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Wave front sets for ultradistribution solutions of linear partial differential operators with coefficients in non–quasianalytic classes

A.A. Albanese[1], **D. Jornet**[2] *, and **A. Oliaro**[3]

[1] Dipartimento di Matematica "E. De Giorgi", Università del Salento, Via Per Arnesano, P.O. Box 193, I-73100 Lecce, Italy
[2] Instituto Universitario de Matemática Pura y Aplicada IUMPA-UPV, Universidad Politécnica de Valencia, C/Camino de Vera, s/n, E-46022 Valencia, Spain
[3] Dipartimento di Matematica, Università di Torino, Via Carlo Alberto, 10, I-10123 Torino, Italy

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We prove the following inclusion

$$WF_*(u) \subset WF_*(Pu) \cup \Sigma, \quad u \in \mathcal{E}'_*(\Omega),$$

where WF_* denotes the non–quasianalytic Beurling or Roumieu wave front set, Ω is an open subset of $\mathbb{R}^n$, P is a linear partial differential operator with suitable ultradifferentiable coefficients, and Σ is the characteristic set of P. The proof relies on some techniques developed in the study of pseudodifferential operators in the Beurling setting. Some applications are also investigated.

1 Introduction

The ultradifferentiable functions are intermediate classes between real analytic and C^∞ functions. Depending on their topological structure they can be classified as classes of Beurling or of Roumieu type. Moreover, according to the Denjoy-Carleman theorem the ultradifferentiable functions can also be classified in quasianalytic and non-quasianalytic classes, where, roughly speaking, the quasianalytic ones are functions whose Fourier transform has a stronger decay at infinity. We refer to [2, 19, 17, 10] for the different ways to introduce these classes and to [8] for an exhaustive comparison between them. We emphasize that the Gevrey classes are particular cases of non-quasianalytic ultradifferentiable functions of Roumieu type.

We establish some basic results on propagation of singularities for solutions of linear partial differential operators with coefficients in certain smooth classes, in a wider setting than the one of classical distributions or, even, ultradistributions of Gevrey type, and from the micro-local point of view, i.e., via wave front sets. The notion of wave front set was introduced by Hörmader in 1970 to simplify the study of the propagation of singularities of (ultra)distribution solutions of linear partial differential operators.

It is known that partial differential operators with ultradifferentiable coefficients, or even pseudodifferential operators of ultradifferentiable type, reduce in most cases the singular support and the wave front set of ultradistributions, [1, 3, 4, 5, 14, 16, 17, 18, 22, 23]. The main aim of this paper is to prove a suitable converse result, more precisely, we prove that if $P = P(x, D)$ is a linear partial differential operator with ultradifferentiable coefficients on $\Omega \subset \mathbb{R}^N$ then the following inclusion holds

$$WF_\omega(u) \subset WF_\omega(Pu) \cup \Sigma, \qquad (1.1)$$

for all ultradistributions with compact support $u \in \mathcal{E}'_\omega(\Omega)$, where Σ is the characteristic manifold of P, i.e., the set where the principal symbol of P vanishes. We cover both Beurling and Roumieu cases in inclusion (1.1). The analogous inclusion for the C^∞-wave front set and classical distributions was proved by Hörmander [16].

* Corresponding author: D. Jornet e-mail: djornet@mat.upv.es, Phone:+34 963 877 000, Fax: +34 963 879 494

Actually, Hörmander [17] (see also Kato [18]) established the validity of inclusion (1.1) also for wave front sets of classical distributions in the setting of certain quasianalytic (and non-quasianalytic) classes of Roumieu type including the spaces of real analytic functions. On the other hand, the present authors extended (1.1) for wave front sets of classical distributions to the Beurling and Roumieu cases at the same time in [1] when the weight functions (quasianalytic or not) and the corresponding spaces of ultradifferentiable functions are considered from the point of view of Braun, Meise and Taylor [10]. We point out that results of this type are a powerful tool in the study of the problem of iterates from a micro-local point of view (see, for example, [3, 4, 5, 23]). Here we mainly deal with non-quasianalytic classes of Beurling type in the sense of Braun, Meise and Taylor [10], and we prove inclusion (1.1) for ultradistributions of Beurling (and of Roumieu) type with compact support. Since the growth of the Fourier transform of compactly supported ultradistributions of Beurling (or even Roumieu) type is rather different to the one of classical distributions, we cannot use here the techniques of Hörmander [17], as we did in [1]. Thus, we need to follow a different approach. Precisely, we use some techniques coming from the theory of pseudodifferential operators of Beurling type and infinite order developed in [13, 14]; observe that in non-quasianalytic classes the use of test functions is allowed and hence, the machinery of [13, 14] can be used. We obtain first the Beurling case. Then, using [14, Proposition 2], a comparison between Beurling and Roumieu wave front sets (such a comparison has been recently extended in [1] also to cover the case of quasianalytic weight functions; see Proposition 2.6 for details), we obtain immediately the corresponding Roumieu version of inclusion (1.1). Because of the topology, in the Beurling case we need to take the coefficients of the operator P in a smaller class. This is not the case in the Roumieu setting. In both cases we need weight functions that are equivalent to sub-additive weight functions. An extension to classical properly supported pseudodifferential operators of the inclusion (1.1) is also obtained. Finally, we mention that the different properties that we prove on pseudodifferential operators are obtained, as in [14], avoiding the difficult techniques of wave front sets of kernels (as it is usual in the literature; see for example, [22, Section 3.4] or [16, Chapter VIII]). An application to the study of wave front sets of solutions of partial differential operators is provided.

2 Notation and preliminaries

In this section we recall the definition of ultradifferentiable classes and ultradistributions of Beurling and Roumieu type, as well as the definition and some needed results concerning wave front sets.

Throughout this article $|\cdot|$ denotes the euclidian norm on $\mathbb{R}^n$ or $\mathbb{C}^n$.

Definition 2.1 A non-quasianalytic $weight$ function is an increasing continuous function $\omega : [0,\infty[\to [0,\infty[$ with the following properties:

(α) there exists $L \geq 0$ such that $\omega(2t) \leq L(\omega(t)+1)$ for all $t \geq 0$,

(β) $\int_1^\infty \frac{\omega(t)}{t^2}dt < \infty$,

(γ) $\log(t) = o(\omega(t))$ as t tends to ∞,

(δ) $\varphi : t \to \omega(e^t)$ is convex.

A weight function ω is equivalent to a sub-additive weight if, and only if, the following property holds:

$$(\alpha_0) \quad \exists D>0 \ \exists t_0>0 \ \forall \lambda \geq 1 \ \forall t \geq t_0 : \ \omega(\lambda t) \leq \lambda D \omega(t).$$

For a weight function ω we define $\tilde{\omega} : \mathbb{C} \to [0,\infty[$ by $\tilde{\omega}(z) := \omega(|z|)$ and again denote this function by ω. The *Young conjugate* $\varphi^* : [0,\infty[\to \mathbb{R}$ of φ is given by

$$\varphi^*(s) := \sup\{st - \varphi(t), t \geq 0\}.$$

There is no loss of generality to assume that ω vanishes on $[0,1]$. Then φ^* has only non-negative values, it is convex and $\varphi^*(t)/t$ is increasing and tends to ∞ as $t \to \infty$ and $\varphi^{**} = \varphi$.

Example 2.2 The following are examples of non-quasianalytic weight functions (eventually after a change on the interval $[0,\delta]$ for a suitable $\delta > 0$):

(1) $\omega(t) = t^\alpha$, $0 < \alpha < 1$;

(2) $\omega(t) = (\log(1+t))^{\beta}$, $\beta > 1$;

(3) $\omega(t) = t\,(\log(e+t))^{-\beta}$, $\beta > 1$;

Definition 2.3 Let ω be a weight function. For an open set $\Omega \subset \mathbb{R}^N$ we let

$$\mathcal{E}_{(\omega)}(\Omega) := \{f \in C^{\infty}(\Omega) : \| f \|_{K,\lambda} < \infty, \text{ for every } K \subset\subset \Omega \text{ and every } \lambda > 0\},$$

and

$$\mathcal{E}_{\{\omega\}}(\Omega) := \quad \{f \in C^{\infty}(\Omega) : \text{for every } K \subset\subset \Omega \text{ there exists } \lambda > 0 \text{ such that } \| f \|_{K,\lambda} < \infty\},$$

where

$$\| f \|_{K,\lambda} := \sup_{x \in K} \sup_{\alpha \in \mathbb{N}_0^N} |f^{(\alpha)}(x)| \exp\left(-\lambda \varphi^{*}\left(\frac{|\alpha|}{\lambda}\right)\right).$$

