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(Article begins on next page)

Bergman–Bianchi identities in field theories*

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Abstract

We relate the generalized Bergman–Bianchi identities for Lagrangian field theories on gauge-natural bundles with the kernel of the associated gauge-natural Jacobi morphism.

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Key words: gauge-natural bundles, Bergman–Bianchi identities, Jacobi morphisms.

1 The Bergman–Bianchi morphism

Our general framework is the calculus of variations on finite order *gauge-natural bundles* [3, 8]. Such geometric structures have been widely recognized to suitably describe so-called gauge-natural field theories, *i.e.* physical theories in which right-invariant infinitesimal automorphisms of the structure bundle $\mathbf{P}$ uniquely define the transformation laws of the fields themselves (see *e.g.* [4] and references quoted therein). We shall in particular consider *finite order variational sequences on gauge-natural bundles*, whereby fundamental objects of calculus of variations such as Lagrangians, Euler–Lagrange and Jacobi morphisms are conveniently represented as quotient morphisms (see *e.g.* [9, 6]). For basic notions and fixing notation we refer to [1, 3, 4, 5, 6, 8, 11] and references therein.

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Recall that generalized Bergman–Bianchi identities for field theories are necessary and (locally) sufficient conditions for the Noether conserved current to be not only closed but also the divergence of a skew-symmetric (tensor) density along solutions of the Euler–Lagrange equations [1]. It was also stressed that in the general theory of relativity these identities coincide with the contracted Bianchi identities for the curvature tensor of the pseudo-Riemannian metric.

Let $\mathbf{Y}_\zeta$ be a gauge-natural bundle and let λ be a gauge-natural Lagrangian [4, 8] on the s -th order prolongation $J_s \mathbf{Y}_\zeta$. Let $\mathcal{A}^{(r,k)}$ be the vector bundle of right-invariant principal automorphisms of the underlying principal structure bundle $\mathbf{P}$. In the following we shall consider variation vector fields which are vertical parts of gauge-natural lifts of a given $\bar{\Xi} \in \mathcal{A}^{(r,k)}$. Let $\mathcal{C}_{2s}^*[\mathcal{A}^{(r,k)}] \simeq J_{2s+1} \mathcal{A}^{(r,k)} \times_{J_{2s} \mathcal{A}^{(r,k)}} V J_{2s} \mathcal{A}^{(r,k)}$. By a slight abuse of notation, we denote by $\mathfrak{G}(\bar{\Xi})_V$ the vertical part – with respect to the contact structure induced by the projections $J_{s+1} \mathbf{Y}_\zeta \rightarrow J_s \mathbf{Y}_\zeta$ – of (jet prolongation of) the gauge-natural lift $\mathfrak{G}(\bar{\Xi})$ [3, 4, 5]. We set

$$\omega(\lambda, \mathfrak{G}(\bar{\Xi})_V) \equiv \mathcal{L}_{\bar{\Xi}} \mathcal{E}_n(\lambda) : J_{2s} \mathbf{Y}_\zeta \rightarrow \mathcal{C}_{2s}^*[\mathcal{A}^{(r,k)}] \otimes \mathcal{C}_0^*[\mathcal{A}^{(r,k)}] \wedge (\wedge^n T^* \mathbf{X}), \quad (1)$$

where $\mathcal{L}_{\bar{\Xi}}$ is the Lie derivative operator acting on sections of the gauge-natural bundle [5], $\lrcorner$ is the interior product and $\mathcal{E}_n(\lambda)$ is the generalized Euler–Lagrange morphism associated with λ [6]. The morphism $\omega(\lambda, \mathfrak{G}(\bar{\Xi})_V)$ so defined is a generalized Lagrangian associated with the field equations of the original Lagrangian λ and it has been considered in applications *e.g.* in General Relativity. By the linearity of $\mathcal{L}$ we can regard $\omega(\lambda, \mathfrak{G}(\bar{\Xi})_V)$ as the extended morphism defined on $J_{2s} \mathbf{Y}_\zeta \times_X V J_{2s} \mathcal{A}^{(r,k)}$. We have $D_H \omega(\lambda, \mathfrak{G}(\bar{\Xi})_V) = 0$, where D_H is the exterior differential; thus, as a consequence of a global decomposition formula for vertical morphisms [7], we can state the following [11].

Lemma 1 *Let $\omega(\lambda, \mathfrak{G}(\bar{\Xi})_V)$ be as above. On the domain of $\omega(\lambda, \mathfrak{G}(\bar{\Xi})_V)$ we have (up to pull-backs):*

$$\omega(\lambda, \mathfrak{G}(\bar{\Xi})_V) = \beta(\lambda, \mathfrak{G}(\bar{\Xi})_V) + F_{\omega(\lambda, \mathfrak{G}(\bar{\Xi})_V)},$$

where

$$\beta(\lambda, \mathfrak{G}(\bar{\Xi})_V) \equiv E_{\omega(\lambda, \mathfrak{G}(\bar{\Xi})_V)}$$

and, locally, $F_{\omega(\lambda, \mathfrak{G}(\bar{\Xi})_V)} = D_H M_{\omega(\lambda, \mathfrak{G}(\bar{\Xi})_V)}$.

Definition 1 We call the global morphism $\beta(\lambda, \mathfrak{G}(\bar{\Xi})_V) := E_{\omega(\lambda, \mathfrak{G}(\bar{\Xi})_V)}$ the *generalized Bergman–Bianchi morphism* associated with the Lagrangian λ and the variation vector field $\bar{\Xi}$. $\square$

Let $\mathfrak{K}$ be the *kernel* of $\mathcal{J}(\lambda, \mathfrak{G}(\bar{\Xi})_V)$. We have the following characterization of the Bergman–Bianchi identities for gauge-natural theories [11].

