The Ornstein-Uhlenbeck process as a model for filtered white noise

The Ornstein-Uhlenbeck process as a model of a low-pass filtered white noise

Enrico Bibbona*

I.N.RI.M - Istituto Nazionale di Ricerca Metrologica

Strada delle Cacce, 91 - 10135 Torino, Italy

Department of Mathematics, University of Torino,

Via Carlo Alberto 10, 10123 Torino, Italy

Gianna Panfilo†

BIPM,

Pavillon de Breteuil,

F-92312 Svres Cedex (France)

Patrizia Tavella‡

I.N.RI.M - Istituto Nazionale di Ricerca Metrologica

Strada delle Cacce, 91 - 10135 Torino, Italy

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Abstract

The Ornstein-Uhlenbeck process is presented with its main mathematical properties and with original results on the first crossing times in case of two threshold barriers. The interpretation as filtered white noise, its stationary spectrum, and Allan variance are also presented for easiness of use in the time and frequency metrology field. An improved simulation scheme for the evaluation of first passage times between two barriers is also introduced.

*Electronic address: enrico.bibbona@unito.it

†Electronic address: gpanfilo@bipm.org

‡Electronic address: p.tavella@inrim.it

I. INTRODUCTION

In time metrology and in particular in the evaluations of the atomic clock behavior, the precision of measurements is often limited by the presence of noises. Typical experimentally observed noises are white (phase or frequency) noises or an integration of white noise leading to a Wiener process. The clock signal mathematical models available in the literature that have been largely used in many different applications (see for example, Ref. [1–4]) generally just partially imbed those types of noise (see Section VI for more details). Moreover the concept of white noise is an idealization and suffers many drawbacks: from the strictly mathematical viewpoint continuous white noise is a rather a pathological object and it does not fulfill the usual definition of stochastic process. From the physical viewpoint white noise do not even exist, and any practical realization appears to be "white" only on a limited band of the frequency spectrum. We therefore say that the white noise is "filtered" either by the measurement device, or by the physical process itself that generates the noise.

The aim of the present paper is to present an improved mathematical model for the white noise that accounts for the filtering of high frequencies such as that obtained with an RC first-order low pass filter. Such a filtering can be observed in the spectrum (provided that the sampling rate is sufficiently high), in the Allan Variance and it brings a non-vanishing correlation between subsequent recordings of the time series of measurements. Of course, the lower is the cut-off frequency of the filter and the longer is the time when correlation is important. The stochastic model we propose is the Ornstein-Uhlenbeck (OU) process. It was historically introduced in order to provide a model for the velocities of the Brownian motions of particles suspended in a fluid more realistic with respect to the one offered by white noise (see for example Refs. [5, 6]).

The importance of a mathematical model for noises in metrology relies on the possibility of evaluating the measurement system capabilities, to understand the impact for example of an atomic clock in a more complex system, or to evaluate the propagation of noise inside a physical device. In several applications it is important to evaluate the probability of the system noise not to exceed a critical value. Suppose for example that at a given time you synchronize a clock in such a way that its error with respect to some reference time is zero. How long will the clock safely work before it accumulates an error that exceeds some given threshold value? The answer of such a question can be given on the theoretical ground. In

mathematics this problem is called first passage time of the process across two barriers. We will exhibit new results on the mean and the variance of the first passage time for the OU process between two barriers.

The paper is organized as follows. In Section II we recall the main properties of the Ornstein-Uhlenbeck process. In Section III we look at its Spectrum, calculate its Allan variance and we interpret a discrete sequence sampled from an OU process as the output of a digital first order Infinite Impulse Response (IIR) low pass filter. In Section IV we study the first passage time problem from a theoretical point of view. In Section V we provide a very efficient technique to simulate first passage times for an Ornstein-Uhlenbeck process and we compare the simulative results with the theoretical ones. On the whole we provide the mathematical background for the use and evaluation of the OU process model in application to atomic clock signals. In Section VI we briefly present possible applications.

II. THE ORNSTEIN-UHLENBECK PROCESS

The Wiener process W_t is the limit of a (discrete) random walk when the steps becomes infinitesimally small and infinitely frequent. In the metrological literature it is often referred to simply as the Random Walk noise. It was introduced as a first mathematical model of the random movement of particles suspended in a fluid (Brownian motion, see for example Ref. [7]). The velocity of a Brownian motion is hence modeled by its derivative that is a continuous white noise. Due to its complicated mathematical description (it is not a stochastic process in the usual sense) and to its physical inconsistencies a different model is usually considered for the velocities of Brownian particles: the so called Ornstein-Uhlenbeck process. The Ornstein-Uhlenbeck process U_t (with $t \geq 0$) is the solution of the following stochastic differential equation named after Langevin (see for example Ref. [8])

$$dU_t = \left(-\frac{U_t}{\tau} + \mu \right) dt + \sigma dW_t \tag{1}$$

where W_t is a Wiener process, $\sigma \geq 0$ is the diffusion coefficient, $\tau \geq 0$ is the time constant and is μ the drift coefficient. To focus on the stochastic component, μ is set to zero for the rest of the paper.

If we disregard the noise ($\sigma = 0$ in addition to $\mu = 0$) equation (1) becomes the same as the one that governs the discharge of a capacitor and the solution is the exponential decay

$U_t = u_0 \mathrm{e}^{-t/\tau}$, where u_0 is the starting point at time $t = 0$. The full solution for the stochastic differential equation (see for example Ref. [9] for the derivation) is the sum

$$U_t = u_0 \mathrm{e}^{-\frac{t}{\tau}} + \mathrm{e}^{-\frac{t}{\tau}} \sigma \int_0^t \mathrm{e}^{-\frac{t}{\tau}} dW_s \tag{2}$$

of the deterministic behavior plus a stochastic term that is a martingale (and thence it has a vanishing expectation, see [8]). If the process is observed at discrete equally spaced times $t_n = nh$, where h is the sampling time, the solution can be put into the following iterative form

$$U_{t_n} = U_{t_{n-1}} \mathrm{e}^{-\frac{h}{\tau}} + Z_n \tag{3}$$

where $Z_n = \sigma \mathrm{e}^{-\frac{t_n}{\tau}} \int_{t_{n-1}}^{t_n} \mathrm{e}^{-\frac{s}{\tau}} dW_s$ is a sequence of independent and identically distributed normal random variables with

$$\mathbb{E}(Z_n) = 0$$
$$\mathrm{Var}(Z_n) = \frac{\sigma^2 \tau}{2}(1 - \mathrm{e}^{-2\frac{h}{\tau}}).$$

Let us apply such an iterative expression in order to interpret the role of the coefficients. In Fig. 1 it is shown how the position U_{t_n} at each instant t_n is related to the previous position according to formula (3).

