ has dimension -1, which is also inherited by χ. With these

[4] Here we are ignoring the (-2) mass dimensions of α'

conventions the action then needs total mass dimension 4. When we introduce the boundary, the distribution Θ is dimensionless and $\delta \sim \partial\Theta$ has dimension 1. Although the fields $\chi_{-1,0,1}(y)$ are the only boundary degrees of freedom which have been truly added, we will still find it convenient to 'package' them in $\chi(x)$, having in mind that the action is only sensitive to the behaviour of $\chi(x)$ infinitesimally close to the boundary.

Notice importantly that the string field χ has degree -1 which means it is ghost-number 1 for closed strings and ghost-number 0 for open strings, just like the gauge parameter Λ. In particular its grassmannality is the opposite of the bulk field Ψ.

3.3.3 Boundary action

By construction the χ variation of S_1 cancels (3.62), but this is not sufficient because S_1 also depends on Ψ, whose gauge variation is still unbalanced. Explicitly we find

$$\delta_g(S_{\rm cyc} + S_1) = \omega(Q\Lambda, \delta\Gamma^* Q\chi)\,. \tag{3.66}$$

This variation can be easily canceled by the pure boundary action

$$S_2(\chi) = -\frac{1}{2}\omega\left(Q\chi, \delta\Gamma^*\, Q\chi\right) = -\frac{1}{2}\omega\left(\chi, Q\delta\Gamma^*\, Q\chi\right)\,. \tag{3.67}$$

Notice that the ghost-number three 'kinetic operator' $Q\delta\Gamma^*\, Q$ is BPZ even

$$(Q\delta\Gamma^*\, Q)^* = -Q^*\Gamma\delta Q^* = -Q\Gamma\delta Q = Q(-B + \delta\Gamma^*)Q = Q\delta\Gamma^*\, Q\,, \tag{3.68}$$

thanks to $Q^2 = 0$ and

$$[Q, B] = [Q, [\Theta, Q]] = \frac{1}{2}[\Theta, [Q, Q]] = 0\,. \tag{3.69}$$

Computing the total gauge variation and using the even BPZ parity of $Q\delta\Gamma^*\, Q$ we then immediately find

$$\delta_g(S_{\rm cyc} + S_1 + S_2) = \omega(Q\Lambda, \delta\Gamma^* Q\chi) - \omega\left(Q\Lambda, \delta\Gamma^*\, Q\chi\right) = 0\,. \tag{3.70}$$

To summarize: we have found the following bulk-boundary gauge-invariant action

$$S_{\text{tot}}(\Psi,\chi)=\frac{1}{2}\omega(\Psi,(\Theta Q-\delta\Gamma^*)\Psi)+\omega(\Psi,\delta\Gamma^* Q\chi)-\frac{1}{2}\omega\left(Q\chi,\delta\Gamma^* Q\chi\right)\,. \tag{3.71}$$

Notice that χ only appears in the combination $Q\chi$ which means that it is defined up to Q-exact terms. Therefore the action is invariant under the following gauge transformation

$$\delta_g\Psi=Q\Lambda\,, \tag{3.72a}$$

$$\delta_g\chi=\Lambda+Q\Upsilon\,, \tag{3.72b}$$

where Υ is a degree -2 (effectively grassmann even) extra gauge parameter, which is simply inherited from the reducibility of the Λ-gauge transformation and the fact the χ is morally the same as Λ at the boundary.

Finally, let us notice that the action (3.71) can be equivalently expressed as

$$S_{\text{tot}}(\Psi,\chi)=\frac{1}{2}\omega\big((\Psi-Q\chi),(\Theta Q-\delta\Gamma^*)(\Psi-Q\chi)\big)\,, \tag{3.73}$$

where gauge-invariance trivially follows from the fact that $(\Psi-Q\chi)$ is itself gauge-invariant. Here it is no longer manifest that χ only contributes through boundary terms: in order to write (3.73) down, one formally needs to extend χ into the bulk of M and notice that the possible bulk terms where χ would appear are vanishing because of $Q^2=0$ and $\Theta Q-\delta\Gamma^*=Q\Theta+\Gamma\delta$. This identity can be also used to show that the boundary part of the action is free from normal derivatives, which is no more apparent at the level of (3.73).

3.3.4 Super-matrix structure

It is convenient to collect the dynamical classical fields Ψ and χ in the "super vector"

$$\mathbf{\Psi}=\begin{pmatrix}\Psi\\ \chi\end{pmatrix}=\begin{pmatrix}\text{bulk mode}\\ \text{boundary mode}\end{pmatrix}\,, \tag{3.74}$$

where the lower entry has one degree less than the upper one and thus opposite grassmannality. We define the (super) degree of $\mathbf{\Psi}$ to be the degree of its top component Ψ. In this notation we can simply write

$$S_{\rm tot} = \frac{1}{2}\omega\left(\mathbf{\Psi}^T, \mathbf{Q}\mathbf{\Psi}\right) , \tag{3.75}$$

where we have introduced the super matrix

$$\mathbf{Q} = \begin{pmatrix} \Theta Q - \delta\Gamma^* & \delta\Gamma^* Q \\ -Q\Gamma\delta & -Q\delta\Gamma^* Q \end{pmatrix} . \tag{3.76}$$

To discuss the BPZ property of this distribution-like operator, we have first of all to specify what we mean by (super)-transposition. In our conventions the super-transpose $(\cdot)^T$ is standard transposition for super-vectors

$$\begin{pmatrix} \Psi \\ \chi \end{pmatrix}^T \equiv (\Psi, \chi) , \tag{3.77}$$

while on a (2×2) supermatrix is defined as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^T \equiv \begin{pmatrix} a & (-1)^{d_c} c \\ b & (-1)^{d_d} d \end{pmatrix} , \tag{3.78}$$

where $d_{(\cdot)}$ is the degree (which is effectively the grassmanality). This definition ensures that given two even supervectors (v, w) and a supermatrix A we have

$$v^T A w = (A^T v)^T w . \tag{3.79}$$

Given this definition, we can easily check that $\mathbf{Q}$ is super-BPZ odd in the sense that

$$\mathbf{Q}^\star \equiv \mathbf{Q}^{*T} = \begin{pmatrix} -(\Theta Q - \delta\Gamma^*) & -\delta\Gamma^* Q \\ Q\Gamma\delta & Q\delta\Gamma^* Q \end{pmatrix} = -\mathbf{Q} . \tag{3.80}$$

In this notation the gauge invariance (3.72a,3.72b) is expressed as

$$\delta_g \mathbf{\Psi} = \mathbf{G}\mathbf{\Lambda}\,, \tag{3.81}$$

where

$$\mathbf{\Lambda} = \begin{pmatrix} \Lambda \\ \Upsilon \end{pmatrix}, \tag{3.82}$$

$$\mathbf{G} = \begin{pmatrix} Q & 0 \\ 1 & Q \end{pmatrix}. \tag{3.83}$$

According to the convention described above, the super-BPZ conjugate of $\mathbf{G}$ is

$$\mathbf{G}^\star = \mathbf{G}^{*T} = \begin{pmatrix} -Q & 1 \\ 0 & Q \end{pmatrix}. \tag{3.84}$$

With this preparation, the gauge variation of the action is given by

$$\delta_g S_{\rm tot} = \omega\left(\mathbf{\Psi}^T, \mathbf{Q}\mathbf{G}\mathbf{\Lambda}\right) = 0\,, \tag{3.85}$$

because, as it easy to check

$$\mathbf{Q}\mathbf{G} = 0\,. \tag{3.86}$$

Notice that $\mathbf{Q}$ is not nilpotent and, being a distribution-like operator containing δ, $\mathbf{Q}^2$ is actually ill-defined. However, as we have just seen this is not a problem since this product does not appear in the proof of gauge invariance. On the other hand the non-distribution-like operator $\mathbf{G}$ *is* nilpotent

$$\mathbf{G}^2 = 0\,, \tag{3.87}$$

which in turns implies that the gauge symmetry (3.81) is infinitely reducible, just as in the boundaryless case.

BV master action

Consider the space of supervectors at degree d

$$\mathbf{H}^{(d)} = \begin{pmatrix} H^{(d)} \\ H^{(d-1)} \end{pmatrix} . \tag{3.88}$$

Together, the $\mathbf{Q}$, $\mathbf{G}$ and $(-\mathbf{G}^\star)$ operators act as a nilpotent map between these spaces

$$\cdots \xrightarrow{\mathbf{G}} \mathbf{H}^{(-2)} \xrightarrow{\mathbf{G}} \mathbf{H}^{(-1)} \xrightarrow{\mathbf{G}} \mathbf{H}^{(0)} \xrightarrow{\mathbf{Q}} \mathbf{H}^{(1)} \xrightarrow{-\mathbf{G}^\star} \mathbf{H}^{(2)} \xrightarrow{-\mathbf{G}^\star} \mathbf{H}^{(3)} \xrightarrow{-\mathbf{G}^\star} \cdots \tag{3.89}$$

This sequence of maps is nilpotent because of the properties

$$\mathbf{Q}^\star = -\mathbf{Q} \,, \tag{3.90a}$$

$$\mathbf{G}^2 = 0 \,, \tag{3.90b}$$

$$\mathbf{QG} = 0 \,, \tag{3.90c}$$

$$-\mathbf{G}^\star\mathbf{Q} = 0 \,, \tag{3.90d}$$

$$(-\mathbf{G}^\star)^2 = 0 \,. \tag{3.90e}$$

This allows us to define the nilpotent super BPZ-odd operator

$$\mathbf{Q}_{BV} \equiv \mathbf{G}\mathbf{P}_{d\leq -1} + \mathbf{Q}\mathbf{P}_{d=0} - \mathbf{G}^\star\mathbf{P}_{d\geq 1} \,, \tag{3.91a}$$

$$\mathbf{Q}_{BV}^2 = 0 \,, \tag{3.91b}$$

$$\mathbf{Q}_{BV}^\star = -\mathbf{Q}_{BV} \,, \tag{3.91c}$$

where $\mathbf{P}_d$ projects on the (super)-degree d subspace of the full Hilbert space.[5]

The construction of the BV master action is now standard. The BV string field is a state in $\mathbf{H} = \oplus_{d\in\mathbb{Z}}\mathbf{H}^{(d)}$

$$\mathbf{\Psi}_{BV} = \begin{pmatrix} \Psi_{BV} \\ \chi_{BV} \end{pmatrix} = \sum_{d\in\mathbb{Z}} \begin{pmatrix} \sum_{s_d} \psi_{s_d}^{(-d)} \, |s_d\rangle \\ \sum_{s_{d-1}} \chi_{s_{d-1}}^{(-d)} \, |s_{d-1}\rangle \end{pmatrix} , \tag{3.92}$$

[5]Notice that $\mathbf{P}_d^\star = \mathbf{P}_{1-d}$ and $\mathbf{Q}_{BV}\mathbf{P}_d = \mathbf{P}_{d+1}\mathbf{Q}_{BV}$.

where $|s_d\rangle$ is a basis of $H^{(d)}$ and the spacetime fields $\psi_{s_d}^{(-d)}$ and $\chi_{s_{d-1}}^{(-d)}$ are the (bulk and boundary) BV fields with BV grading given by $(-d)$. BV spacetime fields with positive or vanishing BV grading are considered as fields while the others are considered as anti-fields. The grassmanality of the spacetime BV fields is given by the BV grading modulo 2. Notice importantly that despite the fact that the basis vectors $|s_d\rangle$ and $|s_{d-1}\rangle$ have opposite grassmanality, the space-time BV fields $\psi_{s_d}^{(-d)}$ and $\chi_{s_{d-1}}^{(-d)}$ have the same grassmanality induced by the overall BV degree. In particular the total BV string field $\mathbf{\Psi}_{BV}$ is a supervector of total degree 0, by summing the super-degree of the basis vectors and the BV degree of the spacetime fields. With this preparation, the classical BV master action is given by the standard expression

$$S_{BV} = \frac{1}{2}\omega\left(\mathbf{\Psi}_{BV}^T, \mathbf{Q}_{BV}\mathbf{\Psi}_{BV}\right) , \tag{3.93}$$

which correctly reduces to the classical gauge-invariant action (3.75) when the anti-fields are set to zero. The classical master equation is then easily shown to be satisfied as

$$\begin{aligned}\left(S_{BV}, S_{BV}\right)_{BV} &= \omega\left(\left(\frac{\delta S_{BV}}{\delta \mathbf{\Psi}_{BV}}\right)^T, \frac{\delta S_{BV}}{\delta \mathbf{\Psi}_{BV}}\right) = \omega((\mathbf{Q}_{BV}\mathbf{\Psi}_{BV})^T, \mathbf{Q}_{BV}\mathbf{\Psi}_{BV}) \\ &= \omega(\mathbf{\Psi}_{BV}^T, \mathbf{Q}_{BV}^\star\mathbf{Q}_{BV}\mathbf{\Psi}_{BV}) = -\omega(\mathbf{\Psi}_{BV}^T, \mathbf{Q}_{BV}^2\mathbf{\Psi}_{BV}) = 0\,, \end{aligned} \tag{3.94}$$

where we have used the super-cyclicity (3.91c) and nilpotency (3.91b) of $\mathbf{Q}_{BV}$.

As a final comment on the construction we have just presented, it is noticeable that the presence of a boundary in target space is only visible in the physical sector at BV degree 0 (i.e. in the gauge-invariant action). At different BV grading, there is no notion of boundary anymore and the corresponding BV fields are defined on $\mathbb{R}^D$. The reason for this is that the 'kinetic' terms of the non-physical BV fields are directly inherited from the structure of the (reducible) gauge transformations (rather than the equations of motion) and, by assumption of our construction, such gauge transformations are essentially the same as in the boundary-less case. To be more precise, the variation of the BV master action with respect to the anti-fields of a given BV degree should give the gauge-transformations (3.81) at the corresponding

degree of reducibility (with the gauge parameter replaced by the corresponding ghost field) and these gauge transformations do not know about the boundary. It would be interesting to investigate whether S_{tot} could be recast in the BV-BFV framework of [81] which is designed to deal with gauge theories on manifolds with boundaries.

3.4 Examples

In this section we will study specific examples of open and closed string backgrounds. For the sake of simplicity, we will specialize on the case of a flat boundary, taking our target space to be the half-space

$$M = \mathbb{R}_+ \times \mathbb{R}^{D-1} \,. \tag{3.95}$$

This is a simple enough choice where we can explicitly see our construction at work in a non-trivial way and at the same time allows for quite manageable examples of bulk+boundary actions. Let z be the global coordinate on $\mathbb{R}_+$ so that the boundary is at $z = 0$. The normal vector to the boundary (pointing outside M) is constant and is given by

$$n^\mu = \big(-1, \mathbf{0}\big) \,. \tag{3.96}$$

The normal derivative is therefore identified with the z-derivative

$$\partial_\perp \equiv -n^\mu \partial_\mu = \partial_z \,. \tag{3.97}$$

The string field in position space $\Psi(x)$ is now understood as

$$\Psi(x) = \Psi(z, y) \,, \tag{3.98}$$

where we will often keep the dependence on the boundary coordinates y^a implicit. In this representation the BRST operator can be written as

$$Q = -c\partial_z^2 - \Omega\partial_z + \tilde{Q} \,, \tag{3.99}$$

where we named Ω^z simply as Ω.

In particular, for the open string, we define

$$c = \alpha' c_0 \,, \tag{3.100a}$$

$$\Omega = i\sqrt{2\alpha'} \sum_{n\neq 0} c_{-n}\alpha_n^z \,, \tag{3.100b}$$

while for the closed string, we write

$$c = \frac{\alpha'}{2} c_0^+ \,, \tag{3.101a}$$

$$\Omega = i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \left(c_{-n}\alpha_n^z + \bar{c}_{-n}\bar{\alpha}_n^z\right) \,, \tag{3.101b}$$

where $\tilde{Q}$ is free of z derivatives (but not of y derivatives). In this representation the algebraic relations which are responsible for the nilpotency of Q are

$$c^2 = 0 \,, \tag{3.102a}$$

$$[c, \tilde{Q}] - \Omega^2 = 0 \,, \tag{3.102b}$$

$$\tilde{Q}^2 = 0 \,, \tag{3.102c}$$

$$[\Omega, c] = 0 \,, \tag{3.102d}$$

$$[\Omega, \tilde{Q}] = 0 \,. \tag{3.102e}$$

We can express the $\mathbb{R}^D$ symplectic form as

$$\omega(\psi_1, \psi_2) = \int_{\mathbb{R}} dz\, \omega_z(\psi_1(z), \psi_2(z)) \,, \tag{3.103}$$

where $\omega_z(\cdot,\cdot)$ takes care of computing the BPZ inner product for all modes, including the center of mass along the boundary but excluding the center of mass in the z-direction, namely

$$\omega_z(\psi_1(z), \psi_2(z)) \equiv \int_{\mathbb{R}^{D-1}} d^{D-1}y\, \omega'(\psi_1(z,y), \psi_2(z,y)) \,. \tag{3.104}$$

Placing the flat boundary at $z = 0$ has the effect of changing the symplectic form as

$$\omega(\psi_1, \psi_2) \longrightarrow \omega_M(\psi_1, \psi_2) \equiv \omega(\psi_1, \Theta\, \psi_2) = \int_{\mathbb{R}^+} dz\, \omega_z(\psi_1(z), \psi_2(z))\,, \tag{3.105}$$

where the distribution-like operator $\Theta(z)$ is just the characteristic function of the positive real axis. We also have

$$[\Theta, Q] = B = c\partial_z\delta(z) + \delta(z)c\partial_z + \delta(z)\Omega \equiv \delta\Gamma^* + \Gamma\delta\,, \tag{3.106}$$

where

$$\Gamma = c\partial_z - \gamma\,, \tag{3.107a}$$

$$\Gamma^* = c\partial_z - \gamma^*\,, \tag{3.107b}$$

and where γ is a ghost-number one operator containing no z derivatives which obeys

$$\gamma + \gamma^* = -\Omega\,. \tag{3.108}$$

For the sake of concreteness, in the explicit examples below, we will be making a particular choice $\gamma = \gamma^*$. In such a case we can write

$$\Gamma^*_{\text{simple}} = c\,\partial_z + \frac{1}{2}\Omega\,. \tag{3.109}$$

We will see that this choice of Γ^* leads to structurally simple forms of boundary terms. This can be traced back to the observation that as soon as we make γ^* BPZ-even, it drops out from the action, as we can clearly see from the factorized form (3.73).

Considering now the gauge-invariant action (3.71)

$$S_{\text{tot}}(\Psi, \chi) = \frac{1}{2}\omega(\Psi, (\Theta Q - \delta\Gamma^*)\Psi) + \omega(\Psi, \delta\Gamma^* Q\chi) - \frac{1}{2}\omega\left(Q\chi, \delta\Gamma^*\, Q\chi\right), \tag{3.110}$$

it depends on the three boundary fields $(\chi_{-1}(y), \chi_0(y), \chi_1(y))$ which were defined by (3.65). These can be equivalently thought of as arising from the Taylor expansion of

$\chi(z)$ around $z = 0$, namely

$$\chi(z) = \chi_{-1} + z\,\chi_0 + \frac{1}{2}z^2\,\chi_1 + O(z^3)\,. \tag{3.111}$$

Analogously, one may expand the gauge parameter near the boundary as

$$\Lambda(z) = \Lambda(0) + z\,\Lambda'(0) + \frac{1}{2}z^2\,\Lambda''(0) + O(z^3)\,, \tag{3.112}$$

where the $'$ on $\Lambda(z)$ stands for ∂_z, so that the Λ-gauge transformation of χ

$$\delta_\Lambda \chi = \Lambda\,, \tag{3.113}$$

translates into the gauge transformation

$$\delta_\Lambda \chi_{-1} = \Lambda(0)\,, \tag{3.114a}$$

$$\delta_\Lambda \chi_0 = \Lambda'(0)\,, \tag{3.114b}$$

$$\delta_\Lambda \chi_1 = \Lambda''(0)\,, \tag{3.114c}$$

of the boundary string fields $\chi_k(y)$.

3.4.1 Massless open string

The free action describing the open-string excitations at the (massless) level 0 can be generated from (3.71) by substituting

$$\Psi(x) = A_\mu(x)\alpha^\mu_{-1}c_1|0\rangle - i\sqrt{\frac{\alpha'}{2}}c_0 B(x)|0\rangle\,, \tag{3.115a}$$

$$\chi_k(y) = \frac{i}{\sqrt{2\alpha'}}\chi^{(k)}(y)|0\rangle\,. \tag{3.115b}$$

for the bulk and boundary string fields. The corresponding gauge-parameter string field giving rise to the gauge transformations of the component fields A_μ, B and χ_k then reads

$$\Lambda(x) = \frac{i}{\sqrt{2\alpha'}}\lambda(x)|0\rangle\,. \tag{3.116}$$

No non-trivial gauge-for-gauge parameter Υ arises at this level. Substituting the gauge-parameter $\Lambda(x)$ given by (3.116) into the general form (3.72) of the gauge transformation of Ψ and χ, one finds

$$\delta_g A_\mu(x) = \partial_\mu \lambda(x)\,, \tag{3.117a}$$

$$\delta_g B(x) = \partial^2 \lambda(x)\,, \tag{3.117b}$$

$$\delta_g \chi_k(y) = \partial_z^{k+1} \lambda(x)|_{z=0}\,. \tag{3.117c}$$

At the same time, the consistent gauge-invariant action (3.71) turns into

$$S(A_\mu, B, \chi^{(k)}) = -\frac{\alpha'}{2}\int d^d x \left(\tfrac{1}{2} F_{\mu\nu}F^{\mu\nu} + \mathcal{K}^2\right) - \frac{\alpha'}{2}\int d^{d-1} y\, \mathcal{R}^{(0)}\mathcal{R}^{(1)}\,, \tag{3.118}$$

where the dependence on auxiliary fields (both in the bulk and on the boundary) was packaged into the gauge-invariant combinations

$$\mathcal{K} = B - \partial_\mu A^\mu\,, \tag{3.119a}$$

$$\mathcal{R}^{(0)} = \chi^{(0)} - A_z\,, \tag{3.119b}$$

$$\mathcal{R}^{(1)} = \chi^{(1)} - \partial^a \partial_a \chi^{(-1)} + 2\partial_a A^a - B\,. \tag{3.119c}$$

Noting that $\mathcal{K}$ can be set to zero by integrating out B and, similarly, recognizing $\mathcal{R}^{(0)}$ as the equation of motion of the auxiliary field $\chi^{(1)}$, we conclude that (3.118) is equivalent to the Maxwell action for the $U(1)$ gauge field A_μ.

3.4.2 First massive level of open string

To test that the free action (3.71) is capable of consistently describing also the massive string excitations, let us expand it in the component fields arising at level 1. In particular, we should obtain an action equivalent to the one already presented in Section 5 of [29] . To this end, we consider the following truncations of the bulk and

boundary string fields

$$\Psi(x) = \frac{1}{2}c_1\varphi_{\mu\nu}(x)\alpha^\mu_{-1}\alpha^\nu_{-1}|0\rangle - ic_0\sqrt{\frac{\alpha'}{2}}C_\mu(x)\alpha^\mu_{-1}|0\rangle + \frac{1}{2}c_{-1}D(x)|0\rangle + \\ + \frac{1}{2}ic_1a_\mu(x)\alpha^\mu_{-2}|0\rangle + c_0b_{-2}c_1\sqrt{\frac{\alpha'}{2}}a(x)|0\rangle\,, \tag{3.120a}$$

$$\chi_k(y) = \frac{i}{\sqrt{2\alpha'}}\chi^{(k)}_\mu(y)\alpha^\mu_{-1}|0\rangle - \frac{1}{\sqrt{2\alpha'}}\chi^{(k)}(y)b_{-2}c_1|0\rangle\,, \tag{3.120b}$$

together with the gauge parameter

$$\Lambda(x) = \frac{i}{\sqrt{2\alpha'}}\xi_\mu(x)\alpha^\mu_{-1}|0\rangle - \frac{1}{\sqrt{2\alpha'}}\xi(x)b_{-2}c_1|0\rangle\,. \tag{3.121}$$

In $d = 26$ dimensions, an action for the massive spin-2 triplet $(\varphi_{\mu\nu}, C_\mu, D)$ of the target fields should be invariant under the gauge transformation

$$\delta_g\varphi_{\mu\nu}(x) = \partial_\mu\xi_\nu + \partial_\nu\xi_\mu - \tfrac{1}{2}\eta_{\mu\nu}\sqrt{\tfrac{2}{\alpha'}}\xi\,, \tag{3.122a}$$

$$\delta_g C_\mu(x) = (\partial^2 - \tfrac{1}{\alpha'})\xi_\mu(x)\,, \tag{3.122b}$$

$$\delta_g D(x) = 2\partial_\mu\xi^\mu - 3\sqrt{\tfrac{2}{\alpha'}}\xi\,, \tag{3.122c}$$

which is indeed generated upon substituting (3.121) into (3.72a). This has to be supplemented by the corresponding gauge transformation

$$\delta_g\chi^{(k)}_\mu(y) = \partial^{k+1}\xi_\mu(x)|_{z=0}\,, \tag{3.123a}$$

$$\delta_g\chi^{(k)}(y) = \partial^{k+1}\xi(x)|_{z=0}\,, \tag{3.123b}$$

of the boundary fields $\chi_\mu^{(k)}(y)$ and $\chi^{(k)}(y)$. After substituting the truncating string fields (3.120) into (3.71), one obtains the gauge-invariant action

$$\begin{aligned}
&S(\varphi_{\mu\nu}, C_\mu, D, a_\mu, a, \chi_\mu^{(k)}, \chi^{(k)}) = \\
&\quad = -\frac{\alpha'}{2}\int_M d^{26}x \Big\{ \tfrac{1}{2}\partial_\rho\varphi_{\mu\nu}\partial^\rho\varphi^{\mu\nu} + \tfrac{1}{2}\tfrac{1}{\alpha'}\varphi^{\mu\nu}\varphi_{\mu\nu} - \tfrac{1}{2}\partial_\mu D\partial^\mu D - \tfrac{1}{2}\tfrac{1}{\alpha'}D^2 + \\
&\qquad\qquad + \partial^\mu\varphi_{\mu\nu}\partial^\nu D - \partial_\nu\varphi_{\mu\rho}\partial^\mu\varphi^{\nu\rho} + \\
&\qquad + \tfrac{1}{4}(\partial^\mu a^\nu - \partial^\nu a^\mu)(\partial_\mu a_\nu - \partial_\nu a_\mu) + \\
&\qquad + \sqrt{\tfrac{1}{2\alpha'}}\Big[2\eta^{\rho\sigma}(\partial_\rho\varphi_{\mu\sigma} - \partial_\mu\varphi_{\rho\sigma}) - \tfrac{5}{2}\partial_\mu(D - \eta_{\rho\sigma}\varphi^{\rho\sigma})\Big]a^\mu + \\
&\qquad - \tfrac{1}{16}\tfrac{1}{\alpha'}(5D - \eta^{\mu\nu}\varphi_{\mu\nu})(D - \eta^{\mu\nu}\varphi_{\mu\nu}) + \mathcal{K}_\nu\mathcal{K}^\nu + 2\mathcal{L}^2 \Big\} \\
&\quad - \frac{\alpha'}{2}\int_{\partial M} d^{25}y \Big\{ \varphi_{za}\Big(2\partial^a\varphi_{zz} - \partial^a D - 2a^a\sqrt{\tfrac{1}{2\alpha'}}\Big) + \\
&\qquad - 2\chi_z^{(-1)}\Big[\partial^a\partial_a(D - \varphi_{zz}) - \partial^a\partial^b\varphi_{ab} + \\
&\qquad\qquad + \partial_a a^a\sqrt{\tfrac{2}{\alpha'}} + \tfrac{1}{4}\tfrac{1}{\alpha'}\eta_{ab}\varphi^{ab} - \tfrac{5}{4}\tfrac{1}{\alpha'}D + \tfrac{5}{4}\tfrac{1}{\alpha'}\varphi_{zz}\Big] + \\
&\qquad + \mathcal{R}_a^{(0)}\mathcal{R}^{(1),a} + \mathcal{R}_z^{(0)}\mathcal{R}_z^{(1)} + 2\mathcal{R}^{(0)}\mathcal{R}^{(1)} \Big\}\,,
\end{aligned} \tag{3.124}$$

where, for the sake of transparency, we have again chosen to package all auxiliary fields in terms of the gauge-invariant combinations

$$\mathcal{K}_\nu = C_\nu - \partial^\mu\varphi_{\mu\nu} + \tfrac{1}{2}\partial_\nu D + \sqrt{\tfrac{1}{2\alpha'}}a_\nu\,, \tag{3.125a}$$

$$\mathcal{L} = a - \tfrac{1}{2}\partial_\mu a^\mu - \tfrac{1}{4}\eta^{\mu\nu}\varphi_{\mu\nu}\sqrt{\tfrac{1}{2\alpha'}} + \tfrac{3}{4}D\sqrt{\tfrac{1}{2\alpha'}}\,, \tag{3.125b}$$

$$\mathcal{R}_a^{(0)} = \chi_a^{(0)} + \partial_a\chi_z^{(-1)} - e_{za}\,, \tag{3.125c}$$

$$\mathcal{R}_a^{(1)} = \chi_a^{(1)} + 2\partial_a\chi_z^{(0)} + 2\partial^b e_{ab} + \tfrac{1}{\alpha'}\chi_a^{(-1)} - \sqrt{\tfrac{2}{\alpha'}}a_a - e_a - \partial_a e - \partial_b\partial^b\chi_a^{(-1)}\,, \tag{3.125d}$$

$$\mathcal{R}_z^{(0)} = \chi_z^{(0)} + \tfrac{1}{2}e - e_{zz} - \partial^a\chi_a^{(-1)} + \sqrt{\tfrac{2}{\alpha'}}\chi^{(-1)}\,, \tag{3.125e}$$

$$\mathcal{R}_z^{(1)} = \chi_z^{(1)} + \partial^a\partial_a\chi_z^{(-1)} - \tfrac{1}{\alpha'}\chi_z^{(-1)} - e_z\,, \tag{3.125f}$$

$$\mathcal{R}^{(0)} = \chi^{(0)} - \tfrac{1}{2}a_z + \tfrac{1}{2}\sqrt{\tfrac{2}{\alpha'}}\chi_z^{(-1)}\,, \tag{3.125g}$$

$$\mathcal{R}^{(1)} = \chi^{(1)} - \partial^a\partial_a\chi^{(-1)} + \tfrac{1}{\alpha'}\chi^{(-1)} + \partial^a a_a - a + \tfrac{1}{4}\sqrt{\tfrac{2}{\alpha'}}(e_{zz} + \eta_{ab}e^{ab} - 3e + 4\chi_z^{(0)})\,. \tag{3.125h}$$

Noticing that $\mathcal{K}_\nu$, $\mathcal{K}$, $\mathcal{R}_a^{(0)}$, $\mathcal{R}_z^{(0)}$ and $\mathcal{R}^{(0)}$ can all be set to zero upon integrating out the auxiliary fields C_ν, a, $\chi_a^{(1)}$, $\chi_z^{(1)}$ and $\chi^{(1)}$, respectively, we conclude that we have indeed recovered the action reported in [29] .

3.4.3 Massless closed string

In this section we will describe the construction of the gauge-invariant action for the massless sector of closed-string. The level-0 component of the bulk closed string field is

$$\Psi(x) = -\frac{1}{2}e_{\mu\nu}(x)\alpha_{-1}^{\mu}\bar{\alpha}_{-1}^{\nu}c_1\bar{c}_1|0\rangle + e(x)c_1c_{-1}|0\rangle + \bar{e}(x)\bar{c}_1\bar{c}_{-1}|0\rangle + \\ + i\sqrt{\frac{\alpha'}{2}}c_0^+(e_\mu(x)c_1\alpha_{-1}^{\mu} + \bar{e}_\mu(x)\bar{c}_1\bar{\alpha}_{-1}^{\mu})|0\rangle\,. \quad (3.126)$$

and the boundary string field is the most generic level-matched ghost number 1 string field at level zero

$$\chi_k(y) = \frac{i}{\sqrt{2\alpha'}}(\chi_\mu^{(k)}(y)c_1\alpha_{-1}^{\mu} - \bar{\chi}_\mu^{(k)}(y)\bar{c}_1\bar{\alpha}_{-1}^{\mu})\,|0\rangle - \chi^{(k)}(y)c_0^+\,|0\rangle\ , \quad (3.127)$$

First, we shall focus on the gauge transformations. To do so, we introduce the string field gauge parameter at ghost number 1

$$\Lambda = \frac{i}{\sqrt{2\alpha'}}(\xi_\mu(x)c_1\alpha_{-1}^{\mu} - \bar{\xi}_\mu(x)\bar{c}_1\bar{\alpha}_{-1}^{\mu})\,|0\rangle - \xi(x)c_0^+\,|0\rangle\ , \quad (3.128)$$

and the string field gauge parameter at ghost number 0

$$\Upsilon = \frac{1}{\alpha'}u(x)\,|0\rangle\ , \quad (3.129)$$

which implements the gauge for gauge transformation $\delta_g\Lambda = Q\Upsilon$. These become relevant when we consider the transformations of boundary modes (3.72b). At the

level of target-space fields, the bulk gauge transformations $\delta_g \Psi = Q\Lambda$ read

$$\delta_g e_{\mu\nu}(x) = \partial_\mu \bar{\xi}_\nu(x) + \partial_\nu \xi_\mu(x)\,, \tag{3.130a}$$

$$\delta_g e_\mu(x) = -\tfrac{1}{2}\partial^2 \xi_\mu(x) - \partial_\mu \xi(x)\,, \tag{3.130b}$$

$$\delta_g \bar{e}_\mu(x) = +\tfrac{1}{2}\partial^2 \bar{\xi}_\mu(x) - \partial_\mu \xi(x)\,, \tag{3.130c}$$

$$\delta_g e(x) = -\tfrac{1}{2}\partial_\mu \xi^\mu(x) - \xi(x)\,, \tag{3.130d}$$

$$\delta_g \bar{e}(x) = +\tfrac{1}{2}\partial_\mu \bar{\xi}^\mu(x) - \xi(x)\,, \tag{3.130e}$$

whereas the transformation laws for the boundary fields (3.72b) become

$$\delta_g \chi_\mu(x) = \xi_\mu(x) - \partial_\mu u(x)\,, \tag{3.131a}$$

$$\delta_g \bar{\chi}_\mu(x) = \bar{\xi}_\mu(x) + \partial_\mu u(x)\,, \tag{3.131b}$$

$$\delta_g \chi(x) = \xi(x) + \tfrac{1}{2}\partial^2 u(x)\,. \tag{3.131c}$$

For the sake of brevity, we wrote the gauge transformations for the boundary fields as if they were bulk fields, meaning that both the left-hand side and the right-hand side must be expanded according to (3.111), thus obtaining the gauge transformation for each Taylor coefficient. For instance the third line of (3.131) yields the following boundary gauge transformations

$$\delta_g \chi_k(y) = \partial_z^{k+1} \xi(x)|_{z=0} + \frac{1}{2}\partial_z^{k+3} u(x)|_{z=0} + \frac{1}{2}\partial_z^{k+1}\partial_a^2 u(x)|_{z=0}\,, \tag{3.132}$$

for $k = -1, 0, 1$.

We now turn to the computation of the gauge-invariant action, obtained by

substituting (3.126) and (3.127) into (3.71). The resulting expression is

$$\begin{aligned}
&S(e_{\mu\nu}, e_\mu, \bar{e}_\mu, e, \bar{e}; \chi^{(k)}, \chi_\mu^{(k)}, \bar{\chi}_\mu^{(k)}) = \\
&\quad = \frac{\alpha'}{16}\int_M d^dx \Big[-\tfrac{1}{4}\partial_\rho e_{\mu\nu}\partial^\rho e^{\mu\nu} + \tfrac{1}{4}\partial_\rho e_{\mu\nu}\partial^\nu e^{\mu\rho} + \tfrac{1}{4}\partial_\rho e_{\nu\mu}\partial^\nu e^{\rho\mu} \\
&\qquad\qquad + \big(\partial_\mu(e-\bar{e})\big)^2 - \tfrac{1}{2}\partial^\nu(e_{\nu\mu}+e_{\mu\nu})\partial^\mu(\bar{e}-e) - \mathcal{K}_\mu\mathcal{K}^\mu - \bar{\mathcal{K}}_\mu\bar{\mathcal{K}}^\mu\Big] \\
&\quad + \frac{\alpha'}{32}\int_{\partial M} d^{d-1}y \Big[(e_{az}+e_{za})\partial^a(\bar{e}-e-e_{zz}) + \\
&\qquad\qquad - \partial_a(\bar{\chi}_z^{(-1)} + \chi_z^{(-1)})\partial_b\big((2\bar{e}-2e-e_{zz})\eta^{ab} - e^{ab}\big) + \\
&\qquad\qquad - \mathcal{R}_a^{(0)}\mathcal{R}^{(1),a} - \bar{\mathcal{R}}_a^{(0)}\bar{\mathcal{R}}^{(1),a} - \mathcal{R}_z^{(0)}\mathcal{R}_z^{(1)} - \bar{\mathcal{R}}_z^{(0)}\bar{\mathcal{R}}_z^{(1)}\Big],
\end{aligned} \tag{3.133}$$

where all dependence on the bulk and boundary auxiliary fields is captured by the gauge-invariant combinations

$$\mathcal{K}_\mu = e_\mu + \tfrac{1}{2}\partial^\nu e_{\mu\nu} - \partial_\mu\bar{e}\,, \tag{3.134a}$$

$$\bar{\mathcal{K}}_\mu = \bar{e}_\mu - \tfrac{1}{2}\partial^\nu e_{\nu\mu} - \partial_\mu e\,, \tag{3.134b}$$

$$\mathcal{R}_a^{(0)} = \chi_a^{(0)} + \partial_a\chi_z^{(-1)} - e_{az}\,, \tag{3.134c}$$

$$\bar{\mathcal{R}}_a^{(0)} = \bar{\chi}_a^{(0)} + \partial_a\chi_z^{(-1)} - e_{za}\,, \tag{3.134d}$$

$$\mathcal{R}_a^{(1)} = \tfrac{1}{2}\chi^{(1)a} + \partial^a\bar{\chi}_z^{(0)} - \partial^a\chi^{(-1)} - \tfrac{1}{2}\partial_b^2\chi^{(-1)a} + e^a - 2\partial^a\bar{e} + \partial_b e^{ab}\,, \tag{3.134e}$$

$$\bar{\mathcal{R}}_a^{(1)} = \tfrac{1}{2}\bar{\chi}^{(1)a} + \partial^a\chi_z^{(0)} + \partial^a\chi^{(-1)} - \tfrac{1}{2}\partial_b^2\bar{\chi}^{(-1)a} - \bar{e}^a + 2\partial^a e + \partial_b e^{ba}\,, \tag{3.134f}$$

$$\mathcal{R}_z^{(0)} = \tfrac{1}{2}\chi_z^{(0)} + \chi^{(-1)} - \tfrac{1}{2}\partial^a\bar{\chi}_a^{(-1)} + \bar{e} - \tfrac{1}{2}e_{zz}\,, \tag{3.134g}$$

$$\bar{\mathcal{R}}_z^{(0)} = \tfrac{1}{2}\bar{\chi}_z^{(0)} - \chi^{(-1)} - \tfrac{1}{2}\partial^a\chi_a^{(-1)} - e - \tfrac{1}{2}e_{zz}\,, \tag{3.134h}$$

$$\mathcal{R}_z^{(1)} = \chi_z^{(1)} + 2\chi^{(0)} + \partial_a^2\chi_z^{(-1)} + 2e_z\,, \tag{3.134i}$$

$$\bar{\mathcal{R}}_z^{(1)} = \bar{\chi}_z^{(1)} - 2\chi^{(0)} + \partial_a^2\bar{\chi}_z^{(-1)} - 2\bar{e}_z\,. \tag{3.134j}$$

In particular, both $\mathcal{K}_\mu$ and $\bar{\mathcal{K}}_\mu$ go to zero when we integrate out classically the two bulk auxiliary fields e_μ and $\bar{e}_\mu$. Similarly, the combinations $\mathcal{R}_a^{(0)}$, $\bar{\mathcal{R}}_a^{(0)}$, $\mathcal{R}_z^{(0)}$ and $\bar{\mathcal{R}}_z^{(0)}$ can be set to zero by solving the equations of motion for the boundary auxiliary fields $\chi_a^{(1)}$, $\bar{\chi}_a^{(1)}$, $\chi_z^{(1)}$ and $\bar{\chi}_z^{(1)}$. Therefore, after having integrated out the auxiliary fields, the only contribution from the boundary modes is due to $\chi_z^{(-1)}$ and $\bar{\chi}_z^{(-1)}$ which only

appear through the combination

$$l(y) = \bar{\chi}_z^{(-1)}(y) + \chi_z^{(-1)}(y)\,, \tag{3.135}$$

which, according to (3.131), transforms as

$$\delta_g l(y) = \left(\xi_z(x) + \bar{\xi}_z(x)\right)|_{z=0}\,. \tag{3.136}$$

Finally, to better understand the structure of (3.133), we split the rank-2 tensor field $e_{\mu\nu}$ in its symmetric and anti-symmetric part

$$e_{\mu\nu} = b_{\mu\nu} + h_{\mu\nu}\,, \qquad b_{\mu\nu} = -b_{\nu\mu}\,, \qquad h_{\mu\nu} = h_{\nu\mu}\,, \tag{3.137}$$

introduce the 3-form

$$H_{\mu\nu\rho} = \partial_\mu b_{\nu\rho} + \partial_\nu b_{\rho\mu} + \partial_\rho b_{\mu\nu}\,, \tag{3.138}$$

and perform the field redefinition

$$\phi = \frac{1}{4}(h^\mu{}_\mu - 2\bar{e} + 2e)\,. \tag{3.139}$$

Therefore, the action (3.133) can be rewritten as

$$\begin{aligned}
&S^*\left(h_{\mu\nu}, b_{\mu\nu}, \phi; l\right) = \\
&\quad = -\frac{\alpha'}{16}\int_M d^dx \left(\tfrac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho}\right) + \\
&\qquad - \frac{\alpha'}{16}\int_M d^dx \left(\tfrac{1}{4}\partial_\rho h_{\mu\nu}\partial^\rho h^{\mu\nu} - \tfrac{1}{2}\partial_\rho h_{\mu\nu}\partial^\nu h^{\mu\rho} + \tfrac{1}{4}\partial^\mu(2h_{\mu\nu} - \eta_{\mu\nu}h^\rho{}_\rho)\partial^\nu h^\lambda{}_\lambda \right. \\
&\qquad\qquad \left. - 2\partial^\mu(h_{\mu\nu} - \eta_{\mu\nu}h^\rho{}_\rho)\partial^\nu\phi - 4\partial_\mu\phi\partial^\mu\phi\right) + \\
&\qquad + \frac{\alpha'}{16}\int_{\partial M} d^{d-1}y \left[\tfrac{1}{2}h_{az}\partial^a(h^b{}_b - h_{zz} - 4\phi) + \tfrac{1}{2}\left((h^c{}_c - 4\phi)\,\eta^{ab} - h^{ab}\right)\partial_a\partial_b l\right],
\end{aligned} \tag{3.140}$$

which is invariant under the gauge transformations

$$\delta_g b_{\mu\nu}(x) = \partial_\mu \lambda_\nu(x) - \partial_\nu \lambda_\mu(x)\,, \tag{3.141a}$$

$$\delta_g h_{\mu\nu}(x) = \partial_\mu \tau_\nu(x) + \partial_\nu \tau_\mu(x)\,, \tag{3.141b}$$

$$\delta_g \phi(x) = 0\,, \tag{3.141c}$$

$$\delta_g l(y) = 2\tau_z(y)\,, \tag{3.141d}$$

where we defined the the rotated gauge parameters as

$$\lambda_\mu = \frac{1}{2}(\bar{\xi}_\mu - \xi_\mu)\,, \tag{3.142a}$$

$$\tau_\mu = \frac{1}{2}(\bar{\xi}_\mu + \xi_\mu)\,. \tag{3.142b}$$

Looking at (3.140), we can recognize the standard linearized low-energy effective string action for the graviton, Kalb-Ramond, and dilaton fields in the bulk, with a boundary term involving an additional boundary degree of freedom l which only couples to the graviton-dilaton sector. Indeed, (3.140) can be obtained by linearizing the standard low-energy effective action for gravity with the Kalb–Ramond and the dilaton, including the Gibbons–Hawking–York term

$$S_{\rm st} = \int_M d^d x\, \sqrt{g}\, e^{-2\phi}\Big(R - \tfrac{1}{12}H^2 + 4(\nabla\phi)^2\Big) + \int_{\partial M} d^{d-1} y\, \sqrt{h}\, e^{-2\phi}\, 2K\,, \tag{3.143}$$

where R is the Ricci scalar, K is the trace of the extrinsic curvature, and the boundary is defined through the implicit equation

$$F(x) = z + \frac{1}{2} l(y) = 0\,, \tag{3.144}$$

which implies that the boundary mode l can be interpreted as the transverse displacement of the fluctuating flat boundary. See Appendix A for more details.

3.4.4 Linear dilaton backgrounds

In this section, we are going to apply our construction in the case of the linear dilaton CFT, with the goal of constructing the gauge-invariant action for the level-0 open and

closed string sector in the half-space. Before doing that, however, it is important to carefully understand how the linear dilaton background arises in General Relativity, in the presence of a (fluctuating) boundary.

Linear dilaton with boundary in General Relativity

From the GR perspective, the linear dilaton background arises as a solution to the equations of motion of the Einstein-Hilbert action with dilaton and a cosmological constant Λ

$$S_{\rm bulk} = \int_M d^d x \, \sqrt{g} \, e^{-2\Phi} \Big(R + 4(\nabla\Phi)^2 + \Lambda \Big) \, . \tag{3.145}$$

This action is the d-dimensional extension of the two-dimensional Callan-Giddings-Harvey-Strominger (CGHS) model [82]. Without a boundary, it admits the following solution

$$g^*_{\mu\nu} = \eta_{\mu\nu} \, , \tag{3.146a}$$

$$\Phi^* = \frac{1}{2} V_\mu x^\mu \, , \tag{3.146b}$$

namely the flat metric and a linear profile for the dilaton, whose slope V_μ is related to the cosmological constant as

$$\Lambda = \eta^{\mu\nu} V_\mu V_\nu \, . \tag{3.147}$$

Interestingly, if we now specify the background to be the flat half-space with a flat fluctuating boundary, it turns out that the usual GHY term is not sufficient anymore to ensure a well-posed variational principle, as we will see shortly. To address this problem, we introduce a boundary cosmological constant $\Lambda_{\rm b}$, leading to the action

$$S_{\rm ld} = \int_M d^d x \, \sqrt{g} \, e^{-2\Phi} \Big(R + 4(\nabla\Phi)^2 + \Lambda \Big) + \int_{\partial M} d^{d-1} y \, \sqrt{h} \, e^{-2\Phi} \, 2(K + \Lambda_{\rm b}) \, . \tag{3.148}$$

In order for the solution (3.146) to continue to extremize the action without the necessity to impose any restrictions on fluctuations at ∂M, it turns out that we must impose the following relation between the dilaton slope and the boundary cosmological constant

$$\Lambda_{\rm b} = \eta^{\mu z} V_\mu \,. \tag{3.149}$$

This becomes clear by studying the fluctuations around the solution (3.146)

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \,, \tag{3.150a}$$

$$\Phi = \frac{1}{2} V_\mu x^\mu + \phi \,, \tag{3.150b}$$

together with the boundary fluctuation l defined through (3.144). Using standard GR manipulations (see appendix F for details) and after a lengthy but straightforward computation, we find that the action receives the following contributions, up to second order in fluctuations. First, a constant term due to the cosmological constants

$$S^{(0)}_{\rm ld} = 2\Lambda \int_M d^d x \, e^{-V_\rho x^\rho} + 2\Lambda_{\rm b} \int_{\partial M} d^{d-1} y \, e^{-V_a x^a} \,, \tag{3.151}$$

then a linear contribution, a potential boundary tadpole

$$S^{(1)}_{\rm ld} = (\Lambda_{\rm b} - V_z) \int_{\partial M} d^{d-1} y \, e^{-V_a x^a} (h^a_a - 4\phi + V_z l) \,. \tag{3.152}$$

Remarkably this vanishes precisely when (3.149) is satisfied, ensuring that (3.146) extremizes the action without restricting the fluctuations h^a_a, ϕ and l.

Finally, one obtains a second-order contribution involving, in particular, the term

$$S^{(2)}_{\rm ld} = \ldots - \frac{1}{2}(\Lambda_{\rm b} - V_z) \int_{\partial M} d^{d-1} y \, e^{-V_a x^a} \, l \, \partial_z (h^a_a - 4\phi) + \ldots \,, \tag{3.153}$$

which, for generic V_z and $\Lambda_{\rm b}$, is plagued by transverse derivatives and thus makes the variational principle ill-defined. But again, the relation (3.149) ensures the consistency of the expanded action. After a series of other cancellations, the full

second-order contribution to the linearized action turns out to read

$$
\begin{aligned}
&S^{(2)}_{\mathrm{ld}} = \\
&-\int_M d^d x\, e^{-V_\rho x^\rho}\Big(\tfrac{1}{4}\partial_\rho h_{\mu\nu}\partial^\rho h^{\mu\nu} - \tfrac{1}{2}\partial_\rho h_{\mu\nu}\partial^\nu h^{\mu\rho} + \tfrac{1}{4}\big(2(\partial^\mu - V^\mu)h_{\mu\nu} - \partial_\nu h^\rho{}_\rho\big)\partial^\nu h^\lambda{}_\lambda \\
&\qquad\qquad - 2\big((\partial^\mu - V^\mu)h_{\mu\nu} - \partial_\nu h^\lambda{}_\lambda\big)\partial^\nu\phi - 4\partial_\mu\phi\partial^\mu\phi\Big) + \\
&\quad + \int_{\partial M} d^{d-1} y\, e^{-V_c y^c}\Big(\tfrac{1}{2}h_{az}\partial^a\big(h^b{}_b - h_{zz} - 4\phi\big) - \tfrac{1}{2}h^{ab}\partial_a\partial_b l - \tfrac{1}{2}\partial^a l\partial_a\big(h^b{}_b - 4\phi\big) \\
&\qquad\qquad - \tfrac{1}{4}V_z h_{zz}^2 - \tfrac{1}{4}V_z\partial_a l\partial^a l\Big)\,.
\end{aligned}
\tag{3.154}
$$

This is a well-defined quadratic action with boundary and it is the expression that has to be compared with our quadratic SFT action defined on the linear dilaton background, with the understood inclusion of the Kalb–Ramond sector, which can be studied separately and is not affected by extra boundary subtleties.

Linear dilaton with boundary in String Field Theory

Let's start by briefly reviewing some basic aspects of the linear dilaton CFT. The main difference with respect to the free boson CFT is that the energy-momentum tensor now has an additional term proportional to the world-sheet second derivative of the scalar fields

$$
T(z) = -\frac{1}{\alpha'}\eta_{\mu\nu} : \partial X^\mu \partial X^\nu : -i\sqrt{\frac{2}{\alpha'}}\beta_\mu \partial^2 X^\mu\,. \tag{3.155}
$$

This gives rise to a cubic pole in the OPE with ∂X^μ, making this field no longer a primary. Indeed, it is possible to show that under the application of a generic conformal map $f(z)$, it obeys the following transformation rule

$$
f \circ \partial X^\mu(z) = f'(z)\partial X^\mu(f(z)) - i\sqrt{\frac{\alpha'}{2}}\beta^\mu \frac{f''(z)}{f'(z)}\,. \tag{3.156}
$$

Consequently, this implies, among other things, that the matter oscillators behave differently under BPZ conjugation. In particular,

$$(\alpha_n^\mu)^* = \begin{cases} -\alpha_{-n}^\mu - 2\beta^\mu \delta_{n,0} & \text{closed string}\,, \\ (-)^{n+1}\alpha_{-n}^\mu - 2\beta^\mu \delta_{n,0} & \text{open string}\,. \end{cases} \tag{3.157}$$

Therefore, from this point of view, the only difference with respect to the free boson CFT is related to the matter zero mode, a.k.a. the momentum

$$(p_\mu)^* = -p_\mu - iV_\mu\,, \tag{3.158}$$

where we introduced the dimensionful background charge

$$V_\mu = \begin{cases} -2i\sqrt{\frac{2}{\alpha'}}\beta_\mu & \text{closed string}\,, \\ -i\sqrt{\frac{2}{\alpha'}}\beta_\mu & \text{open string}\,, \end{cases} \tag{3.159}$$

in order to treat the open and closed string sectors on the same footing. Notice that the square of the background charge is fixed by the criticality condition of the theory

$$c_\beta = d - 12\beta^2 = 26 \quad \longleftrightarrow \quad V^2 = \begin{cases} -\frac{2(d-26)}{3\alpha'} & \text{closed string}\,, \\ -\frac{d-26}{3\alpha'} & \text{open string}\,. \end{cases} \tag{3.160}$$

In the position space representation, (3.158) becomes

$$(\partial_\mu)^* = -\partial_\mu + V_\mu\,. \tag{3.161}$$

This property can be naturally implemented by defining the inner product as follows

$$\omega_V(\Phi,\Psi) = \int_{\mathbb{R}^d} d^d x\, e^{-V_\mu x^\mu}\omega'(\Phi(x),\Psi(x))\,. \tag{3.162}$$

Exactly as in the free boson CFT, also in this framework the BRST charge can be decomposed in the position representation as

$$Q_V = -c\partial_\mu \partial^\mu - \Omega_V^\mu \partial_\mu + Q'_V \,, \tag{3.163}$$

where Ω_V^μ and Q'_V do not contain the position zero mode or its derivatives. These operators satisfy, by construction, the algebra (3.102) and obey the following BPZ properties

$$c^* = -c, \tag{3.164a}$$

$$(\Omega_V^\mu)^* = +\Omega_V^\mu + 2cV^\mu, \tag{3.164b}$$

$$(Q'_V)^* = -Q'_V + \Omega_V^\mu V_\mu + cV^2, \tag{3.164c}$$

which together with (3.161), guarantee that

$$(Q_V)^* = -Q_V \,. \tag{3.165}$$

Note that in this context, the first-order differential operator Γ_V^*, which is decomposed according to (3.57)

$$\Gamma_V^* = c\partial_z - \gamma_V^* \,. \tag{3.166}$$

transforms under BPZ conjugation as

$$\Gamma_V = c\partial_z - V_z c - \gamma_V \,. \tag{3.167}$$

Consequently, the defining relation (3.106) implies

$$\gamma_V + \gamma_V^* = -\Omega_V - V_z c \,, \tag{3.168}$$

which is, as expected, a BPZ even combination. Therefore, analogous to (3.109) we can choose the 'simple' form of γ_V^*

$$\gamma_V^* = \gamma_V = -\frac{1}{2}\left(\Omega_V + V_z c\right) , \tag{3.169}$$

leading to

$$\Gamma_V^* = c\partial_z + \frac{1}{2}\Omega_V + \frac{1}{2}V_z c\,. \tag{3.170}$$

This, unlike its counterpart in the free boson scenario, is not BPZ even. Nevertheless, also in this background, all the requirements for our construction of $S_{\rm tot}$ are satisfied. Therefore, we can compute a gauge-invariant action simply using the symplectic form (3.162) and the operators Q_V and Γ_V in (3.71).

Let us now consider the explicit example of the level-0 open string sector. To ease manipulations, we can consistently truncate the BRST charge by identifying

$$c = \alpha' c_0\,, \tag{3.171a}$$

$$\Omega_V^\mu = i\sqrt{2\alpha'}(c_1\alpha_{-1}^\mu + c_{-1}\alpha_1^\mu) - \alpha' V^\mu c_0\,, \tag{3.171b}$$

$$Q'_V = -2b_0c_{-1}c_1 + i\sqrt{2\alpha'}V_\mu c_{-1}\alpha_1^\mu\,. \tag{3.171c}$$

The field content we consider is the same as discussed in Section 3.4.1 for both the bulk and boundary degrees of freedom. The first difference with respect to the case $V_\mu = 0$ arises in the gauge transformations, which become

$$\delta_g A_\mu(x) = \partial_\mu\lambda(x)\,, \tag{3.172a}$$

$$\delta_g B(x) = (\partial_\mu - V_\mu)\partial^\mu\lambda(x)\,, \tag{3.172b}$$

$$\delta_g \chi^{(k)}(y) = \partial_z^{k+1}\lambda(x)|_{z=0}\,. \tag{3.172c}$$

We now have all the ingredients to evaluate the gauge-invariant action (3.71), which

turns out to be

$$S(A_\mu, B; \chi^{(k)}) = -\frac{\alpha'}{2}\int_M d^d x\, e^{-V_\rho x^\rho}\left(\tfrac{1}{2}F_{\mu\nu}F^{\mu\nu} + \mathcal{K}^2\right) - \frac{\alpha'}{2}\int_{\partial M} d^{d-1} y\, e^{-V_b y^b}\,\mathcal{R}^{(0)}\mathcal{R}^{(1)}\,, \tag{3.173}$$

where we have denoted the gauge-invariant auxiliary structures

$$\mathcal{K} = B - (\partial^\mu - V^\mu)A_\mu\,, \tag{3.174a}$$

$$\mathcal{R}^{(0)} = \chi^{(0)} - A_z\,, \tag{3.174b}$$

$$\mathcal{R}^{(1)} = \chi^{(1)} - (\partial_a - V_a)\partial^a\chi^{(-1)} + 2(\partial_a - V_a)A^a - V_z A_z - B\,. \tag{3.174c}$$

As a sanity check, we can see that this action indeed reproduces (3.118) for $V_\mu = 0$. Furthermore, by solving the equations of motion for the auxiliary fields B and $\chi^{(1)}$, the structures $\mathcal{K}$, $\mathcal{R}^{(0)}$, $\mathcal{R}^{(1)}$ can be removed to obtain

$$S^*(A_\mu) = -\frac{\alpha'}{4}\int_M d^d x\, e^{-V_\rho x^\rho} F_{\mu\nu}F^{\mu\nu}\,. \tag{3.175}$$

The presence of a background charge is entirely encapsulated in the exponential prefactor. The action (3.175) reproduces the result appearing in [83] where, however, boundary terms were not present. This is parallel to what happens to the gauge-field in standard flat space-time.

The situation becomes somewhat more involved for the closed string action at level zero. In this case, the operators that constitute the BRST charge can be consistently truncated to

$$c = \tfrac{\alpha'}{2}c_0^+\,, \tag{3.176a}$$

$$\Omega_V^\mu = i\sqrt{\tfrac{\alpha'}{2}}(c_1\alpha_{-1}^\mu + c_{-1}\alpha_1^\mu + \bar{c}_1\bar{\alpha}_{-1}^\mu + \bar{c}_{-1}\bar{\alpha}_1^\mu) - \tfrac{\alpha'}{2}c_0^+ V^\mu\,, \tag{3.176b}$$

$$Q_V' = -b_0^+(c_{-1}c_1 + \bar{c}_{-1}\bar{c}_1) + i\sqrt{\tfrac{\alpha'}{2}}V_\mu(c_{-1}\alpha_1^\mu + \bar{c}_{-1}\bar{\alpha}_1^\mu)\,. \tag{3.176c}$$

In parallel with the definitions given in Section 3.4.3, namely considering (3.126) and (3.127) as the bulk and boundary string fields, respectively, (3.128) as the gauge string field parameter, and (3.129) as the gauge-for-gauge string field parameter, we

obtain that the gauge transformations (3.72a) and (3.72b) correspond, at the level of the target space fields, to

$$\delta_g e_{\mu\nu}(x) = \partial_\mu \bar{\xi}_\nu(x) + \partial_\nu \xi_\mu(x)\,, \tag{3.177a}$$

$$\delta_g e_\mu(x) = -\tfrac{1}{2}(\partial^\nu - V^\nu)\partial_\nu \xi_\mu(x) - \partial_\mu \xi(x)\,, \tag{3.177b}$$

$$\delta_g \bar{e}_\mu(x) = +\tfrac{1}{2}(\partial^\nu - V^\nu)\partial_\nu \bar{\xi}_\mu(x) - \partial_\mu \xi(x)\,, \tag{3.177c}$$

$$\delta_g e(x) = -\tfrac{1}{2}(\partial_\mu - V_\mu)\xi^\mu(x) - \xi(x)\,, \tag{3.177d}$$

$$\delta_g \bar{e}(x) = +\tfrac{1}{2}(\partial_\mu - V_\mu)\bar{\xi}^\mu(x) - \xi(x)\,, \tag{3.177e}$$

and

$$\delta_g \chi_\mu(x) = \xi_\mu(x) - \partial_\mu u(x)\,, \tag{3.178a}$$

$$\delta_g \bar{\chi}_\mu(x) = \bar{\xi}_\mu(x) + \partial_\mu u(x)\,, \tag{3.178b}$$

$$\delta_g \chi(x) = \xi(x) + \tfrac{1}{2}(\partial^\mu - V^\mu)\partial_\mu u(x)\,. \tag{3.178c}$$

Exactly as discussed in Section 3.4.3, it is useful to split the tensor field $e_{\mu\nu}$ into its symmetric and antisymmetric parts, to define the boundary field l according to (3.135) and to introduce the field ϕ defined in (3.139), which is no longer gauge-invariant. Indeed, by specializing (3.177) for $h_{\mu\nu}$, $b_{\mu\nu}$, l and ϕ, we obtain

$$\delta_g b_{\mu\nu}(x) = \partial_\mu \lambda_\nu(x) - \partial_\nu \lambda_\mu(x)\,, \tag{3.179a}$$

$$\delta_g h_{\mu\nu}(x) = \partial_\mu \tau_\nu(x) + \partial_\nu \tau_\mu(x)\,, \tag{3.179b}$$

$$\delta_g l(y) = 2\tau_z(x)|_{z=0}\,, \tag{3.179c}$$

$$\delta_g \phi(x) = \tfrac{1}{2} V_\mu \tau^\mu(x)\,, \tag{3.179d}$$

where we have redefined the gauge parameters (3.142). Under these definitions, the

gauge-invariant action (3.71) becomes

$$
\begin{aligned}
&S^*\left(h_{\mu\nu}, b_{\mu\nu}, \phi; l\right) = \\
&\quad = -\frac{\alpha'}{16}\int_M d^dx\, e^{-V_\rho x^\rho}\left(\tfrac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho}\right) + \\
&\qquad -\frac{\alpha'}{16}\int_M d^dx\, e^{-V_\rho x^\rho}\Big(\tfrac{1}{4}\partial_\rho h_{\mu\nu}\partial^\rho h^{\mu\nu} - \tfrac{1}{2}\partial_\rho h_{\mu\nu}\partial^\nu h^{\mu\rho} + \tfrac{1}{2}(\partial^\mu - V^\mu)h_{\mu\nu}\partial^\nu h^\lambda{}_\lambda \\
&\qquad\qquad - \tfrac{1}{4}\partial_\nu h^\rho{}_\rho\partial^\nu h^\lambda{}_\lambda - 2\big((\partial^\mu - V^\mu)h_{\mu\nu} - \partial_\nu h^\lambda{}_\lambda\big)\partial^\nu\phi - 4\partial_\mu\phi\partial^\mu\phi\Big) \\
&\qquad + \frac{\alpha'}{16}\int_{\partial M} d^{d-1}y\, e^{-V_c y^c}\Big[\tfrac{1}{2}h_{az}\partial^a\big(h^b{}_b - h_{zz} - 4\phi\big) - \tfrac{1}{2}h^{ab}\partial_a\partial_b l - \tfrac{1}{2}\partial^a l\partial_a\big(h^b{}_b - 4\phi\big) \\
&\qquad\qquad - \tfrac{1}{4}V_z h_{zz}^2 - \tfrac{1}{4}V_z\partial_a l\partial^a l\Big].
\end{aligned}
\tag{3.180}
$$

For clarity, we have omitted the intricate and rich structures associated with the numerous auxiliary fields, which anyway vanish upon solving their corresponding equations of motion. As a consistency check, we observe that the action above reproduces equation (3.140) as we take $V_\mu \to 0$. In this case, the effect of working in the linear dilaton background manifests not only in the exponential prefactor. Indeed, there are additional terms that play a crucial role in restoring gauge invariance. Interestingly, and in contrast to the free boson case, the boundary mode l appears not only linearly in the action but also through a quadratic term proportional to the orthogonal component of the background charge.

CHAPTER 4

Algebraic structure of String Field Theory

4.1 SFT in coalgebra formalism and homotopy algebras

Here, we will discuss the algebraic properties underlying classical open and closed SFT and quantum closed SFT. We will leave the discussion of the full quantum open-closed SFT action to the dedicated Chapter 5.

In particular, the plan is to reformulate the SFT action by introducing the so-called multi-string products. These are maps that take as input a given number n of string states and produce a resulting state whose overlap, through the symplectic form, with another string state yields the (n+1)-string fundamental vertex. These higher-order products are constrained by the BV quantum master equation (2.123), and these conditions define homotopy algebras. Among them we will recognize, at the classical level, the well-known A_∞ and L_∞ algebras first introduced in [84–88] and then studied in the context of bosonic SFT in [10, 58, 89–91] (see also [92–96] for applications in super SFT). At the quantum level, we will describe the closed-loop homotopy algebra first introduced in [97].

For technical reasons, it is advantageous to adopt the tensor co-algebra formalism that we will describe in Section 4.1.2. This approach allows us to package the action into a compact form, making it far more straightforward to impose the BV master equation. Moreover, using co-algebras simplifies the expression of gauge transformations and the equations of motion, and facilitates the construction of the Wilsonian effective action.

4.1.1 Multi-string products

As discussed in Chapter 2, the dynamical closed SFT action can be written as

$$S_{\rm c}(\Phi) = \sum_{g=0}^{\infty} \kappa^{2g-2} \sum_{k=1}^{\infty} \frac{1}{k!} V_k^g(\Phi^{\wedge k}) \,, \tag{4.1}$$

where the extra factor κ^k in (2.187), arising from the Euler characteristic, has been absorbed by redefining the closed string field via

$$\Phi \to \frac{1}{\kappa}\Phi \,, \tag{4.2}$$

this also implies a rescaling of the BV operators

$$(\cdot,\cdot)_{\mathrm{c}} = \kappa^2 \frac{\overleftarrow{\partial}}{\partial\phi^a}(\omega_{\mathrm{c}})^{ab}\frac{\overrightarrow{\partial}}{\partial\phi^b}\,, \tag{4.3a}$$

$$\Delta_{\mathrm{c}} = \frac{\kappa^2}{2}(-)^{\phi^a}(\omega_{\mathrm{c}})^{ab}\frac{\overrightarrow{\partial}}{\partial\phi^a}\frac{\overrightarrow{\partial}}{\partial\phi^b}\,. \tag{4.3b}$$

The vertices $V_k^g(\Phi^{\wedge k})$ can be expressed through the symplectic form ω_{c}, as

$$V_{k+1}^g(\Phi^{\wedge k+1}) \coloneqq \omega_{\mathrm{c}}(\Phi, l_k^{(g)}(\Phi^{\wedge k}))\,, \tag{4.4}$$

where we introduced degree-1 symmetric multilinear maps

$$l_k^{(g)} : \mathcal{H}_{\mathrm{c}}^{\wedge k} \to \mathcal{H}_{\mathrm{c}}\,, \tag{4.5}$$

with

$$l_1^{(0)} = Q_{\mathrm{c}}\,. \tag{4.6}$$

To ensure full symmetry among punctures, the multi-string products must be cyclic,

$$\omega_{\mathrm{c}}\left(\Phi_1, l_{k,0}^{(g)}\left(\Phi_2\wedge\cdots\wedge\Phi_{k+1}\right)\right) = (-1)^{\epsilon}\omega_{\mathrm{c}}\left(\Phi_{k+1}, l_{k,0}^{(g)}\left(\Phi_1\wedge\cdots\wedge\Phi_k\right)\right)\,, \tag{4.7}$$

and BPZ-odd,

$$\omega_{\mathrm{c}}\left(\Phi_1, l_k^{(g)}\left(\Phi_2\wedge\cdots\wedge\Phi_{k+1}\right)\right) = -(-1)^{d(\Phi_1)}\omega_{\mathrm{c}}\left(l_k^{(g)}\left(\Phi_1\wedge\cdots\wedge\Phi_k\right),\Phi_{k+1}\right)\,. \tag{4.8}$$

Thus, the closed string action becomes

$$S_{\mathrm{c}}(\Phi) = \sum_{g=0}^{\infty}\kappa^{2g-2}\sum_{k=0}^{\infty}\frac{1}{(k+1)!}\omega_{\mathrm{c}}(\Phi, l_k^{(g)}(\Phi^{\wedge k}))\,. \tag{4.9}$$

The next step is to require that (4.9) satisfies the BV quantum master equation, which enforces that the multi-string products obey a quantum homotopy algebra, as we will explain in Section 4.1.4.

Let us now turn to the open string sector. In general, a purely open quantum-

consistent SFT action does not exist, because loop amplitudes for purely open string processes suffer from UV divergences. These divergences are commonly understood as IR divergences from the closed string side through the open-closed worldsheet duality. This can be explicitly seen in Appendix E, where we treat the amplitude of a single open string on the annulus. Therefore, to have a consistent quantum treatment of the open string dynamics, we must consider the full open-closed SFT, which we will describe in Chapter 5. However, we can consistently define the classical action describing the dynamics of open strings on the disk

$$S_{\text{disk}}(\Psi) = \kappa^{-1} \sum_{p=2}^{\infty} \frac{1}{p} V_p^{\text{disk}}(\Psi^{\odot p}) \,, \tag{4.10}$$

where the power of the string coupling constant can be understood as follows: according to the topological expansion, the weight of the vertices is $\kappa^{-\chi_{k,p}^{g,b}}$, where $\chi_{k,p}^{g,b}$ is the Euler characteristic of a Riemann surface with genus g, b boundaries, k bulk insertions, and p boundary insertions

$$\chi_{k,p}^{g,b} = 2 - 2g - b - k - \frac{1}{2}p \,. \tag{4.11}$$

In this case we have $\chi_{0,p}^{0,1} = 1 - \frac{1}{2}p$ and then by rescaling the open string field as

$$\Psi \to \frac{1}{\sqrt{\kappa}} \Psi \,, \tag{4.12}$$

we reabsorb the factor $\kappa^{-\frac{1}{2}p}$. This rescaling yields to the following BV differentials

$$(\cdot,\cdot)_{\text{o}} = \kappa \frac{\overleftarrow{\partial}}{\partial \psi^a} (\omega_{\text{o}})^{ab} \frac{\overrightarrow{\partial}}{\partial \psi^b} \,, \tag{4.13a}$$

$$\Delta_{\text{o}} = \frac{\kappa}{2} (-)^{\phi^a} (\omega_{\text{o}})^{ab} \frac{\overrightarrow{\partial}}{\partial \psi^a} \frac{\overrightarrow{\partial}}{\partial \psi^b} \,. \tag{4.13b}$$

Moreover, in (4.10) we introduced the notation $\odot$ to denote the cyclicized tensor product

$$\Psi_1 \odot \cdots \odot \Psi_p \coloneqq \sum_{\sigma \in \mathbf{Z}_p} (-1)^{\epsilon_\sigma} \Psi_{\sigma(1)} \otimes \cdots \otimes \Psi_{\sigma(p)} \,, \tag{4.14}$$

where $\mathbf{Z}_p$ is the group of cyclic permutations. Again, also in this framework we can introduce degree odd multlinear maps

$$m_p : \mathcal{H}_{\rm o}^{\otimes p} \to \mathcal{H}_{\rm o}\,, \tag{4.15}$$

cyclic with respect to $\omega_{\rm o}$

$$\omega_{\rm o}\left(\Psi_1, m_p\left(\Psi_2 \otimes \cdots \otimes \Psi_{p+1}\right)\right) = (-1)^{\epsilon}\omega_{\rm o}\left(\Psi_{p+1}, m_p\left(\Psi_1 \otimes \cdots \otimes \Psi_k\right)\right)\,, \tag{4.16}$$

and BPZ odd

$$\omega_{\rm o}\left(\Psi_1, m_p\left(\Psi_2 \otimes \cdots \otimes \Phi_{p+1}\right)\right) = -(-1)^{d(\Psi_1)}\omega_{\rm o}\left(m_p\left(\Psi_1 \otimes \cdots \otimes \Psi_k\right), \Psi_{p+1}\right)\,, \tag{4.17}$$

defined such that

$$V_{p+1}^{\rm disk}(\Psi^{\odot p+1}) = \omega_{\rm o}(\Psi, m_p(\Psi^{\otimes p}))\,, \tag{4.18}$$

where, in particular

$$m_1 = Q_{\rm o}\,. \tag{4.19}$$

Therefore, (4.10) becomes

$$S_{\rm disk}(\Psi) = \kappa^{-1}\sum_{p=1}^{\infty}\frac{1}{p+1}\omega_{\rm o}(\Psi, m_p(\Psi^{\otimes p}))\,. \tag{4.20}$$

At this point, we might directly require that (4.9) and (4.20) be solutions of the quantum or classical BV master equation, respectively, thereby obtaining the associated homotopy relations. However, this involves a rather long and intricate computation. Therefore, we postpone this task to the next section, where we introduce the co-algebraic formalism, which significantly facilitates the computation.

4.1.2 Co-algebraic formalism

The central idea behind reformulating SFT in the co-algebraic formalism is to promote the maps acting on $\mathcal{H}_{\rm o}$ and $\mathcal{H}_{\rm c}$ to operators on the tensor algebra

$$T\mathcal{H}_{\rm o} = \bigoplus_{p=0}^{\infty} \mathcal{H}_{\rm o}^{\otimes p} = 1_{T\mathcal{H}_{\rm o}} + \mathcal{H}_{\rm o} + \mathcal{H}_{\rm o} \otimes \mathcal{H}_{\rm o} + \dots \,, \tag{4.21}$$

and the symmetrized tensor algebra

$$S\mathcal{H}_{\rm c} = \bigoplus_{k=0}^{\infty} \mathcal{H}_{\rm c}^{\wedge k} = 1_{T\mathcal{H}_{\rm c}} + \mathcal{H}_{\rm c} + \mathcal{H}_{\rm c} \wedge \mathcal{H}_{\rm c} + \dots \,, \tag{4.22}$$

respectively. In these expressions, $1_{T\mathcal{H}_{\rm o}}$ and $1_{T\mathcal{H}_{\rm c}}$ are the identities in the sense that $1_{T\mathcal{H}} \otimes \Psi = \Psi \otimes 1_{T\mathcal{H}} = \Psi$. We then introduce the co-products

$$\Delta_{T\mathcal{H}_{\rm o}} : T\mathcal{H}_{\rm o} \to T\mathcal{H}_{\rm o} \otimes' T\mathcal{H}_{\rm o} \,, \tag{4.23a}$$

$$\Delta_{S\mathcal{H}_{\rm c}} : S\mathcal{H}_{\rm c} \to S\mathcal{H}_{\rm c} \otimes' S\mathcal{H}_{\rm c} \,, \tag{4.23b}$$

defined as

$$\Delta_{T\mathcal{H}_{\rm o}}(\Psi_1 \otimes \cdots \otimes \Psi_p) \coloneqq \sum_{i=0}^{p} (\Psi_1 \otimes \cdots \otimes \Psi_i) \otimes' (\Psi_{i+1} \otimes \cdots \otimes \Psi_p) \,, \tag{4.24}$$

which corresponds to splitting the multi-string state in all possible ways by inserting $\otimes'$, with the rule that for $i = 0$ and $i = p$ the sum gives $1_{T\mathcal{H}_{\rm o}} \otimes' (\Psi_1 \otimes \cdots \otimes \Psi_p)$ and $(\Psi_1 \otimes \cdots \otimes \Psi_p) \otimes' 1_{T\mathcal{H}_{\rm o}}$, respectively. This is crucial for ensuring the co-associativity of the co-product. For the symmetrized tensor algebra, the co-product acts as

$$\Delta_{S\mathcal{H}_{\rm c}}(\Phi_1 \wedge \cdots \wedge \Phi_k) \coloneqq \sum_{\substack{l_1, l_2 \\ l_1 + l_2 = k}} \sum_{\sigma \in \mathbf{S}_{l_1+l_2}} \frac{(-1)^{\epsilon(\sigma)}}{l_1! l_2!} (\Phi_{\sigma(1)} \wedge \cdots \wedge \Phi_{\sigma(l_1)}) \otimes' (\Phi_{\sigma(l_1+1)} \wedge \cdots \wedge \Phi_{\sigma(l_1+l_2)}) \,. \tag{4.25}$$

The pairs $(T\mathcal{H}_{\rm o}, \Delta_{T\mathcal{H}_{\rm o}})$ and $(S\mathcal{H}_{\rm c}, \Delta_{S\mathcal{H}_{\rm c}})$ form a tensor co-algebra and a symmetrized tensor co-algebra, respectively.

It is useful to introduce the projectors

$$\pi_k : T\mathcal{H}_{\rm o} \to \mathcal{H}_{\rm o}^{\otimes k}\,, \tag{4.26}$$

which project onto the subspace of multi-string states composed by the tensor product of k string fields. They act in the same way on $S\mathcal{H}_{\rm c}$.