The space $\mathcal{E}_{(\omega)}(\Omega)$ is Fréchet and nuclear, while $\mathcal{E}_{\{\omega\}}(\Omega)$ is a projective limit of an inductive limits of Banach spaces. The elements of $\mathcal{E}_{(\omega)}(\Omega)$ (resp. $\mathcal{E}_{\{\omega\}}(\Omega)$) are called ultradifferentiable functions of Beurling type (resp. Roumieu type) in Ω. When $\omega(t) = t^d$, $d < 1$, the corresponding Roumieu class $\mathcal{E}_{\{\omega\}}(\Omega)$ is the Gevrey class with exponent $1/d$. Writing $*$ for (ω) or $\{\omega\}$ we denote by $\mathcal{D}_*(K) = \mathcal{E}_*(\Omega) \cap \mathcal{D}(K)$ if K is a compact set. We define $\mathcal{D}_*(\Omega) = \operatorname{ind}_n \mathcal{D}_*(K_n)$, where (K_n) is any compact exhaustion of Ω. Its dual is denoted by $\mathcal{D}'_{(\omega)}(\Omega)$ (resp. $\mathcal{D}'_{\{\omega\}}(\Omega)$) and it is called the space of the ultradistributions of Beurling (resp. Roumieu) type on Ω. Since $\mathcal{D}_{(\omega)}(\Omega) \subset \mathcal{D}_{\{\omega\}}(\Omega)$ and the inclusion is continuous and has dense range, $\mathcal{D}'_{\{\omega\}}(\Omega)$ can be identified with a subset of $\mathcal{D}'_{(\omega)}(\Omega)$. The corresponding spaces of ultradistributions of Beurling and Roumieu type with compact support $\mathcal{E}'_*(\Omega)$ (i.e., $\mathcal{E}'_*(\Omega)$ is the strong dual of $\mathcal{E}_*(\Omega)$) can be considered as well.

Given an ultradistribution $u \in \mathcal{D}'_*(\Omega)$ we define the $*$-singular support of u, denoted by $\operatorname{sing}_* \operatorname{supp} u$, as the complementary in Ω of the biggest open set $U \subset \Omega$ satisfying $u|_U \in \mathcal{E}_*(U)$.

Definition 2.4 Let ω be a weight function and $(x_0, \xi_0) \in \Omega \times (\mathbb{R}^N \setminus \{0\})$.

(i) If $u \in \mathcal{D}'_{(\omega)}(\Omega)$, we define the wave front set $WF_{(\omega)}(u)$ of u to be the complement in $\Omega \times (\mathbb{R}^N \setminus \{0\})$ of the set of points (x_0, ξ_0) such that there exist $\varphi \in \mathcal{D}_{(\omega)}(\Omega)$, $\varphi \equiv 1$ in a neighborhood of x_0, a conic neighborhood Γ of ξ_0, and a sequence $\{C_n\}$ of positive constants satisfying

$$|\widehat{\varphi u}(\xi)| \leq C_n \exp(-n\omega(\xi))$$

for every $\xi \in \Gamma$ and $n \in \mathbb{N}$.

(ii) If $u \in \mathcal{D}'_{\{\omega\}}(\Omega)$, we define the wave front set $WF_{\{\omega\}}(u)$ of u to be the complement in $\Omega \times (\mathbb{R}^N \setminus \{0\})$ of the set of points (x_0, ξ_0) such that there exist $\varphi \in \mathcal{D}_{\{\omega\}}(\Omega)$, $\varphi \equiv 1$ in a neighborhood of x_0, a conic neighborhood Γ of ξ_0, and two constants $C, \epsilon > 0$ satisfying

$$|\widehat{\varphi u}(\xi)| \leq C \exp(-\epsilon\omega(\xi))$$

for every $\xi \in \Gamma$.

Remark 2.5 The following elementary properties are well known (see for example [15]). Let Ω be an open subset in $\mathbb{R}^N$. Then:

(a) Given two weight functions ω and σ such that $\omega(t) = O(\sigma(t))$ as t tends to infinity, we have the following continuous inclusions

$$\mathcal{E}_{(\sigma)}(\Omega) \subset \mathcal{E}_{(\omega)}(\Omega), \quad \mathcal{E}_{\{\sigma\}}(\Omega) \subset \mathcal{E}_{\{\omega\}}(\Omega).$$

(b) If, in addition, $\omega(t) = o(\sigma(t))$ as t tends to infinity, we have the following continuous inclusion:

$$\mathcal{E}_{\{\sigma\}}(\Omega) \subset \mathcal{E}_{(\omega)}(\Omega).$$

(c) If $\omega(t^{1/r}) = o(\sigma(t))$ as t tends to infinity for some constant $0 < r \leq 1$, then for all $\lambda, \mu > 0$ there exists $C_{\lambda,\mu} > 0$ such that for all $j \in \mathbb{N}_0$, we have

$$\lambda\varphi^*_\sigma(\frac{j}{\lambda}) \leq C_{\lambda,\mu} + r\mu\varphi^*_\omega(\frac{j}{\mu}),$$

where φ^*_σ and φ^*_ω are the Young conjugate functions corresponding to the weight functions σ and ω.

We recall some known facts concerning wave front sets (cf., for example, [14]). We observe that if ω and σ are weight functions satisfying $\omega = O(\sigma)$ then

$$WF_{(\omega)}(u) \subset WF_{(\sigma)}(u)$$

for every $u \in \mathcal{D}'_{(\omega)}(\Omega) \subset \mathcal{D}'_{(\sigma)}(\Omega)$, and this inclusion is in general strict.

The Roumieu wave front set can be expressed in terms of Beurling wave front sets in the following way (see [1] for a version with quasianalytic weight functions and classical distributions).

Proposition 2.6 ([14],[1]) *Let us consider $u \in \mathcal{E}'_{\{\omega\}}(\Omega)$. Then there exists a weight function $\sigma_0 = o(\omega)$ such that $u \in \mathcal{E}'_{(\sigma_0)}(\Omega)$ and we have*

$$WF_{\{\omega\}}(u) = \overline{\bigcup_{\sigma\in S} WF_{(\sigma)}(u)}$$

for $S = \{\sigma$ weight function: $\sigma_0 \leq \sigma = o(\omega)\}$.

Analogously to the C^∞ case the following result holds.

Theorem 2.7 *Let us consider $u \in \mathcal{D}'_{(\omega)}(\Omega)$. Then the projection of $WF_{(\omega)}(u)$ on the first variable is the (ω)-singular support of u.*

We now recall the definition of Beurling pseudodifferential operators.

Definition 2.8 ([13]) Let Ω be an open set in $\mathbb{R}^N$, $0 \leq \delta < \rho \leq 1$, $d := \rho - \delta$, and assume that $\omega(t) = o(t^d)$ as $t \to \infty$. An amplitude in $S^{m,\omega}_{\rho,\delta}(\Omega)$ is a function $a(x, y, \xi)$ in $C^\infty(\Omega \times \Omega \times \mathbb{R}^N)$ such that for every compact set $Q \subset \Omega \times \Omega$ there are $R \geq 1$ and a sequence $C_n > 0$, $n \in \mathbb{N}$, with

$$|D^\alpha_x D^\gamma_y D^\beta_\xi a(x, y, \xi)| \leq C_n e^{(\rho-\delta)n\varphi^*(|\beta|/n)} e^{(\rho-\delta)n\varphi^*(|\alpha+\gamma|/n)} e^{m\omega(\xi)} |\xi|^{|\alpha+\gamma|\delta-|\beta|\rho}, \tag{2.1}$$

for every $n \in \mathbb{N}$, $(x, y) \in Q$, and ξ with $\log(\frac{|\xi|}{R}) \geq \frac{n}{|\beta|}\varphi^*(\frac{|\beta|}{n})$.

If we replace $e^{(\rho-\delta)n\varphi^*(|\beta|/n)}$ in the formula (2.1) by the constant $\beta!B^{|\beta|}$ for some $B > 0$ that depends only on Q and on the amplitude, then the corresponding class is written as $AS^{m,\omega}_{\rho,\delta}(\Omega)$. Since $\omega(t) = o(t^d)$ with $d := \rho - \delta$, it follows that $AS^{m,\omega}_{\rho,\delta}(\Omega) \subset S^{m,\omega}_{\rho,\delta}(\Omega)$ (see [13]). An amplitude in $AS^{m,\omega}_{\rho,\delta}(\Omega)$ is said to be of *finite order* if it satisfies the inequality (2.1) with $(1 + |\xi|)^m$ instead of $e^{m\omega(\xi)}$.

In the case $a(x, y, \xi) = p(x, \xi)$, the function $p(x, \xi)$ is usually called *symbol*. The definition of amplitude given in [13] is slightly different from the one given here, but it is equivalent due to the convexity of φ^*.

We recall from [13] that every amplitude $a(x, y, \xi)$ in $S^{m,\omega}_{\rho,\delta}(\Omega)$ defines a continuous and linear operator $A : \mathcal{D}_{(\omega)}(\Omega) \longrightarrow \mathcal{E}_{(\omega)}(\Omega)$ given by the iterated integral

$$(Af)(x) = \int \Big(\int a(x, y, \xi) e^{i(x-y)\xi} f(y)\, dy \Big)\, d\xi, \quad f \in \mathcal{D}_{(\omega)}(\Omega).$$

The operator A is called pseudodifferential operator of (ω) class and type (ρ, δ). In the case $a(x, y, \xi) = p(x, \xi)$, the corresponding pseudodifferential operator $P := P(x, D) : \mathcal{D}_{(\omega)}(\Omega) \to \mathcal{E}_{(\omega)}(\Omega)$ can be written in terms of Fourier transforms in the following way

$$P(x, D)f = \int p(x, \xi) e^{ix\xi} \hat{f}(\xi)\, d\xi\,, \quad f \in \mathcal{D}_{(\omega)}(\Omega).$$

Several examples of amplitudes, symbols, and the corresponding operators in this setting can be found in [13, Example 2.11, Proposition 2.12]. In particular, in [13] the following result has been proved.