To avoid possible misunderstanding we warn the reader that we are now focusing on the *discrete* trajectory of the process (the set of all red dots). The continuos blue line are not part of the trajectory itself but they are plotted in order to illustrate the effect of the deterministic behavior (the first addendum in formula (3), it gives the expectation of the process) between any couple of points that is given by an exponential decay toward zero.

Moreover in each interval between two observations, the stochastic behavior accumulates according to the second summand of formula (3) in independent and identically distributed gaussian jumps of amplitude Z_n (dashed vertical blue lines in the figure). The relative importance between the two contributions depends upon the value of the parameter σ and the ratio $\frac{h}{\tau}$ between the discretization interval h and the time constant τ.

Let us consider the limit behaviors for fixed σ while $\frac{h}{\tau}$ goes to infinity in a case and to zero in the other. If the time constant τ of the exponential decay is very small with respect to the discretization interval h then between any two observed points the decay has already taken place and the independent gaussian jumps always start from zero. The

process in this case reduces to a white noise. On the other hand, when $\frac{h}{\tau} \to \infty$ there is not any deterministic decay between two points and the gaussian jumps are just the stationary independent increments of the process, that reduces to a Wiener process (random walk noise).

To enhance the comparison with the limit behaviors, in Fig. 2 we plot twenty trajectories of a Wiener processes (red) and compare them with twenty Ornstein-Uhlenbeck processes (blue) with the same σ. Trajectories of the Wiener process tend to spread up with time due to its increasing variance, while those of the Ornstein-Uhlenbeck remain confined near the origin. The distribution of the process U_t seems to remain stationary with t.

In Fig. 3 we plot a trajectory of a white noise and one of an Ornstein-Uhlenbeck with the same variance. In the Ornstein-Uhlenbeck noise low frequency oscillations seem to be dominant.

In the original works [5, 6] on the model for the velocity of a Brownian particle, the term $-\frac{U_t}{\tau}$ in equation (1) was interpreted as a viscous friction that slows down the motion proportionally to its velocity. Its presence has two main effect: it keeps the process near the origin and it is linked with the covariance of nearby points.

Let us summarize some properties of the process (2). It is Gaussian and its moments are those reported below. For $t \to \infty$, moreover, the Ornstein-Uhlenbeck process admit a stationary distribution that is again Gaussian and whose moments are also reported below.

$$\begin{aligned}
&\mathbb{E}(U_t) = u_0 \mathrm{e}^{-\frac{t}{\tau}} && t \to \infty \quad \mathbb{E}(U_t) = 0 \\
&\mathrm{Var}(U_t) = \frac{\sigma^2\tau}{2}(1 - \mathrm{e}^{-2\frac{h}{\tau}}) && t \to \infty \quad \mathrm{Var}(U_t) = \frac{\sigma^2\tau}{2} \\
&\mathrm{Cov}(U_{t+h}, U_t) = \frac{\sigma^2\tau}{2}\mathrm{e}^{-\frac{(2t+h)}{\tau}}\left(\mathrm{e}^{\frac{2t}{\tau}} - 1\right) \quad (t \geq s) && t \to \infty \quad \mathrm{Cov}(U_{t+h}, U_t) = \frac{\sigma^2\tau}{2}\mathrm{e}^{-\frac{h}{t}}
\end{aligned} \tag{4}$$

The transition density of the Ornstein-Uhlenbeck process starting at epoch s from position y and arriving in x at epoch t is the following

$$f(x,t|y,s) = \frac{\mathrm{e}^{-\frac{\left(x - y\mathrm{e}^{-(t-s)/\tau}\right)^2}{\sqrt{\sigma^2\tau(1-\mathrm{e}^{-2(t-s)/\tau})}}}}{\pi\sqrt{\sigma^2\tau(1 - \mathrm{e}^{-2(t-s)/\tau})}}$$

III. OU SPECTRUM, ALLAN VARIANCE AND A LOW PASS DIGITAL FILTER

The (two-sided) spectrum of the Ornstein-Uhlenbeck is defined for $t \to \infty$ when the process reaches the stationarity and it was find by the original authors themselves in Ref.[10].

It can be derived as the Fourier transform of the correlation of its stationary distribution and it is equal to

$$S_x(f) = \frac{\sigma^2\tau^2}{1 + 4\pi^2\tau^2 f^2}. \tag{5}$$

Apart from a normalization factor, it is a Lorentzian function that in log-log plot appears as in Fig. 4. It is the same spectrum of a first order RC filter with a cut-off frequency $f_c = \frac{1}{2\pi\tau}$ and this justifies our interpretation of such a stochastic process as a model for a filtered white noise. For low frequencies, in fact, the spectrum is flat, however high frequencies are cutted-off with slope -2. If the filter action is dominating the white noise spectrum (f_c very low), than the spectrum is almost given by $1/f^2$ which corresponds to the spectrum of a random walk (or Wiener) phase process. The OU process has the behavior of a Wiener process (random walk on phase) for high frequencies (short observation times), while it is very close to a white phase noise for low frequencies (long observation times).