Furthermore, the combinations

$$\frac{1}{1-\Psi} \coloneqq \sum_{p=0}^{\infty} \Psi^{\otimes p} = 1_{T\mathcal{H}_{\rm o}} + \Psi + \Psi\otimes\Psi + \dots\,, \tag{4.27}$$

and

$$e^{\wedge\Phi} \coloneqq \sum_{k=0}^{\infty} \frac{1}{k!}\Phi^{\wedge k} = 1_{T\mathcal{H}_{\rm c}} + \Phi + \frac{1}{2!}\Phi\wedge\Phi + \dots\,, \tag{4.28}$$

for $\Psi \in \mathcal{H}_{\rm o}$ and $\Phi \in \mathcal{H}_{\rm c}$ are called group-like elements, as they satisfy

$$\Delta_{T\mathcal{H}_{\rm o}} \frac{1}{1-\Psi} = \frac{1}{1-\Psi} \otimes' \frac{1}{1-\Psi}\,, \tag{4.29}$$

and

$$\Delta_{S\mathcal{H}_{\rm c}} e^{\wedge\Phi} = e^{\wedge\Phi} \otimes' e^{\wedge\Phi}\,. \tag{4.30}$$

Let us now introduce two important concepts: co-homomorphisms and co-derivations. Given a linear map $\boldsymbol{F} : T\mathcal{H}_{\rm o} \to T\mathcal{H}_{\rm o}$[1], we say it is a cohomomorphism if it satisfies the following relation with respect to the co-product

$$\Delta_{T\mathcal{H}_{\rm o}}\boldsymbol{F} = (\boldsymbol{F}\otimes'\boldsymbol{F})\Delta_{T\mathcal{H}_{\rm o}}\,. \tag{4.31}$$

Notably, the composition of two (or more) co-homomorphisms is again a co-homomorphism. Let F be a single input linear map acting on the starting Hilbert space, $F : \mathcal{H}_{\rm c} \to \mathcal{H}_{\rm c}$.

[1] In principle a co-homomorphism can act between two different tensor algebras $\boldsymbol{F} : T\mathcal{H}_1 \to T\mathcal{H}_2$ equipped by two different co-products $\Delta_{T\mathcal{H}_1}$ and $\Delta_{T\mathcal{H}_2}$, in this case the defining relation is $\Delta_{T\mathcal{H}_2}\boldsymbol{F} = (\boldsymbol{F}\otimes'\boldsymbol{F})\Delta_{T\mathcal{H}_1}$.

It can be uplifted to a co-homomorphism through the following relation

$$\boldsymbol{F}\pi_N \coloneqq F^{\otimes N}\pi_N\,, \tag{4.32}$$

which trivially satisfies (4.31).

Given a map $\boldsymbol{c} : T\mathcal{H}_\mathrm{o} \to T\mathcal{H}_\mathrm{o}$, it is said to be a co-derivation if it obeys the co-Leibniz rule

$$\Delta_{T\mathcal{H}_\mathrm{o}}\boldsymbol{c} = (\mathbf{1}_{T\mathcal{H}_\mathrm{o}} \otimes' \boldsymbol{c} + \boldsymbol{c} \otimes' \mathbf{1}_{T\mathcal{H}_\mathrm{o}})\Delta_{T\mathcal{H}_\mathrm{o}}\,, \tag{4.33}$$

where $\mathbf{1}_{T\mathcal{H}_\mathrm{o}}$ is the identity map of $T\mathcal{H}_\mathrm{o}$, viewed as the uplift of the identity map on $\mathcal{H}_\mathrm{o}$, defined by $1_{\mathcal{H}_\mathrm{o}}(\Psi) = \Psi$, acting as a co-homomorphism

$$\mathbf{1}_{T\mathcal{H}_\mathrm{o}}\pi_N = 1_{\mathcal{H}_\mathrm{o}}^{\otimes N}\pi_N\,. \tag{4.34}$$

The composition of two coderivations $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$ is not necessarily a coderivation; however, their graded commutator is, as can be explicitly verified by applying the co-product using the co-Leibniz rule (4.33)

$$\Delta_{T\mathcal{H}_\mathrm{o}}[\boldsymbol{c}_1, \boldsymbol{c}_2] = (\mathbf{1}_{T\mathcal{H}_\mathrm{o}} \otimes' [\boldsymbol{c}_1, \boldsymbol{c}_2] + [\boldsymbol{c}_1, \boldsymbol{c}_2] \otimes' \mathbf{1}_{T\mathcal{H}_\mathrm{o}})\Delta_{T\mathcal{H}_\mathrm{o}}\,. \tag{4.35}$$

Analogously, co-homomorphisms and co-derivations can be defined on a symmetrized co-algebra.

Co-homomorphism and co-derivation acts as follow on the group-like element (4.27)

$$\boldsymbol{F}\frac{1}{1-\Psi} = \frac{1}{1-\pi_1\boldsymbol{F}\frac{1}{1-\Psi}}\,, \tag{4.36a}$$

$$\boldsymbol{c}\frac{1}{1-\Psi} = \frac{1}{1-\Psi} \otimes \pi_1\boldsymbol{c}\frac{1}{1-\Psi} \otimes \frac{1}{1-\Psi}\,, \tag{4.36b}$$

and as

$$\boldsymbol{F}e^{\wedge\Phi} = e^{\wedge\pi_1\boldsymbol{F}e^{\wedge\Phi}}\,, \tag{4.37a}$$

$$\boldsymbol{c}e^{\wedge\Phi} = e^{\wedge\Phi}\wedge\pi_1\boldsymbol{c}e^{\wedge\Phi}\,, \tag{4.37b}$$

on (4.28). Note that (4.36a) and (4.37a) tell us that co-homomorphism maps group-like elements into group-like elements.

We uplift the multi-string products m_p to co-derivations $\boldsymbol{m}_p$ on $T\mathcal{H}_\mathrm{o}$ by defining their action on a collection of states as

$$\boldsymbol{m}_p\pi_N \coloneqq \sum_{n=0}^{N-p}(1_{\mathcal{H}_\mathrm{o}}^{\otimes N-n-p}\otimes m_p\otimes 1_{\mathcal{H}_\mathrm{o}}^{\otimes n})\pi_N\,, \tag{4.38}$$

for $N\geq p$ and zero otherwise. This definition satisfies the co-Leibniz rule (4.33). It is natural to define the total co-derivation as

$$\boldsymbol{m}\coloneqq\sum_{p=1}^{\infty}\boldsymbol{m}_p\,. \tag{4.39}$$

Similarly, in the closed string framework, we promote the multi-string products $l_k^{(g)}$ to co-derivations on $S\mathcal{H}_\mathrm{c}$ with respect to $\Delta_{S\mathcal{H}_\mathrm{c}}$, defined by

$$\boldsymbol{l}_k^{(g)}\pi_N = \left(l_k^{(g)}\wedge 1_{\mathcal{H}_\mathrm{c}}^{\wedge N-k}\right)\pi_N\,. \tag{4.40}$$

Note that in (4.40) there are also the zero-string maps $l_0^{(g)}$, which are vectors in $\mathcal{H}_\mathrm{c}$. The corresponding 0 co-derivations $\boldsymbol{l}_0^{(g)}$ insert these vectors in all possible ways, increasing the number of states $\boldsymbol{l}_0^{(g)}\pi_N : \mathcal{H}_\mathrm{c}^{\wedge N}\to\mathcal{H}_\mathrm{c}^{\wedge N+1}$. In contrast, for $k>0$, the co-derivations $\boldsymbol{l}_k^{(g)}$ always decrease the number of states $\boldsymbol{l}_{k>0}^{(g)}\pi_N : \mathcal{H}_\mathrm{c}^{\wedge N}\to\mathcal{H}_\mathrm{c}^{\wedge N-k+1}$.

Also in this framework, we define the total closed-string co-derivation as

$$\boldsymbol{l}\coloneqq\sum_{g=0}^{\infty}\kappa^{2g}\sum_{k=0}^{\infty}\boldsymbol{l}_k^{(g)}\,. \tag{4.41}$$

The cyclicity properties (4.7), (4.8), (4.16) and (4.17) translate at the level of

coderivations into

$$\omega_{\mathrm{o}}\left(\pi_1 \boldsymbol{c}_1 \frac{1}{1-\Psi}, \pi_1 \boldsymbol{m}\boldsymbol{c_2} \frac{1}{1-\Psi}\right) = -(-)^{c_1}\omega_{\mathrm{o}}\left(\pi_1 \boldsymbol{m}\boldsymbol{c}_1 \frac{1}{1-\Psi}, \pi_1 \boldsymbol{c_2} \frac{1}{1-\Psi}\right) , \quad (4.42a)$$

$$\omega_{\mathrm{c}}\left(\pi_1 \boldsymbol{c}_1 e^{\wedge\Phi}, \pi_1 \boldsymbol{l}\boldsymbol{c_2} e^{\wedge\Phi}\right) = -(-)^{c_1}\omega_{\mathrm{c}}\left(\pi_1 \boldsymbol{l}\boldsymbol{c}_1 e^{\wedge\Phi}, \pi_1 \boldsymbol{c_2} e^{\wedge\Phi}\right) , \quad (4.42b)$$

where $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$ are two generic co-derivations. These identities can be verified by explicitly unpacking the equations above.

4.1.3 Action in the Wess-Zumino-Witten form

In this section, we use the formalism just introduced to rewrite the actions (4.9) and (4.20) in a more compact form, and then require that they be solutions of the corresponding BV master equation. We focus on the closed-string case, though everything can be adapted straightforwardly to the open-string scenario.

The key point in rewriting (4.9) in the so-called Wess–Zumino–Witten (WZW) form is to interpolate the string field $\Phi(t)$ with real parameter $t \in [0,1]$, such that $\Phi(0) = 0$ and $\Phi(1) = \Phi$. By doing so, we can express the action as the integral over t of a total derivative

$$\begin{aligned} S_{\mathrm{c}}(\Phi) &= \int_0^1 dt\, \partial_t S(\Phi(t)) \\ &= \int_0^1 dt\, \frac{\omega_{\mathrm{c}}}{\kappa^2}\left(\dot{\Phi}(t), \sum_{g=0}^{\infty}\kappa^{2g}\sum_{k=0}^{\infty} l_k^{(g)}\left(\frac{1}{k!}\Phi(t)^{\wedge k}\right)\right) , \end{aligned} \quad (4.43)$$

here, in going from the first to the second line, we used cyclicity to combine the $k+1$ equal terms that arise from applying the derivative ∂_t through the Leibnitz rule. Furthermore, we reorganized the expression to make explicit the group-like element (4.28) and the total co-derivation (4.40). Consequently, the action reduces to

$$S_{\mathrm{c}}(\Phi) = \int_0^1 dt\, \frac{\omega_{\mathrm{c}}}{\kappa^2}\left(\pi_1 \boldsymbol{\partial}_t e^{\wedge\Phi(t)}, \pi_1 \boldsymbol{l} e^{\wedge\Phi(t)}\right) , \quad (4.44)$$

where, $\boldsymbol{\partial}_t$ denotes the co-derivation associated with the derivative ∂_t, defined according to (4.33) for a 1-string product, and we introduced the projector π_1 to ensure that the symplectic form takes only single-string states as its inputs.

Similarly, the disk action (4.20) reads

$$S_{\text{disk}}(\Psi) = \int_0^1 dt\, \frac{\omega_{\text{o}}}{\kappa} \left(\pi_1 \boldsymbol{\partial}_t \frac{1}{1-\Psi(t)}, \pi_1 \boldsymbol{m} \frac{1}{1-\Psi(t)} \right) . \tag{4.45}$$

We now aim to determine the conditions on $\boldsymbol{l}$ and $\boldsymbol{m}$ such that (4.44) and (4.45) are solutions of the corresponding BV master action. Again, we focus on the closed-string scenario, which exhibits a richer structure. To apply the BV operators (4.3a) and (4.3b), we must first compute derivatives with respect to the target BV fields

$$\frac{\overrightarrow{\partial}}{\partial \phi^a} S_{\text{c}}(\Phi) = (-)^{\phi^a} \frac{\omega_{\text{c}}}{\kappa^2} \left(c_a, \pi_1 \boldsymbol{l} e^{\wedge \Phi} \right) , \tag{4.46}$$

where using cyclicity (4.42b) we reconstructed a total derivative and then we computed the integral. Therefore, by reconstructing twice the completeness relation (2.73), we obtain

$$\begin{aligned} \frac{1}{2} \left(S_{\text{c}}, S_{\text{c}} \right)_{\text{c}} &= -\frac{1}{2} \frac{\omega_{\text{c}}}{\kappa^2} \left(\pi_1 \boldsymbol{l} e^{\wedge \Phi}, \pi_1 \boldsymbol{l} e^{\wedge \Phi} \right) \\ &= \int_0^1 dt\, \frac{\omega_{\text{c}}}{\kappa^2} \left(\dot{\Phi}(t), \pi_1 \boldsymbol{l}^2 e^{\wedge \Phi} \right) , \end{aligned} \tag{4.47}$$

where in the last line we reintroduced the interpolation $\Phi(t)$ and applied $\int_0^1 dt\, \partial_t$. For the BV Laplacian (4.3b), we find

$$\Delta_{\text{c}} S_{\text{c}} = \frac{1}{2} \omega_{\text{c}} \left(c_a, \pi_1 \boldsymbol{l} \boldsymbol{\partial}_{\phi^a} e^{\wedge \phi} \right) , \tag{4.48}$$

where $\boldsymbol{\partial}_{\phi^a}$ is the co-derivation uplifting of ∂_{ϕ^a}. Let us now proceed by unpacking the coderivations and, using cyclicity in the multi-string products, move the basis vectors

from the first entry to the right-hand side

$$\begin{aligned}\Delta_{\mathrm{c}} S_{\mathrm{c}} &= \frac{1}{2}\sum_{g=0}^{\infty}\kappa^{2g}\sum_{k=2}^{\infty}\frac{1}{(k-1)!}\omega_{\mathrm{c}}(c^a, l_k^{(g)}(c_a \wedge \Phi^{\wedge k-1}))\\ &= \sum_{g=0}^{\infty}\kappa^{2g}\sum_{k=2}^{\infty}\frac{1}{(k-1)!}\omega_{\mathrm{c}}(\Phi, l_k^{(g)}(U_{\mathrm{c}} \wedge \Phi^{\wedge k-2}))\\ &= \int_0^1 dt\,\omega_{\mathrm{c}}\left(\dot{\Phi}(t), \sum_{g=0}^{\infty}\kappa^{2g}\sum_{k=2} l_k^{(g)}\left(U_{\mathrm{c}} \wedge \frac{\Phi^{\wedge k-2}}{(k-2)!}\right)\right)\\ &= \int_0^1 dt\,\omega_{\mathrm{c}}\left(\dot{\Phi}(t), \pi_1 \boldsymbol{l}\left(U_{\mathrm{c}} \wedge e^{\wedge\Phi(t)}\right)\right)\,, \end{aligned} \tag{4.49}$$

in the first line, the sum over k starts from 2 because at $k = 1$ the resulting objects share the same degree structure as in (2.130)(which is the $g = 0$ case), and hence they vanish. For $k \geq 2$ we can always use cyclicity (4.7) and then obtain the second line, where we reconstructed the closed Poisson bi-vector (2.75). In the third line, we introduced the interpolator $\Phi(t)$ to recast the expression as a total derivative. In the fourth line, we reassembled the total coderivation (4.40). Finally, we define

$$\boldsymbol{U}_{\mathrm{c}} \coloneqq \frac{(-)^{c^a}}{2}\boldsymbol{c}_a\boldsymbol{c}^a\,, \tag{4.50}$$

where $\boldsymbol{c}_a$ and $\boldsymbol{c}^a$ are the 0 co-derivations associated with the basis and dual-basis vectors. Therefore, (4.49) becomes

$$\Delta_{\mathrm{c}} S_{\mathrm{c}} = \int_0^1 dt\,\omega_{\mathrm{c}}\left(\dot{\Phi}(t), \pi_1 \boldsymbol{l}\boldsymbol{U}_{\mathrm{c}} e^{\wedge\Phi(t)}\right)\,. \tag{4.51}$$

Note that (4.50) trivially satisfies

$$\boldsymbol{U}_{\mathrm{c}}^2 = \frac{1}{2}[\boldsymbol{U}_{\mathrm{c}}, \boldsymbol{U}_{\mathrm{c}}] = 0\,, \tag{4.52}$$

because it has odd degree, and

$$\pi_1 \boldsymbol{U}_{\mathrm{c}} = 0\,, \tag{4.53}$$

as it always increases the number of states by two. Thanks to these two properties,

the BV quantum master equation can be rewritten as

$$\frac{1}{2}(S_{\rm c}, S_{\rm c})_{\rm c} + \Delta_{\rm c} S_{\rm c} = \int_0^1 dt\, \frac{\omega_{\rm c}}{\kappa^2} \left(\dot{\Phi}(t), \pi_1 \left(\boldsymbol{l} + \kappa^2 \boldsymbol{U}_{\rm c} \right)^2 e^{\wedge \Phi(t)} \right) . \tag{4.54}$$

Therefore, the closed SFT action is quantum-consistent provided the multi-string products satisfy the nilpotency condition

$$\left(\boldsymbol{l} + \kappa^2 \boldsymbol{U}_{\rm c} \right)^2 = 0 \, . \tag{4.55}$$

Equivalently, for the disk action we have

$$\frac{1}{2}(S_{\rm disk}, S_{\rm disk})_{\rm c} = \int_0^1 dt\, \frac{\omega_{\rm o}}{\kappa} \left(\dot{\Psi}(t), \pi_1 \boldsymbol{m}^2 \frac{1}{1 - \Psi(t)} \right) , \tag{4.56}$$

with classical consistency condition

$$\boldsymbol{m}^2 = \frac{1}{2}[\boldsymbol{m}, \boldsymbol{m}] = 0 \, , \tag{4.57}$$

which precisely defines the A_∞ algebra structure, as will be discussed in detail in the next section.

4.1.4 Classical and quantum homotopy algebras

In this section, we expand (4.55) and (4.57) to better understand the structure of these homotopy algebras in terms of the multi-string products, and then explore their geometric interpretation, keeping in mind the link between the differential and the geometric BV operators described in Section (2.3.4).

Let us first consider the closed-string scenario. Expanding (4.55) in powers of the string coupling constant, we obtain:

- $O(\kappa^0)$: The classical strong[2] L_∞ relations that guarantee consistency of the closed-string sector on the sphere

$$[\boldsymbol{l}^{(0)}, \boldsymbol{l}^{(0)}] = 0 \, , \tag{4.58}$$

[2]An L_∞ algebra is called strong if it has no tadpole term ($\boldsymbol{l}_0 = 0$); otherwise, it is referred to as weak.

Evaluated on a fixed number of string states:

- π_0: Trivial, no closed-string tadpole on the sphere.
- π_1: BRST nilpotency

$$Q_{\mathrm{c}}^2 = 0\,, \tag{4.59}$$

- π_2: The Leibnitz rule for Q_{c} acting on the $2-$product

$$Q_{\mathrm{c}} l_2^{(0)}(\Phi_1,\Phi_2) + l_2^{(0)}(Q_{\mathrm{c}}\Phi_1,\Phi_2) + (-)^{\Phi_1} l_2^{(0)}(\Phi_1, Q_{\mathrm{c}}\Phi_2) = 0\,, \tag{4.60}$$

- π_3: The failure of the Leibniz rule of Q_{c} on $l_3^{(0)}$ is compensated by the failure of l_2 to satisfy the super-Jacobi identity

$$\begin{aligned} 0 = {} & Q_{\mathrm{c}} l_3^{(0)}(\Phi_1,\Phi_2,\Phi_3) + l_3^{(0)}(Q_{\mathrm{c}}\Phi_1,\Phi_2,\Phi_3) + (-)^{\Phi_1} l_3^{(0)}(\Phi_1, Q_{\mathrm{c}}\Phi_2,\Phi_3) \\ & + (-)^{\Phi_1+\Phi_2} l_3^{(0)}(\Phi_1,\Phi_2, Q_{\mathrm{c}}\Phi_3) \\ & + l_2^{(0)}\left(l_2^{(0)}(\Phi_1,\Phi_2),\Phi_3\right) + (-)^{\Phi_1(\Phi_2+\Phi_3)} l_2^{(0)}\left(l_2^{(0)}(\Phi_2,\Phi_3),\Phi_1\right) \\ & + (-)^{\Phi_3(\Phi_1+\Phi_2)} l_2^{(0)}\left(l_2^{(0)}(\Phi_3,\Phi_1),\Phi_2\right)\,. \end{aligned} \tag{4.61}$$

- π_4 and beyond: Higher relations describing spheres with more punctures.

- $O(\kappa^2)$: Relations governing closed-string amplitudes on the torus

$$\left[\boldsymbol{l}^{(0)}, \boldsymbol{l}^{(1)} + \boldsymbol{U}_{\mathrm{c}}\right] = 0\,, \tag{4.62}$$

on the identity state (π_0), this gives

$$Q_{\mathrm{c}} l_0^{(1)} + l_2^{(0)}(U_{\mathrm{c}}) = 0\,, \tag{4.63}$$

which expresses the quantum consistency of the one-loop amplitude for a single closed string on the torus, as analyzed in detail in [98]. Geometrically, (4.63) asserts that the boundary of the fundamental domain $\mathcal{V}_1^1$ must coincide with the boundary of the Feynman-diagram region, which is constructed via the non-separating plumbing-fixture procedure on the sphere with three punctures.

- $O(\kappa^4)$: Relations ensuring consistency at two-loop level

$$[\boldsymbol{l}^{(0)}, \boldsymbol{l}^{(2)}] + \frac{1}{2}[\boldsymbol{l}^{(1)}, \boldsymbol{l}^{(1)}] + [\boldsymbol{l}^{(1)}, \boldsymbol{U}_{\rm c}] = 0\,, \tag{4.64}$$

by focusing on the 2-loop tadpole (π_0) we obtain

$$Q_{\rm c} l_0^{(2)} + l_1^{(1)}(l_0^{(1)}) + l_2^{(1)}(U_{\rm c}) = 0\,, \tag{4.65}$$

meaning that the boundary of the genus-2 fundamental domain $\mathcal{V}_1^2$, which corresponds to the first term in the above equation, must coincide with the boundaries of the Feynman region obtained by gluing two tori together and by self-gluing a torus, represented respectively by the second and third terms.

- $O(\kappa^6)$: Relations for genus-3 amplitudes, and so on.

In the open-string framework, we have the A_∞ relations (4.57):

- π_0: Trivial, there is no open-string tadpole on the disk.

- π_1: BRST nilpotency

$$Q_{\rm o}^2 = 0\,. \tag{4.66}$$

- π_2: The Leibnitz rule for $Q_{\rm o}$ acting on the $2-$product

$$Q_{\rm o} m_2(\Psi_1, \Psi_2) + m_2(Q_{\rm c}\Psi_1, \Psi_2) + (-)^{\Psi_1} m_2(\Psi_1, Q_{\rm c}\Psi_2) = 0\,. \tag{4.67}$$

- π_3: The failure of the Leibniz rule of $Q_{\rm o}$ on m_3 is precisely compensated by the failure of m_2 to be associative

$$\begin{aligned} 0 = {} & Q_{\rm o} m_3(\Psi_1, \Psi_2, \Psi_3) + m_3(Q_{\rm o}\Psi_1, \Psi_2, \Psi_3) + (-)^{\Psi_1} m_3(\Psi_1, Q_{\rm o}\Psi_2, \Psi_3) \\ & + (-)^{\Psi_1+\Psi_2} m_3(\Psi_1, \Psi_2, Q_{\rm o}\Psi_3) \\ & + m_2\left(m_2(\Psi_1, \Psi_2), \Psi_3\right) + (-)^{\Psi_1} m_2\left(\Psi_1, m_2(\Psi_2, \Psi_3)\right)\,. \end{aligned} \tag{4.68}$$

- π_4 and beyond: These govern disks amplitudes with five or more punctures and

automatically hold when one truncates to a strictly cubic theory, i.e., setting $m_{k>2} = 0$.

Until now, we have considered the generic case of non-polynomial SFT actions, which lead to an infinite number of multi-string products. However, a compelling open problem is to describe SFT dynamics with a finite number of vertices, namely, to identify multi-string products that satisfy truncated versions of the homotopy algebras just described.

The most fundamental example of a fully gauge-invariant polynomial SFT is due to Witten [9]. He constructed a cubic action that consistently reproduces classical open-string amplitudes. In this framework, one defines a 2-product called the Witten star product

$$m_2(\Psi_1, \Psi_2) = (-)^{\Psi_1} \Psi_1 * \Psi_2 \,, \tag{4.69}$$

which is associative and such that $Q_{\rm o}$ acts as a derivation on it, thereby satisfying the truncated A_∞ algebra for $m_{k>2} = 0$.

4.1.5 Gauge transformations and equation of motions

As anticipated, another powerful application of the co-algebra formalism lies in describing gauge transformations and deriving the equations of motion. In particular, let us focus on the classical sectors represented by the disk and sphere actions, which we report here for completeness

$$S_{\rm disk}(\Psi) = \int_0^1 dt\, \frac{\omega_{\rm o}}{\kappa} \left(\pi_1 \boldsymbol{\partial}_t \frac{1}{1-\Psi(t)}, \pi_1 \boldsymbol{m} \frac{1}{1-\Psi(t)} \right) , \tag{4.70}$$

and

$$S_{\rm sphere}(\Phi) = \int_0^1 dt\, \frac{\omega_{\rm c}}{\kappa^2} \left(\pi_1 \boldsymbol{\partial}_t e^{\wedge \Phi(t)}, \pi_1 \boldsymbol{l}^{(0)} e^{\wedge \Phi(t)} \right) , \tag{4.71}$$

where the multi-string products satisfy the homotopy relations

$$[\boldsymbol{m}, \boldsymbol{m}] = 0 \,, \tag{4.72}$$

and

$$[\boldsymbol{l}^{(0)}, \boldsymbol{l}^{(0)}] = 0\,. \tag{4.73}$$

To understand how an infinitesimal variation $\delta\Psi$ can be described in the coalgebraic formalism, let us first consider a finite field transformation. Since cohomomorphisms map group-like elements to group-like elements (see (4.36a) and (4.37a)), any generic field redefinition can be implemented via a degree-even cohomomorphism as follows

$$\frac{1}{1-\Psi'} = \boldsymbol{F}\frac{1}{1-\Psi}\,. \tag{4.74}$$

Assuming that $\boldsymbol{F}$ admits an expansion in a small parameter ϵ,

$$\boldsymbol{F} = 1_{T\mathcal{H}_{\rm o}} + \epsilon\boldsymbol{\delta} + O(\epsilon^2)\,, \tag{4.75}$$

it follows immediately that the first-order variation is

$$\delta\Psi = \pi_1\boldsymbol{\delta}\frac{1}{1-\Psi}\,, \tag{4.76}$$

where $\boldsymbol{\delta}$ is a degree-even co-derivation. Indeed, from the condition

$$\Delta_{T\mathcal{H}_{\rm o}}\boldsymbol{F} = (\boldsymbol{F}\otimes'\boldsymbol{F})\Delta_{T\mathcal{H}_{\rm o}}\,, \tag{4.77}$$

we can deduce

$$\Delta_{T\mathcal{H}_{\rm o}}\boldsymbol{\delta} = (\mathbf{1}_{T\mathcal{H}_{\rm o}}\otimes'\boldsymbol{\delta} + \boldsymbol{\delta}\otimes'\mathbf{1}_{T\mathcal{H}_{\rm o}})\Delta_{T\mathcal{H}_{\rm o}}\,, \tag{4.78}$$

confirming that $\boldsymbol{\delta}$ is a co-derivation.

Therefore, by varying (4.70) and using cyclicity we obtain

$$\delta S_{\rm disk}(\Psi) = \frac{\omega_{\rm o}}{\kappa}\left(\pi_1\boldsymbol{\delta}\frac{1}{1-\Psi}, \pi_1\boldsymbol{m}\frac{1}{1-\Psi}\right)\,. \tag{4.79}$$

Here, any dependence on the interpolation parameter t drops out, confirming that

the action is independent of the choice of interpolation, even when one considers a t-dependent variation. Assuming that $\boldsymbol{\delta}$ is cyclic, we can reintroduce the interpolation to derive the useful expression

$$\delta S_{\rm disk}(\Psi) = \int_0^1 \frac{\omega_{\rm o}}{\kappa} \left(\dot{\Psi}(t), \pi_1[\boldsymbol{\delta}, \boldsymbol{m}] \frac{1}{1-\Psi(t)} \right) . \tag{4.80}$$

From this and using Jacobi identity and (4.72) one can prove that $S_{\rm disk}$ is invariant under the gauge transformation

$$\boldsymbol{\delta}_g \frac{1}{1-\Psi} = [\boldsymbol{m}, \boldsymbol{\Lambda}_{\rm o}] \frac{1}{1-\Psi} , \tag{4.81}$$

where $\boldsymbol{\Lambda}_{\rm o}$ is the 0 co-derivation built from the degree-(-1) gauge-parameter string field $\Lambda_{\rm o} \in \mathcal{H}_{\rm o}$. Note that by expanding (4.81) we recover the free gauge transformation at first order

$$\delta_g \Psi = Q_{\rm o} \Lambda_{\rm o} + m_2(\Lambda_{\rm o}, \Psi) + m_2(\Psi, \Lambda_{\rm o}) + \dots . \tag{4.82}$$

Moreover, from (4.79), we read off the equation of motion

$$\mathrm{EOM}(\Psi) = \pi_1 \boldsymbol{m} \frac{1}{1-\Psi} = Q_{\rm o} \Psi + m_2(\Psi, \Psi) + \dots . \tag{4.83}$$

Similarly, in the closed-string sector the gauge transformation reads

$$\boldsymbol{\delta}_g e^{\wedge \Phi} = [\boldsymbol{l}^{(0)}, \boldsymbol{\Lambda}_{\rm c}] e^{\wedge \Phi} , \tag{4.84}$$

and the equation of motion is

$$\mathrm{EOM}(\Phi) = \pi_1 \boldsymbol{l}^{(0)} e^{\wedge \Phi} . \tag{4.85}$$

Note that solving (4.83) and (4.85) is equivalent to solve the Maurer-Cartan equations

$$\boldsymbol{m} \frac{1}{1-\Psi} = 0 , \tag{4.86}$$

and

$$\boldsymbol{l}^{(0)} e^{\wedge\Phi} = 0\,, \tag{4.87}$$

as can be seen by using (4.36b) and (4.37b).

Let us now discuss the physical interpretation of solutions to the equations above. As in QFT, non-trivial solutions to the full equations of motion indicate new vacua, around which one can perform perturbation theory. From the perspective of SFT, this means that the Feynman diagram expansion around such a vacuum reproduces, on-shell (with respect to the shifted BRST charge), the amplitudes of a first-quantized ST on a different background, i.e., a different world-sheet (B)CFT.

This observation leads to the so-called background independence conjecture, [15–18]. This conjecture essentially expresses the converse of the previous statement: any two consistent world-sheet (B)CFTs can be connected via classical solutions of the SFT equations of motion. In other words, a SFT action built on one (B)CFT admits a solution corresponding to the second background, in the sense that fluctuation theory around this vacuum reproduces, up to field redefinition, the SFT action that one would obtain starting directly from the second background. In Chapter 7 we will describe how this property makes SFT a powerful alternative approach to standard conformal perturbation theory [99–103].

This statement has been proven at the classical level for nearby backgrounds, namely backgrounds that can be interpolated through a small parameter, in closed SFT in [19] and for generic backgrounds in open SFT in [27, 28]. The proof at the quantum level is more involved, as field redefinitions generally produce a Jacobian factor arising from the measure in the path integral. Nevertheless, the quantum version has been demonstrated for nearby backgrounds in [20].

4.2 Homotopy Transfer and Wilsonian effective actions

In this section, we describe the construction of Wilsonian effective actions in the context of SFT [104]. In QFT, one splits the spectrum into light and heavy modes, then derives the IR effective action from the UV action by integrating out the latter

in the gauge-fixed path integral. This approach is well-motivated at low energies, where such heavy modes cannot appear as external states. Analogously, we split the string field as in (2.160)

$$\Phi = P\Phi + \bar{P}\Phi \coloneqq I\phi + R + \tilde{R}\,, \tag{4.88}$$

where $\tilde{R}$ denotes the out-of-gauge degrees of freedom, as explained in Section 2.2.4. We interpret R as the heavy modes and ϕ as the IR degrees of freedom. Note that, unlike (2.160), here we define $\phi \coloneqq \Pi\Phi$, so that $P\Phi = I\phi$, using the canonical inclusion I and the canonical projector Π introduced in Section 2.2.4.

We thus define the Wilsonian effective action as

$$e^{\tilde{S}_{\mathrm{c}}(\phi)} \coloneqq \int \mathcal{D}R\, e^{S_{\mathrm{c}}(I\phi+R)}\,, \tag{4.89}$$

where the gauge-fixed path integral can be expressed in terms of the BV target fields and antifields, as shown in (2.161). This computation can be performed using standard QFT techniques, which we describe in detail in Appendix D, and the full solution is obtained by summing over all possible Feynman diagrams with ϕ as external legs.

According to the saddle-point expansion, the tree-level contribution is obtained by solving the classical equation of motion for R, and then substituting $\Psi(\phi) = I\phi + R(\phi)$ into the classical sector of the theory, and we refer to this process as integrating out R classically .

The effective action can be formally written in WZW form as

$$\tilde{S}_{\mathrm{c}}(\phi) = \Lambda + \int_0^1 dt\, \frac{\tilde{\omega}_{\mathrm{c}}}{\kappa^2}(\dot{\phi}, \pi_1 \tilde{\boldsymbol{l}} e^{\wedge\phi})\,, \tag{4.90}$$

where Λ denotes the constant arising from vacuum bubble contributions, $\tilde{\omega}_{\mathrm{c}}$ is the effective symplectic form

$$\tilde{\omega}_{\mathrm{c}} : \Pi\mathcal{H}_{\mathrm{c}} \otimes \Pi\mathcal{H}_{\mathrm{c}} \to \mathbb{C}\,, \tag{4.91}$$

defined as

$$\tilde{\omega}_{\mathrm{c}}(\cdot,\cdot) \coloneqq \omega_{\mathrm{c}}(I\cdot, I\cdot)\,, \tag{4.92}$$

and $\tilde{l}$ is the effective total coderivation encoding the IR dynamics, whose explicit form can be obtained through a pure algebraic approach by using the so-called Homotopy Transfer that we will review in the next sections. This technique was first introduced in [105–107] from a purely mathematical perspective and was later reexamined in [108–113] in the context of constructing tree-level effective actions. The quantum generalization was developed in [114, 115] at the level of BV operators and subsequently translated into the co-algebras framework in [33] (see Appendix D).

Furthermore, these techniques have been employed to reinterpret and reformulate various aspects of QFT using co-algebraic language [116–121].

4.2.1 Homotopy Transfer

In this section, we review the construction of the IR multi-string products using the homotopy transfer techniques.

Strong Deformation Retract in the Co-algebra formalism

Given an SDR (see Section 2.2.4)

$$\circlearrowleft(-h)\,(\mathcal{H}, Q) \;\underset{I}{\overset{\Pi}{\rightleftarrows}}\; (\Pi\mathcal{H}, \Pi Q I)\,, \tag{4.93}$$

we can lift all these structures so that they continue to satisfy an SDR on $T\mathcal{H}$ (or $S\mathcal{H}$). In what follows, we focus on the tensor algebra framework, though the construction extends straightforwardly to the symmetrized case.

First, we upgrade both the canonical inclusion I and the canonical projector Π as

cohomomorphism on the tensor algebra

$$\boldsymbol{I}\pi_p = I^{\otimes p}\pi_p\,, \tag{4.94a}$$

$$\boldsymbol{\Pi}\pi_p = \Pi^{\otimes p}\pi_p\,, \tag{4.94b}$$

thus obtaining

$$\boldsymbol{\Pi I} = \mathbf{1}_{T\Pi\mathcal{H}}\,, \tag{4.95a}$$

$$\boldsymbol{I\Pi} = \boldsymbol{P}\,, \tag{4.95b}$$

where $T\Pi\mathcal{H}$ denotes the tensor algebra built from the projected Hilbert space $\Pi\mathcal{H}$, and $\boldsymbol{P}$ is the lift of the projector P to a cohomomorphism on $T\mathcal{H}$

$$\boldsymbol{P}\pi_p = P^{\otimes p}\pi_p\,, \tag{4.96}$$

which satisfies

$$[\boldsymbol{P}, \boldsymbol{Q}] = 0\,. \tag{4.97}$$

The contracting homotopy operator h admits multiple possible extensions to the co-algebraic level that preserve both the Hodge–Kodaira decomposition and the annihilation relations. We adopt the definition introduced in [111], namely

$$\boldsymbol{h}\pi_p = \sum_{i=0}^{p-1} \left(1_{\mathcal{H}}^{\otimes i} \otimes h \otimes P^{\otimes p-i-1}\right) \pi_p\,, \tag{4.98}$$

which ensures

$$[\boldsymbol{Q}, \boldsymbol{h}] = \mathbf{1}_{T\mathcal{H}} - \boldsymbol{P}\,, \tag{4.99}$$

and

$$\boldsymbol{h}^2 = \boldsymbol{hI} = \boldsymbol{\Pi h} = 0\,. \tag{4.100}$$

Note that (4.98) is not a co-derivation. Indeed, under the application of the co-product, it behaves as follows

$$\Delta_{T\mathcal{H}}\boldsymbol{h} = (\boldsymbol{h} \otimes' \boldsymbol{P} + \mathbf{1}_{T\mathcal{H}} \otimes' \boldsymbol{h})\Delta_{T\mathcal{H}}\,. \tag{4.101}$$

Therefore, through these definitions, we have elevated the SDR to the level of the tensor algebra

$$\circlearrowleft(-\boldsymbol{h})\,(T\mathcal{H}, \boldsymbol{Q}) \;\underset{\boldsymbol{I}}{\overset{\boldsymbol{\Pi}}{\rightleftarrows}}\; (T\Pi\mathcal{H}, \boldsymbol{\Pi Q I})\,. \tag{4.102}$$

Homological Perturbation Lemma

Until now, we have considered the free SDR. To introduce interactions, we use the Homological Perturbation Lemma (HPL). This theorem tells us that, given an SDR and a perturbation $\boldsymbol{\Delta}$ of the initial differential such that nilpotency is preserved, i.e.,

$$(\boldsymbol{Q} + \boldsymbol{\Delta})^2 = 0\,, \tag{4.103}$$

we can always construct a new perturbed SDR

$$\circlearrowleft(-\tilde{\boldsymbol{h}})\,(T\mathcal{H}, \boldsymbol{Q} + \boldsymbol{\Delta}) \;\underset{\tilde{\boldsymbol{I}}}{\overset{\tilde{\boldsymbol{\Pi}}}{\rightleftarrows}}\; (TP\Pi\mathcal{H}, \boldsymbol{\Pi Q I} + \tilde{\boldsymbol{\Delta}})\,, \tag{4.104}$$

where the perturbed operators are given by

$$\tilde{\boldsymbol{I}} = \frac{1}{\mathbf{1}_{T\mathcal{H}} + \boldsymbol{h}\boldsymbol{\Delta}}\boldsymbol{I}\,, \tag{4.105a}$$

$$\tilde{\boldsymbol{\Pi}} = \boldsymbol{\Pi}\frac{1}{\mathbf{1}_{T\mathcal{H}} + \boldsymbol{\Delta}\boldsymbol{h}}\,, \tag{4.105b}$$

$$\tilde{\boldsymbol{h}} = \frac{1}{\mathbf{1}_{T\mathcal{H}} + \boldsymbol{h}\boldsymbol{\Delta}}\boldsymbol{h}\,, \tag{4.105c}$$

$$\tilde{\boldsymbol{\Delta}} = \boldsymbol{\Pi}\boldsymbol{\Delta}\frac{1}{\mathbf{1}_{T\mathcal{H}} + \boldsymbol{h}\boldsymbol{\Delta}}\boldsymbol{I}\,, \tag{4.105d}$$

indeed it is possible to prove[3] that they satisfies the SDR properties. In particular, the perturbed inclusion and the perturbed projector obey

$$\tilde{\boldsymbol{\Pi}}\tilde{\boldsymbol{I}} = \mathbf{1}_{T\Pi\mathcal{H}}\,, \tag{4.106a}$$

$$\tilde{\boldsymbol{I}}\tilde{\boldsymbol{\Pi}} = \tilde{\boldsymbol{P}}\,, \tag{4.106b}$$

where the last equation defines the perturbed projector. Moreover, the annihilation relations hold

$$\tilde{\boldsymbol{h}}^2 = \tilde{\boldsymbol{h}}\tilde{\boldsymbol{I}} = \tilde{\boldsymbol{\Pi}}\tilde{\boldsymbol{h}} = 0\,, \tag{4.107}$$

as well as the perturbed Hodge-Kodaira decomposition

$$[\boldsymbol{Q} + \boldsymbol{\Delta}, \tilde{\boldsymbol{h}}] = \mathbf{1}_{T\mathcal{H}} - \tilde{\boldsymbol{P}}\,, \tag{4.108}$$

and the nilpotency of the resulting differential

$$(\boldsymbol{\Pi}\boldsymbol{Q}\boldsymbol{I} + \tilde{\boldsymbol{\Delta}})^2 = 0\,. \tag{4.109}$$

The perturbed differential can be usefully rewritten as

$$\boldsymbol{\Pi}\boldsymbol{Q}\boldsymbol{I} + \tilde{\boldsymbol{\Delta}} = \boldsymbol{\Pi}(\boldsymbol{Q} + \boldsymbol{\Delta})\frac{1}{\mathbf{1}_{T\mathcal{H}} + \boldsymbol{h}\boldsymbol{\Delta}}\boldsymbol{I}\,, \tag{4.110}$$

where we used (4.95) and (4.99).

Finally, the HPL tells us that $\tilde{\boldsymbol{I}}$ and $\tilde{\boldsymbol{\Pi}}$ satisfy the chain-map relations

$$(\boldsymbol{Q} + \boldsymbol{\Delta})\tilde{\boldsymbol{I}} = \tilde{\boldsymbol{I}}(\boldsymbol{\Pi}\boldsymbol{Q}\boldsymbol{I} + \tilde{\boldsymbol{\Delta}})\,, \tag{4.111a}$$

$$\tilde{\boldsymbol{\Pi}}(\boldsymbol{Q} + \boldsymbol{\Delta}) = (\boldsymbol{\Pi}\boldsymbol{Q}\boldsymbol{I} + \tilde{\boldsymbol{\Delta}})\tilde{\boldsymbol{\Pi}}\,, \tag{4.111b}$$

meaning that they are morphisms which transfer the initial homotopy structures defined on $T\mathcal{H}$ to new operators on $T\Pi\mathcal{H}$, preserving the homotopy algebra. This is the core of the Homotopy Transfer.

[3] See, for instance, Appendix A of [111] for the complete proof

Effective multi string products

From the viewpoint of the construction of the Wilsonian effective action, the homotopy structures that describe the IR theory, defined by integrating out the degrees of freedom in $\bar{P}$, are those given by the HPL, considering as perturbation what is needed to restore the nilpotent structure describing the UV theory. Thus, if we consider classical OSFT, we split the total coderivation $\boldsymbol{m} = \boldsymbol{Q}_{\mathrm{o}} + \boldsymbol{\delta m}$ and interpret the interacting component $\boldsymbol{\delta m}$ as the perturbation, obtaining the following free and perturbed SDRs

$$\begin{array}{ccc} \circlearrowleft(-\boldsymbol{h})\,(T\mathcal{H}_{\mathrm{o}}, \boldsymbol{Q}_{\mathrm{o}}) & \underset{\boldsymbol{I}}{\overset{\boldsymbol{\Pi}}{\rightleftarrows}} & (T\Pi\mathcal{H}_{\mathrm{o}}, \boldsymbol{\Pi Q}_{\mathrm{o}}\boldsymbol{I}) \\ \Big\downarrow{\scriptstyle \boldsymbol{\delta m}} & & \Big\downarrow{\scriptstyle \boldsymbol{\delta \tilde{m}}} \\ \circlearrowleft(-\tilde{\boldsymbol{h}})\,(T\mathcal{H}_{\mathrm{o}}, \boldsymbol{m}) & \underset{\tilde{\boldsymbol{I}}}{\overset{\tilde{\boldsymbol{\Pi}}}{\rightleftarrows}} & (T\Pi\mathcal{H}_{\mathrm{o}}, \tilde{\boldsymbol{m}}) \end{array} \tag{4.112}$$

and the IR dynamics is described by the action

$$\tilde{S}_{\mathrm{disk}}(\psi) = S_{\mathrm{disk}}(I\psi + R(\psi)) = \int_0^1 dt\, \frac{\tilde{\omega}_{\mathrm{o}}}{\kappa}\left(\dot{\psi}(t), \pi_1 \tilde{\boldsymbol{m}} \frac{1}{1-\psi(t)}\right), \tag{4.113}$$

where, according to (4.110), the total transferred co-derivation reads

$$\tilde{\boldsymbol{m}} = \boldsymbol{\Pi m} \frac{1}{\mathbf{1}_{T\mathcal{H}_{\mathrm{o}}} + \boldsymbol{h\delta m}} \boldsymbol{I} = \boldsymbol{I Q}_{\mathrm{o}} \boldsymbol{\Pi} + \boldsymbol{\delta \tilde{m}}\,, \tag{4.114}$$

The effective co-derivation is cyclic and nilpotent, thus ensuring that the A_∞ relations are preserved during the process and that $\tilde{S}_{\mathrm{disk}}(\psi)$ is consistent at classical level. Consistently, by expanding (4.114) as a geometric series, we recover all the three level Feynman diagrams with vertices m_p and propagator h. We will explicitly show in Section 4.2.2 that the multi-string products given by the homotopy transfer are the same as those obtained by classically integrating out the $\bar{P}\mathcal{H}_{\mathrm{o}}$ sector.

In the closed string case, we again split the total coderivation as $\boldsymbol{l} = \boldsymbol{Q}_{\mathrm{c}} + \boldsymbol{\delta l}$, but in this case, we must also include the closed Poisson bi-vector in the perturbation,

schematically

$$\begin{array}{ccc} \circlearrowleft(-\boldsymbol{h})\,(S\mathcal{H}_{\rm c},\boldsymbol{Q}_{\rm c}) & \underset{\boldsymbol{I}}{\overset{\boldsymbol{\Pi}}{\rightleftarrows}} & (S\Pi\mathcal{H}_{\rm c},\boldsymbol{\Pi Q}_{\rm c}\boldsymbol{I}) \\ \Big\downarrow{\scriptstyle \boldsymbol{\delta l}+\kappa^2\boldsymbol{U}_{\rm c}} & & \Big\downarrow{\scriptstyle \boldsymbol{\delta\tilde{l}}+\kappa^2\tilde{\boldsymbol{U}}_{\rm c}} \\ \circlearrowleft(-\tilde{\boldsymbol{h}})\,(S\mathcal{H}_{\rm c},\boldsymbol{l}+\kappa^2\boldsymbol{U}_{\rm c}) & \underset{\tilde{\boldsymbol{I}}}{\overset{\tilde{\boldsymbol{\Pi}}}{\rightleftarrows}} & (S\Pi\mathcal{H}_{\rm c},\tilde{\boldsymbol{l}}+\kappa^2\tilde{\boldsymbol{U}}_{\rm c}) \end{array} \tag{4.115}$$

where we defined effective multi-string products

$$\tilde{\boldsymbol{l}}=\boldsymbol{\Pi l}\frac{1}{\mathbf{1}_{S\mathcal{H}_{\rm c}}+\boldsymbol{h}(\boldsymbol{\delta l}+\kappa^2\boldsymbol{U}_{\rm c})}\boldsymbol{I}\,, \tag{4.116}$$

and the projected Poisson bi-vector

$$\tilde{U}_{\rm c}\coloneqq\frac{(-)^{c^i}}{2}\Pi c_i\wedge\Pi c^i\,, \tag{4.117}$$

which inverts the effective symplectic form

$$(\tilde{\omega}_{\rm c}\otimes\mathbf{1})(1\otimes\tilde{U}_{\rm c})=\mathbf{1}_{\Pi\mathcal{H}_{\rm c}}\,. \tag{4.118}$$

This decomposition of the perturbed differential can be verified by manipulating (4.110) adapted to this scenario, see Appendix D, in particular (D.58).

The IR Wilsonian effective action can be written as

$$\tilde{S}_{\rm c}(\phi)=\Lambda+\int_0^1 dt\,\frac{\tilde{\omega}_{\rm c}}{\kappa^2}\left(\dot{\phi}(t),\pi_1\tilde{\boldsymbol{l}}e^{\wedge\phi(t)}\right)\,, \tag{4.119}$$

which is quantum consistent thanks to the transfer of the nilpotent structure guaranteed by the HPL

$$(\boldsymbol{l}+\kappa^2\boldsymbol{U}_{\rm c})^2=0\to(\tilde{\boldsymbol{l}}+\tilde{\boldsymbol{U}}_{\rm c})^2=0\,. \tag{4.120}$$

Here, the relation between the Homotopy Transfer and the Wilsonian effective action makes manifest the interpretation of (4.116) as the sum of all possible Feynman diagrams with propagators h, vertices $l_k^{(g)}$ and loop operators $U_{\rm c}$.

Furthermore, we can notice that by expanding the effective total coderivation in

powers of the string coupling constant,

$$\tilde{\boldsymbol{l}} = \sum_{g=0}^{\infty} \kappa^{2g} \tilde{\boldsymbol{l}}^{(g)} \,, \tag{4.121}$$

according to the topological expansion, we find that the leading contribution is given by

$$\tilde{\boldsymbol{l}}^{(0)} = \boldsymbol{\Pi} \boldsymbol{l}^{(0)} \frac{1}{\mathbf{1}_{S\mathcal{H}_{\mathrm{c}}} + \boldsymbol{h}\boldsymbol{\delta}\boldsymbol{l}^{(0)}} \boldsymbol{I}. \tag{4.122}$$

This is exactly what we would obtain by applying the homotopy transfer to the classical closed string sub-sector

$$\begin{array}{ccc} \circlearrowleft(-\boldsymbol{h})\,(S\mathcal{H}_{\mathrm{c}}, \boldsymbol{Q}_{\mathrm{c}}) & \underset{\boldsymbol{I}}{\overset{\boldsymbol{\Pi}}{\rightleftarrows}} & (S\Pi\mathcal{H}_{\mathrm{c}}, \boldsymbol{\Pi}\boldsymbol{Q}_{\mathrm{c}}\boldsymbol{I}) \\ \Big\downarrow{\scriptstyle \boldsymbol{\delta}\boldsymbol{l}^{(0)}} & & \Big\downarrow{\scriptstyle \boldsymbol{\delta}\boldsymbol{l}^{(0)}} \\ \circlearrowleft(-\tilde{\boldsymbol{h}})\,(S\mathcal{H}_{\mathrm{c}}, \boldsymbol{l}^{(0)}) & \underset{\tilde{\boldsymbol{I}}}{\overset{\tilde{\boldsymbol{\Pi}}}{\rightleftarrows}} & (S\Pi\mathcal{H}_{\mathrm{c}}, \tilde{\boldsymbol{l}}^{(0)}) \end{array} \tag{4.123}$$

and the action

$$\tilde{S}_{\mathrm{sphere}}(\phi) = S_{\mathrm{sphere}}(I\phi + R(\phi)) = \int_0^1 dt\, \frac{\tilde{\omega}_{\mathrm{c}}}{\kappa^2} \left(\dot{\phi}(t), \pi_1 \tilde{\boldsymbol{l}}^{(0)} e^{\wedge \phi(t)}\right) \,, \tag{4.124}$$

corresponds to the classical effective action describing the IR theory, namely, the action obtained by solving the equation of motion for the $\bar{P}\mathcal{H}_{\mathrm{c}}$ sector and plugging it back into the UV action. This is in perfect agreement with the saddle point expansion: the tree-level contribution to the Wilsonian effective action is given by the UV classical sector of the action evaluated on the classical solution,

$$\tilde{S}_{\mathrm{c}}(\phi) = \tilde{S}_{\mathrm{sphere}}(\phi) + O(\kappa^0) \,, \tag{4.125}$$

keeping in mind that $\tilde{S}_{\mathrm{sphere}}(\phi)$ goes as κ^{-2}.

4.2.2 Tree-level effective actions

In this section, we review the proof presented in [111] showing that, at the classical level, the perturbed multi-string product defined in (4.114) exactly reproduces the vertices of the tree-level effective action, thereby confirming (4.113). This approach can be straightforwardly extended to the closed SFT case, thus proving the corresponding result (4.124).

The quantum-level proof is more involved; in [114], the authors establish it at the level of the BV operators. We translate their result into the more familiar co-algebra language in the Appendix D, ending up with (4.119).

Therefore, our task here is to solve the equation of motion for the $\bar{P}\mathcal{H}_{\mathrm{o}}$ sector under the gauge fixing condition $h\Psi = 0$, thus obtaining Ψ as a function of the light modes $\Psi(\phi)$, and then plug it back into the starting action S_{disk}. The first step consists of splitting the gauge-fixed Maurer–Cartan equation (4.86) using the projectors P and $\bar{P}$

$$\mathrm{EOM}_{I\psi}(I\psi + R) = 0 \longleftrightarrow \boldsymbol{Pm}\left(\boldsymbol{I}\frac{1}{1-\psi} + \frac{1}{1-R}\right) = 0\,, \tag{4.126a}$$

$$\mathrm{EOM}_{R}(I\psi + R) = 0 \longleftrightarrow \boldsymbol{\bar{P}m}\left(\boldsymbol{I}\frac{1}{1-\psi} + \frac{1}{1-R}\right) = 0\,, \tag{4.126b}$$

and then solving the second equation. By making the BRST charge explicit in (4.126b) and applying $\boldsymbol{h}$ to both sides, we obtain the recursive relation

$$\frac{1}{1-R} = -\boldsymbol{h\delta m}\left(\boldsymbol{I}\frac{1}{1-\psi} + \frac{1}{1-R}\right)\,, \tag{4.127}$$

where we used that $[\boldsymbol{P}, \boldsymbol{Q}_{\mathrm{o}}] = 0$, the annihilation relation $\boldsymbol{hP} = 0$, and the Hodge-Kodaira decomposition (4.99). We then bring the term involving R to the left-hand side and formally invert the operator $(\mathbf{1}_{T\mathcal{H}_{\mathrm{o}}} + \boldsymbol{h\delta m})$, yielding

$$\frac{1}{1-R} = -\frac{1}{\mathbf{1}_{T\mathcal{H}_{\mathrm{o}}} + \boldsymbol{h\delta m}}\boldsymbol{h\delta m I}\frac{1}{1-\psi} = \left(\frac{1}{\mathbf{1}_{T\mathcal{H}_{\mathrm{o}}} + \boldsymbol{h\delta m}} - \mathbf{1}_{T\mathcal{H}_{\mathrm{o}}}\right)\boldsymbol{I}\frac{1}{1-\psi}\,. \tag{4.128}$$

Therefore, the full string field is given by

$$\frac{1}{1-\Psi(\psi)} = \frac{1}{\mathbf{1}_{T\mathcal{H}_{\mathrm{o}}} + \boldsymbol{h\delta m}} \boldsymbol{I} \frac{1}{1-\psi} = \tilde{\boldsymbol{I}} \frac{1}{1-\psi}\,, \tag{4.129}$$

where in the second step we recognized (4.105a). Substituting this result back into the starting action (4.45), we obtain

$$\begin{aligned} \tilde{S}_{\mathrm{disk}}(\psi) &= \int_0^1 dt\, \frac{\omega_{\mathrm{o}}}{\kappa} \left(\pi_1 \tilde{\boldsymbol{I}} \boldsymbol{\partial}_t \frac{1}{1-\psi(t)}, \pi_1 \boldsymbol{m} \tilde{\boldsymbol{I}} \frac{1}{1-\psi(t)} \right) \\ &= \int_0^1 dt\, \frac{\omega_{\mathrm{o}}}{\kappa} \left(\pi_1 \tilde{\boldsymbol{I}} \boldsymbol{\partial}_t \frac{1}{1-\psi(t)}, \pi_1 \tilde{\boldsymbol{I}} \tilde{\boldsymbol{m}} \frac{1}{1-\psi(t)} \right)\,, \end{aligned} \tag{4.130}$$

where we used the chain-map relation (4.111a).

To conclude the proof, we need to demonstrate that

$$\omega_{\mathrm{o}}(\pi_1 \tilde{\boldsymbol{I}}(\cdot), \pi_1 \tilde{\boldsymbol{I}}(\cdot)) = \omega_{\mathrm{o}}(\pi_1 \boldsymbol{I}(\cdot), \pi_1 \boldsymbol{I}(\cdot)) = \tilde{\omega}_{\mathrm{o}}(\pi_1(\cdot), \pi_1(\cdot))\,, \tag{4.131}$$

where the $(\cdot)$ are placeholders for two generic elements in $T\mathcal{H}_{\mathrm{o}}$. In the last equivalence we obtained the definition of the effective symplectic form using the fact that the cohomomorphism $\boldsymbol{I}$ does not change the number of vectors it acts on, so that $\pi_1 \boldsymbol{I} = I \pi_1$. The first equality can be proved by expanding the perturbed inclusion

$$\begin{aligned} \omega_{\mathrm{o}}(\pi_1 \tilde{\boldsymbol{I}}(\cdot), \pi_1 \tilde{\boldsymbol{I}}(\cdot)) &= \omega_{\mathrm{o}}(\pi_1 \boldsymbol{I}(\cdot), \pi_1 \boldsymbol{I}(\cdot)) \\ &\quad - \omega_{\mathrm{o}}(\pi_1 \boldsymbol{h\delta m I}(\cdot), \pi_1 \boldsymbol{I}(\cdot)) - \omega_{\mathrm{o}}(\pi_1 \boldsymbol{I}(\cdot), \pi_1 \boldsymbol{h\delta m I}(\cdot)) \\ &\quad + \omega_{\mathrm{o}} \left(\pi_1 \boldsymbol{h\delta m} \frac{1}{\mathbf{1}_{T\mathcal{H}_{\mathrm{o}}} + \boldsymbol{h\delta m}} \boldsymbol{I}(\cdot), \pi_1 \boldsymbol{h\delta m} \frac{1}{\mathbf{1}_{T\mathcal{H}_{\mathrm{o}}} + \boldsymbol{h\delta m}} \boldsymbol{I}(\cdot) \right) \\ &= \omega_{\mathrm{o}}(\pi_1 \boldsymbol{I}(\cdot), \pi_1 \boldsymbol{I}(\cdot))\,, \end{aligned} \tag{4.132}$$

noting that only the identity contribution survives due to the fact that $\boldsymbol{h}$ is BPZ even and the annihilation relations $\boldsymbol{h}^2 = \boldsymbol{hI} = 0$. Consequently, the effective action is given by

$$\tilde{S}_{\mathrm{disk}}(\psi) = \int_0^1 dt\, \frac{\tilde{\omega}_{\mathrm{o}}}{\kappa} \left(\dot{\psi}(t), \pi_1 \tilde{\boldsymbol{m}} \frac{1}{1-\psi(t)} \right)\,, \tag{4.133}$$

as desired. Notice that to prove the consistency of the above action, we must still

verify the cyclicity of $\tilde{\boldsymbol{m}}$ with respect to $\tilde{\omega}_{\mathrm{o}}$, which, together with the nilpotency guaranteed by the homotopy transfer, implies that (4.133) is a solution to the BV classical master equation in $\Pi\mathcal{H}$. This means that, we want to check that

$$\tilde{\omega}_{\mathrm{o}}\left(\pi_1\boldsymbol{c}_1\frac{1}{1-\psi},\pi_1\tilde{\boldsymbol{m}}\boldsymbol{c_2}\frac{1}{1-\psi}\right)=-(-)^{c_1}\tilde{\omega}_{\mathrm{o}}\left(\pi_1\tilde{\boldsymbol{m}}\boldsymbol{c}_1\frac{1}{1-\psi},\pi_1\boldsymbol{c_2}\frac{1}{1-\psi}\right)\,, \quad (4.134)$$

for two generic co-derivations $\boldsymbol{c}_1$ and $\boldsymbol{c}_1$. Rewriting the left-hand side in terms of the symplectic form on $\mathcal{H}_{\mathrm{o}}$, we get

$$\begin{aligned}\tilde{\omega}_{\mathrm{o}}\left(\pi_1\boldsymbol{c}_1\frac{1}{1-\psi},\pi_1\tilde{\boldsymbol{m}}\boldsymbol{c_2}\frac{1}{1-\psi}\right)&=\omega_{\mathrm{o}}\left(\pi_1\boldsymbol{I}\boldsymbol{c}_1\frac{1}{1-\psi},\pi_1\boldsymbol{I}\tilde{\boldsymbol{m}}\frac{1}{1-\psi}\right)\\&=\omega_{\mathrm{o}}\left(\pi_1\tilde{\boldsymbol{I}}\boldsymbol{c}_1\frac{1}{1-\psi},\pi_1\boldsymbol{m}\tilde{\boldsymbol{I}}\frac{1}{1-\psi}\right)\\&=-(-)^{c_1}\omega_{\mathrm{o}}\left(\pi_1\tilde{\boldsymbol{I}}\tilde{\boldsymbol{m}}\boldsymbol{c}_1\frac{1}{1-\psi},\pi_1\tilde{\boldsymbol{I}}\frac{1}{1-\psi}\right)\\&=-(-)^{c_1}\tilde{\omega}_{\mathrm{o}}\left(\pi_1\tilde{\boldsymbol{m}}\boldsymbol{c}_1\frac{1}{1-\psi},\pi_1\boldsymbol{c_2}\frac{1}{1-\psi}\right)\,, \quad (4.135)\end{aligned}$$

where in the second line we used property (4.131) and the chain-map relation (4.111a), then moved $\boldsymbol{m}$ to the left using cyclicity, and finally applied (4.111a) and (4.131) again to reconstruct $\tilde{\boldsymbol{m}}$ and absorb $\tilde{\boldsymbol{I}}$ into the symplectic form, thus proving (4.134).

Finally, note that, as discussed in Section 2.2.3, having fixed the gauge, we in principle also have the out-of-gauge equations of motion that must be imposed as side conditions on the gauge-fixed action. However, we observe that such constraints are automatically satisfied once we consider solutions to the effective equation of motion

$$\mathrm{eom}(\psi)=\tilde{\boldsymbol{m}}\frac{1}{1-\psi}\,, \quad (4.136)$$

meaning that no further constraints are needed. Indeed the total equation of motion is

$$\mathrm{EOM}(\Psi(\psi))=\boldsymbol{m}\frac{1}{1-\Psi(\psi)}\,, \quad (4.137)$$

and we can rewrite the right hand side as

$$\boldsymbol{m}\frac{1}{1-\Psi(\psi)} = \tilde{\boldsymbol{I}}\tilde{\boldsymbol{m}}\frac{1}{1-\psi}\,, \tag{4.138}$$

where we used (4.129) and (4.111a). Therefore, we have that

$$\mathrm{eom}(\psi) = 0 \longrightarrow \mathrm{EOM}(\Psi(\psi)) = 0\,, \tag{4.139}$$

as expected.

CHAPTER 5

The nilpotent structure of Open-Closed SFT and the large N limit

5.1 Open-closed SFT and the D-brane backreaction

In this chapter, based on our works [31] and [32], we describe the construction of the open–closed SFT action, emphasizing the homotopy structure that governs its quantum consistency. As previously outlined, the open–closed world-sheet duality relating an open-string quantum loop to a classical closed-string exchange implies that there is no classical open–closed SFT involving propagating open and closed strings. Consequently, the theory is intrinsically quantum mechanical.

The initial formulation of open–closed SFT within the BV framework was developed by Zwiebach [11, 122]. Building on this, the first step toward identifying the homotopy structure of this theory was achieved by Kajiura and Stasheff, who defined the Open–Closed Homotopy Algebra (OCHA) [123, 124][1] describing the consistency of the truncated sector of the theory consisting only of spheres and disks, while excluding configurations with closed-string punctures on the disk. This framework was extended in [126] to a full Sphere–Disk Homotopy Algebra (SDHA) by incorporating all possible couplings on these topologies.

The first attempt to describe the full quantum homotopy–algebraic structure of open–closed SFT is due to Muenster and Sachs [127], where the authors used open and closed IBL_∞ algebras and the morphisms between them.[2] Although mathematically interesting, this framework treats mixed open–closed couplings differently from purely open or purely closed ones, making it less practical when one wishes to handle all path-integral variables and their interactions on an equal footing. A more unified formulation was later proposed in [129], in the context of constructing the $1\mathrm{PI}$ action for open–closed super SFT, based on common open–closed multi-string products. However, in that approach open strings are treated fully symmetrically, and important structures such as color ordering are lost.

The strategy we will adopt to describe the homotopy structures of the full open–closed SFT action is completely analogous to what was discussed in Chapter 4. In particular, we first reformulate open–closed SFT in terms of open–closed multi-string products, which are then upgraded to generalized co-derivations with the

[1] The Kajiura–Stasheff homotopy structure has been generalized to the RNS superstring in [125].
[2] This structure and its relation to our formalism were recently examined in [128].

aim of writing the action in WZW form. Imposing the quantum BV master equation on the action then induces relations among the multi-string products, which constitute the quantum homotopy algebra governing the consistency of open–closed SFT. Understanding the geometric meaning of this structure will be one of our main goals.

After that, we will use this construction to study the D-brane back-reaction from the SFT perspective. This problem sits at the heart of ST and it has led to the gauge/gravity duality (see [130, 131] for review or original papers [132–134]), a profound conjecture whose proof is however still a challenge. In the absence of a full microscopic understanding in the critical-string setting, this problem has been studied in specific simplified examples where the background is such that the local string dynamics is drastically simplified or absent, for example in the topological string [135–137] or in the non-critical string [138–141] and also in some specific time-dependent backgrounds [142, 143] of critical ST. In these cases it has been indeed possible to reconstruct the full closed string backreaction from the open strings (or, more precisely, the D-branes) moduli. The typical key move to access this has been to start from a large number N of D-branes and then focus on the 't Hooft limit $N \to \infty$ while keeping the 't Hooft coupling $\lambda = \kappa N$ fixed (or sending it to a critical value λ_c, [139]). Since in this limit κ is guaranteed to be very small, the world-sheet description is valid and trustable. In this limit the relevant string amplitudes are dominated by genus zero surfaces with an arbitrary number of boundaries, weighted by the 't Hooft coupling. Resumming the boundaries should then result in pure closed string amplitudes on the backreacted background as in the paradigmatic example of the Gopakumar-Vafa conifold transition [136].

Before going into the details of our construction, let us give a schematic overview of the main steps in our procedure to describe the D-brane backreaction:

1. Define a consistent quantum open–closed string field action

$$S_{\rm oc}(\Phi, \Psi)\,, \tag{5.1}$$

on the initial background

$$\mathrm{CFT}_0 \otimes \mathrm{BCFT}_0 \,. \tag{5.2}$$

2. Perform the large-N limit, resulting in a theory composed of a purely classical closed string sector and a quantum but planar open string sector

$$S_{\mathrm{oc}}(\Phi, \Psi) \to S_{\mathrm{pl}}(\Phi, \Psi) \,. \tag{5.3}$$

3. Integrate out the open string sector to obtain a classical closed SFT with a tadpole

$$S_{\mathrm{pl}}(\Phi, \Psi) \to \tilde{S}(\Phi) \,. \tag{5.4}$$

4. Perform the vacuum shift to eliminate the tadpole

$$\tilde{S}(\Phi) \to S_{\mathrm{BR}}(\varphi) \,, \tag{5.5}$$

and describe the dynamics in the new backreacted background

$$\mathrm{CFT}_* \,. \tag{5.6}$$

5.2 Open-closed string field theory action

The BV quantum master action can be written as

$$S_{\mathrm{oc}}(\Phi, \Psi) = \sum_{\substack{g,b,k=0 \\ \{p_1,\dots,p_b\}=0}}^{\infty} \frac{\kappa^{2g+b-2}}{b!k!(p_1)\cdots(p_b)} V^{g,b}_{k;\{p_1,\dots,p_b\}} \left(\Phi^{\wedge k} \otimes' \Psi^{\odot p_1} \wedge' \dots \wedge' \Psi^{\odot p_b}\right) , \tag{5.7}$$

where $(p) \coloneqq p + \delta_{p,0}$. Without too much specification the vertices $V^{g,b}_{k;\{p_1,\dots,p_b\}}$ are the open-closed string extension of the fundamental vertices described in Section 2.3 associated with Riemann surfaces of genus g and b boundaries with k bulk punctures and p_i boundary punctures on the i-th boundary. Notice that there can be boundaries

with no open-string insertions. Every vertex is weighted by a corresponding power of the string coupling constant κ according to the worldsheet topological expansion (4.11) and the string field rescaling (4.2) and (4.12). We recall that the symmetrized tensor products $\wedge$ symmetrize over bulk punctures

$$\Phi_1 \wedge \cdots \wedge \Phi_n \coloneqq \sum_{\sigma \in \mathbf{S}_n} (-1)^{\epsilon_\sigma} \Phi_{\sigma(1)} \otimes \cdots \otimes \Phi_{\sigma(n)} \,, \tag{5.8}$$

where $\mathbf{S}_n$ is the group of permutations. The direct sums of such multi-string states are elements of the symmetric tensor algebra $S\mathcal{H}_{\mathrm{c}}$, defined as

$$S\mathcal{H}_{\mathrm{c}} \coloneqq \bigoplus_{n \geq 0} \mathcal{H}_{\mathrm{c}}^{\wedge n} \,. \tag{5.9}$$

On every boundary open strings are inserted according to the cyclic tensor product

$$\Psi_1 \odot \cdots \odot \Psi_n \coloneqq \sum_{\sigma \in \mathbf{Z}_n} (-1)^{\epsilon_\sigma} \Psi_{\sigma(1)} \otimes \cdots \otimes \Psi_{\sigma(n)} \,, \tag{5.10}$$

where $\mathbf{Z}_n$ is the group of cyclic permutations. The vertices defined by $V^{g,b}_{k;\{p_1,\ldots,p_b\}}$ are therefore color-ordered. We can define cyclic tensor algebra $C\mathcal{H}_{\mathrm{o}}$ as follows

$$C\mathcal{H}_{\mathrm{o}} \coloneqq \bigoplus_{n \geq 0} \mathcal{H}_{\mathrm{o}}^{\odot n} \,. \tag{5.11}$$

This cyclic ordering is present on every boundary and all the boundaries are finally symmetrized by another symmetrized tensor product $\wedge'$. The extra tensor product $\otimes'$ separates closed and open strings.

Given the closed and open symplectic forms ω_{c} and ω_{o} it is useful to write the $V^{g,b}_{k;\{p_1,\ldots,p_b\}}$ using multi-string products which are defined as follows. First of all let us consider amplitudes with no open strings but not restricted by the genus or by the number of boundaries. These vertices can be expressed as

$$V^{g,b}_{k+1}(\Phi_1 \wedge \cdots \wedge \Phi_{k+1}) \coloneqq \omega_{\mathrm{c}} \left(\Phi_1, l^{(g,b)}_{k,0} \left(\Phi_2 \wedge \cdots \wedge \Phi_{k+1} \right) \right) \,, \tag{5.12}$$

where we have defined the degree odd multi string product

$$l_{k,0}^{(g,b)} : \mathcal{H}_{\rm c}^{\wedge k} \to \mathcal{H}_{\rm c} \,, \tag{5.13}$$

which is cyclic with respect to $\omega_{\rm c}$

$$\omega_{\rm c}\left(\Phi_1, l_{k,0}^{(g,b)}\left(\Phi_2 \wedge \cdots \wedge \Phi_{k+1}\right)\right) = (-1)^{\epsilon}\omega_{\rm c}\left(\Phi_{k+1}, l_{k,0}^{(g,b)}\left(\Phi_1 \wedge \cdots \wedge \Phi_k\right)\right) \,, \tag{5.14}$$

and is BPZ odd

$$\omega_{\rm c}\left(\Phi_1, l_{k,0}^{(g,b)}\left(\Phi_2 \wedge \cdots \wedge \Phi_{k+1}\right)\right) = -(-1)^{d(\Phi_1)}\omega_{\rm c}\left(l_{k,0}^{(g,b)}\left(\Phi_1 \wedge \cdots \wedge \Phi_k\right), \Phi_{k+1}\right) \,. \tag{5.15}$$

All the remaining vertices have at least one open string in at least one boundary and therefore they can be written using the open string symplectic form, by picking an arbitrary boundary with at least one open string puncture. This, thanks to full boundary symmetrization, can always be conventionally taken to correspond to the last in the arguments of $V^{g,b}$, that is the b-th argument with $p_b \geq 1$

$$\begin{aligned} &V_{k;\{p_1,\ldots,p_b\}}^{g,b}\left(\Phi_1 \wedge \cdots \wedge \Phi_k \otimes' \Psi_{1,1} \odot \cdots \odot \Psi_{1,p_1} \wedge' \ldots \wedge' \Psi_{b,1} \odot \cdots \odot \Psi_{b,p_b}\right) \\ :=&\omega_{\rm o}\left(\Psi_{b,1}, m_{k[p_1,\ldots,p_{b-1}]p_b-1}^{(g,b)}\left((\Phi)_k \otimes' [\Psi]_{p_1} \wedge' \cdots \wedge' [\Psi]_{p_{b-1}} \otimes'' \Psi_{b,2} \otimes \cdots \otimes \Psi_{b,p_b}\right)\right) \,, \end{aligned} \tag{5.16}$$

where to lighten the notation we have written

$$\begin{aligned} (\Phi)_k &:= \Phi_1 \wedge \cdots \wedge \Phi_k \,, \\ [\Psi]_{p_j} &:= \Psi_{j,1} \odot \cdots \odot \Psi_{j,p_j} \,, \end{aligned} \tag{5.17}$$

and the normalizations due to cyclicity are defined by $(p) := p + \delta_{p,0}$. Notice that we have introduced the $\otimes''$ to separate the special boundary from the others. The multi-string products with open string output $m^{(g,b)}$ are odd linear maps

$$m_{k[p_1,\ldots,p_{b-1}]p_b}^{(g,b)} : \mathcal{H}_{\rm c}^{\wedge k} \otimes' \mathcal{H}_{\rm o}^{\odot p_1} \wedge' \cdots \wedge' \mathcal{H}_{\rm o}^{\odot p_{b-1}} \otimes'' \mathcal{H}_{\rm o}^{\otimes p_b} \to \mathcal{H}_{\rm o} \,, \tag{5.18}$$

which are cyclic

$$\begin{aligned}
&\omega_{\mathrm{o}}\left(\Psi_{b,1}, m^{(g,b)}_{k[p_1,\dots,p_{b-1}]p_b-1}\left((\Phi)_k \otimes' [\Psi]_{p_1} \wedge' \cdots \wedge' [\Psi]_{p_{b-1}} \otimes'' \Psi_{b,2} \otimes \cdots \otimes \Psi_{b,p_b}\right)\right) \\
&= (-1)^{\epsilon} \\
&\omega_{\mathrm{o}}\left(\Psi_{b,p_b}, m^{(g,b)}_{k[p_1,\dots,p_{b-1}]p_b-1}\left((\Phi)_k \otimes' [\Psi]_{p_1} \wedge' \cdots \wedge' [\Psi]_{p_{b-1}} \otimes'' \Psi_{b,1} \otimes \cdots \otimes \Psi_{b,p_b-1}\right)\right),
\end{aligned} \tag{5.19}$$

and BPZ odd

$$\begin{aligned}
&\omega_{\mathrm{o}}\left(\Psi_{b,1}, m^{(g,b)}_{k[p_1,\dots,p_{b-1}]p_b-1}\left((\Phi)_k \otimes' [\Psi]_{p_1} \wedge' \cdots \wedge' [\Psi]_{p_{b-1}} \otimes'' \Psi_{b,2} \otimes \cdots \otimes \Psi_{b,p_b}\right)\right) \\
&= -(-1)^{\epsilon} \\
&\omega_{\mathrm{o}}\left(m^{(g,b)}_{k[p_1,\dots,p_{b-1}]p_b-1}\left((\Phi)_k \otimes' [\Psi]_{p_1} \wedge' \cdots \wedge' [\Psi]_{p_{b-1}} \otimes'' \Psi_{b,1} \otimes \cdots \otimes \Psi_{b,p_b-1}\right), \Psi_{b,p_b}\right).
\end{aligned} \tag{5.20}$$

Obviously there is nothing special about the choice of the special boundary b and this has the consequence that the $m^{(g,b)}$ products have extra properties obtained by expressing the same amplitude with different choice of special boundary

$$\begin{aligned}
&\omega_{\mathrm{o}}\left(\Psi_{b,1}, m^{(g,b)}_{k[p_1,\dots,p_{b-1}]p_b-1}\left((\Phi)_k \otimes' [\Psi]_{p_1} \wedge' \cdots \wedge' [\Psi]_{p_{b-1}} \otimes'' \Psi_{b,2} \otimes \cdots \otimes \Psi_{b,p_b}\right)\right) \\
=&(-1)^{\epsilon}\,\omega_{\mathrm{o}}\left(\Psi_{1,1}, m^{(g,b)}_{k[p_2,\dots,p_b]p_1-1}\left((\Phi)_k \otimes' [\Psi]_{p_2} \wedge' \cdots \wedge' [\Psi]_{p_b} \otimes'' \Psi_{1,2} \otimes \cdots \otimes \Psi_{1,p_1}\right)\right) \\
=&V^{g,b}_{k;\{p_1,\dots,p_b\}}\left((\Phi)_k \otimes' [\Psi]_{p_1} \wedge' \dots \wedge' [\Psi]_{p_b}\right).
\end{aligned} \tag{5.21}$$

Finally, it is useful to define dual products $l^{(g,b)}$ with a closed string output such that the same open-closed amplitude can be computed with the open or closed symplectic form

$$\begin{aligned}
&\omega_{\mathrm{o}}\left(\Psi_{b,1}, m^{(g,b)}_{k[p_1,\dots,p_{b-1}]p_b-1}\left((\Phi)_k \otimes' [\Psi]_{p_1} \wedge' \cdots \wedge' [\Psi]_{p_{b-1}} \otimes'' \Psi_{b,2} \otimes \cdots \otimes \Psi_{b,p_b}\right)\right) \\
:=&(-1)^{\epsilon}\omega_{\mathrm{c}}\left(\Phi_1, l^{(g,b)}_{k-1,[p_1,\cdots,p_b]}\left(\Phi_2 \wedge \cdots \wedge \Phi_k \otimes' [\Psi]_{p_1} \wedge' \cdots \wedge' [\Psi]_{p_b}\right)\right) \\
=&V^{g,b}_{k;\{p_1,\dots,p_b\}}\left((\Phi)_k \otimes' [\Psi]_{p_1} \wedge' \dots \wedge' [\Psi]_{p_b}\right).
\end{aligned} \tag{5.22}$$

With the definitions and properties given above, the open-closed BV action (5.7) can be written as

$$S_{\rm oc}(\Phi,\Psi)=\sum_{g,b=0}^{\infty}\kappa^{2g+b-2}\sum_{k=0}^{\infty}\frac{1}{b!}\Bigg[\frac{1}{(k+1)!}\omega_{\rm c}\left(\Phi,l_{k,0}^{(g,b)}\left(\Phi^{\wedge k}\right)\right)+\sum_{\substack{\{p_1,\dots,p_{b-1}\}=0\\ p_b=1}}^{\infty}\frac{C(p_1,\dots,p_b)}{k!\,(p_1)\cdots(p_b)}\,\omega_{\rm o}\left(\Psi,m^{(g,b)}_{k[p_1,\dots,p_{b-1}]p_b-1}\left(\Phi^{\wedge k}\otimes'\Psi^{\odot p_1}\dots\Psi^{\odot p_{b-1}}\otimes''\Psi^{\otimes(p_b-1)}\right)\right)\Bigg]\,. \tag{5.23}$$

Note that since we were forced to restrict the sum over p_b to start from 1 (so that the special boundary contains always at least one insertion), the summation would no longer automatically produce the correct symmetry factor in front of couplings containing at least one empty boundary. This needs to be corrected by including into the action the factor $C(p_1,\dots,p_b)$ which we define as

$$C(p_1,\dots,p_b)=\frac{\#\text{ of inequiv. rearrangements of the } b\text{-tuple } (p_1,\dots,p_b)}{\#\text{ of inequiv. rearrangements of the } b\text{-tuple } (p_1,\dots,p_b)\text{ s.t. the last entry}\neq 0}\,. \tag{5.24}$$

Using elementary combinatorics, one can show that $C(p_1,\dots,p_b)$ can in fact be expressed as

$$C(p_1,\dots,p_b)=\frac{b}{b-b_0}\,, \tag{5.25}$$

where $0\leq b_0<b$ denotes the number of boundaries with zero open-string field insertions. We will observe below that the presence of this combinatorial factor will be automatically accounted for when working in the WZW co-derivation formalism.