Theorem 2.9 ([13]) *The pseudodifferential operator A associated to the amplitude $a(x,y,\xi)$ in $S^{m,\omega}_{\rho,\delta}(\Omega)$ admits a unique continuous and linear extension*

$$\tilde{A} : \mathcal{E}'_{(\omega)}(\Omega) \longrightarrow \mathcal{D}'_{(\omega)}(\Omega)\,.$$

An important property is that pseudodifferential operators defined by amplitudes are (ω)-pseudolocal, i.e., they shrink the (ω)-singular support.

Theorem 2.10 ([13]) *Let $A : \mathcal{E}'_{(\omega)}(\Omega) \longrightarrow \mathcal{D}'_{(\omega)}(\Omega)$ be the pseudodifferential operator associated to the amplitude $a(x,y,\xi)$ in $S^{m,\omega}_{\rho,\delta}(\Omega)$. Then*

$$\mathrm{sing}_{(\omega)}\,\mathrm{supp}\,(A\mu) \subseteq \mathrm{sing}_{(\omega)}\,\mathrm{supp}\,(\mu),$$

for every $\mu \in \mathcal{E}'_{(\omega)}(\Omega)$.

More recently, in [14] it was proved that pseudodifferential operators defined by symbols are (ω)-micro-pseudolocal, i.e., they also shrink the (ω)-wave front set. The proof in [14] does not use the machinery of wave front sets for kernels, but it relies on some L_2 techniques.

Theorem 2.11 ([14]) *Let $P(x,D) : \mathcal{E}'_{(\omega)}(\Omega) \to \mathcal{D}'_{(\omega)}(\Omega)$ be a pseudodifferential operator of class (ω) and type (ρ,δ) given by a symbol in $S^{m,\omega}_{\rho,\delta}(\Omega)$. Then*

$$WF_{(\omega)}\big(P(x,D)u\big) \subset WF_{(\omega)}(u) \tag{2.2}$$

for every $u \in \mathcal{E}'_{(\omega)}(\Omega)$.

For each $\ell \in \mathbb{N}$, we set

$$\mathcal{E}_\ell(\mathbb{R}^N) = \Big\{ f \in C^\infty(\mathbb{R}^N) \,:\, |f|_\ell := \sup_{x\in\mathbb{R}^N} \sup_{\alpha\in\mathbb{N}_0^N} \big|f^{(\alpha)}(x)\big| \exp\big(-\ell\varphi^*(|\alpha|/\ell)\big) < +\infty \Big\}, \tag{2.3}$$

where φ^* is the Young conjugate corresponding to the weight function ω.

Let $G(z) \in \mathcal{H}(\mathbb{C}^N)$ be an entire function such that $\log|G(z)| = O(\omega(z))$ when $|z|$ tends to infinity. Then, the continuous and linear convolution operator $G(D) : \mathcal{D}'_{(\omega)}(\mathbb{R}^N) \to \mathcal{D}'_{(\omega)}(\mathbb{R}^N)$ defined by

$$\big(G(D)f\big)(x) = \sum_{\alpha\in\mathbb{N}_0^N} i^{|\alpha|}\frac{G^{(\alpha)}(0)}{\alpha!} f^{(\alpha)}(x),$$

is called (ω)-ultradifferential operator.

Now, we can recall the following extension of Komatsu's second structure theorem for ultradistributions due to Braun [9] (see also Langenbruch [20]).

Theorem 2.12 *Given an ultradistribution with compact support $\mu \in \mathcal{E}'_{(\omega)}(\mathbb{R}^N)$ and $\ell \in \mathbb{N}$, there exist an elliptic (ω)-ultradifferential operator $G(D)$ and a function $f \in \mathcal{E}_\ell(\mathbb{R}^N)$ such that*

$$\mu = G(D)f.$$

3 Micro-local symbolic calculus and wave front sets

We begin stating (and proving) some auxiliary definitions and results.

Proposition 3.1 *Given an amplitude $a(x,y,\xi)$ in $S^{m,\omega}_{\rho,\delta}(\Omega)$, a test function $\varphi \in \mathcal{D}_{(\omega)}(\Omega)$, a compact set K in Ω, $n \in \mathbb{N}$ and $A > 0$, there exist $C_n > 0$ and a constant $\tilde{A}$ such that for $|\eta| \le A|\xi|$ and $y \in K$ the following inequality is satisfied*

$$\left| D_y^\alpha \int e^{ix\xi} a(x,y,\eta)\varphi(x)dx \right| \le C_n |\varphi|_n \frac{e^{m\omega(\eta)+\log(\tilde{A}\xi)}}{e^{n(\rho-\delta)\omega(\tilde{A}\xi)}} e^{n\varphi^*(|\alpha|/n)},$$

where $|\varphi|_n$ is defined according to (2.3).

P r o o f. The proof follows first by integrating by parts with respect to x and then proceeding as in [14, Proposition 4]. □

Definition 3.2 Let $a(x, y, \xi)$ be an amplitude in $S^{m,\omega}_{\rho,\delta}(\Omega)$. The pseudodifferential operator A associated to $a(x, y, \xi)$ is said to be (ω)-micro-regularizing in $\Omega \times \Gamma_0$ for some open cone $\Gamma_0 \subset \mathbb{R}^N$ if

$$(\Omega \times \Gamma_0) \cap WF_{(\omega)}(Au) = \emptyset \quad \text{for all } \; u \in \mathcal{E}'_{(\omega)}(\Omega).$$

In the next result we give a sufficient condition for a pseudodifferential operator A to be (ω)-micro-regularizing.

Theorem 3.3 *Let $a(x, y, \xi)$ be an amplitude in $S^{m,\omega}_{\rho,\delta}(\Omega)$. If $a = 0$ in $\Omega \times \Omega \times \Gamma_0$ for some open cone $\Gamma_0 \subset \mathbb{R}^N$, then the pseudodifferential operator A associated to $a(x, y, \xi)$ is (ω)-micro-regularizing in $\Omega \times \Gamma_0$.*

P r o o f. We fix an ultradistribution $u \in \mathcal{E}'_{(\omega)}(\Omega)$ and $\ell \in \mathbb{N}$. By Theorem 2.12 we can find $f \in \mathcal{E}_\ell(\mathbb{R}^N)$ and an elliptic $(\omega)-$ultradifferential operator $G(D)$ such that $u = G(D)f$ in Ω.

We observe that if $G(D)$ is given by the entire function $G(z) = \sum_\alpha a_\alpha z^\alpha$, then $G(D)f = \sum_\alpha i^{|\alpha|} a_\alpha f^{(\alpha)}$ in $\mathcal{D}'_{(\omega)}(\Omega)$.

We now fix a test function $\chi \in \mathcal{D}_{(\omega)}(\Omega)$ equal to 1 in a neighborhood of $\operatorname{supp} u$. Then we have $Au = \sum_\alpha i^{|\alpha|} a_\alpha A(\chi f^{(\alpha)})$ weakly in $\mathcal{D}'_{(\omega)}(\Omega)$ as, for every $\varphi \in \mathcal{D}_{(\omega)}(\Omega)$,

$$\begin{aligned}\langle Au, \varphi\rangle &= \langle u, A^t\varphi\rangle = \langle u, \chi A^t \varphi\rangle = \langle \sum_\alpha i^{|\alpha|} a_\alpha f^{(\alpha)}, \chi A^t\varphi\rangle = \\ &= \sum_\alpha i^{|\alpha|} a_\alpha \langle \chi f^{(\alpha)}, A^t\varphi\rangle = \sum_\alpha i^{|\alpha|} a_\alpha \langle A(\chi f^{(\alpha)}), \varphi\rangle.\end{aligned}$$

Since $\mathcal{E}_{(\omega)}(\Omega)$ is a Fréchet-Montel space, it follows that $(\varphi Au)\hat{}(\xi) = \sum_\alpha i^{|\alpha|} a_\alpha \left(\varphi A(\chi f^{(\alpha)})\right)\hat{}(\xi)$ uniformly on the compacts subsets of $\mathbb{R}^N$ for every $\varphi \in \mathcal{D}_{(\omega)}(\Omega)$ (see [10, 21]). So, to conclude the proof, it suffices to estimate each one of the terms $\left(\varphi A(\chi f^{(\alpha)})\right)\hat{}(\xi)$ for ξ in an appropriate cone and an arbitrary fixed function $\varphi \in \mathcal{D}_{(\omega)}(\Omega)$, and then to sum in α. In order to do this, we first show, for ξ in a certain cone, that

$$\left(\varphi A(\chi f^{(\alpha)})\right)\hat{}(\xi) = \iint e^{-iy\eta} \sigma(y, \xi, \eta)\chi(y) f^{(\alpha)}(y) dy d\eta, \tag{3.1}$$

where $\sigma(y, \xi, \eta) = \int e^{ix(\eta-\xi)} a(x, y, \eta)\varphi(x)dx$.