Let us derive the Allan deviation $\sigma_y(h)$ [11] of an OU process. The Allan deviation is defined in terms of phase or normalized frequency deviations as follows. Let's X_t be the phase deviation of a clock and $\bar{Y}_t$ the average frequency deviation with respect to a certain reference clock. The following relationship holds $\bar{Y}_t = \frac{X_t - X_{t-h}}{h}$ where h is the "discretization step" or the "observation interval". In this paper we use h instead of the more familiar τ used in time metrology for coherence with the OU literature. The Allan deviation is defined as:

$$\sigma_y^2(h) = \frac{1}{2}\mathbb{E}\left[\left(\bar{Y}_{t+h} - \bar{Y}_t\right)^2\right]$$

or, in terms of phase data X_t, as

$$\sigma_y^2(h) = \frac{1}{2h^2}\mathbb{E}\left((X_{t+h} - 2X_t + X_{t-h})^2\right). \tag{6}$$

Let's consider that the phase deviation of a clock is affected by an OU process (already in its stationary phase). We can substitute the asymptotic expression of the moments (4) into equation (6) getting

$$\begin{aligned}
\sigma_y^2(h) &= \frac{1}{2h^2}\mathbb{E}\left(U_{t+h}^2 + 4U_t^2 + U_{t-h}^2 - 4U_{t+h}U_t - 4U_tU_{t-h} + 2U_{t+h}U_{t-h}\right) \\
&= \frac{1}{2h^2}\left(6\frac{\sigma^2}{2\alpha} - 8\frac{\sigma^2}{2\alpha}\mathrm{e}^{-\alpha h} + 2\frac{\sigma^2}{2\alpha}\mathrm{e}^{-2\alpha h}\right) \\
&= \frac{\sigma^2}{2\alpha h^2}\left(3 - 4\mathrm{e}^{-\alpha h} + \mathrm{e}^{-2\alpha h}\right).
\end{aligned} \tag{7}$$

The function (7) is plotted in log-log scale in Fig. 5 . The Allan variance for very long observation intervals (corresponding to low frequencies) has the same slope (h^{-2} for the variance, h^{-1} for the Allan deviation) of a white phase noise, while for $h \to 0$ (correspondent to short observation intervals and to high frequencies behavior) the slope is h^{-1} for the Allan variance and $h^{-1/2}$ for the Allan deviation as it is observed in the case of a random walk (Wiener) phase noise corresponding to a white frequency noise.

Another derivation for the same result can be obtained by the following relationship [11]

$$\sigma_y^2(h) = \int_0^\infty 2(2\pi f)^2 |H(f)|^2 S_x(f)$$

that links the two-sided spectrum $S_x(f)$ of the phase noise to the Allan variance. The function $H(f)$ is the so called "transfer function" of the Allan variance and it has the following squared modulus

$$|H(f)|^2 = \frac{2\sin^4(\pi f h)}{(\pi f h)^2}.$$

For the Ornstein-Uhlenbeck process we then have

$$\sigma_y^2(h) = \frac{16\sigma^2}{h^2} \int_0^\infty \frac{2\sin^4(\pi f h)}{\alpha^2 + 4\pi^2 f^2} = \frac{\sigma^2}{2\alpha h} \left(3 - 4\mathrm{e}^{-\alpha h} + \mathrm{e}^{-2\alpha h}\right).$$

Let us conclude this section with a further reading of equation (3). Such a formula can be interpreted considering Z_n as a white noise input to a digital first order Infinite Impulse Response (IIR) filter whose output is the sequence U_n of the discretized Ornstein-Uhlenbeck process. In Fig. 6 we depicted the scheme of a general IIR filter that manipulates a white discretized noise X_n providing a discrete output Y_n according to the following formula

$$Y_n = aY_{n-1} + bX_n \tag{8}$$

where a and b are suitable constants of the filter design that in our case take the following values: $a = \mathrm{e}^{-\frac{h}{\tau}}$ and $b = 1$. The transfer function of such a filter in the z-domain is the following (see Ref.[12])

$$G(z) = \frac{1}{1 - \mathrm{e}^{-\frac{h}{\tau}} z^{-1}}$$

and if we pose $z = \mathrm{e}^{i2\pi h f}$ in order to come back to Fourier frequencies and take the squared modulus we get

$$|G(z)|^2 = \frac{1}{2\left(1 - 2\mathrm{e}^{-\frac{h}{\tau}}\cos(2\pi h f) + \mathrm{e}^{-2\frac{h}{\tau}}\right)},$$

that is the power spectrum of the digital filter in the range $-\frac{1}{2h} < f < \frac{1}{2h}$. In the same range the plots of $|G(z)|^2$ and that of $S_x(f)$ from (5), normalized over their values in $f = 0$, are almost indistinguishable.

The sequence U_n of the discretized Ornstein-Uhlenbeck process can be thence interpreted as the output of a first order low pass IIR digital filter applied to a white sequence and this fact is a further confirmation of the validity of our model for a discretized sequence coming from a low-pass filtered white noise.

IV. THE FIRST PASSAGE TIME PROBLEM

In this Section we consider the problem of the first passage time across two constant absorbing boundaries for an Ornstein-Uhlenbeck process U_t.

Let U_t start at $-S < u_0 < L$ at time $t = 0$ and let $T(u_0, -S, L)$ be the first time such that the process takes value beyond the interval $(-S, L)$. The *first passage time* $T(u_0, -S, L)$ is a random variable and the goal of this section is to find its mean and variance.

Let the *survival probability* $P_s(u_0, t)$ be the probability that the process starting in u_0 has not already crossed any of the thresholds at time t, or in other word that the first passage time occurs later then time t.

If we interpret the process as the error of an atomic clock, the survival probability at time t is the probability that the clock error has not yet exceeded a tolerable limit at epoch t after synchronization.

Let us remark that the Ornstein-Uhlenbeck process admits a stationary distribution with vanishing expectation which means that we are almost sure that after having escaped outside the barriers the process will quite soon go back near the origin. This behavior could erroneously suggest that the crossing of the barriers is not a very important fact. However, one can easily imagine applications where an exceedingly large error could not be tolerable even if it persists for a short duration.

Many different techniques have been developed in the literature (see Ref. [13] for a review) to find first passage time distributions, their moments and the survival probability density.

We choose here the method of the infinitesimal generator (see Ref. [8] for a more detailed introduction) that reduces these problems to the solution of suitable partial differential equations. To our knowledge the moments of the first passage time distribution for the OU process across two barriers have not yet been studied besides in the unpublished master thesis Ref. [14] that was advised by one of the present authors.