Notice that, using the open-closed relation (5.22) it is possible to write the action in different equivalent ways. For example, we can isolate the terms with only open

strings

$$
\begin{aligned}
&S_{\mathrm{oc}}(\Phi,\Psi) = \\
&\sum_{g,b=0}^{\infty} \kappa^{2g+b-2}\frac{1}{b!}\Bigg[\\
&\quad + \sum_{\{p_1,\dots,p_b\}=0}^{\infty} \frac{1}{(p_1)\cdots(p_b)} \sum_{k=0}^{\infty} \frac{1}{(k+1)!}\omega_{\mathrm{c}}\left(\Phi, l^{(g,b)}_{k,[p_1,\cdots,p_b]}\left(\Phi^{\wedge k}\otimes \Psi^{\odot p_1}\dots\Psi^{\odot p_b}\right)\right) \\
&\quad + \sum_{\substack{\{p_1,\dots,p_{b-1}\}=0 \\ p_b=1}}^{\infty} \frac{C(p_1,\dots,p_b)}{(p_1)\cdots(p_b)} \\
&\qquad\qquad \omega_{\mathrm{o}}\left(\Psi, m^{(g,b)}_{0[p_1,\dots,p_{b-1}]p_b-1}\left(\Psi^{\odot p_1}\dots\Psi^{\odot p_{b-1}}\otimes'' \Psi^{\otimes(p_b-1)}\right)\right)\Bigg].
\end{aligned}
\tag{5.26}
$$

Other rewritings (for example choosing differently the special boundary) are also possible. It is one of the aim of this chapter to give a presentation of the OC action which is invariant under all of the above-mentioned rewritings, based on an appropriately defined symplectic form.

5.3 Co-algebraic open-closed structure

To start with we define the open-closed tensor algebra as

$$
S\mathcal{H}_{\mathrm{c}} \otimes' SC\mathcal{H}_{\mathrm{o}} \coloneqq \left[\bigoplus_{k\geq 0}\mathcal{H}_{\mathrm{c}}^{\wedge k}\right] \otimes' \left[\bigoplus_{b\geq 0}\left(\bigoplus_{p_1\geq 0}\mathcal{H}_{\mathrm{o}}^{\odot p_1}\right)\wedge'\cdots\wedge'\left(\bigoplus_{p_b\geq 0}\mathcal{H}_{\mathrm{o}}^{\odot p_b}\right)\right], \tag{5.27}
$$

this formula gives a definition of $SC\mathcal{H}_{\mathrm{o}}$ as the direct sum of the symmetric tensor product of cyclic tensor algebras, defined in (5.11). To manipulate the action it is useful to introduce the open-closed group-like element

$$
\mathcal{G} \coloneqq e^{\wedge\Phi} \otimes' e^{\wedge' C(\Psi)}, \tag{5.28}
$$

where we have defined the cyclic group-like element[3]

$$\mathcal{C}(\Psi) \coloneqq \sum_{l=0}^{\infty} \frac{1}{(l)} \Psi^{\odot l} = 1 + \log(1 - \odot\Psi)\,. \tag{5.29}$$

Explicitly we can write

$$\mathcal{G} = \left[\sum_{k\geq 0} \frac{1}{k!}\Phi^{\wedge k}\right] \otimes' \left[\sum_{b\geq 0} \frac{1}{b!} \left(\sum_{p_1\geq 0} \frac{1}{(p_1)}\Psi^{\odot p_1}\right) \wedge' \cdots \wedge' \left(\sum_{p_b\geq 0} \frac{1}{(p_b)}\Psi^{\odot p_b}\right)\right]\,. \tag{5.30}$$

As we are now going to see $\mathcal{G}$ is the correct reservoir for symmetrized closed strings, together with cyclicized open strings taken from symmetrized boundaries, that appear in the action.

The multi-string products $l^{(g,b)}$ and $m^{(g,b)}$ can be upgraded to odd co-derivation-like maps on $S\mathcal{H}_{\rm c} \otimes' S\mathcal{CH}_{\rm o}$. This is straightforward for the $l^{(g,b)}$ products because their inputs are already in $S\mathcal{H}_{\rm c} \otimes' S\mathcal{CH}_{\rm o}$ and we have

$$\boldsymbol{l}^{(g,b)}_{k[p_1,\cdots,p_b]} : S\mathcal{H}_{\rm c} \otimes' S\mathcal{CH}_{\rm o} \to S\mathcal{H}_{\rm c} \otimes' S\mathcal{CH}_{\rm o}\,, \tag{5.31}$$

where the defining action on the group-like element $\mathcal{G}$ is as follows

$$\begin{aligned}\boldsymbol{l}^{(g,b)}_{k[p_1,\cdots,p_b]}\mathcal{G} = \sum_{k'\geq k}^{\infty}\sum_{b'\geq b}^{\infty} \sum_{\{p'_{b+1},\ldots,p'_{b'}\}\geq 0} \frac{1}{b'!k'!\left((p_1)\cdots(p_b)(p'_{b+1})\cdots(p'_{b'})\right)} \binom{k'}{k}\binom{b'}{b} \\ \left(l^{(g,b)}_{k[p_1,\cdots,p_b]}\left(\Phi^{\wedge k}\otimes' \Psi^{\odot p_1}\wedge'\cdots\wedge'\Psi^{\odot p_b}\right) \wedge \Phi^{\wedge(k'-k)} \otimes' \Psi^{\odot p'_{b+1}} \wedge' \cdots \wedge' \Psi^{\odot p'_{b'}}\right)\,,\end{aligned} \tag{5.32}$$

where the binomial coefficients account for the choice of k bulk punctures out of k' and b boundaries out of b'. Notice in particular that it behaves as a co-derivation for the closed string inputs and for the open strings on the boundaries which are not acted upon. On the other hand the number of open string insertions on the boundaries that enter into the multi-string product has to precisely match the indices $\{p_1,\cdots,p_b\}$. The reader can easily extend this definition to a generic element of $S\mathcal{H}_{\rm c} \otimes' S\mathcal{CH}_{\rm o}$.

[3]The symmetrized and cyclicized group like elements $e^{\wedge\chi}$ and $\mathcal{C}(\chi)$ are the same when expressed in terms of the simple tensor product $\otimes$, $e^{\wedge\chi} = \mathcal{C}(\chi) = \frac{1}{1-\otimes\chi} = \sum_{k\geq 0}\chi^{\otimes k}$.

We can then sum on all possible inputs and, at fixed genus and boundaries, define

$$\boldsymbol{l}^{(g,b)} := \sum_{k,\{p_1,\dots,p_b\}} \boldsymbol{l}^{(g,b)}_{k[p_1,\dots,p_b]} \,. \tag{5.33}$$

This is an example of an odd cyclic closed string co-derivation which obeys

$$\omega_{\rm c}\left(\pi_{10}\,\boldsymbol{l}^{(g,b)}\,\boldsymbol{a}_{\rm c}\,\mathcal{G}\,,\,\pi_{10}\boldsymbol{b}_{\rm c}\,\mathcal{G}\right) = -(-1)^{d(a)}\omega_{\rm c}\left(\pi_{10}\,\boldsymbol{a}_{\rm c}\,\mathcal{G}\,,\,\pi_{10}\boldsymbol{l}^{(g,b)}\boldsymbol{b}_{\rm c}\,\mathcal{G}\right)\,, \tag{5.34}$$

where π_{10} projects onto a single copy of $\mathcal{H}_{\rm c}$ inside $S\mathcal{H}_{\rm c}\otimes' SC\mathcal{H}_{\rm o}$ and $(\boldsymbol{a}_{\rm c},\boldsymbol{b}_{\rm c})$ are any two closed string co-derivations (*i.e.* with closed string output). This is a consequence of a relation analogous to (5.15) which holds for the products $l^{(g,b)}$.

We would now like to upgrade also the products $m^{(g,b)}$ (with open string output) to $S\mathcal{H}_{\rm c}\otimes' SC\mathcal{H}_{\rm o}$. An immediate difficulty for doing so is that the products $m^{(g,b)}_{k[p_1,\cdots,p_{b-1}]p_b}$ are defined with non-cyclic inputs on the special boundary b, (5.18). However it is easy to realize that co-derivations on the tensor algebra are automatically co-derivations on the cyclic tensor algebra. In particular if $\boldsymbol{c}_i$ is a co-derivation on $\mathcal{TH}$ it will also act as a coderivation-like map on $C\mathcal{H}$: for $i\leq j$ we have

$$\begin{aligned}
\boldsymbol{c}_i\left(\Psi_1\odot\cdots\odot\Psi_j\right) &= \boldsymbol{c}_i\left(\sum_{\sigma\in\mathbf{Z}_j}\Psi_{\sigma(1)}\otimes\cdots\otimes\Psi_{\sigma(j)}\right) = \\
&= \sum_{\sigma\in\mathbf{Z}_j} c_i\left(\Psi_{\sigma(1)}\otimes\cdots\otimes\Psi_{\sigma(i)}\right)\otimes\Psi_{\sigma(i+1)}\otimes\cdots\otimes\Psi_{\sigma(l)}+ \\
&\quad\ \vdots \\
&+ \sum_{\sigma\in\mathbf{Z}_j}\Psi_{\sigma(1)}\otimes\cdots\otimes\Psi_{\sigma(j-i)}\otimes c_i\left(\Psi_{\sigma(j-i+1)}\otimes\cdots\otimes\Psi_{\sigma(j)}\right) = \\
&= c_i\left(\Psi_1\otimes\cdots\otimes\Psi_i\right)\odot\Psi_{i+1}\odot\cdots\odot\Psi_j \\
&+ c_i\left(\Psi_2\otimes\cdots\otimes\Psi_{i+1}\right)\odot\Psi_{i+2}\odot\cdots\Psi_j\odot\Psi_1 \\
&\quad\ \vdots \\
&+ c_i\left(\Psi_j\otimes\Psi_1\otimes\cdots\otimes\Psi_{i-1}\right)\odot\Psi_i\odot\cdots\odot\Psi_{j-1},
\end{aligned} \tag{5.35}$$

where we have assumed degree even entries (otherwise some obvious signs appear).

Therefore, one can consistently restrict

$$\begin{array}{c} \boldsymbol{c}_i : \mathcal{TH} \to \mathcal{TH} \\ \downarrow \\ \boldsymbol{c}_i : C\mathcal{H} \to C\mathcal{H}. \end{array} \tag{5.36}$$

Notice in particular that for $i \leq j$ we have

$$\boldsymbol{c}_i \left(\Psi^{\odot j} \right) = (j)\, c_i \left(\Psi^{\otimes i} \right) \odot \Psi^{\odot j - i}. \tag{5.37}$$

This suggests to define the odd co-derivation with open string output

$$\boldsymbol{m}^{(g,b)}_{k[p_1,\cdots,p_{b-1}]p_b} : S\mathcal{H}_{\rm c} \otimes' SC\mathcal{H}_{\rm o} \to S\mathcal{H}_{\rm c} \otimes' SC\mathcal{H}_{\rm o}, \tag{5.38}$$

in such a way that its action on the open-closed group-like element will be

$$\begin{aligned} &\boldsymbol{m}^{(g,b)}_{k[p_1,\ldots,p_{b-1}]p_b} \mathcal{G} = \\ &= \sum_{k' \geq k} \sum_{b' \geq b} \sum_{p'_b \geq p_b} \sum_{\{p'_{b+1},\ldots,p'_{b'}\} \geq 0} \frac{1}{b'!k'!} \, \frac{1}{(p_1)\cdots(p_{b-1})} \, \frac{1}{(p'_b)} \, \frac{1}{(p'_{b+1})\cdots(p'_{b'})} \times \\ &\qquad \times \binom{k'}{k} \binom{b'}{b} b\,(p'_b) \times \\ &\times \Bigg(\Phi^{\wedge k'-k} \otimes' \\ &\qquad \otimes' \, m^{(g,b)}_{k[p_1,\ldots,p_{b-1}]p_b} \left(\Phi^{\wedge k} \otimes' \Psi^{\odot p_1} \wedge' \ldots \wedge' \Psi^{\odot p_{b-1}} \otimes'' \Psi^{\otimes p_b} \right) \odot \Psi^{\odot p'_b - p_b} \wedge' \\ &\qquad\qquad \wedge' \, \Psi^{\odot p'_{b+1}} \wedge' \ldots \wedge' \Psi^{\odot p'_{b'}} \Bigg), \end{aligned} \tag{5.39}$$

where, to help the visualization, the first line contains the coefficients of the open-closed group-like element, the second line contains the combinatorial factors associated to choosing k closed strings out of k', b boundaries out of b' and choosing one special boundary out of b. Finally the coefficient (p'_b) is the result of (5.37). Notice the cancellation $\frac{1}{(p'_b)}\,(p'_b) = 1$. The explicit action on a generic element of $S\mathcal{H}_{\rm c} \otimes' SC\mathcal{H}_{\rm o}$ is straightforwardly obtained given the understood symmetrizations

and cyclicizations. It is however rather cloggy and not particularly instructive to write it down.

Analogously to what we have done with the closed string products $\boldsymbol{l}^{(g,b)}$, we can now sum over the full open and closed insertions at fixed genus and boundaries

$$\boldsymbol{m}^{(g,b)} := \sum_{k\geq 0} \sum_{p_1 \cdots p_b \geq 0} \boldsymbol{m}^{(g,b)}_{k[p_1,\ldots,p_{b-1}]p_b}. \tag{5.40}$$

The cyclicity properties further imply the relations

$$\omega_{\mathrm{o}}\left(\pi_{01}\boldsymbol{m}^{(g,b)}\boldsymbol{a}_{\mathrm{o}}\,\mathcal{G}, \pi_{01}\boldsymbol{b}_{\mathrm{o}}\,\mathcal{G}\right) = -(-1)^{d(a)}\omega_{\mathrm{o}}\left(\pi_{01}\boldsymbol{a}_{\mathrm{o}}\,\mathcal{G}, \pi_{01}\boldsymbol{m}^{(g,b)}\boldsymbol{b}_{\mathrm{o}}\,\mathcal{G}\right), \tag{5.41}$$

for any pair of open string co-derivations $(\boldsymbol{a}_{\mathrm{o}}, \boldsymbol{b}_{\mathrm{o}})$. This is a consquence of (5.20) and (5.21).

By construction the obtained open and closed string co-derivations $\boldsymbol{m}^{(g,b)}$ and $\boldsymbol{l}^{(g,b)}$ are not independent but obey the relations

$$\omega_{\mathrm{c}}\left(\pi_{10}\,\boldsymbol{l}^{(g,b)}\,\boldsymbol{a}_{\mathrm{o}}\,\mathcal{G}\,,\ \pi_{10}\,\boldsymbol{b}_{\mathrm{c}}\,\mathcal{G}\right) = -(-1)^{d(a)}\omega_{\mathrm{o}}\left(\pi_{01}\,\boldsymbol{a}_{\mathrm{o}}\,\mathcal{G}\,,\ \pi_{01}\,\boldsymbol{m}^{(g,b)}\,\boldsymbol{b}_{\mathrm{c}}\,\mathcal{G}\right) \tag{5.42}$$

$$\omega_{\mathrm{o}}\left(\pi_{01}\,\boldsymbol{m}^{(g,b)}\,\boldsymbol{a}_{\mathrm{c}}\,\mathcal{G}\,,\ \pi_{01}\,\boldsymbol{b}_{\mathrm{o}}\,\mathcal{G}\right) = -(-1)^{d(a)}\omega_{\mathrm{c}}\left(\pi_{10}\,\boldsymbol{a}_{\mathrm{c}}\,\mathcal{G}\,,\ \pi_{10}\,\boldsymbol{l}^{(g,b)}\,\boldsymbol{b}_{\mathrm{o}}\,\mathcal{G}\right), \tag{5.43}$$

where $(\boldsymbol{a}_{\mathrm{o,c}}, \boldsymbol{b}_{\mathrm{o,c}})$ are open or closed co-derivations. This is a consequence of (5.22).

5.3.1 BV Action in WZW form

To write the action in a compact form we use a standard trick. We first define an interpolation for both the open and closed string fields: namely $\Phi(t)$ such that $\Phi(0) = 0$, $\Phi(1) = \Phi$ and $\Psi(t)$ such that $\Psi(0) = 0$, $\Psi(1) = 1$. Then we write the action (5.23) as the integral of a total derivative by inserting $\int_0^1 dt\partial_t$ in front. Let us start

with the simpler term containing only closed strings

$$\begin{aligned}
&\sum_{g,b=0}^{\infty} \kappa^{2g+b-2} \sum_{k=0}^{\infty} \frac{1}{(k+1)!b!} \left(\int_0^1 dt \partial_t \right) \omega_{\mathrm{c}} \left(\Phi, l_{k,0}^{(g,b)} \left(\Phi^{\wedge k} \right) \right) \\
&= \sum_{g,b=0}^{\infty} \kappa^{2g+b-2} \sum_{k=0}^{\infty} \int_0^1 dt \frac{k+1}{(k+1)!b!} \omega_{\mathrm{c}} \left(\dot{\Phi}, l_{k,0}^{(g,b)} \left(\Phi^{\wedge k} \right) \right) \\
&= \sum_{g,b=0}^{\infty} \kappa^{2g+b-2} \frac{1}{b!} \int_0^1 dt \omega_{\mathrm{c}} \left(\dot{\Phi}, \left[\sum_{k=0}^{\infty} \frac{1}{k!} l_{k,0}^{(g,b)} \left(\Phi^{\wedge k} \right) \right] \right) . \qquad (5.44)
\end{aligned}$$

We keep it as it is for the time being and we consider the other term in (5.23) containing open strings (excluding for now the sum over topologies)

$$\begin{aligned}
&\sum_{\substack{k,\{p_1,\ldots,p_{b-1}\}=0 \\ p_b=1}}^{\infty} \frac{C(p_1,\ldots,p_b)}{b!k!(p_1)\cdots(p_b)} \\
&\left(\int_0^1 dt \partial_t \right) \omega_{\mathrm{o}} \left(\Psi, m_{k[p_1,\ldots,p_{b-1}]p_b-1}^{(g,b)} \left(\Phi^{\wedge k} \otimes' \Psi^{\odot p_1} \ldots \Psi^{\odot p_{b-1}} \otimes'' \Psi^{\otimes (p_b-1)} \right) \right) . \qquad (5.45)
\end{aligned}$$

The t-derivative now acts separately on $\Phi(t)$ and $\Psi(t)$. We start writing down the contribution from $\Phi(t)$ which is given by

$$\begin{aligned}
&\sum_{\substack{k=1,\{p_1,\ldots,p_{b-1}\}=0 \\ p_b=1}}^{\infty} \frac{kC(p_1,\ldots,p_b)}{b!k!(p_1)\cdots(p_b)} \\
&\int_0^1 dt\, \omega_{\mathrm{o}} \left(\Psi, m_{k[p_1,\ldots,p_{b-1}]p_b-1}^{(g,b)} \left(\dot{\Phi} \wedge \Phi^{\wedge k-1} \otimes' \Psi^{\odot p_1} \ldots \Psi^{\odot p_{b-1}} \otimes'' \Psi^{\otimes (p_b-1)} \right) \right) , \qquad (5.46)
\end{aligned}$$

now we use (5.22) and rewrite it using the closed symplectic form as

$$\begin{aligned}
&\sum_{k=1,\{p_1,\ldots,p_b\}=0} \frac{k}{b!k!(p_1)\cdots(p_b)} \\
&\qquad \int_0^1 dt\, \omega_{\mathrm{c}} \left(\dot{\Phi}, l_{k-1[p_1,\ldots,p_{b-1},p_b]}^{(g,b)} \left(\Phi^{\wedge k-1} \otimes' \Psi^{\odot p_1} \ldots \Psi^{\odot p_b} \right) \right) . \qquad (5.47)
\end{aligned}$$

Notice that the combinatorial factor $C(p_1,\dots,p_b)$ disappeared from the last expression because now all the summation indices $p_1,\dots,p_b$ are allowed to be zero. We then realize that we can combine this term (adding the understood sum over topologies $\sum_{g,b}\kappa^{2g+b-2}$) with (5.44) to obtain

$$\begin{aligned}(5.44)+\sum_{g,b}\kappa^{2g+b-2}(5.47) &= \int_0^1 dt\,\sum_{g,b}\kappa^{2g+b}\,\frac{\omega_{\rm c}}{\kappa^2}\left(\pi_{10}\boldsymbol{\partial}_t\,\mathcal{G}\,,\,\pi_{10}\boldsymbol{l}^{(g,b)}\,\mathcal{G}\right)\\ &= \int_0^1 dt\,\frac{\omega_{\rm c}}{\kappa^2}\left(\pi_{10}\boldsymbol{\partial}_t\,\mathcal{G}\,,\,\pi_{10}\boldsymbol{l}\,\mathcal{G}\right)\,, \end{aligned}\tag{5.48}$$

where we have defined the total closed string co-derivation

$$\boldsymbol{l}\coloneqq\sum_{g,b}\kappa^{2g+b}\boldsymbol{l}^{(g,b)}\,.\tag{5.49}$$

After this treatment we remain with the ∂_t acting on $\Psi(t)$ in (5.45). This will of course act only on the $b-b_0$ boundaries which contain at least one open-string field insertion. Using the formula (5.25) for $C(p_1,\dots,p_b)$, we observe that acting with the derivative ∂_t therefore turns the combinatorial factor $C(p_1,\dots,p_b)$ into a factor of b. Thanks to the cyclicity relations (5.20) and (5.21) and due to the explicit summations on open string insertions we therefore produce an overall factor $b(p_b)$ obtaining in total, cfr (5.39)

$$\begin{aligned}&\sum_{g,b}\sum_{k,\{p_1,\dots,p_b,p_b\geq 1\}}\frac{\kappa^{2g+b-2}\,b(p_b)}{b!k!(p_1)\cdots(p_b)}\\ &\qquad\int_0^1 dt\;\omega_{\rm o}\left(\dot\Psi,m^{(g,b)}_{k[p_1,\dots,p_{b-1}]p_b-1}\left(\Phi^{\wedge k}\otimes'\Psi^{\odot p_1}\dots\Psi^{\odot p_{b-1}}\otimes''\Psi^{\otimes(p_b-1)}\right)\right)\\ =&\sum_{g,b}\kappa^{2g+b-1}\int_0^1 dt\,\frac{\omega_{\rm o}}{\kappa}\left(\pi_{01}\,\boldsymbol{\partial}_t\,\mathcal{G}\,,\,\pi_{01}\boldsymbol{m}^{(g,b)}\,\mathcal{G}\right)\\ =&\int_0^1 dt\,\frac{\omega_{\rm o}}{\kappa}\left(\pi_{01}\,\boldsymbol{\partial}_t\,\mathcal{G}\,,\,\pi_{01}\boldsymbol{m}\,\mathcal{G}\right),\end{aligned}\tag{5.50}$$

where we have defined the total open string co-derivation

$$\boldsymbol{m}\coloneqq\sum_{g,b}\kappa^{2g+b-1}\boldsymbol{m}^{(g,b)}\,.\tag{5.51}$$

The full BV master action can thus be written as

$$S_{\rm oc}(\Phi,\Psi)=\int_0^1 dt\left(\frac{\omega_{\rm c}}{\kappa^2}\left(\pi_{10}\boldsymbol{\partial}_t\,\mathcal{G}\,,\,\pi_{10}\boldsymbol{l}\,\mathcal{G}\right)+\frac{\omega_{\rm o}}{\kappa}\left(\pi_{01}\,\boldsymbol{\partial}_t\,\mathcal{G}\,,\,\pi_{01}\boldsymbol{m}\,\mathcal{G}\right)\right)\,, \tag{5.52}$$

At last we point out that, as expected from the WZW form, the variation of the action is a pure t-boundary term that can be explicitly evaluated as

$$\begin{aligned}\delta S_{\rm oc}=&\frac{\omega_{\rm o}}{\kappa}\left(\pi_{01}\boldsymbol{\delta}_o\mathcal{G},\pi_{01}\boldsymbol{m}\,\mathcal{G}\right)\Big|_0^1+\frac{\omega_{\rm c}}{\kappa^2}\left(\pi_{10}\boldsymbol{\delta}_c\,\mathcal{G},\pi_{10}\boldsymbol{l}\,\mathcal{G}\right)\Big|_0^1 &(5.53)\\ =&\frac{\omega_{\rm o}}{\kappa}\left(\delta\Psi,\pi_{01}\boldsymbol{m}\,\mathcal{G}\right)+\frac{\omega_{\rm c}}{\kappa^2}\left(\delta\Phi,\pi_{10}\boldsymbol{l}\,\mathcal{G}\right)\,, &(5.54)\end{aligned}$$

where the generic variation δ has been naturally upgraded to a co-derivation which variates every entry of the interpolated group-like element according to the Leibniz rule on the tensor product such that $\delta\Phi(0)=0=\delta\Psi(0)$ and can be naturally split into its open- and closed-string valued part $\delta\to\boldsymbol{\delta}=\boldsymbol{\delta}_{\rm c}+\boldsymbol{\delta}_{\rm o}$.

5.3.2 Open-closed co-derivations and repackaging

The form of the action (5.52) is clearly suggesting an even more compact formulation. First of all we can define a (κ-dependent) symplectic form in the space $\mathcal{H}_{\rm c}\oplus\mathcal{H}_{\rm o}$ as

$$\hat{\omega}(\Phi_1+\Psi_1\,,\,\Phi_2+\Psi_2)\coloneqq\frac{\omega_{\rm c}}{\kappa^2}(\Phi_1,\Phi_2)+\frac{\omega_{\rm o}}{\kappa}(\Psi_1,\Psi_2)\,. \tag{5.55}$$

Next we realize that calling $\chi\coloneqq\Phi+\Psi$ the open-closed string field, this is simply extracted from the open-closed group-like element $\mathcal{G}$ by the projection on a single string $\pi_1\coloneqq\pi_{10}+\pi_{01}$ as

$$\pi_1\mathcal{G}=\chi=\Phi+\Psi\,. \tag{5.56}$$

Then we simply observe that for open and closed string co-derivations $(\boldsymbol{l},\boldsymbol{m})$ we obviously have

$$\pi_{01}\boldsymbol{l}=0, \tag{5.57}$$

$$\pi_{10}\boldsymbol{m}=0 \tag{5.58}$$

so that we can define the open-closed co-derivation

$$\boldsymbol{n} \coloneqq \boldsymbol{l} + \boldsymbol{m}\,, \tag{5.59}$$

to rewrite the action (5.52) simply as

$$S_{\mathrm{oc}}(\Phi,\Psi) = S(\chi) = \int_0^1 dt\, \hat{\omega}\left(\pi_1 \boldsymbol{\partial}_t\, \mathcal{G}\,,\, \pi_1\, \boldsymbol{n}\, \mathcal{G}\right)\,. \tag{5.60}$$

In the total open-closed Hilbert space $\mathcal{H}_{\mathrm{c}} \oplus \mathcal{H}_{\mathrm{o}}$ it is important to realize that $\boldsymbol{n}$ is a cyclic co-derivation with respect to $\hat{\omega}$, meaning that given any two open-closed co-derivations $\boldsymbol{a} = \boldsymbol{a}_{\mathrm{c}} + \boldsymbol{a}_{\mathrm{o}}$ and $\boldsymbol{b} = \boldsymbol{b}_{\mathrm{c}} + \boldsymbol{b}_{\mathrm{o}}$, as a consequence of (5.34, 5.41, 5.42, 5.43), we have that

$$\hat{\omega}\left(\pi_1\, \boldsymbol{n}\, \boldsymbol{a}\, \mathcal{G}\,,\, \pi_1\, \boldsymbol{b}\, \mathcal{G}\right) = -(-1)^{d(a)} \hat{\omega}\left(\pi_1\, \boldsymbol{a}\, \mathcal{G}\,,\, \pi_1\, \boldsymbol{n}\, \boldsymbol{b}\, \mathcal{G}\right)\,. \tag{5.61}$$

The variation of the action can also be written in compact form by repackaging (5.53) as

$$\delta S_{\mathrm{oc}} = \hat{\omega}\left(\pi_1 \boldsymbol{\delta}\, \mathcal{G}\,,\, \pi_1 \boldsymbol{n}\, \mathcal{G}\right)\Big|_0^1 = \hat{\omega}\left(\delta\chi\,,\, \pi_1 \boldsymbol{n}\, \mathcal{G}\right)\,. \tag{5.62}$$

5.4 BV quantum master equation and homotopy relations

Now we would like to establish under which conditions the open-closed SFT action (5.7) is consistent, i.e. it is a solution to the BV master equation ($\hbar = 1$)

$$(S,S) + 2\Delta S = (S,S)_{\mathrm{c}} + (S,S)_{\mathrm{o}} + 2\Delta_{\mathrm{c}} S + 2\Delta_{\mathrm{o}} S = 0\,, \tag{5.63}$$

where the open and closed BV structures are given defined in (4.3a), (4.3b), (4.13a) and (4.13b). These objects can be unified in open-closed structures by defining the open-closed BV string field

$$\chi = \chi^a f_a = \Phi^a c_a + \Psi^a o_a\,. \tag{5.64}$$

The total BV bracket is then given by

$$(\,\cdot\,,\,\cdot\,) = \frac{\overleftarrow{\partial}}{\partial\chi^a}\,\hat{\omega}^{ab}\,\frac{\overrightarrow{\partial}}{\partial\chi^b}\,, \tag{5.65}$$

and the symplectic laplacian

$$\Delta = \frac{1}{2}(-)^{\chi^a}\,\hat{\omega}^{ab}\frac{\overrightarrow{\partial}}{\partial\chi^a}\frac{\overrightarrow{\partial}}{\partial\chi^b}\,. \tag{5.66}$$

Notice that in these unified open-closed structures the dependence on the string coupling constant is hidden inside the open-closed symplectic form $\hat{\omega}$ (5.55). Now the completeness relation is [4]

$$\mathbf{1}_{\mathcal{H}_{\rm c}\oplus\mathcal{H}_{\rm o}} = f_a\hat{\omega}\,(f^a,\cdot) = -f^a\hat{\omega}\,(f_a,\cdot)\;, \tag{5.67}$$

and the open-closed Poisson bi-vector is given by

$$U = \kappa^2\,U_{\rm c} + \kappa\,U_{\rm o} = \frac{(-1)^{f^a}}{2}f_a\wedge f^a\,, \tag{5.68}$$

which is easily shown to obey

$$(\hat{\omega}\otimes\mathbf{1})(1\otimes U) = \mathbf{1}_{\mathcal{H}_{\rm c}\oplus\mathcal{H}_{\rm o}}\,. \tag{5.69}$$

[4] Notice that $f^a = (\kappa^2\,c^a, \kappa\,o^a)$.

5.4.1 BV master equation

Let us see how we can evaluate the master equation directly at the unified open-closed co-algebra level. Starting with the BV bracket we find

$$
\begin{aligned}
(S_{\mathrm{oc}}, S_{\mathrm{oc}}) =& S_{\mathrm{oc}} \frac{\overleftarrow{\partial}}{\partial \chi^a} \hat{\omega}^{ab} \frac{\overrightarrow{\partial}}{\partial \chi^b} S_{\mathrm{oc}} && (5.70)\\
=& (-1)^{\chi^b} \hat{\omega} \left(\pi_1 \boldsymbol{\partial}_{\chi^a} \mathcal{G} \,,\, \pi_1 \boldsymbol{n}\, \mathcal{G}\right) \hat{\omega}^{ab} \hat{\omega} \left(\pi_1 \boldsymbol{\partial}_{\chi^b} \mathcal{G} \,,\, \pi_1 \boldsymbol{n}\, \mathcal{G}\right)\\
=& (-1)^{\chi^b} \hat{\omega} \left(f_a \,,\, \pi_1 \boldsymbol{n}\, \mathcal{G}\right) \hat{\omega}^{ab} \hat{\omega} \left(f_b \,,\, \pi_1 \boldsymbol{n}\, \mathcal{G}\right)\\
=& - \hat{\omega} \left(\pi_1 \boldsymbol{n}\, \mathcal{G} \,,\, \pi_1 \boldsymbol{n}\, \mathcal{G}\right) && (5.71)\\
=& - \int_0^1 dt\, \partial_t \hat{\omega} \left(\pi_1 \boldsymbol{n}\, \mathcal{G}(t) \,,\, \pi_1 \boldsymbol{n}\, \mathcal{G}(t)\right)\\
=& - 2 \int_0^1 dt\, \hat{\omega} \left(\pi_1 \boldsymbol{n} \boldsymbol{\partial}_t \mathcal{G} \,,\, \pi_1 \boldsymbol{n}\, \mathcal{G}\right)\\
=& 2 \int_0^1 dt\, \hat{\omega} \left(\pi_1 \boldsymbol{\partial}_t \mathcal{G} \,,\, \pi_1 \boldsymbol{n}\boldsymbol{n}\, \mathcal{G}\right) = 2 \int_0^1 dt\, \hat{\omega} \left(\dot{\chi},\, \pi_1 \boldsymbol{n}^2\, \mathcal{G}\right) \,, && (5.72)
\end{aligned}
$$

where we have used the variation of the action (5.62), the completeness relation (5.67) and we have finally rewritten everything in interpolated way to be able to bring the co-derivation $\boldsymbol{n}$ to the other side.

Consider now the symplectic laplacian

$$
\begin{aligned}
\Delta S_{\mathrm{oc}} =& \frac{1}{2} (-)^{\chi^a} \hat{\omega}^{ab} \frac{\overrightarrow{\partial}}{\partial \chi^a} \frac{\overrightarrow{\partial}}{\partial \chi^b} S_{\mathrm{oc}}\\
=& - \frac{1}{2} \hat{\omega}^{ab} \hat{\omega} \left(\pi_1 \boldsymbol{\partial}_{\chi^a} \mathcal{G} \,,\, \pi_1 \boldsymbol{n} \boldsymbol{\partial}_{\chi^b} \mathcal{G}\right)\\
=& \frac{1}{2} \hat{\omega} \left(f^a \,,\, \pi_1 \boldsymbol{n} \boldsymbol{f}_a \mathcal{G}\right) \,, && (5.73)
\end{aligned}
$$

where we have introduced the 0-product co-derivation $\boldsymbol{f}_a$ which inserts the open-closed basis element f_a in $\mathcal{G}$ as an open-closed co-derivation. At this point, we can proceed analogously to (4.49), using the consequences of open-closed duality to

write

$$\begin{aligned}
&\frac{1}{2}\,\hat{\omega}\,(f^a\,,\,\pi_1\boldsymbol{n}\boldsymbol{f}_a\,\mathcal{G})\\
=&\frac{1}{2}(-)^{f^a}\int_0^1\,\hat{\omega}\,(\dot{\chi}\,,\,\pi_1\boldsymbol{n}\boldsymbol{f}_a\boldsymbol{f}^a\,\mathcal{G}(t))\\
=&\int_0^1\,\hat{\omega}\,(\dot{\chi}\,,\,\pi_1\boldsymbol{n}\,\boldsymbol{U}\,\mathcal{G}(t))\;,
\end{aligned}\tag{5.74}$$

where we have defined the higher order odd co-derivation associated to the open-closed Poisson bi-vector

$$\boldsymbol{U}\coloneqq\frac{1}{2}(-)^{f^a}\boldsymbol{f}_a\boldsymbol{f}^a,\tag{5.75}$$

which trivially obeys

$$\boldsymbol{U}^2=0,\tag{5.76}$$

$$\pi_1\boldsymbol{U}=0.\tag{5.77}$$

In total we have thus shown

$$\begin{aligned}
\frac{1}{2}(S_{\rm oc},S_{\rm oc})+\Delta S_{\rm oc}&=\int_0^1\,\hat{\omega}\,(\dot{\chi}\,,\,\pi_1\boldsymbol{n}\,(\boldsymbol{n}+\boldsymbol{U})\,\mathcal{G}(t))\\
&=\int_0^1\,\hat{\omega}\left(\dot{\chi}\,,\,\pi_1(\boldsymbol{n}+\boldsymbol{U})^2\,\mathcal{G}(t)\right)\,.
\end{aligned}\tag{5.78}$$

Therefore the BV quantum master equation is satisfied if the open-closed products assembled in the co-derivation $\boldsymbol{n}$ obey

$$(\boldsymbol{n}+\boldsymbol{U})^2=0\,.\tag{5.79}$$

5.4.2 Topological expansion

It is instructive to partially un-package this construction by making explicit the open and closed sectors. In this case the quantum BV master equation takes the form

$$\begin{aligned}&\frac{1}{2}(S_{oc},S_{oc})+\Delta S_{oc}=\\&=\frac{1}{\kappa^2}\int_0^1 dt\,\omega_{\rm c}\left(\dot\Phi,\pi_{10}\left\{\frac{1}{2}[\boldsymbol{l},\boldsymbol{l}]+\boldsymbol{l}\boldsymbol{m}+\left[\boldsymbol{l},\kappa\boldsymbol{U}_o+\kappa^2\boldsymbol{U}_c\right]\right\}\mathcal{G}(t)\right)+\\&+\frac{1}{\kappa^2}\int_0^1 dt\,\omega_{\rm o}\left(\dot\Psi,\kappa\pi_{01}\left\{\frac{1}{2}[\boldsymbol{m},\boldsymbol{m}]+\boldsymbol{m}\boldsymbol{l}+\left[\boldsymbol{m},\kappa\boldsymbol{U}_o+\kappa^2\boldsymbol{U}_c\right]\right\}\mathcal{G}(t)\right)\,,\end{aligned}\tag{5.80}$$

where we have defined the higher order co-derivations associated to the closed and open Poisson bi-vectors

$$\boldsymbol{U}=\kappa^2\,\boldsymbol{U}_{\rm c}+\kappa\,\boldsymbol{U}_{\rm o}\,,\tag{5.81}$$

$$\boldsymbol{U}_{\rm c}=\frac{(-)^{c^a}}{2}\boldsymbol{c}^a\boldsymbol{c}_a\,,\tag{5.82}$$

$$\boldsymbol{U}_{\rm o}=\frac{(-)^{o^a}}{2}\boldsymbol{o}^a\boldsymbol{o}_a\,.\tag{5.83}$$

We already know that the above expression is equal to zero if the products satisfy the nilpotency condition (5.79) which now reads

$$\left[\boldsymbol{l}+\boldsymbol{m}+\kappa\boldsymbol{U}_{\rm o}+\kappa^2\boldsymbol{U}_{\rm c},\boldsymbol{l}+\boldsymbol{m}+\kappa\boldsymbol{U}_{\rm o}+\kappa^2\boldsymbol{U}_{\rm c}\right]=0\,.\tag{5.84}$$

Now we would like to split (5.84) into the open and closed sectors by applying π_{10} and π_{01} and then expand in powers of κ. First of all we immediately notice that

$$\begin{aligned}&\pi_{10}\left[\boldsymbol{l}+\boldsymbol{m}+\kappa\boldsymbol{U}_{\rm o}+\kappa^2\boldsymbol{U}_{\rm c},\boldsymbol{l}+\boldsymbol{m}+\kappa\boldsymbol{U}_{\rm o}+\kappa^2\boldsymbol{U}_{\rm c}\right]\\&\qquad=\pi_{10}\left\{\frac{1}{2}[\boldsymbol{l},\boldsymbol{l}]+\boldsymbol{l}\boldsymbol{m}+\left[\boldsymbol{l},\kappa\boldsymbol{U}_o+\kappa^2\boldsymbol{U}_c\right]\right\}\,,\end{aligned}\tag{5.85}$$

$$\begin{aligned}&\kappa\pi_{01}\left[\boldsymbol{l}+\boldsymbol{m}+\kappa\boldsymbol{U}_{\rm o}+\kappa^2\boldsymbol{U}_{\rm c},\boldsymbol{l}+\boldsymbol{m}+\kappa\boldsymbol{U}_{\rm o}+\kappa^2\boldsymbol{U}_{\rm c}\right]\\&\qquad=\kappa\pi_{01}\left\{\frac{1}{2}[\boldsymbol{m},\boldsymbol{m}]+\boldsymbol{m}\boldsymbol{l}+\left[\boldsymbol{m},\kappa\boldsymbol{U}_o+\kappa^2\boldsymbol{U}_c\right]\right\}\,.\end{aligned}\tag{5.86}$$

Notice that the factor of κ in front of π_{01} is necessary to work homogeneously in κ and thus to get an homogeneous dependence on the topology of the involved Riemann surfaces when we switch from the open to closed symplectic forms and viceversa. If we now expand in κ using (5.104, 5.105) we get

- $O(\kappa^0)$: the L_∞ relations controlling the consistency of closed string amplitudes on the sphere
  - $\kappa\pi_{01}$:
    $$\text{Nothing.} \tag{5.87}$$
  - π_{10}:
    $$\frac{1}{2}[\boldsymbol{l}^{(0,0)}, \boldsymbol{l}^{(0,0)}] = 0\,. \tag{5.88}$$
- $O(\kappa)$: These are relations controlling the consistency of open-closed amplitudes on the disk, resulting in the SDHA of [126]
  - $\kappa\pi_{01}$: this sector deals with disk amplitudes with at least one open string and arbitrary number of closed strings. The reader can recognize Kajiura-Stasheff OCHA[5] [123, 124]
    $$\frac{1}{2}[\boldsymbol{m}^{(0,1)}, \boldsymbol{m}^{(0,1)}] + [\boldsymbol{m}^{(0,1)}, \boldsymbol{l}^{(0,0)}] = 0\,. \tag{5.89}$$
  - π_{10}: this sector is a dual version of the OCHA (meaning no new constraints) when there is at least one open string input. However it also contains the case with only closed strings on the disk which are not included in the OCHA
    $$[\boldsymbol{l}^{(0,0)}, \boldsymbol{l}^{(0,1)}] + \pi_{1,0}\, \boldsymbol{l}^{(0,1)} \boldsymbol{m}^{(0,1)} = 0\,, \tag{5.90}$$
- $O(\kappa^2)$: These are relations dealing with surfaces of vanishing Euler number: the torus (purely closed string amplitudes) and annulus (open-closed amplitudes)

[5] Notice that here we had to compensate the removal of the projector π_{01} from in front of the homotopy relation by formally turning the $\boldsymbol{m}^{(0,1)}\boldsymbol{l}^{(0,0)}$ term into a full graded commutator in order to remove the contributions where the multi-string products are not nested into each other. In this particular case this is an allowed operation because the $\boldsymbol{l}^{(0,0)}\boldsymbol{m}^{(0,1)}$ term would never contribute with nested products.

– $\kappa\pi_{01}$:

$$[\boldsymbol{m}^{(0,2)},\boldsymbol{m}^{(0,1)}]+\pi_{0,1}\,\boldsymbol{m}^{(0,1)}\boldsymbol{l}^{(0,1)}+[\boldsymbol{m}^{(0,2)},\boldsymbol{l}^{(0,0)}]+[\boldsymbol{m}^{(0,1)},\boldsymbol{U}_{\rm o}]=0\,. \quad (5.91)$$

– π_{10}:

$$\frac{1}{2}[\boldsymbol{l}^{(0,1)},\boldsymbol{l}^{(0,1)}]+[\boldsymbol{l}^{(0,0)},\boldsymbol{l}^{(0,2)}+\boldsymbol{l}^{(1,0)}]+[\boldsymbol{l}^{(0,1)},\boldsymbol{U}_{\rm o}]+[\boldsymbol{l}^{(0,0)},\boldsymbol{U}_{\rm c}]=0\,, \quad (5.92)$$

- $O(\kappa^3)$: Surfaces of Euler number $\chi=-1$, and so on...

Thus in total, paying attention that the open and closed symplectic forms are normalized with different powers of κ, we see that the nilpotency relation (5.79) has a precise perturbative expansion in κ which correctly follows the expansion in the Euler number. Notice however that this expansion is not a real topological expansion because different topologies can have the same Euler number (for example the torus and the annulus). In the next section we will fully resolve the topology by formulating OC-SFT on a set of N identical D-branes.

5.5 Open-Closed string field theory in the large N-limit

In this section we will reformulate what we have just described in the case of a background characterized by a stack of N identical D-branes in the large N limit. In other words, given BCFT$_0$ which describes the initial open-closed background, represented in the closed string channel by a given boundary state $|B_0'\rangle$ we want to consider this theory defined on N identical BCFT$_0$ D-branes, described by the boundary state $N\,|B_0'\rangle$. This means to add $N\times N$ Chan-Paton factors to the open string field Ψ, which can be now decomposed as

$$\Psi=\psi^a_{ij}\,t^{ij}o_a, \quad (5.93)$$

where $\{t^{ij}\}$ is a basis for $N\times N$ matrices which for definiteness we take as

$$(t^{ij})_{pq}=\delta^i_p\,\delta^j_q\,. \quad (5.94)$$

Therefore, the basis vectors are normalized as

$$\omega_{\rm o}(t_{ij}o^a, t^{pq}o_b) = \delta^a_b\,\delta^p_i\,\delta^q_j\,, \tag{5.95}$$

where we we have defined the dual matrices

$$t_{ij} \coloneqq t^{ji}\,. \tag{5.96}$$

5.5.1 The planar action

To describe the action (5.7) in the large N limit it is useful to rewrite it in terms of objects which remain finite as the limit $N \to \infty$ is taken. To do so it is convenient to define the normalized trace for $N \times N$ matrices

$$\mathrm{Tr}'[M] \coloneqq \frac{1}{N}\,\mathrm{Tr}[M], \tag{5.97}$$

in such a way that, for example, $\mathrm{Tr}'[\mathbf{1}_N] = 1$.

The vertex $V^{g,b}$ contains a number of traces over CP factors which equals the number of boundaries. We can thus define normalized vertices as

$$V'^{g,b} \coloneqq \frac{1}{N^b}\,V^{g,b}, \tag{5.98}$$

which remain finite when $N \to \infty$. If we write the action (5.7) in terms of normalized CP traces we thus get

$$\begin{aligned} S_{\rm oc}(\Phi,\Psi) &= \sum_{\substack{g,b,k=0\\ \{p_1,\ldots,p_b\}=0}}^{\infty} \frac{\kappa^{2g+b-2}N^b}{b!k!(p_1)\cdots(p_b)} V'^{g,b}_{k;\{p_1,\ldots,p_b\}}\left(\Phi^{\wedge k}\otimes'\,\Psi^{\odot p_1}\wedge'\ldots\wedge'\,\Psi^{\odot p_b}\right) \\ &= \sum_{\substack{g,b,k=0\\ \{p_1,\ldots,p_b\}=0}}^{\infty} \frac{\kappa^{2g-2}\lambda^b}{b!k!(p_1)\cdots(p_b)} V'^{g,b}_{k;\{p_1,\ldots,p_b\}}\left(\Phi^{\wedge k}\otimes'\,\Psi^{\odot p_1}\wedge'\ldots\wedge'\,\Psi^{\odot p_b}\right)\,, \end{aligned} \tag{5.99}$$

where we have defined the 't Hooft coupling

$$\lambda \coloneqq \kappa N. \tag{5.100}$$

Alternatively we can also write

$$S_{\mathrm{oc}}(\Phi,\Psi) = \sum_{\substack{g,b,k=0\\ \{p_1,\dots,p_b\}=0}}^{\infty} \frac{N^{2-2g}\lambda^{2g+b-2}}{b!k!(p_1)\cdots(p_b)} V'^{g,b}_{k;\{p_1,\dots,p_b\}} \left(\Phi^{\wedge k}\otimes' \Psi^{\odot p_1}\wedge' \dots \wedge' \Psi^{\odot p_b}\right). \tag{5.101}$$

We are interested in the open-closed action in the large N limit at fixed 't Hooft coupling λ. In this limit we notice that only genus zero surfaces survive

$$\begin{aligned} S_{\mathrm{pl}}(\Phi,\Psi) &\coloneqq \lim_{N\to\infty}\frac{1}{N^2}S_{\mathrm{oc}}(\Phi,\Psi)\\ &= \sum_{\substack{g,b,k=0\\ \{p_1,\dots,p_b\}=0}}^{\infty} \frac{\lambda^{b-2}}{b!k!(p_1)\cdots(p_b)} V'^{g=0,b}_{k;\{p_1,\dots,p_b\}} \left(\Phi^{\wedge k}\otimes' \Psi^{\odot p_1}\wedge' \dots \wedge' \Psi^{\odot p_b}\right), \end{aligned} \tag{5.102}$$

where "pl" stands for *planar*. Furthermore, by introducing the normalized open symplectic form with a normalized Chan-Paton trace

$$\omega'_{\mathrm{o}} \coloneqq \frac{\omega_{\mathrm{o}}}{N}, \tag{5.103}$$

and wring the total coderivation in terms of normalized multi-sting products

$$\boldsymbol{l} \coloneqq \sum_{g,b}\kappa^{2g+b}\boldsymbol{l}^{(g,b)} \coloneqq \sum_{g,b}\kappa^{2g+b}N^b\,\boldsymbol{l}'^{(g,b)} = \sum_{g,b}\kappa^{2g}\lambda^b\,\boldsymbol{l}'^{(g,b)}, \tag{5.104}$$

$$\boldsymbol{m} \coloneqq \sum_{g,b}\kappa^{2g+b-1}\boldsymbol{m}^{(g,b)} \coloneqq \sum_{g,b}\kappa^{2g+b-1}N^{b-1}\,\boldsymbol{m}'^{(g,b)} = \sum_{g,b}\kappa^{2g}\lambda^{b-1}\,\boldsymbol{m}'^{(g,b)}, \tag{5.105}$$

the action in the WZW form (5.52) can be rewritten as

$$S_{\mathrm{oc}}(\Phi,\Psi) = N^2\int_0^1 dt\left(\frac{\omega_{\mathrm{c}}}{\lambda^2}\left(\dot\Phi\,,\,\pi_{10}\boldsymbol{l}\,\mathcal{G}\right)+\frac{\omega'_{\mathrm{o}}}{\lambda}\left(\dot\Psi\,,\,\pi_{01}\boldsymbol{m}\,\mathcal{G}\right)\right). \tag{5.106}$$

Therefore, in the 't Hooft limit $N\to\infty$ with λ constant the planar action (5.102) is

thus written as

$$S_{\rm pl}(\Phi,\Psi)=\int_0^1 dt\left(\frac{\omega_{\rm c}}{\lambda^2}\left(\dot\Phi\,,\,\pi_{10}\boldsymbol{l}^{(p)}\,\mathcal{G}\right)+\frac{\omega_{\rm o}'}{\lambda}\left(\dot\Psi\,,\,\pi_{01}\boldsymbol{m}^{(p)}\,\mathcal{G}\right)\right)\,, \tag{5.107}$$

where we have defined the planar coderivations $\boldsymbol{l}^{(p)}$ and $\boldsymbol{m}^{(p)}$ as the genus zero contributions of $\boldsymbol{l}$ and $\boldsymbol{m}$.

$$\boldsymbol{l}^{(p)}\coloneqq\sum_b\lambda^b\,\boldsymbol{l}'^{(0,b)}, \tag{5.108}$$

$$\boldsymbol{m}^{(p)}\coloneqq\sum_b\lambda^{b-1}\,\boldsymbol{m}'^{(0,b)}. \tag{5.109}$$

Notice that, like $\omega_{\rm o}'$, these objects are finite in the 't Hooft limit, contrary to the higher genus contributions which are suppressed by $\kappa^{2g}\sim N^{-2g}$.

5.5.2 Large N BV master equation

Following the definitions just introduced, the BV brackets become

$$(\cdot,\cdot)_{\rm c}=\kappa^2\frac{\overleftarrow{\partial}}{\partial\phi^a}\omega_{\rm c}^{ab}\frac{\overrightarrow{\partial}}{\partial\phi^b}=\lambda^2N^{-2}\frac{\overleftarrow{\partial}}{\partial\phi^a}\omega_{\rm c}^{ab}\frac{\overrightarrow{\partial}}{\partial\phi^b}\,, \tag{5.110}$$

$$(\cdot,\cdot)_{\rm o}=\kappa\frac{\overleftarrow{\partial}}{\partial\psi^a_{ij}}(\omega_{\rm o})^{ab}_{ij,kl}\frac{\overrightarrow{\partial}}{\partial\psi^b_{kl}}=\lambda N^{-2}\frac{\overleftarrow{\partial}}{\partial\psi^a_{ij}}(\omega_{\rm o}')^{ab}_{ij,kl}\frac{\overrightarrow{\partial}}{\partial\psi^b_{kl}}\,. \tag{5.111}$$

Similarly, the symplectic Laplacians become

$$\Delta_{\rm c}=\kappa^2(-)^{\phi^a}\omega_{\rm c}^{ab}\frac{\overrightarrow{\partial}}{\partial\phi^a}\frac{\overrightarrow{\partial}}{\partial\phi^b}=\frac{\lambda^2N^{-2}}{2}(-)^{\phi^a}\omega_{\rm c}^{ab}\frac{\overrightarrow{\partial}}{\partial\phi^a}\frac{\overrightarrow{\partial}}{\partial\phi^b}\,, \tag{5.112}$$

$$\Delta_{\rm o}=\kappa(-)^{\psi^a_{ij}}(\omega_{\rm o})^{ab}_{ij,kl}\frac{\overrightarrow{\partial}}{\partial\psi^a_{ij}}\frac{\overrightarrow{\partial}}{\partial\psi^b_{kl}}=\frac{\lambda N^{-2}}{2}(-)^{\psi^a_{ij}}(\omega_{\rm o}')^{ab}_{ij,kl}\frac{\overrightarrow{\partial}}{\partial\psi^a_{ij}}\frac{\overrightarrow{\partial}}{\partial\psi^b_{kl}}\,. \tag{5.113}$$

We have defined

$$\omega_{\rm o}(t^{ij}o_a,t^{pq}o_b)\coloneqq(\omega_{\rm o})^{ij,pq}_{ab}\,, \tag{5.114}$$

$$\omega_{\rm o}(t_{ij}o^a,t_{pq}o^b)\coloneqq(\omega_{\rm o})^{ab}_{ij,pq}\,. \tag{5.115}$$

In the large N limit it is important to split the open string laplacian $\Delta_{\rm o}$ into two contributions where the two derivatives pick open strings from the same or from different boundaries

$$\Delta_{\rm o} = \Delta_{\rm o}^{(1)} + \Delta_{\rm o}^{(2)}, \tag{5.116}$$

where

$$\Delta_{\rm o}^{(1)} = \frac{\lambda N^{-2}}{2}(-)^{\psi^a_{ij}}(\omega'_{\rm o})^{ab}_{ij,kl}\frac{\overrightarrow{\partial}}{\partial\psi^a_{ij}}\frac{\overrightarrow{\partial}}{\partial\psi^b_{kl}}\Big|_{\text{same boundary}} \tag{5.117}$$

$$\Delta_{\rm o}^{(2)} = \frac{\lambda N^{-2}}{2}(-)^{\psi^a_{ij}}(\omega'_{\rm o})^{ab}_{ij,kl}\frac{\overrightarrow{\partial}}{\partial\psi^a_{ij}}\frac{\overrightarrow{\partial}}{\partial\psi^b_{kl}}\Big|_{\text{different boundaries}}. \tag{5.118}$$

Geometrically $\Delta_{\rm o}^{(1)}$ glues two punctures on the same boundary thus adding one boundary to the surface. On the other hand $\Delta_{\rm o}^{(2)}$ glues two punctures from different boundaries, thus reducing by one the number of boundaries but increasing the genus by one. These geometrical operations are reflected on the Chan-Paton matrices by the following trace identities which are easy to verify

$$\Delta_{\rm o}^{(1)}: \qquad \mathrm{Tr}\left(t_{ij}At^{ij}B\right) = \mathrm{Tr}(A)\mathrm{Tr}(B), \tag{5.119}$$

$$\Delta_{\rm o}^{(2)}: \qquad \mathrm{Tr}\left(t_{ij}A\right)\mathrm{Tr}\left(t^{ij}B\right) = \mathrm{Tr}(AB), \tag{5.120}$$

which we write also using normalized traces

$$\Delta_{\rm o}^{(1)}: \qquad \frac{1}{N}\mathrm{Tr}'\left(t_{ij}At^{ij}B\right) = \mathrm{Tr}'(A)\mathrm{Tr}'(B), \tag{5.121}$$

$$\Delta_{\rm o}^{(2)}: \qquad N\mathrm{Tr}'\left(t_{ij}A\right)\mathrm{Tr}'\left(t^{ij}B\right) = \mathrm{Tr}'(AB). \tag{5.122}$$

In total both $\Delta_{\rm o}^{(1,2)}$ have the effect of decreasing the Euler number by one but topologically they are very different operations, which are finally distinguished when the single parameter κ is resolved to κ and λ, thanks to the new parameter N.

Similarly, also the open Poisson bi-vector

$$\boldsymbol{U}_{\mathrm{o}} = \frac{(-1)^{o^a}}{2}\, \boldsymbol{o}_a^{ij} \boldsymbol{o}_{ij}^a \,, \tag{5.123}$$

can be slitted in two components

$$\boldsymbol{U}_{\mathrm{o}}^{(1)} = \frac{(-)^{o^a}}{2} \boldsymbol{o}_a^{ij} \boldsymbol{o}_{ij}^a \Big|_{\text{same boundary}} \tag{5.124}$$

$$\boldsymbol{U}_{\mathrm{o}}^{(2)} = \frac{(-)^{o^a}}{2} \boldsymbol{o}_a^{ij} \boldsymbol{o}_{ij}^a \Big|_{\text{different boundaries}} \,, \tag{5.125}$$

$$\tag{5.126}$$

which become clear by studying the action of $\boldsymbol{U}_{\mathrm{o}}$ on the group like element

$$\begin{aligned}
\boldsymbol{U}_{\mathrm{o}}\mathcal{G} &= \frac{(-)^{o^a}}{2} \boldsymbol{o}_{ij}^a \boldsymbol{o}_a^{ij} \sum_{b,k,\{l_1,\ldots,l_b\}}^{\infty} \frac{1}{b!k!(l_1)\cdots(l_b)} \left(\Phi^{\wedge k} \otimes' \Psi^{\odot l_1} \wedge' \cdots \wedge' \Psi^{\odot l_b}\right) \\
&= \frac{(-)^{o^a}}{2} \boldsymbol{o}_{ij}^a \sum_{b,k,\{l_1,\ldots,l_b\}}^{\infty} \frac{b(l_1)}{b!k!(l_1)\cdots(l_b)} \left(\Phi^{\wedge k} \otimes' o_a^{ij} \odot \Psi^{\odot l_1} \wedge' \cdots \wedge' \Psi^{\odot l_b}\right) \\
&= \frac{(-)^{o^a}}{2} \sum_{b,k,\{l_1,\ldots,l_b\}}^{\infty} \frac{b(l_1)}{b!k!(l_1)\cdots(l_b)} \\
&\qquad \left(\Phi^{\wedge k} \otimes' \left[\sum_{n=0}^{l_1} o_{ij}^a \odot \Psi^{\odot n} \odot o_a^{ij} \odot \Psi^{\odot l_1 - n}\right] \wedge' \cdots \wedge' \Psi^{\odot l_b}\right) \\
&+ \frac{(-)^{o^a}}{2} \sum_{b,k,\{l_1,\ldots,l_b\}}^{\infty} \frac{b(b-1)(l_1)(l_2)}{b!k!(l_1)\cdots(l_b)} \left(\Phi^{\wedge k} \otimes' o_a^{ij} \odot \Psi^{\odot l_1} \wedge' o_{ij}^a \odot \Psi^{\odot l_2} \cdots \wedge' \Psi^{\odot l_b}\right) \\
&:= \boldsymbol{U}_{\mathrm{o}}^{(1)}\mathcal{G} + \boldsymbol{U}_{\mathrm{o}}^{(2)}\mathcal{G}.
\end{aligned} \tag{5.127}$$

Again it is useful to define normalized versions (which have a finite large N limit). Taking into account (5.121,5.122) we see that they are given by

$$\boldsymbol{U}_o'^{(1)} := \frac{1}{N} \boldsymbol{U}_o^{(1)} \tag{5.128}$$

and

$$\boldsymbol{U}_o'^{(2)} := N \boldsymbol{U}_o^{(2)}. \tag{5.129}$$

Finally we can write the relations with the open string laplacians as

$$\Delta_{\rm o}^{(1)} \to \lambda \boldsymbol{U}_{\rm o}^{\prime(1)} \tag{5.130}$$

and

$$\Delta_{\rm o}^{(2)} \to \frac{\lambda}{N^2} \boldsymbol{U}_{\rm o}^{\prime(2)}. \tag{5.131}$$

Now we can clearly see that in the limit $N \to \infty$ both $\Delta_{\rm c}$ and $\Delta_{\rm o}^{(2)}$ are suppressed as N^{-2}. This is in perfect agreement with the fact that the open-closed action (5.106) is dominated by genus zero vertices in the large N limit and therefore the BV structures that increase the genus (which are precisely $\Delta_{\rm c}$ and $\Delta_{\rm o}^{(2)}$) should be suppressed as well.

5.5.3 Planar homotopy relations and nilpotency

With these preparation we can now read-off from Section 5.4.1 the following results for the open-closed action (5.106)

$$\frac{1}{2}(S_{\rm oc}, S_{\rm oc})_{\rm c} = N^2 \int_0^1 dt \left(\frac{\omega_{\rm c}}{\lambda^2} \left(\dot{\Phi}, \pi_{10}\, \boldsymbol{l}\boldsymbol{l}\, \mathcal{G} \right) + \frac{\omega_{\rm o}'}{\lambda} \left(\dot{\Psi}, \pi_{01} \boldsymbol{m}\boldsymbol{l}\, \mathcal{G} \right) \right), \tag{5.132}$$

$$\frac{1}{2}(S_{\rm oc}, S_{\rm oc})_{\rm o} = N^2 \int_0^1 dt \left(\frac{\omega_{\rm c}}{\lambda^2} \left(\dot{\Phi}, \pi_{10}\, \boldsymbol{l}\boldsymbol{m}\, \mathcal{G} \right) + \frac{\omega_{\rm o}'}{\lambda} \left(\dot{\Psi}, \pi_{01} \boldsymbol{m}\boldsymbol{m}\, \mathcal{G} \right) \right), \tag{5.133}$$

$$\Delta_{\rm c} S_{\rm oc} = N^2 \int_0^1 dt \left(\frac{\omega_{\rm c}}{\lambda^2} \left(\dot{\Phi}, \pi_{10}\, \frac{\lambda^2}{N^2} \boldsymbol{l}\boldsymbol{U}_{\rm c}\, \mathcal{G} \right) + \frac{\omega_{\rm o}'}{\lambda} \left(\dot{\Psi}, \pi_{01}\, \frac{\lambda^2}{N^2} \boldsymbol{m}\boldsymbol{U}_{\rm c}\, \mathcal{G} \right) \right), \tag{5.134}$$

$$\Delta_{\rm o}^{(1)} S_{\rm oc} = N^2 \int_0^1 dt \left(\frac{\omega_{\rm c}}{\lambda^2} \left(\dot{\Phi}, \pi_{10}\, \lambda \boldsymbol{l}\boldsymbol{U}_{\rm o}^{\prime(1)}\, \mathcal{G} \right) + \frac{\omega_{\rm o}'}{\lambda} \left(\dot{\Psi}, \pi_{01}\, \lambda \boldsymbol{m}\boldsymbol{U}_{\rm o}^{\prime(1)}\, \mathcal{G} \right) \right), \tag{5.135}$$

$$\Delta_{\rm o}^{(2)} S_{\rm oc} = N^2 \int_0^1 dt \left(\frac{\omega_{\rm c}}{\lambda^2} \left(\dot{\Phi}, \pi_{10}\, \frac{\lambda}{N^2} \boldsymbol{l}\boldsymbol{U}_{\rm o}^{\prime(2)}\, \mathcal{G} \right) + \frac{\omega_{\rm o}'}{\lambda} \left(\dot{\Psi}, \pi_{01}\, \frac{\lambda}{N^2} \boldsymbol{m}\boldsymbol{U}_{\rm o}^{\prime(2)}\, \mathcal{G} \right) \right). \tag{5.136}$$

If we now consider the planar action (5.107) we see that its master equation can

be explicitly expressed as

$$\frac{1}{2}(S_{\rm pl},S_{\rm pl})_{\rm c}+\frac{1}{2}(S_{\rm pl},S_{\rm pl})_{\rm o}+\Delta_{\rm o}^{(1)}S_{\rm pl} \tag{5.137}$$

$$=\int_0^1 dt\,\frac{\omega_{\rm c}}{\lambda^2}\left(\dot\Phi,\pi_{10}\left[\boldsymbol{l}^{(p)}\boldsymbol{l}^{(p)}+\boldsymbol{l}^{(p)}\boldsymbol{m}^{(p)}+\lambda\,\boldsymbol{l}^{(p)}\boldsymbol{U}_{\rm o}^{\prime(1)}\right]\mathcal{G}\right) \tag{5.138}$$

$$+\int_0^1 dt\,\frac{\omega_{\rm o}'}{\lambda}\left(\dot\Psi,\pi_{01}\left[\boldsymbol{m}^{(p)}\boldsymbol{m}^{(p)}+\boldsymbol{m}^{(p)}\boldsymbol{l}^{(p)}+\lambda\,\boldsymbol{m}^{(p)}\boldsymbol{U}_{\rm o}^{\prime(1)}\right]\mathcal{G}\right). \tag{5.139}$$

Just as for the total open-closed action we can repackage everything into a single structure

$$S_{\rm pl}(\Phi,\Psi)=\int_0^1\hat\omega'\left(\dot\chi\,,\,\pi_1\boldsymbol{n}^{(p)}\,\mathcal{G}\right), \tag{5.140}$$

where

$$\chi\coloneqq\Phi+\Psi\;=(\pi_{10}+\pi_{01})\,\mathcal{G}=\pi_1\,\mathcal{G}\,, \tag{5.141}$$

$$\hat\omega'(\chi_1,\chi_2)\coloneqq\frac{\omega_{\rm c}}{\lambda^2}(\Phi_1,\Phi_2)+\frac{\omega_{\rm o}'}{\lambda}(\Psi_1,\Psi_2)\,, \tag{5.142}$$

$$\boldsymbol{n}^{(p)}\coloneqq\boldsymbol{l}^{(p)}+\boldsymbol{m}^{(p)}\,. \tag{5.143}$$

Then it is easy to realize that the planar BV master equation is solved provided

$$(\boldsymbol{n}^{(p)}+\boldsymbol{U}^{(p)})^2=0, \tag{5.144}$$

where we have defined $\boldsymbol{U}^{(p)}\coloneqq\lambda\boldsymbol{U}_{\rm o}^{\prime(1)}$. Notice indeed that (upon use of the trivial vanishing of $\pi_1\boldsymbol{U}^{(p)}$ and $\left(\boldsymbol{U}^{(p)}\right)^2$) we have that

$$\pi_{10}(\boldsymbol{n}^{(p)}+\boldsymbol{U}^{(p)})^2=\pi_{10}\left[\boldsymbol{l}^{(p)}\boldsymbol{l}^{(p)}+\boldsymbol{l}^{(p)}\boldsymbol{m}^{(p)}+\lambda\,\boldsymbol{l}^{(p)}\boldsymbol{U}_{\rm o}^{\prime(1)}\right], \tag{5.145}$$

$$\pi_{01}(\boldsymbol{n}^{(p)}+\boldsymbol{U}^{(p)})^2=\pi_{01}\left[\boldsymbol{m}^{(p)}\boldsymbol{m}^{(p)}+\boldsymbol{m}^{(p)}\boldsymbol{l}^{(p)}+\lambda\,\boldsymbol{m}^{(p)}\boldsymbol{U}_{\rm o}^{\prime(1)}\right]. \tag{5.146}$$

This homotopy structure (which was in fact already noticed by Zwiebach years ago [11], without however an interpretation of its physical meaning) describes a (large N) theory where closed strings are purely classical while open strings are fully quantum but restricted to the planar sector. It appears rather clear that this

is a privileged starting point to understand microscopically how a large number of D-branes can back-react on the initial classical closed string geometry to a new classical closed string geometry, possibly without D-branes anymore. In the next section, purely driven by the consistency of the homotopy structure, we will attempt to make some first step in this direction.

5.6 Closed string backreaction

The way to the closed string backreaction is composed of two steps, both of which require some assumption on the initial background. We first integrate out the open string degrees of freedom to end up with an intermediate closed ST which (second step) has to be stabilized by canceling the tadpole induced by the world-sheet boundaries.

5.6.1 Integrating out open strings

After having constructed the planar open-closed action (5.140) we would like to integrate out the open string sector. Formally this can be done using the homotopy transfer 4.2.1 with a projector $P_{\rm c} = 1_{\mathcal{H}_{\rm c}} \oplus 0_{\mathcal{H}_{\rm o}}$

$$P_{\rm c}\left(\mathcal{H}_{\rm c} \oplus \mathcal{H}_{\rm o}\right) = \mathcal{H}_{\rm c}\,. \tag{5.147}$$

Notice that obviously

$$[Q, P_{\rm c}] = 0\,, \tag{5.148}$$

where $Q = Q_{\rm c} + Q_{\rm o}$. To run the homotopy transfer we need a propagator $h_{\rm o}$ such that the Hodge-Kodaira decomposition is satisfied

$$[Q, h_{\rm o}] = 1 - P_{\rm c} = P_{\rm o} = 0_{\mathcal{H}_{\rm c}} \oplus 1_{\mathcal{H}_{\rm o}}\,. \tag{5.149}$$

This equation may look surprising because $P_{\rm o}$ projects on the full open string Hilbert space where there is also the open string cohomology which evidently cannot be

exact (in the form of $[Q, h_{\rm o}]$). Nevertheless this is the needed projection to integrate out the open string sector. Here we are assuming to deal with a peculiar open-closed background such that the open string cohomology (if present) is prevented from propagating inside generic amplitudes with external closed strings. This is certainly a rather special situation that is not realized in generic D-brane systems. However there are cases in which the open string cohomology is really only present in the form of external states and the interactions of the theory are such that on-shell open strings are never produced inside a diagram. In the bosonic string this is explicitly realized for example in the minimal $(2,1)$ string with FZZT branes, as discussed extensively by Gaiotto and Rastelli [138], see also the discussion in the conclusions. A similar situation, although in the framework of the topological string, is expected to be realized in the Gopakumar-Vafa conifold transition [136].[6]

This is our working hypothesis without which it would not be possible to integrate out open strings completely, at least not in this simple way.[7] Therefore we will just assume that our open-closed background allows for the possibility of integrating out open strings in the sense described above. That said, we can think of $h_{\rm o}$ as $\frac{b_0}{L_0}$ but we do not necessarily want to commit to this choice. Without loss of generality, we choose $h_{\rm o}$ so that $h_{\rm o} P_{\rm c} = P_{\rm c} h_{\rm o} = 0$ as well as $h_{\rm o}^2 = 0$. We also write

$$\Pi_{\rm c} I_{\rm c} = 1_{\mathcal{H}_{\rm c}}\,, \tag{5.150}$$

$$I_{\rm c} \Pi_{\rm c} = P_{\rm c}\,, \tag{5.151}$$

where $\Pi_{\rm c}$ and $I_{\rm c}$ are the canonical projection and inclusion between $\mathcal{H}_{\rm c} \oplus \mathcal{H}_{\rm o}$ and $\mathcal{H}_{\rm c}$. These objects define the SDR

$$\circlearrowleft \big(-h_{\rm o}\big)\, \big(\mathcal{H}_{\rm c} \oplus \mathcal{H}_{\rm o}, Q_{\rm c} + Q_{\rm o}\big) \underset{I_{\rm c}}{\overset{\Pi_{\rm c}}{\rightleftarrows}} \big(\mathcal{H}_{\rm c}, Q_{\rm c}\big)\,. \tag{5.152}$$

These operators can be upgraded to operators on the open-closed tensor algebra

[6] In a slightly more generic situation we expect that the non-propagation of open string cohomology inside a diagram could be achieved by a suitable decoupling limit.

[7] There could be instances in which the open string cohomology is present and propagating, but still it will have to be integrated out at the end, to end up with amplitudes with external closed strings only. This is realized for example in D-instanton contributions (see for example [144]), in order to eventually obtain closed string momentum conservation.