We fix $\xi_0 \in \Gamma_0$. We will estimate $|D^\alpha_y \sigma(y, \xi, \eta)|$ when $y \in \operatorname{supp}\chi$, $\eta \in \mathbb{R}^N$ and ξ is in a neighborhood Γ'_0 of ξ_0 such that $\Gamma'_0 \subset\subset \Gamma_0$. For $\eta \in \Gamma_0$, $a(x, y, \eta) = 0$ by hypothesis, and for $\eta \notin \Gamma_0$ and $\xi \in \Gamma'_0$ we can find $\lambda > 0$ with $|\xi - \eta| \geq \lambda(|\xi| + |\eta|)$. By the Proposition 3.1, for every $n, s \in \mathbb{N}$,

$$|D^\alpha_y \sigma(y, \xi, \eta)| \leq C_{sn}|\varphi|_{sn} \frac{e^{m\omega(\eta)+\log(\tilde{A}|\xi-\eta|)}}{e^{sn(\rho-\delta)\omega(\tilde{A}(\xi-\eta))}} e^{sn\varphi^*(|\alpha|/(sn))}, \tag{3.2}$$

where C_{ns} and $\tilde{A}$ are suitable positive constants depending only on n, s and λ. Since $sn\varphi^*(|\alpha|/(sn)) \leq n\varphi^*(|\alpha|/n)$, we obtain (changing $s \in \mathbb{N}$ and the constant C_{sn} if necessary), for $\xi \in \Gamma'_0$ and $\eta \notin \Gamma_0$,

$$|D^\alpha_y \sigma(y, \xi, \eta)| \leq \tilde{C}_{sn} e^{-sn\omega(\eta)-sn\omega(\xi)} e^{n\varphi^*(|\alpha|/n)} \tag{3.3}$$

This implies that the equality (3.1) is satisfied for all $\eta \in \mathbb{R}^N$ and $\xi \in \Gamma'_0$.

Integrating by parts with respect to y in (3.1) and using that $|\eta|^{|\alpha_1|} \leq e^{n\varphi^*(|\alpha_1|/n)} e^{n\omega(\eta)}$, we obtain, for s sufficiently large,

$$\begin{aligned}&\left|\left(\varphi A(\chi f^{(\alpha)})\right)\hat{}(\xi)\right| = \\ &\quad = \left|\sum_{\alpha_1+\alpha_2+\alpha_3=\alpha} \binom{\alpha}{\alpha_1,\, \alpha_2,\, \alpha_3} \iint e^{-iy\eta}(-i\eta)^{\alpha_1} \partial^{\alpha_2}_y \sigma(y, \xi, \eta)\chi^{(\alpha_3)}(y) f(y) dy d\eta\right| \leq \\ &\quad \leq m(\operatorname{supp}\chi) D_{sn} e^{-sn\omega(\xi)} 3^{|\alpha|} e^{n\varphi^*(|\alpha|/n)} \int e^{-(s-1)n\omega(\eta)} d\eta,\end{aligned}$$

where $m(\cdot)$ denotes the Lebesgue measure. We know from the properties of $G(z)$ that there exists $k \in \mathbb{N}$ such that $|a_\alpha| \leq e^k e^{-k\varphi^*(|\alpha|/k)}$ for each α. So, taking $n = L^2 k$ in the last inequalities, we obtain, for $\xi \in \Gamma'_0$ and s large enough,

$$\begin{aligned} |(\varphi A u)\hat{}(\xi)| &\leq \sum_\alpha |a_\alpha| \left| \left(\varphi A(\chi f^{(\alpha)})\right)\hat{}(\xi)\right| \leq \\ &\leq e^{-s\omega(\xi)} \tilde{D}_{sk} \sum_\alpha |a_\alpha| 3^{|\alpha|} e^{L^2 k \varphi^*(|\alpha|/(L^2 k))} \leq \\ &\leq E_{ks} e^{-s\omega(\xi)} \sum_\alpha (3/e^2)^{|\alpha|}. \end{aligned}$$

The proof is now complete. □

In order to study the propagation of singularities, we introduce a micro-local version of the symbolic calculus for pseudo-differential operators as follows. We consider neighborhoods of the type $\Lambda = U \times U \times \Gamma$ or $\Lambda = U \times \Gamma$ where U is a relatively compact open set of $\mathbb{R}^N$ and Γ is an open cone in $\mathbb{R}^N \setminus \{0\}$. We fix also $\Lambda^* = U^* \times U^* \times \Gamma^*$ as a set of the same type (or $\Lambda^* = U^* \times \Gamma^*$) such that $\overline{U} \subset U^*$ and $\overline{\Gamma} \cap S^{N-1} \subset \Gamma^*$. This notation will be used throughout the rest of the section.

Definition 3.4 Let $0 \leq \delta < \rho \leq 1$, $d := \rho - \delta$, and assume that $\omega(t) = o(t^d)$ as $t \to \infty$. We denote by $MS^{m,\omega}_{\rho,\delta}(\Lambda)$ the space of all amplitudes $a(x,y,\xi) \in C^\infty(\Lambda^*)$, where $\Lambda = U \times U \times \Gamma$ and $\Lambda^* = U^* \times U^* \times \Gamma^*$ are as in the notation above, with $\Gamma \subset\subset \Gamma^*$, satisfying the following condition: there are $R \geq 1$, $B \geq 1$ and a sequence $C_n > 0$, $n \in \mathbb{N}$, with the property

$$|D^\alpha_x D^\gamma_y D^\beta_\xi a(x,y,\xi)| \leq C_n B^{|\beta|} \beta! e^{(\rho-\delta)n\varphi^*(\frac{|\alpha+\gamma|}{n})} e^{m\omega(\xi)} |\xi|^{|\alpha+\gamma|\delta - |\beta|\rho} \tag{3.4}$$

for every $n \in \mathbb{N}$, $(x,y,\xi) \in \Lambda^*$, with $\log(\frac{|\xi|}{R}) \geq \frac{n}{|\beta|}\varphi^*(\frac{|\beta|}{n})$.

If $a(x,y,\xi) = p(x,\xi)$, we have a similar definition for symbols, and we anyway write that $p(x,\xi) \in MS^{m,\omega}_{\rho,\delta}(\Lambda)$.

We want now to construct pseudodifferential operators for such type of symbols and amplitudes. So, we assume $\omega(t) = o(t^r)$, $0 < r \leq 1$, $\rho \leq r$. Moreover, we take $\psi(\xi) \in \mathcal{E}_{\{t^r\}}(\mathbb{R}^N)$ such that $\psi \equiv 1$ in a neighborhood of Γ, and $\operatorname{supp}\psi \subset \Gamma^*$ for large $|\xi|$, satisfying

$$|D^\beta \psi(\xi)| \leq D_n \frac{B^{|\beta|}\beta!}{|\xi|^{r|\beta|}} \tag{3.5}$$

for $\log(\frac{|\xi|}{R}) \geq \frac{n}{|\beta|}\varphi^*(\frac{|\beta|}{n})$ (we refer to [13, Example 2.11.(2)] for a description of $\varphi^*_\omega(t)$ when the corresponding weight function is a Gevrey weight of type $\omega(t) = t^r$, $0 < r \leq 1$, and also to [22]). For a weight σ with $\omega(t^{1/d}) = o(\sigma(t))$, we also fix $\phi \in \mathcal{D}_{\{\sigma\}}(U^*)$ with $\phi \equiv 1$ in a neighborhood of U. Finally, we set for $p(x,\xi) \in MS^{m,\omega}_{\rho,\delta}(\Lambda)$,

$$P(x,D)f(x) = \int e^{ix\xi}\phi(x)p(x,\xi)\psi(\xi)(\phi f)\hat{}(\xi)d\xi, \tag{3.6}$$

and more generally for $a(x,y,\xi) \in MS^{m,\omega}_{\rho,\delta}(\Lambda)$,

$$\big(Af\big)(x) = \iint e^{i(x-y)\xi}\phi(x)a(x,y,\xi)\psi(\xi)\phi(y)f(y)dyd\xi. \tag{3.7}$$

The following lemma explains why we select ϕ and ψ is this a way, and it will be useful in the following results.