To any one dimensional diffusion process X_t solution of a stochastic differential equation of the form

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t$$

we can associate the following differential operator

$$\mathbb{L}_t = \frac{1}{2}\sigma^2(X_t)\frac{\partial^2}{\partial x^2} + \mu(X_t)\frac{\partial}{\partial x} \tag{9}$$

called the *infinitesimal generator* of the diffusion. Kolmogorov forward equation for the transition density $f(x, s|y, t)$ can be written as $\mathbb{L}_t f = \frac{\partial f}{\partial t}$ and if one can solve this differential equation the transition density is obtained. The survival probability $P_s(u_0, t)$ between two constant boundary at $-S$ and L, moreover, has to be a solution of the following partial differential equation with boundary conditions:

$$\begin{cases} \mathbb{L}_t P_s(u_0, t) = \dfrac{\partial}{\partial t} P_s(u_0, t) \\ P(u_0, 0) = 1_{(-S,L)} \\ P(-S, t) = 0 \\ P(L, t) = 0 \end{cases} \tag{10}$$

where the derivatives with respect to x in (9) have to be understood as derivatives with respect to the initial position u_0.

This equation, however, is not easy to be solved analytically for the Ornstein-Uhlenbeck process, thus we move to the easier problem of finding the moments of the first passage time.

The expectation value as a function of the initial point u_0 is denoted by $m(u_0) = \mathbb{E}[T(u_0, -S, L)]$ and it can be obtained by solving the following partial differential equation:

$$\begin{cases} \mathbb{L}_t m(u_0) = -1 \\ m(-S) = 0 \\ m(L) = 0 \end{cases} \tag{11}$$

that for an Ornstein-Uhlenbeck process reduces to

$$\begin{cases} \dfrac{\sigma^2}{2}\dfrac{\partial^2 m}{\partial {u_0}^2} - \dfrac{x}{\tau}\dfrac{\partial m}{\partial u_0} = -1 \\ m(-S) = 0 \\ m(L) = 0 \end{cases}$$

Separation of variables allows to find the following solution

$$m(u_0) = \frac{2}{\sigma}\left\{ \left[\int_{-S}^{L} \mathrm{e}^{\frac{-z^2}{\sigma^2\tau}} dz - \frac{\int_{-S}^{L}\left(\int_{-S}^{z}\mathrm{e}^{\frac{u^2}{\sigma^2\tau}}du\right)\mathrm{e}^{\frac{-z^2}{\sigma^2\tau}}dz}{\int_{-S}^{L}\mathrm{e}^{\frac{z^2}{\sigma^2\tau}}dz} \right] \int_{-S}^{u_0}\mathrm{e}^{\frac{z^2}{\sigma^2\tau}}dz + \right.$$
$$\left. + \int_{-S}^{u_0}\left(\int_{-S}^{z}\mathrm{e}^{\frac{u^2}{\sigma^2\tau}}du\right)\mathrm{e}^{\frac{-z^2}{\sigma^2\tau}}dz - \int_{-S}^{u_0}\mathrm{e}^{\frac{u^2}{\sigma^2\tau}}du\int_{-S}^{u_0}\mathrm{e}^{\frac{-z^2}{\sigma^2\tau}}dz \right\}. \quad (12)$$

In the metrological application we assume that the threshold values for the tolerability of the synchronization error are symmetric and take $S = L$. Once we fix the parameters of the process, numerical integration provides the evaluation of the expected value.

Let us generalise this result to higher order moments $m_n(u_0)$. We have to solve iteratively the following partial differential equations:

$$\frac{\sigma^2}{2}\frac{\partial^2 m_n}{\partial {u_0}^2} - \frac{x}{\tau}\frac{\partial m_n}{\partial u_0} = -1 \quad (13)$$

with the same boundary conditions considered in (11).

the following solutions may be obtained by separation of variables

$$m_n(u_0) = \frac{\nu_n(L)}{\int_{-S}^{L}\mathrm{e}^{\frac{u^2}{\sigma^2\tau}}du}\int_{-S}^{u_0}\mathrm{e}^{\frac{u^2}{\sigma^2\tau}}du + \nu_n(u_0).$$

The function $\nu_n(x)$ is defined by

$$\nu_n(u_0) = \frac{2n}{\sigma^2}\left(\int_{-S}^{u_0} m_{n-1}(u)\mathrm{e}^{-\frac{u^2}{\sigma^2\tau}}\int_{-S}^{u}\mathrm{e}^{\frac{z^2}{\sigma^2\tau}}dzdu - \int_{-S}^{u_0}\mathrm{e}^{\frac{u^2}{\sigma^2\tau}}du\int_{-S}^{u_0} m_{n-1}(u)\mathrm{e}^{-\frac{u^2}{\sigma^2\tau}}du\right)$$

In particular we can give an analytical expression for the variance of the first passage time between two constant barriers as follows

$$\mathrm{Var}\,[T(u_0, -S, L)] = m_2(u_0) - m(u_0)^2. \quad (14)$$

Numerical integration provides the values of such an expression once we fix the values of the parameters. In the next Section a comparison between the analytical expressions of

expectation and variance of the first passage time between two symmetric thresholds and their simulative counterparts is discussed.

V. SIMULATION OF FIRST PASSAGE TIMES BETWEEN TWO BARRIERS

A simulation method for the paths of an Ornstein-Uhlenbeck process is easily derived by the iterative formula (3). Each point U_n is in fact calculated (see again Fig. 1) by considering the deterministic exponential decay $U_{n-1}\mathrm{e}^{-\frac{h}{\tau}}$ from the previous point (h is the discretization interval) and adding to it a realization of a normal random variable Z_n with zero mean and variance equal to $\frac{\sigma^2\tau}{2}(1-\mathrm{e}^{-2\frac{h}{\tau}})$. The random variables Z_n are independent and they can be simulated by standard techniques.

First passage times between two symmetric thresholds S and $-S$ can be evaluated, as a first approximation, stopping the simulation the first time that U_n is greater than S or smaller than $-S$ and considering the first passage time T as $T = nh$, where h is the discretization interval. A short discretization interval is required to obtain reasonable results. In our case we took $h = 10^{-4}s$.

In particular we evaluated the mean and the variance of the first passage time simulating 10^5 trajectories starting at $u_0 = 0$ with OU parameters $\tau = 1s$ and $\sigma = 1.5$. Different symmetric thresholds S and $-S$ have been imposed ($S = 0.3 \quad 0.5 \quad 0.7 \quad 0.8 \quad 1 \quad 1.2$).