$S\mathcal{H}_{\mathrm{c}} \otimes' S\mathcal{CH}_{\mathrm{o}}$

$$Q \to \boldsymbol{Q}\,, \tag{5.153a}$$

$$P_{\mathrm{c}} \to \boldsymbol{P}_{\mathrm{c}}\,, \tag{5.153b}$$

$$\Pi_{\mathrm{c}} \to \boldsymbol{\Pi}_{\mathrm{c}}\,, \tag{5.153c}$$

$$I_{\mathrm{c}} \to \mathbf{I}_{\mathrm{c}}\,, \tag{5.153d}$$

$$h_{\mathrm{o}} \to \boldsymbol{h}_{\mathrm{o}}\,, \tag{5.153e}$$

in such a way that

$$[\boldsymbol{Q}, \boldsymbol{h}_{\mathrm{o}}] = \mathbf{1}_{S\mathcal{H}_{\mathrm{c}} \otimes' S\mathcal{CH}_{\mathrm{o}}} - \boldsymbol{P}_{\mathrm{c}}\,, \tag{5.154}$$

$$\boldsymbol{\Pi}_{\mathrm{c}}\mathbf{I}_{\mathrm{c}} = \mathbf{1}_{\mathcal{SH}_{\mathrm{c}}}\,, \tag{5.155}$$

$$\mathbf{I}_{\mathrm{c}}\boldsymbol{\Pi}_{\mathrm{c}} = \boldsymbol{P}_{\mathrm{c}}\,. \tag{5.156}$$

See appendix C for the explicit construction. This allows to upgrade (5.152) to $S\mathcal{H}_{\mathrm{c}} \otimes' S\mathcal{CH}_{\mathrm{o}}$ and then use the homotopy transfer (see Section 4.2.1)

$$\begin{array}{ccc} \circlearrowleft(-\boldsymbol{h}_{\mathrm{o}})\,(S\mathcal{H}_{\mathrm{c}} \otimes' S\mathcal{CH}_{\mathrm{o}}, \boldsymbol{Q}_{\mathrm{c}} + \boldsymbol{Q}_{\mathrm{o}}) & \underset{\boldsymbol{I}_{\mathrm{c}}}{\overset{\boldsymbol{\Pi}_{\mathbf{c}}}{\rightleftarrows}} & (S\mathcal{H}_{\mathrm{c}}, \boldsymbol{Q}_{\mathrm{c}}) \\ \Big\downarrow{\scriptstyle \boldsymbol{\delta n}^{(p)} + \boldsymbol{U}^{(p)}} & & \Big\downarrow{\scriptstyle \boldsymbol{\delta \tilde{n}}^{(p)} + \tilde{\boldsymbol{U}}^{(p)}} \\ \circlearrowleft(-\tilde{\boldsymbol{h}}_{\mathrm{o}})\,(S\mathcal{H}_{\mathrm{c}} \otimes' S\mathcal{CH}_{\mathrm{o}}, \boldsymbol{n}^{(p)} + \boldsymbol{U}^{(p)}) & \underset{\tilde{\boldsymbol{I}}_{\mathrm{c}}}{\overset{\tilde{\boldsymbol{\Pi}}_{\mathrm{c}}}{\rightleftarrows}} & (S\mathcal{H}_{\mathrm{c}}, \tilde{\boldsymbol{n}}^{(p)} + \tilde{\boldsymbol{U}}^{(p)}) \end{array} \tag{5.157}$$

with perturbation $\delta\boldsymbol{n}^{(p)} + \boldsymbol{U}^{(p)}$, where $\delta\boldsymbol{n}^{(p)} \coloneqq \boldsymbol{n}^{(p)} - \boldsymbol{Q}$, to compute the effective

multi-string products. The perturbed operators are

$$\tilde{\boldsymbol{I}}_{\rm c} = \frac{1}{\mathbf{1}_{S\mathcal{H}_{\rm c}\otimes' SC\mathcal{H}_{\rm o}} + \boldsymbol{h}_{\rm o}(\delta\boldsymbol{n}^{(p)} + \boldsymbol{U}^{(p)})}\boldsymbol{I}_{\rm c}\,, \tag{5.158}$$

$$\tilde{\boldsymbol{\Pi}}_{\rm c} = \boldsymbol{\Pi}_{\rm c}\frac{1}{\mathbf{1}_{S\mathcal{H}_{\rm c}\otimes' SC\mathcal{H}_{\rm o}} + (\delta\boldsymbol{n}^{(p)} + \boldsymbol{U}^{(p)})\boldsymbol{h}_{\rm o}}\,, \tag{5.159}$$

$$\tilde{\boldsymbol{h}}_{\rm o} = \frac{1}{\mathbf{1}_{S\mathcal{H}_{\rm c}\otimes' SC\mathcal{H}_{\rm o}} + \boldsymbol{h}_{\rm o}(\delta\boldsymbol{n}^{(p)} + \boldsymbol{U}^{(p)})}\boldsymbol{h}_{\rm o}\,, \tag{5.160}$$

$$\tilde{\boldsymbol{l}}^{(p)} = \boldsymbol{\Pi}_{\rm c}\boldsymbol{n}^{(p)}\frac{1}{\mathbf{1}_{S\mathcal{H}_{\rm c}\otimes' SC\mathcal{H}_{\rm o}} + \boldsymbol{h}_{\rm o}(\delta\boldsymbol{n}^{(p)} + \boldsymbol{U}^{(p)})}\boldsymbol{I}_{\rm c}\,, \tag{5.161}$$

according to the HPL (4.105). Furthermore, we can notice that

$$\tilde{\boldsymbol{U}}^{(p)} = 0\,, \tag{5.162}$$

because $\boldsymbol{U}^{(p)}$ creates open string states which are killed by the projection $\Pi_{\rm c}$. Therefore, $\tilde{\boldsymbol{l}}^{(p)}$ is actually a nilpotent coderivation

$$[\tilde{\boldsymbol{l}}^{(p)}, \tilde{\boldsymbol{l}}^{(p)}] = 0\,, \tag{5.163}$$

This means that $\tilde{\boldsymbol{l}}^{(p)}$ describes an L_∞ algebra and thus the effective action

$$\tilde{S}(\Phi) = \int_0^1 dt\,\frac{\omega_{\rm c}}{\lambda^2}\left(\dot{\Phi}\,,\,\pi_1\tilde{\boldsymbol{l}}\,e^{\wedge\Phi}\right) + \Lambda_{\rm open} + \Lambda_0\,, \tag{5.164}$$

is a BV classical master action. In (5.164) we also reported the constant terms $\Lambda_{\rm open}$, which is generated by the path integral from the kinetic term and the zero products, and $\Lambda_{\rm o}$ the constant of the initial microscopic action, which we did not write explicitly in (5.7) as it is not important for the local open-closed dynamics. The effective action (5.164) is structurally the same as (5.102), with all the open strings set to zero (but not the boundaries)

$$\tilde{S}(\Phi) = \sum_{b=0}^{\infty}\lambda^{b-2}\sum_{k=0}^{\infty}\frac{1}{b!k!}\tilde{V}_k^{g=0,b}\left(\Phi^{\wedge k}\right)\,, \tag{5.165}$$

$$\tilde{V}_{k+1}^{g=0,b}(\Phi_1\wedge\cdots\wedge\Phi_{k+1}) = \omega_{\rm c}\left(\Phi_1, \tilde{l}_k^{(b)}\left(\Phi_2\wedge\cdots\wedge\Phi_{k+1}\right)\right)\,, \tag{5.166}$$

$$\tilde{l}_k = \pi_1\tilde{\boldsymbol{l}}\pi_k = \sum_{b=0}^{\infty}\lambda^b\,\tilde{l}_k^{(b)}\,. \tag{5.167}$$

The difference is that now the fundamental vertices $\tilde{V}_k^{g=0,b}$ are integrated in moduli space also in the regions of open string degeneration, thanks to the integration-out of the open strings that we have just performed. The remaining part of integration towards closed string degeneration will be performed by the Feynman diagrams of the obtained effective action, through the closed string propagators.

5.6.2 Eliminating the closed string tadpole

Notice that the effective closed ST we have just obtained is based on a *weak* L_∞-algebra. Indeed there is clearly a tadpole represented by a genus zero surface with a closed string puncture and an arbitrary number of boundaries weighted by the 't Hooft coupling. In particular, in a specific choice of SFT data we can write

$$\tilde{l}_0 = \lambda e^{-sL_0^+} \, |B_0'\rangle + O(\lambda^2) \,, \tag{5.168}$$

where $|B_0'\rangle$ is the initial (normalized) boundary state. The full tadpole can be written as a power series expansion in the 't Hooft coupling as

$$\tilde{l}_0 = \sum_{b=1}^{\infty} \lambda^b \, \tilde{l}_0^{(b)} \,. \tag{5.169}$$

Notice that there is no $b = 0$ contribution (the original bulk CFT is assumed to have no sphere tadpole). In fact the same expansion in boundaries can be done for all other closed strings products

$$\tilde{l}_k = \sum_{b=0}^{\infty} \lambda^b \, \tilde{l}_k^{(b)} \,, \tag{5.170}$$

where this time also the sphere $b = 0$ contributions are contained. Because of the tadpole we cannot say that we have obtained a classical closed string background. To achieve this we have to solve the tadpole-sourced closed SFT equation of motion[8]

$$\sum_{k=1}^{\infty} \frac{1}{k!} \tilde{l}_k(\Phi^{\wedge k}) = -\tilde{l}_0 \,, \tag{5.171}$$

[8] A very similar equation has been considered in [145] as a way to regulate the collisions of the deformed world-sheet sigma model with boundaries.

which, if expanded in terms of the 't Hooft coupling, gives rise to the recursive equations

$$\Phi = \sum_{n=1}^{\infty} \lambda^n \, \Phi_n \,, \tag{5.172}$$

$$O(\lambda): \quad Q_{\rm c}\Phi_1 = -\tilde{l}_0^{(1)} \tag{5.173}$$

$$O(\lambda^2): \quad Q_{\rm c}\Phi_2 = -\frac{1}{2}\tilde{l}_2^{(0)}(\Phi_1^{\wedge 2}) - \tilde{l}_1^{(1)}(\Phi_1) - \tilde{l}_0^{(2)} \,, \tag{5.174}$$

$$\begin{aligned} O(\lambda^3): \quad Q_{\rm c}\Phi_3 = &-\frac{1}{6}\tilde{l}_3^{(0)}(\Phi_1^{\wedge 3}) - \tilde{l}_2^{(0)}(\Phi_1 \wedge \Phi_2) - \frac{1}{2}\tilde{l}_2^{(1)}(\Phi_1^{\wedge 2}) - \tilde{l}_1^{(1)}(\Phi_2) \\ &- \tilde{l}_1^{(2)}(\Phi_1) - \tilde{l}_0^{(3)} \,, \end{aligned} \tag{5.175}$$

$$\vdots \qquad .$$

The $O(\lambda)$ equation that we found has been already discussed in the past. For example (implicitly) in [146], to extract the large distance behaviour of the supergravity p-brane metric from the boundary state, or in [147] to read-off the emitted closed string radiation from the boundary state of a decaying D-brane.

This equation is rather subtle: indeed already at $O(\lambda)$, (5.173) is apparently obstructed since $\tilde{l}_0^{(1)} \sim |B\rangle$ is clearly in the closed string cohomology and therefore it cannot be equal to $Q\Psi_1$ if Ψ_1 has to be a regular string field. However the obstruction is in the zero momentum sector and if the boundary state has enough Dirichlet directions, the obstruction will be contained in the continuous momentum integral of the boundary state and therefore can be avoided by continuity. This is in fact fully analogous to how the Coulomb potential is formed as a vacuum shift solution of electromagnetism in presence of a delta-function source and the meaning of the obstruction is essentially that there is no Coulomb potential if the dimensions transverse to the charge are compact, so that the zero momentum part of the source would give an isolated equation without solution. On the other hand, the higher order λ-corrections seem to mark an important difference from the simple backreaction of a source, which is a solution with a singularity at the position of the source itself. Indeed we expect that if the full vacuum shift solution exists it should not be singular at the locus of the initial D-branes. In this sense perhaps an expansion in λ is not the best way to proceed, since every power of λ may be singular but the whole

resummation may be regular[9]. Understanding the structure of this equation and its possible solution seems a rather fundamental problem that is universal in every situation in which a large number of D-branes backreacts on the initial closed-string background.

Suppose however that we have found a solution $\Phi_*(\lambda)$ to (5.171). Then this will give rise to a new closed string background without a tadpole anymore

$$S_{\rm BR}(\varphi) \coloneqq \tilde{S}(\Phi_*(\lambda)+\varphi) = \int_0^1 dt\, \frac{\omega_{\rm c}}{\lambda^2}\left(\dot{\varphi}\,,\, \pi_1 \tilde{\boldsymbol{l}}_*\, e^{\wedge\varphi}\right) + \Lambda_{\rm open} + \Lambda_0 + \Lambda_{\rm closed}\,, \tag{5.176}$$

where the shifted coderivation $\tilde{\boldsymbol{l}}_*$ now describes a strong L_∞ structure

$$[\tilde{\boldsymbol{l}}_*, \tilde{\boldsymbol{l}}_*] = 0\,, \tag{5.177}$$

$$\tilde{\boldsymbol{l}}_*\pi_0 = 0\,. \tag{5.178}$$

Notice that the closed string shift generates a new piece of vacuum energy so that in total we now have

$$\Lambda_{\rm tot} = \Lambda_{\rm open} + \Lambda_0 + \Lambda_{\rm closed} = \sum_{b=1}^{\infty} \lambda^{b-2}\Lambda_b\,. \tag{5.179}$$

The meaning of this full cosmological constant should be the genus zero partition function summed over all possible boundaries, with a full moduli space integration. In particular Λ_0 integrates in the middle of moduli space, $\Lambda_{\rm open}$ (generated by the open string integration-out) takes care of the open-string degeneration regions and finally $\Lambda_{\rm closed}$ (generated by the closed string vacuum shift) integrates towards closed string degenerations. It would be interesting to study this vacuum energy more concretely.

The next term in the obtained stable closed string action is the kinetic term

$$S_2(\varphi) = \frac{1}{2\lambda^2}\omega_{\rm c}\left(\varphi, \tilde{l}_1^*\varphi\right)\,, \tag{5.180}$$

[9]Think for example about the regular "instanton" $\frac{\lambda}{x^2+\lambda^2}$, when it is expanded in $\lambda = 0$.

where

$$\tilde{l}_1^* = \pi_1 \tilde{\boldsymbol{l}}_* \pi_1 = Q_{\rm c} + \sum_{b=1}^{\infty} \lambda^b \, \tilde{l}_1^{*(b)} \, . \tag{5.181}$$

The cohomology of this operator can be studied in an expansion in λ for example following section 7 of [148] (see also appendix B of [126]) and this is expected to give the physical closed string spectrum on the 'dual' closed string geometry. Continuing in this way we can describe the new closed string background in powers of λ. It would be clearly very instructive to test this in some explicit and tractable example.

CHAPTER 6

Adding stubs to quantum String Field Theories

6.1 Stub deformation

In this chapter, based on [33], we present a systematic algebraic procedure for adding stubs to a generic quantum SFT. We extend the homotopy-transfer approach for adding stubs, originally pointed out by Schnabl–Stettinger [149][1] and Erbin–Firat [152], to the quantum scenario.

Before going into the details of our construction, let us first give a brief review of the concept of stub deformation. As anticipated in Section 2.3, there are infinitely many possible SFT actions related by field redefinitions that can be obtained by choosing different sections $\mathcal{S}$ of the space of decorated Riemann surfaces $\mathcal{P}$. Given a well-defined SFT action constructed on a certain section, consistently decomposed into the fundamental region $\mathcal{V}$ and the Feynman region $\mathcal{F}$, we can construct another section $\mathcal{S}'$ by redefining the local coordinate maps as

$$z_a = f_a(w_a) \to z_a = f'_a(w_a) \coloneqq f_a(e^{-\lambda} w_a)\,, \tag{6.1}$$

where $\lambda \in [0, \infty)$ is the stub parameter. This is equivalent to rescaling the local coordinate patches with the canonical stub operators $e^{-\lambda L_0}$ (for open strings) or $e^{-\lambda(L_0+\bar{L}_0)}$ (for closed strings).

The effect of this deformation is to identify a new section characterized by a larger fundamental region and reduced Feynman domains

$$\mathcal{V} \subseteq \mathcal{V}'\,, \tag{6.2}$$

$$\mathcal{F}' \subseteq \mathcal{F}\,. \tag{6.3}$$

Indeed, by considering the plumbing fixture procedure in Section 2.3.3 on two insertions with local coordinates w_1 and w_2, we obtain

$$w_1 w_2 = e^{-s} \longleftrightarrow f_1^{-1}(z_1)\, f_2^{-1}(z_2) = e^{-s-2\lambda}\,. \tag{6.4}$$

Therefore, by following the procedure explained in Section 2.3.3, this leads to

[1] For recent developments of these techniques, see also [150], and for the extension in the Sliver frame, see [151].

the propagator

$$h_0' = \int_{2\lambda}^{\infty} ds\, b_0\, e^{-sL_0} = \frac{b_0}{L_0} e^{-2\lambda L_0}\,, \tag{6.5}$$

in the open string scenario, which clearly implies that $\mathcal{F}' \subseteq \mathcal{F}$. The remaining part of the moduli space that was originally covered by the Siegel propagator

$$h_0 - h_0' = \int_{0}^{2\lambda} ds\, b_0\, e^{-sL_0} = \frac{b_0}{L_0}(1 - e^{-2\lambda L_0})\,, \tag{6.6}$$

is now encoded in the new fundamental domain $\mathcal{V}' \supseteq \mathcal{V}$.

In particular for very large λ, almost all the moduli space is covered by the fundamental vertices except the regions where the surface degenerates. This possibility of varying λ is the stub deformation.

This deformation can be also understood as a smooth way of integrating out off-shell degrees freedom, in that in the $\lambda \to \infty$ limit the stub operator $e^{-\lambda L_0}$ approaches the projector P_0 into the kernel of L_0 [2]

$$\lim_{\lambda \to +\infty} e^{-2\lambda L_0} \sim P_0\,, \tag{6.7}$$

and the additional contribution to the fundamental vertices associated with (6.6) in this limit corresponds to a Feynman diagram with the regulated Siegel propagator

$$\lim_{\lambda \to \infty} \frac{b_0}{L_0}(1 - e^{-2\lambda L_0}) \sim \frac{b_0}{L_0}(1 - P_0)\,. \tag{6.8}$$

A qualitative 'physical' effect of this deformation is to push (for large λ) the effect of the higher level states inside the fundamental vertices, leaving the Feynman diagrams to be dominated by the contribution of the massless fields. However, for any finite λ, the off-shell degrees of freedom are strictly conserved and therefore one should not expect a qualitative change in the theory, in particular in the vacuum structure [153].

[2] This is certainly true as long as we act on $h \geq 0$ states. When tachyons are present, the stub deformation amplifies them, instead of acting as a suppression factor. It would be interesting to define a different stub operator which can also suppress tachyons. An example would be $1/(1 + \lambda L_0)$ which however doesn't work well when there are tachyonic nearly-marginal states which accumulate at $L_0 \to 0^-$.

In this respect the stub deformation appears to be conceptually different from the standard integration-out although the two operations clearly share similarities.

This similarity with the integrating-out process has been a key idea in the recent works [149, 152], which provided two related algebraic descriptions of the deformation in the context of (classical) Witten OSFT using the homotopy transfer machinery. In particular, the qualitative idea behind this approach is that, in the large-stub limit, the operators (6.7) and (6.8), together with the BRST charge, form an SDR, allowing one to apply homotopy transfer. The resulting theory is then the effective theory obtained by integrating out the massive degrees of freedom, namely the stubbed theory for λ strictly equal to ∞. Thus, the strategy for finding the stubbed theory at finite λ is to extend the construction so that the operators discussed above form an SDR also for finite values of the stub parameter, and then apply homotopy transfer.

We will devote the remaining part of the chapter to formalizing and making this idea concrete for both classical and quantum SFTs.

6.2 Systematic approach for adding stubs

In this section we provide a systematic algebraic procedure for introducing stubs into a generic quantum theory within the BV formalism. Then we will apply it to classical and quantum closed SFT and finally to open-closed SFT.

6.2.1 Classical stubs

We begin by considering a generic (euclidean for definiteness) classical SFT described by the following action

$$S(\rho) = \frac{1}{2}\omega(\rho, Q\rho) + S_{\text{int}}(\rho)\,, \tag{6.9}$$

where ρ is a degree zero string field belonging to the Hilbert space $\mathcal{H}$, Q is nilpotent and $S_{\text{int}}(\rho)$ is the interacting part of the starting un-stubbed theory. To introduce stubs we consider an operator $c_\lambda : \mathcal{H} \to \mathcal{H}$, which can be written as

$$c_\lambda = e^{-\lambda L}\,, \tag{6.10}$$

where λ is the stub parameter and $L : \mathcal{H} \to \mathcal{H}$ is a Grassmann even operator satisfying $[L, Q] = 0$[3] . In this completely general context, we also require L to be BPZ even, in the sense that [4]

$$\omega(\rho_1, L\rho_2) = \omega(L\rho_1, \rho_2)\,. \tag{6.11}$$

Furthermore, having in mind the case when $L = L_0$ or L_0^+, we require the existence of a Grassmann odd operator b such that $[Q, b] = L$. Moreover we also assume that $b = 0$ is a 'good' gauge fixing, meaning that the cohomology of Q can be systematically searched for inside $\ker(b)$ and therefore is contained in $\ker(L)$.
Let us now introduce a decoupled auxiliary field χ, without cohomology. This condition can be achieved by requiring that $\chi \in (1 - P)\mathcal{H} \coloneqq \bar{P}\mathcal{H}$, where P projects onto $\ker(L)$. With this trivial addiction the action 6.9 becomes

$$S'(\rho, \chi) = \frac{1}{2}\omega(\rho, Q\rho) + \frac{1}{2}\omega(\chi, Q\chi) + S_{\text{int}}(\rho)\,. \tag{6.12}$$

Notice that this is not adding any new propagating degrees of freedom because $\chi \in \bar{P}\mathcal{H}$, where the cohomology is absent. In particular, classically integrating out χ (i.e. solving its equation of motion) is the same as setting it to zero.

Now, let us do a trivial field redefinition, a rotation induced by the orthogonal matrix

$$\begin{pmatrix} \rho \\ \chi \end{pmatrix} = O_\lambda \begin{pmatrix} \psi \\ \Sigma \end{pmatrix} = \begin{pmatrix} c_\lambda & -\sqrt{1 - c_\lambda^2} \\ \sqrt{1 - c_\lambda^2} & c_\lambda \end{pmatrix} \begin{pmatrix} \psi \\ \Sigma \end{pmatrix}\,, \tag{6.13}$$

which leaves the kinetic terms unchanged but mixes the interactions

$$S(\psi, \Sigma) \coloneqq S'(\rho, \chi) = \frac{1}{2}\omega(\psi, Q\psi) + \frac{1}{2}\omega(\Sigma, Q\Sigma) + S_{\text{int}}\left(c_\lambda \psi - \sqrt{1 - c_\lambda^2}\Sigma\right)\,. \tag{6.14}$$

Notice that since $\chi \in \bar{P}\mathcal{H}$, then $\Sigma \in \bar{P}\mathcal{H}$ as well, given that $\sqrt{1 - c_\lambda^2}\,\psi \sim (2\lambda L)^{1/2}\,\psi \in \bar{P}\mathcal{H}$. Therefore Σ can be integrated out completely without loosing physical degrees of freedom. This time however this is not the same as setting it equal to zero due

[3] In this chapter we consider for definiteness $c_\lambda = e^{-\lambda L}$ but it is clear that one can consider more general stubs of the kind $c(\lambda, L)$, see [152] for explicit examples.

[4] In [150] this condition has been relaxed, it would be interesting to see how our formalism can be accordingly extended to this case.

to the non-trivial couplings between Σ and ψ. However, since Σ doesn't contain physical degrees of freedom, classically integrating it out (i.e. solving the Σ equation of motion as a function of ψ) should give a theory with the same physical content as the initial one. This new theory is by definition the (classical) stubbed theory

$$\tilde{S}(\psi) = S(\psi, \Sigma(\psi)) = \frac{1}{2}\omega(\psi, Q\psi) + \tilde{S}_{\text{int}}(\psi)\,. \tag{6.15}$$

The procedure we have outlined is just a convenient re-writing of what has been done by Erbin and Fırat in [152] in the particular context of Witten theory. Indeed in this case (6.14) reduces to the extended theory described in [152] by renaming $\Sigma_{\text{EF}} = \sqrt{1-c_\lambda^2}\Sigma \in \bar{P}\mathcal{H}$, namely

$$\begin{aligned} S_{\text{EF}}(\psi, \Sigma_{\text{EF}}) &\coloneqq S(\psi, \Sigma) \\ &= \frac{1}{2}\omega\left(\psi, Q\psi\right) + \frac{1}{2}\omega\left(\Sigma_{\text{EF}}, \frac{Q}{1-c_\lambda^2}\Sigma_{\text{EF}}\right) \\ &- \frac{1}{3}\omega\left((\Sigma_{\text{EF}} - c_\lambda\psi), m_2\left((\Sigma_{\text{EF}} - c_\lambda\psi), (\Sigma_{\text{EF}} - c_\lambda\psi)\right)\right)\,. \end{aligned} \tag{6.16}$$

Furthermore in [152] the authors consider a reduction of the theory (6.16) onto a "line" defined by the linear relation

$$\Sigma_{\text{EF}} = \left(c_\lambda - c_\lambda^{-1}\right)\psi \longleftrightarrow \Sigma = -\frac{\sqrt{1-c_\lambda^2}}{c_\lambda}\psi\,, \tag{6.17}$$

which gives Witten's theory without stubs. In our language this means to undo the field redefinition 6.13 and then integrate out the free field χ (which is the same as setting it to zero) thus getting back the original theory without stubs. Thanks to our reformulation, we will be able to extend these results to the quantum theory in a particularly transparent way.

6.2.2 Quantum stubs

Let us now repeat the above procedure in the quantum case. This time the integration out of the auxiliary fields really means to perform the path integral over χ or the 'stub-rotated' Σ. Again we assume to start from a theory without stubs described

by the action $S(\rho)$, which we suppose satisfies the quantum BV master equation. Then we add the non-interacting auxiliary field χ. As a simple warm-up let us review how to perform the BV path integral over the free auxiliary field χ. Remember that $\chi \in \bar{P}\mathcal{H}$, which can be decomposed as

$$\bar{P}\mathcal{H} = hQ\mathcal{H} \oplus Qh\mathcal{H}\,, \tag{6.18}$$

where we have defined the homotopy operator / propagator

$$h \coloneqq \frac{b}{L}\,\bar{P}\,. \tag{6.19}$$

The BV path integral has to be done over a lagrangian submanifold (a maximal subspace where the symplectic form ω vanishes) of the total space $\bar{P}\mathcal{H}$. A gauge fixing is a choice of lagrangian submanifold. In the case at hand we use the obvious gauge fixing $h\chi = 0$ and therefore the lagrangian submanifold is the (component-dual of the) vector space $hQ\mathcal{H}$, on which the symplectic form correctly vanishes. Let $\{c_i\}$ be a basis of $hQ\mathcal{H}$ and $\{b_j\}$ a basis of $Qh\mathcal{H}$. Notice that only the c_i's give a non-trivial contribution to the kinetic term, so that expressing $\chi = \gamma^i c_i + \beta^j b_j$, we remain with

$$\omega(\chi, Q\chi) = \gamma^i Q_{ij} \gamma^j\,, \tag{6.20}$$

where $Q_{ij} = \omega(c_i, Qc_j)$, and the coefficients β^i and γ^i have a non-trivial degree assigned such that $d(\chi) = d(\beta^i) + d(b_i) = d(\gamma^i) + d(c_i) = 0$. Doing the BV path integral thus means to perform the (super) gaussian integration over the BV fields γ^i according to their grassmanality

$$\int_{h\chi=0} \mathcal{D}\chi\, e^{S(\rho)+\frac{1}{2}\omega(\chi,Q\chi)} = e^{\mathcal{A}+S(\rho)}\,, \tag{6.21}$$

where $\mathcal{A} = \frac{1}{2}\log(\mathrm{sdet}(|Q_{ij}|)) = \frac{1}{2}\mathrm{str}(\log(|Q_{ij}|))$, and the supertrace of a super matirx M is defined as $\mathrm{str}(M) = \sum_i (-)^i M_{ii}$, (for more details on the complete derivation

of $\mathcal{A}$ see [62, 154]). Therefore, if we start from the extended action

$$S'(\rho,\chi) = \frac{1}{2}\omega(\rho, Q\rho) + \frac{1}{2}\omega(\chi, Q\chi) + S_{\rm int}(\rho) - \mathcal{A}\,, \tag{6.22}$$

the BV path integral over χ will precisely reproduce the original action $S(\rho)$. This sets the difference in vacuum energies between the initial and the extended theory, which are now fully equivalent also at the quantum level.

We proceed now with the second step, namely redefining the fields as described in (6.13), obtaining

$$\begin{aligned} S(\psi,\Sigma) &\coloneqq S'(\rho(\psi,\Sigma),\chi(\psi,\Sigma)) \\ &= \frac{1}{2}\omega(\psi, Q\psi) + \frac{1}{2}\omega(\Sigma, Q\Sigma) + S_{\rm int}\left(c_\lambda\psi - \sqrt{1-c_\lambda^2}\Sigma\right) - \mathcal{A}\,. \end{aligned} \tag{6.23}$$

The stubbed theory is given by integrating out Σ

$$\tilde{S}(\psi) = \log\int_{h\Sigma=0}\mathcal{D}\Sigma\, e^{S'(\psi,\Sigma)} = \Lambda + \frac{1}{2}\omega(\psi, Q\psi) + \tilde{S}_{\rm int}(\psi)\,, \tag{6.24}$$

where Λ is a constant associated with the vacuum bubbles. Notice that the 1-loop bubble that one gets by integrating out Σ is just the same as the one obtained by integrating out χ (which has been subtracted away in the definition of the extended action), so the vacuum energy Λ only includes (interacting) vacuum diagrams from two loops onwards.

From now on, we will ignore these extra constants and focus on the dynamical part of the stubbed action, which will obviously depend on the peculiar interactions of the original theory.

6.2.3 Example: adding quantum stubs to a cubic theory

In this subsection, we would like to apply the procedure just outlined to the simplest non-trivial example, focusing on a cubic theory. This also allows to make direct contact with the classical analysis of the previous works [149] and [152]. Therefore we start with

$$S(\rho) = \frac{1}{2}\omega(\rho, Q\rho) + \frac{1}{3}\omega\left(\rho, m_2(\rho,\rho)\right)\,, \tag{6.25}$$

which we have written starting from a cyclic differential graded associative algebra (Q, m_2), the simplest non-trivial example of a cyclic A_∞ algebra (see Section 4.1.4). The action satisfies the classical BV master equation thanks to the A_∞ relations

$$Q^2 = 0\,, \tag{6.26a}$$

$$Qm_2 + m_2 Q = 0\,, \tag{6.26b}$$

$$m_2 m_2 = 0\,. \tag{6.26c}$$

But, in general, it may not satisfy the quantum BV master equation. Focusing on the BV laplacian, we get

$$\Delta S(\rho) = \omega\left(\rho, m_2(U)\right)\,, \tag{6.27}$$

Since, thanks to the associativity of m_2, we have $(S, S) = 0$, the quantum BV master equation reduces to

$$\frac{1}{2}(S, S) + \Delta S = \omega\left(\rho, m_2(U)\right)\,, \tag{6.28}$$

and therefore is satisfied iff $m_2(U) = 0$. In what follows we will assume that this is true for the theory at hand. The only scope of this assumption is to see our formalism at work in the simplest possible example and leave more realistic scenarios for later. Indeed, as it is well-known (see for example [35]) this quantity is non-vanisihing and ill-defined in Witten theory. Formally it receives contributions from the emission of physical closed string states, which appear as poles in the one-loop one point open string amplitude [155]. But, for the sake of simplicity (and leaving open the possibility that there could be very peculiar D-branes in very peculiar closed string backgrounds where Witten theory makes sense quantum-mechanically), we assume here that we have an associative two-product m_2 such that $m_2(U) = 0$. In later examples we will work with proper solutions to the quantum master equation, but the structures we will encounter will be essentially the same as in this simpler example, after we assume $m_2(U) = 0$. That said, let us then introduce the decoupled auxiliary field χ

$$S'(\rho, \chi) = \frac{1}{2}\omega(\rho, Q\rho) + \frac{1}{2}\omega(\chi, Q\chi) + \frac{1}{3}\omega\left(\rho, m_2(\rho, \rho)\right)\,. \tag{6.29}$$

This action can be conveniently packaged as

$$S'(R) = \frac{1}{2}\Omega\left(R, m_1'(R)\right) + \frac{1}{3}\Omega\left(R, m_2'(R,R)\right) \,, \tag{6.30}$$

by organizing the string field and the auxiliary field in a vector

$$R \coloneqq \begin{pmatrix} \rho \\ \chi \end{pmatrix} \in \mathcal{H}' \coloneqq \mathcal{H} \oplus \bar{P}\mathcal{H} \,, \tag{6.31}$$

and redefining the symplectic form and the string products as follows

$$m_1'(R) \coloneqq \begin{pmatrix} Q\rho \\ Q\chi \end{pmatrix} \,, \tag{6.32a}$$

$$m_2'(R_1, R_2) \coloneqq \begin{pmatrix} m_2(\rho_1, \rho_2) \\ 0 \end{pmatrix} \,, \tag{6.32b}$$

$$\Omega(R_1, R_2) \coloneqq \mathrm{Tr} \begin{pmatrix} \omega(\rho_1, \rho_2) & 0 \\ 0 & \omega\left(\chi_1, \chi_2\right) \end{pmatrix} \,. \tag{6.32c}$$

Let's also define the Poisson bi-vector U' associated to the symplectic form Ω on $\mathcal{H}'$, which satisfies

$$(\Omega \otimes \mathbf{1}_{\mathcal{H}'})(1_{\mathcal{H}'} \otimes U') = \mathbf{1}_{\mathcal{H}'} \,, \tag{6.33}$$

and is explicitly written as

$$U' = \frac{(-1)^{v'^a}}{2} v_a' \wedge v'^a \in \mathcal{H}'^{\wedge 2} \,, \tag{6.34}$$

where the $\{v_a'\}$ and $\{v'^a\}$ are two canonically normalized basis of $\mathcal{H}'$

$$\Omega(v'^a, v_b') = -\Omega(v_b', v'^a) = \delta_b^a \,, \tag{6.35}$$

from which we can write the completeness relation

$$1_{\mathcal{H}'} = v_a' \Omega(v'^a, \cdot) = -v'^a \Omega(v_a', \cdot) \,. \tag{6.36}$$

Now, let us perform the field redefinition (6.13), $R = O_\lambda \Psi$, obtaining

$$S(\Psi) \coloneqq S'(O_\lambda \Psi) = \frac{1}{2}\Omega\left(\Psi, m_1'(\Psi)\right) + \frac{1}{3}\Omega\left(\Psi, O_\lambda^{-1} m_2'(O_\lambda\Psi, O_\lambda\Psi)\right) , \tag{6.37}$$

where we used the fact that we can move O_λ from the left to the right of Ω by taking its inverse (since L is bpz even)

$$\Omega\left(O_\lambda\Psi_1, \Psi_2\right) = \Omega\left(\Psi_1, O_\lambda^{-1}\Psi_2\right) , \tag{6.38}$$

and the fact that this field redefinition leaves the kinetic term unchanged $m_1' = O_\lambda^{-1} m_1' O_\lambda$. Therefore, by defining

$$M_1 \coloneqq m_1' \tag{6.39a}$$

$$\begin{aligned} M_2(\Psi_1, \Psi_2) &\coloneqq O_\lambda^{-1} m_2'(O_\lambda\Psi_1, O_\lambda\Psi_2) \\ &= \begin{pmatrix} c_\lambda m_2\left(c_\lambda\psi - \sqrt{1-c_\lambda^2}\Sigma, c_\lambda\psi - \sqrt{1-c_\lambda^2}\Sigma\right) \\ -\sqrt{1-c_\lambda^2} m_2\left(c_\lambda\psi - \sqrt{1-c_\lambda^2}\Sigma, c_\lambda\psi - \sqrt{1-c_\lambda^2}\Sigma\right) \end{pmatrix} , \end{aligned} \tag{6.39b}$$

the action (6.37) becomes

$$S(\Psi) = \frac{1}{2}\Omega\left(\Psi, M_1(\Psi)\right) + \frac{1}{3}\Omega\left(\Psi, M_2(\Psi,\Psi)\right) . \tag{6.40}$$

Let us now uplift this formalism to the level of tensor co-algebras, by extending the structures to the tensor product space $T\mathcal{H}'$, as explained in detail in Section 4.1.2. In particular, the string products on $\mathcal{H}'$ are promoted to co-derivations and the map O_λ to a co-homorphism. This means that if we introduce the co-product $\Delta_{T\mathcal{H}'} : T\mathcal{H}' \to T\mathcal{H}' \otimes' T\mathcal{H}'$ we have the following relations

$$\Delta_{T\mathcal{H}'}\boldsymbol{m}' = \left(\boldsymbol{m}' \otimes' \mathbf{1}_{T\mathcal{H}'} + \mathbf{1}_{T\mathcal{H}'} \otimes' \boldsymbol{m}'\right)\Delta_{T\mathcal{H}'} , \tag{6.41}$$

$$\Delta_{T\mathcal{H}'}\boldsymbol{O}_\lambda = (\boldsymbol{O}_\lambda \otimes' \boldsymbol{O}_\lambda)\Delta_{T\mathcal{H}'} , \tag{6.42}$$

where we defined $\boldsymbol{m}' = \boldsymbol{m}'_1 + \boldsymbol{m}'_2$ and

$$\boldsymbol{m}'_1 \pi_k = \sum_{i=0}^{k-1} 1^{\otimes i} \otimes m'_1 \otimes 1^{\otimes(k-i-1)} \,, \tag{6.43}$$

$$\boldsymbol{m}'_2 \pi_k = \sum_{i=0}^{k-2} 1^{\otimes i} \otimes m'_2 \otimes 1^{\otimes(k-i-2)} \,, \tag{6.44}$$

where π_k projects onto the k-th tensor power of $T\mathcal{H}'$. Similarly, the co-homomorphism $\boldsymbol{O}_\lambda$ is defined as

$$\boldsymbol{O}_\lambda \pi_k \coloneqq (O_\lambda)^{\otimes k} \,. \tag{6.45}$$

The action (6.30) can be rewritten in Wess-Zumino-Witten (WZW) form

$$S'(R) = \int_0^1 dt\, \Omega \left(\pi_1 \boldsymbol{\partial}_t \frac{1}{1-R(t)} \,,\, \pi_1 \boldsymbol{m}' \frac{1}{1-R(t)} \right) \,, \tag{6.46}$$

and, following Section 4.1.4, the BV master equation can be expressed as

$$\frac{1}{2}(S', S') + \Delta S' = \int_0^1 dt\, \Omega \left(\pi_1 \boldsymbol{\partial}_t \frac{1}{1-R(t)} \,,\, \pi_1 (\boldsymbol{m}'^2 + \boldsymbol{m}' \boldsymbol{U}') \frac{1}{1-R(t)} \right) \,. \tag{6.47}$$

Upon use of the classical A_∞ relations $(\boldsymbol{m}')^2 = 0$, the quantum consistency reduces to $\boldsymbol{m}' \boldsymbol{U}' = 0$, whose only non-trivial implication is $m_2(U) = 0$, which is the quantum consistency of the initial un-stubbed theory we are assuming since the beginning.
In this formalism, the field redefinition (6.13) can be written as a map between group like elements, via the co-homomorphism $\boldsymbol{O}_\lambda$

$$\frac{1}{1-\Psi} = \boldsymbol{O}_\lambda \frac{1}{1-R} \,, \tag{6.48}$$

and thus, using (6.38), the above action becomes

$$S'(O_\lambda \Psi) = S(\Psi) = \int_0^1 dt\, \Omega \left(\pi_1 \boldsymbol{\partial}_t \frac{1}{1-\Psi(t)} \,,\, \pi_1 \boldsymbol{M} \frac{1}{1-\Psi(t)} \right) \,, \tag{6.49}$$

where we consistently defined

$$\boldsymbol{M} \coloneqq \boldsymbol{O}_\lambda^{-1}\boldsymbol{m}'\boldsymbol{O}_\lambda = \boldsymbol{m}_1' + \boldsymbol{O}_\lambda^{-1}\boldsymbol{m}_2'\boldsymbol{O}_\lambda\,, \tag{6.50}$$

that is a co-derivation

$$\Delta_{T\mathcal{H}'}\boldsymbol{M} = (\boldsymbol{M} \otimes' \mathbf{1}_{T\mathcal{H}'} + \mathbf{1}_{T\mathcal{H}'} \otimes' \boldsymbol{M})\,\Delta_{T\mathcal{H}'}\,, \tag{6.51}$$

as a consequence of (6.41) and (6.42).
Clearly, the coderivation $\boldsymbol{M}$ satisfies the A_∞ relations $\boldsymbol{M}^2 = 0$, whereas verifying that $\boldsymbol{M}\boldsymbol{U}' = 0$ requires just few further steps. The first essential property is that in complete generality, given an even cohomorphism constructed from a linear operator F on $\mathcal{H}$, $\boldsymbol{F} = \oplus_k F^{\otimes k}\pi_k$, we easily find

$$\boldsymbol{F}\boldsymbol{U} = \boldsymbol{U}_F\boldsymbol{F}\,, \tag{6.52}$$

where $\boldsymbol{U}_F$ corresponds to the co-algebra uplifting of the bi-vector

$$U_F \coloneqq \boldsymbol{F}U = \frac{(-)^{v^a}}{2} Fv_a \wedge Fv^a \coloneqq \frac{(-)^{f^a}}{2} f_a \wedge f^a\,, \tag{6.53}$$

namely

$$\boldsymbol{U}_F = \frac{(-)^{f^a}}{2}\boldsymbol{f}_a\boldsymbol{f}^a\,. \tag{6.54}$$

In our context we are interested in U'_{O_λ} which can be manipulated using the completeness relation (6.36) and (6.38)

$$\begin{aligned} U'_{O_\lambda} &= \frac{(-)^{v'^a}}{2} O_\lambda v'_a \wedge O_\lambda v'^a \\ &= \frac{(-)^{v'^a}}{2} v'_b \Omega(v'^b, O_\lambda v'_a) \wedge O_\lambda v'^a \\ &= \frac{(-)^{v'_a}}{2} v'_b \wedge O_\lambda v'^a \Omega(v'_a, O_\lambda^{-1} v'^b) \\ &= \frac{(-)^{v'^b}}{2} v'_b \wedge O_\lambda O_\lambda^{-1} v'^b \\ &= U'\,. \end{aligned} \tag{6.55}$$

Therefore, from these two results we can state that

$$\boldsymbol{O}_\lambda \boldsymbol{U}' = \boldsymbol{U}' \boldsymbol{O}_\lambda \,, \tag{6.56}$$

and similarly

$$\boldsymbol{O}_\lambda^{-1} \boldsymbol{U}' = \boldsymbol{U}' \boldsymbol{O}_\lambda^{-1} \,. \tag{6.57}$$

Thus, we can easily check that (6.49) is consistent at quantum level

$$\boldsymbol{M}\boldsymbol{U}' = \boldsymbol{O}_\lambda^{-1} \boldsymbol{m}' \boldsymbol{O}_\lambda \boldsymbol{U}' = \boldsymbol{O}_\lambda^{-1} \boldsymbol{m}' \boldsymbol{U}' \boldsymbol{O}_\lambda = 0 \,, \tag{6.58}$$

because $\boldsymbol{m}'\boldsymbol{U}' = 0$, as a consequence of the original assumption $m_2(U) = 0$.
Now we want to integrate-out the auxiliary field Σ as in (6.24). As discussed in detail in appendix D (see in particular the final equation (D.59)), this will generate constant terms (which we ignore here) and effective string products for the remaining string fields ψ, which can be conveniently obtained by the (quantum) homotopy transfer, starting from the free strong deformation retract

$$\circlearrowleft (-h) \, (\mathcal{H}', M_1) \; \underset{\iota}{\overset{\pi}{\rightleftarrows}} \; (\mathcal{H}, Q) \,, \tag{6.59}$$

where the canonical projector π and the canonical inclusion ι are defined as follows

$$\pi(\Psi) = \psi \in \mathcal{H} \,, \tag{6.60a}$$

$$\iota(\psi) = \begin{pmatrix} \psi \\ 0 \end{pmatrix} \in \mathcal{H}' \,, \tag{6.60b}$$

and clearly satisfy $(\iota\pi)^2 = \iota\pi$ and $\pi\iota = \mathbf{1}$. The needed propagator h is defined as

$$h(\Psi) = \begin{pmatrix} 0 \\ \frac{b}{L}\Sigma \end{pmatrix} \,, \tag{6.61}$$

which obeys the annihilation relations and the Hodge-Kodaira decomposition

$$\pi h = h\iota = h^2 = 0\,, \tag{6.62a}$$

$$M_1 h + h M_1 = 1_{\mathcal{H}'} - \iota\pi\,. \tag{6.62b}$$

Notice that in (6.61) the propagator is well defined even without the $\bar{P}$ because $\Sigma \in \bar{P}\mathcal{H}$.

Now, we can promote all these objects at the co-algebra level. In doing this π and ι are promoted to co-homomorphisms and the propagator is uplifted as

$$\boldsymbol{h}\pi_n = \sum_{i=0}^{n} 1_{\mathcal{H}'}^{\otimes i} \otimes h \otimes (\iota\pi)^{\otimes n-i-1}\pi_n\,. \tag{6.63}$$

This establishes the existence of the strong deformation retract at the level of the tensor co-algebra

$$\circlearrowleft(-\boldsymbol{h})\,(T\mathcal{H}', \boldsymbol{M}_1) \underset{\boldsymbol{\iota}}{\overset{\boldsymbol{\pi}}{\rightleftarrows}} (T\mathcal{H}, \boldsymbol{Q})\,, \tag{6.64}$$

The result of integrating out Σ from the interacting theory is obtained by deforming the above strong deformation retract through $\boldsymbol{M}_2 + \boldsymbol{U}'$ and the quantum homological perturbation lemma gives us the explicit expression for the effective products, that is

$$\tilde{\boldsymbol{m}} = \boldsymbol{\pi}(\boldsymbol{M}_1+\boldsymbol{M}_2)\frac{1}{\mathbf{1}_{T\mathcal{H}} + \boldsymbol{h}\,(\boldsymbol{M}_2+\boldsymbol{U}')}\boldsymbol{\iota} = \boldsymbol{Q}+\boldsymbol{\pi}\boldsymbol{M}_2\frac{1}{\mathbf{1}_{T\mathcal{H}} + \boldsymbol{h}\,(\boldsymbol{M}_2+\boldsymbol{U}')}\boldsymbol{\iota}\,. \tag{6.65}$$

As explained in the appendix D, these products define the dynamical part of the effective action, which can be written as

$$\tilde{S}(\psi) = \log\int_{h\Sigma=0} \mathcal{D}\Sigma\, e^{S(\psi,\Sigma)} = \sum_{n=0}^{\infty}\frac{1}{n+1}\omega\left(\psi, \tilde{m}_n\left(\psi^{\otimes n}\right)\right) + \Lambda\,, \tag{6.66}$$

where

$$\tilde{m}_n = \pi_1\tilde{\boldsymbol{m}}\pi_n\,, \tag{6.67}$$

and Λ contains the (connected) vacuum bubbles generated by the path integral.

Let us now study in detail the effective products by making explicit the dependence on the starting coderivations $\boldsymbol{m'}$, which refers to the theory without stubs (with the addition of the trivial free and auxiliary field χ)

$$\begin{aligned}\tilde{\boldsymbol{m}} &= \boldsymbol{Q} + \boldsymbol{\pi}\boldsymbol{M}_2 \frac{1}{\mathbf{1}_{T\mathcal{H}'} + \boldsymbol{h}\left(\boldsymbol{M}_2 + \boldsymbol{U}'\right)} \boldsymbol{\iota} \\ &= \boldsymbol{Q} + \boldsymbol{\pi}\boldsymbol{O}_\lambda^{-1}\boldsymbol{m}_2'\boldsymbol{O}_\lambda \frac{1}{\mathbf{1}_{T\mathcal{H}'} + \boldsymbol{h}\left(\boldsymbol{O}_\lambda^{-1}\boldsymbol{m}_2'\boldsymbol{O}_\lambda + \boldsymbol{U}'\right)} \boldsymbol{\iota} \\ &= \boldsymbol{Q} + \boldsymbol{\pi}\boldsymbol{O}_\lambda^{-1}\boldsymbol{m}_2' \frac{1}{\mathbf{1}_{T\mathcal{H}'} + \boldsymbol{O}_\lambda\boldsymbol{h}\boldsymbol{O}_\lambda^{-1}\left(\boldsymbol{m}_2' + \boldsymbol{O}_\lambda\boldsymbol{U}'\boldsymbol{O}_\lambda^{-1}\right)} \boldsymbol{O}_\lambda\boldsymbol{\iota} \\ &= \boldsymbol{Q} + \boldsymbol{\pi}\boldsymbol{O}_\lambda^{-1}\boldsymbol{m}_2' \frac{1}{\mathbf{1}_{T\mathcal{H}'} + \boldsymbol{O}_\lambda\boldsymbol{h}\boldsymbol{O}_\lambda^{-1}\left(\boldsymbol{m}_2' + \boldsymbol{U}'\right)} \boldsymbol{O}_\lambda\boldsymbol{\iota}\,,\end{aligned} \tag{6.68}$$

where in the last step we used (6.56). Therefore, we can consistently define

$$\boldsymbol{\pi}_\lambda \coloneqq \boldsymbol{\pi}\boldsymbol{O}_\lambda^{-1}\,, \tag{6.69a}$$

$$\boldsymbol{\iota}_\lambda \coloneqq \boldsymbol{O}_\lambda\boldsymbol{\iota}\,, \tag{6.69b}$$

$$\boldsymbol{h}_\lambda \coloneqq \boldsymbol{O}_\lambda\boldsymbol{h}\boldsymbol{O}_\lambda^{-1}\,, \tag{6.69c}$$

obtaining

$$\tilde{\boldsymbol{m}} = \boldsymbol{Q} + \boldsymbol{\pi}_\lambda\boldsymbol{m}_2' \frac{1}{\mathbf{1}_{T\mathcal{H}'} + \boldsymbol{h}_\lambda\left(\boldsymbol{m}_2' + \boldsymbol{U}'\right)} \boldsymbol{\iota}_\lambda\,. \tag{6.70}$$

Notice that $\boldsymbol{\pi}_\lambda$ and $\boldsymbol{\iota}_\lambda$ are cohomomorphisms because are given by the composition of two cohomomorphisms and they trivially satisfy the retract property, namely $\boldsymbol{\pi}_\lambda\boldsymbol{\iota}_\lambda = 1_{T\mathcal{H}}$ and

$$\boldsymbol{P}_\lambda \coloneqq \boldsymbol{\iota}_\lambda\boldsymbol{\pi}_\lambda\,, \tag{6.71}$$

is a projector. In particular, $\boldsymbol{P}_\lambda$ is the cohomomorphism associated with the following projector acting on $\mathcal{H}'$

$$P_\lambda = \begin{pmatrix} c_\lambda^2 & c_\lambda\sqrt{1-c_\lambda^2} \\ c_\lambda\sqrt{1-c_\lambda^2} & 1-c_\lambda^2 \end{pmatrix}\,. \tag{6.72}$$

Let us now focus on the propagator $\boldsymbol{h}_\lambda$, which clearly satisfies the annihilation

relations $\boldsymbol{h}_\lambda^2 = \boldsymbol{\pi}_\lambda \boldsymbol{h}_\lambda = \boldsymbol{h}_\lambda \boldsymbol{\iota}_\lambda = 0$, and it acts on a bunch of n states as follows

$$\boldsymbol{h}_\lambda \pi_n = \sum_{i=0}^{n} 1_{T\mathcal{H}'}^{\otimes i} \otimes h_\lambda \otimes P_\lambda^{\otimes n-i-1} \pi_n \,, \tag{6.73}$$

that is the usual structure of a propagator lifted to the co-algebras formalism. Moreover, the explicit expression of the propagator is

$$h_\lambda = \begin{pmatrix} \frac{b}{L}(1-c_\lambda^2) & -\frac{b}{L} c_\lambda \sqrt{1-c_\lambda^2} \\ -\frac{b}{L} c_\lambda \sqrt{1-c_\lambda^2} & \frac{b}{L} c_\lambda^2 \end{pmatrix} . \tag{6.74}$$

Therefore, adding stubs through the procedure described above can be equivalently implemented by applying this peculiar homotopy transfer referred to the action (6.30), namely considering the following strong deformation retract

$$\circlearrowleft \left(-\boldsymbol{h}_\lambda\right) \left(T\mathcal{H}', \boldsymbol{m}_1'\right) \underset{\boldsymbol{\iota}_\lambda}{\overset{\boldsymbol{\pi}_\lambda}{\rightleftarrows}} \left(T\mathcal{H}, \boldsymbol{Q}\right), \tag{6.75}$$

and deforming it with $\boldsymbol{m_1}' \to \boldsymbol{m}' + \boldsymbol{U}'$. This describes the transfer between the two nilpotent structures

$$\tilde{\boldsymbol{m}} + \boldsymbol{U} = \boldsymbol{\pi}_\lambda (\boldsymbol{m}' + \boldsymbol{U}') \frac{1}{\mathbf{1}_{T\mathcal{H}'} + \boldsymbol{h}_\lambda \left(\boldsymbol{m}_2' + \boldsymbol{U}'\right)} \boldsymbol{\iota}_\lambda \tag{6.76}$$

$$= \boldsymbol{\pi} (\boldsymbol{M} + \boldsymbol{U}') \frac{1}{\mathbf{1}_{T\mathcal{H}'} + \boldsymbol{h} \left(\boldsymbol{M}_2 + \boldsymbol{U}'\right)} \boldsymbol{\iota} \,. \tag{6.77}$$

Notice that, as explained in appendix Section 4.2.1, the transfer of $\boldsymbol{U}'$ just gives $\boldsymbol{U}$.

It is important to realize that, although the quantum cubic action is supposed to be the same as the classical one, the resulting quantum action with stubs is different from its classical counterpart [149, 152]. This is because it has to satisfy the quantum BV master equation (under the assumption that $m_2(U) = 0$) and not the classical. For instance, we can notice a tadpole term that is not present in the classical treatment but which is necessary for the final theory to satisfy the BV quantum master equation. In particular if we consider the action weighted by the string coupling constant κ

according to the topological expansion we get

$$\begin{aligned}\tilde{m}_0 &= -\kappa\boldsymbol{\pi}_\lambda \boldsymbol{m}_2' \boldsymbol{h}_\lambda \boldsymbol{U}' \mathbf{1}_{T\mathcal{H}'} + O(\kappa^2) \\ &= -\frac{1}{2}\kappa\boldsymbol{\pi}_\lambda \begin{pmatrix} m_2\left(v^i \wedge \frac{b}{L}(1-c_\lambda^2)v_i\right) \\ 0 \end{pmatrix} + O(\kappa^2) \\ &= -\kappa m_2\left(v^i, \frac{b}{L}\left(1-c_\lambda^2\right)v_i\right) + O(\kappa^2)\,,\end{aligned} \tag{6.78}$$

where in the last step we used the completeness relation to move $\sqrt{1-c_\lambda^2}$ from the left to the right in the Poisson bi-vector and the fact that c_λ commutes with the Siegel propagator. This correction is expected: in the quantum Witten theory the one-loop tadpole is fully constructed from a Feynman diagram, but after adding stubs a part of moduli space (far from open string degeneration) is eliminated in the Feynman diagram and the elementary vertex $\tilde{m}_0$ is added to compensate. This vertex explicitly contains the closed string degeneration region but, just like the original Feynman diagram, it doesn't tell us how to deal with it. This appears to be general: adding open string stubs does not help in dealing with closed string degeneration. This problem can be fully dealt with in open-closed SFT, as we will explicitly see in the following.

6.2.4 Closed string field theory with stubs

In this section we will apply the prescription for adding stubs described above to the case of closed SFT, which is a theory described by a truly quantum BV master action. The action can be written as

$$S_{\rm c}(\rho_{\rm c}) = \sum_{g=0}^{\infty}\kappa^{2g-2}\sum_{k=0}^{\infty}\frac{1}{k!}\omega_{\rm c}\left(\rho_{\rm c}, l_k^{(g)}(\rho_{\rm c}^{\wedge k})\right) = \frac{1}{2\kappa^2}\omega_{\rm c}(\rho_{\rm c}, Q_{\rm c}\rho_{\rm c}) + S_{\rm int}(\rho_{\rm c})\,, \tag{6.79}$$

where $\rho_{\rm c} = r_{\rm c}^i c_i \in \mathcal{H}_{\rm c}$ is the closed string field, c_i the basis vectors and $Q_{\rm c}$ the closed BRST charge. As discussed in the previous section we introduce an auxiliary field $\chi_c \in \bar{P}_{\rm c}\mathcal{H}_{\rm c}$, where $P_{\rm c}$ projects onto $\ker(L_{\rm c})$ (the standard choice being $L_{\rm c} = L_0 + \bar{L}_0$),

as described in (6.12), obtaining

$$S'_{\rm c}(\rho_{\rm c},\chi_{\rm c}) = \frac{1}{2\kappa^2}\omega_{\rm c}\left(\chi_{\rm c}, Q_{\rm c}\chi_{\rm c}\right) + \frac{1}{2\kappa^2}\omega_c(\rho_{\rm c}, Q_{\rm c}\rho_{\rm c}) + S_{\rm int}(\rho_{\rm c})\,. \tag{6.80}$$

Again we organize the two fields in a vector

$$R_{\rm c} \coloneqq \begin{pmatrix}\rho_{\rm c}\\ \chi_{\rm c}\end{pmatrix} \in \mathcal{H}'_{\rm c} \coloneqq \mathcal{H}_{\rm c}\oplus\bar{P}_{\rm c}\mathcal{H}_{\rm c}\,, \tag{6.81}$$

and introduce a collective symplectic form and its corresponding Poisson bi-vector

$$\Omega_{\rm c}(R_1,R_2) \coloneqq {\rm Tr}\begin{pmatrix}\omega_{\rm c}(\rho_1,\rho_2) & 0\\ 0 & \omega_{\rm c}\left(\chi_1,\chi_2\right)\end{pmatrix}\,, \tag{6.82}$$

$$U'_{\rm c} = \left[\frac{(-)^{c^i}}{2}\begin{pmatrix}c_i\\0\end{pmatrix}\wedge\begin{pmatrix}c^i\\0\end{pmatrix} + \frac{(-)^{\bar{c}^j}}{2}\begin{pmatrix}0\\\bar{c}_j\end{pmatrix}\wedge\begin{pmatrix}0\\\bar{c}^j\end{pmatrix}\right] \in \mathcal{H}'^{\wedge 2}_{\rm c}\,, \tag{6.83}$$

such that

$$(\Omega_{\rm c}\otimes\mathbf{1})(1\otimes U'_{\rm c}) = \mathbf{1}_{\mathcal{H}'_{\rm c}}\,. \tag{6.84}$$

Let us then extend the multi-string products on the new Hilbert space $\mathcal{H}'_{\rm c}$

$$l'^{(g)}_n(R_1,\cdots,R_n) \coloneqq \begin{pmatrix}l'^{(g)}_n(\rho_1,\cdots,\rho_n)\\0\end{pmatrix}\,, \tag{6.85}$$

which can be uplifted to the co-algebras formalism as co-derivations acting on the symmetrized closed tensor algebra

$$\boldsymbol{l}' \coloneqq \sum_g \kappa^{2g}\sum_n \boldsymbol{l}'^{(g)}_n : S\mathcal{H}'_{\rm c}\to S\mathcal{H}'_{\rm c}\,. \tag{6.86}$$

Therefore, the action (6.80) can be rewritten in the WZW form as follows

$$S'(R_{\rm c}) = \int_0^1 dt\,\frac{\Omega_{\rm c}}{\kappa^2}\left(\pi_1\boldsymbol{\partial}_t\, e^{\wedge R_{\rm c}(t)}\,,\,\pi_1\boldsymbol{l}'e^{\wedge R_{\rm c}(t)}\right)\,, \tag{6.87}$$

where π_1 projects onto the single closed string state sector.
Moreover, the multi-string products satisfy the quantum L_∞ homotopy algebra, which in this formalism reads

$$(\boldsymbol{l}' + \boldsymbol{U}'_{\rm c})^2 = 0\,, \tag{6.88}$$

and this guarantees the validity of the quantum BV master equation (see Section 4.1.4). Let us now perform the field redefinition (6.13) adapted to this case

$$R_{\rm c} = \begin{pmatrix} \rho_{\rm c} \\ \chi_{\rm c} \end{pmatrix} = O_\lambda^{({\rm c})} \begin{pmatrix} \phi \\ \Sigma_{\rm c} \end{pmatrix} = \begin{pmatrix} c_\lambda^{({\rm c})} & -\sqrt{1-c_\lambda^{({\rm c})2}} \\ \sqrt{1-c_\lambda^{({\rm c})2}} & c_\lambda^{({\rm c})} \end{pmatrix} \begin{pmatrix} \phi \\ \Sigma_{\rm c} \end{pmatrix} \coloneqq \Phi\,, \tag{6.89}$$

where we can standardly choose $c_\lambda^{({\rm c})} = e^{\lambda L_{\rm c}} \to e^{\lambda L_0^+}$. This rotation uplifts to a cohomomorphism between two group-like elements

$$e^{\wedge R_{\rm c}} = \boldsymbol{O}_\lambda^{({\rm c})} e^{\wedge \Phi}\,. \tag{6.90}$$

Thus, by using the property (6.38) the action can be rewritten as follows

$$S'(\Phi) = \int_0^1 dt\, \frac{\Omega_{\rm c}}{\kappa^2} \left(\pi_1 \boldsymbol{\partial}_t\, e^{\Phi(t)}\,,\, \pi_1 \boldsymbol{L} e^{\Phi(t)} \right)\,, \tag{6.91}$$

where we defined

$$\boldsymbol{L} \coloneqq \boldsymbol{O}_\lambda^{({\rm c})-1} \boldsymbol{l}' \boldsymbol{O}_\lambda^{({\rm c})}\,, \tag{6.92}$$

that is clearly a co-derivation that satisfies the quantum L_∞ relations, indeed

$$(\boldsymbol{L} + \boldsymbol{U}'_{\rm c})^2 = \boldsymbol{O}_\lambda^{({\rm c})-1} \left(\boldsymbol{l}' + \boldsymbol{O}_\lambda^{({\rm c})} \boldsymbol{U}'_{\rm c} \boldsymbol{O}_\lambda^{({\rm c})-1} \right)^2 \boldsymbol{O}_\lambda^{({\rm c})} = \boldsymbol{O}_\lambda^{({\rm c})-1} (\boldsymbol{l}' + \boldsymbol{U}'_{\rm c})^2 \boldsymbol{O}_\lambda^{({\rm c})} = 0\,, \tag{6.93}$$

where in the last step we used (6.56).
Since the products satisfy the quantum homotopy relations we can consistently apply the homological perturbation lemma to integrate out the auxiliary field Σ ending up on a quantum stubbed closed SFT action. We therefore introduce the objects which define the free strong deformation retract, namely the canonical inclusion $\iota_{\rm c}$, the

canonical projector π_{c} and the propagator h_{c} as follows

$$\pi_{\mathrm{c}}(\Phi) = \phi\,, \tag{6.94a}$$

$$\iota_{\mathrm{c}}(\phi) = \begin{pmatrix} \phi \\ 0 \end{pmatrix}\,, \tag{6.94b}$$

$$h_{\mathrm{c}}(\Phi) = \begin{pmatrix} 0 \\ \frac{b_c}{L_c}\Sigma_{\mathrm{c}} \end{pmatrix}\,, \tag{6.94c}$$

which clearly satisfy the retract properties $\pi_{\mathrm{c}}\iota_{\mathrm{c}} = 1_{\mathcal{H}_{\mathrm{c}}}$ and $(\iota_{\mathrm{c}}\pi_{\mathrm{c}})^2 = \iota_{\mathrm{c}}\pi_{\mathrm{c}}$, the annihilation relations $h_{\mathrm{c}}\iota_{\mathrm{c}} = \pi_{\mathrm{c}}h_{\mathrm{c}} = h_{\mathrm{c}}^2 = 0$ and the Hodge-Kodaira decomposition

$$L_1^{(0)}h_{\mathrm{c}} + h_{\mathrm{c}}L_1^{(0)} = 1_{\mathcal{H}'_{\mathrm{c}}} - \iota_{\mathrm{c}}\pi_{\mathrm{c}}\,. \tag{6.95}$$

We can then promote such objects to the language of co-algebras and apply the homological perturbation lemma, which tells us that the effective products that will describe the transferred theory take the following form

$$\tilde{\boldsymbol{l}} = \boldsymbol{Q}_{\mathrm{c}} + \boldsymbol{\pi}_{\mathrm{c}}\boldsymbol{\delta L}\frac{1}{\mathbf{1}_{S\mathcal{H}'_{\mathrm{c}}} + \boldsymbol{h}_{\mathrm{c}}\left(\boldsymbol{\delta L} + \kappa^2\boldsymbol{U'}_{\mathrm{c}}\right)}\boldsymbol{\iota}_{\mathrm{c}} = \tag{6.96}$$

$$= \boldsymbol{Q}_{\mathrm{c}} + \boldsymbol{\pi}_{\mathrm{c}}\boldsymbol{O}_\lambda^{(\mathrm{c})-1}\boldsymbol{\delta l'}\frac{1}{\mathbf{1}_{S\mathcal{H}'} + \boldsymbol{O}_\lambda^{(\mathrm{c})}\boldsymbol{h}_{\mathrm{c}}\boldsymbol{O}_\lambda^{(\mathrm{c})-1}\left(\boldsymbol{\delta l'} + \kappa^2\boldsymbol{U}'_{\mathrm{c}}\right)}\boldsymbol{O}_\lambda^{(\mathrm{c})}\boldsymbol{\iota}_{\mathrm{c}} \tag{6.97}$$

$$= \boldsymbol{Q}_{\mathrm{c}} + \boldsymbol{\pi}_\lambda^{(\mathrm{c})}\boldsymbol{\delta l'}\frac{1}{\mathbf{1}_{S\mathcal{H}'} + \boldsymbol{h}_\lambda^{(\mathrm{c})}\left(\boldsymbol{\delta l'} + \kappa^2\boldsymbol{U}'_{\mathrm{c}}\right)}\boldsymbol{\iota}_\lambda^{(\mathrm{c})}\,, \tag{6.98}$$

in which we consistently defined

$$\boldsymbol{\pi}_\lambda^{(\mathrm{c})} \coloneqq \boldsymbol{\pi}_{\mathrm{c}}\boldsymbol{O}_\lambda^{(\mathrm{c})-1}\,, \tag{6.99a}$$

$$\boldsymbol{\iota}_\lambda^{(\mathrm{c})} \coloneqq \boldsymbol{O}_\lambda^{(\mathrm{c})}\boldsymbol{\iota}_{\mathrm{c}}\,, \tag{6.99b}$$

$$\boldsymbol{h}_\lambda^{(\mathrm{c})} \coloneqq \boldsymbol{O}_\lambda^{(\mathrm{c})}\boldsymbol{h}_{\mathrm{c}}\boldsymbol{O}_\lambda^{(\mathrm{c})-1}\,. \tag{6.99c}$$

Therefore, the action for the subbed closed string field (ignoring the vacuum

bubbles, which are not captured by the homotopy transfer) reads

$$\tilde{S}_{\mathrm{c}}(\phi) = \int_0^1 dt\, \frac{\omega_{\mathrm{c}}}{\kappa^2} \left(\pi_1 \boldsymbol{\partial}_t\, e^{\wedge\phi(t)}\,,\, \pi_{10} \tilde{\boldsymbol{l}}\, e^{\wedge\phi(t)} \right) , \tag{6.100}$$

where $\tilde{\boldsymbol{l}} = \sum_{g=0}^{\infty} \kappa^{2g} \tilde{\boldsymbol{l}}^{(g)}$. In particular the truncation to $g = 0$ is the classical stubbed closed SFT, which obeys the classical master equation.

6.2.5 Open-closed string field theory with stubs

In this section we will apply our method to open-closed SFT (see Chapter 5). Following the general strategy, we isolate the two kinetic terms and we introduce an auxiliary field for each sector

$$\begin{aligned} S'_{\mathrm{oc}}(\rho_{\mathrm{o}}, \chi_{\mathrm{o}}, \rho_{\mathrm{c}}, \chi_{\mathrm{c}}) = {} & \frac{1}{2\kappa} \omega_{\mathrm{o}}(\rho_{\mathrm{o}}, Q_{\mathrm{o}} \rho_{\mathrm{o}}) + \frac{1}{2\kappa} \omega_{\mathrm{o}}\left(\chi_{\mathrm{o}}, Q_{\mathrm{o}} \chi_{\mathrm{o}}\right) \\ & + \frac{1}{2\kappa^2} \omega_{\mathrm{c}}(\rho_{\mathrm{c}}, Q_{\mathrm{c}} \rho_{\mathrm{c}}) + \frac{1}{2\kappa^2} \omega_{\mathrm{c}}\left(\chi_{\mathrm{c}}, Q_{\mathrm{c}} \chi_{\mathrm{c}}\right) \\ & + S_{\mathrm{int}}(\rho_{\mathrm{o}}, \rho_{\mathrm{c}})\,. \end{aligned} \tag{6.101}$$

in which $\rho_{\mathrm{o}} \in \mathcal{H}_{\mathrm{o}}$, $\rho_{\mathrm{c}} \in \mathcal{H}_{\mathrm{c}}$, $\chi_{\mathrm{o}} \in \bar{P}_{\mathrm{o}} \mathcal{H}_{\mathrm{o}}$ and $\chi_{\mathrm{c}} \in \bar{P}_{\mathrm{c}} \mathcal{H}_{\mathrm{c}}$ where P_{o} and P_{c} project onto $\ker(L_{\mathrm{o}})$ and $\ker(L_{\mathrm{c}})$, respectively. The standard choice of stubs is $L_{\mathrm{c}} = L_0^+$ and $L_{\mathrm{o}} = L_0$.

We then define

$$R_{\mathrm{c}} \coloneqq \begin{pmatrix} \rho_{\mathrm{c}} \\ \chi_{\mathrm{c}} \end{pmatrix} \in \mathcal{H}'_{\mathrm{c}} \coloneqq \mathcal{H}_{\mathrm{c}} \oplus \bar{P}_{\mathrm{c}} \mathcal{H}_{\mathrm{c}}\,, \tag{6.102a}$$

$$R_{\mathrm{o}} \coloneqq \begin{pmatrix} \rho_{\mathrm{o}} \\ \chi_{\mathrm{o}} \end{pmatrix} \in \mathcal{H}'_{\mathrm{o}} \coloneqq \mathcal{H}_{\mathrm{o}} \oplus \bar{P}_{\mathrm{o}} \mathcal{H}_{\mathrm{o}}\,, \tag{6.102b}$$

and

$$\Omega_{\mathrm{o}}(R_{\mathrm{o}(1)}, R_{\mathrm{o}(2)}) \coloneqq \mathrm{Tr} \begin{pmatrix} \omega_{\mathrm{o}}(\rho_{\mathrm{o}(1)}, \rho_{\mathrm{o}(2)}) & 0 \\ 0 & \omega_{\mathrm{o}}\left(\chi_{\mathrm{o}(1)}, \chi_{\mathrm{o}(2)}\right) \end{pmatrix} , \tag{6.103a}$$

$$\Omega_{\mathrm{c}}(R_{\mathrm{c}(1)}, R_{\mathrm{c}(2)}) \coloneqq \mathrm{Tr} \begin{pmatrix} \omega_{\mathrm{c}}(\rho_{\mathrm{c}(1)}, \rho_{\mathrm{c}(2)}) & 0 \\ 0 & \omega_{\mathrm{c}}\left(\chi_{\mathrm{c}(1)}, \chi_{\mathrm{c}(2)}\right) \end{pmatrix} , \tag{6.103b}$$

$$U'_{\mathrm{o}} = \left[\frac{(-)^{o^i}}{2} \begin{pmatrix} o_i \\ 0 \end{pmatrix} \wedge \begin{pmatrix} o^i \\ 0 \end{pmatrix} + \frac{(-)^{\bar{o}^j}}{2} \begin{pmatrix} 0 \\ \bar{o}_j \end{pmatrix} \wedge \begin{pmatrix} 0 \\ \bar{o}^j \end{pmatrix} \right] \in \mathcal{H}'^{\wedge 2}_{\mathrm{o}} , \tag{6.104a}$$

$$U'_{\mathrm{c}} = \left[\frac{(-)^{c^i}}{2} \begin{pmatrix} c_i \\ 0 \end{pmatrix} \wedge \begin{pmatrix} c^i \\ 0 \end{pmatrix} + \frac{(-)^{\bar{c}^j}}{2} \begin{pmatrix} 0 \\ \bar{c}_j \end{pmatrix} \wedge \begin{pmatrix} 0 \\ \bar{c}^j \end{pmatrix} \right] \in \mathcal{H}'^{\wedge 2}_{\mathrm{c}} , \tag{6.104b}$$

such that

$$(\Omega_{\mathrm{o}} \otimes \mathbf{1})(1 \otimes U'_{\mathrm{o}}) = \mathbf{1}_{\mathcal{H}'_{\mathrm{o}}} , \tag{6.105a}$$

$$(\Omega_{\mathrm{c}} \otimes \mathbf{1})(1 \otimes U'_{\mathrm{c}}) = \mathbf{1}_{\mathcal{H}'_{\mathrm{c}}} . \tag{6.105b}$$

Now, we promote the multi-string products on the new Hilbert spaces

$$l'^{(g,b)}_{k[p_1,\cdots,p_b]}(R_{\mathrm{c}}^{\wedge k} \otimes' R_{\mathrm{o}}^{\odot p_b} \wedge' \cdots \wedge' R_{\mathrm{o}}^{\odot p_1}) \coloneqq \begin{pmatrix} l^{(g,b)}_{k[p_1,\cdots,p_b]}(\rho_{\mathrm{c}}^{\wedge k} \otimes' \rho_{\mathrm{o}}^{\odot p_b} \wedge' \cdots \wedge' \rho_{\mathrm{o}}^{\odot p_1}) \\ 0 \end{pmatrix} , \tag{6.106a}$$

$$m'^{(g,b)}_{k[p_1,\cdots,p_{b-1}]p_b}(R_{\mathrm{c}}^{\wedge k} \otimes' R_{\mathrm{o}}^{\odot p_1} \wedge' \cdots \wedge' R_{\mathrm{o}}^{\odot p_{b-1}} \otimes'' R_{\mathrm{o}}^{\otimes p_b}) \coloneqq \begin{pmatrix} m^{(g,b)}_{k[p_1,\cdots,p_{b-1}]p_b}(\rho_{\mathrm{c}}^{\wedge k} \otimes' \rho_{\mathrm{o}}^{\odot p_1} \wedge' \cdots \wedge' \rho_{\mathrm{o}}^{\odot p_{b-1}} \otimes'' \rho_{\mathrm{o}}^{\otimes p_b}) \\ 0 \end{pmatrix} , \tag{6.106b}$$

which can be uplifted to the co-algebras formalism in the sense described in Chapter 5

$$\boldsymbol{l}' = \sum_{g,b} \kappa^{2g+b} \sum_{k,\{p_1,\ldots,p_b\}} \boldsymbol{l}^{(g,b)}_{k[p_1,\ldots,p_b]} : S\mathcal{H}'_{\mathrm{c}} \otimes' SC\mathcal{H}'_{\mathrm{o}} \to S\mathcal{H}'_{\mathrm{c}} \otimes' SC\mathcal{H}'_{\mathrm{o}} , \tag{6.107a}$$

$$\boldsymbol{m}' = \sum_{g,b} \kappa^{2g+b-1} \sum_{k,\{p_1,\ldots,p_b\}} \boldsymbol{m}^{(g,b)}_{k[p_1,\ldots,p_{b-1}]p_b} : S\mathcal{H}'_{\mathrm{c}} \otimes' SC\mathcal{H}'_{\mathrm{o}} \to S\mathcal{H}'_{\mathrm{c}} \otimes' SC\mathcal{H}'_{\mathrm{o}} . \tag{6.107b}$$

Therefore, by defining an interpolation for each field, namely $R_{\mathrm{o}}(t)$ and $R_{\mathrm{c}}(t)$ such that for $t = 0$ they return 0 and for $t = 1$ R_{o} and R_{c}, respectively, we can rewrite

the open-closed SFT action with free auxiliary fields in the WZW form

$$S'(R_{\rm c},R_{\rm o})=\int_0^1 dt\,\frac{\Omega_{\rm c}}{\kappa^2}\left(\pi_{10}\boldsymbol{\partial}_t\,\mathcal{G}(R_{\rm c}(t),R_{\rm o}(t))\right),\,\pi_{10}\boldsymbol{l}'\,\mathcal{G}(R_{\rm c}(t),R_{\rm o}(t)))+ \\ \frac{\Omega_{\rm o}}{\kappa}\left(\pi_{01}\,\boldsymbol{\partial}_t\,\mathcal{G}(R_{\rm c}(t),R_{\rm o}(t))\right),\,\pi_{01}\boldsymbol{m}'\,\mathcal{G}(R_{\rm c}(t),R_{\rm o}(t)))\,, \quad (6.108)$$

where π_{10} and π_{01} returns a single copy of $\mathcal{H}'_{\rm c}$ and $\mathcal{H}'_{\rm o}$, respectively. In addition, the group like element is defined as in (5.30)

$$\mathcal{G}(R_{\rm c},R_{\rm o})=\sum_{k,b,p_1,\ldots,p_b\geq 0}\frac{1}{k!b!(p_1)\cdots(p_b)}\left(R_{\rm c}^{\wedge k}\otimes' R_{\rm o}^{\odot p_1}\wedge'\cdots\wedge' R_{\rm o}^{\odot p_b}\right). \quad (6.109)$$

Furthermore, the products satisfy the quantum open-closed homotopy algebra, namely

$$\pi_{10}\left\{\frac{1}{2}[\boldsymbol{l}',\boldsymbol{l}']+\boldsymbol{l}'\boldsymbol{m}'+\left[\boldsymbol{l}',\kappa\boldsymbol{U}'_o+\kappa^2\boldsymbol{U}'_c\right]\right\}=0\,, \quad (6.110)$$

$$\kappa\pi_{01}\left\{\frac{1}{2}[\boldsymbol{m}',\boldsymbol{m}']+\boldsymbol{m}'\boldsymbol{l}'+\left[\boldsymbol{m}',\kappa\boldsymbol{U}'_o+\kappa^2\boldsymbol{U}'_c\right]\right\}=0\,, \quad (6.111)$$

which can be succinctly written as

$$(\boldsymbol{l}'+\boldsymbol{m}'+\kappa\boldsymbol{U}'_o+\kappa^2\boldsymbol{U}'_c)^2=0.$$

Let us now implement the field redefinition (6.13), with two independent stub parameters $\lambda_{\rm c}$ and $\lambda_{\rm o}$

$$R_{\rm c}=\begin{pmatrix}\rho_{\rm c}\\ \chi_{\rm c}\end{pmatrix}=O^{({\rm c})}_{\lambda_{\rm c}}\begin{pmatrix}\phi\\ \Sigma_{\rm c}\end{pmatrix}=\begin{pmatrix}c^{({\rm c})}_{\lambda_{\rm c}} & -\sqrt{1-c^{({\rm c})2}_{\lambda_{\rm c}}}\\ \sqrt{1-c^{({\rm c})2}_{\lambda_{\rm c}}} & c^{({\rm c})}_{\lambda_{\rm c}}\end{pmatrix}\begin{pmatrix}\phi\\ \Sigma_{\rm c}\end{pmatrix}\coloneqq\Phi\,, \quad (6.112a)$$

$$R_{\rm o}=\begin{pmatrix}\rho_{\rm o}\\ \chi_{\rm o}\end{pmatrix}=O^{({\rm o})}_{\lambda_{\rm o}}\begin{pmatrix}\psi\\ \Sigma_{\rm o}\end{pmatrix}=\begin{pmatrix}c^{({\rm o})}_{\lambda_{\rm o}} & -\sqrt{1-c^{({\rm o})2}_{\lambda_{\rm o}}}\\ \sqrt{1-c^{({\rm o})2}_{\lambda_{\rm o}}} & c^{({\rm o})}_{\lambda_{\rm o}}\end{pmatrix}\begin{pmatrix}\psi\\ \Sigma_{\rm o}\end{pmatrix}\coloneqq\Psi\,. \quad (6.112b)$$

As discussed in the previous sections the orthogonal matrices which define the above field redefinitions can be upgraded to the co-algebras formalism as an open-closed

cohomomorphism-like object acting on the group-like element

$$\mathcal{G}(\Phi,\Psi) = \boldsymbol{O}_{\lambda_{\rm o},\lambda_{\rm c}}\mathcal{G}(R_{\rm c},R_{\rm o})\,, \tag{6.113}$$

where

$$\boldsymbol{O}_{\lambda_{\rm o},\lambda_{\rm c}} = \boldsymbol{O}^{(\rm c)}_{\lambda_{\rm c}} \otimes' \boldsymbol{O}^{(\rm o)}_{\lambda_{\rm o}} : S\mathcal{H}'_{\rm c} \otimes' SC\mathcal{H}'_{\rm o} \to S\mathcal{H}_{\rm c} \otimes' SC\mathcal{H}_{\rm o}\,. \tag{6.114}$$

Therefore, thanks to (6.38) the action (6.108) can be rewritten as

$$\begin{aligned} S'(\Phi,\Psi) = \int_0^1 dt\, \frac{\Omega_{\rm c}}{\kappa^2}\left(\pi_{10}\boldsymbol{\partial}_t\,\mathcal{G}(\Phi(t),\Psi(t))\right),\, \pi_{10}\boldsymbol{L}\,\mathcal{G}(\Phi(t),\Psi(t))) + \\ \frac{\Omega_{\rm o}}{\kappa}\left(\pi_{01}\,\boldsymbol{\partial}_t\,\mathcal{G}(\Phi(t),\Psi(t))\right),\, \pi_{01}\boldsymbol{M}\,\mathcal{G}(\Phi(t),\Psi(t)))\,, \end{aligned} \tag{6.115}$$

where we defined

$$\boldsymbol{L} \coloneqq \boldsymbol{O}^{-1}_{\lambda_{\rm o},\lambda_{\rm c}}\boldsymbol{l}'\boldsymbol{O}_{\lambda_{\rm o},\lambda_{\rm c}}\,, \tag{6.116}$$

$$\boldsymbol{M} \coloneqq \boldsymbol{O}^{-1}_{\lambda_{\rm o},\lambda_{\rm c}}\boldsymbol{m}'\boldsymbol{O}_{\lambda_{\rm o},\lambda_{\rm c}}\,, \tag{6.117}$$

Moreover, the new multi-string products satisfy the quantum homotopy algebra

$$\pi_{10}\left\{\frac{1}{2}[\boldsymbol{L},\boldsymbol{L}] + \boldsymbol{L}\boldsymbol{M} + \left[\boldsymbol{L},\kappa\boldsymbol{U}'_o + \kappa^2\boldsymbol{U}'_c\right]\right\} = 0\,, \tag{6.118}$$

$$\kappa\pi_{01}\left\{\frac{1}{2}[\boldsymbol{M},\boldsymbol{M}] + \boldsymbol{M}\boldsymbol{L} + \left[\boldsymbol{M},\kappa\boldsymbol{U}'_o + \kappa^2\boldsymbol{U}'_c\right]\right\} = 0\,, \tag{6.119}$$

and this can be seen by following the same steps as in (6.93). Therefore, (6.115) satisfies the quantum master equation, and we can consistently apply the quantum homotopy transfer with the goal of integrating out the two auxiliary fields $\Sigma_{\rm o}$ and $\Sigma_{\rm c}$. Furthermore, precisely because the products $\boldsymbol{L}$ and $\boldsymbol{M}$ define a quantum nilpotent structure

$$(\boldsymbol{L} + \boldsymbol{M} + \kappa\boldsymbol{U}'_o + \kappa^2\boldsymbol{U}'_c)^2 = 0,$$

we can assert that the effective theory will also be a solution of the corresponding quantum master equation.