Lemma 3.5 *Let Ω be an open subset of $\mathbb{R}^N$ with $U^* \subset \Omega$. If the amplitude $a(x,y,\xi)$ is in $MS^{m,\omega}_{\rho,\delta}(\Lambda)$, then the function $b(x,y,\xi) = \phi(x)\phi(y)a(x,y,\xi)\psi(\xi)$ is an amplitude in $AS^{m,\omega}_{\rho,\delta}(\Omega)$.*

P r o o f. Since $U^* \subset \Omega$ and $\phi \in \mathcal{D}_{\{\sigma\}}(\Omega)$, in view of Remark 2.5, for each $n \in \mathbb{N}$ there is $E_n > 0$ such that

$$\sup_{x\in\mathbb{R}^N} |D^\alpha \phi(x)| \leq E_n e^{(\rho-\delta)n\varphi^*_\omega(\frac{|\alpha|}{n})}. \tag{3.8}$$

We write $\varphi^*_\omega = \varphi^*$, and $Q = \operatorname{supp}\phi$. Then, using equations (3.4) and (3.5), the convexity of φ^*, and the fact that $\sum_{\alpha_1\leq\alpha}\binom{\alpha}{\alpha_1} \leq 2^{|\alpha|}$, we obtain, for $x, y \in Q$ and $n \in \mathbb{N}$,

$$\begin{aligned}
&|D^\alpha_x D^\gamma_y D^\beta_\xi b(x,y,\xi)| \\
&\leq \sum_{\alpha_1\leq\alpha}\sum_{\beta_1\leq\beta}\sum_{\gamma_1\leq\gamma}\binom{\alpha}{\alpha_1}\binom{\beta}{\beta_1}\binom{\gamma}{\gamma_1}\left|D^{\alpha_1}\phi(x)D^{\gamma_1}\phi(y)D^{\beta_1}\psi(\xi)D_x^{\alpha-\alpha_1}D_y^{\gamma-\gamma_1}D_\xi^{\beta-\beta_1}a(x,y,\xi)\right| \\
&\leq C_n E_n^2 D_n 2^{|\alpha+\gamma|} e^{(\rho-\delta)n\varphi^*(\frac{|\alpha+\gamma|}{n})}\cdot\sum_{\beta_1\leq\beta}\binom{\beta}{\beta_1}\beta! B^{|\beta|}|\xi|^{\delta|\alpha+\gamma|-\rho|\beta-\beta_1|-r|\beta_1|},
\end{aligned}$$

for $\xi \in \mathbb{R}^N$ with $\log(\frac{|\xi|}{R}) \geq \frac{n}{|\beta|}\varphi^*(\frac{|\beta|}{n})$. An application of [13, Lemma 1.3.(2)] and the fact that $\rho \leq r$ concludes the proof. □

We will see that the pseudodifferential operators defined in cones have similar properties to the standard ones. We will check first that a pseudodifferential operator corresponding to a standard amplitude shrinks the (ω)-wave front set when the operator acts on relatively compact open subsets. For this, we recall from [13, Example 2.11.(4)] that if Ω is an open set of $\mathbb{R}^N$ then a continuous linear operator $T : \mathcal{D}_{(\omega)}(\Omega) \to \mathcal{E}_{(\omega)}(\Omega)$ admiting a continuous and linear extension $\tilde{T} : \mathcal{E}'_{(\omega)}(\Omega) \to \mathcal{E}_{(\omega)}(\Omega)$ is called (ω)-smoothing operator. These operators are exactly the integral operators defined by kernels in $\mathcal{E}_{(\omega)}(\Omega\times\Omega)$.

Lemma 3.6 *Let Ω be a relatively compact open subset of $\mathbb{R}^N$. Let $a(x,y,\xi) \in S^{m,\omega}_{\rho,\delta}(\Omega)$ be given. Then, the corresponding pseudodifferential operator $A : \mathcal{E}'_{(\omega)}(\Omega) \to \mathcal{D}'_{(\omega)}(\Omega)$ shrinks the (ω)-wave front set, that is,*

$$WF_{(\omega)}(Au) \subset WF_{(\omega)}(u),$$

for every $u \in \mathcal{E}'_{(\omega)}(\Omega)$.

P r o o f. The restriction of A to $\mathcal{D}_{(\omega)}(\Omega)$, by [13, Theorem 3.13], can be decomposed as the sum $A = P + R$, where P is a pseudodifferential operator given by a symbol in $S^{m,\omega}_{\rho,\delta}(\Omega)$ and $R : \mathcal{E}'_{(\omega)}(\Omega) \to \mathcal{E}_{(\omega)}(\Omega)$ is a (ω)-smoothing operator. Since the extension of A to $\mathcal{E}'_{(\omega)}(\Omega)$ is unique, we have that $A = P + R$ also in $\mathcal{E}'_{(\omega)}(\Omega)$. Now, it suffices to apply Theorem 2.11 to conclude. □

We observe that in the following result we do not use techniques of wave front set of kernels, as it usual in the literature for such type of results.

Theorem 3.7 *Let Ω be an open subset of $\mathbb{R}^N$ with $U^* \subset \Omega$. Then A in (3.7) satisfies the following properties:*

(i) *A is a properly supported linear continuous map from $\mathcal{D}_{(\omega)}(\Omega)$ to $\mathcal{D}_{(\omega)}(\Omega)$, and admits a continuous and linear extension from $\mathcal{D}'_{(\omega)}(\Omega)$ to $\mathcal{D}'_{(\omega)}(\Omega)$.*

(ii) *A shrinks the (ω)-wave front set in Ω. In particular, it shrinks also the (ω)-singular support in Ω.*

P r o o f. We first observe that $\tilde{a}(x,y,\xi) = a(x,y,\xi)\psi(\xi)$ defines an amplitude in $AS^{m,\omega}_{\rho,\delta}(U^*)$ by the properties of ψ (see Lemma 3.5). Moreover, the map A is the composition $B \circ T \circ B$, where B is the multiplication operator by the fixed test function ϕ, and T is the pseudodifferential operator associated to the amplitude $\tilde{a}(x,y,\xi) = a(x,y,\xi)\psi(\xi)$ in $U^* \times U^* \times \mathbb{R}^N$.

Suppose that $\operatorname{supp}\phi \subset \Omega_1 \subset U^*$, being Ω_1 a relatively compact open subset of Ω. Then the multiplication operator by ϕ maps $\mathcal{D}'_{(\omega)}(\Omega)$ into $\mathcal{E}'_{(\omega)}(\Omega_1)$ continuously and hence, the operator A can be viewed as the following composition of continuous and linear operators

$$\mathcal{D}'_{(\omega)}(\Omega) \xrightarrow{B} \mathcal{E}'_{(\omega)}(\Omega_1) \xrightarrow{T} \mathcal{D}'_{(\omega)}(\Omega_1) \xrightarrow{B} \mathcal{E}'_{(\omega)}(\Omega),$$

where we use the fact that T can also be extended as a linear and continuous operator from $\mathcal{E}'_{(\omega)}(\Omega_1)$ to $\mathcal{D}'_{(\omega)}(\Omega_1)$ as we have the natural inclusion $\mathcal{D}_{(\omega)}(\Omega_1) \subset \mathcal{D}_{(\omega)}(\Omega)$ (Theorem 2.9). The last arrow follows from the fact that the restriction mapping $\mathcal{E}_{(\omega)}(\Omega) \to \mathcal{E}_{(\omega)}(\Omega_1)$ is continuous and hence, the inclusion $\mathcal{E}'_{(\omega)}(\Omega_1) \subset \mathcal{E}'_{(\omega)}(\Omega)$ is continuous. Then, A is linear and continuous operator acting from $\mathcal{D}'_{(\omega)}(\Omega)$ to $\mathcal{D}'_{(\omega)}(\Omega)$. Similarly, we can prove that A is also linear and continuous operator from $\mathcal{D}_{(\omega)}(\Omega)$ to $\mathcal{D}_{(\omega)}(\Omega)$. Consequently, A is properly supported. Then property (i) is proved. By Lemma 3.6, all the operators involved, even the last inclusion, shrink the (ω)-wave front set. An application of Theorem 2.7 gives the same conclusion for (ω)-singular supports, which proves (ii) and concludes the proof. □

The next proposition shows that the action of A does not depend on the choice of ψ and ϕ.

Proposition 3.8 *Let ψ', ϕ' denote other functions with the properties of ψ, ϕ after Definition 3.4, and let A' the corresponding operator defined according to* (3.7). *Then the map $A - A'$ is (ω)-micro-regularizing in Λ.*

P r o o f. Lemma 3.5 shows that the functions

$$b(x, y, \xi) = \phi(x)\phi(y)a(x, y, \xi)\psi(\xi)$$

and

$$b'(x, y, \xi) = \phi'(x)\phi'(y)a(x, y, \xi)\psi'(\xi)$$

are amplitudes in $AS^{m,\omega}_{\rho,\delta}(\Omega)$ corresponding to the operators A and A', respectively. Since $\omega(t) = o(t^d)$, $d = \rho - \delta$, these amplitudes are also in $S^{m,\omega}_{\rho,\delta}(\Omega)$. On the other hand, $b(x, y, \xi) - b'(x, y, \xi) = 0$ in Λ. Therefore, to conclude the proof we only have to apply Theorem 3.3. □

Now, we can develop a micro-local symbolic calculus for this type of operators, that will allow us to construct parametrices in this setting (see also [13, 3.1]).