Results are presented in Fig. 7 and Fig. 8 where they are compared with the theoretical values obtained by numerical integration of the formulae (12) and (14). Despite the very short discretization interval, the first passage time evaluated from simulations appears to be always overestimated. This effect can be explained as follows: the process is continuos in time while we are observing it just at discrete intervals. There is a non-vanishing probability that a passage occurs between two points that are both below the threshold (see Fig. 9). These *hidden passages* cannot be observed by this method and the simulation continues also if a crossing may be occurred. The estimated first passage time is thence longer than the true value.

However in [15] a method was developed in the case of a single threshold S in order to take into account hidden passages in the simulations. The idea is simple and we summarize it without entering computational details. For an Ornstein-Uhlenbeck process with fixed values $U_{n-1} = u_{n-1}$ and $U_n = u_n$ at times t_{n-1} and t_n the probability P_h of an hidden

passage between the two fixed points solves an integral equation that was firstly derived in Ref. [16]. In [15] an algorithm based on a computationally efficient approximation of this probability for small discretization steps was presented that allows to simulate with high precisions the statistics of first passage times taking into accounts the actual probability of hidden crossings. Let us describe the algorithm. At each simulation step one start from the given value u_{n-1} below the threshold. Then U_n is generated according to the iterative formula (3). If it happens to be below the threshold, instead of simply going on with the next point, one evaluate the probability P_h of an hidden crossing between our two points. Then one generates a realization a of a uniform random variable on the interval $[0, 1]$. If $a \geq P_h$ then one proceeds as if no crossing has taken place and go on with the next point. Otherwise one stops and takes $T = nh$ as the first passage time for the considered trajectory. This technique was proved to be very efficient in the evaluation of first passage times across a single threshold.

To adapt the method to the case of two symmetric barriers we used an approximation by considering the crossing of the lower and of the upper barrier as two independent processes and just summing up the two separate crossing probabilities. Even if this method is not completely rigorous, when the two barriers are far from each other and the time step is small, the approximation give very good results as we are going to illustrate.

The results of the mean and the variance of the first passage time corrected by this method are plotted in Fig. 10 and Fig. 11 and again compared with theoretical values. The agreement is very good and the validity of the approximations is confirmed.

VI. POSSIBLE APPLICATIONS

The range of possible applications of the Ornstein-Uhlenbeck model is very wide and it embraces every branch of metrology where white noise plays a role. As already stated every white noise necessarily comes with a filtering and the Ornstein-Uhlenbeck process is expected to fit such a kind of data.

To restrict the focus on time metrology and on atomic clocks we refer to the available mathematical models of the atomic clock behavior (cf. Refs. [1–4]). Atomic clocks are typically affected by the following five classes of noise:

- white phase modulation (WPM)

- flicker phase modulation (FPM)
- white frequency modulation (WFM), which produces a random walk on phase (RWPM)
- flicker frequency modulation (FFM)
- random walk frequency modulation (RWFM).

While WFM and RWFM are already embraced in the known models (cf. [3, 4]) WPM is not and the Ornstein-Uhlenbeck process is a good candidate to model such a noise (that necessarily is somewhere filtered). A secondary application could be to WFM. In fact WFM noise was previously modeled as a random walk on the phase (its integral), however the white frequency noise is again not perfectly white and maybe an Ornstein-Uhlenbeck process (or its integral if seen on the phase) could better fit the data. This second application however requires more work as some of the expressions here presented are specifically related to a filtered white noise on the phase. The clock model can be thence enhanced by considering an Ornstein-Uhlenbeck contribution to the already modeled noises. The addition of a further component to the model may bring a non-trivial complication and some analytical results would not be available any more. Let us however remark that the simulation scheme here presented could be directly used to simulate the further component of the noise.

The OU process is somehow similar to a Wiener process but with a constraining force that keeps it close to zero. In this perspective it can also be seen as a model for a *constrained* clock, for example a steered frequency standard whose time or frequency offset would naturally evolve according to a Wiener process but which is constrained by a steering action to remain close to a predefined value. The interpretation of the OU as the output of a low pass filter according to the equations given in Sec. III is another hint for applications. Developing the equation (8) with the given values of the coefficients we can interpret the OU process as the result of an exponentially weighted moving average of past data, in time and frequency. An average of this kind is used for the prediction of future values or for interpolating missing data. The OU process would be a good model for such estimated values.

We want moreover to observe that in the metrological literature (see for example [17, 18]) filtered white noise can often be found and it is usually dealt with starting from its power spectrum or its autocorrelation function. The modeling through a OU process and the

analytical results we have presented here could be of advantages also in those cases.

VII. CONCLUSIONS

This paper presents the mathematical properties of the Ornstein-Uhlenbeck process including results on the first crossing time between two barries, the spectrum, the Allan variance and an improved simulation scheme for the statistics of first crossing times between two barriers. The OU process may be interpreted as a white noise filtered by a Lorenzian filter, therefore this process is deemed useful for modeling atomic clock behaviour and to understand how it may affect more complex systems.

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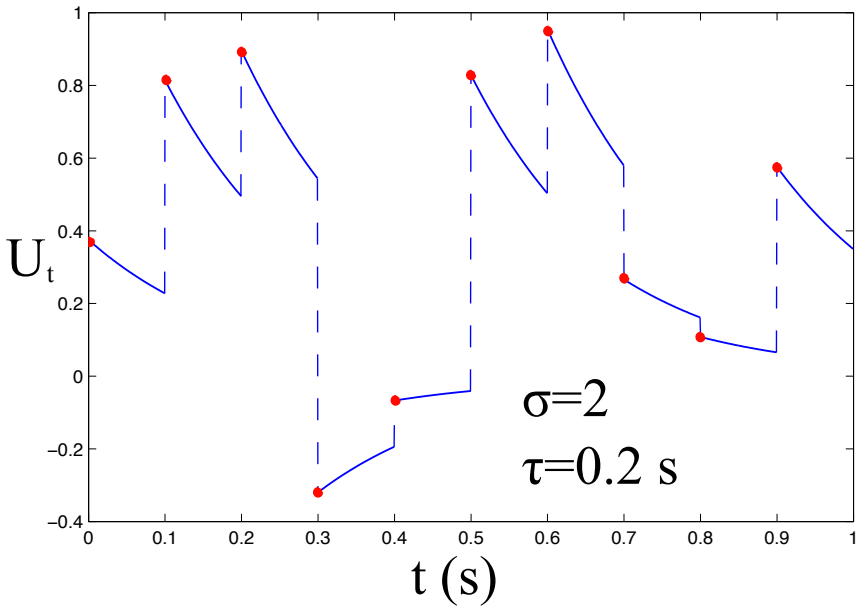

Figure 1: A discretized trajectory (red dots) of an Ornstein-Uhlenbeck process is plotted according to the iterative formula (3). Each point is related to the previous one by a deterministic decay towards the origin (continuos blue line, not part of the trajectory) added to the stochastic behavior that accumulates during the discretization intervals bringing i.i.d. gaussian jumps having zero mean and constant variance at each step.