Let us then define the desired strong deformation retract by introducing a canonical

projector and a canonical inclusion promoted to the co-algebras formalism as co-homorphisms

$$\boldsymbol{\pi} = \boldsymbol{\pi}_{\mathrm{c}} \otimes' \boldsymbol{\pi}_{\mathrm{o}} \,, \tag{6.120a}$$

$$\boldsymbol{\iota} = \boldsymbol{\iota}_{\mathrm{c}} \otimes' \boldsymbol{\iota}_{\mathrm{o}} \,, \tag{6.120b}$$

where

$$\pi_{\mathrm{c}}(\Phi) = \phi \,, \tag{6.121a}$$

$$\iota_{\mathrm{c}}(\phi) = \begin{pmatrix} \phi \\ 0 \end{pmatrix} \,, \tag{6.121b}$$

the same for π_{o} and ι_{o}. Hence, if we apply $\boldsymbol{\pi}$ to the group like element we get

$$\boldsymbol{\pi}\mathcal{G}(\Phi,\Psi) = \sum_{k,b,p_1,\ldots,p_b\geq 0} \frac{1}{k!b!(p_1)\cdots(p_b)} \left(\phi^{\wedge k} \otimes' \psi^{\odot p_1} \wedge' \cdots \wedge' \psi^{\odot p_b}\right) = \mathcal{G}(\phi,\psi) \coloneqq \mathcal{G} \,. \tag{6.122}$$

Let us now define the propagator

$$\boldsymbol{h} = \boldsymbol{h}_c \otimes' \boldsymbol{\iota}_{\mathrm{o}}\boldsymbol{\pi}_{\mathrm{o}} + \mathbf{1}_{S\mathcal{H}'_{\mathrm{c}}} \otimes' \boldsymbol{h}_{\mathrm{o}} \,, \tag{6.123}$$

where

$$h_{\mathrm{c}}(\Phi) = \begin{pmatrix} 0 \\ \frac{b_{\mathrm{c}}}{L_{\mathrm{c}}}\Sigma_{\mathrm{c}} \end{pmatrix} \,, \tag{6.124a}$$

$$h_{\mathrm{o}}(\Psi) = \begin{pmatrix} 0 \\ \frac{b_{\mathrm{o}}}{L_{\mathrm{o}}}\Sigma_{\mathrm{o}} \end{pmatrix} \,. \tag{6.124b}$$

Finally, by applying the quantum homological perturbation lemma we get the

following action for the stubbed open-closed theory

$$\tilde{S}_{\mathrm{oc}}(\phi,\psi)=\int_0^1 dt\left(\frac{\omega_{\mathrm{c}}}{\kappa^2}\left(\pi_{10}\boldsymbol{\partial}_t\,\mathcal{G}\,,\,\pi_{10}\tilde{\boldsymbol{l}}\,\mathcal{G}\right)+\frac{\omega_{\mathrm{o}}}{\kappa}\left(\pi_{01}\,\boldsymbol{\partial}_t\,\mathcal{G}\,,\,\pi_{01}\tilde{\boldsymbol{m}}\,\mathcal{G}\right)\right)\,. \tag{6.125}$$

where the effective products are

$$\tilde{\boldsymbol{l}}+\tilde{\boldsymbol{m}}=$$
$$\boldsymbol{Q}_{\mathrm{c}}+\boldsymbol{Q}_{\mathrm{o}}+\boldsymbol{\pi}_{\lambda_{\mathrm{o}},\lambda_{\mathrm{c}}}(\boldsymbol{\delta m}'+\boldsymbol{\delta l}')\frac{1}{\mathbf{1}_{\mathcal{SH}_{\mathrm{c}}\otimes'\mathcal{SCH}_{\mathrm{o}}}+\boldsymbol{h}_{\lambda_{\mathrm{o}},\lambda_{\mathrm{c}}}\left(\boldsymbol{\delta m}'+\boldsymbol{\delta l}'+\kappa\boldsymbol{U}'_{\mathrm{o}}+\kappa^2\boldsymbol{U}'_{\mathrm{c}}\right)}\boldsymbol{\iota}_{\lambda_{\mathrm{o}},\lambda_{\mathrm{c}}} \tag{6.126}$$

and we have introduced

$$\boldsymbol{\pi}_{\lambda_{\mathrm{o}},\lambda_{\mathrm{c}}}\coloneqq\boldsymbol{\pi}\boldsymbol{O}^{-1}_{\lambda_{\mathrm{o}},\lambda_{\mathrm{c}}}\,, \tag{6.127a}$$

$$\boldsymbol{\iota}_{\lambda_{\mathrm{o}},\lambda_{\mathrm{c}}}\coloneqq\boldsymbol{O}_{\lambda_{\mathrm{o}},\lambda_{\mathrm{c}}}\boldsymbol{\iota}\,, \tag{6.127b}$$

$$\boldsymbol{h}_{\lambda_{\mathrm{o}},\lambda_{\mathrm{c}}}\coloneqq\boldsymbol{O}_{\lambda_{\mathrm{o}},\lambda_{\mathrm{c}}}\boldsymbol{h}\boldsymbol{O}^{-1}_{\lambda_{\mathrm{o}},\lambda_{\mathrm{c}}}\,. \tag{6.127c}$$

6.3 Equivalence of amplitudes

In this section we will show that the on-shell amplitudes for a theory with and without stubs are equivalent at the quantum level. For the sake of clarity, we will consider 'quantum' A_∞ theories (of the kind analyzed in section 6.2.3), but the proof can be straightforwardly generalized to quantum L_∞ theories and to open-closed SFT.
Our proof will rely on showing that the two theories give rise to the same 'almost' minimal model, namely the effective action for fields in the image of P, properly containing the cohomology. Then amplitudes will trivially be the same, by further restricting the external legs to the cohomology. Let's start with the original theory without stubs (trivially extended by the free auxiliary field χ, which certainly does not change the amplitudes). In this case the minimal model effective products (the

amplitudes) are given by the following perturbed SDR

$$\begin{array}{ccc} \circlearrowleft\,(-\boldsymbol{h}'_0)\,(T\mathcal{H}',\boldsymbol{m}'_1) & \underset{\boldsymbol{\iota}'_0}{\overset{\boldsymbol{\pi}'_0}{\rightleftarrows}} & (T\pi_0\mathcal{H},\boldsymbol{\pi}_0\boldsymbol{Q}\boldsymbol{\iota}_0) \\ \Big\downarrow{\scriptstyle \boldsymbol{\delta m}'+\boldsymbol{U}'} & & \Big\downarrow{\scriptstyle \boldsymbol{\delta}\hat{\boldsymbol{m}}+\boldsymbol{U}_{\pi_0}} \\ \circlearrowleft\,(-\hat{\boldsymbol{h}}'_0)\,(T\mathcal{H}',\boldsymbol{m}'+\boldsymbol{U}') & \underset{\hat{\boldsymbol{\iota}}'_0}{\overset{\hat{\boldsymbol{\pi}}'_0}{\rightleftarrows}} & (T\pi_0\mathcal{H},\hat{\boldsymbol{m}}+\boldsymbol{U}_{\pi_0}) \end{array} \tag{6.128}$$

where the (un-perturbed) projectors are defined as follows

$$\boldsymbol{\pi}'_0 \coloneqq \boldsymbol{\pi}_0\boldsymbol{\pi} : \mathcal{T}\mathcal{H}' \to \mathcal{T}\pi_0\mathcal{H}\,, \tag{6.129}$$

$$\boldsymbol{\iota}'_0 \coloneqq \boldsymbol{\iota}\boldsymbol{\iota}_0 : \mathcal{T}\pi_0\mathcal{H} \to \mathcal{T}\mathcal{H}'\,, \tag{6.130}$$

and the propagator $\boldsymbol{h}'_0$ is the co-algebraic uplifting (6.73) of h'_0 which acts on an element of $\mathcal{H}'$ as

$$h'_0 \coloneqq \begin{pmatrix} h_0 & 0 \\ 0 & \frac{b}{L} \end{pmatrix} \coloneqq \begin{pmatrix} \frac{b}{L}\bar{P} & 0 \\ 0 & \frac{b}{L} \end{pmatrix}\,. \tag{6.131}$$

Notice that $\bar{P}$ is not needed in the second slot because there are no states belonging to $\ker(L)$.

Using the homotopy transfer, the effective products read

$$\hat{\boldsymbol{m}} = \boldsymbol{\pi}'_0\boldsymbol{m}'\frac{1}{\mathbf{1}_{\mathcal{T}\mathcal{H}'}+\boldsymbol{h}'_0(\boldsymbol{\delta m}'+\boldsymbol{U}')}\boldsymbol{\iota}'_0\,. \tag{6.132}$$

Let's now discuss the amplitudes of the stubbed theory, defined by the homotopy transfer (6.75). To obtain the almost minimal model in this scenario, we must apply a further integration-out. Therefore, computing amplitudes for the stubbed theory corresponds to applying two consecutive homotopy transfers to the starting theory (with the non-propagating free field χ), without stubs

$$\begin{array}{ccccc} \circlearrowleft\,(-\boldsymbol{h}_\lambda)\,(T\mathcal{H}',\boldsymbol{m}'_1) & \underset{\boldsymbol{\iota}_\lambda}{\overset{\boldsymbol{\pi}_\lambda}{\rightleftarrows}} & \circlearrowleft\,(-\boldsymbol{h}_0)\,(T\mathcal{H},\boldsymbol{Q}) & \underset{\boldsymbol{\iota}_0}{\overset{\boldsymbol{\pi}_0}{\rightleftarrows}} & (T\pi_0\mathcal{H},\boldsymbol{\pi}_0\boldsymbol{Q}\boldsymbol{\iota}_0) \\ \Big\downarrow{\scriptstyle \boldsymbol{\delta m}'+\boldsymbol{U}'} & & \Big\downarrow{\scriptstyle \boldsymbol{\delta}\tilde{\boldsymbol{m}}+\boldsymbol{U}} & & \Big\downarrow{\scriptstyle \boldsymbol{\delta}\hat{\boldsymbol{m}}_\lambda+\boldsymbol{U}_{\pi_0}} \\ \circlearrowleft\,(-\tilde{\boldsymbol{h}}_\lambda)\,(T\mathcal{H}',\boldsymbol{m}'+\boldsymbol{U}') & \underset{\tilde{\boldsymbol{\iota}}_\lambda}{\overset{\tilde{\boldsymbol{\pi}}_\lambda}{\rightleftarrows}} & \circlearrowleft\,(-\hat{\boldsymbol{h}}_0)\,(T\mathcal{H},\tilde{\boldsymbol{m}}+\boldsymbol{U}) & \underset{\hat{\boldsymbol{\iota}}_0}{\overset{\hat{\boldsymbol{\pi}}_0}{\rightleftarrows}} & (T\pi_0\mathcal{H},\hat{\boldsymbol{m}}_\lambda+\boldsymbol{U}_{\pi_0}) \end{array} \tag{6.133}$$

This is an occurrence of the *horizontal composition* described in detail in [111], which tells us that we can write

$$\hat{\boldsymbol{m}}_\lambda = \boldsymbol{\pi}_0\boldsymbol{\pi}_\lambda \boldsymbol{m}' \frac{1}{\mathbf{1}_{\mathcal{T}\mathcal{H}'} + (\boldsymbol{h}_\lambda + \boldsymbol{\iota}_\lambda \boldsymbol{h}_0 \boldsymbol{\pi}_\lambda)\left(\boldsymbol{\delta}\boldsymbol{m}' + \boldsymbol{U}'\right)} \boldsymbol{\iota}_\lambda \boldsymbol{\iota}_0 \,. \tag{6.134}$$

This expression (6.134) has to be compared with (6.132). The two expressions apparently differ because of the π projectors, the ι inclusions and the propagators. Let's start by analyzing the apparent differences in the propagators. First of all we have to understand that, as already pointed out in [111], the composite co-algebraic propagator $\boldsymbol{h}_\lambda + \boldsymbol{\iota}_\lambda \boldsymbol{h}_0 \boldsymbol{\pi}_\lambda$ does not appear to be of the typical form (6.73), indeed its action is given by

$$\begin{aligned}(\boldsymbol{h}_\lambda + \boldsymbol{\iota}_\lambda \boldsymbol{h}_0 \boldsymbol{\pi}_\lambda)\, \pi_n = \\ \sum_{k=0}^{n-1} \left[1_{\mathcal{H}'}^{\otimes k} \otimes h_\lambda \otimes P_\lambda^{\otimes(n-k-1)} + P_\lambda^{\otimes k} \otimes \iota_\lambda h_0 \pi_\lambda \otimes (\iota_\lambda P \pi_\lambda)^{\otimes(n-k-1)} \right] \pi_n \,.\end{aligned} \tag{6.135}$$

Nevertheless, this composite propagator satisfies (by construction) the Hodge-Kodaira decomposition and thus (6.135) is just an alternative equivalent way to uplift the propagator $h_\lambda + \iota_\lambda h_0 \pi_\lambda$ from $\mathcal{H}'$ to $\mathcal{T}\mathcal{H}'$. Additionally, it is possible to show that after unpackaging (6.134) the effective products are given in terms of $h_\lambda + \iota_\lambda h_0 \pi_\lambda$ exactly as if we had upgraded the 'propagator' $h_\lambda + \iota_\lambda h_0 \pi_\lambda$ according to the prescription (6.73). Therefore, we can directly compare the propagators in (6.132) and (6.134) as operators on $\mathcal{H}'$.

Let us then compute explicitly the composite propagator using the definitions (6.69a),

(6.69b) and (6.69c)

$$\begin{aligned} h_\lambda + \iota_\lambda h_0 \pi_\lambda &= O_\lambda \left(h_0 + \iota h_0 \pi\right) O_\lambda^{-1} \\ &= O_\lambda \begin{pmatrix} \frac{b}{L}\bar{P} & 0 \\ 0 & \frac{b}{L} \end{pmatrix} O_\lambda^{-1} \\ &= \begin{pmatrix} \frac{b}{L}\bar{P}c_\lambda^2 + \frac{b}{L}(1-c_\lambda^2) & \frac{b}{L}\bar{P}c_\lambda\sqrt{1-c_\lambda^2} - \frac{b}{L}c_\lambda\sqrt{1-c_\lambda^2} \\ \frac{b}{L}\bar{P}c_\lambda\sqrt{1-c_\lambda^2} - \frac{b}{L}c_\lambda\sqrt{1-c_\lambda^2} & \frac{b}{L}\bar{P}(1-c_\lambda^2) + \frac{b}{L}c_\lambda^2 \end{pmatrix} \\ &= \begin{pmatrix} \frac{b}{L}\bar{P} & 0 \\ 0 & \frac{b}{L} \end{pmatrix} = h_0' \,, \end{aligned} \tag{6.136}$$

where in the last step we used the fact that $\bar{P}(1-c_\lambda^2) = (1-c_\lambda^2)$ or equivalently $\bar{P}\sqrt{1-c_\lambda^2} = \sqrt{1-c_\lambda^2}$, because $\sqrt{1-c_\lambda^2}\rho \in \bar{P}\mathcal{H} \quad \forall \rho \in \mathcal{H}$, as explained in section 6.2.1.

Let us now study the composite canonical projection $\pi_0\pi_\lambda$ by applying it on a test state $R \in \mathcal{H}'$

$$\begin{aligned} \pi_0\pi_\lambda R &= \pi_0 \pi O_\lambda^{-1} \begin{pmatrix} \rho \\ \chi \end{pmatrix} \\ &= \pi_0 \left(c_\lambda \rho + \sqrt{1-c_\lambda^2}\chi \right) \\ &= \pi_0 c_\lambda \rho \\ &= \pi_0 \rho \,, \end{aligned} \tag{6.137}$$

where the third line is justified by the fact that $\sqrt{1-c_\lambda^2}\,\chi$ does not belong to $\ker(L)$ and thus it is annihilated by π_0. Moreover, in the last step we noticed that $\pi_0 c_\lambda \rho = \pi_0 \rho$ because π_0 returns a non-vanishing result when $\rho \in \ker(L)$ and clearly in such a situation $c_\lambda \rho = e^{-\lambda L}\rho = \rho$. Therefore, the composite canonical projector acts effectively as

$$\pi_0\pi_\lambda = \pi_0\pi = \pi_0' \,. \tag{6.138}$$

Similarly, it is possible to show that the composite canonical inclusion $\iota_\lambda \iota_0 = \iota'_0$ and thus we can conclude that

$$\hat{\boldsymbol{m}}_\lambda = \hat{\boldsymbol{m}}\,. \tag{6.139}$$

6.3.1 Example: open string one-loop tadpole in OC-SFT

After the general proof we have just given we can concretely test the equivalence between amplitudes in open-closed SFT with and without stubs. To compute an amplitude in OC-SFT, as in the previous section, we apply the homological perturbation lemma to the following strong deformation retract

$$\circlearrowleft (\boldsymbol{H}_0)\,(\mathcal{SH}_\mathrm{c} \otimes' \mathcal{SCH}_\mathrm{o}, \boldsymbol{Q}_\mathrm{c} + \boldsymbol{Q}_\mathrm{o}) \;\underset{\boldsymbol{I}_0}{\overset{\boldsymbol{\Pi}_0}{\rightleftarrows}}\; (\boldsymbol{\Pi}_0(S\mathcal{H}_\mathrm{c} \otimes' S\mathcal{CH}_\mathrm{o}), \boldsymbol{\Pi}_0\,(\boldsymbol{Q}_\mathrm{c} + \boldsymbol{Q}_\mathrm{o})\,\boldsymbol{I}_0)\,, \tag{6.140}$$

in which the projector $\boldsymbol{I_0\Pi_0}$ is defined as

$$\boldsymbol{I}_0\boldsymbol{\Pi}_0 = \boldsymbol{P}_0^+ \otimes' \boldsymbol{P}_0\,, \tag{6.141}$$

where P_0 and P_0^+ project onto the $\ker(L_0)$ and $\ker(L_0^+)$, respectively. Furthermore, we define the closed string field projected by P_0^+ as $\varphi_\mathrm{c} \coloneqq P_0^+\rho_\mathrm{c}$, and the component in $\ker(L_0)$ of the open string field as $\varphi_\mathrm{o} \coloneqq P_0\rho_\mathrm{o}$. As discussed in C, the open-closed co-algebraic propagator $\boldsymbol{H}_0$ can be chosen as

$$\boldsymbol{H}_0 = \boldsymbol{h}_0^+ \otimes' \boldsymbol{P}_0 + \mathbf{1}_{S\mathcal{H}'_\mathrm{c}} \otimes' \boldsymbol{h}_0\,, \tag{6.142}$$

where $\boldsymbol{h}_0^+$ and $\boldsymbol{h}_0$ are the co-algebraic promotion of the Siegel gauge propagators

$$h_0^+ = \frac{b_0^+}{L_0^+}\bar{P}_0^+\,, \tag{6.143}$$

$$h_0 = \frac{b_0}{L_0}\bar{P}_0\,, \tag{6.144}$$

see appendix C for more details. Therefore, the effective products of the almost minimal model, which define the amplitudes are

$$\left(\hat{\boldsymbol{l}}+\hat{\boldsymbol{m}}\right)=\boldsymbol{\pi}_0\left(\boldsymbol{l}+\boldsymbol{m}\right)\frac{1}{\mathbf{1}_{\mathcal{SH}_{\rm c}\otimes'\mathcal{SCH}_{\rm o}}+\boldsymbol{H}_0\left(\boldsymbol{\delta m}+\boldsymbol{\delta l}+\kappa\boldsymbol{U}_{\rm o}+\kappa^2\boldsymbol{U}_{\rm c}\right)}\boldsymbol{\iota}_0\,. \tag{6.145}$$

It is particularly illuminating, as a concrete example, to specifically discuss the open string one-loop tadpole, which is simple enough and yet non-trivially tests the quantum structure of our construction.[5] We denote this amplitude as $A^{(0,2)}_{0;1}(\varphi_{\rm o})$. In this particular case, the relevant homotopy relation is

$$[\boldsymbol{m}^{(0,2)},\boldsymbol{m}^{(0,1)}]+\pi_{0,1}\boldsymbol{m}^{(0,1)}\boldsymbol{l}^{(0,1)}+[\boldsymbol{m}^{(0,2)},\boldsymbol{l}^{(0,0)}]+[\boldsymbol{m}^{(0,1)},\boldsymbol{U}_{\rm o}]=0\,, \tag{6.146}$$

and if we apply it to the identity we get

$$m^{(0,1)}_{1,0}\left(l^{(01)}_{0,0}\right)+Q_{\rm o}m^{(0,2)}_{0,0}+m^{(0,1)}_{0,2}(U_{\rm o})=0\,. \tag{6.147}$$

Using 6.145 we can find the expression for the desired amplitudes without and with stubs

$$\mathcal{A}^{0,2}_{0;1}(\varphi_{\rm o})=\omega_{\rm o}\left(\varphi_{\rm o},m^{0,2}_{0,0}\right)-\omega_{\rm o}\left(\varphi_{\rm o},m^{(0,1)}_{1,0}\left(\frac{b_0^+}{L_0^+}\bar{P}_0^+l^{(0,1)}_{0,0}\right)\right)-\omega_{\rm o}\left(\varphi_{\rm o},m^{(0,1)}_{0,2}\left(\boldsymbol{h}_0U_{\rm o}\right)\right), \tag{6.148}$$

and

$$A^{0,2}_{\lambda\,0;1}(\varphi_{\rm o})=\omega_{\rm o}\left(\varphi_{\rm o},\tilde{m}^{0,2}_{0,0}\right)-\omega_{\rm o}\left(\varphi_{\rm o},\tilde{m}^{(0,1)}_{1,0}\left(\frac{b_0^+}{L_0^+}\bar{P}_0^+\tilde{l}^{(0,1)}_{0,0}\right)\right)-\omega_{\rm o}\left(\varphi_{\rm o},\tilde{m}^{(0,1)}_{0,2}\left(\boldsymbol{h}_0U_{\rm o}\right)\right). \tag{6.149}$$

As before, let us proceed by extracting the vertices with stubs that appear in the

[5] Another simple (but classical) example we have studied is the closed string two-point amplitude on the disk, which was analyzed in detail in [126]. In this case the stub deformation creates an explicit fundamental vertex in the middle of moduli space, which was absent (with a specific choice of $SL(2,\mathbb{C})$ vertices) in [126].

previous expression from equation 6.126 which we list below

$$\tilde{l}^{(0,1)}_{0,0} = c^{(\mathrm{c})}_{\lambda_\mathrm{c}} l^{(0,1)}_{0,0}\,, \tag{6.150a}$$

$$\tilde{m}^{(0,1)}_{1,0}(\cdot) = c^{(\mathrm{c})}_{\lambda_\mathrm{c}} m^{(0,1)}_{1,0}(c^{(\mathrm{o})}_{\lambda_\mathrm{o}}\cdot)\,, \tag{6.150b}$$

$$\tilde{m}^{(0,1)}_{0,2}(\cdot,\cdot) = c^{(\mathrm{o})}_{\lambda_\mathrm{o}} m^{(0,1)}_{0,2}(c^{(\mathrm{o})}_{\lambda_\mathrm{o}}\cdot, c^{(\mathrm{o})}_{\lambda_\mathrm{o}}\cdot)\,, \tag{6.150c}$$

$$\tilde{m}^{(0,2)}_{0,0} = c^{(\mathrm{o})}_{\lambda_\mathrm{o}} m^{(0,2)}_{0,0} - c^{(\mathrm{o})}_{\lambda_\mathrm{o}} m^{(0,1)}_{1,0}\left(H_\mathrm{c} l^{(0,1)}_{0,0}\right) - c^{(\mathrm{o})}_{\lambda_\mathrm{o}} m^{(0,1)}_{0,2}\left(\boldsymbol{H}_\mathrm{o} U_\mathrm{o}\right)\,. \tag{6.150d}$$

Therefore, the stubbed amplitude becomes

$$\begin{aligned} A^{0,2}_{\lambda\,0;1}(\varphi_\mathrm{o}) &= \omega_\mathrm{o}\left(\varphi_\mathrm{o}, m^{(0,2)}_{0,0}\right) - \omega_\mathrm{o}\left(\varphi_\mathrm{o}, m^{(0,1)}_{1,0}\left(\left[H_\mathrm{c} + \frac{b^+_0}{L^+_0}\bar{P}^+_0 c^{(\mathrm{c})2}_{\lambda_\mathrm{c}}\right] l^{(0,1)}_{0,0}\right)\right) \\ &\quad - \omega_\mathrm{o}\left(\varphi_\mathrm{o}, m^{(0,1)}_{0,2}\left(\left[\boldsymbol{H}_\mathrm{o} + \boldsymbol{c}^{(\mathrm{o})}_{\lambda_\mathrm{o}}\boldsymbol{h}_0\right] U_\mathrm{o}\right)\right) \\ &= \omega_\mathrm{o}\left(\varphi_\mathrm{o}, m^{(0,2)}_{0,0}\right) - \omega_\mathrm{o}\left(\varphi_\mathrm{o}, m^{(0,1)}_{1,0}\left(\frac{b^+_0}{L^+_0}\bar{P}^+_0 l^{(0,1)}_{0,0}\right)\right) \\ &\quad - \omega_\mathrm{o}\left(\varphi_\mathrm{o}, m^{(0,1)}_{0,2}\left(\left[\boldsymbol{H}_\mathrm{o} + \boldsymbol{c}^{(\mathrm{o})}_{\lambda_\mathrm{o}}\boldsymbol{h}_0\right] U_\mathrm{o}\right)\right)\,. \end{aligned} \tag{6.151}$$

Thus, following the general picture we have previously discussed, the two amplitudes are equivalent if

$$\left[\boldsymbol{H}_\mathrm{o} + \boldsymbol{c}^{(\mathrm{o})}_{\lambda_\mathrm{o}}\boldsymbol{h}_0\right] U_\mathrm{o} = \boldsymbol{h}_0 U_\mathrm{o}\,, \tag{6.152}$$

which is simply demonstrated as

$$\begin{aligned} \left[\boldsymbol{H}_\mathrm{o} + \boldsymbol{c}^{(\mathrm{o})}_{\lambda_\mathrm{o}}\boldsymbol{h}_0\right] U_\mathrm{o} &= \frac{1}{2} o^i \wedge H_o o_i + \boldsymbol{c}^{(\mathrm{o})}_{\lambda_\mathrm{o}}\left(\frac{1}{2} o^i \wedge h_o o_i\right) \\ &= \frac{1}{2} o^i \wedge \left[H_\mathrm{o} + \frac{b_0}{L_0}\bar{P}_0 c^{(\mathrm{o})2}_{\lambda_\mathrm{o}}\right] o_i \\ &= \frac{1}{2} o^i \wedge \frac{b_0}{L_0}\bar{P}_0 o_i = \boldsymbol{h}_0 U_\mathrm{o}\,, \end{aligned} \tag{6.153}$$

where we inserted the completeness relation to move operators from the basis vector on the left to the basis vectors on the right.

It is also instructive to provide a geometric picture of the stub deformation in this example. This is possible by explicitly constructing the OC-SFT vertices that realize this amplitude. We have done this is in Appendix E, by explicitly solving the homotopy relation (6.147). Using these results, we can rewrite the stubbed and the un-stubbed

amplitude in terms of surface states

$$\begin{aligned}\mathcal{A}^{0,2}_{0;1}(\varphi_{\rm o}) = &-\frac{1}{2\pi i}\int_0^\infty ds\langle S^{(0,1)}_{1,1}|\left(1_{\mathcal{H}_{\rm o}}\otimes b_0^+ e^{-sL_0^+}\right)\left(\varphi_{\rm o}\otimes l^{(0,1)}_{0,0}\right)\\ &+\int_{t_{\rm min}}^{t_{\rm max}} dt\langle S^{(0,2)}_{0,1}(t)|B^t(t)\varphi_{\rm o}\\ &-\int_0^\infty ds\langle S^{(0,1)}_{0,3}|\left(1_{\mathcal{H}_{\rm o}}\otimes 1_{\mathcal{H}_{\rm o}}\otimes b_0 e^{-sL_0}\right)\left(o^i\otimes\varphi_{\rm o}\otimes o_i\right),\end{aligned} \tag{6.154}$$

and

$$\begin{aligned}\mathcal{A}^{0,2}_{\lambda\,0;1}(\varphi_{\rm o}) = &-\frac{1}{2\pi i}\langle S^{(0,1)}_{1,1}|\left(1_{\mathcal{H}_{\rm o}}\otimes h_0^+ c^{({\rm c})2}_{\lambda_{\rm c}}\right)\left(\varphi_{\rm o}\otimes l^{(0,1)}_{0,0}\right)\\ &-\frac{1}{2\pi i}\langle S^{(0,1)}_{1,1}|\left(1_{\mathcal{H}_{\rm o}}\otimes H_{\rm c}\right)\left(\varphi_{\rm o}\otimes l^{(0,1)}_{0,0}\right)\\ &+\int_{t_{\rm min}}^{t_{\rm max}} dt\langle S^{(0,2)}_{0,1}(t)|B^t(t)\varphi_{\rm o}\\ &-\langle S^{(0,1)}_{0,3}|\left(1_{\mathcal{H}_{\rm o}}\otimes 1_{\mathcal{H}_{\rm o}}\otimes h_0 c^{({\rm o})2}_{\lambda_{\rm o}}\right)\left(o^i\otimes\varphi_{\rm o}\otimes o_i\right)\\ &-\langle S^{(0,1)}_{0,3}|\left(1_{\mathcal{H}_{\rm o}}\otimes 1_{\mathcal{H}_{\rm o}}\otimes H_{\rm o}\right)\left(o^i\otimes\varphi_{\rm o}\otimes o_i\right).\end{aligned} \tag{6.155}$$

Now, using the standard Schwinger representation for the propagators, we get

$$\begin{aligned}\mathcal{A}^{0,2}_{\lambda\,0;1}(\varphi_{\rm o}) = &-\frac{1}{2\pi i}\int_{2\lambda_{\rm c}}^\infty ds\langle S^{(0,1)}_{1,1}|\left(1_{\mathcal{H}_{\rm o}}\otimes b_0^+ e^{-sL_0^+}\right)\left(\varphi_{\rm o}\otimes l^{(0,1)}_{0,0}\right)\\ &-\frac{1}{2\pi i}\int_0^{2\lambda_{\rm c}} ds\langle S^{(0,1)}_{1,1}|\left(1_{\mathcal{H}_{\rm o}}\otimes b_0^+ e^{-sL_0^+}\right)\left(\varphi_{\rm o}\otimes l^{(0,1)}_{0,0}\right)\\ &+\int_{t_{\rm min}}^{t_{\rm max}} dt\langle S^{(0,2)}_{0,1}(t)|B^t(t)\varphi_{\rm o}\\ &-\int_0^{2\lambda_{\rm o}} ds\langle S^{(0,1)}_{0,3}|\left(1_{\mathcal{H}_{\rm o}}\otimes 1_{\mathcal{H}_{\rm o}}\otimes b_0 e^{-sL_0}\right)\left(o^i\otimes\varphi_{\rm o}\otimes o_i\right)\\ &-\int_{2\lambda_{\rm o}}^\infty ds\langle S^{(0,1)}_{0,3}|\left(1_{\mathcal{H}_{\rm o}}\otimes 1_{\mathcal{H}_{\rm o}}\otimes b_0 e^{-sL_0}\right)\left(o^i\otimes\varphi_{\rm o}\otimes o_i\right),\end{aligned} \tag{6.156}$$

that is clearly equivalent to the amplitude without stubs. Furthermore, we can easily interpret the reorganization of the terms which build-up the amplitude after having introduced the stubs. In particular, we have a new fundamental vertex given by the sum of the second, third and fourth line of the above expression (6.156) and thus now the fundamental vertex involves integration over a bigger region of moduli space. Conversely, the portion of moduli space covered by Feynman diagrams is

reduced. Indeed the amplitude can be rewritten as

$$\begin{aligned}\mathcal{A}^{0,2}_{\lambda\,0;1}(\varphi_{\mathrm{o}}) = &-\int_{2\lambda_{\mathrm{o}}}^{\infty} ds\langle S^{(0,1)}_{0,3}|\left(1_{\mathcal{H}_{\mathrm{o}}}\otimes 1_{\mathcal{H}_{\mathrm{o}}}\otimes b_0 e^{-sL_0}\right)\left(o^i\otimes\varphi_{\mathrm{o}}\otimes o_i\right)\\ &+\int_{t_{\mathrm{c}}(e^{-\lambda_{\mathrm{o}}})}^{t_{\mathrm{o}}(e^{-\lambda_{\mathrm{o}}})} dt\langle S^{(0,2)}_{0,1}(t)|B^t(t)\varphi_{\mathrm{o}}\\ &-\frac{1}{2\pi i}\int_{2\lambda_{\mathrm{c}}}^{\infty} ds\langle S^{(0,1)}_{1,1}|\left(1_{\mathcal{H}_{\mathrm{o}}}\otimes b_0^+ e^{-sL_0^+}\right)\left(\varphi_{\mathrm{o}}\otimes l^{(0,1)}_{0,0}\right),\end{aligned} \tag{6.157}$$

where $t_{\mathrm{o}}(q)$ and $t_{\mathrm{c}}(q)$ are defined in (E.14) and (E.29). In particular the local coordinate map associated with the surface state $\langle\Sigma^{(0,2)}_{0,1}(t)|$ is defined as follows

$$\hat{F}_t(w) = \begin{cases} G_{\mathrm{c}}(w) & t < t_{\min} \\ F_t(w) & t_{\min} < t < t_{\max} \\ G_{\mathrm{o}}(w) & t > t_{\max} \end{cases} \tag{6.158}$$

where the maps G_{c}, G_{c}, F_t are defined in Appendix E.

CHAPTER 7

Bulk-induced D-brane deformations and the string coupling constant

7.1 SFT approach to Conformal Perturbation Theory

In this chapter, based on [34], we describe the short RG flow triggered by a slightly relevant bulk operator $\mathbb{V}$ of conformal weight $(1-y, 1-y)$, with y small and positive, in the presence of a D-brane system, with the aim of computing the shift in the g-function and in the string coupling constant κ induced by such a deformation. As anticipated in Section 4.1.5, the possibility of approaching Conformal Perturbation Theory (CPT) problems using SFT relies on background independence, which tells us that we can connect different 2d (B)CFTs via classical solutions [46, 47, 156, 157]. Following the analysis of [48, 67, 158], we can describe the RG flow of a 2d (B)CFT by considering the UV theory as the starting background and then finding the classical solution describing the IR theory.

The advantage of SFT over CPT is that it provides an automatically consistent way of regulating singularities arising from collisions of perturbing fields by recognizing such collisions as originating from degenerations of punctured Riemann surfaces (in this particular application, spheres and disks). In this picture, the collisions of the perturbing operators occur at a corner of moduli space that is always associated with an SFT Feynman diagram, with the SFT propagator taking care of the degenerating region in a way analogous to the $i\epsilon$ prescription of quantum field theory, as described in Sections 2.2.3 and 2.3.3. In this way, all contact divergences are systematically regulated.

In particular, once the UV (B)CFT is embedded into a world-sheet (B)CFT, which we will call $\mathrm{CFT}^{(0)}_{\mathrm{TOT}}$, with boundary conditions encoded in the boundary state $\|B\rangle\rangle$, the starting open-closed string background is characterized by the triplet

$$(\mathrm{CFT}^{(0)}_{\mathrm{TOT}}, \|\mathrm{B}\rangle\rangle, \kappa)\,. \tag{7.1}$$

Therefore, we can consistently construct an open-closed SFT action $S^{(0)}_{\mathrm{oc}}(\Phi, \Psi)$, which admits a classical solution (Φ^*, Ψ^*) that characterizes the IR background

$$(\mathrm{CFT}^{(*)}_{\mathrm{TOT}}, \|\mathrm{B}^{(*)}\rangle\rangle, \kappa^*)\,, \tag{7.2}$$

as schematically illustrated in the Figure 7.2

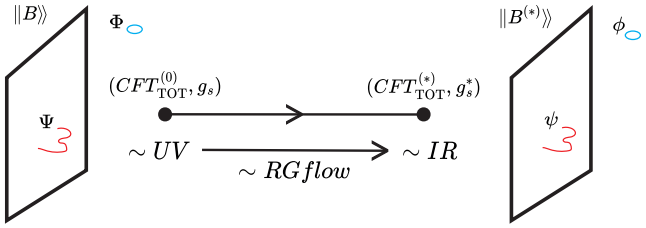


Figure 7.1: Schematic representation of the RG flow from the SFT perspective. On the left, the initial background, interpreted as the UV theory, and on the right, the final IR background.

Note that in this process, the string coupling constant can also change, an a priori unexpected result that was discovered in [126]. We will return to this point later. According to background independence, the open-closed SFT disk action constructed in the background (7.2), $S^{(*)}_{\rm disk}(\phi^{(*)}, \psi^{(*)})$, when restricted to the closed string field solution $\phi^{(*)} = 0$ (which corresponds to the solution $\Phi = \Phi^*$ in the starting background), is related to the original action expanded around the open string solution, $S^{(0)}_{\rm disk}(\Phi^*, \Psi^* + \psi)$, via field redefinition $\psi^{(*)} = \psi^{(*)}(\psi)$

$$S^{(0)}_{\rm disk}(\Phi^*, \Psi^* + \psi) = S^{(*)}_{\rm disk}(0, \psi^{(*)}(\psi)) \,. \tag{7.3}$$

Note that there are no contributions from the path integral measure [20], because, as we will explain in more detail later, the disk action evaluated on a closed string solution satisfies the classical BV master equation. The importance of (7.3) lies in the fact that it makes possible to compute observables for the IR theory using only UV data. In particular, by evaluating (7.3) at $\psi = 0$ and noting that $\psi^{(*)}(0) = 0$, we obtain

$$S^{(0)}_{\rm disk}(\Phi^*, \Psi^*) = S^{(*)}_{\rm disk}(0, 0) \,. \tag{7.4}$$

In the following, we will see that this equation allows us to compute the shift in the g-function and in the string coupling constant by evaluating the on-shell disk action in the UV theory.

Moreover, it is clear that to approach this problem, we must solve the non-polynomial equations of motion that describe the dynamics of the classical sector of

open-closed SFT. For this reason, it is crucial to consider short RG flows characterized by a small parameter, in this case, y, which can be used to solve the equations of motion perturbatively.

7.2 The on-shell disk action

In this section we will review the main results and observations of [126] relating to the disk part of the open-closed SFT action evaluated on a classical closed string-field configuration Φ^*, as well as on an open-string field configuration Ψ^* which can be found as a classical solution of open-string equations of motion on the closed-string background given by Φ^*.

7.2.1 Generalised Sen's conjecture

In the context of pure open-SFT, it has been conjectured by Ashoke Sen [13] that the on-shell action gives a direct measure of the difference between the g-functions (tensions) of the D-brane associated with the perturbative vacuum $\Psi = 0$ and the D-brane which is represented by a classical open-string field configuration Ψ^*. We will now discuss a generalisation of this statement to the situations where one is allowed to also change the closed-string background by turning on a non-trivial classical closed-string field configuration $\Phi^* \neq 0$.

Let us start by writing down the pure-closed classical SFT action (4.71)

$$S_{\text{sphere}}(\Phi) = Z_{\text{sphere}} + \frac{1}{\kappa^2}\sum_{k=1}^{\infty}\frac{1}{(k+1)!}\omega_{\text{c}}(\Phi, l_k(\Phi^{\wedge k}))\,, \tag{7.5}$$

where $l_k \coloneqq l_k^{(0)}$ and satisfy the L_∞ relations (4.58)

$$\sum_{k=1}^{r} l_k l_{r+1-k} = 0\,. \tag{7.6}$$

Following the Feynman rules dictated by the action (7.5), one can compute tree-level on-shell closed-string amplitudes around the chosen background. On the other hand,

varying this action with respect to Φ yields the equation of motion

$$\sum_{k=1}^{\infty} \frac{1}{k!} l_k(\Phi^{\wedge k}) = 0\,. \tag{7.7}$$

A solution Φ^* of (7.7) should then be thought of as representing a new consistent classical closed-string background described in terms of a new worldsheet theory CFT^* and, in general, a new value κ^* of the string coupling constant. By the token of the background-independence conjecture, on-shell closed-string amplitudes on this new background can be computed by expanding the action (7.5) in small fluctuations around Φ^* and applying a suitable field redefinition.

To describe the D-branes backreacton, encoded in the boundary state $\|B\rangle\!\rangle$, on a given closed-string background specified by a worldsheet CFT and string coupling constant κ we need the full open-closed SFT action (5.52) which contains the sphere part (7.5), as well as the disk part

$$\begin{aligned} S_{\mathrm{disk}}(\Phi,\Psi) = Z_{\mathrm{disk}} + \frac{1}{\kappa}\sum_{k=0}^{\infty}\frac{1}{(k+1)!}\omega_{\mathrm{c}}(\Phi, l_{k,0}(\Phi^{\wedge k}))+ \\ + \frac{1}{\kappa}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{1}{k!}\frac{1}{l+1}\omega_{\mathrm{o}}(\Psi, m_{k,l}(\Phi^{\wedge k};\Psi^{\otimes l}))\,, \end{aligned} \tag{7.8}$$

where we introduced the notation $l^{(0,1)}_{k,p} \coloneqq l_{k,l}$ and $m^{(0,1)}_{k,l} \coloneqq m_{k,l}$ to denote the multi-string products describing the disk open-closed SFT action which satisfy the SDHA (see (5.89) and (5.90))

$$\sum_{k=1}^{r} m_{k,s-1} l_{r+1-k} + \sum_{k=0}^{r}\sum_{n=1}^{s} m_{k,n} m_{r-k,s-n} = 0\,, \tag{7.9a}$$

$$\sum_{k=1}^{r} [l_k, l_{r-k,0}] + \sum_{k=1}^{r-1} l_{k-1,1} m_{r-k,0} = 0\,, \tag{7.9b}$$

Furthermore, the constant term in (7.8) is the disk partition function

$$Z_{\mathrm{disk}} = -\frac{1}{2\pi^2}\frac{g}{\kappa}\,, \tag{7.10}$$

where $g = \langle 0 \| B \rangle\!\rangle$ denotes the g-function of the consistent conformal boundary state $\|B\rangle\!\rangle$.

Restricting the closed string field Φ to satisfy the classical closed-string equation of motion (7.7), one can show that the action

$$S_{\Phi^*}(\Psi) = S_{\rm disk}(\Phi^*, \Psi)\,, \tag{7.11}$$

satisfies the classical open-string BV master equation

$$\big(S_{\Phi^*}, S_{\Phi^*}\big)_{\rm open} = 0\,. \tag{7.12}$$

This means that (7.11) represents a consistent *classical* action for the open-string field Ψ on a closed-string background configuration given by Ψ^*. In particular, one should emphasize that the action (7.11) is therefore well-suited for the description of classical dynamics of D-branes on a changing closed-string background in the *probe approximation*: while S_{Φ^*} will be able to encompass the response of a given D-brane system to the change in the closed-string background given by the classical closed-string field configuration Φ^*, it will not provide any information about the backreaction of the D-branes on the bulk. The vertices of the action $S_{\Phi^*}(\Psi)$ can be conveniently parametrized in terms of open-string valued products $\tilde{m}_l : (\mathcal{H}_{\rm o})^{\otimes l} \longrightarrow \mathcal{H}_{\rm o}$ defined as

$$\tilde{m}_l(\Psi^{\otimes l}) = \sum_{k=0}^{\infty} \frac{1}{k!} m_{k,l}((\Phi^*)^{\wedge k}; \Psi^{\otimes l})\,. \tag{7.13}$$

One can then write the action (7.11) in the form

$$S_{\Phi^*}(\Psi) = Z_{\rm disk} + \Lambda_{\rm c}(\Phi^*) + \frac{1}{\kappa} \sum_{l=0}^{\infty} \omega_{\rm o}\big(\Psi, \tilde{m}_l(\Psi^{\otimes l})\big)\,, \tag{7.14}$$

where we have isolated the contribution

$$\Lambda_{\rm c}(\Phi^*) = \frac{1}{\kappa} \sum_{k=0}^{\infty} \frac{1}{(k+1)!} \omega_{\rm c}(\Phi^*, l_{k,0}((\Phi^*)^{\wedge k})) \tag{7.15}$$

of purely-closed disk vertices into the *constant* part of S_{Φ^*}. The BV relation (7.12)

is then equivalent to stating that the products $\tilde{m}_l$ satisfy an A_∞ algebra. Crucially, this algebra is *weak*, because the product $\tilde{m}_0$ is generally non-zero and gives rise to a tadpole in (7.14). In particular, this means that $\Psi = 0$ no longer represents a critical open-string background. So as to remove the tadpole and hence restore the full open-closed criticality, one has to re-expand the action (7.14) in small fluctuations around an open-string field configuration Ψ^*, which solves the equation of motion

$$\sum_{l=0}^{\infty} \tilde{m}_l(\Psi^{\otimes l}) = 0\,. \tag{7.16}$$

The pair (Φ^*, Ψ^*) of string fields should then be thought of as interpolating between the original perturbative vacuum $\Phi = \Psi = 0$ and a new consistent classical open-closed background described in terms of a new bulk worldsheet theory CFT^*, new string coupling κ^* and the corresponding response $\|B^*\rangle\!\rangle$ of the D-brane system to the change of the bulk. In order to gain some mileage on *what* exactly the new background is, one has to formulate (and then, ideally, calculate) some observable quantities. To this end, we will be interested in the constant part of the action $S_{\Phi^*}(\Psi)$ expanded around Ψ^*, which is given simply by the *on-shell disk action*

$$S_{\Phi^*}(\Psi^*) = S_{\mathrm{disk}}(\Phi^*, \Psi^*) = Z_{\mathrm{disk}} + \Lambda(\Phi^*, \Psi^*)\,, \tag{7.17}$$

where we have denoted

$$\Lambda(\Phi^*, \Psi^*) = \Lambda_{\mathrm{c}}(\Phi^*) + \frac{1}{\kappa}\sum_{l=0}^{\infty} \omega_{\mathrm{o}}(\Psi^*, \tilde{m}_l((\Psi^*)^{\otimes l})) \tag{7.18a}$$

$$= \frac{1}{\kappa}\sum_{k=0}^{\infty} \frac{1}{(k+1)!}\omega_{\mathrm{c}}(\Phi^*, l_{k,0}((\Phi^*)^{\wedge k}))+ \\ + \frac{1}{\kappa}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{1}{k!}\frac{1}{l+1}\omega_{\mathrm{o}}(\Psi^*, m_{k,l}((\Phi^*)^{\wedge k}; (\Psi^*)^{\otimes l}))\,. \tag{7.18b}$$

Strong evidence for the observability of the on-shell disk action (7.17) is provided by the fact that the quantity $\Lambda(\Phi^*, \Psi^*)$ can be shown [126] to be invariant not only with respect to the (weak) A_∞ gauge-variation of the open-string solution Ψ^*, but also

with respect to the L_∞ gauge-variation of the closed string solution Φ^*. Following the spirit of background independence (7.3), one is then led to conjecture that the disk action $S_{\rm disk}(\Phi^*, \Psi^*)$ should be identified as the disk partition function $Z^*_{\rm disk}$ of the new open-closed background $(\mathrm{CFT}^*, \kappa^*, \|B^*\rangle\!\rangle)$. Expressing the original, as well as the new disk partition function in terms of the g-function and the string coupling constant, we finally arrive at the *generalised Sen's conjecture*

$$-\frac{1}{2\pi^2}\left(\frac{g^*}{\kappa^*}-\frac{g}{\kappa}\right)=\Lambda(\Phi^*,\Psi^*)\,. \tag{7.19}$$

In the special class of cases when the closed-string background does *not* undergo any change (that is $\Phi^*=0$), one has to put $\kappa^*=\kappa$ while, so that the l.h.s. of (7.19) is directly proportional to the change in the g-function of the D-brane system. Since, at the same time, the r.h.s. of (7.19) then reduces to the on-shell OSFT action, one recovers the classical Sen's conjecture. In the more general case when $\Phi^*\neq 0$, the conjecture (7.19) not only provides a way of tracking the change in the g-function of probe D-branes as the bulk is deformed, but it also represents an indirect method of measuring any potential changes in the string coupling constant: as the new value κ^* should depend neither on the choice of the initial boundary state $\|B\rangle\!\rangle$, nor on the open-string solution Ψ^*, one should be able to isolate the change in the string coupling constant by analyzing the r.h.s. of (7.19) for a number of distinct open-string backgrounds and assuming that one *knows* the expected change in the g-function for some of these. This demonstrates how coupling the bulk closed-ST to probe D-branes may be beneficial for providing observables which help with keeping track of pure closed-string dynamics.

7.2.2 Bulk marginal deformations

The formal statement of the generalised Sen's conjecture can be illustrated by taking the closed-string solution Φ^* to represent a marginal deformation.

Perturbative solutions $\Phi^*(\mu)$ and $\Psi^*(\mu)$

We will consider an exactly marginal deformation of the closed-string background which, at the leading order of the (continuous) deformation parameter μ, is given by an on-shell ghost-number 2 operator V. We will mostly focus on the cases when $V = c\bar{c}\mathbb{V}$, where $\mathbb{V}$ is a matter primary with dimensions $(h, \bar{h}) = (1, 1)$. The exact marginality of the deformation needs to be guaranteed order by order in μ [126]. For instance, at second order, this entails requiring that

$$P_0^+ l_2(\mu V, \mu V) = 0 \tag{7.20}$$

up to $Q_{\rm c}$-exact terms. The full unobstructed solution $\Phi^*(\mu)$ can then be expanded as

$$\Phi^*(\mu) = \mu V - \frac{1}{2!}\frac{b_0^+}{L_0^+}\bar{P}_0^+ l_2(\mu V, \mu V) + \mathcal{O}(\mu^3)\,. \tag{7.21}$$

Furthermore, let us assume that given the exactly marginal deformation $\Phi^*(\mu)$, one can find a corresponding solution $\Psi^*(\mu)$ to the open-string equation of motion (7.16) which remains perturbatively close to the original background at $\mu = 0$. This again necessitates clearing possible cohomological obstructions order by order in μ, starting with

$$P_0 m_{1,0}(\mu V) = 0 \tag{7.22}$$

up to $Q_{\rm o}$-exact terms. Failure of these would signify an inability of the D-brane system at hand to adapt to the marginal deformation which we are turning on in the bulk. Instead, one would then have to search for a non-perturbative solution to (7.16) in order to restore criticality of the background. In the absence of any obstructions, the open-string solution $\Psi^*(\mu)$ would be expanded in μ as

$$\Psi^*(\mu) = -\frac{b_0}{L_0}\bar{P}_0 m_{1,0}(\mu V) + \mathcal{O}(\mu^2)\,. \tag{7.23}$$

The deformed disk partition function

Substituting the solutions (7.21) and (7.23) into (7.18b) yields a μ-expansion of the change in the disk partition function. Combining this with the statement (7.19) of the generalised Sen's conjecture, one obtains

$$\begin{aligned}\frac{1}{2\pi^2}\Big[1-\frac{g^*(\mu)}{g}\Big(\frac{\kappa^*(\mu)}{\kappa}\Big)^{-1}\Big] =\\ =\frac{1}{g}\omega_{\mathrm{c}}(\mu V,l_{0,0})+\frac{1}{2}\Big[\frac{1}{g}\omega_{\mathrm{c}}\Big(l_{0,0},\frac{b_0^+}{L_0^+}\bar{P}_0^+ l_2(\mu V,\mu V)\Big)+\\ +\frac{1}{g}\omega_{\mathrm{c}}(\mu V,l_{1,0}(\mu V))+\frac{1}{g}\omega_{\mathrm{o}}\Big(m_{1,0}(\mu V),\frac{b_0}{L_0}\bar{P}_0 m_{1,0}(\mu V)\Big)\Big]+\mathcal{O}(\mu^3)\,.\end{aligned} \tag{7.24}$$

To evaluate the r.h.s. of (7.24), it is practical to make concrete assumptions about the form of the SFT vertices. See [126] for a particular choice of vertices which are based on $SL(2,\mathbb{C})$ maps and which will be also put to use in later sections of this chapter when discussing relevant deformations. One should nonetheless keep in mind that as the disk action is an observable quantity, the endpoint of calculating the r.h.s. of (7.24) should be manifestly independent of any off-shell SFT data (such as the local coordinate maps around punctures and their parameters). This serves as a useful consistency check. Following the steps outlined in [126], keeping in mind that here we are working using the normalization (2.13), one can then arrive at the expression

$$\begin{aligned}\frac{g^*(\mu)}{g}\Big(\frac{\kappa^*(\mu)}{\kappa}\Big)^{-1} =\\ 1+2\,\frac{1}{g}\big\langle\mu\mathbb{V}(i,\bar{i})\big\rangle_{\mathrm{UHP}}+\int\limits_{\substack{0\\ \text{reg.}}}^{1} ds\,(s^2-1)\,\frac{1}{g}\big\langle\mu\mathbb{V}(i,\bar{i})\,\mu\mathbb{V}(is,\bar{is})\big\rangle_{\mathrm{UHP}}+\mathcal{O}(\mu^3)\,,\end{aligned} \tag{7.25}$$

for the evolution of the worldhseet disk partition function with μ. The integral in the second term runs from $s=0$ (open-string degeneration) to $s=1$ (closed-string degeneration) and needs to be regulated at both endpoints so as to avoid divergences coming from the propagation of the open- and closed-string tachyons. In particular,

we define [126, 159]

$$\int\limits_{\substack{0\\ \text{reg.}}}^{1} ds \;=\; \lim_{\substack{\epsilon_{\text{c}}\to 0\\ \epsilon_{\text{o}}\to 0}} \Big[\int_0^a ds\, s^{\epsilon_{\text{o}}} + \int_a^1 ds \left(\frac{1-\sqrt{s}}{1+\sqrt{s}}\right)^{\epsilon_{\text{c}}} \Big]\,, \tag{7.26}$$

where the parameter $a \in (0,1)$ serves to separate the two degeneration regions (and can be related to the open-string stub parameter λ_{o} as $a = 1/\lambda_{\text{o}}^2$). The result of the integration is however always independent of a, as can be readily checked by differentiating with respect to it. Also note that the function appearing in the second term of (7.26) regulating the closed-string collision can be naturally interpreted in terms of the radial coordinate y on the disk, so that near the closed-string degeneration ($y = 0$), the tachyon-divergence is regulated with $y^{\epsilon_{\text{c}}}$, as appropriate. Finally, note that the integral is not endangered by any divergences due to propagation of massless modes as these are absent precisely by virtue of the conditions (7.20) (closed-string channel) and (7.22) (open-string channel).

7.2.3 Example: navigating the Narain moduli space

As an illustration, we will now consider computing the on-shell disk action in the case when the on-shell closed-string field Φ^* represents a finite deformation of the Narain modulus $E_{\mu\nu} = G_{\mu\nu} + B_{\mu\nu}$ of a toroidal compactification of d free bosons. We will assume that our probe D-brane is represented by an elementary matter boundary state which satisfies the Ishibashi conditions

$$\big[(\alpha^{\text{L}}_n)_\mu + \Omega_\mu{}^\nu (\alpha^{\text{R}}_{-n})_\nu\big] \|B\rangle\!\rangle = 0\,, \tag{7.27}$$

where the gluing automorphism matrix Ω is constrained to satisfy $\Omega G \Omega^T = G$. Such boundary state is built up from $U(1)^d$ Ishibashi states, which are labelled by the left- and right-moving momenta

$$k^{\text{L}} = k + Ew\,, \tag{7.28a}$$

$$k^{\text{R}} = k - E^T w\,, \tag{7.28b}$$

where the momentum k_μ and winding w^ν are both $\mathbb{Z}^d$-valued. Specializing the relation (7.27) for the zero-modes $n = 0$, one learns that k and w need to be restricted so that

$$(1+\Omega)k + (E - \Omega E^T)w = 0\,. \tag{7.29}$$

Requiring that (7.29) is solved by a d-dimensional sublattice of $\mathbb{Z}^d \oplus \mathbb{Z}^d$ imposes further restrictions on Ω, which in fact make the set of allowed gluing matrices Ω discrete.

BCFT expectation

Let us first discuss what change in the g-function of a Narain boundary state one should expect (based on pure BCFT considerations) as we deform the Narain modulus by sending $E \to E + \epsilon_\sigma$. In conformal perturbation theory, this change can be implemented by the marginal deformation

$$S(X;E) \longrightarrow S(X;E) + \frac{1}{4\pi}\int d^2z\, \mathbb{V}^{\epsilon_\sigma}(z,\bar z)\,, \tag{7.30}$$

where $S(X;E)$ is the action of d free bosons X^μ compactified on lattice with Narain modulus $E_{\mu\nu}$ and

$$\mathbb{V}^{\epsilon_\sigma}(z,\bar z) = (\epsilon_\sigma)_{\mu\nu}\,\partial X^\mu(z)\,\bar\partial X^\nu(\bar z) \tag{7.31}$$

is the deforming operator. To first order in ϵ_σ, a one-loop calculation in conformal perturbation theory reveals that the g function changes as

$$\frac{\delta^{(1)}g}{g} = \frac{1}{4}\,\mathrm{Tr}\left[\Omega^T G^{-1}\epsilon_\sigma\right]. \tag{7.32}$$

Furthermore, varying the relation (7.29) with respect to E, it is straightforward to realise that

$$\delta^{(1)}\Omega = \frac{1}{2}(\epsilon_\sigma - \Omega\,\epsilon_\sigma^T)G^{-1}(1+\Omega)\,. \tag{7.33}$$

Given the results (7.32) and (7.33), one can straightforwardly find the second variation $\delta^{(2)}g$ of g with respect to E. Combining this with the first variation (7.32), we learn that the finite change Δg of the g-function can be expanded to second order in ϵ_σ as

$$1+\frac{\Delta g}{g}=1+\frac{\delta^{(1)}g}{g}+\frac{1}{2}\frac{\delta^{(2)}g}{g}+\mathcal{O}(\epsilon_\sigma^3) \tag{7.34a}$$

$$\begin{aligned}&=1+\frac{1}{4}\operatorname{Tr}\left[\Omega^T G^{-1}\epsilon_\sigma\right]+\frac{1}{32}\operatorname{Tr}\left[\Omega^T G^{-1}\epsilon_\sigma\right]^2-\frac{1}{16}\operatorname{Tr}\left[(\Omega^T G^{-1}\epsilon_\sigma)^2\right]+\\&\quad-\frac{1}{8}\operatorname{Tr}\left[\Omega^T G^{-1}\epsilon_\sigma G^{-1}\epsilon_\sigma\right]+\frac{1}{16}\operatorname{Tr}\left[G^{-1}\epsilon_\sigma^T G^{-1}\epsilon_\sigma\right]+\mathcal{O}(\epsilon_\sigma^3)\,.\end{aligned} \tag{7.34b}$$

Notice that all terms in (7.34b) depend on the gluing matrix Ω (and thus on the boundary condition) except for the last one, which is proportional to the two-point function coefficient $C^{\epsilon_\sigma}_{\mathbb{V}\mathbb{V}\mathbb{1}}$ of the deforming operator $\mathbb{V}^{\epsilon_\sigma}(z,\bar z)$ on the sphere. Indeed, substituting from (7.31), one has

$$\left\langle \mathbb{V}^{\epsilon_\sigma}(z,\bar z)\,\mathbb{V}^{\epsilon_\sigma}(w,\bar w)\right\rangle_{\mathbb{C}}=\frac{C^{\epsilon_\sigma}_{\mathbb{V}\mathbb{V}\mathbb{1}}}{|z-w|^4}\,, \tag{7.35}$$

where

$$C^{\epsilon_\sigma}_{\mathbb{V}\mathbb{V}\mathbb{1}}=\frac{1}{4}\operatorname{Tr}\left[G^{-1}\epsilon_\sigma^T G^{-1}\epsilon_\sigma\right]. \tag{7.36}$$

SFT calculation

We will now implement the marginal deformation which sends E to $E+\epsilon_\sigma$ as a classical solution (of the form (7.21)) in closed SFT. To this end, we put

$$\mu\mathbb{V}(z,\bar z)\equiv\mathbb{V}^{\epsilon}(z,\bar z)\equiv\epsilon_{\mu\nu}\,\partial X^\mu(z)\,\bar\partial X^\nu(\bar z)\,. \tag{7.37}$$

It is not difficult to check that for such a marginal state, the condition (7.20) holds, which in turn guarantees exact marginality of the deformation to second order in ϵ. First, one can compute the change in conformal weights of the bulk momentum plane waves induced by the solution (7.21). This calculation proceeds by analyzing the cohomology of the BRST operator shifted around the classical solution. Comparing

the result with the shift in conformal weights which is expected upon deforming the Narain modulus E by sending $E \to E + \epsilon_\sigma$, we can fix the relation between the SFT deformation parameter ϵ and the CFT deformation ϵ^σ to be [126]

$$\epsilon = \epsilon_\sigma - \frac{1}{2}\epsilon_\sigma G^{-1}\epsilon_\sigma + \mathcal{O}(\epsilon_\sigma^3)\,. \tag{7.38}$$

Second, considering any Dp-brane described by a gluing matrix Ω, we would like to evaluate the disk action (7.25) of the open-string solution (7.23) computed on a classical closed-string background given by the solution (7.21). This is a healthy solution, as one can readily verify that the condition (7.22) holds for any Ω. It describes how the D-brane adapts as we perturb the Narain modulus of the bulk CFT. Evaluating the relevant UHP correlators, we obtain

$$\frac{1}{g}\big\langle \mathbb{V}^\epsilon(i,\bar{i})\big\rangle_{\text{UHP}} = \frac{1}{8}\,\text{Tr}[\epsilon\Omega^T G^{-1}]\,, \tag{7.39a}$$

$$\frac{1}{g}\big\langle \mathbb{V}^\epsilon(i,\bar{i})\,\mathbb{V}^\epsilon(is,\bar{is})\big\rangle_{\text{UHP}} = \frac{1}{4}\bigg(\frac{\text{Tr}[\epsilon\Omega^T G^{-1}]^2}{16s^2} + \frac{\text{Tr}[\epsilon G^{-1}\epsilon^T G^{-1}]}{(1-s)^4} + \frac{\text{Tr}[\epsilon\Omega^T G^{-1}\epsilon\Omega^T G^{-1}]}{(1+s)^4}\bigg)\,. \tag{7.39b}$$

Using the prescription (7.26), we can also evaluate

$$\int\limits_{\substack{0\\ \text{reg.}}}^{1} ds\,\frac{s^2-1}{16s^2} = +\frac{1}{8}\,, \tag{7.40a}$$

$$\int\limits_{\substack{0\\ \text{reg.}}}^{1} ds\,\frac{s+1}{(s-1)^3} = +\frac{1}{8}\,, \tag{7.40b}$$

$$\int\limits_{\substack{0\\ \text{reg.}}}^{1} ds\,\frac{s-1}{(s+1)^3} = -\frac{1}{4}\,. \tag{7.40c}$$

Finally, substituting the results (7.38), (7.39) and (7.40) into the expression (7.25) for the disk action, we obtain

$$\begin{aligned}
&\left(1+\frac{\Delta g}{g}\right)\left(1+\frac{\Delta\kappa}{\kappa}\right)^{-1} = \\
&= \left(1+\frac{1}{4}\,\mathrm{Tr}[\epsilon_\sigma\Omega^T G^{-1}]+\frac{1}{32}\,\mathrm{Tr}[\epsilon_\sigma\Omega^T G^{-1}]^2-\frac{1}{16}\,\mathrm{Tr}[(\epsilon_\sigma\Omega^T G^{-1})^2]+\right. \\
&\quad \left.-\frac{1}{8}\,\mathrm{Tr}[\epsilon_\sigma G^{-1}\epsilon_\sigma\Omega^T G^{-1}]+\frac{1}{16}\,\mathrm{Tr}[\epsilon_\sigma G^{-1}\epsilon_\sigma^T G^{-1}]\right)\left(1+\frac{1}{32}\,\mathrm{Tr}[\epsilon_\sigma G^{-1}\epsilon_\sigma^T G^{-1}]\right)^{-1} \\
&\quad +\mathcal{O}(\epsilon_\sigma^3)\,. \qquad (7.41)
\end{aligned}$$

Quite non-trivially, on the r.h.s. of (7.41) we recover the expected response (7.34b) of the g-function of the Dp-brane boundary state to the bulk deformation $E \to E+\varepsilon_\sigma$. This is multiplied by a universal factor which is independent of Ω. Attributing this factor to a change $\Delta\kappa$ in the string coupling constant κ which is induced by the closed-string solution (7.21), we can write

$$1+\frac{\Delta\kappa}{\kappa} = 1+\frac{1}{8}C^{\epsilon_\sigma}_{\mathbb{V}\mathbb{V}\mathbb{1}}+\mathcal{O}(\epsilon_\sigma^3)\,, \qquad (7.42)$$

where the two-point function coefficient $C^{\epsilon_\sigma}_{\mathbb{V}\mathbb{V}\mathbb{1}}$ was computed in (7.36).

7.3 Short bulk RG flows with boundaries

In this section, we will apply the open-closed SFT framework to investigate how relevant bulk CFT perturbations affect boundary states. In particular, we will construct the classical open SFT solutions that describe D-branes after the closed-string background has undergone a 'short' RG flow. The possibility of studying such closed-string perturbations via SFT was considered long ago by Mukherji and Sen [47] and more recently in [67] and [48]. Here we would like to extend this analysis to the presence of world-sheet boundaries representing D-branes.

We will begin our exposition by describing the pure 2d CFT setup and briefly reviewing the results of [67]. We will then proceed with an explicit construction of the open SFT solutions.

7.3.1 Conformal perturbation theory description

Let us start with an overview of conformal perturbation theory (CPT) for a theory CFT_0 with boundary described by a boundary state $\|B_0\rangle\!\rangle$. In general, bulk deformations of 2d CFTs with boundaries can induce boundary RG flows. To write down the RG equations, we consider a generic bulk and boundary deformation given in terms of bulk primary fields $\mathbb{O}_k$ with scaling dimensions Δ_k and boundary primary fields $\mathbb{o}_l$ with scaling dimensions h_l as

$$S_{\mathrm{CFT}_0} \to S_{\mathrm{CFT}_0} + \sum_i \lambda_i \epsilon^{\Delta_i - 2} \int d^2 z\, \mathbb{O}_i(z,\bar{z}) + \sum_j \mu_j \epsilon^{h_j - 1} \int dx\, \mathbb{o}_i(x)\,, \tag{7.43}$$

where λ_i and μ_j are dimensionless coupling constants, and ϵ is the length scale used as a natural cutoff to treat the contact divergences arising in correlators of the perturbed theory. Following the usual CPT approach (see, for example, [160]), we obtain the following RG equations

$$\frac{d\lambda_k}{d\log(\epsilon)} = (2-\Delta_k)\lambda_k + \sum_{ij} \pi C_{ijk} \lambda_i \lambda_j + \mathcal{O}(\lambda^3)\,, \tag{7.44a}$$

$$\frac{d\mu_l}{d\log(\epsilon)} = (1-h_l)\mu_l + \sum_i \frac{1}{2}\frac{B_{il}}{g}\lambda_i + \mathcal{O}(\lambda\mu) + \mathcal{O}(\lambda^2) + \mathcal{O}(\mu^2)\,, \tag{7.44b}$$

where g is the g-function of the initial BCFT and C_{ijk} and B_{il}/g are the OPE coefficients

$$\mathbb{O}_i(z,\bar{z})\,\mathbb{O}_j(w,\bar{w}) = \sum_k \frac{C_{ijk}}{|z-w|^{\Delta_i+\Delta_j-\Delta_k}} \mathbb{O}_k(w,\bar{w}) + \dots\,, \tag{7.45a}$$

$$\mathbb{O}_i(x+is, x-is) = \sum_l \frac{1}{(2s)^{\Delta_i - h_l}} \frac{B_{il}}{g} \mathbb{o}_l(x) + \dots\,. \tag{7.45b}$$

In particular, we will consider a perturbation triggered by a nearly marginal primary bulk field $\mathbb{V}(z,\bar{z})$ which has conformal dimensions $(1-y, 1-y)$ with y small and positive. We will also assume that the corresponding Virasoro representation satisfies the fusion rule

$$\mathbb{V} \times \mathbb{V} = 1 + \mathbb{V} + \dots\,, \tag{7.46}$$

where the dots represent possible irrelevant contributions. The bulk and the bulk-boundary OPEs of $\mathbb{V}$ can then be written as

$$\mathbb{V}(z,\bar{z})\mathbb{V}(w,\bar{w}) \overset{z\to w}{\sim} \frac{1}{|z-w|^{4(1-y)}} + \frac{1}{|z-w|^{2(1-y)}} C_{\mathbb{V}\mathbb{V}\mathbb{V}}\mathbb{V}(w,\bar{w}) + \text{reg}\,, \tag{7.47}$$

$$\mathbb{V}(x+is,x-is) \overset{s\to 0}{\sim} \frac{1}{(2s)^{2(1-y)}}\frac{B_{\mathbb{V}\mathbb{1}}}{g} + \frac{1}{(2s)^{1-y}}\frac{B_{\mathbb{V}\mathbb{v}}}{g}\mathbb{v}(x) + \text{reg}\,, \tag{7.48}$$

where we normalised the nearly marginal field as $\langle\mathbb{V}|\mathbb{V}\rangle = C_{\mathbb{V}\mathbb{V}\mathbb{1}} = 1$. We have also introduced the boundary field $\mathbb{v}$ of dimension $h = 1-y$ transforming in the same Virasoro representation as the deforming bulk field $\mathbb{V}$. Note that sometimes, it may happen that this representation is absent from the spectrum of the boundary fields as determined by the Cardy condition (annulus crossing equation). In such cases, the OPE coefficient $B_{\mathbb{V}\mathbb{v}}/g$ has to vanish.

Under these assumptions the RG equations (7.44a) and (7.44b) can be rewritten as

$$\frac{dt}{d\log(\epsilon)} = 2yt - C_{\mathbb{V}\mathbb{V}\mathbb{V}}t^2 + \mathcal{O}(t^3)\,, \tag{7.49a}$$

$$\frac{d\tau}{d\log(\epsilon)} = y\tau + \frac{B_{\mathbb{V}\mathbb{v}}}{g}\frac{t}{2} + \mathcal{O}(t\tau) + \mathcal{O}(t^2) + \mathcal{O}(\tau^2)\,, \tag{7.49b}$$

where $\tau \coloneqq -\mu_{\mathbb{v}}\pi$. In turn, from (7.49a) and (7.49b) we can generally read off the fixed points at leading order in y as

$$t^*(y) = \frac{2y}{C_{\mathbb{V}\mathbb{V}\mathbb{V}}} + \mathcal{O}(y^2)\,, \tag{7.50a}$$

$$\tau^*(y) = -\frac{1}{C_{\mathbb{V}\mathbb{V}\mathbb{V}}}\frac{B_{\mathbb{V}\mathbb{v}}}{g} + \mathcal{O}(y)\,. \tag{7.50b}$$

Looking at the first line, we can notice that y adjusts the length of the bulk RG flow by making it short in the $y \to 0$ limit, in the sense that we have a perturbative bulk RG fixed point $t^*(y)$. On the other hand, in the case with boundaries, we have a perturbative fixed point (with $\tau^*(0) = 0$) only if

$$\left.\frac{B_{\mathbb{V}\mathbb{v}}}{g}\right|_{y=0} = 0\,. \tag{7.51}$$

Therefore, to have a short bulk RG flow with boundaries, we must work under the assumption $B_{\mathbb{V}_\mathbb{v}}/g \overset{y\to 0}{\sim} \mathcal{O}(y)$. This includes the cases when the bulk-broundary OPE coefficient $B_{\mathbb{V}_\mathbb{v}}/g$ vanishes exactly. In such situations, one has a choice of not triggering the boundary RG flow at all. Finally, in the cases when the field $\mathbb{v}$ is not present in the boundary spectrum, the boundary RG equation (7.49b) becomes vacuous and there is no boundary RG flow to be induced by the bulk deformation triggered by $\mathbb{V}$.

As we are now going to discuss, from the SFT perspective, the bulk-boundary perturbation triggered by $\mathbb{V}$ can be described by 1. starting with a consistent open-closed world-sheet background incorporating the initial matter theory CFT_0 and boundary state $\|B_0\rangle\rangle$, 2. solving perturbatively the closed-string equation of motion for a classical solution $\Phi^*(y)$, 3. shifting the open-closed background by this solution and then, finally, 4. solving the tadpole-sourced open-string equation of motion to find the classical solution $\Psi^*(y)$ which captures the changes in the D-brane system along the RG flow.