Definition 3.9 We denote by $FMS^{m,\omega}_{\rho,\delta}(\Lambda)$ the set of all formal sums $\sum_{j\in\mathbb{N}_0} a_j(x, y, \xi)$ such that $a_j(x, y, \xi) \in C^\infty(\Lambda^*)$, with Λ and Λ^* defined as in Definition 3.4, and there are $R, B \geq 1$ and a sequence $C_n > 0$, $n \in \mathbb{N}$, of constants satisfying

$$\begin{aligned}&\left|D^\alpha_x D^\gamma_y D^\beta_\xi a_j(x, y, \xi)\right| \\ &\qquad \leq C_n B^{|\beta|}\beta! e^{(\rho-\delta)n\varphi^*((|\alpha+\gamma|+j)/n)} e^{m\omega(\xi)} |\xi|^{|\alpha+\gamma|\delta-|\beta|\rho-(\rho-\delta)j}\end{aligned}$$

for every $j \in \mathbb{N}_0$, $(x, y, \xi) \in \Lambda^*$ with $\log(|\xi|/R) \geq \frac{n}{|\beta|+j}\varphi^*\big(\frac{|\beta|+j}{n}\big)$.

In a similar way we define the equivalence between formal sums, and, as a particular case, between symbols and formal sums, via a natural identification between symbols and formal sums.

Definition 3.10 Two formal sums $\sum a_j$ and $\sum b_j$ in $FMS^{m,\omega}_{\rho,\delta}(\Lambda)$ are said to be equivalent, and we write $\sum_{j\geq 0} a_j \sim \sum_{j\geq 0} b_j$, if for Λ and Λ^* defined as in Definition 3.4 there are $R, B \geq 1$ and two sequences $C_n > 0$ and $N_n > 0$, $n \in \mathbb{N}$, of constants satisfying

$$\begin{aligned}&\Big|D^\alpha_x D^\gamma_y D^\beta_\xi \sum_{j<M} (a_j - b_j)(x, y, \xi)\Big| \\ &\qquad \leq C_n B^{|\beta|}\beta! e^{(\rho-\delta)n\varphi^*((|\alpha+\gamma|+M)/n)} e^{m\omega(\xi)} |\xi|^{|\alpha+\gamma|\delta-|\beta|\rho-(\rho-\delta)M}\end{aligned}$$

for every $(x, y, \xi) \in \Lambda^*$, $M \geq N_n$, with $\log(|\xi|/R) \geq \frac{n}{|\beta|+M}\varphi^*\big(\frac{|\beta|+M}{n}\big)$.

Proposition 3.11 *Consider $p(x, \xi) \in MS^{m,\omega}_{\rho,\delta}(\Lambda)$ and assume that $p(x, \xi) \sim 0$ in $FMS^{m,\omega}_{\rho,\delta}(\Lambda)$. Then the operator $P(x, D)$ defined as in* (3.6) *after Definition 3.4 is (ω)-micro-regularizing in Λ.*

P r o o f. We choose ϕ, ψ as in the definition (3.6). Proceeding as in the proof of Lemma 3.5 we can see that the symbol $\tilde{p}(x, \xi) = \phi(x)p(x, \xi)\psi(\xi)$ in $AS^{m,\omega}_{\rho,\delta}(\Omega)$ is equivalent to 0 as a formal sum in $FAS^{m,\omega}_{\rho,\delta}(\Omega)$ (see [13, Definition 3.1] for the definition of the class $FAS^{m,\omega}_{\rho,\delta}(\Omega)$). By [13, Theorem 3.5], the corresponding operator $\tilde{P}(x, D)$ is (ω)-smoothing in Ω. But $P(x, D) = \tilde{P}(x, D) - (P(x, D) - \tilde{P}(x, D))$ is the sum of a (ω)-smoothing operator in Ω and a (ω)-micro-regularizing operator in Λ by Proposition 3.8, which gives the conclusion. □

The proof of the following two results is similar. In particular, we assume that $\omega(t) = o(t^d)$, for $d \le \rho - \delta$, $d < 1$, and that Ω is a relatively compact open set.

Theorem 3.12 *Let $a(x,y,\xi) \in MS^{m,\omega}_{\rho,\delta}(\Lambda)$ be given and let A be defined as in (3.7). Let $p(x,\xi) \in MS^{m,\omega}_{\rho,\delta}(\Lambda)$ with $p \sim \sum_{j\ge 0} p_j$, where*

$$p_j(x,\xi) = \sum_{|\alpha|=j} (\alpha!)^{-1} \partial_\xi^\alpha D_y^\alpha a(x,y,\xi)|_{y=x}.$$

If $P = P(x,D)$ is given as in (3.6), then the map $T = A - P$ is (ω)-micro-regularizing in Λ.

P r o o f. We consider the amplitude $\tilde{a}(x,y,\xi) = \phi(x)\phi(y)a(x,y,\xi)\psi(\xi)$ in $AS^{m,\omega}_{\rho,\delta}(\Omega)$ and define $\tilde{p}_j(x,\xi) = \sum_{|\alpha|=j}(\alpha!)^{-1}\partial_\xi^\alpha D_y^\alpha \tilde{a}(x,y,\xi)|_{y=x}$. By [13, Theorem 3.13] we obtain a pseudodifferential operator $\tilde{P}$ with symbol $\tilde{p} \sim \sum_{j\ge 0}\tilde{p}_j$, such that if $\tilde{A}$ is the pseudodifferential operator associated to $\tilde{a}$, then $\tilde{T} = \tilde{A} - \tilde{P}$ is (ω)-smoothing in Ω. Since $T - \tilde{T}$ is (ω)-micro-regularizing in Λ, we deduce that T is also (ω)-micro-regularizing in Λ. □

The formal sum $p \circ q$ can be also defined (see [13, Definition 3.15]) and the following result holds.

Theorem 3.13 *Consider $p(x,\xi), q(x,\xi) \in MS^{m,\omega}_{\rho,\delta}(\Lambda)$ and let $P = P(x,D), Q = Q(x,D)$ be defined according to (3.6). Then $P \circ Q = R + T$, where T is (ω)-micro-regularizing in Λ and $R = R(x,D)$ is defined as in (3.6) for $r(x,\xi) \in MS^{2m,\omega}_{\rho,\delta}(\Lambda)$ with $r(x,\xi) \sim (2\pi)^N p(x,\xi) \circ q(x,\xi)$ in $FMS^{2m,\omega}_{\rho,\delta}(\Lambda)$.*

P r o o f. It suffices to proceed as in Theorem 3.12 and apply [13, Theorem 3.18]. □

We can now state and prove a sufficient condition for a linear partial differential operator to be (ω)-micro-hypoelliptic at some point $(x_0,\xi_0) \in \Omega \times (\mathbb{R}^N \setminus \{0\})$, via the construction of a micro-local parametrix. In this case we assume that σ is a weight function satisfying property (α_0) (see after Definition 2.1), and then σ turns to be equivalent to a sub-additive weight function.

Theorem 3.14 *Let Ω be a relatively compact open subset of $\mathbb{R}^N$, and $0 \le \delta < \rho \le 1$, ω a weight function and σ a weight function satisfying (α_0) such that $\omega(t^{1/(\rho-\delta)}) = o(\sigma(t))$ as $t \to \infty$. Let $P = P(x,D)$ be a partial differential operator with symbol $p(x,\xi) \in AS^{m,\omega}_{\rho,\delta}(\Omega)$. Let U be a neighborhood of x_0 in Ω and Γ an open conic neighborhood of $\xi_0 \ne 0$. Assume there exist some other neighborhoods U^*, Γ^* as in Definition 3.4, with $U^* \subset \Omega$, and constants $A, C > 0$ and $n \in \mathbb{N}$ such that*

$$|p(x,\xi)| \ge \frac{1}{A} e^{-m\omega(\xi)} \text{ for } x \in U^*, \xi \in \Gamma^*, |\xi| > A, \tag{3.9}$$

$$|D_x^\alpha D_\xi^\beta p(x,\xi)| \le C^{|\alpha|+|\beta|}\beta! e^{\frac{1}{n}\varphi_\sigma^*(|\alpha|n)}|p(x,\xi)|(1+|\xi|)^{\delta|\alpha|-\rho|\beta|} \tag{3.10}$$

for all $(\alpha,\beta) \ne 0$, $x \in U^*$, $\xi \in \Gamma^*$, $|\xi| > A$. Then there is a properly supported map Q with kernel in $\mathcal{D}'_{(\omega)}(\Omega \times \Omega)$ such that Q is (ω)-micro-pseudolocal in Ω and $Q \circ P = I + T$, where I is the identity operator and T is (ω)-micro-regularizing in $\Lambda = U \times \Gamma$. In particular, the following holds

$$\Lambda \cap WF_{(\omega)}(Pu) = \Lambda \cap WF_{(\omega)}(u) \text{ for all } u \in \mathcal{E}'_{(\omega)}(\Omega).$$

P r o o f. Arguing as in the proof of [12, Theorem 3.4] we can construct $q(x,\xi) \in MS^{m,\omega}_{\rho,\delta}(\Lambda)$ such that $q \circ p \sim 1$ in $FMS^{m,\omega}_{\rho,\delta}(\Lambda)$. So, we can define $Q = Q(x,D)$ according to (3.6) using the symbol $q(x,\xi)$. By Theorem 3.13 we deduce $(\frac{1}{(2\pi)^N}Q) \circ P = R + T$, where T is (ω)-micro-regularizing in Λ. Moreover, if $R = R(x,D)$ is the pseudodifferential operator defined by $r(x,\xi) \in MS^{m,\omega}_{\rho,\delta}(\Lambda)$, we have $r(x,\xi) - 1 \sim 0$ in $FMS^{m,\omega}_{\rho,\delta}(\Lambda)$. An application of Proposition 3.11 gives the conclusion. □

Since linear partial differential operators always shrink the support, we can now obtain the main result of the paper for arbitrary open sets.