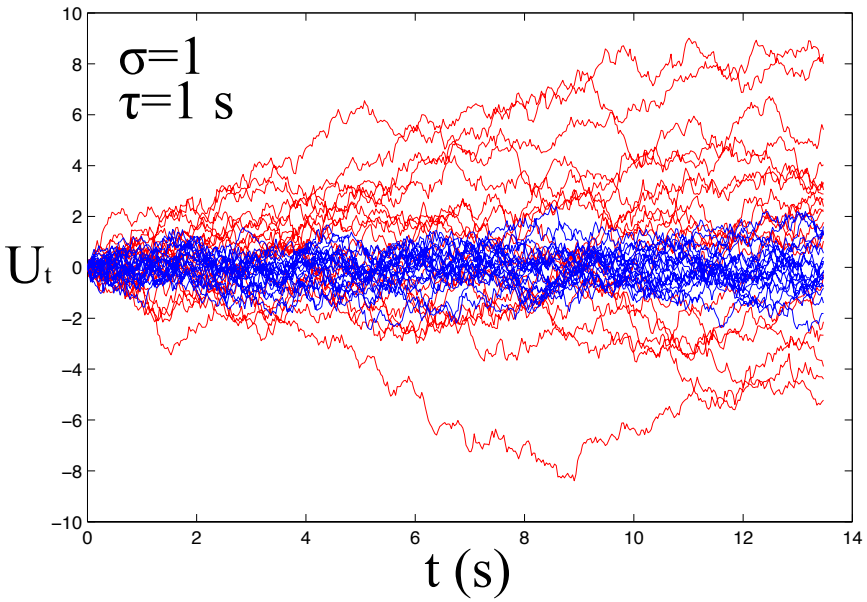

Figure 2: Trajectories of an Ornstein-Uhlenbeck (in blue) are compared with trajectories of a Wiener process (in red). The former admit a stationary distribution, while the variance of the latter is increasing with time.

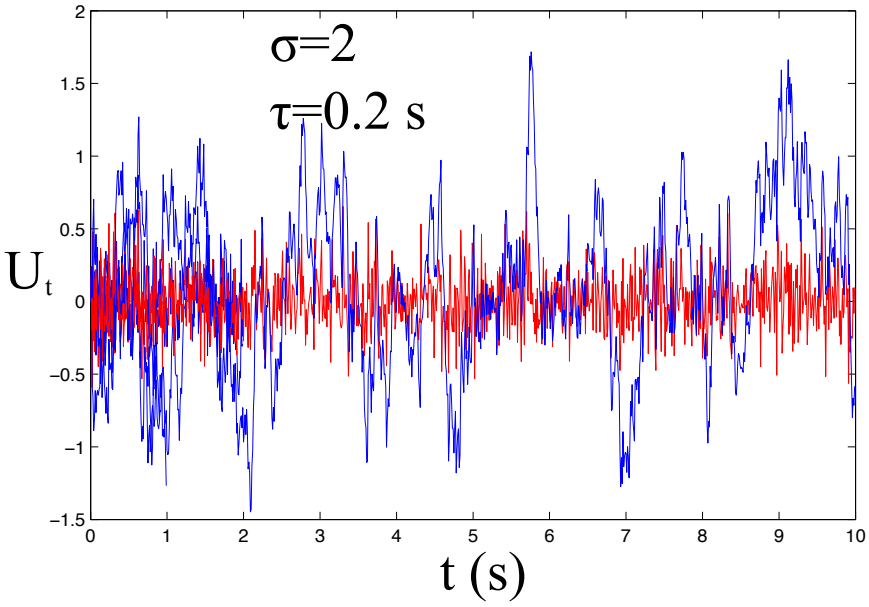

Figure 3: A trajectory of an Ornstein-Uhlenbeck (in blue) is compared with one of a white noise (in red) with the same variance. In the former low frequency oscillations are dominant.

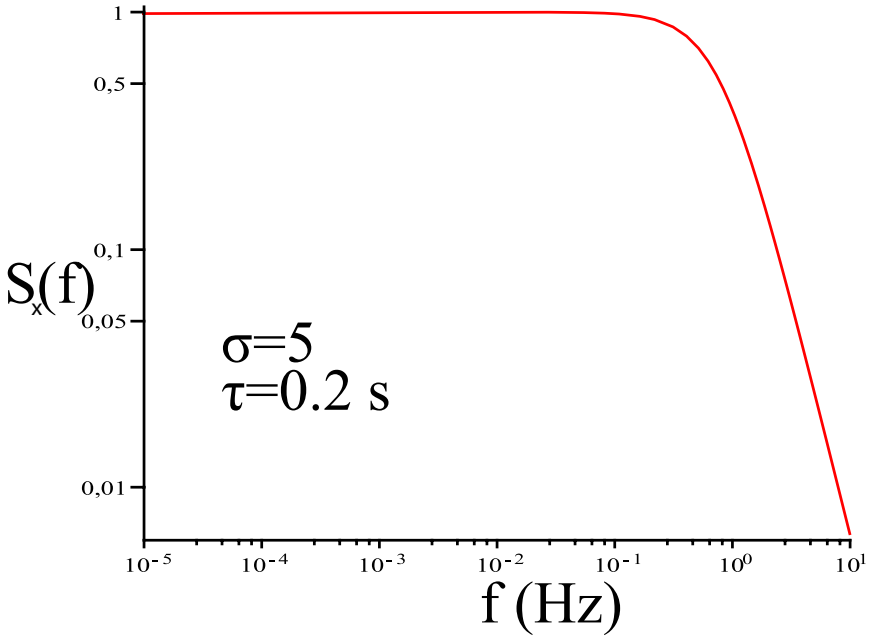

Figure 4: The spectrum of stationary Ornstein-Uhlenbeck (cf. equation (5) process is plotted in log-log scale.