7.3.2 Short bulk RG flows from SFT

Starting with the bulk RG flows, we embed the matter theory CFT_0 with central charge c, into a worldsheet CFT given by the tensor product of three sectors

$$\mathrm{CFT}^{(0)}_{\mathrm{tot}} = \mathrm{CFT}_0 \otimes \mathrm{CFT}_{\mathrm{aux}} \otimes \mathrm{CFT}_{\mathrm{gh}}\,, \tag{7.52}$$

where $\mathrm{CFT}_{\mathrm{gh}}$ is the usual bc-ghost system with central charge $c_{\mathrm{gh}} = -26$ and $\mathrm{CFT}_{\mathrm{aux}}$ is an auxiliary CFT with $c_{\mathrm{aux}} = 26 - c$, which ensures that the theory is critical,

$$c_{\mathrm{tot}} = c + c_{\mathrm{aux}} + c_{\mathrm{gh}} = 0\,. \tag{7.53}$$

Now, our aim is to perturbatively solve the classical closed-string equation of motion (7.7). Doing so, one should find an explicit expression for the solution $\Phi^*(y)$ which describes the critical closed-string background corresponding to fixed point (7.50a) of the short RG flow. Since our eventual goal will be to evaluate the the on-shell disk action (7.17) up to quadratic order, we will only be interested in finding $\Phi^*(y)$

up to second order in y. In CPT, this would correspond to a rather hard two-loop computation. Also recall that as we are neglecting backreaction of the D-branes on the bulk, the closed-string equation of motion is unaffected by the presence of the boundaries, exactly in the spirit of the bulk RG equations (7.44a).

Closed-string equation of motion and obstructions

Following the strategy outlined in [47] and [67], we introduce a projector P that projects onto the space of eigenstates of the operators L_0 and $\bar{L}_0$ which have eigenvalues $(0,0)$ and $(-y,-y)$. It is not hard to see that P commutes with the BRST charge, that is $[P, Q_\mathrm{c}] = 0$. Afterwards, we split the closed string field through the action of P and $\bar{P} = 1 - P$ by defining

$$\Phi = P\Phi + \bar{P}\Phi \coloneqq W + R\,. \tag{7.54}$$

For the "tachyon" component W of the string field, we then consider the ansatz

$$W = t(y)T\,, \tag{7.55}$$

where $T = c\bar{c}\mathbb{V}$ is the nearly on-shell tachyon and $t(y)$ is a function that will be fixed by imposing the closed-string equation of motion. In particular, this function can be expanded as

$$t(y) = t_1 y + t_2 y^2 + \mathcal{O}(y^3)\,. \tag{7.56}$$

Our aim will be to explicitly find t_1 and t_2 in terms of CFT data.

First, let us project the full equation of motion

$$\mathrm{EOM}_\mathrm{c}[\Phi] = Q_\mathrm{c}\Phi + \frac{1}{2}l_2(\Phi,\Phi) + \frac{1}{3!}l_3(\Phi,\Phi,\Phi) + \mathcal{O}(y^4) \tag{7.57}$$

into the two (orthogonal) subspaces given as the image of P and $\bar{P}$, namely

$$P\,\mathrm{EOM}_\mathrm{c}[\Phi] = Q_\mathrm{c}W + \frac{1}{2}Pl_2(\Phi,\Phi) + \frac{1}{3!}Pl_3(\Phi,\Phi,\Phi) + \mathcal{O}(y^4)\,, \tag{7.58a}$$

$$\bar{P}\,\mathrm{EOM}_\mathrm{c}[\Phi] = Q_\mathrm{c}R + \frac{1}{2}\bar{P}l_2(\Phi,\Phi) + \frac{1}{3!}\bar{P}l_3(\Phi,\Phi,\Phi) + \mathcal{O}(y^4)\,. \tag{7.58b}$$

Since it is possible to invert $Q_{\rm c}$ on the image of $\bar{P}$, this splitting enables us to write down a recursive solution for R in terms of W. At leading order, this reads (in the Siegel gauge $b_0^+ = 0$)

$$R(W) = -\frac{1}{2}\frac{b_0^+}{L_0^+}\bar{P}l_2(W,W) + \mathcal{O}(y^3)\,. \tag{7.59}$$

Plugging (7.59) back into (7.58a), the component of the full equation of motion which lies in the image of P can be recast purely in terms of W as

$$\begin{aligned} P\,\mathrm{EOM}_{\rm c}[\Phi] = Q_{\rm c}W + \frac{1}{2}Pl_2(W,W) - \frac{1}{2}Pl_2\left(W, \frac{b_0^+}{L_0^+}\bar{P}l_2(W,W)\right) + \frac{1}{3!}Pl_3(W,W,W) \\ + \mathcal{O}(y^4)\,. \end{aligned} \tag{7.60}$$

Looking in detail at the r.h.s. of (7.60), we can notice that there is an obstruction in solving the equation of motion. Indeed, the last term turns out to give rise to terms proportional to the anti-ghost dilaton, $c_0^+\left(c\partial^2 c - \bar{c}\bar{\partial}^2\bar{c}\right)$ at cubic order in y, which cannot be balanced by any of the remaining terms on the r.h.s. of (7.60). The meaning of this obstruction is related to the fact that the deformation by $\mathbb{V}$ gives rise to an $\mathcal{O}(y^3)$ change in the central charge of the matter CFT. With no further provisions, this would have resulted into a non-critical total world-sheet CFT.

A solution to this problem was suggested already in [47] and very recently concretised in [48]. The key point is to modify the auxiliary sector of the initial worldsheet CFT by adding a linear dilaton sector $\mathrm{CFT}_{\mathrm{D}_\beta}$. This has the central charge $c_\beta = 1 - 12\beta^2$ where β denotes the background charge. Hence, the total worldsheet theory $\mathrm{CFT}^{(0)}_{\rm tot}$ now factorizes as

$$\mathrm{CFT}^{(0)}_{\rm tot} = \mathrm{CFT}_0 \otimes \mathrm{CFT}_{\mathrm{D}_\beta} \otimes \mathrm{CFT}_{\rm aux} \otimes \mathrm{CFT}_{\rm gh}\,. \tag{7.61}$$

Denoting by Y the scalar field of the linear dilaton CFT, the anti-ghost dilaton can then be BRST-trivialized by the field

$$\Theta = \frac{1}{2}\left(c\partial^2 c - \bar{c}\bar{\partial}^2\bar{c}\right)Y + c_0^+(c\partial Y - \bar{c}\bar{\partial}Y)\,. \tag{7.62}$$

Indeed, we have

$$Q_{\rm c}\Theta = -i\sqrt{2\alpha'}\beta c_0^+ \left(c\partial^2 c - \bar{c}\bar{\partial}^2\bar{c}\right) . \tag{7.63}$$

This makes it possible to compensate for the obstruction by introducing a term proportional to Θ in the SFT solution, namely

$$W = t(y)T + \theta(y)\Theta\,, \tag{7.64}$$

where $\theta(y)$ is a function which starts at the order y^3. From the worldsheet CFT viewpoint, this corresponds to implementing the RG flow both in the matter and in the linear dilaton sector. The latter will induce a change in the background charge and, therefore, in c_β. This, in turn, exactly compensates for the variation of the matter central charge, resulting in a total worldsheet CFT that remains critical. We therefore recover

$$c^*_{\rm tot} = c^* + c^*_\beta + c_{\rm aux} + c_{\rm gh} = 0\,, \tag{7.65}$$

where $*$ indicates that the corresponding quantities refer to the perturbed theory.

Finally, having discussed the subtleties of searching for a closed-string solution which changes the matter central charge, we recall that our primary focus in this chapter is on computing the disk action on the classical solution up to *second* order in y. Hence, the third-order term $\theta(y)\Theta$ will not be of any practical interest in the remainder of our analysis.

The classical closed-string solution

Let us now impose the equation of motion (7.60) and substitute the ansatz (7.55). We obtain

$$Q_{\rm c}T + \frac{t(y)}{2}Pl_2(T,T) + \frac{t(y)^2}{3!}\left(Pl_3(T,T,T) - 3Pl_2\left(T, \frac{b_0^+}{L_0^+}\bar{P}l_2(T,T)\right)\right) + \mathcal{O}(y^3) = 0\,. \tag{7.66}$$

By calculating the symplectic form of T against the l.h.s. of (7.66), we can write

$$\omega_{\rm c}\left(T, Q_{\rm c}T\right) + \frac{t(y)}{2}\omega_{\rm c}\left(T, l_2(T,T)\right) + \frac{t(y)^2}{3!}\mathcal{A}_{TTTT} + \mathcal{O}(y^3) = 0\,, \tag{7.67}$$

where we defined

$$\mathcal{A}_{TTTT} \coloneqq \omega_{\mathrm{c}}\left(T, l_3(T,T,T)\right) + 3\,\omega_{\mathrm{c}}\left(l_2(T,T), \frac{b_0^+}{L_0^+}\bar{P} l_2(T,T)\right) . \tag{7.68}$$

The quantity $\mathcal{A}_{TTTT}$ is proportional to the zero-momentum amplitude of four slightly off-shell tachyons on the sphere.

Let us now separately describe the three terms on the l.h.s. of (7.67). Starting with the first one, we can write

$$\omega_{\mathrm{c}}(T, Q_{\mathrm{c}}T) = \langle c\bar{c}\mathbb{V}(0,\bar{0})|c_0^-(c_0 L_0 + \bar{c}_0\bar{L}_0)|c\bar{c}\mathbb{V}(0,\bar{0})\rangle = y\,, \tag{7.69}$$

in which we used the fact that we are working in the Siegel gauge and with operators normalized so that $\langle 0|c_{-1}\bar{c}_{-1}c_0\bar{c}_0c_1\bar{c}_1|0\rangle = -1$ and $\langle\mathbb{V}|\mathbb{V}\rangle = 1$. To evaluate the second term in (7.67), we must introduce the cubic string product l_2. In particular, we will use the $\mathrm{SL}(2,\mathbb{C})$ vertices defined in [126]. We will describe these in more completeness in the following section. For now, we will just provide the definition of l_2, namely

$$|l_2(\Phi_1,\Phi_2)\rangle = b_0^-\delta(L_0^-) f_1\circ\Phi_1(0,\bar{0}) f_2\circ\Phi_2(0,\bar{0})|0\rangle_{\mathrm{SL}(2,\mathbb{C})}\,, \tag{7.70}$$

with

$$f_1(w) = \frac{1}{\lambda_{\mathrm{c}}}\frac{w-\lambda_{\mathrm{c}}}{3\lambda_{\mathrm{c}}+w} = -f_2(w)\,, \tag{7.71}$$

where $\lambda_{\mathrm{c}} > 1$ is a tunable stub parameter. Using this definition to compute the second term in (7.67), we get

$$\omega_{\mathrm{c}}\left(T, l_2(T,T)\right) = \langle c\bar{c}\mathbb{V}(0,\bar{0})|c_0^-b_0^-\delta(L_0^-)|f_1\circ c\bar{c}\mathbb{V}(0,\bar{0}) f_2\circ c\bar{c}\mathbb{V}(0,\bar{0})\rangle \tag{7.72a}$$

$$= \langle I_{\mathrm{c}}\circ c\bar{c}\mathbb{V}(0,\bar{0}) f_1\circ c\bar{c}\mathbb{V}(0,\bar{0}) f_2\circ c\bar{c}\mathbb{V}(0,\bar{0})\rangle_{\mathbb{C}} \tag{7.72b}$$

$$= -C_{\mathbb{V}\mathbb{V}\mathbb{V}}\left(\frac{3}{2}\lambda_{\mathrm{c}}\right)^{6y}\,, \tag{7.72c}$$

where, in the first line, we have reabsorbed the projector onto the level-matched Hilbert space in the bra, and, in the second line, we introduced the closed-string BPZ

inverse map defined as $I_{\rm c}(w)=\frac{1}{w}$.

Summarizing our progress up to this point, plugging back (7.69) and (7.72) into (7.67) and expressing $t(y)$ as a power-series expansion, we obtain

$$y-\frac{1}{2}(t_1 y+t_2 y^2)\left(1+6y\log\left(\frac{3}{2}\lambda_{\rm c}\right)\right)C_{\mathbb{V}\mathbb{V}\mathbb{V}}+\frac{1}{3!}t_1^2 y^2\mathcal{A}_{TTTT}+\mathcal{O}(y^3)=0\,. \quad (7.73)$$

This enables one to determine the coefficients t_1 and t_2 as

$$t_1=\frac{2}{C_{\mathbb{V}\mathbb{V}\mathbb{V}}}\,, \quad (7.74a)$$

$$t_2=-\left(\frac{2}{C_{\mathbb{V}\mathbb{V}\mathbb{V}}}\right)^2\left(3C_{\mathbb{V}\mathbb{V}\mathbb{V}}\log\left(\frac{3}{2}\lambda_{\rm c}\right)\right)+\frac{1}{3!}\left(\frac{2}{C_{\mathbb{V}\mathbb{V}\mathbb{V}}}\right)^3\mathcal{A}_{TTTT}|_{y=0}\,. \quad (7.74b)$$

It remains to evaluate the amplitude $\mathcal{A}_{TTTT}$ of four slightly relevant operators. This is one of the main results of [67]. It turns out that in the large stub limit $\lambda_{\rm c}\to\infty$, one can write it as

$$\mathcal{A}_{TTTT}=12\,(C_{\mathbb{V}\mathbb{V}\mathbb{V}})^2\log\left(\frac{3}{2}\lambda_{\rm c}\right)+\mathcal{A}^{\rm f.p.}_{TTTT}\,, \quad (7.75)$$

where $\mathcal{A}^{\rm f.p.}_{TTTT}$ is a finite number. Indeed, in computing this amplitude, there are contact divergences which can be regularized through a cutoff naturally provided by the closed stub parameter in the limit $\lambda_{\rm c}\to\infty$. The idea is then to expand the four-tachyon fundamental vertex in conformal blocks and then add and subtract divergent terms to obtain a finite part $\mathcal{A}^{\rm f.p.}_{TTTT}$ in which we can truly perform the limit $\lambda_{\rm c}\to\infty$. The remaining divergent terms explicitly depend on the closed-string stub parameter $\lambda_{\rm c}$. Note that by considering the limit $\lambda_{\rm c}\to\infty$, we do not lose generality because any observable which we compute must be independent of the off-shell SFT data. We will explicitly see this at work in the following sections, where we will prove that the classical on-shell disk action does not depend on $\lambda_{\rm c}$.

Finally, by inserting (7.75) into (7.74b), we obtain

$$t_2=\left(\frac{\mathcal{A}^{\rm f.p.}_{TTTT}}{3C_{\mathbb{V}\mathbb{V}\mathbb{V}}}+C_{\mathbb{V}\mathbb{V}\mathbb{V}}\log\left(\frac{3}{2}\lambda_{\rm c}\right)\right)\left(\frac{2}{C_{\mathbb{V}\mathbb{V}\mathbb{V}}}\right)^2\,. \quad (7.76)$$

Hence, using (7.54), (7.55), (7.59), (7.74a) and (7.76), we find that the classical closed-string solution expanded up to quadratic order in y reads

$$\Phi^*(y) = \left[\frac{2y}{C_{\mathbb{V}\mathbb{V}\mathbb{V}}} + \left(\frac{\mathcal{A}^{\text{f.p.}}_{TTTT}}{3C_{\mathbb{V}\mathbb{V}\mathbb{V}}} + C_{\mathbb{V}\mathbb{V}\mathbb{V}}\log\left(\frac{3}{2}\lambda_{\text{c}}\right)\right)\left(\frac{2y}{C_{\mathbb{V}\mathbb{V}\mathbb{V}}}\right)^2\right] c\bar{c}\mathbb{V} + \\ - \frac{1}{2}\left(\frac{2y}{C_{\mathbb{V}\mathbb{V}\mathbb{V}}}\right)^2 \frac{b_0^+}{L_0^+}\bar{P}l_2(c\bar{c}\mathbb{V}, c\bar{c}\mathbb{V}) + \mathcal{O}(y^3)\,. \tag{7.77}$$

When one substitutes the solution (7.77) into the sphere part of the SFT action, all dependence on the stub parameter λ_{c} indeed cancels, as it should, since the on-shell value of the action is a gauge-invariant (observable) quantity. In [67], this quantity has been conjectured to be related to the change in the central charge of the matter CFT under the short bulk RG flow triggered by $\mathbb{V}$, and this has been non-trivially verified up to $\mathcal{O}(y^4)$ in the case of Zamolodchikov flows of large m minimal models.

7.3.3 Fate of probe D-branes under short bulk RG flows

In this section, we will make a key intermediate step towards computing the disk part of the on-shell action in the situation when the closed-string sector (which is coupled to a probe D-brane system) is undergoing a short RG flow in the matter sector: we will compute the solution $\Psi^*(y)$ to the tadpole-sourced open-string equation of motion (7.16). We will see that since we are interested in expanding the disk action up to second order in y, we can truncate the open-string solution at first order in y. This is because for a critical open-string background, the disk open-string tadpole vanishes, so that second order terms in the open-string solution $\Psi^*(y)$ only start contributing into the on-shell disk action at third order in y.

The initial open-string background can be defined by a consistent boundary CFT. This in turn can characterized by a boundary state which can be written as a tensor product of boundary states for each sector of the bulk theory, that is

$$\|B_{\text{tot}}\rangle\rangle = \|B_0\rangle\rangle \otimes \|B_{\text{aux}}\rangle\rangle \otimes \|B_{\text{gh}}\rangle\rangle\,. \tag{7.78}$$

In order to find $\Psi^*(y)$, we will follow a strategy similar to the above-described case of the pure closed-string background. Namely, we will split the open-string field into two orthogonal components through the application of the projector P, which projects onto the sub-space of the open-string Hilbert space which is spanned by the L_0 eigenvectors with eigenvalues 0 and $-y$. In other words, we write

$$\Psi = P\Psi + \bar{P}\Psi \coloneqq w + r\,. \tag{7.79}$$

For the component w (which lies in the image of P), we consider an ansatz proportional to the nearly marginal boundary field

$$w = \tau(y)\, c\, \mathbb{v}\,. \tag{7.80}$$

To obtain an equation fixing the function $\tau(y)$, we consider the tadpole-sourced open-string equation of motion evaluated on the classical closed-string background $\Phi^*(y)$ which is given by (7.77). Assuming that the open-string solution Ψ^* will start at order y, this can be truncated as

$$\begin{aligned}\mathrm{EOM}_{\Phi^*(y)}[\Psi] = {}& Q_{\mathrm{o}}\Psi + m_{0,2}(\Psi,\Psi) + \\ & + m_{1,0}(\Phi^*(y)) + m_{1,1}(\Phi^*(y),\Psi) + m_{2,0}(\Phi^*(y),\Phi^*(y)) + \mathcal{O}(y^3)\,.\end{aligned} \tag{7.81}$$

Solving this equation can be facilitated by considering its projections into the image of P and its orthogonal complement, namely

$$\begin{aligned}P\,\mathrm{EOM}_{\Phi^*(y)}[\Psi] = {}& Q_{\mathrm{o}}w + Pm_{0,2}(\Psi,\Psi) + \\ & + Pm_{1,0}(\Phi^*(y)) + Pm_{1,1}(\Phi^*(y),\Psi) + Pm_{2,0}(\Phi^*(y),\Phi^*(y)) + \mathcal{O}(y^3)\,,\end{aligned} \tag{7.82a}$$

$$\begin{aligned}\bar{P}\,\mathrm{EOM}_{\Phi^*(y)}[\Psi] = {}& Q_{\mathrm{o}}r + \bar{P}m_{0,2}(\Psi,\Psi) + \\ & + \bar{P}m_{1,0}(\Phi^*(y)) + \bar{P}m_{1,1}(\Phi^*(y),\Psi) + \bar{P}m_{2,0}(\Phi^*(y),\Phi^*(y)) + \mathcal{O}(y^3)\,.\end{aligned} \tag{7.82b}$$

Since the open-string BRST operator Q_{o} is invertible on the image of $\bar{P}$, the second equation can now be recursively solved in the Siegel gauge $b_0 = 0$ as

$$r^*(y) = -\frac{b_0}{L_0}\bar{P}m_{1,0}(\Phi^*(y)) + \mathcal{O}(y^2) = -\frac{2y}{C_{\mathbb{V}\mathbb{V}\mathbb{V}}}\frac{b_0}{L_0}\bar{P}m_{1,0}(c\bar{c}\mathbb{V}) + \mathcal{O}(y^2)\,. \tag{7.83}$$

Notice that, in contrast to the closed-string case, the component of the open-string solution lying in the image of $\bar{P}$ now starts at order y and not y^2. This is due to the presence of a closed-string sourced open-string tadpole in the disk action.

It remains to solve the equation of motion (7.82a) for w (and therefore for $\tau(y)$). First, note that when the slightly relevant boundary field $\mathbb{v}$ is not part of the spectrum of the matter BCFT, the image of the projector P can be taken as empty.[1] In such cases, the equation of motion (7.82a) is simply absent and the open-string solution $\Psi^*(y)$ lies purely in the image of $\bar{P}$. On the other hand, when the boundary field $\mathbb{v}$ *is* part of the matter BCFT, then, under our assumption (7.51), we can see that the P-component w of $\Psi^*(y)$ will start at $\mathcal{O}(y)$. Indeed, the contribution

$$Q_{\mathrm{o}}w = \tau(y)Q_{\mathrm{o}}c\,\mathbb{v} = y\tau(y)\,c\partial c\,\mathbb{v} = \mathcal{O}(y^2)\,. \tag{7.84}$$

to the r.h.s. of (7.82a) then generally comes at the same order as $Pm_{0,2}(\Psi^*(y),\Psi^*(y))$, as well as $Pm_{1,0}(\Phi^*(y))$ which acts as a source. Hence, if we put $w(y) = \tau_1 y\,c\,\mathbb{v} + \mathcal{O}(y^2)$, the equation of motion (7.82a) can be balanced and, at least in principle, solved for the coefficients τ_i. This is how our assumption (7.51) guarantees perturbativity of the bulk-induced boundary deformation.

However, it turns out that in order to achieve our goal of finding the on-shell disk action up to second order in y, we do not have to be interested in finding the explicit expression for the solution $w^*(y) = P\Psi^*(y)$ because it only would have started to contribute at $\mathcal{O}(y^3)$. Indeed, the leading contribution to (7.18b) involving the component of the open-string solution lying in the image of P would go as

$$\omega_{\mathrm{o}}\big[w^*(y), m_{1,0}(\Phi^*(y))\big] \sim y^2\frac{B_{\mathbb{V}\mathbb{v}}}{g} + \cdots \sim \mathcal{O}(y^3) \tag{7.85}$$

[1] This is provided that we choose to work with vertices which do not generate the pure-ghost boundary field ∂c.

where two powers of y come from expanding the string fields $w^*(y)$ and $\Phi^*(y)$ in y, while another power of y comes from the bulk-boundary OPE coefficient $B_{\mathbb{V}_\mathrm{v}}/g \sim \mathcal{O}(y)$ as per our assumption (7.51). On the other hand, the $\bar{P}$ component of the solution contributes to (7.18b) at order y^2

$$\omega_\mathrm{o}[r^*(y), m_{1,0}(\Phi^*(y))] \sim y^2 \frac{(B_{\mathbb{V}\mathbb{1}})^2}{g} + \cdots \sim \mathcal{O}(y^2)\,, \tag{7.86}$$

because generally $B_{\mathbb{V}\mathbb{1}}(y)/g(y) \sim \mathcal{O}(y^0)$.

To summarize, for the purposes of evaluating the on-shell disk action up to second order in y, we may take the open string solution to read

$$\Psi^*(y) = -\frac{2y}{C_{\mathbb{V}\mathbb{V}\mathbb{V}}} \frac{b_0}{L_0} \bar{P} m_{1,0}(c\bar{c}\mathbb{V}) + \dots\,, \tag{7.87}$$

where the dots $\dots$ represent not only $\mathcal{O}(y^2)$ terms, but also terms at order y which, however, would have contributed at cubic order into the on-shell disk action.

7.4 Disk action of short RG flows

In this section, we will compute (up to quadratic order in y) the disk action of open-closed SFT evaluated on the classical solutions $\Phi^*(y)$ and $\Psi^*(y)$ which were constructed in section 7.3 to describe short RG-flows in the matter (B)CFT. According to the generalised Sen's conjecture (7.19), this will allow us to obtain the induced change in the ratio of the worldsheet boundary state g-function and the string coupling constant in terms of CFT data.

7.4.1 Choice of vertices and initial setup

In order to evaluate the on-shell disk action, the first step is to properly define the vertices that appear in (7.77), (7.87), as well as on the r.h.s. of (7.18b). Specifically, we will use the definition of multi-string products given in [126], which is based on the construction of fundamental vertices using $\mathrm{SL}(2,\mathbb{C})$ maps. The reason why this construction is particularly useful is that it enables us to set the multi-string product $l_{1,0}$ to zero, as demonstrated in [126]. Indeed, in such a setup, the interior of the

moduli space, associated with the amplitude with two closed-string punctures on the disk, is fully covered through Feynman diagrams given by fundamental vertices arising at lower order in string perturbation theory. In other words, the $\mathrm{SL}(2,\mathbb{C})$ products turn out to satisfy the corresponding homotopy relation without $l_{1,0}$, namely

$$l_2 l_{0,0} + l_{0,1} m_{1,0} = 0 \,. \tag{7.88}$$

Let us then explicitly list the string products which appear in (7.77), (7.87) and (7.18b). These were defined in [126] as

$$|l_2(\Phi_1,\Phi_2)\rangle = b_0^- \delta(L_0^-) f_1 \circ \Phi_1(0,\bar{0}) f_2 \circ \Phi_2(0,\bar{0})|0\rangle_{\mathrm{SL}(2,\mathbb{C})} \,, \tag{7.89a}$$

$$|l_{0,0}\rangle = \frac{1}{(2\pi i)^2} \lambda_{\mathrm{b}}^{-L_0^+} \|B_0\rangle\rangle \,, \tag{7.89b}$$

$$(-)^{d(\Phi)} |m_{1,0}(\Phi)\rangle = \frac{1}{2\pi i} \big[m \circ \widetilde{\Phi(0,\bar{0})} \big] |0\rangle_{\mathrm{SL}(2,\mathbb{R})} \,, \tag{7.89c}$$

where $\lambda_{\mathrm{b}} > 1$ is a tunable stub parameter and $\widetilde{(\cdots)}$ means that the bulk-boundary OPE is understood to have been performed. Finally, the $\mathrm{SL}(2,\mathbb{C})$ maps $f_1(w)$, $f_2(w)$ and $m(w)$ are defined as

$$f_1(w) = \frac{1}{\lambda_{\mathrm{c}}} \frac{w - \lambda_{\mathrm{c}}}{3\lambda_{\mathrm{c}} + w} = -f_2(w) \,, \tag{7.90a}$$

$$m(w) = \frac{i}{\lambda_{\mathrm{o}}} \frac{1 + \frac{w}{\beta_2}}{1 + \frac{w}{\beta_1}} \,, \tag{7.90b}$$

in which $\lambda_{\mathrm{c}} > 1$ is a free stub parameter whereas λ_{o}, β_1 and β_2 are fixed by the homotopy relation (7.88) as

$$\lambda_{\mathrm{o}} = \frac{3\lambda_{\mathrm{b}}\lambda_{\mathrm{c}} + 1}{3\lambda_{\mathrm{b}}\lambda_{\mathrm{c}} - 1} \,, \tag{7.91a}$$

$$\beta_1 = \frac{3\lambda_{\mathrm{b}}\lambda_{\mathrm{c}} + 1}{3\lambda_{\mathrm{b}}\lambda_{\mathrm{c}} - 1} \lambda_{\mathrm{c}} \,, \tag{7.91b}$$

$$\beta_2 = \frac{3\lambda_{\mathrm{b}}\lambda_{\mathrm{c}} - 1}{3\lambda_{\mathrm{b}}\lambda_{\mathrm{c}} + 1} \lambda_{\mathrm{c}} \,. \tag{7.91c}$$

Notice that (7.91a) gives a relation between the open-string and closed-string stub parameters, which is explicitly showing how the open-closed SFT naturally interpol-

ates between the two limits where the dynamics is mostly dominated by the open and closed strings, respectively [161, 162]. Indeed, the large closed-string stub limit $\lambda_{\mathrm{c}} \to \infty$ implies the short open-string stub limit $\lambda_{\mathrm{o}} \to \mathcal{O}(1)$, which means that the moduli space will be mostly covered by open-string propagators and closed-string fundamental vertex regions. Conversely, if $\lambda_{\mathrm{o}} \to \infty$, then $\lambda_{\mathrm{c}} \to \mathcal{O}(1)$, so that the moduli space is mostly covered by closed-string propagators and open-string fundamental vertices. As we have advertised in the previous sections, we will focus on the former limit.

Given this particular choice of SFT vertices, we can further simplify the expression (7.18b) by setting $l_{1,0} = 0$, namely

$$\frac{1}{2\pi^2}\left[1 - \frac{g^*}{g}\left(\frac{\kappa^*}{\kappa}\right)^{-1}\right] = \frac{1}{g}\,\omega_{\mathrm{c}}\left(\Phi^*(y), l_{0,0}\right) + \frac{1}{2}\frac{1}{g}\,\omega_{\mathrm{o}}\left(\Psi^*(y), m_{1,0}(\Phi^*(y))\right) + \mathcal{O}(y^3)\,. \tag{7.92}$$

Substituting the explicit expressions (7.77) and (7.87) for the classical solutions $\Phi^*(y)$ and $\Psi^*(y)$ into the r.h.s. of (7.92) and carefully collecting all contributions, we obtain

$$\begin{aligned}\frac{1}{2\pi^2}\left[1 - \frac{g^*}{g}\left(\frac{\kappa^*}{\kappa}\right)^{-1}\right] = {}& \left[\frac{2y}{C_{\mathbb{V}\mathbb{V}\mathbb{V}}} + \left(\frac{\mathcal{A}^{\mathrm{f.p.}}_{TTTT}}{3C_{\mathbb{V}\mathbb{V}\mathbb{V}}} + C_{\mathbb{V}\mathbb{V}\mathbb{V}}\log\left(\frac{3}{2}\lambda_{\mathrm{c}}\right)\right)\left(\frac{2y}{C_{\mathbb{V}\mathbb{V}\mathbb{V}}}\right)^2\right]\mathcal{A}^{\mathrm{disk}}_T \\ & - \frac{1}{2}\left(\frac{2y}{C_{\mathbb{V}\mathbb{V}\mathbb{V}}}\right)^2 \mathcal{A}^{\mathrm{disk}}_{TT} + O(y^3)\,,\end{aligned} \tag{7.93}$$

where $\mathcal{A}^{\mathrm{disk}}_T$ and $\mathcal{A}^{\mathrm{disk}}_{TT}$ are the zero-momentum one- and two-point amplitudes of the nearly on-shell tachyon $c\bar{c}\mathbb{V}$ on the disk, which are rescaled by the ratio of the initial g-function and the string coupling constant. That is to say, we define

$$\mathcal{A}^{\mathrm{disk}}_T \coloneqq \frac{1}{g}\,\omega_{\mathrm{c}}\left(c\bar{c}\mathbb{V}, l_{0,0}\right)\,, \tag{7.94a}$$

$$\mathcal{A}^{\mathrm{disk}}_{TT} \coloneqq \frac{1}{g}\,\omega_{\mathrm{c}}\left(\frac{b_0^+}{L_0^+}\bar{P}l_2(c\bar{c}\mathbb{V}, c\bar{c}\mathbb{V}), l_{0,0}\right) + \frac{1}{g}\,\omega_{\mathrm{o}}\left(\frac{b_0}{L_0}\bar{P}m_{1,0}(c\bar{c}\mathbb{V}), m_{1,0}(c\bar{c}\mathbb{V})\right)\,. \tag{7.94b}$$

Looking at (7.93), it would naively appear that the on-shell disk action could depend on the stub parameter λ_{c}. This would stand in contradiction with the interpretation

of the on-shell disk action as an observable quantity because as such, it should not depend on the choice of the local coordinate maps. Therefore, it will be crucial to prove that, after careful evaluation of the two amplitudes $\mathcal{A}_T^{\text{disk}}$ and $\mathcal{A}_{TT}^{\text{disk}}$, all dependence on λ_{c} drops out, so that one ends up with an expression that is independent of the off-shell SFT data. In the following subsection, we will explicitly compute these two amplitudes in the large closed-string stub limit $\lambda_{\text{c}} \to \infty$.

7.4.2 Computing $\mathcal{A}_T^{\text{disk}}$ and $\mathcal{A}_{TT}^{\text{disk}}$

Let us begin by calculating the amplitude $\mathcal{A}_T^{\text{disk}}$ of one tachyon on the disk. By substituting into (7.94a) the explicit expression (7.89b) for the closed-string tadpole and rewriting the symplectic form as the BPZ inner product, we obtain

$$\mathcal{A}_T^{\text{disk}} = -\frac{1}{4\pi^2 g}\langle c\bar{c}\mathbb{V}(0,\bar{0})|c_0^- \lambda_{\text{b}}^{-L_0^+}\|B_0\rangle\rangle = -\frac{1}{4\pi^2}\frac{B_{\mathbb{V}\mathbb{1}}}{g}\lambda_{\text{b}}^{2y}, \tag{7.95}$$

where in the second equality, we acted with the BPZ-even operator $\lambda_{\text{b}}^{L_0^+}$ on the bra and we also computed the ghost correlator

$$\langle c\bar{c}(0,\bar{0})|c_0^-\|B_{\text{ghost}}\rangle\rangle = \frac{1}{8}\langle c(i)c(-i)\left(\partial c(i) - \partial c(-i)\right)\rangle_{\text{UHP}} = 1\,. \tag{7.96}$$

As far as the matter contribution is concerned, we only have to deal with the one-point function on the disk $\langle\mathbb{V}(0,\bar{0})\|B_0\rangle\rangle$, which is identically equal to $B_{\mathbb{V}\mathbb{1}}$. Finally, expanding the stub contribution up to first order in y, we end up with the expression

$$\mathcal{A}_T^{\text{disk}} = -\frac{1}{4\pi^2}\left[\frac{B_{\mathbb{V}\mathbb{1}}}{g} + \left(\frac{1}{2}\frac{B_{\mathbb{V}\mathbb{1}}}{g}C_{\mathbb{V}\mathbb{V}\mathbb{V}}\log\left(\lambda_{\text{b}}\right)\right)\left(\frac{2y}{C_{\mathbb{V}\mathbb{V}\mathbb{V}}}\right)\right] + \mathcal{O}(y^2)\,, \tag{7.97}$$

which is exact up to linear order in y. This precision is sufficient as we recall that $\mathcal{A}_T^{\text{disk}}$ is already multiplied by y in the expression (7.93) for the on-shell disk action.

Second, let us deal with the amplitude $\mathcal{A}_{TT}^{\text{disk}}$ which was defined in (7.94b). In this case, we will only be interested in the leading order $\mathcal{O}(y^0)$ because in (7.93), $\mathcal{A}_{TT}^{\text{disk}}$ appears already multiplied by y^2. Focusing first on the closed-string exchange,

we can write

$$\frac{1}{g}\,\omega_{\rm c}\left(\frac{b_0^+}{L_0^+}\bar{P}l_2(c\bar{c}\mathbb{V},c\bar{c}\mathbb{V}),l_{0,0}\right)=$$

$$=\frac{1}{g}\langle l_{0,0}|c_0^-\frac{b_0^+}{L_0^+}\bar{P}|l_2(c\bar{c}\mathbb{V}(0,\bar{0}),c\bar{c}\mathbb{V}(0,\bar{0}))\rangle \tag{7.98a}$$

$$=-\frac{1}{4\pi^2 g}\langle\!\langle B_0\|c_0^-\lambda_{\rm b}^{-L_0^+}\frac{b_0^+}{L_0^+}\bar{P}b_0^-c_0^-|l_2(c\bar{c}\mathbb{V}(0,\bar{0}),c\bar{c}\mathbb{V}(0,\bar{0}))\rangle \tag{7.98b}$$

$$=-\frac{1}{4\pi^2 g}\langle\!\langle B_0\|c_0^-\lambda_{\rm b}^{-L_0^+}|\xi_i\rangle\langle\xi_i^c|\frac{b_0^+}{L_0^+}\bar{P}b_0^-|\xi_j^c\rangle\langle\xi_j|c_0^-|l_2(c\bar{c}\mathbb{V}(0,\bar{0}),c\bar{c}\mathbb{V}(0,\bar{0}))\rangle\,. \tag{7.98c}$$

Here, in the second equality, we wrote out the closed-string tadpole in its explicit form (7.89b) and we also made manifest the projection onto the b_0^- exact states in the intermediate channel. Finally, in the last equality, we made two insertions of the identity operator $1_{\mathcal{H}_{\rm c}}=|\xi_i\rangle\langle\xi_i^c|=|\xi_i^c\rangle\langle\xi_i|$ on the Hilbert space $\mathcal{H}_{\rm c}$ of the level-matched closed-string states (where sums over repeated indices are understood). The basis vectors $|\xi_i\rangle$ and their BPZ duals $|\xi_i^c\rangle$ are normalized so that $\langle\xi_i^c|\xi_j\rangle=\langle\xi_i|\xi_j^c\rangle=\delta_{ij}$.

Following the argument described in [67], thanks to the projector $\bar{P}$ in front of the closed-string propagator, the only contribution received by (7.98) in the large stub limit comes from the propagation of the universal tachyon $|\xi_j\rangle=|\xi_i\rangle=|c_1\bar{c}_1\rangle$. Hence, the part (7.98) of the amplitude $\mathcal{A}_{TT}^{\rm disk}$ associated with the closed-string exchange can be rewritten as

$$\frac{1}{g}\,\omega_{\rm c}\left(\frac{b_0^+}{L_0^+}\bar{P}l_2\left(c\bar{c}\mathbb{V},c\bar{c}\mathbb{V}\right),l_{0,0}\right)=$$

$$=-\frac{1}{4\pi^2 g}\langle\!\langle B_0\|c_0^-\lambda_{\rm b}^{-L_0^+}|c_1\bar{c}_1\rangle\langle c_0\bar{c}_0c_1\bar{c}_1|\frac{b_0^+}{L_0^+}\bar{P}b_0^-|c_0\bar{c}_0c_1\bar{c}_1\rangle$$

$$\times\,\langle c_1\bar{c}_1|c_0^-|l_2(c\bar{c}\mathbb{V}(0,\bar{0}),c\bar{c}\mathbb{V}(0,\bar{0}))\rangle\,. \tag{7.99}$$

Let us now separately evaluate the three factors on the r.h.s. of (7.99). The computation of the first one is entirely analogous to what was done in (7.95) provided that we replace $\mathbb{V}$ with the identity in the matter sector. We then obtain

$$-\frac{1}{4\pi^2 g}\,\langle\!\langle B_0\|c_0^-\lambda_{\rm b}^{-L_0^+}|c_1\bar{c}_1\rangle=-\frac{1}{4\pi^2}\,\lambda_{\rm b}^2\,. \tag{7.100}$$

Regarding the second factor, we trivially obtain

$$\langle c_0\bar{c}_0c_1\bar{c}_1|\frac{b_0^+}{L_0^+}\bar{P}b_0^-|c_0\bar{c}_0c_1\bar{c}_1\rangle = 1\,, \tag{7.101}$$

while the third factor can be computed using (7.89a) and (7.90a) as

$$\begin{aligned}\langle c_1\bar{c}_1|c_0^-|l_2(c\bar{c}\mathbb{V}(0,\bar{0}),c\bar{c}\mathbb{V}(0,\bar{0}))\rangle &= \langle c_1\bar{c}_1|c_0^-b_0^-\delta(L_0^-)|f_1\circ c\bar{c}\mathbb{V}(0,\bar{0})f_2\circ c\bar{c}\mathbb{V}(0,\bar{0})\rangle \\ &= \big\langle I_{\rm c}\circ c\bar{c}(0,\bar{0})\,f_1\circ c\bar{c}\mathbb{V}(0,\bar{0})\,f_2\circ c\bar{c}\mathbb{V}(0,\bar{0})\big\rangle_{\mathbb{C}} \\ &= -\left(\frac{3}{2}\lambda_{\rm c}\right)^2+\mathcal{O}(y)\,,\end{aligned} \tag{7.102a}$$

where $I_{\rm c}(w)=\frac{1}{w}$ denotes the closed-string BPZ inversion. Putting all these results together, we get

$$\frac{1}{g}\,\omega_{\rm c}\left(\frac{b_0^+}{L_0^+}\bar{P}l_2(c\bar{c}\mathbb{V},c\bar{c}\mathbb{V}),l_{0,0}\right)=\frac{1}{4\pi^2}\left(\frac{3}{2}\lambda_{\rm c}\lambda_{\rm b}\right)^2+\mathcal{O}(y) \tag{7.103}$$

for the closed-string channel.

Let us now focus on computing the channel of $\mathcal{A}_{TT}^{\rm disk}$ which is associated with the open-string exchange (second term in (7.94b)). Substituting the explicit form (7.89c) of the open-closed product $m_{1,0}$, we can first express

$$\begin{aligned}&\frac{1}{g}\,\omega_{\rm o}\left(\frac{b_0}{L_0}\bar{P}m_{1,0}\left(c\bar{c}\mathbb{V}\right),m_{1,0}\left(c\bar{c}\mathbb{V}\right)\right)= \\ &\qquad=-\frac{1}{4\pi^2 g}\int_0^1\frac{dt}{t}t^{\epsilon_o}\big\langle I_{\rm o}\circ m\circ c\bar{c}\mathbb{V}(0,\bar{0})\,b_0\,t^{L_0}\,m\circ c\bar{c}\mathbb{V}(0,\bar{0})\big\rangle_{\rm UHP}\Big|_{\epsilon_o\to 0}\,,\end{aligned} \tag{7.104}$$

where we wrote the Siegel propagator in the Schwinger representation and introduced the open-string BPZ inversion $I_{\rm o}(w)=-\frac{1}{w}$. Furthermore, we introduced the factor t^{ϵ_o} to regulate the divergences which generally arise near the open-string degeneration $t\to 0$ due to propagation of open-string tachyons. In our particular case, the only divergence will come from the identity channel in the bulk-boundary OPE of the matter field $\mathbb{V}$. Also notice that on the r.h.s. of (7.104), we have dropped the projector $\bar{P}$. This is possible by realizing that our assumption (7.51) guarantees

that at the order $\mathcal{O}(y^0)$, no states in the image of P propagate in the open-string channel of the amplitude $\mathcal{A}_{TT}^{\text{disk}}$ so that the $\bar{P}$ insertion in front of the open-string propagator on the l.h.s. of (7.104) becomes indistinguishable from the identity. If (7.51) were not assumed, the states in the image of P propagating in the open-string channel would have to be removed by hand. With these provisions in mind, we can implement the action of the conformal maps (7.90b) on the r.h.s. of (7.104) and compute the ghost-part of the correlator. Keeping only the contributions at the leading order in y, we eventually get

$$\frac{1}{g}\,\omega_{\text{o}}\left(\frac{b_0}{L_0}\bar{P}m_{1,0}\left(c\bar{c}\mathbb{V}\right),m_{1,0}\left(c\bar{c}\mathbb{V}\right)\right)= \\ \frac{1}{4\pi^2 g}\int_0^{\frac{1}{\lambda_{\text{o}}^2}} ds\, s^{\epsilon_o}\, 4(s^2-1)\big\langle \mathbb{V}(i,\bar{i})\,\mathbb{V}(is,\bar{is})\big\rangle_{\text{UHP}}\Big|_{\epsilon_o\to 0}\,, \tag{7.105}$$

where the details can be checked by following an analogous computation which was presented in [126].

Open and closed degenerations in $\mathcal{A}_{TT}^{\text{disk}}$

At this point, we will focus on making explicit the terms on the r.h.s. of (7.105) which diverge as we remove the regulators λ_{o} and ϵ_{o}, putting particular emphasis on the regime of closed-string degeneration. This regime can be achieved as the Schwinger parameter s approaches 1 and therefore becomes available as we send $\lambda_{\text{o}} \to 1$. As showcased by the relation (7.91a), this is equivalent to the limit $\lambda_{\text{b}}\lambda_{\text{c}} \to \infty$ in which the closed-string stub parameters become large. By subtracting and adding the divergent contributions, we will be able to isolate the explicit dependence of (7.105) on the stub parameters λ_{b}, λ_{c}. This will eventually facilitate the proof that the value of the on-shell disk action does not depend on the SFT data.

Let us start by discussing the open-string degeneration $s \to 0$. In this limit, the operator insertion $\mathbb{V}(is,-is)$ in (7.105) can be replaced by the corresponding

bulk-boundary OPE, namely

$$\mathbb{V}(is,-is) \overset{s\to 0}{\sim} \frac{1}{(2s)^2}\frac{B_{\mathbb{V}\mathbb{1}}}{g} + \frac{1}{2s}\frac{B_{\mathbb{V}\mathbb{v}}}{g}\mathbb{v}(0) + \text{reg}\,. \tag{7.106}$$

Notice that the second term of the above relation would have been annihilated by the $\bar{P}$ appearing in front of the Siegel-gauge propagator on the l.h.s. of (7.104) and as such, it would have to be removed by hand when evaluating the open-string channel of the amplitude. However, as per our assumption (7.51), we have $\frac{B_{\mathbb{V}\mathbb{v}}}{g} \sim \mathcal{O}(y)$, so that at the order $\mathcal{O}(y^0)$, this is taken care of automatically. As a result, the only divergence arising is the one due to the identity channel, which can be regulated through the parameter ϵ_o in (7.105). Consequently, in the limit $s \to 0$, one can isolate the divergent contribution to the correlator as

$$\langle \mathbb{V}(i,\bar{i})\mathbb{V}(is,\bar{is})\rangle_{\text{UHP}} \overset{s\to 0}{\sim} \frac{1}{(2s)^2}\frac{B_{\mathbb{V}\mathbb{1}}}{g}\langle \mathbb{V}(i,\bar{i})\rangle_{\text{UHP}} = \frac{1}{(4s)^2}\frac{(B_{\mathbb{V}\mathbb{1}})^2}{g}\,. \tag{7.107}$$

For the sake of later convenience, let us also introduce an explicit notation

$$\text{div}_{\text{o}}(s) \coloneqq \frac{1}{(4s)^2}\frac{(B_{\mathbb{V}\mathbb{1}})^2}{g} \tag{7.108}$$

for the open-string divergence in the 2-point correlator of $\mathbb{V}$.

The closed string degeneration occurs for $s \to 1$, and it is therefore regulated through the stub parameter λ_{o}^{-2}. This regulator can be lifted by taking the limit $\lambda_{\text{c}} \to \infty$ which, according to (7.91a), corresponds to $\lambda_{\text{o}} \to 1$. As the Schwinger parameter s approaches 1, we observe a collision between the two bulk fields and, therefore, we can read off the divergent contribution of the correlator to the amplitude $\mathcal{A}_{TT}^{\text{disk}}$ by using the bulk OPE

$$\langle \mathbb{V}(i,\bar{i})\,\mathbb{V}(is,\bar{is})\rangle_{\text{UHP}} \overset{s\to 1}{\sim} \frac{1}{(1-s)^4}\,g + \frac{1}{4(1-s)^2}\,C_{\mathbb{V}\mathbb{V}\mathbb{V}}B_{\mathbb{V}\mathbb{1}}\,. \tag{7.109}$$

Note that here we are ignoring the simple pole in $1-s$ because thanks to the c-ghost measure $4(s^2-1)$, this gives rise to a finite contribution to the amplitude. In an exact

analogy with the open-string channel, we also introduce the notation

$$\mathrm{div_c}(s) \coloneqq \frac{1}{(1-s)^4} g + \frac{1}{4(1-s)^2} C_{\mathbb{V}\mathbb{V}\mathbb{V}} B_{\mathbb{V}\mathbb{1}} \tag{7.110}$$

for the closed-string divergence.

Let us then subtract and add (7.108) and (7.110) to the 2-point correlator of $\mathbb{V}$ when evaluating the integral over s in (7.105). This gives

$$\begin{aligned} &\frac{1}{g}\,\omega_{\mathrm{o}}\left(\frac{b_0}{L_0}\bar{P} m_{1,0}\left(c\bar{c}\mathbb{V}\right), m_{1,0}\left(c\bar{c}\mathbb{V}\right)\right) = \\ &\qquad = \frac{1}{4\pi^2}\mathcal{A}_{TT}^{\mathrm{f.p.}} + \frac{1}{4\pi^2 g}\int_0^{\frac{1}{\lambda_{\mathrm{o}}^2}} ds\, s^{\epsilon_o} 4(s^2-1)\left(\mathrm{div_o}(s) + \mathrm{div_c}(s)\right)\Big|_{\epsilon_o \to 0}, \end{aligned} \tag{7.111}$$

where we define a finite contribution

$$\mathcal{A}_{TT}^{\mathrm{f.p.}} \coloneqq \frac{1}{g}\int_0^1 ds\, 4(s^2-1)\Big[\langle \mathbb{V}(i,\bar{i})\mathbb{V}\left(is, \bar{is}\right)\rangle_{\mathrm{UHP}} - \mathrm{div_o}(s) - \mathrm{div_c}(s)\Big], \tag{7.112}$$

to the amplitude $\mathcal{A}_{TT}^{\mathrm{disk}}$, in which all regulators can be removed as we are subtracting all sources of divergent behaviour. The value of $\mathcal{A}_{TT}^{\mathrm{f.p.}}$ will of course depend on the details of the particular 2d CFT in which the short RG flow is considered. On the other hand, the second term in (7.111) can be evaluated as

$$\begin{aligned} &-\frac{1}{4\pi^2 g}\int_0^{\frac{1}{\lambda_{\mathrm{o}}^2}} ds\, s^{\epsilon_o} 4(s^2-1)\big[\mathrm{div_o}(s) + \mathrm{div_c}(s)\big]\Big|_{\epsilon_o\to 0} = \\ &= -\frac{1}{4\pi^2 g} \\ &\quad \times \int_0^{\frac{1}{\lambda_{\mathrm{o}}^2}} ds\, s^{\epsilon_o} 4(s^2-1)\left[\frac{1}{(4s)^2}\frac{\left(B_{\mathbb{V}\mathbb{1}}\right)^2}{g} + \frac{1}{(1-s)^4} g + \frac{1}{4(1-s)^2} C_{\mathbb{V}\mathbb{V}\mathbb{V}} B_{\mathbb{V}\mathbb{1}}\right]\Big|_{\epsilon_o\to 0} \\ &= -\frac{1}{4\pi^2} \\ &\quad \times \left[\frac{1}{2}\left(\frac{B_{\mathbb{V}\mathbb{1}}}{g}\right)^2 + \frac{1}{2} + C_{\mathbb{V}\mathbb{V}\mathbb{V}}\frac{B_{\mathbb{V}\mathbb{1}}}{g}\left(1 + 2\log(2) - 2\log\left(\frac{3}{2}\lambda_{\mathrm{b}}\lambda_{\mathrm{c}}\right)\right) - \left(\frac{3}{2}\lambda_{\mathrm{b}}\lambda_{\mathrm{c}}\right)^2\right]. \end{aligned} \tag{7.113a}$$

Notice that the last term in the square brackets of (7.113a) exactly cancels out the

closed-string channel contribution (7.103).

In total, the amplitude $\mathcal{A}_{TT}^{\text{disk}}$ of two nearly on-shell tachyons on the disk at the order $\mathcal{O}(y^0)$ becomes

$$\begin{aligned}\mathcal{A}_{TT}^{\text{disk}} &= \frac{1}{4\pi^2}\mathcal{A}_{TT}^{\text{f.p.}} \\ &\quad + \frac{1}{4\pi^2}\left[\frac{1}{2}\left(\frac{B_{\mathbb{V}\mathbb{1}}}{g}\right)^2 + \frac{1}{2} + C_{\mathbb{V}\mathbb{V}\mathbb{V}}\frac{B_{\mathbb{V}\mathbb{1}}}{g}\left(1+2\log(2)-2\log\left(\frac{3}{2}\lambda_{\text{b}}\lambda_{\text{c}}\right)\right)\right] .\end{aligned} \tag{7.114}$$

We observe that all remaining dependence of $\mathcal{A}_{TT}^{\text{disk}}$ on the off-shell SFT data is encapsulated in the logarithmic term

$$\frac{1}{2\pi^2}\, C_{\mathbb{V}\mathbb{V}\mathbb{V}}\frac{B_{\mathbb{V}\mathbb{1}}}{g}\log\left(\frac{3}{2}\lambda_{\text{b}}\lambda_{\text{c}}\right) . \tag{7.115}$$

The final formula for the on-shell disk action

At this point, we have in place all the quantities needed for writing down the on-shell disk action in terms of CFT data. By substituting the results (7.97) and (7.114) for the amplitudes $\mathcal{A}_{T}^{\text{disk}}$ and $\mathcal{A}_{TT}^{\text{disk}}$ into the expression (7.93) for the disk action, we can explicitly see that the stub-dependent part (7.115) of the amplitude $\mathcal{A}_{TT}^{\text{disk}}$ exactly cancels with the $\log(\frac{3}{2}\lambda_{\text{c}})$ term in the prefactor of $\mathcal{A}_{T}^{\text{disk}}$ in (7.93) coming from the closed-string solution (7.77), as well as with the $\log(\lambda_{\text{b}})$ term in the result (7.97) for $\mathcal{A}_{T}^{\text{disk}}$. We therefore obtain the following expression for the on-shell open-closed SFT disk action

$$\frac{g^*}{g}\left(\frac{\kappa^*}{\kappa}\right)^{-1} = 1 + \frac{B_{\mathbb{V}\mathbb{1}}}{g}\left(\frac{y}{C_{\mathbb{V}\mathbb{V}\mathbb{V}}}\right) + \left(\frac{B_{\mathbb{V}\mathbb{1}}}{g}\frac{2\mathcal{A}_{TTTT}^{\text{f.p.}}}{3C_{\mathbb{V}\mathbb{V}\mathbb{V}}} + \tilde{\mathcal{A}}_{TT}^{\text{f.p.}}\right)\left(\frac{y}{C_{\mathbb{V}\mathbb{V}\mathbb{V}}}\right)^2 + \mathcal{O}(y^3) , \tag{7.116}$$

where the shifted finite part $\tilde{\mathcal{A}}_{TT}^{\text{f.p.}}$ is defined as

$$\tilde{\mathcal{A}}_{TT}^{\text{f.p.}} \coloneqq \mathcal{A}_{TT}^{\text{f.p.}} + \frac{1}{2}\left(\frac{B_{\mathbb{V}\mathbb{1}}}{g}\right)^2 + \frac{1}{2} + C_{\mathbb{V}\mathbb{V}\mathbb{V}}\frac{B_{\mathbb{V}\mathbb{1}}}{g}\left(1+2\log(2)\right) . \tag{7.117}$$

This result is manifestly independent of any off-shell SFT data and is expressed purely

in terms of 2d CFT quantities.

7.5 Example: Virasoro minimal models

In this section, we will present an application of the above-described construction of open-closed SFT solutions to the explicit example of Virasoro minimal models at c close to 1. From the pure 2d CFT point of view, this setup has already been famously analyzed in the case of pure-bulk RG flows by Zamolodchikov in [100] and, in the case of pure-boundary perturbations, by Recknagel, Roggenkamp and Schomerus in [163]. Here, our aim will be to showcase how open-closed SFT can be conveniently used to evaluate the response of conformal boundaries to the Zamolodchikov perturbation in the bulk, to the order which would correspond to a *two-loop* calculation in conformal perturbation theory. Apart from confirming the conjectured flows of [164] at subleading order in perturbation theory, we will also show that the classical SFT solution describing the Zamolodchikov bulk RG flow changes the string coupling constant at a rate which is given by the same formula (in terms of the sphere 2-point function of the deforming operator) as in the case of the Narain marginal deformations discussed in section 7.2.

7.5.1 Review of bulk and boundary data

As we have advertised in the introductory paragraph, in this section we will assume that the matter sector of the world-sheet CFT contains a factor given by a unitary A-series Virasoro minimal model. These are 2d CFTs arising at a central charge $0 < c < 1$ which can be discretely parametrized as

$$c_m = 1 - \frac{6}{m(m+1)}\,, \tag{7.118}$$

where $m = 3, 4, 5, \ldots$ For such values of c, the allowed highest weight representations of the Virasoro algebra can, in turn, be parametrized by the Kac labels (r, s) where $1 \leq r \leq m-1$ and $1 \leq s \leq m$, with the identification $(r, s) \sim (m-r, m+1-s)$. The

corresponding weights $h_{(r,s)}$ are given as

$$h_{(r,s)} = \frac{[(m+1)r - ms]^2 - 1}{4m(m+1)} \,. \tag{7.119}$$

As we take $m \to \infty$, the leading contribution to the weights is determined purely by $|r-s|$, as we can write $h_{(r,s)} = \frac{1}{4}(r-s)^2 + \mathcal{O}(\frac{1}{m})$.

In the bulk, the highest-weight representations (r,s) give rise to the (diagonal) primary fields $\phi_{(r,s)}$ with scaling dimensions $\Delta_{(r,s)} = 2h_{(r,s)} = \frac{1}{2}(r-s)^2 + \mathcal{O}(\frac{1}{m})$. In particular, to implement a short RG flow in the bulk, one can study the relevant perturbation induced by the bulk field $\mathbb{V} = \phi_{(1,3)}$. Indeed, since its scaling dimension $\Delta_{(1,3)}$ reads

$$\Delta_{(1,3)} = 2 - \frac{4}{m+1} \coloneqq 2(1-y) \,, \tag{7.120}$$

we can see that $\phi_{(1,3)}$ approaches marginality as $m \to \infty$ (or, equivalently, as $c_m \to 1$). The expansion parameter y of the corresponding short RG flow (which measures the failure of the relevant perturbing operator $\mathbb{V}$ to be marginal) can thus be identified as

$$y = \frac{2}{m+1} \,. \tag{7.121}$$

Furthermore, it can be shown that the representation $(1,3)$ satisfies the fusion algebra

$$(1,3) \times (1,3) = (1,1) + (1,3) + (1,5) \,. \tag{7.122}$$

Since the representation $(1,5)$ gives rise to an irrelevant bulk field (with scaling dimension $\Delta_{(1,5)} = 8 + \mathcal{O}(y)$), this implies that the nearly marginal bulk field $\phi_{(1,3)}$ satisfies a fusion rule of the type (7.46), which is a necessary condition for the existence of a short RG flow. Additionally, normalizing the two-point function coefficients of all bulk primaries to unity, the three-point function coefficient $C_{(1,3)(1,3)(1,3)}(m)$ of three $\phi_{(1,3)}$ bulk fields turns out to be [100]

$$C_{(1,3)(1,3)(1,3)}(m) \equiv C_{\mathbb{V}\mathbb{V}\mathbb{V}} = \frac{4}{\sqrt{3}} \left(1 - \frac{3}{2}y + \mathcal{O}(y^2)\right) \tag{7.123}$$

in the small y limit (or, equivalently, in the large m limit by using (7.121)). Since

this is non-zero, all conditions for the existence of a nearby fixed point of the RG flow triggered by $\phi_{(1,3)}$ are met. As we have mentioned earlier, in the pure-bulk case, this RG flow was described already by Zamolodchikov in [100]. His analysis showed that flowing from the m^{th} minimal model in the UV, one ends up with the $(m-1)^{\text{th}}$ minimal model in the IR. This results in a change in the central charge

$$c^* - c = -\frac{3}{2}y^3 + \mathcal{O}(y^4)\,, \tag{7.124}$$

which starts at cubic order in y.

Regarding the open-string background, we will assume that the worldsheet matter boundary state factorizes into a minimal-model boundary state and some boundary state of the auxiliary CFT. The most general elementary boundary state in a diagonal Virasoro minimal model can be labeled by two integers $\boldsymbol{a} := (a_1, a_2)$, which run over the same range as the Kac labels (r,s) of highest-weight representations and are subject to the same identifications. As argued by Cardy [165], these boundary states can be expanded as

$$\|\boldsymbol{a}\rangle\!\rangle_m = \sum_{(r,s)} \frac{S^{(r,s)}_{(a_1,a_2)}}{\sqrt{S^{(r,s)}_{(1,1)}}} |r,s\rangle\!\rangle\,, \tag{7.125}$$

where the Ishibashi states $|r,s\rangle\!\rangle$ [166] are normalized as

$$\lim_{\tau\to\infty} \langle\!\langle r,s|e^{-2\pi\tau(L_0+\bar{L}_0-\Delta_{(r,s)})}|r',s'\rangle\!\rangle = \delta_{r,r'}\delta_{s,s'}\,, \tag{7.126}$$

and $S^{(r,s)}_{(a_1,a_2)}$ are the components of the modular S-matrix

$$S^{(r,s)}_{(a_1,a_2)}(m) = (-1)^{1+a_1 s+a_2 r}\sqrt{\frac{8}{m(m+1)}}\sin\left(\frac{m+1}{m}\pi a_1 r\right)\sin\left(\frac{m}{m+1}\pi a_2 s\right)\,. \tag{7.127}$$

To diagnose the effect of a short bulk RG flow on a boundary state, it will be crucial to investigate the structure of the bulk-boundary OPE of the perturbing bulk operator. In particular, for the $\phi_{(1,3)}$ bulk field in the presence of a general Cardy boundary

state $\boldsymbol{a}$, we can write

$$\phi_{(1,3)}(x+is,x-is) \overset{s\to 0}{\sim} \frac{1}{(2s)^{2(1-\frac{2}{m+1})}}\frac{B^{(\boldsymbol{a})}_{(1,3)(1,1)}(m)}{g^{(\boldsymbol{a})}(m)} + \frac{1}{(2s)^{1-\frac{2}{m+1}}}\frac{B^{(\boldsymbol{a})}_{(1,3)(1,3)}(m)}{g^{(\boldsymbol{a})}(m)}\,\psi_{(1,3)}(x) + \text{reg}\,. \tag{7.128}$$

where $\psi_{(1,3)} \coloneqq \mathbb{v}$ is the boundary field transforming in the $(1,3)$ representation of Virasoro algebra. Here $g^{(\boldsymbol{a})}$ denotes the g-function (boundary entropy) of the boundary state, which can be identified with the coefficient in the expansion (7.125) in front of the $(1,1)$ (identity) Ishibashi state. This means that we can write

$$g^{(\boldsymbol{a})}(m) \coloneqq \langle 0\|\boldsymbol{a}\rangle\!\rangle_m = \frac{S^{(1,1)}_{(a_1,a_2)}(m)}{\sqrt{S^{(1,1)}_{(1,1)}(m)}}\,. \tag{7.129}$$

Second, the coefficient $B^{(\boldsymbol{a})}_{(1,3)(1,1)}$ of the identity channel in the bulk-boundary OPE (7.128) can be equated with the coefficient of the $(1,3)$ Ishibashi states in (7.125), meaning that in terms of the modular S-matrix, we can write

$$\frac{B^{(\boldsymbol{a})}_{(1,3)(1,1)}(m)}{g^{(\boldsymbol{a})}(m)} \coloneqq \frac{\langle\phi_{(1,3)}\|\boldsymbol{a}\rangle\!\rangle_m}{g^{(\boldsymbol{a})}(m)} = \sqrt{\frac{S^{(1,1)}_{(1,1)}(m)}{S^{(1,3)}_{(1,1)}(m)}}\frac{S^{(1,3)}_{(a_1,a_2)}(m)}{S^{(1,1)}_{(a_1,a_2)}(m)}\,. \tag{7.130}$$

Finally, the OPE coefficient $B^{(\boldsymbol{a})}_{(1,3)(1,3)}(m)$ in front of $\psi_{(1,3)}$ can generally be computed using the 2-point function bootstrap on the upper-half plane. For arbitrary m, this was achieved in [167], where all structure constants of the A-series of Virasoro minimal models were expressed in terms of the fusing matrix F and the modular S-matrix. In appendix F, we derive the corresponding crossing equation in the strict $m\to\infty$ limit (see (F.13)), which suffices for our needs. In particular, note that for the boundary states of the type $\boldsymbol{a}=(1,a_2)$, the representation $(1,3)$ is actually not present in the spectrum of boundary fields, as can be explicitly checked using the Cardy condition. In such cases, the coefficient $B^{(\boldsymbol{a})}_{(1,3)(1,3)}(m)$ identically vanishes to all orders in y.

More generally, we will be interested in such boundary states $\boldsymbol{a}$ for which the ratio $B^{(\boldsymbol{a})}_{(1,3)(1,3)}/g^{(\boldsymbol{a})}$ vanishes at the leading order $\mathcal{O}(y^0)$. Recalling our discussion in section 7.3, this will guarantee perturbativity of the boundary deformation induced by the bulk RG flow triggered by $\phi_{(1,3)}$. As analyzed in more detail in appendix F, setting $B^{(\boldsymbol{a})}_{(1,3)(1,3)}$ to zero in the crossing relation (F.13) yields two distinct solutions for the coefficient $B^{(\boldsymbol{a})}_{(1,3)(1,1)}$ which, in turn, can be associated with two distinct classes of boundary states for which the bulk-induced boundary deformation remains perturbative. These classes will be described in more detail below.

7.5.2 Short RG flows in Virasoro minimal models with boundaries

In this subsection, we will briefly review the exhaustive analysis of [164], where the authors study the RG flows in the Virasoro minimal models with boundaries, which are induced by the nearly-marginal bulk field $\phi_{(1,3)}$, as well as by the nearly-marginal boundary field $\psi_{(1,3)}$ in the limit of large m. One of the main results of their work is a diagram which we reproduce in figure 7.2 and which summarizes the possible RG flows in this setup. In particular, the upper horizontal line in this diagram refers to the minimal model m, while the lower horizontal line refers to the minimal model $m-1$. This means that the horizontal arrows indicate pure-boundary RG flows while the vertical ones denote RG flows induced by the bulk field $\phi_{(1,3)}$. We can notice that the RG flow chain naturally organizes itself horizontally so that the value of a_2 increases from right to left. On the other hand, by changing the value of a_1, one would generally produce a separate chain.

In summary, starting from a boundary state with labels (a_1, a_2) in minimal model m, one can identify essentially two distinct sequences of deformations.

The first one connects the initial boundary state (a_1, a_2) with the fixed points (I), (II) and (III) (see figure 7.2). This proceeds 1. via a pure-boundary RG flow triggered by $\psi_{(1,3)}$ in the minimal model m which takes us from (a_1, a_2) to the fixed point (I), then 2. via a pure-bulk flow induced by $\phi_{(1,3)}$ which is interpolating between the fixed points (I) and (II) and then finally, 3. via a pure-boundary flow, again triggered by $\psi_{(1,3)}$ but this time in the minimal model $m-1$, mapping between the fixed points (II) and (III). Identifying the fixed points with specific boundary states in the two

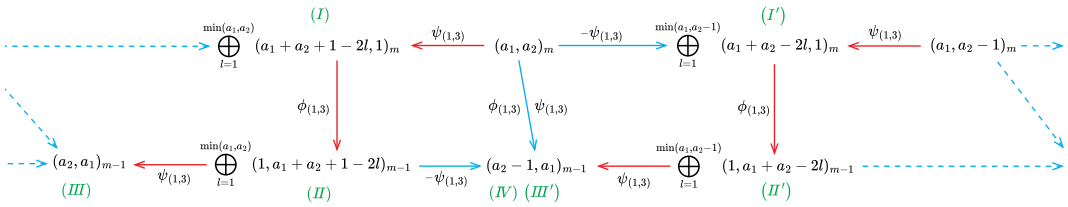


Figure 7.2: The chain of RG flows in Virasoro minimal models with boundaries. The upper horizontal line consists of pure-boundary RG flows in the minimal model m, while the lower line contains pure-boundary RG flows in the minimal model $m-1$. Vertical lines indicate pure bulk or bulk-boundary RG flows which provide the transition from the minimal model m to the minimal model $m-1$. In particular, here we display the piece of the chain which is generated by RG flows starting from the Cardy boundary state (a_1, a_2) where both a_1 and a_2 are kept fixed as we take the limit $m \to \infty$. Red lines denote short RG flows, blue lines indicate non-perturbative RG flows, and green Roman numerals mark specific RG fixed points referred to in the text.

respective minimal models, we can write this sequence of deformations as

$$\begin{array}{ccc} (a_1,a_2)_m & \longrightarrow & \bigoplus_{l=1}^{\min(a_1,a_2)} (a_1+a_2+1-2l,1)_m \\ & & \downarrow \\ & & \bigoplus_{l=1}^{\min(a_1,a_2)} (1,a_1+a_2+1-2l)_{m-1} \longrightarrow (a_2,a_1)_{m-1}\,, \end{array} \tag{7.131}$$

Crucially, for values a_1 and a_2 which are kept fixed as we take the limit $m \to \infty$, this sequence of deformations is *perturbative*. Correspondingly, in figure 7.2 it is highlighted in red color. In more detail, to explicitly confirm perturbativity of the sequence (7.131), we can use (7.129) to compute the corresponding change in the g-function. This gives

$$\frac{g^{(a_2,a_1)}(m-1)}{g^{(a_1,a_2)}(m)} = \sqrt{\frac{S^{(1,1)}_{(1,1)}(m)}{S^{(1,1)}_{(1,1)}(m-1)}}\,\frac{S^{(1,1)}_{(a_2,a_1)}(m-1)}{S^{(1,1)}_{(a_1,a_2)}(m)} = 1+\frac{3y}{4}+\frac{21y^2}{32}+\mathcal{O}(y^3)\,, \tag{7.132}$$

which goes to 1 as y approaches 0, consistently with our statement. Also notice that since, we are interested in computing the variation of the g-function only up to second order in y, our analysis will not be sensitive to those steps in the sequence (7.131) which consist of pure-boundary flows, namely steps 1 and 3, which gives

contribution at $\mathcal{O}(y^3)$. From the point of view of constructing the corresponding SFT solutions, these two steps would be reflected by turning on a perturbative open-string solution corresponding to a short boundary RG flow both at the level of the initial perturbative background around the boundary state $(a_1, a_2)_m$, as well as at the level of the small fluctuations around the vacuum-shift solution (7.87) which describes the fixed point (II). Hence, to the order in y up to which we are working, the on-shell disk action of the open-closed SFT solution computed in section 7.4 should encode the difference of g-functions associated with the deformation

$$(a_1, a_2)_m \to (a_2, a_1)_{m-1} \,, \tag{7.133}$$

which can be viewed as a streamlined version of the step-by-step deformation (7.131). As we have advertised in the previous section, this class of perturbative deformations is associated with one of the two solutions (F.14) for $B^{(\boldsymbol{a})}_{(1,3)(1,1)}/g^{(\boldsymbol{a})}$ of the crossing equation (F.13) where we put $B^{(\boldsymbol{a})}_{(1,3)(1,3)}$ to be $\mathcal{O}(y)$: indeed, for arbitrary labels a_1 and a_2, as long as we keep them fixed as we send $m \to \infty$, it follows from (7.130) that $B^{(\boldsymbol{a})}_{(1,3)(1,1)}/g^{(\boldsymbol{a})} = \sqrt{3} + \mathcal{O}(y^2)$.

The second type of deformation in this setup brings us from the boundary state $(a_1, a_2)_m$ to the fixed point (IV). As we can see on figure 7.2, this can proceed either via the fixed points (I) and (II) (from where a pure-boundary flow triggered by $\psi_{(1,3)}$ with the opposite sign must be followed), or, via the fixed points (I') and (II') by first triggering a pure-boundary flow by $-\psi_{(1,3)}$ in the minimal model m, then following a pure-bulk flow induced by $\phi_{(1,3)}$ and then finally, riding down yet another pure-boundary flow in the minimal model $m-1$. One can also follow an intermediate path by simultaneously turning a combination of bulk and boundary deformations starting from $(a_1, a_2)_m$. Altogether, this gives us the flow

$$(a_1, a_2)_m \to (a_2 - 1, a_1)_{m-1} \,, \tag{7.134}$$

which, if we keep the labels a_1 and a_2 fixed as we take $m \to \infty$, is *non-perturbative*. Indeed, in this case we have

$$\begin{aligned}\frac{g^{(a_2-1,a_1)}(m-1)}{g^{(a_1,a_2)}(m)} &= \sqrt{\frac{S^{(1,1)}_{(1,1)}(m)}{S^{(1,1)}_{(1,1)}(m-1)}}\frac{S^{(1,1)}_{(a_2-1,a_1)}(m-1)}{S^{(1,1)}_{(a_1,a_2)}(m)} \\ &= 1 - \frac{1}{a_2} + \frac{3(a_2-1)y}{4a_2} + \mathcal{O}(y^2)\,, \end{aligned} \tag{7.135}$$

which does *not* approach 1 in the small y limit. On the other hand, near the middle of the chain displayed in figure 7.2, specifically for $a_2 = \frac{m+1}{2} + \alpha$, where we are fixing α as we take $m \to \infty$, perturbativity of the flow (7.134) is achieved, as can be seen by analyzing the g-functions

$$\begin{aligned}\frac{g^{((m-1)/2+\alpha,a_1)}(m-1)}{g^{(a_1,(m+1)/2+\alpha)}(m)} &= \sqrt{\frac{S^{(1,1)}_{(1,1)}(m)}{S^{(1,1)}_{(1,1)}(m-1)}}\frac{S^{(1,1)}_{((m-1)/2+\alpha,a_1)}(m-1)}{S^{(1,1)}_{(a_1,(m+1)/2+\alpha)}(m)} \\ &= 1 - \frac{y}{4} - \frac{3y^2}{32} + \mathcal{O}(y^3)\,. \end{aligned} \tag{7.136}$$

Therefore, near the middle of the chain, a second independent *short* RG flow

$$\left(a_1, \tfrac{m+1}{2} + \alpha\right)_m \to \left(\tfrac{m-1}{2} + \alpha, a_1\right)_{m-1} \tag{7.137}$$

becomes available. It can be readily checked using (7.130) that for such values of the labels a_1 and a_2, the bulk-boundary OPE coefficient of the bulk field $\phi_{(1,3)}$ in front of the boundary identity channel becomes $B^{(\boldsymbol{a})}_{(1,3)(1,1)}/g^{(\boldsymbol{a})} = -\frac{1}{\sqrt{3}} + \mathcal{O}(y^2)$, which can be identified with the second solution of the crossing equation (F.13).

7.5.3 SFT analysis

Our final goal in this chapter will be to confirm the two independent short RG flows (7.133) and (7.137) up to second order in y by computing the change in the g-functions using the SFT framework which we described in great detail in sections 7.3 and 7.4. In conformal perturbation theory, such a calculation would require going to the two-loop level. On the other hand, in our SFT scheme, one only needs to

compute manifestly finite one-dimensional integrals, as it is apparent from the r.h.s. of the final formula (7.116) for the on-shell disk action of the SFT solution describing short bulk RG flows in the presence of conformal boundaries.

Also recall that the quantity which the on-shell SFT disk action computes is really the difference in the worldsheet disk partition functions between the corresponding two string backgrounds. As such, it is generally sensitive not only to a change in the boundary-state g-function, but also to a possible change in the value of the string coupling constant κ. This is clearly manifested on the l.h.s. of (7.116). Thus, we need to devise a method of separating these two pieces of information. As in the case of the Narain deformations of compact free bosons which we discussed in section 7.2, our strategy will be to show that for both of the aforementioned classes (7.133) and (7.137) of short RG flows in Virasoro minimal models, the SFT on-shell disk action (7.116) can indeed be identified with the ratio $\frac{g^*}{g}$ of the g-functions corresponding to the endpoints of the flows *times* a universal factor, which is the same for both classes of flows (thus independent of the open-string solution $\Psi^*(y)$ and only sensitive to the closed-string solution $\Phi^*(y)$). This universal factor will then be identified with the ratio $(\frac{g_{\rm s}^*}{\kappa})^{-1}$. Such a strategy is equivalent to *assuming* that the change in the g-function is known for one of the flows and using this knowledge to fix the change in the string coupling constant. Then the SFT on-shell disk action *predicts* the change in the g-function for the second flow and thus serves as an indepedent check of the (conjectured) RG flow.

Let us start by quantifying various ingredients needed for evaluation of the on-shell disk action, as given by the final result (7.116) of section 7.4. Recall that the main non-trivial ingredient entering the subleading order of this calculation was the finite part $\mathcal{A}_{TT}^{\rm f.p.}$ of the two-point disk amplitude of two slightly off-shell tachyons, introduced in (7.112). Given the analysis of appendix F, we can now substitute the explicit expression for the regulated correlator (F.16) of two bulk fields $\phi_{(1,3)}$ on the UHP, obtaining

$$\mathcal{A}_{TT}^{\rm f.p.} = -1 - \frac{B_{\mathbb{V}\mathbb{1}}}{g}\, C_{\mathbb{V}\mathbb{V}\mathbb{V}}\, (1 + 2\log 2)\,. \tag{7.138}$$

At the same time, we recall the result $\mathcal{A}_{TTTT}^{\rm f.p.} = -8$ of [67] for the finite part of the sphere amplitude of four slightly off-shell tachyons. Finally, we also remember the

y-expansion (7.123) of the three-point structure constant $C_{(1,3)(1,3)(1,3)}$, as well as the the observation that in the case of both flows (7.133) and (7.137), the $m \to \infty$ limit of the bulk-boundary coefficient $B^{(\boldsymbol{a})}_{(1,3)(1,1)}/g^{(\boldsymbol{a})}$ only starts receiving finite m corrections at the order $\mathcal{O}(y^2)$.