Theorem 3.15 *Let Ω be an arbitrary open set in $\mathbb{R}^N$, ω and σ be two weight functions such that $\omega(t) = o(\sigma(t))$ as t tends to infinity and σ satisfies property (α_0). Let $P = P(x,D) = \sum_{|\alpha|\leq m} a_\alpha(x)D^\alpha$ be a linear partial differential operator with coefficients in $\mathcal{E}_{\{\sigma\}}(\Omega)$. Then we have*

$$WF_{(\omega)}(u) \subset WF_{(\omega)}(Pu) \cup \Sigma,$$

for all $u \in \mathcal{E}'_{(\omega)}(\Omega)$, where Σ is the characteristic manifold of the principal symbol of P.

P r o o f. Since $\operatorname{supp}(Pu) \subset \operatorname{supp} u$ and u is compactly supported, without loss of generality we can assume that Ω is relatively compact. Let $(x_0,\xi_0) \notin \Sigma$ and $p_m(x,\xi)$ be the principal symbol of P. We fix neighborhoods $\Lambda = U \times \Gamma$, $\Lambda^* = U^* \times \Gamma^*$ of (x_0,ξ_0) conic with respect to the second variable, with U^* and Γ^* satisfying the conditions of Definition 3.4, and sufficiently small such that $p_m(x,\xi) \neq 0$ in Λ^*.

Denote by $p(x,\xi)$ the symbol of P. By homogeneity $c|\xi|^m \leq |p_m(x,\xi)|$ for $(x,\xi) \in \Lambda^*$ for some constant $c > 0$ (U^* is relatively compact). On the other hand, for some constants $A, C > 0$,

$$|p(x,\xi) - p_m(x,\xi)| \leq C(1+|\xi|)^{m-1}, \quad (x,\xi) \in \Lambda^*, \ \ |\xi| > A,$$

thereby obtaining, if A is large enough,

$$c(1+|\xi|)^m/2 \leq |p(x,\xi)|, \quad (x,\xi) \in \Lambda^*, \ \ |\xi| > A, \tag{3.11}$$

which proves (3.9) in Theorem 3.14. By hypothesis

$$|D_x^\alpha D_\xi^\beta p(x,\xi)| \leq D\beta! e^{\frac{1}{n}\varphi_\sigma^*(|\alpha|n)}(1+|\xi|)^{m-|\beta|}, \tag{3.12}$$

for $x \in U^*$ and $\xi \in \mathbb{R}^N$. Combining (3.11) and (3.12) we obtain (3.10) of Theorem 3.14 for $(x,\xi) \in \Lambda^*$ and $|\xi| > A$. So, we can apply Theorem 3.14 for $\rho = 1$ and $\delta = 0$ to obtain the existence of a micro-local parametrix Q. Then, for every $(x_0,\xi_0) \notin \Sigma$ we find a neighborhood Λ of this point, conic with respect to the second variable, such that

$$\Lambda \cap WF_{(\omega)}(Pu) = \Lambda \cap WF_{(\omega)}(u) \ \text{ for all } \ u \in \mathcal{E}'_{(\omega)}(\Omega),$$

which concludes the proof. □

The Roumieu version of the result follows as a corollary. We observe that in this case we can avoid to take the coefficients of P in a smaller class.

Corollary 3.16 *Let Ω be an arbitrary open set in $\mathbb{R}^N$. Let ω be a weight function satisfying property (α_0). Let $P = P(x,D) = \sum_{|\alpha|\leq m} a_\alpha(x)D^\alpha$ be a linear partial differential operator with coefficients in $\mathcal{E}_{\{\omega\}}(\Omega)$. Then we have*

$$WF_{\{\omega\}}(u) \subset WF_{\{\omega\}}(Pu) \cup \Sigma,$$

for all $u \in \mathcal{E}'_{\{\omega\}}(\Omega)$, where Σ is the characteristic manifold of the principal symbol of P.

P r o o f. In view of Proposition 2.6 there exists a weight function σ_0 such that $\sigma_0 = o(\omega)$, $u \in \mathcal{E}'_{(\sigma_0)}(\Omega)$ and

$$WF_{\{\omega\}}(u) = \overline{\bigcup_{\sigma\in S} WF_{(\sigma)}u} \tag{3.13}$$

for $S = \{\sigma \text{ weight function: } \sigma_0 \leq \sigma = o(\omega)\}$. Since $u \in \mathcal{E}'_{(\sigma)}(\Omega)$ for all $\sigma \in S$, we can apply Theorem 3.15 to obtain

$$WF_{(\sigma)}(u) \subset WF_{(\sigma)}(Pu) \cup \Sigma,$$

for all $\sigma \in S$. The conclusion follows directly from (3.13). □

Remark 3.17 We recall that a pseudodifferential operator $P(x,D)$ with symbol in $AS^{m,\omega}_{\rho,\delta}(\Omega)$ is said to be classical if its symbol $p(x,\xi)$ admits an asymptotic expansion of the kind

$$p(x,\xi) \sim \sum_{j=0}^{+\infty} p_{m-j}(x,\xi),$$

where $p_{m-j}(x,\xi)$ is positively homogeneous of order $m-j$ with respect to ξ. The results of Theorem 3.15 and Corollary 3.16 continue to hold for classical properly supported pseudodifferential operators with symbol in $AS^{m,\omega}_{\rho,\delta}(\Omega)$, $0 \le \delta < \rho \le 1$. In fact, let us consider $p(x,\xi) \in AS^{m,\omega}_{\rho,\delta}(\Omega)$. We observe at first that if the corresponding pseudodifferential operator $P(x,D)$ is classical then Σ is well defined. Moreover, since $P(x,D)$ is properly supported we have

$$P(x,D) : \mathcal{E}'_{(\omega)}(\Omega) \to \mathcal{E}'_{(\omega)}(\Omega). \tag{3.14}$$

The proof of Theorem 3.14 holds for general pseudodifferential operators with symbol in $AS^{m,\omega}_{\rho,\delta}(\Omega)$. Concerning Theorem 3.15, we have only to show that Ω can be assumed to be relatively compact, and then the proof works in the same way. For every fixed $u \in \mathcal{E}'_{(\omega)}(\Omega)$, from (3.14) we have that also $P(x,D)u$ has compact support and hence, we can take Ω as a relatively compact open set containing both $\operatorname{supp} u$ and $\operatorname{supp}(P(x,D)u)$. Corollary 3.16 can then be deduced for pseudodifferential operators in analogous way from Theorem 3.15.

4 Some applications

The aim of this section is to give some applications of the results obtained in the paper. First, we obtain the relation between ellipticity and hypoellipticity as an immediate consequence of Theorem 3.15. Let us consider an elliptic linear partial differential operator $P = P(x,D)$ satisfying the hypotheses of Theorem 3.15. Then the characteristic set of P is empty and hence, we have

$$WF_{(\omega)}(u) \subset WF_{(\omega)}(Pu)$$

for every $u \in \mathcal{E}'_{(\omega)}(\Omega)$. Since the opposite inclusion is also true, P then satisfies

$$WF_{(\omega)}(u) = WF_{(\omega)}(Pu)$$

for every $u \in \mathcal{E}'_{(\omega)}(\Omega)$. This means that $P(x,D)$ is (ω)-micro-hypoelliptic and then (ω)-hypoelliptic, too, in view of Theorem 2.7. The same conclusion holds in the Roumieu setting by using Corollary 3.16.

We want now to study the wave front set of the solutions for the following (non hypoelliptic) partial differential operator of principal type in $\mathbb{R}^N$:

$$P = \frac{\partial}{\partial x_N}.$$

We write $*$ for (ω) or $\{\omega\}$. Observe that the characteristic set of P is

$$\Sigma = \{(x,\xi) \in \mathbb{R}^{2N} : \xi_N = 0,\ \xi \neq 0\}.$$

Moreover, we point out that $u \in \mathcal{D}'_*(\mathbb{R}^N)$ is a solution of $Pu = 0$ if, and only if, $u = v \otimes \mathbf{1}$ for some $v \in \mathcal{D}'_*(\mathbb{R}^{N-1})$, being $\mathbf{1}$ the function identically 1 in the x_N-variable. Indeed, if u is of the form $v \otimes \mathbf{1}$, then $\frac{\partial}{\partial x_N}(v \otimes \mathbf{1}) = 0$. On the other hand, if $Pu = 0$, then u satisfies $\tau_h u = u$ for every $h = (0,\ldots,0,h_N)$, where $\tau_h u$ denotes the h-translation of the distribution u (see, for example, [22]). From this fact, by an approximation procedure, it is easy to conclude that u must be of the form $v \otimes \mathbf{1}$, for some distribution $v \in \mathcal{D}'_*(\mathbb{R}^{N-1})$.