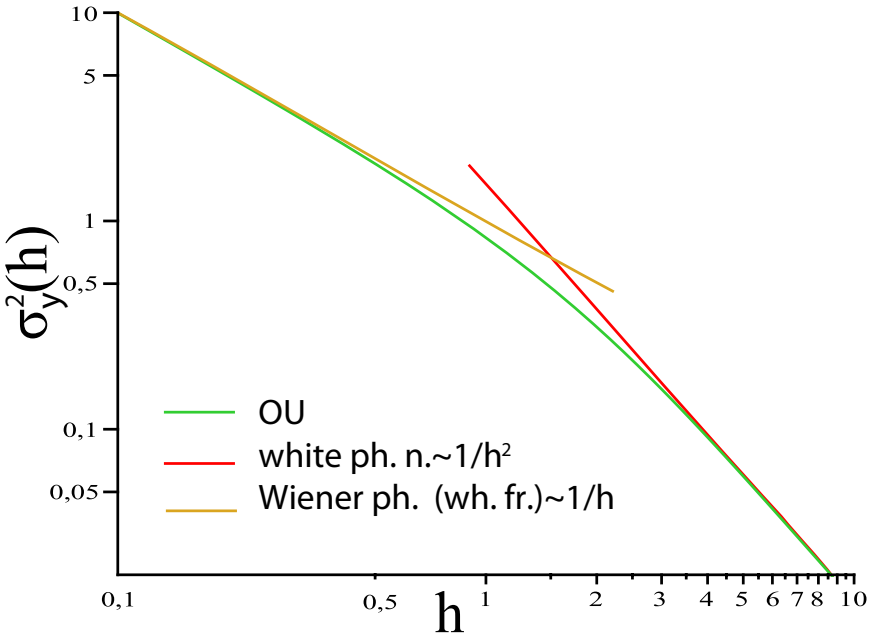

Figure 5: The Allan variance of the frequency a phase Ornstein-Uhlenbeck process (cf. equation (7) is plotted in log-log scale.

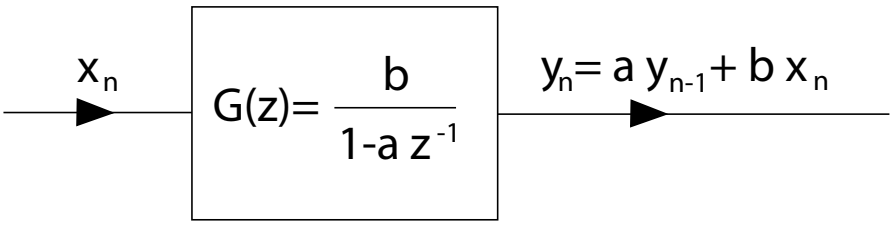

Figure 6: The scheme of a low-pass first-order lIR digital filter.

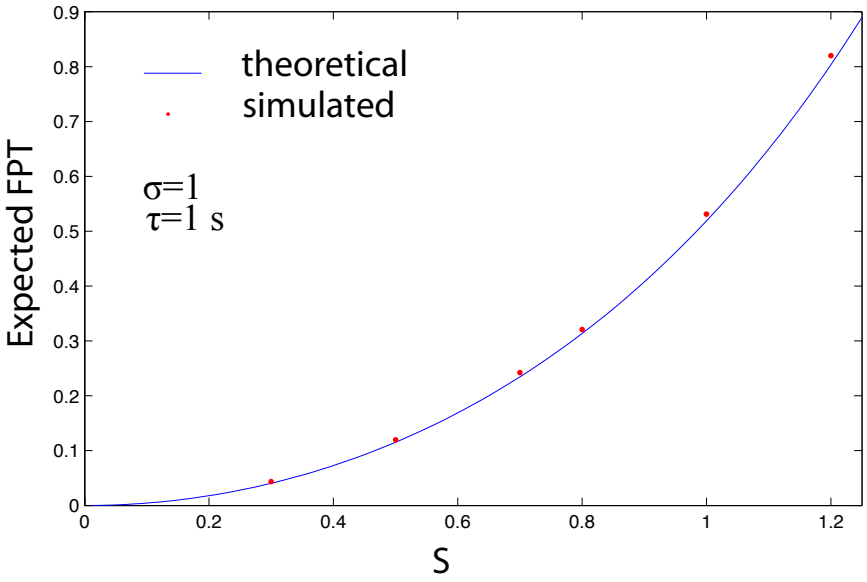

Figure 7: The mean first passage times between two symmetric barriers at $-S$ and S are computed for different values of S. The initial position is taken as $u_0 = 0$. Theoretical values are compared with those arising from simulations. First passage times are overestimated from standard simulations due to the presence of hidden passages.

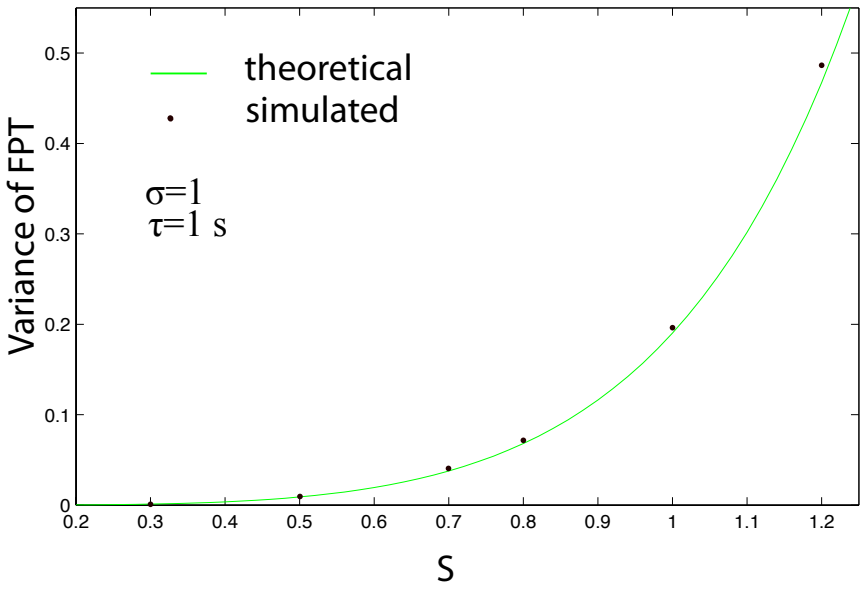

Figure 8: The variances of the first passage times between two symmetric barriers at $-S$ and S are computed for different values of S. The initial position is taken as $u_0 = 0$. Theoretical values are compared with those arising from simulations. The variances of the first passage times are overestimated from standard simulations due to the presence of hidden passages.