In view of these results, the general formula (7.116) becomes

$$\begin{aligned}\frac{g^*}{g}\left(\frac{\kappa^*}{\kappa}\right)^{-1} &= \\ &= \left(1+\frac{\sqrt{3}}{4}\frac{B_{\mathbb{V}\mathbb{1}}}{g}\,y+\frac{3}{8}\frac{B_{\mathbb{V}\mathbb{1}}}{g}\left(\frac{1}{\sqrt{3}}+\frac{1}{4}\frac{B_{\mathbb{V}\mathbb{1}}}{g}\right)y^2+\mathcal{O}(y^3)\right)\left(1+\frac{3y^2}{32}+\mathcal{O}(y^3)\right)^{-1},\end{aligned} \tag{7.139}$$

where, at this point, the ratio $B_{\mathbb{V}\mathbb{1}}/g$ can be replaced with its strict $m \to \infty$ limit. Also, we have rearranged the expression so that the $\mathcal{O}(y^2)$ term which does not depend on the boundary conditions is explicitly factored away from the rest, in a hope that we will be able to attribute this universal factor to the ratio $(\frac{\kappa^*}{\kappa})^{-1}$ of the old and the new string coupling constant. We will now provide an unambiguous check for this expectation by showing that the first factor on the r.h.s. of (7.139) precisely reproduces the ratio $\frac{g^*}{g}$ of the g-functions of the endpoints of the conjectured flows (7.133) and (7.137).

Considering first the flow (7.133), which starts from the boundary state $(a_1,a_2)_m$ with both a_1, a_2 fixed as we take $m \to \infty$, we can substitute

$$\frac{B_{\mathbb{V}\mathbb{1}}}{g} \to \frac{B^{(a_1,1)}_{(1,3)(1,1)}(m)}{g^{(a_1,1)}(m)} = \sqrt{\frac{S^{(1,1)}_{(1,1)}(m)}{S^{(1,3)}_{(1,1)}(m)}\frac{S^{(1,3)}_{(a_1,1)}(m)}{S^{(1,1)}_{(a_1,1)}(m)}} = \sqrt{3}+\mathcal{O}(y^2)\,, \tag{7.140}$$

to show that the first factor in (7.139) evaluates to

$$1+\frac{3y}{4}+\frac{21y^2}{32}+\mathcal{O}(y^3)\,, \tag{7.141}$$

which stands in precise agreement with the large-m expansion (7.132) of the ratio $\frac{g^{(a_2,a_1)}(m-1)}{g^{(a_1,a_2)}(m)}$ of g-functions for the expected endpoints of the flow (7.133).

Similarly, we can consider the remaining class of flows (7.137), for which the UV BCFT is given by the boundary state $(a_1,a_2)_m$ with $a_2 = \frac{m+1}{2}+\alpha$ with both a_1 and α

kept fixed in the limit $m \to \infty$. In this case, substituting

$$\frac{B_{\mathbb{V}\mathbb{1}}}{g} \to \frac{B^{(a_1,(m+1)/2+\alpha)}_{(1,3)(1,1)}(m)}{g^{(a_1,(m+1)/2+\alpha)}(m)} = \sqrt{\frac{S^{(1,1)}_{(1,1)}(m)}{S^{(1,3)}_{(1,1)}(m)}}\frac{S^{(1,3)}_{(a_1,(m+1)/2+\alpha)}(m)}{S^{(1,1)}_{(a_1,(m+1)/2+\alpha)}(m)} = -\frac{1}{\sqrt{3}} + \mathcal{O}(y^2)\,, \tag{7.142}$$

into the first factor in (7.139) yields

$$1 - \frac{y}{4} - \frac{3y^2}{32} + \mathcal{O}(y^3)\,, \tag{7.143}$$

which exactly matches the large-m expansion (7.136) of the ratio $\frac{g^{((m-1)/2+\alpha,a_1)}(m-1)}{g^{(a_1,(m+1)/2+\alpha)}(m)}$.

The results (7.141) and (7.143) confirm that the value of the SFT on-shell disk action factorizes 1. into the ratio $\frac{g^*}{g}$ of the g-functions of the initial BCFT and the BCFT describing the classical solution and 2. into the ratio $(\frac{\kappa^*}{\kappa})^{-1}$ of the corresponding string coupling constants. At the same time, the latter can be isolated as

$$\frac{\kappa^*}{\kappa} = 1 + \frac{3y^2}{32} + \mathcal{O}(y^3) = 1 + \frac{1}{8}\left(\frac{2y}{C_{\mathbb{V}\mathbb{V}\mathbb{V}}}\right)^2 + \mathcal{O}(y^3) = 1 + \frac{1}{8}t(y)^2 + \mathcal{O}(y^3)\,. \tag{7.144}$$

Finally, we note that the quantity $t(y)^2$ can be identified with the sphere two-point function of the deforming matter CFT operator $t(y)\,\mathbb{V}$. Thus, comparing (7.144) with (7.42), we conclude that, at least to the leading order in y, the change in the string coupling constant is described by the *same* expansion in terms of the sphere two-point function of the deforming operator as in the case of Narain deformations.

CHAPTER 8

Conclusion

8.1 Summary and outlook

This thesis has presented several developments in bosonic SFT, addressing problems related to boundary terms, D-brane backreaction, stub deformations, and bulk-induced RG flows.

In Chapter 3, we constructed a gauge-invariant kinetic term for open and closed bosonic SFT on a flat target space with boundary. Gauge invariance was restored by promoting the gauge parameter at the boundary to a dynamical boundary string field, whose transformation cancels the boundary variation. The construction was illustrated with explicit examples in both the open and closed string sectors, including the linear dilaton background. In these examples, we restricted ourselves to half-Euclidean flat space with boundary given by the hyperplane at $z = 0$. It would be natural to revisit the computation for more general boundary geometries. Our construction applies to arbitrary boundary shapes, but as shown in Appendix A, the standard Einstein–Hilbert action with the Gibbons–Hawking–York term produces a boundary tadpole proportional to

$$S_{\text{tadpole}}(h) = \int_{\partial M} d^{d-1}y \, \sqrt{\gamma} \, (K\gamma^{\alpha\beta} - K^{\alpha\beta}) \, h_{\alpha\beta} \,, \tag{8.1}$$

which does not vanish even in flat space. Such a contribution may play a crucial role for gauge invariance once interactions are included. It remains an open problem to understand how our action describes free strings on manifolds with generic boundaries, given that it apparently requires neither tadpoles nor interactions to restore gauge invariance.

A second interesting future direction is to apply our construction to black-hole-type backgrounds with the ambitious goal of describing black holes from the SFT perspective. The first natural example would be the 2d CGHS black hole solution with linear dilaton, characterized by the metric

$$ds^2 = (1 - \lambda e^{V\varphi})dt^2 + (1 - \lambda e^{V\varphi})^{-1} d\varphi^2 \,, \tag{8.2}$$

where t is the Euclidean time, φ is the unique spatial direction, V is the background

charge, and the parameter λ is proportional to the black hole mass. It is possible to show that this metric is a solution to the equation of motion of (3.145) for a dilaton with linear profile

$$\Phi^* = \frac{1}{2} V \varphi \,. \tag{8.3}$$

This is the perfect example to begin with because it is known how this background can be obtained from the SFT perspective. Indeed, in [46] the authors presented the perturbative construction of the classical closed-string solution describing this 2d black hole. Therefore, we can in principle compare the results coming from the theory of fluctuations of the CGHS model around the black hole solution with a flat boundary to the results coming from our SFT action expanded around the solution described in [46]. One immediately notices a complication arising from the CGHS side, where a boundary tadpole appears (which cannot be removed simply by introducing a boundary cosmological constant following the strategy described in 3.4.4). Thus, similarly to what was discussed above in the context of more general boundary geometries, we must understand in what sense our construction provides a consistent gauge-invariant formulation without reproducing this type of term.

A third and more fundamental direction concerns the extension of our construction to the full interacting SFT action. From this perspective, it will be crucial to understand how the requirement of a well-defined variational principle should be adapted once interactions are included, where the presence of the string coupling constant leads to intrinsically non-local effects and makes the problem substantially more challenging.

In Chapter 5, we reformulated open–closed SFT in the coalgebraic language, deriving the nilpotent structure that governs quantum consistency within the BV framework. Using this construction, we described D-brane backreaction in the large-N limit and explicitly constructed the classical closed SFT describing the resulting background without branes. We also observed that this double process of integration-out and vacuum shift is potentially obstructed by massless propagating open-string states. These can be absent in very special string backgrounds, such as topological or minimal ST, but otherwise must be carefully handled—via a not yet fully understood

"decoupling limit"—as in the standard AdS/CFT case starting with D3-branes in flat space.

The natural next step in this line of research would be to apply the method described in 5, focusing on finding an explicit example in which the open-string sector can be fully integrated out. A promising candidate is the $(2,1)$ minimal ST with N FZZT branes (see [168, 169]). As described in [138], the backreaction of the FZZT branes in this case can be completely absorbed by changing the closed-string background. Specifically, the authors of [138] noted that the resulting bulk background is a $(2,2k+1)$ minimal string, depending on the boundary cosmological constants that characterize the FZZT branes. Another potential model of interest is the recently proposed Complex Liouville string by S. Collier et al. [170].

Furthermore, the construction presented in Chapter 5 is based on the bosonic string, but the homotopy structure we discussed is expected to be realized in the superstring as well (although it has not yet been explicitly constructed; see [125, 171] for recent progress). Therefore, our picture is expected to be universal for any ST containing D-branes. This includes theories that are consistent at the quantum level, such as the $c=1$ and the minimal string theories (which are specific examples of bosonic strings), the topological string, and of course Type IIA/IIB superstring theory. By generalizing our construction to non-orientable world-sheets [129], we could also treat, along the same lines, the $SO(32)$ Type I superstring.

In Chapter 6, we developed a systematic procedure for adding stubs to generic classical and quantum SFTs, proving that tree- and loop-level amplitudes remain invariant under such deformations.

According to our picture, adding stubs also generates constant terms in the action through the path integral over the stub-rotated auxiliary field Σ. In the main text we ignored these constants, but it is important to understand their origin. The first contribution is the one-loop determinant of the Σ kinetic term. Since this coincides with the one-loop determinant of the χ kinetic term, it must be canceled by a counterterm in the original extended action (6.12) if integrating out χ is to reproduce the initial theory exactly. Additional vacuum terms arise from the interactions between Σ and ψ and start at two loops. Their presence ensures that

the stubbed and original actions lead to the same vacuum energy once reduced to the (almost) minimal model, i.e. after integrating out the full $\bar{P}\mathcal{H}$. This is consistent with (quantum) background independence and the invariance of the path-integral measure under field redefinitions [20]. We expect this equivalence to hold only after including the constant terms generated by integrating out Σ (apart from the one-loop determinant). It would be valuable to check this systematically, likely extending the arguments of Section 6.3. Technically, it would also be interesting to understand how these vacuum bubbles can be described using the same co-algebraic homotopy transfer that generates amplitudes with external legs.

Another aspect that deserves further study is the invertibility of stub deformations. Although the stub deformation is formally a field redefinition, it is not strictly invertible: geometrically, the inverse operator $e^{+\lambda L_0}$ removes a strip of worldsheet, an operation not always well defined. Still, one might expect the inverse stub to make sense within a restricted domain—for example, it should be possible to remove stubs from a theory to which they have just been added, provided the inverse stub is shorter than the original. Introducing inverse stubs may also have useful applications. In closed SFT, inserting inverse stubs in the kinetic operator could regulate the one-loop determinant, which otherwise reproduces the torus vacuum amplitude infinitely many times over equivalent fundamental domains [172]. Likewise, in open–closed SFT, the vacuum energy obtained after integrating out the open strings and performing the closed-string vacuum shift, as discussed in Chapter 5, seems to overcount the annulus moduli space. Inverse open-string stubs could help resolve this redundancy. More broadly, "quantum stubs" might offer new tools for addressing the long-standing problem of the cosmological constant in SFT.

Finally, open–closed SFT admits a natural "interpolating" form [161], where in different regions of the interpolation the moduli space is predominantly covered by open or closed string propagators [162, 173]. It would be interesting to relate these interpolations to open and closed stub deformations, which again seems to require the ability to simultaneously lengthen closed-string stubs while shortening open-string stubs (and vice versa).

Finally, in chapter 7 we computed the on-shell disk action at next-to-leading order

for the perturbative solution associated with a short RG flow triggered by a nearly marginal bulk operator in a matter bulk CFT with a probe D-brane. This computation allows us to determine the shift in the g-function at second order in y, circumventing the notoriously difficult two-loop calculation required by the standard CPT approach.

We further found that, both in this scenario and in the case of exactly marginal deformations, there is a shift in the string coupling constant proportional to the sphere two-point function of the perturbing matter bulk operator T

$$\frac{\kappa^*}{\kappa} = 1 + \frac{1}{8}\frac{\langle T, T\rangle}{\langle 0|0\rangle}\,, \tag{8.4}$$

This appears to be a universal feature, although the underlying reason remains an open question.

An interesting direction for future work is to investigate whether the change in the string coupling constant can be captured using purely closed string methods. Indeed, while we have employed D-branes as probes to detect this effect, it originates entirely from the closed-string sector. It should therefore be possible, at least in principle, to reproduce this shift within the framework of closed SFT alone.

Overall, this thesis has advanced our understanding of bosonic SFT by providing the construction of a well-defined gauge-invariant quadratic action on a target space with boundary, clarifying the structure of open–closed SFT and D-brane backreaction, developing systematic stub deformations, and applying SFT to compute bulk-induced RG flows.

Beyond the specific results presented here, the broader message emerging from our analysis is the remarkable power of SFT as a conceptually robust and versatile framework for addressing fundamental problems in ST and CPT. In this sense, the achievements of the previous chapters should not be regarded as isolated technical results, but rather as concrete demonstrations of how SFT can open new and fertile lines of research.

Among these, two directions appear particularly promising. The first is the extension of our construction for the quadratic action in a target space with boundary to the full interacting SFT action. Such a development would represent a major conceptual step. A complete understanding of boundary effects in interacting SFT

would provide the tools to better interpret the physical meaning of the closed SFT action evaluated on classical solutions, even those that are not well behaved at infinity. In particular, it could help to reconcile the apparently discordant results of [66] and [67]: the former showing that the closed SFT action vanishes on a solution in the absence of boundary terms, and the latter demonstrating that the action evaluated on the non-normalizable solution (7.3.2) computes the shift in the central charge. Furthermore, because gravity is a subsector of SFT, achieving full control over interactions in the presence of boundaries could grant access to genuinely gravitational measurable quantities, like the ADM observables [174] which, for black hole solutions, are fundamentally captured by the GHY term.

The second major direction concerns the use of SFT as a framework for describing RG flows. As demonstrated in Chapter 7, SFT provides an efficient and conceptually clear method for computing observables in RG flows that are difficult to obtain using conventional CPT techniques. The natural next step is to extend this approach so as to access additional observables, both for boundary and bulk flows and for a wider class of string backgrounds. Very recent progress in open SFT—such as the general formulae for the IR boundary state, boundary spectrum, and structure constants in short boundary RG flows [175]—reinforces this expectation and points to a rich landscape of applications.

APPENDIX A

Linearizing gravity around flat space with a flat boundary

Let us now explore in detail how the gauge-invariant massless spin-2 actions on a flat half-space with a flat boundary which we presented in Sections 3.4.3 arise from linearizing the Einstein-Hilbert action of general relativity supplemented by the Gibbons-Hawking-York boundary term. For simplicity, we set the Kalb–Ramond field and the dilaton to zero. The generalization to the full case is similar and adds no new conceptual insight.

A.1 Rigid boundary

We will start our presentation by computing the second variation of the pure-gravity action

$$S_{\rm GR} = S_{\rm EH} + S_{\rm GHY} = \int_M d^d x\, \sqrt{g}\, R + \int_{\partial M} d^{d-1} y\, \sqrt{\gamma}\, 2K\,. \tag{A.1}$$

with respect to the bulk metric $g_{\mu\nu}$, keeping the boundary ∂M of the (topological) manifold M rigid.

Let us begin by considering the first variation of (A.1) with respect to the bulk metric $g_{\mu\nu}$, namely

$$\delta_g S_{\rm GR} = \int_M d^d x\, \delta_g(\sqrt{g} R) + \int_{\partial M} d^{d-1} y\, \delta_g(\sqrt{\gamma}\, 2K)\,. \tag{A.2}$$

We will mostly follow the detailed exposition of [176]. Performing standard manipulations, one first finds that the variation of the Einstein-Hilbert lagrangian takes the form

$$\delta_g(\sqrt{g}R) = \sqrt{g}\left(\frac{1}{2}Rg^{\alpha\beta} - R^{\alpha\beta}\right)\delta g_{\alpha\beta} + \sqrt{g}\,\nabla_\mu(g^{\alpha\beta}\delta_g\Gamma^\mu_{\alpha\beta} - g^{\alpha\mu}\delta_g\Gamma^\beta_{\beta\alpha})\,. \tag{A.3}$$

meaning that the r.h.s. of (A.2) becomes

$$\begin{aligned}\delta_g S_{\rm GR} = \int_M d^d x\, \sqrt{g}\left(\tfrac{1}{2}Rg^{\alpha\beta} - R^{\alpha\beta}\right)\delta g_{\alpha\beta} + \\ + \int_{\partial M} d^{d-1} y \sqrt{\gamma}\, n_\mu\Big(g^{\alpha\beta}\delta_g\Gamma^\mu_{\alpha\beta} - g^{\alpha\mu}\delta_g\Gamma^\beta_{\beta\alpha}\Big) + \int_{\partial M} d^{d-1} y\, \delta_g(\sqrt{\gamma}\, 2K)\,.\end{aligned} \tag{A.4}$$

As expected, the first variation is proportional to the Einstein field equations up to a boundary term.

A.1.1 Essential hypersurface toolkit

Our goal is now to show that we can rewrite the second line of (A.4) in terms of the symmetric extrinsic curvature

$$K_{\alpha\beta} = \nabla_\alpha n_\beta - n_\alpha a_\beta \,, \tag{A.5}$$

where we have introduced the acceleration 1-form $a_\beta = n^\lambda \nabla_\lambda n_\beta$ which is tangential to ∂M. The induced metric γ_{ab} on the boundary can be introduced as

$$\gamma_{ab} = e^\alpha_a \, e^\beta_b \, g_{\alpha\beta} \,, \tag{A.6}$$

where, defining the boundary in terms of an embedding map $x^\alpha(y^a)$, the transformation matrices e^α_a can be defined as

$$e^\alpha_a = \frac{\partial x^\alpha(y)}{\partial y^a} \,. \tag{A.7}$$

At the same time, these give the components of the $d-1$ vector fields which are tangent to ∂M. The inverse induced metric γ^{ab} also satisfies

$$e^\alpha_a \, e^\beta_b \gamma^{ab} = g^{\alpha\beta} - n^\alpha n^\beta \coloneqq \gamma^{\alpha\beta} \,. \tag{A.8}$$

The trace K of $K_{\alpha\beta}$ can then be expressed as

$$K = g^{\alpha\beta} K_{\alpha\beta} = \gamma^{\alpha\beta} K_{\alpha\beta} = \nabla_\alpha n^\alpha \,, \tag{A.9}$$

since $n^\alpha K_{\alpha\beta} = n^\alpha a_\alpha = 0$. Furthermore, note that defining the hypersurface ∂M implicitly as

$$F(x) = 0 \tag{A.10}$$

for some function F on M, the expression

$$n_\alpha = \frac{\partial_\alpha F}{\sqrt{g^{\rho\sigma}\partial_\rho F \partial_\sigma F}}\,. \tag{A.11}$$

defines the unit form field normal to ∂M. Finally, the *intrinsic* covariant derivative of a one-form field $A_a = e^\alpha_a A_\alpha$ where A_α is tangential to ∂M (that is $n^\alpha A_\alpha = 0$) can be defined as

$$D_a A_b = e^\alpha_a e^\beta_b \nabla_\alpha A_\beta\,. \tag{A.12}$$

This can be shown to coincide with the covariant derivative on ∂M defined in terms of the Levi-Civita connection compatible with the induced metric γ_{ab} which is used to raise and lower latin indices. Given $D_a A_b$, let us also define the induced covariant derivative

$$D^\alpha A^\beta \coloneqq e^\alpha_a e^\beta_b D^a A^b = \gamma^\alpha{}_\mu\, \gamma^\beta{}_\nu \nabla^\mu A^\nu\,. \tag{A.13}$$

In particular, we note the relation

$$D_\alpha A^\alpha = g_{\alpha\beta} e^\alpha_a e^\beta_b D^a A^b = \gamma_{ab} D^a A^b = D_a A^a \tag{A.14}$$

between the induced and the intrinsic covariant divergences.

A.1.2 Boundary terms of the first variation

Given this preparation, we first notice that we can rewrite the variations of the Levi-Civita connection appearing in (A.4) as

$$n_\mu \left(g^{\alpha\beta}\delta_g \Gamma^\mu_{\alpha\beta} - g^{\alpha\mu}\delta_g \Gamma^\beta_{\beta\alpha} \right) = \nabla_\alpha\, \delta u^\alpha - 2\,\delta_g K + \nabla_\alpha n_\beta\, \delta_g g^{\alpha\beta} \tag{A.15}$$

where we have introduced a tangent vector field

$$\delta u^\alpha = \delta_g n^\alpha + g^{\alpha\beta}\delta_g n_\beta\,, \tag{A.16}$$

and recalled the form $K = \nabla_\mu n^\mu$ for the trace of the extrinsic curvature tensor of ∂M. At the same time, after some straightforward manipulations, we can rewrite

$$D_a \delta u^a = D_\alpha \delta u^\alpha = \nabla_\alpha \delta u^\alpha + \delta u^\alpha a_\alpha \,. \tag{A.17}$$

This enables us to further rewrite the r.h.s. of (A.15) as

$$n_\mu \Big(g^{\alpha\beta} \delta_g \Gamma^\mu_{\alpha\beta} - g^{\alpha\mu} \delta_g \Gamma^\beta_{\beta\alpha} \Big) = D_a \delta u^a - 2\, \delta_g K - \delta u^\alpha \, a_\alpha + \nabla_\alpha n_\beta \, \delta_g g^{\alpha\beta} \,. \tag{A.18}$$

Finally, we can compute

$$\delta u^\alpha \, a_\alpha = a_\alpha n_\beta \, \delta_g g^{\alpha\beta} + 2 a^\beta \delta n_\beta \,, \tag{A.19}$$

where, assuming that the boundary is rigid, the variation $\delta_g n_\beta$ can be expressed as (recalling the form (A.11) of n_β)

$$\delta_g n_\beta = -\frac{1}{2} n_\beta \, \frac{\partial_\rho F \, \partial_\sigma F \, \delta g^{\rho\sigma}}{\partial_\rho F \, \partial_\sigma F \, g^{\rho\sigma}} \,. \tag{A.20}$$

As we can see that $\delta_g n_\beta$ is parallel to n_β, we conclude that the last term of (A.19) vanishes. Hence, we can finally rewrite

$$n_\mu \Big(g^{\alpha\beta} \delta_g \Gamma^\mu_{\alpha\beta} - g^{\alpha\mu} \delta_g \Gamma^\beta_{\beta\alpha} \Big) = D_a \delta u^a - 2\, \delta_g K + K_{\alpha\beta} \, \delta_g g^{\alpha\beta} \tag{A.21}$$

where we recognized the extrinsic curvature tensor $K_{\alpha\beta} = \nabla_\alpha n_\beta - n_\alpha a_\beta$. Substituting the result (A.21) back into (A.4), we therefore find

$$\begin{aligned} \delta_g S_{\rm GR} = & \int_M d^d x \, \sqrt{g} \left(\tfrac{1}{2} R g^{\alpha\beta} - R^{\alpha\beta} \right) \delta g_{\alpha\beta} + \\ & - \int_{\partial M} d^{d-1} y \sqrt{\gamma} \, K^{\alpha\beta} \, \delta g_{\alpha\beta} + \int_{\partial M} d^{d-1} y \, \delta_g(\sqrt{\gamma}) \, 2K + \int_{\partial M} d^{d-1} y \, \sqrt{\gamma} \, D_a \delta u^a \,, \end{aligned} \tag{A.22}$$

where we will assume that the last term can be dropped without picking any contributions from the boundaries of ∂M (corners of M). It remains to express

$$\delta_g\sqrt{\gamma} = \frac{1}{2}\sqrt{\gamma}\,\gamma^{ab}\delta_g\gamma_{ab} = \frac{1}{2}\sqrt{\gamma}\,\gamma^{ab}e^{\alpha}_{a}e^{\beta}_{b}\delta g_{\alpha\beta} = \frac{1}{2}\sqrt{\gamma}\,\gamma^{\alpha\beta}\delta g_{\alpha\beta}\,, \tag{A.23}$$

where, since the boundary ∂M is being kept rigid, we have used that $\delta_g e^{\alpha}_{a} = 0$. Therefore, the first variation of the action (A.1) with respect to the bulk metric $g_{\mu\nu}$ takes the final form

$$\delta_g S_{\rm GR} = \int_M d^dx\,\sqrt{g}\,(\tfrac{1}{2}Rg^{\alpha\beta} - R^{\alpha\beta})\,\delta g_{\alpha\beta} + \int_{\partial M} d^{d-1}y\,\sqrt{\gamma}\,(K\gamma^{\alpha\beta} - K^{\alpha\beta})\,\delta g_{\alpha\beta}\,. \tag{A.24}$$

A.1.3 Second variation in flat half-space

We can now proceed with varying the action with respect to the bulk metric once more. In order not to get overwhelmed with complicated expressions and, at the same time, having in mind our final goal of finding an action for massless spin-2 fluctuations around a flat half-space bounded by a flat boundary, we will be substituting all bulk and boundary quantities with their flat-space values as we go. In particular, using the cartesian coordinates $x^\mu = (y^a, z)$ introduced in Section 3.4, we will be setting

$$\overline{g}_{\mu\nu} = \eta_{\mu\nu} = \text{diag}[+1, \cdots, +1]\,, \tag{A.25a}$$

$$\overline{R} = 0\,, \tag{A.25b}$$

$$\overline{K} = 0\,, \tag{A.25c}$$

$$\overline{n}_\mu = (0, \ldots, 0, 1) = \delta^z_\mu\,. \tag{A.25d}$$

The flat half-space M will be taken to extend for $z < 0$. Notice that this is indeed a consistent critical background as it makes the first variation (A.24) vanishing.

Let us start by writing down the second variation as

$$\delta_g^2 S_{\rm GR}|_{\rm flat} = \int_M d^d x \left(\tfrac{1}{2}\eta^{\alpha\beta}\delta_g R - \delta_g R^{\alpha\beta}\right)\delta g_{\alpha\beta} + \\ + \int_{\partial M} d^{d-1}y \left[(\eta^{\alpha\beta} - \delta_z^\alpha \delta_z^\beta)\delta_g K - \delta_g K^{\alpha\beta}\right]\delta g_{\alpha\beta}\,, \tag{A.26}$$

where we have noted that the variation has to hit the curvatures, otherwise we get a zero upon substituting for the background values (A.25). To evaluate the r.h.s. of (A.26), we have to compute the following

$$\delta_g R|_{\rm flat} = \partial^\rho\partial^\sigma\delta g_{\rho\sigma} - \eta^{\rho\sigma}\partial^2\delta g_{\rho\sigma}\,, \tag{A.27a}$$

$$\delta_g R^{\alpha\beta}|_{\rm flat} = \frac{1}{2}\eta^{\alpha\nu}\eta^{\beta\sigma}\Big(\partial^\rho\partial_\nu\delta g_{\rho\sigma} + \partial^\rho\partial_\sigma\delta g_{\rho\nu} - \partial^2\delta g_{\nu\sigma} - \partial_\sigma\partial_\nu\eta^{\lambda\mu}\delta g_{\lambda\mu}\Big)\,, \tag{A.27b}$$

as well as

$$\delta_g n_\beta|_{\rm flat} = -\frac{1}{2}\delta^z_\beta\,\delta g^{zz} = +\frac{1}{2}\delta^z_\beta\,\delta g_{zz}\,, \tag{A.28}$$

where we have used A.20 to derive the last result. Notice that $\delta_g n_\beta|_{\rm flat}$ is non-vanishing only along the z component. Besides that, we will need the result

$$\delta_g a_\beta|_{\rm flat} = \delta^\nu_z(\partial_\nu\delta_g n_\beta - \delta^z_\rho\,\delta_g\Gamma^\rho{}_{\nu\beta}) = \frac{1}{2}\delta^z_\beta\partial_z\delta g_{zz} - \delta_g\Gamma^z_{z\beta} = \frac{1}{2}\delta^z_\beta\partial_z\delta g_{zz} - \frac{1}{2}\partial_\beta\delta g_{zz}\,. \tag{A.29}$$

This enables us to compute the variation of the extrinsic curvature, namely

$$\delta_g K_{\alpha\beta}|_{\rm flat} = \partial_\alpha\delta_g n_\beta - \delta_g\Gamma^\rho_{\alpha\beta}\delta^z_\rho - \delta^z_\alpha\delta_g a_\beta \tag{A.30a}$$

$$= \frac{1}{2}\delta^z_\beta\partial_\alpha\delta g_{zz} - \delta_g\Gamma^z_{\alpha\beta} - \frac{1}{2}\delta^z_\alpha(\delta^z_\beta\partial_z\delta g_{zz} - \partial_\beta\delta g_{zz})\,, \tag{A.30b}$$

or, equivalently,

$$\delta_g K_{ab}\big|_{\text{flat}} = -\frac{1}{2}(\partial_a \delta g_{zb} + \partial_b \delta g_{za} - \partial_z \delta g_{ab})\,, \tag{A.31a}$$

$$\delta_g K_{zz}\big|_{\text{flat}} = 0\,, \tag{A.31b}$$

$$\delta_g K_{za}\big|_{\text{flat}} = 0\,, \tag{A.31c}$$

as well as the variation of its trace

$$\delta_g K\big|_{\text{flat}} = -\partial^b \delta g_{zb} + \frac{1}{2}\eta^{ab}\partial_z \delta g_{ab}\,. \tag{A.32}$$

Putting together the results (A.31) and (A.32), we can, after some algebra, express the integrand of the boundary term in (A.26) as

$$\left[(\eta^{\alpha\beta} - \delta^\alpha_z \delta^\beta_z)\delta_g K - \delta_g K^{\alpha\beta}\right]\delta g_{\alpha\beta} = (\eta^{ac}\eta^{bd} - \eta^{ab}\eta^{cd})\Big(\partial_c \delta g_{za} - \frac{1}{2}\partial_z \delta g_{cd}\Big)\,\delta g_{ab}\,. \tag{A.33}$$

At this point, let us identify the variation $\delta g_{\alpha\beta}$ with the linearized fluctuation $h_{\alpha\beta}$ of the bulk metric, namely

$$h_{\alpha\beta} \coloneqq \delta g_{\alpha\beta}\,. \tag{A.34}$$

Substituting then for the bulk curvature variations (A.27), as well as for the boundary-term variation (A.33) into (A.26), we have

$$\begin{aligned}\frac{1}{2}\delta_g^2 S_{\text{GR}}\big|_{\text{flat}}(h_{\alpha\beta}) = \frac{1}{4}\int_M d^d x\, h_{\alpha\beta}\Big(\eta^{\alpha\beta}(\partial^\rho\partial^\sigma \delta g_{\rho\sigma} - \eta^{\rho\sigma}\partial^2 h^\lambda{}_\lambda) - 2\partial_\lambda \partial^\alpha h^{\lambda\beta} + \\ + \partial^2 h^{\alpha\beta} + \partial^\alpha\partial^\beta h^\lambda{}_\lambda\Big) + \\ + \frac{1}{4}\int_{\partial M} d^{d-1}y\, h_{ab}\,(\eta^{ac}\eta^{bd} - \eta^{ab}\eta^{cd})\Big(2\partial_c h_{za} - \partial_z h_{cd}\Big)\,.\end{aligned} \tag{A.35}$$

We notice that 1. in the bulk term of (A.35), there are a number of terms containing second derivatives with respect to the z coordinate, and, 2. in the boundary term of (A.35), there are terms with first derivatives with respect to z. These would render

the variational principle given by the action (A.35) inconsistent. Luckily though, all these turn out to cancel among themselves upon integrating some of the derivatives in the bulk terms by parts. Indeed, after a few straightforward steps, the action (A.35) becomes

$$\begin{aligned}\frac{1}{2}\delta_g^2 S_{\rm GR}\big|_{\rm flat}(h_{\alpha\beta}) = \\ = -\frac{1}{2}\int_M d^dx\left(\frac{1}{2}\partial^\lambda h^{\alpha\beta}\partial_\lambda h_{\alpha\beta} - \frac{1}{2}\partial^\lambda h^\rho{}_\rho\partial_\lambda h^\sigma{}_\sigma - \partial^\alpha h^{\lambda\beta}\partial_\lambda h_{\alpha\beta} + \partial^\beta h^\rho{}_\rho\partial^\alpha h_{\alpha\beta}\right)+ \\ - \frac{1}{2}\int_{\partial M} d^{d-1}y\, h_{zd}\,\partial^d(2h_{zz} - h^\rho{}_\rho)\,,\end{aligned} \tag{A.36}$$

which constitutes a well-defined variational principle for linearized massless spin-2 fluctuations $h_{\alpha\beta}$ on the given background. Furthermore, (A.36) is invariant under the linearized gauge transformation

$$\delta_\xi h_{\alpha\beta} = \partial_\alpha\xi_\beta + \partial_\beta\xi_\alpha \tag{A.37}$$

provided that we restrict the gauge parameter ξ_α to satisfy

$$n^\alpha\xi_\alpha = \xi_z = 0 \tag{A.38}$$

at ∂M. As the gauge transformation (A.37) descends from the linearized trasformation of the bulk metric under the infinitesimal diffeomorphism

$$x^\mu \longrightarrow (x')^\mu = x^\mu - \xi^\mu\,, \tag{A.39}$$

the meaning of (A.38) is clear: we are confining ourselves to considering diffeomorphisms which preserve the hyperplane at $z = 0$. The action (A.36) matches with the massless spin-2 actions obtained at level 0 of the closed string and level 1 of the tensionless open string upon setting both the boundary degree of freedom $l(y^a)$ and the dilaton $\phi(x^\mu)$ to zero.

A.2 Fluctuating boundary

We will now show that the *whole* gauge-invariant massless spin-2 action (including non-zero boundary field $l(y^a)$) obtained from SFT can be recovered by linearizing GR around a flat half-space: we will see that the field $l(y^a)$ should be interpreted as being proportional to a transverse displacement of the boundary.

A.2.1 Parametrization of ∂M

The boundary ∂M can be generally described by an implicit equation of the form (A.10) in terms of a function F on M. Keeping in mind the concrete example of linearized fluctuations around a flat hyperplane bounding a half-space inside $\mathbb{R}^d$, we will consider F to have the particular form

$$F(x^\mu) = z - f(y^a)\,. \tag{A.40}$$

Here, as in the main body of the Chapter 3, y^a are some coordinates along the hyperplane while z is the coordinate running in the normal direction. The function $f(y^a)$ then parametrizes a class of smooth deformations of the hyperplane at $z = 0$. Upon enacting the infinitesimal diffeomorphism (A.39) of M, the function $f(y^a)$ obeys the linearized transformation law (dropping $\mathcal{O}(\xi f)$ terms)

$$\delta_\xi f(y^a) = -\xi_z(y^a)\,, \tag{A.41}$$

where we have used the shortcut $\xi_z(y^a) = \xi_z(x^\mu(y^a))$. The hypersurface ∂M given by (A.40) can be alternatively described by the embedding map

$$x^\mu(y^c) = \delta^\mu_b\, y^b + \delta^\mu_z f(y^c)\,. \tag{A.42}$$

When we vary with respect to f, the embedding map then varies as

$$\delta_f\big(x^\mu(y^c)\big) = \delta^\mu_z\, \delta f(y^c)\,. \tag{A.43}$$

Correspondingly, for the transition maps e^μ_a and their variations $\delta_f e^\mu_a$, we can write

$$e^\mu_a(y^c) = \delta^\mu_a + \delta^\mu_z\, \partial_a f(y^c)\,, \tag{A.44a}$$

$$\delta_f(e^\mu_a(y^c)) = \delta^\mu_z\, \partial_a \delta f(y^c)\,. \tag{A.44b}$$

A.2.2 Linearized action: summary

Given this setup, the action (A.1) can be considered as a functional of both the bulk metric $g_{\mu\nu}$ as well as of the function $f(y^a)$ parametrizing the boundary manifold ∂M. Varying the action with respect to $f(y^a)$ will therefore yield additional terms into the linearized action (A.36) for massless spin-2 fluctuations. These will involve the displacement $\delta f(y^a)$ of the boundary. Introducing a rescaled fluctuation

$$l(y^a) \coloneqq -2\,\delta f(y^a) \tag{A.45}$$

and given the transformation property (A.41) of $f(y^a)$, we can observe that the field $l(y^a)$ transforms as

$$\delta_\xi l(y^a) = 2\,\xi_z(y^a)\,. \tag{A.46}$$

This exactly matches the gauge transformation of the corresponding boundary degree of freedom entering the massless spin-2 actions derived from SFT. Indeed, we will see below that varying the GR action (A.1) once with respect to f, once with respect to $g_{\alpha\beta}$, substituting for the flat background $g_{\mu\nu} = \eta_{\mu\nu}$ with flat boundary $f(y^a) = 0$ and, finally, using the redefinitions (A.34), (A.45), we will obtain a boundary term which 1. restores the full invariance of the action (A.36) under the gauge transformation (A.37) and 2. exactly agrees with the term proportional to $l(y^a)$ in the linearised SFT actions (3.140) upon setting the dilaton $\phi = 0$. That is to say, we will show that

$$\frac{1}{2}\delta_f(\delta_g S_{\rm GR})|_{\rm flat}(h_{\alpha\beta}, l) = \frac{1}{2}\delta_g(\delta_f S_{\rm GR})|_{\rm flat}(h_{\alpha\beta}, l) \tag{A.47a}$$

$$= -\frac{1}{4}\int_{\partial M} d^{d-1}y\, (h^{ab} - \eta^{ab}h^c{}_c)\, \partial_a\partial_b l\,. \tag{A.47b}$$

The interchangeability of the variations with respect to $g_{\mu\nu}$ and f was of course to be expected since the action (A.1) is a smooth functional of $g_{\mu\nu}$ and f. We will also show that

$$\delta_f^2 S_{\rm GR}|_{\rm flat}(h_{\alpha\beta}, l) = 0\,, \tag{A.48}$$

meaning that all terms in the linearized action which are quadratic in the field $l(y^a)$ vanish. Altogether, the full linearized action therefore reads

$$\begin{aligned} &S(h_{\alpha\beta}, l) = \\ &= \frac{1}{2}\delta_g^2 S_{\rm GR}|_{\rm flat}(h_{\alpha\beta}) + \delta_f\delta_g S_{\rm GR}|_{\rm flat}(h_{\alpha\beta}, l) + \frac{1}{2}\delta_f^2 S_{\rm GR}|_{\rm flat}(h_{\alpha\beta}, l) && \text{(A.49a)} \\ &= -\frac{1}{2}\int_M d^d x \left(\frac{1}{2}\partial^\lambda h^{\alpha\beta}\partial_\lambda h_{\alpha\beta} - \frac{1}{2}\partial^\lambda h^\rho{}_\rho \partial_\lambda h^\sigma{}_\sigma - \partial^\alpha h^{\lambda\beta}\partial_\lambda h_{\alpha\beta} + \partial^\beta h^\rho{}_\rho \partial^\alpha h_{\alpha\beta}\right) + \\ &\qquad - \frac{1}{2}\int_{\partial M} d^{d-1} y \left[h_{zd}\,\partial^d(2h_{zz} - h^\rho{}_\rho) - (\eta^{ab}h^c{}_c - h^{ab})\,\partial_a\partial_b l\right]. && \text{(A.49b)} \end{aligned}$$

This is in exact agreement with the pure-graviton part of the closed SFT action (3.140).

A.2.3 Linearized action: detailed calculation

Let us start by varying the action (A.1) once with respect to f. Recall that since f was defined to be a function of y only, the variation δf does not depend on the variable z.

Collecting first the necessary ingredients, we start by computing the variation of the induced metric

$$\delta_f \gamma_{ab}(y) = \delta_f\big(e_a^\alpha(y)\, e_b^\beta(y)\, g_{\alpha\beta}(x(y))\big)\,. \tag{A.50}$$

Substituting from (A.43) and (A.44b) for the variations of the embedding functions and the transition matrices e_a^α, we end up with

$$\delta_f \gamma_{ab} = \delta f\, e_a^\alpha\, e_b^\beta\, \partial_z g_{\alpha\beta} + \big(e_a^\alpha \partial_b \delta f + e_b^\alpha \partial_a \delta f\big)\, g_{\alpha z}\,. \tag{A.51}$$

It follows that for the variation of the determinant of the induced metric, we can write

$$\delta_f\sqrt{\gamma} = \frac{1}{2}\sqrt{\gamma}\Big[\delta f\,\gamma^{\alpha\beta}\partial_z g_{\alpha\beta} + \gamma^{ab}(e^\alpha_a\partial_b\delta f + e^\alpha_b\partial_a\delta f)\,g_{\alpha z}\Big]\,. \tag{A.52}$$

Notice that when we substitute for the flat background (A.25) (which, in particular, amounts to setting $f=0$), both $\delta_f\gamma_{ab}$ and $\delta_f\sqrt{\gamma}$ vanish, that is

$$\delta_f\gamma_{ab}\big|_{\text{flat}} = 0\,, \tag{A.53a}$$

$$\delta_f\sqrt{\gamma}\big|_{\text{flat}} = 0\,. \tag{A.53b}$$

We will also need to vary the unit form n_μ normal to ∂M. Varying the r.h.s. of (A.11), we obtain

$$\delta_f n_\mu(x(y)) = -\frac{\delta^a_\mu\partial_a\delta f}{\sqrt{g^{\rho\sigma}\partial_\rho F\partial_\sigma F}} + n_\mu\bigg(\frac{g^{\rho a}\partial_\rho F\,\partial_a\delta f}{g^{\rho\sigma}\partial_\rho F\partial_\sigma F} - \frac{1}{2}\delta f\,\frac{\partial_z g^{\rho\sigma}\partial_\rho F\partial_\sigma F}{g^{\rho\sigma}\partial_\rho F\partial_\sigma F}\bigg)\,. \tag{A.54}$$

In the flat background (A.25), this gives simply

$$\delta_f n_\mu(x(y))\big|_{\text{flat}} = -\partial_\mu\delta f\,. \tag{A.55}$$

Finally, the trace of the extrinsic curvature varies with f as

$$\begin{aligned}\delta_f K(x(y)) = \nabla_\mu\bigg(n^\mu\frac{g^{\rho a}\partial_\rho F\,\partial_a\delta f}{g^{\rho\sigma}\partial_\rho F\partial_\sigma F} - \frac{\delta^\mu_a\partial^a\delta f}{\sqrt{g^{\rho\sigma}\partial_\rho F\partial_\sigma F}}\bigg) + \\ + \delta f\,\Big(\partial_z g^{\mu\nu}\,\nabla_\mu n_\nu - g^{\mu\nu}\partial_z\Gamma^\rho_{\mu\nu}\,n_\rho\Big) - \frac{1}{2}\nabla_\mu\bigg(n^\mu\delta f\,\frac{\partial_z g^{\rho\sigma}\partial_\rho F\partial_\sigma F}{g^{\rho\sigma}\partial_\rho F\partial_\sigma F}\bigg)\,.\end{aligned} \tag{A.56}$$

This reduces to

$$\delta_f K(x(y))\big|_{\text{flat}} = -\partial^a\partial_a\delta f \tag{A.57}$$

in the flat half-space.

Hence, varying the action (A.1) once with respect to f, we obtain

$$\delta_f S_{\rm GR} = \int_{\partial M} d^{d-1}y\,\sqrt{g}\,R\,\delta f + \int_{\partial M} d^{d-1}y\,\sqrt{\gamma}\,(K\,\gamma^{ab}\delta_f\gamma_{ab} + 2\delta_f K) \tag{A.58}$$

where in the first term, we have remembered that the limits of the bulk (Einstein-Hilbert) integral vary with f. We have also concluded that explicitly substituting for the variations $\delta_f\gamma_{ab}$ and $\delta_f K$ would not have been particularly illuminating at this point. It is nevertheless reassuring to note that evaluating (A.58) in the flat half-space, we obtain

$$\delta_f S_{\rm GR}|_{\rm flat} = -2\int_{\partial M} d^{d-1}y\,\partial^a\partial_a\delta f = 0 \tag{A.59}$$

assuming, as usual, that the boundary of ∂M can be ignored. This means that the flat half-space extremizes the action (A.1) even with respect to the transverse displacement f of the boundary ∂M.

Let us now vary with respect to f once more, this time immediately evaluating the results for the flat half-space. First, we can quickly note that

$$\delta_f R(x(y))\big|_{\rm flat} = \delta f\,\partial_z R|_{\rm flat} = 0\,, \tag{A.60}$$

meaning that the first term in (A.58) will not contribute. For the second term, the only new necessary object to compute is $\delta_f^2 K|_{\rm flat}$ because the would-be contribution of $\delta_f^2\gamma_{ab}|_{\rm flat}$ is pre-multiplied by K which kills it. After an uneventful tour through the terms appearing on the r.h.s. of (A.56), we obtain

$$\delta_f^2 K|_{\rm flat} = \partial_\mu(-\delta^\mu_z\partial^a\delta f\partial_a\delta f) = 0\,, \tag{A.61}$$

where we have recalled that the variation δf does not depend on z. Hence, since we can write

$$\delta_f^2 S_{\rm GR}\big|_{\rm flat} = 2\int_{\partial M} d^{d-1}y\,\sqrt{\gamma}\,\delta_f^2 K|_{\rm flat} = 0\,, \tag{A.62}$$

we confirm our claim that the linearized free action (A.49) does not contain any

terms which would be quadratic in the boundary displacement.

Let us proceed with computing the mixed variations evaluated in flat half-space, starting with the f-variation of (A.24). The only new ingredients needed here are

$$\delta_f a_\mu(x(y))\big|_{\text{flat}} = \partial_z(\delta_f n_\mu|_{\text{flat}}) = -\partial_\mu\partial_z\delta f = 0\,, \tag{A.63}$$

which can be used to compute

$$\delta_f K_{\mu\nu}(x(y))\big|_{\text{flat}} = -\delta^a_\mu\,\delta^b_\nu\,\partial_a\partial_b\delta f\,. \tag{A.64}$$

Furthermore, when we vary the limits of the bulk term in (A.24), we get a zero upon substituting for the flat half-space background since it solves the Einstein field equation in the bulk. Hence, we can write

$$\frac{1}{2}\delta_f(\delta_g S_{\text{GR}})\big|_{\text{flat}} = \frac{1}{2}\int_{\partial M} d^{d-1}y\,\left((\eta^{\alpha\beta}-\delta^\alpha_z\delta^\beta_z)\,\delta_f K\big|_{\text{flat}} - \delta_f K^{\alpha\beta}\big|_{\text{flat}}\right)\delta g_{\alpha\beta} \tag{A.65a}$$

$$= \frac{1}{2}\int_{\partial M} d^{d-1}y\,(\delta g_{ab} - \eta_{ab}\,\eta^{cd}\delta g_{cd})\partial^a\partial^b\delta f\,, \tag{A.65b}$$

so that upon substituting for δg_{ab} and δf in terms of h_{ab} and l, we exactly recover the r.h.s. of (A.47b). It remains to verify that the $g_{\mu\nu}$-variation of (A.58) gives the same result. Since the variations with respect to f and $g_{\mu\nu}$ should commute, this is really just a sanity check. First, we write

$$\frac{1}{2}\delta_g(\delta_f S_{\text{GR}})\big|_{\text{flat}} = \frac{1}{2}\int_{\partial M} d^{d-1}y\,\delta_g R\big|_{\text{flat}}\,\delta f + \int_{\partial M} d^{d-1}y\,\delta_g(\sqrt{\gamma}\delta_f K)\big|_{\text{flat}}\,, \tag{A.66}$$

where we have tacitly used the result (A.53a). Since $\delta_g R\big|_{\text{flat}}$ was already computed

in (A.27), we only have to find

$$\delta_g(\sqrt{\gamma}\delta_f K)\big|_{\text{flat}} =$$

$$= \eta^{\mu\nu}\delta_g\Gamma^a_{\mu\nu}\,\partial_a\delta f + \delta g_{\mu a}\,\partial^\mu\partial^a\delta f - \partial_z\delta g_{za}\,\partial^a\delta f + \\ - \frac{1}{2}\partial^a(\partial_a\delta f\,\delta g_{zz}) - \delta f\,\eta^{\mu\nu}\,\partial_z\delta_g\Gamma^z_{\mu\nu} + \frac{1}{2}\delta f\,\partial_z^2\,\delta g_{zz} - \frac{1}{2}\partial^a\partial_a\delta f\,\eta^{ab}\,\delta g_{ab} \tag{A.67a}$$

$$= \left(\partial_z\delta g_{cz} - \tfrac{1}{2}\partial_c\delta g_{zz}\right)\partial^c\delta f + \frac{1}{2}\delta f\,\eta^{ab}\partial_z^2\delta g_{ab} + \\ + \partial^a\Big(\delta g_{ab}\,\partial^b\delta f - \tfrac{1}{2}\partial_a\delta f\,\delta g_{zz} - \partial_z\delta g_{za}\,\delta f - \tfrac{1}{2}\partial_a\delta f\,\eta^{cd}\delta g_{cd}\Big)\,, \tag{A.67b}$$

where the very last term, since it is a total derivative along ∂M, will drop upon substituting into (A.66). In total, we obtain

$$\frac{1}{2}\delta_g(\delta_f S_{\text{GR}})\big|_{\text{flat}} = \\ = \frac{1}{2}\int_{\partial M} d^{d-1}y\,\left(\partial^a\partial^b\delta g_{ab} + 2\partial^a\partial^z\delta g_{az} - \eta^{ab}\partial^c\partial_c\delta g_{ab} - \eta^{ab}\partial_z^2\delta g_{ab} - \partial^a\partial_a\delta g_{zz}\right)\delta f + \\ + \frac{1}{2}\int_{\partial M} d^{d-1}y\left[\left(2\partial_z\delta g_{cz} - \partial_c\delta g_{zz}\right)\partial^c\delta f + \eta^{ab}\partial_z^2\delta g_{ab}\,\delta f\right], \tag{A.68}$$

where we can observe exact cancellation among all terms containing derivatives with respect to z. After straightforward manipulations, we end up with

$$\frac{1}{2}\delta_g(\delta_f S_{\text{GR}})\big|_{\text{flat}} = \frac{1}{2}\int_{\partial M} d^{d-1}y\,(\delta g_{ab} - \eta_{ab}\,\eta^{cd}\delta g_{cd})\partial^a\partial^b\delta f = \frac{1}{2}\delta_f(\delta_g S_{\text{GR}})\big|_{\text{flat}}\,, \tag{A.69}$$

thus confirming our claim (A.47b) and hence, finally, the result (A.49).

APPENDIX **B**

Equivalent forms of the action

In this appendix we will present two actions which we found prior to the discovery of (3.71) and which are both closely related to it. We do this partly for keeping track of the route that brought us to (3.71) and also because these alternative forms may turn useful in the future.

B.1 First alternative action

Since in (3.71) χ only appears as $Q\chi$, we can define a new boundary string field Σ such that

$$\Sigma = Q\chi\,, \tag{B.1}$$

so that Σ formally gauge-transforms in the same way as Ψ does, namely

$$\delta_g \Sigma = Q\Lambda \,. \tag{B.2}$$

We can now re-write (3.71) as

$$S^{(1)}(\Psi,\Sigma) = \frac{1}{2}\omega(\Psi,(\Theta Q - \delta\Gamma^*)\Psi) + \omega(\Psi,\delta\Gamma^*\Sigma) - \frac{1}{2}\omega\left(\Sigma,\delta\Gamma^*\,\Sigma\right) \tag{B.3}$$

and we can simply conclude that the gauge-variation is given by

$$\delta_g S^{(1)} = -\frac{1}{2}\omega\left(\Lambda, B\,Q\Sigma\right). \tag{B.4}$$

This is zero if we assume that $Q\Sigma = 0$ which is obviously implied by (B.1).

In the case of the string propagating on the half-space $M = \mathbb{R}_+ \times \mathbb{R}^{D-1}$ (see section 3.4 for details), the string field Σ is understood as a collection of boundary fields

$$\Sigma(x) = \Sigma_1(y) + z\Sigma_2(y) + \frac{1}{2}z^2\Sigma_3(y) + O(z^3)\,. \tag{B.5}$$

It is not difficult to realize that the action only depends on Σ_1 and Σ_2, but still Σ_3 is needed to realize the constraint $Q\Sigma = 0$ which in terms of the boundary fields Σ_i reads

$$\begin{pmatrix} \tilde{Q} & -\Omega & -c \\ 0 & \tilde{Q} & -\Omega \\ 0 & 0 & \tilde{Q} \end{pmatrix} \begin{pmatrix} \Sigma_1 \\ \Sigma_2 \\ \Sigma_3 \end{pmatrix} = 0\,. \tag{B.6}$$

Explicitly the action in half-space takes the form

$$\begin{aligned} S^{(1)}(\Psi;\Sigma_1,\Sigma_2) = \frac{1}{2}\omega_M(\Psi,Q\Psi) - \frac{1}{2}\omega_{\partial M}(\Psi,c\partial_z\Psi) + \\ + \frac{1}{2}\omega_{\partial M}\left(\Psi,\Omega\Sigma_1 + 2c\Sigma_2\right) + \omega_{\partial M}\left(\Sigma_1,c\Sigma_2\right)\,. \end{aligned} \tag{B.7}$$

The gauge transformations of the Σ_i's are simply given

$$\delta_g \begin{pmatrix} \Sigma_1 \\ \Sigma_2 \\ \Sigma_3 \end{pmatrix} = \begin{pmatrix} \tilde{Q} & -\Omega & -c \\ 0 & \tilde{Q} & -\Omega \\ 0 & 0 & \tilde{Q} \end{pmatrix} \begin{pmatrix} \Lambda \\ \Lambda' \\ \Lambda'' \end{pmatrix} , \tag{B.8}$$

and are obviously compatible with the (gauge-invariant) constraint (B.6) thanks to

$$\begin{pmatrix} \tilde{Q} & -\Omega & -c \\ 0 & \tilde{Q} & -\Omega \\ 0 & 0 & \tilde{Q} \end{pmatrix}^2 = 0 \quad \longleftrightarrow \quad Q^2 = 0 \, . \tag{B.9}$$

In principle this action is more general than (3.71), which corresponds to Σ being Q-exact and not just Q-closed. It is not clear to us at the moment whether a cohomological non-trivial Σ could play some non-trivial role and in all concrete examples we have analyzed: so far we have always found that (3.71) is fully sufficient for describing a satisfactory free action for all string modes.

B.2 Second alternative action

The last action we would like to mention is infact the first one which we found and is structurally more similar to the action we used in [29]. Again, we present it in the half-space $M = \mathbb{R}_+ \times \mathbb{R}^{D-1}$, in the conventions of section 3.4

$$\begin{aligned} S^{(2)}(\Psi; \tilde{\Sigma}_0, \tilde{\Sigma}_1) = \frac{1}{2}\omega_M(\Psi, Q\Psi) - \frac{1}{2}\omega_{\partial M}(\Psi, c\partial_z \Psi) + \\ - \frac{1}{2}\omega_{\partial M}\left(\Psi, \Omega\tilde{\Sigma}_1 - 2\tilde{Q}\tilde{\Sigma}_0\right) + \omega_{\partial M}\left(\tilde{\Sigma}_1, \tilde{Q}\tilde{\Sigma}_0\right) . \end{aligned} \tag{B.10}$$

Similarly to the action described in the previous subsection B.1, the boundary fields $\tilde{\Sigma}_{1,0}(y)$ are constrained by the relation

$$\begin{pmatrix} c & \Omega & -\tilde{Q} \\ 0 & c & \Omega \\ 0 & 0 & c \end{pmatrix} \begin{pmatrix} \tilde{\Sigma}_1 \\ \tilde{\Sigma}_0 \\ \tilde{\Sigma}_{-1} \end{pmatrix} \equiv \boldsymbol{C\Sigma} = 0 \, , \tag{B.11}$$

where $\tilde{\Sigma}_{-1}$ is a generic boundary field with mass dimension -1, which decouples from the action, but which is needed to express the constraint (B.11). Then the action is invariant under the gauge transformation

$$\delta_g \Psi = Q\Lambda \,, \tag{B.12a}$$

$$\delta_g \boldsymbol{\Sigma} = \boldsymbol{C}\boldsymbol{\Lambda} \,, \tag{B.12b}$$

after assuming the (gauge-invariant) constraint (B.11) and using the nilpotency relation

$$\boldsymbol{C}^2 = 0 \quad \longleftrightarrow \quad Q^2 = 0 \,, \tag{B.13}$$

which is easy to verify.

Although it does not look so at first sight, this action is also very closely related to $S_{\rm tot}$ (3.71) and is in fact equivalent to it. The key for understanding this is to realize that, in this particular case, the constraints (B.11) can be explicitly solved by decomposing the $\Sigma_{1,0,-1}$ boundary fields using the ghost zero-modes basic relation

$$bc + cb = 1 \,, \tag{B.14}$$

where $b = b_0$ for the open string and $b = b_0 + \bar{b}_0$ for the closed string. Then, in full generality we can write

$$\begin{pmatrix} \tilde{\Sigma}_1 \\ \tilde{\Sigma}_0 \\ \tilde{\Sigma}_{-1} \end{pmatrix} = \begin{pmatrix} \sigma_1 + c\,\eta_1 \\ \sigma_0 + c\,\eta_0 \\ \sigma_{-1} + c\,\eta_{-1} \end{pmatrix} , \tag{B.15}$$

where

$$b\,\sigma_i = b\,\eta_i = 0 \,. \tag{B.16}$$

It is not difficult to see that the constraints (B.11) can be uniquely solved by express-

ing the σ_i's in terms of the η_j's. One then obtains

$$\begin{pmatrix} \tilde{\Sigma}_1 \\ \tilde{\Sigma}_0 \\ \tilde{\Sigma}_{-1} \end{pmatrix} = \begin{pmatrix} -bc\tilde{Q}\,\eta_{-1} + \Omega\,\eta_0 + c\,\eta_1 \\ \Omega\,\eta_{-1} + c\,\eta_0 \\ c\,\eta_{-1} \end{pmatrix} . \tag{B.17}$$

The action $S^{(2)}$ can thus be expressed as

$$S^{(2)}(\Psi, \tilde{\Sigma}_i) \longrightarrow S^{(2)}(\Psi, \tilde{\Sigma}_i(\eta_j)) \equiv \tilde{S}^{(2)}(\Psi; \eta_{-1}, \eta_0, \eta_1)\,. \tag{B.18}$$

We can find the gauge transformations of the η fields by realizing from (B.17) that we can write

$$\eta_i = b\,\tilde{\Sigma}_i\,, \tag{B.19}$$

which implies

$$\delta_g \eta_{-1} = b(c\Lambda)\,, \tag{B.20a}$$

$$\delta_g \eta_0 = b(c\Lambda' + \Omega\Lambda)\,, \tag{B.20b}$$

$$\delta_g \eta_1 = b(c\Lambda'' + \Omega\Lambda' - \tilde{Q}\Lambda)\,. \tag{B.20c}$$

Notice that the η fields have degree (-1) and opposite grassmannality wrt the bulk string field Ψ. This already suggests of a possible relation between the three η fields in $\tilde{S}^{(2)}$ and the three χ fields of $S_{\rm tot}$ (3.71), whose Λ-gauge transformations (3.114b) are however rather different from (B.20b). Moreover, the η fields are annihilated by b while this is not true for the χ fields. The key point here is that χ has an extra gauge symmetry given by $Q\Upsilon$ (3.72b) and this can be used to put χ in Siegel gauge without touching the Λ gauge-symmetry. To do this we use the b-c decomposition

$$\chi_i = \beta_i + c\gamma_i\,, \tag{B.21}$$

(where $b\beta_i = b\gamma_i = 0$) and search for a gauge parameter

$$\Upsilon(z) = \frac{1}{2}z^2\Upsilon_{-1} + \frac{1}{3!}z^3\Upsilon_0 + \frac{1}{4!}z^4\Upsilon_1\,, \tag{B.22}$$

to put $\chi_{-1,0,1}$ in Siegel gauge.[1]

By explicitly computing $Q\Upsilon$ we get

$$\delta_\Upsilon\chi_{-1} = -c\,\Upsilon_{-1}\,, \tag{B.23a}$$

$$\delta_\Upsilon\chi_0 = -c\,\Upsilon_0 - \Omega\,\Upsilon_{-1}\,, \tag{B.23b}$$

$$\delta_\Upsilon\chi_1 = -c\,\Upsilon_1 - \Omega\,\Upsilon_0 + \tilde{Q}\,\Upsilon_{-1}\,. \tag{B.23c}$$

We can then easily determine $\Upsilon_{-1,0,1}$ so that

$$\chi \longrightarrow \chi + Q\Upsilon = \hat{\beta} + c\hat{\gamma}, \quad \text{with } \hat{\gamma} = 0\,. \tag{B.24}$$

By simple matching, we end up with

$$\hat{\beta}_{-1} = \beta_{-1}\,, \tag{B.25a}$$

$$\hat{\beta}_0 = \beta_0 - \Omega\,\gamma_{-1}\,, \tag{B.25b}$$

$$\hat{\beta}_1 = \beta_1 - \Omega\,\gamma_0 + bc\tilde{Q}\,\gamma_{-1}\,. \tag{B.25c}$$

These explicit relations allow to determine how the Λ-gauge transformation acts on the new variables $\hat{\beta}_i$, by tracing back its action on the original $\beta = bc\,\chi$ and $\gamma = b\chi$ variables. This gives

$$\delta_\Lambda\hat{\beta}_{-1} = bc\,\Lambda\,, \tag{B.26a}$$

$$\delta_\Lambda\hat{\beta}_0 = b(c\,\Lambda' + \Omega\,\Lambda)\,, \tag{B.26b}$$

$$\delta_\Lambda\hat{\beta}_1 = b(c\,\Lambda'' + \Omega\,\Lambda' - \tilde{Q}cb\,\Lambda)\,. \tag{B.26c}$$

These gauge transformations are now rather similar to the gauge transformations of

[1] Other z powers in Υ are not needed for bringing $\chi_{-1,0,1}$ in Siegel gauge.

the η fields (B.20b) and become identical after the field redefinition

$$\hat{\beta}_{-1} = \eta_{-1}\,, \tag{B.27a}$$

$$\hat{\beta}_{0} = \eta_{0}\,, \tag{B.27b}$$

$$\hat{\beta}_{1} = \eta_{1} - b\tilde{Q}\,\eta_{-1}\,. \tag{B.27c}$$

This gives the final relation between the η fields and the χ fields and establishes that

$$\tilde{S}^{(2)}(\Psi,\eta) = S_{\mathrm{tot}}(\Psi,\hat{\beta}(\eta))\,. \tag{B.28}$$

APPENDIX **C**

Action of the propagator on $\mathcal{SH}_{\mathrm{C}} \otimes' \mathcal{SCH}_{\mathrm{O}}$

In this appendix we want to promote a generic propagator h defined on $\mathcal{H}_{\mathrm{c}} \oplus \mathcal{H}_{\mathrm{o}}$ to $\mathcal{SH}_c \otimes' \mathcal{SCH}_o$ and prove the validity of the Hodge-Kodaira decomposition in that case. To start with, we recall the well-known definitions of the propagator $\boldsymbol{h}_{TH} : \mathcal{TH} \longrightarrow \mathcal{TH}$ and $\boldsymbol{h}_{SH} : \mathcal{SH} \longrightarrow \mathcal{SH}$, see for example [111]

$$\boldsymbol{h}_{TH}\pi_n = \sum_{i=0}^{n-1} \left(1^{\otimes i} \otimes h \otimes P^{\otimes n-i-1}\right) \pi_n, \tag{C.1}$$

$$\boldsymbol{h}_{SH}\pi_n = \frac{1}{n!} \sum_{i=0}^{n-1} \left(1^{\wedge i} \wedge h \wedge P^{\wedge n-i-1}\right) \pi_n, \tag{C.2}$$

which satisfy the corresponding Hodge-Kodaira decomposition

$$[\boldsymbol{h}_{TH}, \boldsymbol{Q}_{TH}] = 1_{TH} - \boldsymbol{P}_{TH}, \tag{C.3}$$

$$[\boldsymbol{h}_{SH}, \boldsymbol{Q}_{SH}] = 1_{SH} - \boldsymbol{P}_{SH}, \tag{C.4}$$

where $\boldsymbol{Q}$ and $\boldsymbol{P}$ are defined as usual i.e., as coderivation and cohomomorphism respectively on the reference space

$$\boldsymbol{P}_{TH}\pi_n = P^{\otimes n}\pi_n, \tag{C.5}$$

$$\boldsymbol{P}_{SH}\pi_n = \frac{1}{n!}P^{\wedge n}\pi_n, \tag{C.6}$$

$$\boldsymbol{Q}_{TH}\pi_n = \sum_{i=0}^{n-1}\left(1^{\otimes i}\otimes Q\otimes 1^{\otimes n-i-1}\right)\pi_n, \tag{C.7}$$

$$\boldsymbol{Q}_{SH}\pi_n = \frac{1}{n!}\sum_{i=0}^{n-1}\left(1^{\wedge i}\wedge Q\wedge 1^{\wedge n-i-1}\right)\pi_n. \tag{C.8}$$

Quite naturally we extend these definitions to the cyclic tensor algebra $\mathcal{CH}$

$$\boldsymbol{h}_{CH}\pi_n = \frac{1}{n}\sum_{i=0}^{n-1}\left(1^{\odot i}\odot h\odot P^{\odot n-i-1}\right)\pi_n, \tag{C.9}$$

$$\boldsymbol{P}_{CH}\pi_n = \frac{1}{n}P^{\odot n}\pi_n, \tag{C.10}$$

$$\boldsymbol{Q}_{CH}\pi_n = \frac{1}{n}\sum_{i=0}^{n-1}\left(1^{\odot i}\odot Q\odot 1^{\otimes n-i-1}\right)\pi_n. \tag{C.11}$$

We then proceed by demonstrating the validity of the Hodge-Kodaria decomposition

$$\begin{aligned}\boldsymbol{Q}_{CH}\boldsymbol{h}_{CH}\pi_n &= \frac{1}{n}\sum_{i=0}^{n-1}\boldsymbol{Q}_{CH}\left(1^{\odot i}\odot h\odot P^{\odot n-i-1}\right)\pi_n \\ &= \frac{1}{n}\sum_{i=0}^{n-1}\Bigg\{\sum_{j=0}^{i-1}\left(1^{\odot j}\odot Q\odot 1^{\odot i-j-1}\odot h\odot P^{\odot n-i-1}\right)+ \\ &\qquad + 1^{\odot i}\odot Qh\odot P^{\odot n-i-1}+ \\ &\qquad - \sum_{j=0}^{n-i-2}1^{\odot i}\odot h\odot P^{\odot j}\odot PQ\odot P^{\odot n-i-j-2}\Bigg\}\pi_n,\end{aligned} \tag{C.12}$$

$$\begin{aligned}\boldsymbol{h}_{CH}\boldsymbol{Q}_{CH}\pi_n &= \frac{1}{n}\sum_{i=0}^{n-1}\left(1^{\odot i}\odot h\odot P^{\odot n-i-1}\right)\boldsymbol{Q}_{CH}\pi_n \\ &= \frac{1}{n}\sum_{i=0}^{n-1}\Big\{-\sum_{j=0}^{i-1}\left(1^{\odot j}\odot Q\odot 1^{\odot i-j-1}\odot h\odot P^{\odot n-i-1}\right)+ \\ &\qquad +1^{\odot i}\odot Qh\odot P^{\odot n-i-1}+ \\ &\qquad +\sum_{j=0}^{n-i-2}1^{\odot i}\odot h\odot P^{\odot j}\odot PQ\odot P^{\odot n-i-j-2}\Big\}\pi_n,\end{aligned} \tag{C.13}$$

so that we get

$$\begin{aligned}\left[\boldsymbol{Q}_{CH},\boldsymbol{h}_{CH}\right]\pi_n &= \frac{1}{n}\sum_{i=0}^{n-1}\left(1^{\odot i}\odot[Q,h]\odot P^{\odot n-i-1}\right)\pi_n \\ &= \frac{1}{n}\sum_{i=0}^{n-1}\left(1^{\odot i+1}\odot P^{\odot n-i-1}-1^{\odot i}\odot P^{\odot n-i}\right)\pi_n \\ &= \frac{1}{n}\left(1^{\odot n}-P^{\odot n}\right)\pi_n=\left(1_{CH}-\boldsymbol{P}_{CH}\right)\pi_n.\end{aligned} \tag{C.14}$$

Now having defined such operators we can easily extend them when applied to spaces obtained by tensoring $\mathcal{CH}$,$\mathcal{SH}$ and/or $\mathcal{TH}$. For example let us consider $\mathcal{SCH}$

$$\boldsymbol{P}_{SCH}\hat{\pi}_b=\frac{1}{b!}\boldsymbol{P}_{CH}^{\wedge' b}\hat{\pi}_b, \tag{C.15}$$

$$\boldsymbol{Q}_{SCH}\hat{\pi}_b=\frac{1}{b!}\sum_{i=0}^{b-1}\left(1_{CH}^{\wedge' i}\wedge'\boldsymbol{Q}_{CH}\wedge' 1_{CH}^{\wedge' b-i-1}\right)\hat{\pi}_b, \tag{C.16}$$

$$\boldsymbol{h}_{SCH}\hat{\pi}_b=\frac{1}{b!}\sum_{i=0}^{b-1}\left(1_{CH}^{\wedge' i}\wedge'\boldsymbol{h}_{CH}\wedge'\boldsymbol{P}_{CH}^{\wedge' b-i-1}\right)\hat{\pi}_b, \tag{C.17}$$

where $\hat{\pi}_b$ is the projector that selects the terms in which there are b $\wedge'$-products of $\mathcal{CH}$ (or in other words all possible states to be inserted in b boundaries). Following the same steps illustrated above, it can be shown that the Hodge-Kodaira decomposition is also satisfied here

$$\left[\boldsymbol{h}_{SCH},\boldsymbol{Q}_{SCH}\right]=1_{SCH}-\boldsymbol{P}_{SCH}. \tag{C.18}$$

Finally, it is easy to treat the case of our interest i.e. the definition of the Hodge-Kodaira operators on $\mathcal{SH}_{\rm c} \otimes' \mathcal{SCH}_{\rm o}$

$$\boldsymbol{P} = \boldsymbol{P}_{SH} \otimes' \boldsymbol{P}_{SCH}, \tag{C.19}$$

$$\boldsymbol{Q} = \boldsymbol{Q}_{SH} \otimes' 1_{SCH} + 1_{SH} \otimes' \boldsymbol{Q}_{SCH}, \tag{C.20}$$

$$\boldsymbol{h} = \boldsymbol{h}_{SH} \otimes' \boldsymbol{P}_{SCH} + 1_{SH} \otimes' \boldsymbol{h}_{SCH}, \tag{C.21}$$

and obviously it is true that

$$[\boldsymbol{Q}, \boldsymbol{h}] = 1_{SH \otimes' SCH} - \boldsymbol{P}. \tag{C.22}$$

Notice that these operators simplify when we choose $P = P_{\rm c}$ as in the main text, to integrate out completely the open string sector, because in this case every entry of the open string tensor algebra (except the zeroth tensor power) is killed by P.

APPENDIX D

Path integral and co-algebraic homotopy transfer

In this section we will prove explicitly that the homotopy transfer at the level of co-algebras (that we heavily use in the main text) performs the path integral over the degrees of freedom which we want to integrate out. This amounts in showing that the transferred products given by the homotopy transfer precisely reproduce the effective vertices obtained by standard QFT manipulations. The equivalence of the homotopy transfer to the path integral has been discussed in detail in [114], at the level of Batalin-Vilkovisky formalism. Here we will translate this language to the 'dual' co-algebra language, on which our SFT formalism is based. In order to be as self-contained as possible we will start from the very beginning.

Let us then consider a generic SFT action $S(\Phi)$ that satisfies the BV quantum

master equation, where $\Phi \in \mathcal{H}$ can be decomposed as $\Phi = \phi^i f_i$ where the f_i's are the basis vectors of a graded vector space $\mathcal{H}$, the space of string fields.
Let's furthermore distinguish between the free kinetic term and the interactions

$$S[\Phi] = S_{\text{free}}[\Phi] + S_{\text{int}}[\Phi]\,, \tag{D.1}$$

and assume a kinetic term of the form

$$S_{\text{free}}(\Phi) = \frac{1}{2}\omega\left(\Phi, Q\Phi\right) = \frac{1}{2}\omega(f_i, Qf_j)\phi^i\phi^j = \frac{1}{2}Q_{ij}\phi^i\phi^j, \tag{D.2}$$

where ω is a symplectic form on $\mathcal{H}$, Q is a BRST operator (an odd nilpotent linear operator on $\mathcal{H}$) and Q_{ij} is defined as

$$Q_{ij} \coloneqq \frac{1}{2}\omega(f_i, Qf_j)\,, \tag{D.3}$$

with

$$Q_{ij} = (-)^{f_j}Q_{ji}\,, \tag{D.4}$$

where $(-)^v$ accounts for the degree of the vector v. Let us then introduce a projector $P = \iota\pi$ and a propagator h satisfying the Hodge-Kodaira decomposition

$$P + hQ + Qh = 1, \tag{D.5}$$

as well as $h^2 = Ph = hP = 0$, so that (P, hQ, Qh) are orthogonal projectors which can be used to decompose the Hilbert space as

$$A \coloneqq P\mathcal{H} = \text{Span}\{a_i\}, \tag{D.6}$$

$$B \coloneqq Qh\mathcal{H} = \text{Span}\{b_i\}, \tag{D.7}$$

$$C \coloneqq hQ\mathcal{H} = \text{Span}\{c_i\}\,, \tag{D.8}$$

and according to the notation introduced in Section 2.2.4 the string field can be

written as

$$\Phi = \iota\varphi + \tilde{R} + R = \alpha^i a_i + \beta^i b_i + \gamma^i c_i \,. \tag{D.9}$$

Our aim is to find the Wilsonian effective action $\tilde{S}(\varphi)$ by integrating out $R + \tilde{R}$ in the gauge fixed BV path integral, namely

$$e^{\tilde{S}(\varphi)} = \int \mathcal{D}R e^{S(\iota\varphi+R)} = \int \prod_i d\gamma^j e^{S(\iota\varphi+R)} \,, \tag{D.10}$$

according to 4.2.