Proposition 4.1 *Let ω be a weight satisfying property (α_0), and write, as usual, $*$ for (ω) or $\{\omega\}$. Let $u \in \mathcal{E}'_*(\mathbb{R}^N)$ be a solution of the equation $Pu = 0$. If $(\overline{x},\overline{\xi}) \in WF_*(u)$, then $(\overline{x},\overline{\xi}) \in \Sigma$, and splitting $\mathbb{R}^N \ni x = (x', x_N) = (x_1,\ldots,x_{N-1},x_N)$, we have that the straight line*

$$L = \{(\overline{x}', x_N, \overline{\xi}),\ x_N \in \mathbb{R}\}$$

is contained in $WF_*(u)$. Moreover, for every $(\overline{x},\overline{\xi}) \in \Sigma$ there exists a solution $u \in \mathcal{E}'_*(\mathbb{R}^N)$ of $Pu = 0$, whose $*$-wave front set is given by the straight line L.

P r o o f. We will only deal with the Beurling case, as in the Roumieu case the result follows using the same arguments and Corollary 3.16.

Let $u \in \mathcal{E}'_{(\omega)}(\mathbb{R}^N)$ be a solution of $Pu = 0$. Then by Theorem 3.15 we have

$$WF_{(\omega)}(u) \subset \Sigma,$$

and $u = v \otimes \mathbf{1}$ for some $v \in \mathcal{E}'_{(\omega)}(\mathbb{R}^{N-1})$. We claim that

$$WF_{(\omega)}(u) = \{(x, \xi) \in \Sigma : (x', \xi') \in WF_{(\omega)}(v)\}. \tag{4.1}$$

To prove (4.1) we proceed as follows.

Let us fix $(\overline{x}, \overline{\xi}) \in \Sigma$. If $(\overline{x}', \overline{\xi}') \notin WF_{(\omega)}(v)$, then, by Definition 2.4, there exists $\varphi \in \mathcal{D}_{(\omega)}(\mathbb{R}^{N-1})$, $\varphi \equiv 1$ in a neighborhood of $\overline{x}'$, and a conic neighborhood Γ' of $\overline{\xi}'$ such that for every $m \in \mathbb{N}$ there exists a positive constant C_m satisfying

$$|\widehat{\varphi v}(\xi')| \le C_m e^{-m\omega(\xi')}$$

for every $\xi' \in \Gamma'$. Let $\chi \in \mathcal{D}_{(\omega)}(\mathbb{R})$ be a function equal to 1 in a neighborhood I of $\overline{x}_N$. Let Γ be a conic neighborhood of $(\overline{\xi}, 0)$ ($(\overline{x}, \overline{\xi}) \in \Sigma$, so we have $\overline{\xi}_N = 0$) with $\Gamma \cap \{\xi_N = 0\} \subset \Gamma'$. Then there exists a positive constant c_1 such that

$$|\xi_N| \le c_1 |\xi'|$$

in Γ. Then we obtain, for every $m \in \mathbb{N}$,

$$|((\varphi \otimes \chi)u)^\wedge(\xi)| = |\widehat{\varphi v}(\xi')\hat{\chi}(\xi_N)| \le C'_m e^{-m(\omega(\xi') + \omega(\xi_N))}. \tag{4.2}$$

Since ω satisfies the property (α_0), we have $\omega(\xi) \le \omega(|\xi'| + |\xi_N|) \le \omega((1 + c_1)|\xi'|) \le (1 + c_1)D\omega(\xi') \le C(\omega(\xi') + \omega(\xi_N))$ for some positive constant C independent of ξ, and for every $\xi \in \Gamma$. Thus, by (4.2) we deduce that for every $m \in \mathbb{N}$ there exists a constant $C''_m > 0$ such that

$$|((\varphi \otimes \chi)u)^\wedge(\xi)| \le C''_m e^{-m\omega(\xi)},$$

for every $\xi \in \Gamma$. This means that $(\overline{x}, \overline{\xi}) \notin WF_{(\omega)}(u)$. So, we have proved that

$$\{(x, \xi) \in \Sigma : (x', \xi') \in WF_{(\omega)}(v)\} \subset WF_{(\omega)}(u).$$

In order to obtain (4.1) we have now to prove the opposite inclusion, i.e., if $(\overline{x}, \overline{\xi}) \in \Sigma$ and $(\overline{x}, \overline{\xi}) \notin WF_{(\omega)}(u)$ then $(\overline{x}', \overline{\xi}') \notin WF_{(\omega)}(v)$.

Let $(\overline{x}, \overline{\xi}) \notin WF_{(\omega)}(u)$. Then there exists $\phi \in \mathcal{D}_{(\omega)}(\mathbb{R}^N)$, $\phi \equiv 1$ in a neighborhood $\overline{U}$ of $\overline{x}$, and a conic neighborhood Γ of $\overline{\xi}$, such that for every $m \in \mathbb{N}$ there exists a positive constant C_m such that

$$|\widehat{\phi u}(\xi)| \le C_m e^{-m\omega(\xi)}$$

for every $\xi \in \Gamma$. Without loss of generality, we can assume that $\phi(x) = \phi_1(x')\phi_2(x_N)$, with $\hat{\phi}_2(0) \ne 0$, eventually by multiplying by a tensor product test function with support contained in $\overline{U}$. Then, we have

$$|\widehat{\phi_1 v}(\xi')\hat{\phi}_2(\xi_N)| \le C_m e^{-m\omega(\xi)}$$

for $\xi \in \Gamma$ and hence,

$$|\widehat{\phi_1 v}(\xi')| \le \frac{C_m}{|\hat{\phi}_2(0)|} e^{-m\omega(\xi')},$$

for $\xi' \in \Gamma' = \Gamma \cap \{\xi_N = 0\}$. It follows that $(\overline{x}', \overline{\xi}') \notin WF_{(\omega)}(v)$. This concludes the proof of (4.1).

Applying (4.1), we immediately conclude that if $(\overline{x}, \overline{\xi}) \in WF_{(\omega)}(u)$, u being a solution of $Pu = 0$, then every point of the kind $(\overline{x}', x_N, \overline{\xi})$ with $x_N \in \mathbb{R}$ belongs to $WF_{(\omega)}(u)$, i.e., the straight line L is contained in $WF_{(\omega)}(u)$.

Let us fix now $(\overline{x},\overline{\xi})\in\Sigma$. We want to construct a solution $u\in\mathcal{E}'_*(\mathbb{R}^N)$ of $Pu=0$, $*=(\omega)$ or $\{\omega\}$, such that

$$WF_*(u)=\{(x,\xi)\in\mathbb{R}^N\times\mathbb{R}^N\setminus\{0\}: x=(\overline{x}',x_N),\ \xi=\lambda\overline{\xi},\ x_N\in\mathbb{R},\ \lambda>0\}.$$

Since $u=v\otimes\mathbf{1}$, by (4.1) it is enough to construct $v\in\mathcal{E}'_*(\mathbb{R}^{N-1})$ such that

$$WF_*(v)=\{(x',\xi')\in\mathbb{R}^{N-1}\times\mathbb{R}^{N-1}\setminus\{0\}: x'=\overline{x}',\ \xi'=\lambda\overline{\xi}',\ \lambda>0\}. \tag{4.3}$$

We know from Example 1 in [14] that it is possible to construct $\tilde{u}\in\mathcal{E}'_{(\omega)}(\mathbb{R})$ whose Beurling wave front set is given by $\{0\}\times(0,+\infty)$. The same construction holds for the Roumieu case, and can be easily extended to dimension greater than 1. Therefore, there exists $\tilde{v}\in\mathcal{E}'_*(\mathbb{R}^{N-1})$ satisfying

$$WF_*(\tilde{v})=\{(x',\xi')\in\mathbb{R}^{N-1}\times\mathbb{R}^{N-1}\setminus\{0\}: x'=0,\ \xi'=(0,\dots,0,\xi_{N-1}),\ \xi_{N-1}>0\}.$$

By a linear change of variable and a translation we then find v satisfying (4.3). □

Observe that an analogous result holds for the equation $Pu=f$, where we can take for example $f\in C^\infty(\mathbb{R}^N)$. Indeed, every solution of $Pu=f$ can be written as

$$u(x)=u_0(x)+\int_0^{x_N} f(x',t)\,dt,\quad x=(x',x_N)\in\mathbb{R}^N,$$

where u_0 is some solution of $Pu=0$. If $(\overline{x},\overline{\xi})\in\Sigma$ and $(\overline{x},\overline{\xi})\notin WF_*(f)$, then $(\overline{x},\overline{\xi})\in WF_*(u)$ implies that $(\overline{x}',x_N,\overline{\xi})\in WF_*(u)$ for x_N in a suitable interval I containing x_N. In fact, if $(\overline{x},\overline{\xi})\notin WF_*(f)$ then there exists a neighborhood U of $(\overline{x},\overline{\xi})$ with empty intersection with $WF_*(f)$. Consequently, in a neighborhood of $\overline{x}$, the wave front set of u is determined by the wave front set of u_0.

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