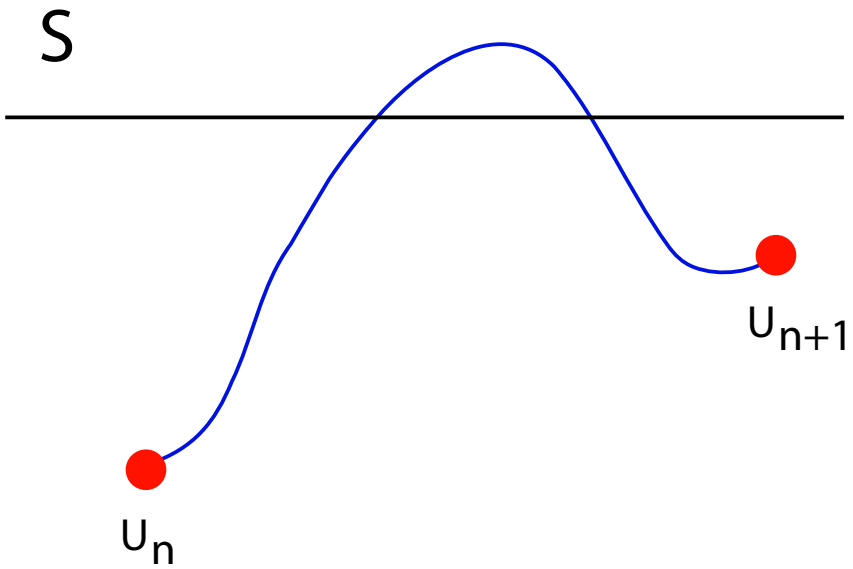

Figure 9: The process is continuos but we observe it at discrete intervals. Hidden passages may occur between two observations.

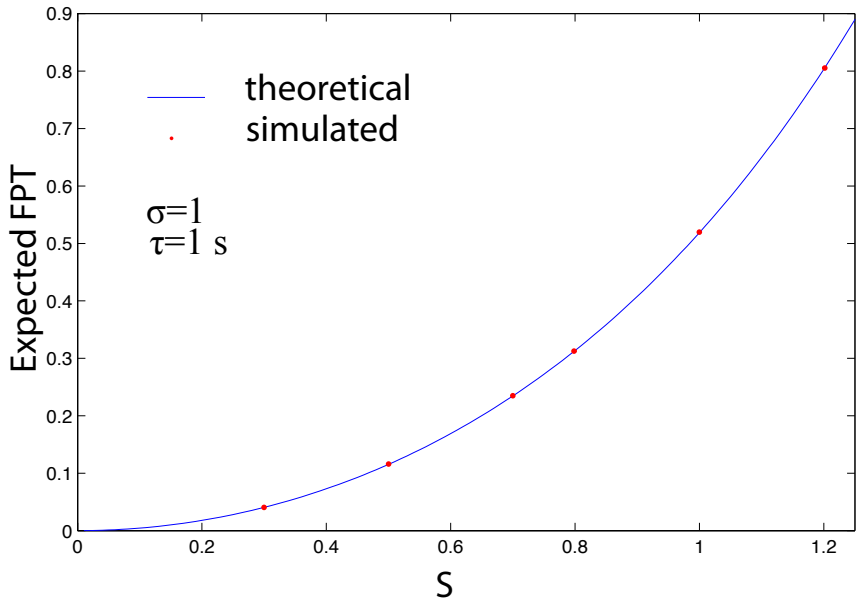

Figure 10: The improved method for the simulations of first passages between two symmetric thresholds provides estimated values for the first passage times that are in very close agreement with the theoretical predictions.

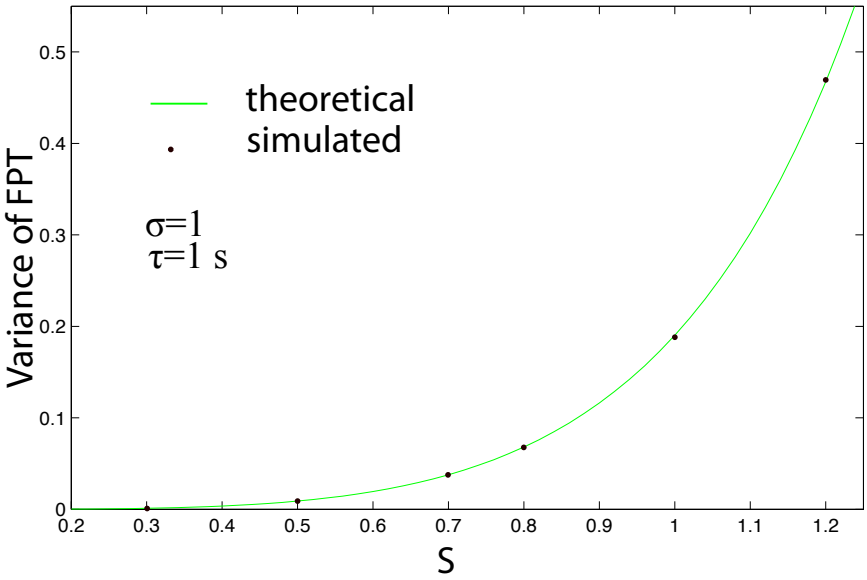

Figure 11: The improved method for the simulations of first passages between two symmetric barriers provides estimated values for the variance of the first passage times that are in very close agreement with the theoretical predictions.