To perform the above integral we introduce a source term $J_i\gamma^i$, with $d(J_i) = d(\gamma^i)$, and we isolate the kinetic term of the C sector

$$\begin{aligned} e^{\tilde{S}(\varphi)} &= \int \prod_i d\gamma^i e^{\frac{1}{2}Q_{ij}\gamma^i\gamma^j + S_{\text{free}}(\iota\varphi) + S_{\text{int}}(\iota\varphi+R) + J_i\gamma^i}\big|_{J_i=0} \\ &= e^{S_{\text{free}}(\iota\varphi) + S_{\text{int}}(\iota\varphi + \partial_{J_i} c_i)} \int \prod_i d\gamma^i e^{\frac{1}{2}Q_{ij}\gamma^i\gamma^j + J_i\gamma^i}\big|_{J_i=0} \,. \end{aligned} \tag{D.11}$$

The problem then reduces to the computation of a Gaussian integral. Assuming positivity of Q_{ij} (also by deforming possible 'tachyonic' contours in the complex plane so that, for grassmann even variables, the integral is done on a steepest path), the integral converges when Q_{ij} is non-degenerate. This condition is satisfied if $\ker(Q)$ is contained in $\mathrm{Im}(P)$. Under this assumption we can readily compute the path integral by completing the square. To do this, we define

$$h^{ij} := (-)^{b_i}\omega(b^i, hb^j) \,, \tag{D.12}$$

that satisfies the following properties

$$h^{ij} = (-)^{b^i} h^{ji} \,, \tag{D.13a}$$

$$Q_{ji}h^{jk} = \delta_i^k \,, \tag{D.13b}$$

$$Q_{ij}h^{jk} = (-)^{c_i}\delta_i^k \,. \tag{D.13c}$$

Then we can easily verify

$$\frac{1}{2}Q_{ij}\gamma^i\gamma^j + J_i\gamma^i = \frac{1}{2}\left[Q_{ij}(\gamma^i - h^{ik}J_k)(\gamma^j - h^{jl}J_l)\right] + \frac{1}{2}h^{ij}J_iJ_j\,, \tag{D.14}$$

and thus, by making the change of variables $x^i = \gamma^i - h^{ik}J_k$ we get

$$\begin{aligned} e^{\tilde{S}[\varphi]} &= e^{S_{\text{free}}(\iota\varphi)+S_{\text{int}}(\iota\varphi)+\partial_{J_i}c_i]}e^{\frac{1}{2}h^{ij}J_iJ_j}\int\prod_i dx^i e^{\frac{1}{2}Q_{ij}x^ix^j}\big|_{J_i=0} \\ &= \text{sdet}(|Q_{ij}|)^{-\frac{1}{2}}e^{S_{\text{free}}(\iota\varphi)+S_{\text{int}}(\iota\varphi+\partial_{J_i}c_i)}e^{\frac{1}{2}h^{ij}J_iJ_j}\big|_{J_i=0}\,, \end{aligned} \tag{D.15}$$

Therefore, the effective action can be written as

$$\tilde{S}(\varphi) = \mathcal{A} + \log\left(e^{S_{\text{free}}(\iota\varphi)-S_{\text{int}}(\iota\varphi+\partial_{J_i}c_i)}e^{\frac{1}{2}h^{ij}J_iJ_j}\right)\big|_{J_i=0}\,. \tag{D.16}$$

Additionally, we can introduce a further source term $J_i\gamma^i$, getting

$$\begin{aligned} \tilde{S}(\varphi) &= \mathcal{A} + \log\left(e^{S_{\text{free}}(\iota\varphi)+S_{\text{int}}(\iota\varphi+\partial_{J_i}c_i)}e^{\frac{1}{2}h^{ij}J_iJ_j}e^{J_i\gamma^i}\right)\big|_{J_i=0,\gamma^i=0} \\ &= \mathcal{A} + \log\left(e^{S_{\text{free}}(\iota\varphi)+S_{\text{int}}(\iota\varphi+\partial_{J_i}c_i)}e^{\frac{1}{2}h^{ij}\partial_{\gamma^i}\partial_{\gamma^j}}e^{J_i\gamma^i}\right)\big|_{J_i=0,\gamma^i=0} \\ &= \mathcal{A} + \log\left(e^{\frac{1}{2}h^{ij}\partial_{\gamma^i}\partial_{\gamma^j}}e^{S_{\text{free}}(\iota\varphi)+S_{\text{int}}(\iota\varphi+\partial_{J_i}c_i)}e^{J_i\gamma^i}\right)\big|_{J_i=0,\gamma^i=0} \\ &= \mathcal{A} + \log\left(e^{\frac{1}{2}h^{ij}\partial_{\gamma^i}\partial_{\gamma^j}}e^{S_{\text{free}}(\iota\varphi)+S_{\text{int}}(\iota\varphi+\gamma^ic_i)}\right)\big|_{\gamma^i=0}\,. \end{aligned} \tag{D.17}$$

Let us then introduce a canonical projector $\Pi : \mathcal{F}(\mathcal{H}) \to \mathcal{F}(A)$, where $\mathcal{F}(\mathcal{H})$ is the functional space over $\mathcal{H}$ and $\mathcal{F}(A)$ the same over A. This projector is defined such that

$$\Pi(\alpha^i) = \alpha^i, \tag{D.18}$$

$$\Pi(\beta^i) = \Pi(\gamma^i) = 0, \tag{D.19}$$

and it is generalized on a generic functional $f \in \mathcal{F}(\mathcal{H})$ by Taylor expanding the functional in powers of α, β and γ. With this understanding, the condition $\gamma^i = 0$ in the effective action can be implemented through the introduction of Π

$$\tilde{S}[\varphi] = \mathcal{A} + \log\left(\Pi e^{\frac{1}{2}h^{ij}\partial_{\gamma^i}\partial_{\gamma^j}}e^{S_{\text{free}}(\iota\varphi)+S_{\text{int}}(\iota\varphi+\gamma^ic_i)}\right)\,. \tag{D.20}$$

Furthermore let us define

$$\tilde{\Pi} \coloneqq \Pi e^{\frac{1}{2}h^{ij}\partial_{\gamma^i}\partial_{\gamma^j}} , \tag{D.21}$$

thus obtaining

$$\tilde{S}(\varphi) = \mathcal{A} + \log\left(\tilde{\Pi} e^{S_{\text{free}}(\iota\varphi]+S_{\text{int}}(\iota\varphi+\gamma^i c_i)}\right) . \tag{D.22}$$

We introduced $\tilde{\Pi}$ to establish a connection between this standard QFT procedure and [114], where the authors demonstrate that $\tilde{\Pi}$ represents the transferred projector derived from the homological perturbation lemma (HPL) at the functional level, applied to the following free SDR

$$\circlearrowleft\,(-H)\,(\mathcal{F}(\mathcal{H}), d) \underset{I}{\overset{\Pi}{\rightleftarrows}} (\mathcal{F}(A), \Pi d I) , \tag{D.23}$$

in which the starting differential and the propagator are defined as

$$d \coloneqq (S_{\text{free}}[\varphi + R'], \cdot) = \gamma^i \omega(c^j, Q c_i)\frac{\overrightarrow{\partial}}{\partial\beta^j} + \alpha^i\omega(a^j, Q a_i)\frac{\overrightarrow{\partial}}{\partial\alpha^j} , \tag{D.24}$$

$$H \coloneqq \frac{1}{\#_{\beta+\gamma}}\beta^i\omega(b^j, h b_i)\frac{\overrightarrow{\partial}}{\partial\gamma^j} , \tag{D.25}$$

where $\#_{\beta+\gamma}$ counts the number of occurrence of the variables β and γ in each monomial on which H acts. The inclusion I trivially embeds a function of α into the space of functions of (α, β, γ). It is easy to see that the propagator satisfies the Hodge-Kodaira decomposition

$$dH + Hd = 1 - I\Pi , \tag{D.26}$$

Next, we consider adding a deformation given by the BV Laplacian $\delta_1 = \Delta$

$$\circlearrowleft\,(-\tilde{H})\,(\mathcal{F}(\mathcal{H}), d + \Delta) \underset{\tilde{I}}{\overset{\tilde{\Pi}}{\rightleftarrows}} (\mathcal{F}(A), \tilde{d}) , \tag{D.27}$$

then by the homological perturbation lemma the transferred canonical projector will be given by

$$\tilde{\Pi} = \Pi\frac{1}{1+\delta_1 H} . \tag{D.28}$$

An important result of [114] (section 4.1.1) is that D.21 and D.28 are in fact the same projector, when applied to a generic functional $f \in \mathcal{F}(\mathcal{H})$, namely

$$\Pi e^{\frac{1}{2}h^{ij}\partial_{\gamma^i}\partial_{\gamma^j}} f = \Pi \frac{1}{1+\delta_1 H} f \,. \tag{D.29}$$

This result allows us to draw a connection between the more standard path integral approach and the application of the HPL to the free strong deformation retract D.23. Moreover it ensures that $\tilde{\Pi}$ defined as D.21 satisfies all the properties required by the HPL.

Notice that, as it is evident from D.17, this path integral will also compute the vacuum bubbles, which will appear as constant terms in the effective action. In particular if we set φ to zero in the effective action (D.22) we get

$$\tilde{S}(0) = \mathcal{A} + \log\left(\tilde{\Pi} e^{S_{\text{int}}[\gamma^i c_i]}\right) , \tag{D.30}$$

and the second term will produce the (interacting) vacuum bubbles, in addition to the one-loop determinant $\mathcal{A}$. We can isolate the bubbles by adding and subtracting them from D.22

$$\tilde{S}(\varphi) = \mathcal{A} + \log\left(\tilde{\Pi} e^{-S_{\text{int}}[\gamma^i c_i]}\right) + \log\left(\frac{\tilde{\Pi} e^{S_{\text{free}}(\iota\varphi)+S_{\text{int}}(\iota\varphi+\gamma^i c_i)}}{\tilde{\Pi} e^{-S_{\text{int}}(\gamma^i c_i)}}\right) . \tag{D.31}$$

The last term

$$\tilde{S}'(\varphi) = \log\left(\frac{\tilde{\Pi} e^{S_{\text{free}}(\iota\varphi)+S_{\text{int}}(\iota\varphi+\gamma^i c_i)}}{\tilde{\Pi} e^{S_{\text{int}}(\gamma^i c_i)}}\right) , \tag{D.32}$$

is then the effective action without the vacuum bubbles, which correctly vanishes when $\varphi = 0$. In order to extract this normalized action and concretely link it to the products defining the initial UV action, we follow again [114], who applied the HPL to the free SDR D.23, but with a different perturbation δ_2 defined as

$$\delta_2 = (S_{\text{int}}[\iota\varphi + R], \cdot) + \Delta \,. \tag{D.33}$$

This gives the following perturbed SDR

$$\circlearrowleft \left(-\tilde{H}_2\right)\left(\mathcal{F}(\mathcal{H}), d+\delta_2\right) \underset{\tilde{I}_2}{\overset{\tilde{\Pi}_2}{\rightleftarrows}} \left(\mathcal{F}(A), \tilde{d}_2\right), \tag{D.34}$$

and the HPL dictates that

$$\tilde{\Pi}_2 = \Pi \frac{1}{1+\delta_2 H}. \tag{D.35}$$

Focusing now on the transferred differential

$$\tilde{d}_2 = \Pi d I + \Pi \delta_2 \frac{1}{1+H\delta_2} I, \tag{D.36}$$

it is possible to show (section 4.2 of [114]) that $\tilde{d}_2$ can be independently written as

$$\tilde{d}_2 = (\tilde{S}'(\varphi), \cdot)_{\alpha\alpha} + \Delta_{\alpha\alpha}, \tag{D.37}$$

where the subscript $(\cdot)_{\alpha\alpha}$ means that the BV bracket and BV laplacian only contain derivatives in α, namely

$$(\cdot,\cdot)_{\alpha\alpha} = \frac{\overleftarrow{\partial}}{\partial\alpha^i}(\omega_A)^{ij}\frac{\overrightarrow{\partial}}{\partial\alpha^j}, \tag{D.38a}$$

$$\Delta_{\alpha\alpha} = \frac{1}{2}(-)^{\alpha^i}(\omega_A)^{ij}\frac{\overrightarrow{\partial}}{\partial\alpha^i}\frac{\overrightarrow{\partial}}{\partial\alpha^j}. \tag{D.38b}$$

Notice that any possible constant part of $\tilde{S}'[\varphi]$ in (D.37) would be eliminated by the BV bracket. We can then determine the effective products of $\tilde{S}'[\varphi]$ by imposing the equality of D.36 and D.37

$$\begin{aligned}
(\tilde{S}'(\varphi), \cdot)_{\alpha\alpha} + \Delta_{\alpha\alpha} &= \Pi d I + \Pi\delta_2 \frac{1}{1+H\delta_2} I \\
&= \Pi d I + \Pi\delta_2 I - \Pi\delta_2 \frac{1}{1+H\delta_2} H\delta_2 I \\
&= \Pi(S(\iota\varphi + R), \cdot) I + \Pi\Delta I - \Pi\delta_2 \frac{1}{1+H\delta_2} H\delta_2 I \\
&= \Pi(S(\Phi), \cdot) I + \Delta_{\alpha\alpha} - \Pi \frac{1}{1+\delta_2 H}\delta_2 H \delta_2 I,
\end{aligned} \tag{D.39}$$

where we named $\Phi = \varphi + R$. The above expression allows us to establish a relation

between the BV bracket of the effective theory and the starting action, that can be further simplified as

$$(\tilde{S}'(\varphi),\cdot)_{\alpha\alpha} = \Pi(S(\Phi),\cdot)I - \Pi\frac{1}{1+\delta_2 H}\delta_2 H\delta_2 I\,. \tag{D.40}$$

By applying this expression to φ we will now show that we precisely recover the effective vertices as obtained by the HPL at the level of the Hilbert space $\mathcal{H}$ and the co-algebraic description used in the main text.

In order to find the effective products we assume that the starting action and the effective action can be written in WZW form as follows:

$$S(\Phi) = \int_0^1 dt\,\omega(\dot{\Phi}, \pi_1 \boldsymbol{l} e^{\wedge\Phi})\,, \tag{D.41}$$

$$\tilde{S}'(\varphi) = \int_0^1 dt\,\tilde{\omega}(\dot{\varphi}, \pi_1 \tilde{\boldsymbol{l}} e^{\wedge\varphi})\,, \tag{D.42}$$

where $\tilde{\omega}(\cdot,\cdot) \coloneqq \omega(\iota\cdot,\iota\cdot)$ and both $\boldsymbol{l}$ and $\tilde{\boldsymbol{l}}$ are assumed to be cyclic, with respect to the corresponding symplectic form. Then, if we apply D.40 to φ we get

$$(\tilde{S}'(\varphi),\varphi)_{\alpha\alpha} = \Pi(S[\Phi], I\varphi) - \Pi\frac{1}{1+\delta_2 H}\delta_2 H\delta_2 I\varphi\,. \tag{D.43}$$

The left hand side can be obtained by a direct computation

$$(\tilde{S}'(\varphi),\varphi)_{\alpha\alpha} = \tilde{S}'(\varphi)\frac{\overleftarrow{\partial}}{\partial\alpha^i}\omega_A^{ij}\frac{\overrightarrow{\partial}}{\partial\alpha^j}(\alpha^k a_k) = (-)^{\alpha^i}\frac{\overrightarrow{\partial}}{\partial\alpha^i}\tilde{S}'(\varphi)\omega_A^{ij}a_j\,. \tag{D.44}$$

The derivative acts on the action as

$$\begin{aligned}\frac{\overrightarrow{\partial}}{\partial\alpha^j}\tilde{S}'(\varphi) &= \frac{\overrightarrow{\partial}}{\partial\alpha^j}\int_0^1 dt\,\omega(\pi_1\boldsymbol{\partial}_t e^{\wedge\varphi}, \pi_1\tilde{\boldsymbol{l}}e^{\wedge\varphi}) \\ &= (-)^{\alpha^i}\int_0^1 dt\,\omega(\pi_1\boldsymbol{\partial}_t\boldsymbol{\partial}_{\alpha_i}e^{\wedge\varphi}, \pi_1\tilde{\boldsymbol{l}}e^{\wedge\varphi}) + \int_0^1 dt\,\omega(\pi_1\boldsymbol{\partial}_t e^{\wedge\varphi}, \pi_1\tilde{\boldsymbol{l}}\boldsymbol{\partial}_{\alpha_i}e^{\wedge\varphi})\,,\end{aligned} \tag{D.45}$$

where $\boldsymbol{\partial}_{\alpha_i}$ should be considered as a coderivation on the same footing as $\boldsymbol{\partial}_t$. Then

using the cyclic property of $\tilde{\boldsymbol{l}}$ we can rewrite the second term as

$$\omega(\pi_1 \boldsymbol{\partial}_t e^{\wedge\varphi}, \pi_1 \tilde{\boldsymbol{l}} \boldsymbol{\partial}_{\alpha_i} e^{\wedge\varphi}) = (-)^{\alpha_i} \omega(\pi_1 \boldsymbol{\partial}_{\alpha^i} e^{\wedge\varphi}, \pi_1 \boldsymbol{\partial}_t \tilde{\boldsymbol{l}} e^{\wedge\varphi}) \,, \tag{D.46}$$

so that the two terms in D.45 can be resummed using the Leibniz rule

$$\begin{aligned} \frac{\overrightarrow{\partial}}{\partial \alpha^j} \tilde{S}'(\varphi) &= (-)^{\alpha^i} \int_0^1 dt \, \partial_t \, \omega(\pi_1 \boldsymbol{\partial}_{\alpha^i} e^{\wedge\varphi}, \pi_1 \tilde{\boldsymbol{l}} e^{\wedge\varphi}) \\ &= (-)^{\alpha^i} \omega(\pi_1 \boldsymbol{\partial}_{\alpha^i} e^{\wedge\varphi}, \pi_1 \tilde{\boldsymbol{l}} e^{\wedge\varphi}) = (-)^{\alpha^i} \omega(a_i, \pi_1 \tilde{\boldsymbol{l}} e^{\wedge\varphi}) \,. \end{aligned} \tag{D.47}$$

Plugging this back into D.44 we can use the completeness relation to build

$$(\tilde{S}'(\varphi), \varphi)_{\alpha\alpha} = \omega(a_i, \pi_1 \tilde{\boldsymbol{l}} e^{\wedge\varphi}) \omega_A^{ij} a_j = \pi_1 \tilde{\boldsymbol{l}} e^{\wedge\varphi} \,. \tag{D.48}$$

Let's now turn to the first term on the right hand side of D.43. Since the β, γ part of the Poisson bracket vanishes when acting on φ, we are left with

$$\Pi(S(\Phi), I\varphi) = \Pi \left[S(\Phi) \frac{\overleftarrow{\partial}}{\partial \alpha^i} \omega_A^{ij} \frac{\overrightarrow{\partial}}{\partial \alpha^j} I\varphi \right] \,, \tag{D.49}$$

where $I\varphi$ must be understood now as an element of $\mathcal{F}(\mathcal{H})$. Following analogous steps as in D.48 we get

$$\Pi(S(\Phi), I\varphi) = \Pi \left[\omega(a_i, \pi_1 \boldsymbol{l} e^{\wedge\Phi}) \right] \omega_A^{ij} a_j \,. \tag{D.50}$$

The symplectic form gives a non zero result only if $\pi_1 \boldsymbol{l} e^{\wedge\Phi} \in A$ so the previous expression is equivalent to

$$\Pi(S(\Phi), I\varphi) = \Pi \left[\omega(a_i, \pi_1 \boldsymbol{\pi} \boldsymbol{l} e^{\wedge\Phi}) \right] \omega_A^{ij} a_j = \Pi[\pi_1 \boldsymbol{\pi} \boldsymbol{l} e^{\wedge\Phi}] \,. \tag{D.51}$$

As Π kills all β's and γ's we can trade it with the $\boldsymbol{\iota}$ immersion acting on φ and write

$$\Pi(S(\Phi), I\varphi) = \pi_1 \boldsymbol{\pi} \boldsymbol{l} \boldsymbol{\iota} e^{\wedge\varphi} \,. \tag{D.52}$$

The second term in the RHS of D.48 can be computed trough similar steps, giving

the following chain of relations

$$\begin{aligned}\delta_2 I\varphi &= \pi_1 \boldsymbol{\delta l} e^{\wedge\Phi}\,,\\ H\delta_2 I\varphi &= (-)^{\gamma^j}\beta^j \pi_1 \boldsymbol{\delta l}(hb_j) e^{\wedge\Phi}\,,\\ \delta_2 H\delta_2 I\varphi &= \pi_1 \boldsymbol{\delta l}(\boldsymbol{h\delta l} + \boldsymbol{hU}) e^{\wedge\Phi}\,,\end{aligned} \tag{D.53}$$

where (hb_j) is understood as the zero-coderivation associated with the state hb_j. Additionally, we can notice that each application of a $\delta_2 H$ on $\delta_2 H\delta_2\varphi$ consists in inserting a $(\boldsymbol{h\delta l} + \boldsymbol{hU})$ in front of $e^{\wedge\Phi}$, namely

$$(\delta_2 H)^n \delta_2 H\delta_2 I\varphi = \pi_1 \boldsymbol{\delta l}(\boldsymbol{h\delta l} + \boldsymbol{hU})^{n+1} e^{\wedge\Phi}\,, \tag{D.54}$$

Therefore D.43 becomes

$$\begin{aligned}\pi_1 \tilde{\boldsymbol{l}} e^{\wedge\varphi} &= \pi_1 \boldsymbol{\pi l \iota} e^{\wedge\varphi} - \pi_1 \boldsymbol{\pi\delta l}\frac{1}{1+\boldsymbol{h}(\boldsymbol{\delta l}+\boldsymbol{U})}\boldsymbol{h}(\boldsymbol{\delta l}+\boldsymbol{U})\boldsymbol{\iota} e^{\wedge\varphi}\\ &= \pi_1 \boldsymbol{\pi l}\frac{1}{1+\boldsymbol{h}(\boldsymbol{\delta l}+\boldsymbol{U})}\boldsymbol{\iota} e^{\wedge\varphi}\,,\end{aligned} \tag{D.55}$$

where we have also traded the Π in the second term of D.43 with a $\boldsymbol{\iota}$ acting on $e^{\wedge\varphi}$, following the same reasoning adopted in D.52. This establishes that the HPL at the co-algebraic level which we use in the main text is the result of the path integral.

Moreover, we can remove the π_1 from (D.55) and add the projected Poisson bi-vector $\boldsymbol{U}_\pi$ on both sides of the above expression, obtaining

$$\tilde{\boldsymbol{l}} + \boldsymbol{U}_\pi = \boldsymbol{\pi l}\frac{1}{1+\boldsymbol{h}(\boldsymbol{\delta l}+\boldsymbol{U})}\boldsymbol{\iota} + \boldsymbol{U}_\pi\,. \tag{D.56}$$

Remarkably this is equivalent to what we would get by applying the homotopy transfer to the nilpotent structure $(\boldsymbol{l}+\boldsymbol{U})$, that is

$$\tilde{\boldsymbol{l}} + \boldsymbol{U}_\pi = \boldsymbol{\pi}(\boldsymbol{l}+\boldsymbol{U})\frac{1}{1+\boldsymbol{h}(\boldsymbol{\delta l}+\boldsymbol{U})}\boldsymbol{\iota}\,. \tag{D.57}$$

To see this equivalence, let us manipulate

$$\begin{aligned}
&\boldsymbol{\pi}(\boldsymbol{l}+\boldsymbol{U})\frac{1}{1+\boldsymbol{h}(\boldsymbol{\delta l}+\boldsymbol{U})}\boldsymbol{\iota} \\
&\qquad = \boldsymbol{\pi l}\frac{1}{1+\boldsymbol{h}(\boldsymbol{\delta l}+\boldsymbol{U})}\boldsymbol{\iota} + \boldsymbol{\pi U}\frac{1}{1+\boldsymbol{h}(\boldsymbol{\delta l}+\boldsymbol{U})}\boldsymbol{\iota} \\
&\qquad = \boldsymbol{\pi l}\frac{1}{1+\boldsymbol{h}(\boldsymbol{\delta l}+\boldsymbol{U})}\boldsymbol{\iota} + \boldsymbol{U}_\pi\boldsymbol{\pi}\frac{1}{1+\boldsymbol{h}(\boldsymbol{\delta l}+\boldsymbol{U})}\boldsymbol{\iota} \\
&\qquad = \boldsymbol{\pi l}\frac{1}{1+\boldsymbol{h}(\boldsymbol{\delta l}+\boldsymbol{U})}\boldsymbol{\iota} + \boldsymbol{U}_\pi\boldsymbol{\pi\iota} - \boldsymbol{U}_\pi\boldsymbol{\pi h}(\boldsymbol{\delta l}+\boldsymbol{U})\frac{1}{1+\boldsymbol{h}(\boldsymbol{\delta l}+\boldsymbol{U})}\boldsymbol{\iota} \\
&\qquad = \boldsymbol{\pi l}\frac{1}{1+\boldsymbol{h}(\boldsymbol{\delta l}+\boldsymbol{U})}\boldsymbol{\iota} + \boldsymbol{U}_\pi = \tilde{\boldsymbol{l}} + \boldsymbol{U}_\pi\,,
\end{aligned} \tag{D.58}$$

where in the second line we have used (6.52) and in the third line $\boldsymbol{\pi\iota} = 1$ and $\boldsymbol{\pi h} = 0$.

So, in conclusion, we have proven the following general formula for the Wilsonian effective action

$$\tilde{S}(\varphi) = \mathcal{A} + \log\left(e^{\frac{1}{2}h^{ij}\partial_{\gamma^i}\partial_{\gamma^j}}e^{S_{\rm int}(\gamma^i c_i)}\right)\Big|_{\gamma^i=0} + \int_0^1 dt\,\omega(\dot{\varphi},\pi_1\tilde{\boldsymbol{l}}e^{\varphi})\,, \tag{D.59}$$

which describes the transfer between the nilpotent structures

$$(\boldsymbol{l}+\boldsymbol{U})^2 = 0 \quad \to \quad (\tilde{\boldsymbol{l}}+\boldsymbol{U}_\pi)^2 = 0\,, \tag{D.60}$$

which, as expected, guarantees that the effective action is a solution to the BV quantum master equation, provided this is true for the initial UV action. The first two terms in (D.59) give the expression of the (connected) vacuum bubbles, that are the additive constants in the effective action. These constants do not enter in the BV quantum master equation, yet they are important for the correct computation of observables. The dynamical part of the action, on the right, is given by the homotopy transfer at the co-algebra level. It would be interesting and useful to understand if also the constant part of the action could be written at the co-algebra level, encoding the vacuum amplitudes that one can construct using the interaction vertices and the propagators.

APPENDIX **E**

$SL(2,\mathbb{C})$ vertices for the open string one-loop tadpole

In this appendix, we will construct the OC-SFT vertices needed to compute the open string one-loop tadpole. The moduli space associated with this amplitude is one dimensional. For the sake of simplicity we will use $SL(2,\mathbb{C})$ vertices as much as we can.[1] A similar construction has been discussed in [177], but the closed string degeneration region was not addressed. Specifically, we can characterize the annulus as a cylinder, $C_{(\pi,2\pi t)}$, with height π and circumference $2\pi t$, where $t \in [0,\infty)$ parameterizes the moduli space. Let's denote z the complex coordinate on the

[1] It would be also interesting to use hyperbolic vertices [162, 173].

cylinder, which is subject to the identification

$$z \simeq z + 2\pi t, . \tag{E.1}$$

Within this parameterization, we have the open string degeneration in the limit $t \to \infty$, which corresponds to an extremely wide and flattened cylinder that leads to the collision of boundaries. On the other hand, the limit $t \to 0$ is the closed string degeneration, represented by a very narrow tube.
As anticipated in the main text, this amplitude receives contribution from three Feynman diagrams, the first one containing the open string degeneration, the second one containing the closed string degeneration and the third one which is a fundamental vertex covering the interior of the moduli space

$$\mathcal{A}^{0,2}_{0;1}(\varphi_{\rm o}) = -\omega_{\rm o}\left(\varphi_{\rm o}, m^{(0,1)}_{0,2}\left(\boldsymbol{h}_0 U_{\rm o}\right)\right) - \omega_{\rm o}\left(\varphi_{\rm o}, m^{(0,1)}_{1,0}\left(\frac{b^+_0}{L^+_0}\bar{P}^+_0 l^{(0,1)}_{0,0}\right)\right) + \omega_{\rm o}\left(\varphi_{\rm o}, m^{(0,2)}_{0,0}\right), \tag{E.2}$$

where $\varphi_{\rm o}$ is the open string field projected in $\ker(L_0)$.

E.1 Open string non-separating plumbing fixture

Let us begin by describing the open channel

$$-\omega_{\rm o}\left(\varphi_{\rm o}, m^{(0,1)}_{0,2}\left(\boldsymbol{h}_0 U_{\rm o}\right)\right) .$$

To start with we give an explicit expression to the product $m^{(0,1)}_{0,2}$

$$\omega_{\rm o}\left(\Psi_1, m^{(0,1)}_{0,2}(\Psi_2,\Psi_3)\right) = (-)^{\Psi_1+\Psi_2}\langle f_{-\xi}\circ\Psi_1(0) f_0\circ\Psi_2(0) f_\xi\circ\Psi_3(0)\rangle_{\rm UHP}\,, \tag{E.3}$$

and we define the f-maps as generic $SL(2,\mathbb{R})$ transformations which insert the punctures on the real axis of the UHP in $-\xi$, 0 and $\xi > 0$ respectively, namely $f_\xi(0) = \xi$, $f_0(0) = 0$ and $f_{-\xi}(0) = -\xi$. We impose cyclicity by considering the $SL(2,\mathbb{R})$ transformation g that cyclically permutes the location of the punctures,

$g(0) = -\xi$, $g(\xi) = 0$ and $g(-\xi) = \xi$ and enforcing the following conditions

$$g \circ f_{-\xi} = f_{\xi}\,, \qquad g \circ f_0 = f_{-\xi}\,, \qquad g \circ f_{\xi} = f_0\,. \tag{E.4}$$

Additionally we also impose twist symmetry $f_{\xi}(-w) = -f_{-\xi}(w)$ and $f_0(-w) = -f_0(w)$. This gives a one-parameter (l) family of maps

$$f_{\xi}(w_1) = \frac{\xi^2 + l\xi w_1}{\xi - 3lw_1}\,, \tag{E.5a}$$

$$f_0(w_2) = lw_2\,, \tag{E.5b}$$

$$f_{-\xi}(w_3) = \frac{l\xi w_3 - \xi^2}{\xi + 3lw_3}\,. \tag{E.5c}$$

The two positive real parameters l and ξ must satisfy $\xi > 3l$ in order to avoid the overlap of the local coordinate patches.
Presently, our aim is to implement the plumbing fixture procedure, identifying the local coordinates associated with the punctures located in ξ and $-\xi$ as follows

$$w_1 \simeq -\frac{q}{w_3}\,, \tag{E.6}$$

where $q \in [0, 1]$. By inverting (E.5a) and (E.5c) we can rewrite (E.6) getting an identification on the UHP (with coordinate u) which defines an annulus

$$u \simeq f_{-\xi} \circ I_{-q} \circ f_{\xi}^{-1}(u)\,, \tag{E.7}$$

where we have denoted $I_{-q}(w) = -q/w$. In the following, we will use the symbol $\Sigma^{\text{p.f.}}_{\text{open}}(q)$ to denote this surface, which is reported in figure E.1. Note that identification (E.7) corresponds to composition of $SL(2,\mathbb{R})$ maps, and therefore, following [177], we can find the similarity transformation that 'diagonalizes' the previous expression. In other words, we introduce another $SL(2,\mathbb{R})$ map

$$u = p(z') = \xi\sqrt{\frac{\xi^2 - ql^2}{\xi^2 - 9ql^2}}\left(\frac{1 - z'}{1 + z'}\right)\,, \tag{E.8}$$

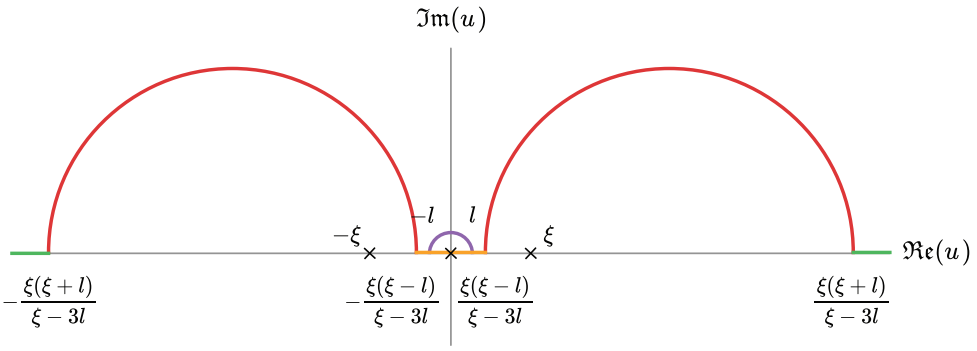


Figure E.1: Annulus given by the plumbing fixture procedure with boundaries in green and orange. The red curves, identified through the equivalence relation (E.6), delineate the contours of the local patches associated with maps $f_{-\xi}$ and f_{ξ}. The $\times$ denote the open string punctures and the purple half-circle the contour of the local patch related to f_0.

such that

$$z' \simeq p^{-1} \circ f_{-\xi} \circ I_{-q} \circ f_{\xi}^{-1} \circ p(z') = \Lambda z' \,, \tag{E.9}$$

where the real number Λ is the ratio of the two eigenvalues of the diagonal matrix associated with

$$p^{-1} \circ f_{-\xi} \circ I_{-q} \circ f_{\xi}^{-1} \circ p \,. \tag{E.10}$$

Its explicit expression is

$$\Lambda(l,\xi,q) = \frac{\xi^2 + 3l^2 q - \sqrt{\xi^4 + 9l^4 q^2 - 10 l^2 \xi^2 q}}{\xi^2 + 3l^2 q + \sqrt{\xi^4 + 9l^4 q^2 - 10 l^2 \xi^2 q}} \,, \tag{E.11}$$

which is real thanks to the non-overlap condition $\xi > 3l$. The resulting surface is shown in figure E.2.

Now it is easy to map to the frame where the annulus is described by (E.1). Indeed, defining

$$z = -\log(z') \,, \tag{E.12}$$

we get

$$z \simeq z + \log\left(\frac{1}{\Lambda}\right) \,. \tag{E.13}$$

Therefore, comparing with (E.1), we can read off how the modulus is related to the

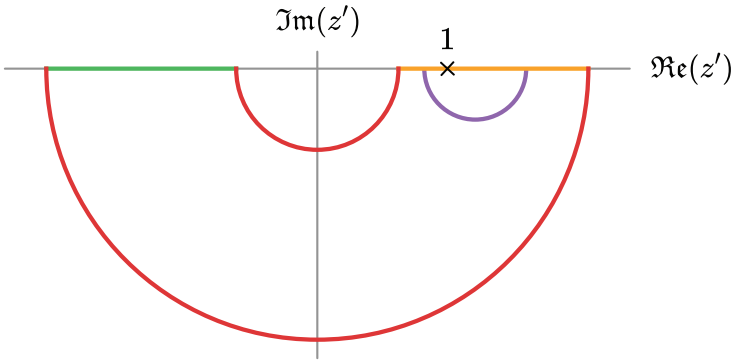


Figure E.2: Annulus in the frame $z' = p^{-1}(u)$.

plumbing fixture parameter q and the SFT data ξ and l

$$t_{\mathrm{o}}(q,l,\xi) = \frac{1}{2\pi}\log\left(\frac{\xi^2+3l^2q+\sqrt{\xi^4+9l^4q^2-10l^2\xi^2q}}{\xi^2+3l^2q-\sqrt{\xi^4+9l^4q^2-10l^2\xi^2q}}\right). \tag{E.14}$$

Notice that by varying the plumbing fixture parameter in $[0,1]$ the modulus changes in the interval

$$t_{\mathrm{o}}(q,l,\xi) \in [t_{\max}(l,\xi),\infty)\,, \tag{E.15}$$

in which we defined $t_{\max}(l,\xi) \coloneqq t_{\mathrm{o}}(q=1,l,\xi)$. This is the portion of moduli space covered by the non-separating open string Feynman diagram, which depends on the data ξ and l.

Finally, we can easily find the local coordinate map associated to the open string puncture on $C_{(\pi,2\pi t)}$, figure E.3

$$\begin{aligned} G_{\mathrm{o}}(w) &\coloneqq g_{\mathrm{o}}\circ f_0(w) \\ &= \log\left(\frac{\xi\sqrt{\xi^4+9l^4q^2-10l^2\xi^2q}+(\xi^2-9l^2q)w}{\xi\sqrt{\xi^4+9l^4q^2-10l^2\xi^2q}-(\xi^2-9l^2q)w}\right) \\ &= 2\tanh^{-1}\left(\frac{1}{\xi}\sqrt{\frac{\xi^2-9l^2q}{\xi^2-l^2q}}w\right). \end{aligned} \tag{E.16}$$

where we defined $g_{\mathrm{o}}(u) \coloneqq -\log \circ p^{-1}(u)$.

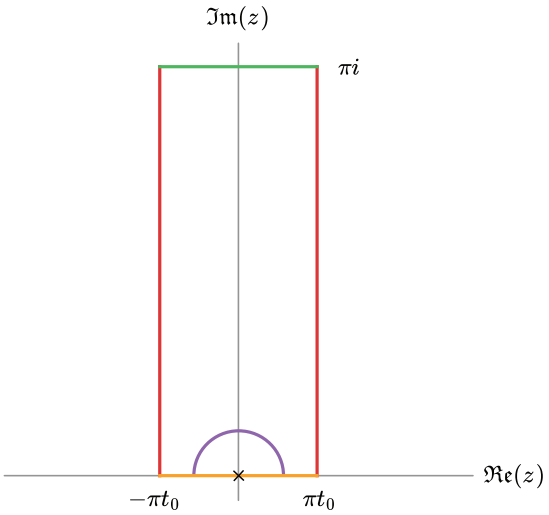


Figure E.3: Annulus in the frame $z = -\log(z')$.

E.2 Closed string separating plumbing fixture

Regarding the closed channel

$$-\omega_{\mathrm{o}}\left(\varphi_{\mathrm{o}}, m_{1,0}^{(0,1)}\left(\frac{b_0^+}{L_0^+}\bar{P}_0^+ l_{0,0}^{(0,1)}\right)\right),$$

this time we have to perform the plumbing fixture on the following two fundamental vertices

$$\omega_{\mathrm{c}}\left(l_{0,0}^{(0,1)}, \Phi\right) = \frac{1}{4\pi^2}\langle f_b \circ c_0^- \Phi(0,\bar{0})\rangle_{\mathrm{disk}}\,, \tag{E.17}$$

and

$$\omega_{\mathrm{o}}\left(\Psi, m_{1,0}^{(0,1)}(\Phi)\right) = \frac{(-)^{\Phi+\Psi}}{2\pi i}\langle d \circ f_{\mathrm{o}} \circ \Psi(0) d \circ f_{\mathrm{c}} \circ \Phi(0,\bar{0})\rangle_{\mathrm{disk}}\,, \tag{E.18}$$

where, following [126], f_b, f_c and f_o are the $SL(2,\mathbb{C})$ maps

$$f_{\rm b}(w_1) = \frac{w_1}{l_{\rm b}}\,, \tag{E.19a}$$

$$f_{\rm o}(w_2) = \frac{w_2}{l_{\rm o}}\,, \tag{E.19b}$$

$$f_{\rm c}(w_3) = i\frac{1+\frac{w_3}{\beta_1}}{1+\frac{w_3}{\beta_2}}\,, \tag{E.19c}$$

where $l_{\rm b} > 1$ and $l_{\rm o} > 1$ are tunable parameters such as β_1 and β_2 which are chosen to obey $\beta_2 > \beta_1 > 1$. Additionally, imposing the non-overlap condition implies the following constraint

$$l_{\rm o} > \frac{\beta_1(1-\beta_2)}{\beta_2(1-\beta_1)}\,. \tag{E.20}$$

The two punctured disks are reported in figure E.4.

Furthermore we called d the Cayley function, $d(z) = \frac{i-z}{i+z}$, which maps the UHP into

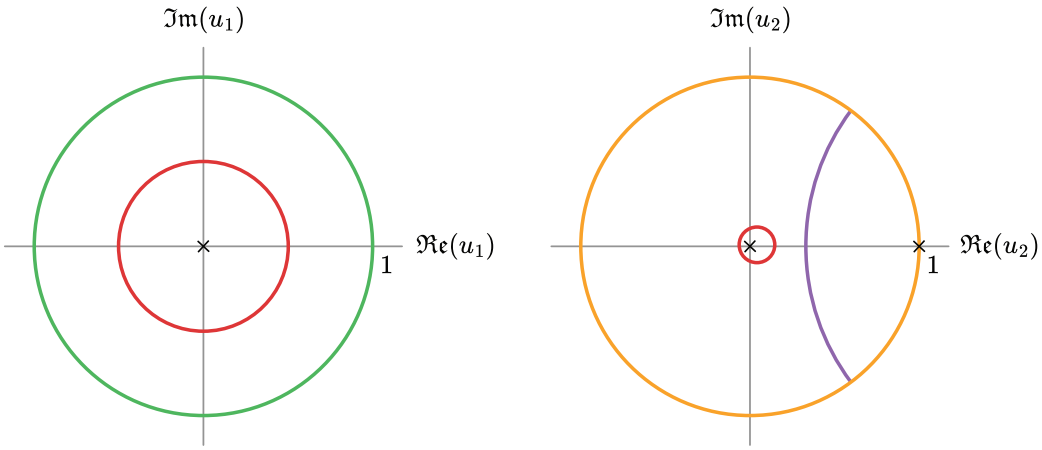


Figure E.4: On the left, the unit circle with an inserction of a closed string at 0 and local patches corresponding to a circle of radius $l_{\rm b}^{-1}$. On the right, the unit disk with a closed string puncture and an open string puncture, with local patches represented in red and purple, respectively. The colors used are consistent with the previous section. The plumbing fixture glues the two disks along the red curves, resulting in an annulus with orange and green boundaries.

the unitary disk in the standard way $d(0) = 1$, $d(i) = 0$, $d(\infty) = -1$. Therefore the local coordinate maps on the disk are

$$u_1 = f_{\rm b}(w_1), \qquad u_2 = d \circ f_{\rm o}(w_2), \qquad u_2 = d \circ f_{\rm c}(w_3), \tag{E.21}$$

Now, let us perform the closed-string separating plumbing fixture, identifying the local coordinates associated with the closed punctures on the two disks using the relationship

$$w_1 \simeq \frac{q}{w_3}, \tag{E.22}$$

where $q \in [0,1]$.

Inverting (E.21) and inserting it into (E.22) allows us to derive the relation between the coordinates on the two disks

$$u_2 = d \circ f_{\rm c} \circ I_q \circ f_{\rm b}^{-1}(u_1)\,, \tag{E.23}$$

we will call this surface as $\Sigma^{\rm p.f.}_{\rm closed}(q)$, figure E.5. In terms of the u_2 coordinate, we are

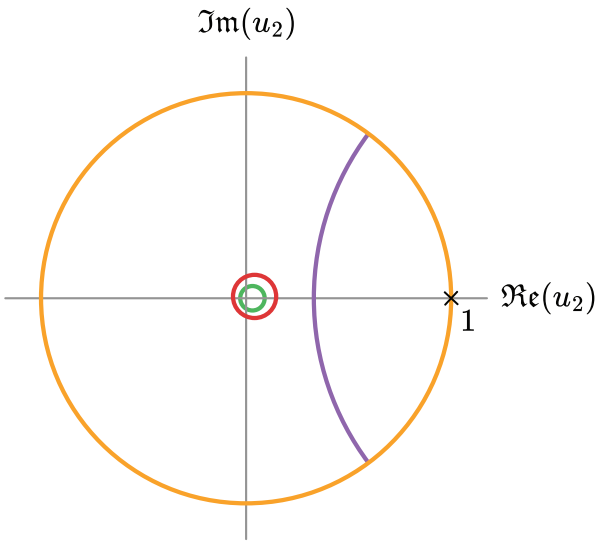


Figure E.5: Boundary-punctured annulus given by the separating closed plumbing fixture

delineating an annulus with an outer boundary of unit radius with the open puncture and an inner boundary consisting of a circle with radius R,

$$R(l_{\rm b},\beta_1,\beta_2,q) = \frac{2q\beta_1\beta_2(\beta_2-\beta_1)l_{\rm b}}{4{\beta_1}^2{\beta_2}^2{l_{\rm b}}^2 - q^2(\beta_1+\beta_2)^2}\,, \tag{E.24}$$

centered at C

$$C(l_{\rm b},\beta_1,\beta_2,q)=\frac{q^2(\beta_2^2-\beta_1^2)}{4{\beta_1}^2{\beta_2}^2{l_{\rm b}}^2-q^2(\beta_1+\beta_2)^2}\,. \tag{E.25}$$

Therefore, to obtain the annulus in the frame where (E.1) holds, it is necessary to apply an additional transformation to center both boundaries at zero. This is done as follows: we start with the generic real Möbius map that leaves the unit circle unchanged

$$h_\alpha(u_2)=\frac{u_2-\alpha}{1-\alpha u_2}\,, \tag{E.26}$$

with $\alpha\in\mathbb{R}$, and we fix α in order to center the inner circle on the real axis. This is achieved by the condition

$$h_\alpha\left(C+R\right)=-h_\alpha\left(C-R\right), \tag{E.27}$$

thus getting

$$\alpha_\pm=\frac{q^2\left({\beta_1}^2+{\beta_2}^2\right)\pm 2\beta_1\beta_2\left(\sqrt{(q^2-\beta_1^2l_b^2)(q^2-\beta_2^2l_b^2)}\mp\beta_1\beta_2{l_{\rm b}}^2\right)}{q^2(\beta_1^2-\beta_2^2)}\,, \tag{E.28}$$

notice that both solutions are real because $1<\frac{\beta_1 l_{\rm b}}{q}<\frac{\beta_2 l_{\rm b}}{q}$ by assumption. These two solutions identify the two possible maps which center the boundary without the open insertion by leaving the unit circle unchanged. In particular h_{α_+} maps the interior of the unit circle into the interior of the unit circle, whereas h_{α_-} sends the interior of the unit circle into the exterior of the unit circle. We will consider $z'=h_{\alpha_-}(u_1)$ and thus the case in which the punctured circle becomes the inner boundary, as shown in figure E.6. Finally, by applying the logarithm, we obtain the cylinder, which, however, is rotated by $\pi/2$ and has a boundary length of 2π, as reported in the first figure of E.7. To achieve the desired surface it is needed to further apply a rotation and a dilatation, second figure of E.7. At this point, it is possible to read off the modulus by

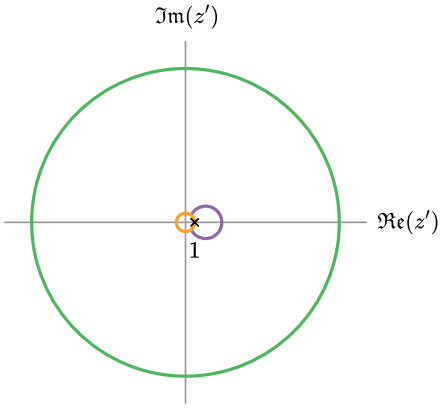


Figure E.6: Annulus in the frame $z' = h_{\alpha_-}(u_2)$.

comparing with (E.1)

$$t_{\rm c}(q; l_{\rm b}, \beta_1, \beta_2) = \pi \log \left(\frac{\beta_1\beta_2 l_b^2 - q^2 + \sqrt{(q^2 - \beta_1^2 l_b^2)\,(q^2 - \beta_2^2 l_b^2)}}{q l_{\rm b}(\beta_2 - \beta_1)} \right)^{-1} , \tag{E.29}$$

notice that by varying the plumbing fixture parameter in [0, 1] the modulus changes as follows

$$t_{\rm c}(q, l_{\rm b}, \beta_1, \beta_2) \in [0, t_{\rm min}(l_{\rm b}, \beta_1, \beta_2)] \,, \tag{E.30}$$

where we defined $t_{\rm min}(l_{\rm b}, \beta_1, \beta_2) = t_{\rm c}(q = 1, l_{\rm b}, \beta_1, \beta_2)$.
As expected in the closed string channel we are able to cover a sub-region of moduli space containing the closed string degeneration $t = 0$.
The local coordinate associated with the open string insertion on the canonical cylinder is given by

$$\begin{aligned} G_{\rm c}(w) &\coloneqq g_{\rm c} \circ d \circ f_{\rm o}(w) \\ &= i t_{\rm c} \log \left(\frac{q^2\left(\beta_1^2 w + i\beta_2^2 l_o\right) - \beta_1\beta_2(w + i l_o)\left(\beta_1\beta_2 l_b^2 + \sqrt{(q^2 - \beta_1^2 l_b^2)(q^2 - \beta_2^2 l_b^2)}\right)}{q^2\left(-\beta_1^2 w + i\beta_2^2 l_o\right) + \beta_1\beta_2(w - i l_o)\left(\beta_1\beta_2 l_b^2 + \sqrt{(q^2 - \beta_1^2 l_b^2)(q^2 - \beta_2^2 l_b^2)}\right)} \right) \\ &= 2 t_{\rm c} \tan^{-1}\left(-\frac{1}{l_{\rm o}} \frac{q\beta_1 + l_{\rm b} e^{\pi/t_{\rm c}}}{q\beta_2 - l_{\rm b} e^{\pi/t_{\rm c}}} w \right) . \end{aligned} \tag{E.31}$$

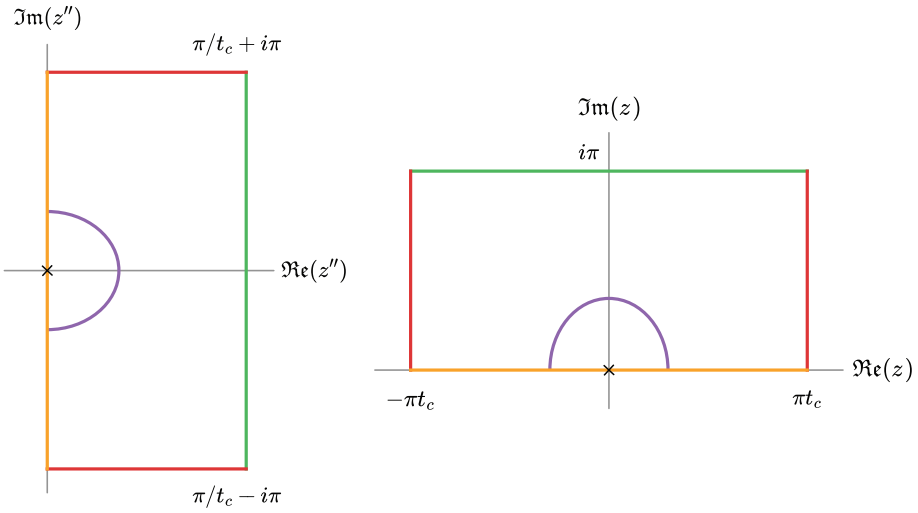


Figure E.7: Annulus in the frames $z'' = \log(z')$ and $z = it_c z''$.

where we have introduced

$$g_c(u_2) := it_c \log \circ h_{\alpha_-}(u_2)\,. \tag{E.32}$$

E.3 Fundamental vertex and decoupling of BRST exact states

It is important to notice the following fact. Given the reality of the geometrical parameters $(l, \xi, l_b, l_o, \beta_1, \beta_2)$, together with the non-overlap conditions $\xi > 3l$ and (E.20) (which ensure that the various square roots in (E.16) and (E.31) are real and positive), the local coordinate map in the open string channel (E.16) can never coincide with the same quantity in the closed string channel (E.31). This is in particular true when the open and closed parameters are chosen such that $t_c = t_o$. Therefore, even if the moduli space can in principle be covered with the two Feynman diagrams containing the degenerations, this would violate the one-punctured annulus homotopy relations (6.147). Therefore, differently from the disk amplitude with two closed strings that was analyzed in [126], this time we need a fundamental vertex covering the interior of moduli space.

In order to do so we will rewrite the amplitudes (E.2) in terms of surface states,

by making explicit the integration over the moduli space. From now on, for the sake of simplicity, we will assume that the string field $\varphi_{\rm o}$ has even degree, which is true in the case of a physical tadpole amplitude.
Let us start with the open channel

$$\begin{aligned}
-\omega_{\rm o}\left(\varphi_{\rm o}, m_{0,2}^{(0,1)}\left(\boldsymbol{h}_0 U_{\rm o}\right)\right) &= -(-)^{o^i}\omega_{\rm o}\left(\varphi_{\rm o}, m_{0,2}^{(0,1)}\left(h_0 o_i, o^i\right)\right) \\
&= -(-)^{o^i}\omega_{\rm o}\left(o^i, m_{0,2}^{(0,1)}\left(\varphi_{\rm o}, h_0 o_i\right)\right) \\
&= -\langle S_{0,3}^{(0,1)}| o^i \otimes \varphi_{\rm o} \otimes h_0 o_i \\
&= -\int_0^1 \frac{dq}{q}\langle S_{0,3}^{(0,1)}|\left(1_{\mathcal{H}_{\rm o}}\otimes 1_{\mathcal{H}_{\rm o}}\otimes b_0 q^{L_0}\right)\left(o^i\otimes\varphi_{\rm o}\otimes o_i\right),
\end{aligned} \tag{E.33}$$

where in the second step we used cyclicity and in the third line we introduced the surface state $\langle S_{0,3}^{(0,1)}|$ that is defined as follows

$$\begin{aligned}
\langle S_{0,3}^{(0,1)}|\Psi_1\otimes\Psi_2\otimes\Psi_3 &\coloneqq \langle f_{-\xi}\circ\Psi_1(0) f_0\circ\Psi_2(0) f_\xi\circ\Psi_3(0)\rangle_{\rm UHP} \\
&= (-)^{\Psi_1+\Psi_2}\omega_{\rm o}\left(\Psi_1, m_{0,2}^{0,1}(\Psi_2,\Psi_3)\right).
\end{aligned} \tag{E.34}$$

Note that this surface state differs from the one discussed in Section 2.3.3, because it also contains information about the local coordinate maps. As for the closed string channel we have

$$\begin{aligned}
-\omega_{\rm o}\left(\varphi_{\rm o}, m_{1,0}^{(0,1)}\left(h_0^+ l_{0,0}^{(0,1)}\right)\right) &= -\frac{1}{2\pi i}\langle S_{1,1}^{(0,1)}|\varphi_{\rm o}\otimes h_0^+ l_{0,0}^{(0,1)} \\
&= -\frac{1}{2\pi i}\int_0^1\frac{dq}{q}\langle S_{1,1}^{(0,1)}|\left(1_{\mathcal{H}_{\rm o}}\otimes b_0^+ q^{L_0^+}\right)\left(\varphi_{\rm o}\otimes l_{0,0}^{(0,1)}\right),
\end{aligned} \tag{E.35}$$

where we have defined the surface state $\langle S_{1,1}^{(0,1)}|$ as follows

$$\langle S_{1,1}^{(0,1)}|\Psi\otimes\Phi \coloneqq \langle d\circ f_{\rm o}\circ\Psi(0) d\circ f_{\rm c}\circ\Phi(0,\bar 0)\rangle_{\rm disk} = (-)^{\Phi+\Psi}2\pi i\,\omega_{\rm o}\left(\Psi, m_{1,0}^{(0,1)}(\Phi)\right). \tag{E.36}$$

Now, our attention turns to the fundamental vertex

$$\omega_{\mathrm{o}}\left(\varphi_{\mathrm{o}}, m_{0,0}^{(0,2)}\right) .$$

Although it has not been explicitly constructed yet, we know that it corresponds to an integral over a piece of a section of the fiber bundle having the moduli space as its base and the local coordinate map as its fiber. Clearly such subsection can be parameterized using the same variable t used to characterize the moduli space. Then the region covered by $m_{0,0}^{(0,2)}$ has to align with the interval $t \in [t_{\min}, t_{\max}]$, namely the portion that cannot be covered by the two previous Feynman diagrams. In our specific case, based on the definitions provided in the previous section, it is actually possible to choose the parameters such that $t_{\min} = t_{\max}$. However it does not appear to be possible to match the local coordinates and therefore the fundamental vertex is needed, with our choice of lower dimensional vertices.[2]
In general we thus have

$$\omega_{\mathrm{o}}(\varphi_{\mathrm{o}}, m_{0,0}^{(0,2)}) = \int_{t_{\min}}^{t_{\max}} dt \langle S_{0,1}^{(0,2)}(t)| B^t(t)\varphi_{\mathrm{o}} \,, \tag{E.37}$$

where the surface state is defined as

$$\langle S_{0,1}^{(0,2)}(t)|\Psi \coloneqq \langle F_t \circ \Psi(0)\rangle_{C_{(\pi,2\pi t)}} . \tag{E.38}$$

The local coordinate map $F_t(w)$ will be fixed shortly by imposing BRST decoupling (which is equivalent to the homotopy relations), and $B^t(t)$ is the Beltrami form which in this case reads

$$B^t(t) = \oint_0 \frac{dw}{2\pi i} b(w) \frac{\partial F_t^{-1}}{\partial t}\left(F_t(w)\right) , \tag{E.39}$$

where the contour surrounds the puncture in the local coordinate frame Section 2.3.

Crucially, the surface state satisfies

$$\frac{d}{dt}\langle S_{0,1}^{(0,2)}(t)| = -\langle S_{0,1}^{(0,2)}(t)| B^t(t) Q_{\mathrm{o}} \,, \tag{E.40}$$

[2]When $t_{\min} = t_{\max}$ the fundamental vertex would be associated to a *vertical integration* [178] changing the local coordinate while not moving in moduli space. This degenerate situation would require an ad hoc treatment which we do not consider in this thesis.

as a consequence of (2.182), see [10, 38, 39] for more details. With these premises the full annulus amplitude (E.2) becomes

$$\begin{aligned}\mathcal{A}^{0,2}_{0;1}(\varphi_{\rm o}) = &-\frac{1}{2\pi i}\int_0^1 \frac{dq}{q}\langle S^{(0,1)}_{1,1}|\left(1_{\mathcal{H}_{\rm o}}\otimes b_0^+ q^{L_0^+}\right)\left(\varphi_{\rm o}\otimes l^{(0,1)}_{0,0}\right)\\ &+\int_{t_{\min}}^{t_{\max}} dt\langle S^{(0,2)}_{0,1}(t)|B^t(t)\varphi_{\rm o}\\ &-\int_0^1\frac{dq}{q}\langle S^{(0,1)}_{0,3}|\left(1_{\mathcal{H}_{\rm o}}\otimes 1_{\mathcal{H}_{\rm o}}\otimes b_0 q^{L_0}\right)\left(o^i\otimes\varphi_{\rm o}\otimes o_i\right).\end{aligned} \tag{E.41}$$

Now, we want to verify if and how BRST exact states decouple. To do so let us consider $\varphi_{\rm o}=Q_{\rm o}\Lambda$

$$\begin{aligned}\mathcal{A}^{0,2}_{0;1}(Q_{\rm o}\Lambda) = &-\frac{1}{2\pi i}\int_0^1 \frac{dq}{q}\langle S^{(0,1)}_{1,1}|\left(1_{\mathcal{H}_{\rm o}}\otimes b_0^+ q^{L_0^+}\right)\left(Q_{\rm o}\Lambda\otimes l^{(0,1)}_{0,0}\right)\\ &+\int_{t_{\min}}^{t_{\max}} dt\langle S^{(0,2)}_{0,1}(t)|B^t(t)Q_{\rm o}\Lambda\\ &-\int_0^1\frac{dq}{q}\langle S^{(0,1)}_{0,3}|\left(1_{\mathcal{H}_{\rm o}}\otimes 1_{\mathcal{H}_{\rm o}}\otimes b_0 q^{L_0}\right)\left(o^i\otimes Q_{\rm o}\Lambda\otimes o_i\right),\end{aligned} \tag{E.42}$$

by using the fact that the surface states are BRST invariant, the property (E.40), and the relations $\boldsymbol{Q}_{\rm o}\boldsymbol{U}_{\rm o}=0$ and $Q_{\rm c}l^{(0,1)}_{0,0}=0$ we get

$$\begin{aligned}&\mathcal{A}^{0,2}_{0;1}(Q_{\rm o}\Lambda) =\\ &+\frac{1}{2\pi i}\int_0^1\frac{dq}{q}\langle S^{(0,1)}_{1,1}|\left(1_{\mathcal{H}_{\rm o}}\otimes\left[b_0^+q^{L_0^+},Q_{\rm c}\right]\right)\left(\Lambda\otimes l^{(0,1)}_{0,0}\right)\\ &-\int_{t_{\min}}^{t_{\max}}dt\frac{d}{dt}\langle S^{(0,2)}_{0,1}(t)|\Lambda\\ &+\int_0^1\frac{dq}{q}\langle S^{(0,1)}_{0,3}|\left(1_{\mathcal{H}_{\rm o}}\otimes 1_{\mathcal{H}_{\rm o}}\otimes\left[b_0q^{L_0},Q_{\rm o}\right]\right)\left(o^i\otimes\Lambda\otimes o_i\right)\\ &=-\frac{1}{2\pi i}\int_0^\infty ds\frac{d}{ds}\langle S^{(0,1)}_{1,1}|\left(1_{\mathcal{H}_{\rm o}}\otimes e^{-sL_0^+}\right)\left(\Lambda\otimes l^{(0,1)}_{0,0}\right)\\ &+\langle S^{(0,2)}_{0,1}(t_{\min})|\Lambda-\langle S^{(0,2)}_{0,1}(t_{\max})|\Lambda\\ &-\int_0^\infty ds\frac{d}{ds}\langle S^{(0,1)}_{0,3}|\left(1_{\mathcal{H}_{\rm o}}\otimes 1_{\mathcal{H}_{\rm o}}\otimes e^{-sL_0}\right)\left(o^i\otimes\Lambda\otimes o_i\right)\\ &=\frac{1}{2\pi i}\langle S^{(0,1)}_{1,1}|\left(\Lambda\otimes l^{(0,1)}_{0,0}\right)+\langle S^{(0,2)}_{0,1}(t_{\min})|\Lambda-\langle S^{(0,2)}_{0,1}(t_{\max})|\Lambda+\langle S^{(0,1)}_{0,3}|\left(o^i\otimes\Lambda\otimes o_i\right),\end{aligned} \tag{E.43}$$

where we made the change of variable $q = e^{-s}$ and we ignored the contributions at the boundary of moduli space (open and closed string degeneration). Notice that the last line is equivalent to the homotopy relation (6.147). Therefore, the amplitude vanishes for all Λ if we impose

$$\langle S_{0,3}^{(0,1)}| \left(o^i \otimes 1_{\mathcal{H}_\mathrm{o}} \otimes o_i \right) = \langle S_{0,1}^{(0,2)}(t_{\max})| \,, \tag{E.44}$$

$$-\frac{1}{2\pi i} \langle S_{1,1}^{(0,1)}| \left(1_{\mathcal{H}_\mathrm{o}} \otimes l_{0,0}^{(0,1)} \right) = \langle S_{0,1}^{(0,2)}(t_{\min})| \,. \tag{E.45}$$

These two relations allow us to determine the local coordinate F_t and thus to fully define the fundamental vertex. Notice that the surface states in the l.h.s are obtained respectively trough the open and closed plumbing fixture, with $q = 1$. Similarly, the surface states associated to the plumbing fixture for generic q can be written as

$$\langle S_{\mathrm{open}}^{\mathrm{p.f.}}(q)| = \langle S_{0,3}^{(0,1)}| \left(o^i \otimes 1_{\mathcal{H}_\mathrm{o}} \otimes q^{L_0} o_i \right) , \tag{E.46}$$

$$\langle S_{\mathrm{closed}}^{\mathrm{p.f.}}(q)| = -\frac{1}{2\pi i} \langle S_{1,1}^{(0,1)}| \left(1_{\mathcal{H}_\mathrm{o}} \otimes q^{L_0^+} l_{0,0}^{(0,1)} \right) \tag{E.47}$$

and we have

$$\langle S_{\mathrm{open}}^{\mathrm{p.f.}}(q)|\Psi = \langle f_0 \circ \Psi(0) \rangle_{\Sigma_{\mathrm{open}}^{\mathrm{p.f.}}(q)} = \langle g_\mathrm{o} \circ f_0 \circ \Psi(0) \rangle_{C_{\pi, 2\pi t_\mathrm{o}}} = \langle G_\mathrm{o} \circ \Psi(0) \rangle_{C_{\pi, 2\pi t_\mathrm{o}}} , \tag{E.48}$$

$$\langle S_{\mathrm{closed}}^{\mathrm{p.f.}}(q)|\Psi = \langle d \circ f_\mathrm{o} \circ \Psi(0) \rangle_{\Sigma_{\mathrm{closed}}^{\mathrm{p.f.}}(q)} = \langle g_\mathrm{c} \circ d \circ f_\mathrm{o} \circ \Psi(0) \rangle_{C_{\pi, 2\pi t_\mathrm{c}}} = \langle G_\mathrm{c} \circ \Psi(0) \rangle_{C_{\pi, 2\pi t_\mathrm{c}}} . \tag{E.49}$$

Therefore, considering the b-ghost insertions the amplitude becomes

$$\begin{aligned} \mathcal{A}_{0;1}^{0,2}(\varphi_\mathrm{o}) = &+ \int_0^1 \frac{dq}{q} \langle d \circ f_\mathrm{o} \circ \varphi_\mathrm{o}(0) d \circ f_\mathrm{c} \circ b_o \rangle_{\Sigma_{\mathrm{closed}}^{\mathrm{p.f.}}(q)} \\ &+ \int_{t_{\min}}^{t_{\max}} dt \langle B^t(t) F_t \circ \varphi_\mathrm{o}(0) \rangle_{C_{\pi, 2\pi t}} \\ &- \int_0^1 \frac{dq}{q} \langle f_0 \circ \varphi_\mathrm{o}(0) f_\xi \circ b_0 \rangle_{\Sigma_{\mathrm{open}}^{\mathrm{p.f.}}(q)} , \end{aligned} \tag{E.50}$$

Finally, let us focus on the local coordinate map F_t. In particular, to ensure (E.44)

and (E.45), we need

$$\left\{\begin{array}{l} \langle S^{\rm p.f}_{\rm open}(q=1)|\Psi = \langle S^{(0,2)}_{0,1}(t_{\rm max})|\Psi \\ \langle S^{\rm p.f}_{\rm closed}(q=1)|\Psi = \langle S^{(0,2)}_{0,1}(t_{\rm min})|\Psi \end{array}\right. \longrightarrow \left\{\begin{array}{l} G_{\rm o}(w)|_{q=1} = F_t(w)|_{t=t_{\rm max}} \\ G_{\rm c}(w)|_{q=1} = F_t(w)|_{t=t_{\rm min}} \end{array}\right. \tag{E.51}$$

thus F_t can be any holomorphic function which continuously interpolates $G_{\rm c}(w)|_{q=1}$ and $G_{\rm o}(w)|_{q=1}$ in the interval $t \in [t_{\rm min}, t_{\rm max}]$.

APPENDIX F

2-point function of $\phi_{(1,3)}$ on the UHP

In this appendix, we will compute the large-m limit of the 2-point correlation function

$$\frac{1}{g^{(\boldsymbol{a})}} \left\langle \phi_{(1,3)}(i,\bar{i})\, \phi_{(1,3)}(is,\bar{is}) \right\rangle^{(\boldsymbol{a})}_{\mathrm{UHP}} , \tag{F.1}$$

where $\phi_{(1,3)}$ is the primary with Kac label $(1,3)$ in the unitary Virasoro minimal model $m = 3, 4, \ldots$ and the parameter s runs from 0 to 1. The conformal boundary conditions along the real line will be specified by a Cardy boundary state with Kac labels $\boldsymbol{a} = (a_1, a_2)$ and boundary entropy $g^{(\boldsymbol{a})} = \langle 1 \rangle^{(\boldsymbol{a})}_{\mathrm{UHP}}$. To evaluate the correlation function (F.1), our strategy will be to exploit the existence of a null state at level 3 in the degenerate module $(1,3)$ in order to show that (F.1) satisfies a third-order

differential equation. Using the known values of bulk and boundary OPE coefficients for the A-series of Virasoro minimal models, we will then fix a particular solution to this equation which has the correct factorization properties in the bulk and boundary channels.

Let us start by noting that one can use the doubling trick to rewrite (F.1) in terms of a chiral 4-point function on the full complex plane

$$\langle \phi_{(1,3)}(z_1)\, \phi_{(1,3)}(z_2)\, \phi_{(1,3)}(z_3)\, \phi_{(1,3)}(z_4) \rangle_{\mathbb{C}} = (z_{12} z_{34})^{-2h}\, \mathcal{G}(\eta)\,, \tag{F.2}$$

subject to assigning $z_1 = i$, $z_2 = -i$, $z_3 = is$, $z_4 = -is$ and $h = h_{(1,3)} = \frac{m-1}{m+1} = 1 - y$. Note that on the right-hand side of (F.2), we have made use of global conformal invariance to recast the correlator in terms of a function $\mathcal{G}(\eta)$ of the cross-ratio

$$\eta = \frac{z_{12} z_{34}}{z_{13} z_{24}} = -\frac{4s}{(1-s)^2}\,. \tag{F.3}$$

We will see below that sometimes, it may be more convenient to work with a related function $\tilde{\mathcal{G}}(\tilde{\eta}) \equiv \mathcal{G}(\eta(\tilde{\eta}))$ which is expressed in terms of the redefined parameter

$$\tilde{\eta}(\eta) = \frac{\eta^2}{1-\eta}\,. \tag{F.4}$$

One can then exploit the existence of the null-state

$$\left[(h+2) L_{-3} - 2 L_{-1} L_{-2} + \frac{1}{h+1} (L_{-1})^3 \right] |\phi_{(1,3)}\rangle \tag{F.5}$$

to infer that the function $\tilde{\mathcal{G}}(\tilde{\eta})$ satisfies the third-order differential equation

$$\begin{aligned} 0 = (\tilde{\eta}^3 + 4\tilde{\eta}^2) \frac{d^3 \tilde{\mathcal{G}}}{d\tilde{\eta}^3} + \left[(4-2h)\tilde{\eta}^2 + (10-8h)\tilde{\eta} \right] \frac{d^2 \tilde{\mathcal{G}}}{d\tilde{\eta}^2} + \\ + \left[(2 - 5h - h^2)\tilde{\eta} + 2 - 7h + 3h^2 \right] \frac{d\tilde{\mathcal{G}}}{d\tilde{\eta}} + 2h^2(1+h)\tilde{\mathcal{G}}\,. \end{aligned} \tag{F.6}$$

Equation (F.6) admits a solution for $\tilde{\mathcal{G}}(\tilde{\eta})$ which, for general h, can be explicitly written as a linear combination of hypergeometric functions of the type ${}_3F_2$ in the

variable $-\frac{\tilde{\eta}}{4}$. However, as we are really interested only in the leading contribution to the correlator (F.1) as we take the limit $m \to \infty$, we can set $h = 1$ and instead consider solving the simpler differential equation

$$0 = \eta\,(1-\eta)^3\,\frac{d^3\mathcal{G}}{d\eta^3} - 2(1-\eta)^2(1+\eta)\,\frac{d^2\mathcal{G}}{d\eta^2} + 2(1-\eta)(1-2\eta)\frac{d\mathcal{G}}{d\eta} + 4(2-\eta)\mathcal{G} \quad \text{(F.7)}$$

for $\mathcal{G}(\eta)$. Doing so, we learn that the 2-point correlator (F.1) takes the form of a rational function in s

$$\begin{aligned}\frac{1}{g^{(\boldsymbol{a})}}\,\langle\phi_{(1,3)}(i,\bar{i})\,\phi_{(1,3)}(is,\bar{is})\rangle^{(\boldsymbol{a})}_{\text{UHP}} = \\ = \frac{1}{s^2(1-s^2)^4}\Big[K_1(s^8+28s^6-314s^4+28s^2+1)+ \\ + K_2(s^8+28s^6-826s^4+28s^2+1) + K_3(s^7+7s^5+7s^3+s)\Big]\,,\end{aligned} \quad \text{(F.8)}$$

where K_1, K_2 and K_3 are some integration constants. These can be fixed by identifying the states propagating in the bulk channel and the boundary channel of the correlator (taking the limits $s \to 1$ and $s \to 0$, respectively, on the r.h.s. of (F.8)) and then matching the coefficients in front of poles with the corresponding OPE structure constants.

First, in the bulk channel, one can expand the r.h.s. of (F.8) as

$$\frac{-16K_1-48K_2+K_3}{(1-s)^4} + \frac{32K_1+64K_2+K_3}{2(1-s)^2} + \mathcal{O}\big[(1-s)^{-1}\big]\,. \quad \text{(F.9)}$$

Here, the quartic pole is to be associated with the propagation of the bulk identity $\phi_{(1,1)}$ in the bulk channel, while the quadratic pole arises due to propagation of $\phi_{(1,3)}$. Comparing (F.9) with the result of taking the bulk OPE of the two bulk operators $\phi_{(1,3)}$ inside the correlator (F.1), we obtain the constraints

$$-1 = 16K_1+48K_2-K_3\,, \quad \text{(F.10a)}$$

$$\frac{1}{4}C_{(1,3)(1,3)(1,3)}\frac{B^{(\boldsymbol{a})}_{(1,3)(1,1)}}{g^{(\boldsymbol{a})}} = 16K_1+32K_2+\frac{K_3}{2}\,. \quad \text{(F.10b)}$$

We recall that in the strict limit $m \to \infty$, the bulk 3-point structure constant $C_{(1,3)(1,3)(1,3)}$ is equal to $\frac{4}{\sqrt{3}}$, while the value of the disk one-point function coefficient $B^{(\boldsymbol{a})}_{(1,3)(1,1)}$ will generally depend on the particular choice $\boldsymbol{a}$ of the Cardy boundary condition.

Second, expanding the r.h.s. of (F.8) in the limit $s \to 0$ (boundary channel), one gets

$$\frac{K_1 + K_2}{s^2} + \frac{K_3}{s} + \mathcal{O}(s^0)\,, \tag{F.11}$$

where the quadratic and simple poles are due to propagation of the identity and the $(1,3)$ module in the boundary channel, respectively. This prompts us to identify

$$\frac{1}{16}\left[\frac{B^{(\boldsymbol{a})}_{(1,3)(1,1)}}{g^{(\boldsymbol{a})}}\right]^2 = K_1 + K_2\,, \tag{F.12a}$$

$$\frac{1}{4}\left[\frac{B^{(\boldsymbol{a})}_{(1,3)(1,3)}}{g^{(\boldsymbol{a})}}\right]^2 = K_3\,, \tag{F.12b}$$

where $B^{(\boldsymbol{a})}_{(1,3)(1,3)}$ denotes the coefficient which enters the two-point function between a $(1,3)$ field in the bulk and a $(1,3)$ field on the boundary.

By solving the constraints (F.10a), (F.10b) and (F.12a), one can fix the integration constants K_1, K_2 and K_3 (and hence the correlator (F.1)) in terms of $C_{(1,3)(1,3)(1,3)}$, $B^{(\boldsymbol{a})}_{(1,3)(1,1)}$ and $g^{(\boldsymbol{a})}$. The remaining constraint (F.12b) then provides a non-trivial bulk-boundary crossing relation for the structure constant $B^{(\boldsymbol{a})}_{(1,3)(1,3)}$, namely

$$\left[\frac{B^{(\boldsymbol{a})}_{(1,3)(1,3)}}{g^{(\boldsymbol{a})}}\right]^2 = 2 - 2\left[\frac{B^{(\boldsymbol{a})}_{(1,3)(1,1)}}{g^{(\boldsymbol{a})}}\right]^2 + C_{(1,3)(1,3)(1,3)}\,\frac{B^{(\boldsymbol{a})}_{(1,3)(1,1)}}{g^{(\boldsymbol{a})}}\,. \tag{F.13}$$

Validity of this relation can, in principle, be checked against the known values of the structure constants in minimal models [167] in the limit $m \to \infty$.

To facilitate applications to the SFT computation of the change in the g-function of Cardy boundary conditions upon introducing a bulk perturbation (section 7.5), we should keep in mind our assumption (7.51), which ensures that the deformation triggered by the bulk field $\phi_{(1,3)}$ on the boundary remains perturbative. From now on, we will therefore fix the Cardy boundary condition $\boldsymbol{a}$ so that the l.h.s. of (F.13)

vanishes. This implies two distinct solutions

$$\frac{B^{(\boldsymbol{a})}_{(1,3)(1,1)}}{g^{(\boldsymbol{a})}} = \begin{cases} \sqrt{3} + \mathcal{O}(y) \\ \\ -\frac{1}{\sqrt{3}} + \mathcal{O}(y) \end{cases} \tag{F.14}$$

of the crossing equation (F.13) for the structure constant $B^{(\boldsymbol{a})}_{(1,3)(1,1)}$.

The first solution $\sqrt{3}$ can be realized by taking the Kac labels (a_1, a_2) of the Cardy boundary state *arbitrary but fixed* when taking the limit $m \to \infty$. Correspondingly, one can establish that for such boundary states, the bulk field $\phi_{(1,3)}$ can indeed be used to trigger a sequence of perturbative bulk and boundary deformations. These turn out to relate the initial boundary condition (a_1, a_2) in the minimal model m to the boundary condition (a_2, a_1) in the minimal model $m - 1$. This is true in particular for the boundary conditions of the type $(a_1, 1)$ for which the module $(1, 3)$ is absent from the boundary spectrum so that the structure constant $B^{(\boldsymbol{a})}_{(1,3)(1,3)}$ vanishes identically to all orders in y. In these cases, there is a direct RG flow taking the boundary condition $(a_1, 1)_m$ to $(1, a_1)_{m-1}$. See [164], as well as section 7.5 for details.

On the other hand, the second solution $-\frac{1}{\sqrt{3}}$ arises for the boundary states with Kac labels (a_1, a_2), where a_1 is kept fixed while we put $a_2 = \frac{m+1}{2} + \alpha$ for m odd (and for a fixed integer α). This is in precise agreement with the existence of an RG flow triggered by the bulk field $\phi_{(1,3)}$ taking the boundary condition $(a_1, a_2)_m$ directly to the boundary condition $(a_2 - 1, a_1)_{m-1}$. In the aformentioned regime of the Kac labels (a_1, a_2), such a flow is indeed perturbative.

Focusing our attention on those structure constants $B^{(\boldsymbol{a})}_{(1,3)(1,1)}$ which solve the crossing equation (F.13) with zero l.h.s. (consistent with our assumption (7.51)), then, in the strict $m \to \infty$ limit, the desired two-point correlator can finally be fixed

as

$$\frac{1}{g^{(\boldsymbol{a})}}\,\langle\phi_{(1,3)}(i,\bar{i})\,\phi_{(1,3)}(is,\bar{is})\rangle^{(\boldsymbol{a})}_{\text{UHP}} = \\ = C_{(1,3)(1,3)(1,3)}\,\frac{B^{(\boldsymbol{a})}_{(1,3)(1,1)}}{g^{(\boldsymbol{a})}}\,\frac{s^4+30s^2+1}{32s^2\,(s^2-1)^2}+\frac{(s^4+14s^2+1)^2}{16s^2(s^2-1)^4}\,. \tag{F.15}$$

Furthermore, subtracting the bulk and the boundary channel divergences in a way described in section 7.4, we obtain

$$\frac{1}{g^{(\boldsymbol{a})}}\,\Big[\langle\phi_{(1,3)}(i,\bar{i})\,\phi_{(1,3)}(is,\bar{is})\rangle^{(\boldsymbol{a})}_{\text{UHP}}-\text{div}_{\text{c}}(s)-\text{div}_{\text{o}}(s)\Big] = \\ = \frac{1}{(1+s)^4}+C_{(1,3)(1,3)(1,3)}\,\frac{B^{(\boldsymbol{a})}_{(1,3)(1,1)}}{g^{(\boldsymbol{a})}}\,\frac{s+3}{4(1-s)(1+s)^2}\,. \tag{F.16}$$

This is the function which is to be integrated against the c-ghost measure $4(s^2-1)$ in (7.112) when calculating the contribution $\mathcal{A}^{\text{f.p.}}_{TT}$ to the on-shell disk action at subleading order in y in section 7.5.

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