Theorem 1 *The generalized Bergman–Bianchi morphism is globally vanishing for the variation vector field $\bar{\Xi}$ if and only if $\delta_{\mathfrak{G}}^2 \lambda \equiv \mathcal{J}(\lambda, \mathfrak{G}(\bar{\Xi})_V) = 0$, i.e. if and only if $\mathfrak{G}(\bar{\Xi})_V \in \mathfrak{K}$.*

From now on we shall write $\omega(\lambda, \mathfrak{K})$ to denote $\omega(\lambda, \mathfrak{G}(\bar{\Xi})_V)$ when $\mathfrak{G}(\bar{\Xi})_V$ belongs to $\mathfrak{K}$. Analogously for β and other morphisms.

First of all let us make the following important consideration. Let $\mathcal{L}_{j_s \bar{\Xi}}$ be the variational Lie derivative operator [6] acting on generalized variational morphisms.

Proposition 1 *For each $\bar{\Xi} \in \mathcal{A}^{(r,k)}$ such that $\bar{\Xi}_V \in \mathfrak{K}$, we have*

$$\mathcal{L}_{j_s \bar{\Xi}_H} \omega(\lambda, \mathfrak{K}) = -D_H(-j_s \mathcal{L}_{\bar{\Xi}}] p_{D_V \omega(\lambda, \mathfrak{K})}). \quad (2)$$

PROOF. We have

$$\mathcal{L}_{j_s \bar{\Xi}_V} \omega(\lambda, \mathfrak{K}) = \mathcal{L}_{j_s \bar{\Xi}_V} \mathcal{L}_{j_s \bar{\Xi}} \lambda = \mathcal{L}_{j_s [\bar{\Xi}_V, \bar{\Xi}_H]} \lambda.$$

On the other hand it is also easy to verify that

$$\mathcal{L}_{j_s \bar{\Xi}_H} \omega(\lambda, \mathfrak{K}) = \mathcal{L}_{j_s [\bar{\Xi}_H, \bar{\Xi}_V]} \lambda = -\mathcal{L}_{j_s \bar{\Xi}_V} \omega(\lambda, \mathfrak{K}).$$

Since

$$\begin{aligned} \mathcal{L}_{j_s \bar{\Xi}_V} \omega(\lambda, \mathfrak{K}) &= -\mathcal{L}_{\bar{\Xi}}] \mathcal{E}_n(\omega(\lambda, \mathfrak{K})) + D_H(-j_s \mathcal{L}_{\bar{\Xi}}] p_{D_V \omega(\lambda, \mathfrak{K})}) = \\ &= \beta(\lambda, \mathfrak{K}) + D_H(-j_s \mathcal{L}_{\bar{\Xi}}] p_{D_V \omega(\lambda, \mathfrak{K})}), \end{aligned}$$

from the Theorem above we get the assertion. $\square$

The new generalized Lagrangian $\omega(\lambda, \mathfrak{K})$ is gauge-natural invariant too, i.e. $\mathcal{L}_{j_s \bar{\Xi}} \omega(\lambda, \mathfrak{K}) = 0$.

Even more, we can state the following

Proposition 2 *Let $\bar{\Xi}_V \in \mathfrak{K}$. We have*

$$\mathcal{L}_{j_s \bar{\Xi}_H} \omega(\lambda, \mathfrak{K}) = 0. \quad (3)$$

Corollary 1 *Let $\bar{\Xi}_V \in \mathfrak{K}$. We have the covariant conservation law*

$$D_H(-j_s \mathcal{L}_{\bar{\Xi}}] p_{D_V \omega(\lambda, \mathfrak{K})}) = 0. \quad (4)$$

Definition 2 We define the covariantly conserved current

$$\mathcal{H}(\lambda, \mathfrak{K}) = -j_s \mathcal{L}_{\bar{\Xi}}] p_{D_V \omega(\lambda, \mathfrak{K})}, \quad (5)$$

to be a Hamiltonian form for $\omega(\lambda, \mathfrak{K})$ (in the sense of [10]). $\square$

References

- [1] P.G. BERGMAN: Conservation Laws in General relativity as the Generators of Co-ordinate Transformations, *Phys. Rev.* **112** (1) (1958) 287–289.
- [2] B. CASCIARO, M. FRANCAVIGLIA: A new variational characterization of Jacobi fields along geodesics. *Ann. Mat. Pura Appl.* (4) **172** (1997), 219–228.
- [3] D.J. ECK: Gauge-natural bundles and generalized gauge theories, *Mem. Amer. Math. Soc.* **247** (1981) 1–48.
- [4] L. FATIBENE, M. FRANCAVIGLIA: *Natural and gauge natural formalism for classical field theories. A geometric perspective including spinors and gauge theories*; Kluwer Academic Publishers, Dordrecht, 2003.
- [5] L. FATIBENE, M. FRANCAVIGLIA, M. PALESE: Conservation laws and variational sequences in gauge-natural theories, *Math. Proc. Camb. Phil. Soc.* **130** (2001) 555–569.
- [6] M. FRANCAVIGLIA, M. PALESE, R. VITOLO: Symmetries in Finite Order Variational Sequences, *Czech. Math. J.* **52(127)** (2002) 197–213.
- [7] I. KOLÁŘ: A Geometrical Version of the Higher Order Hamilton Formalism in Fibred Manifolds, *J. Geom. Phys.*, **1** (2) (1984) 127–137.
- [8] I. KOLÁŘ, P.W. MICHOR, J. SLOVÁK: *Natural Operations in Differential Geometry*, (Springer–Verlag, N.Y., 1993).
- [9] D. KRUPKA: Variational Sequences on Finite Order Jet Spaces, *Proc. Diff. Geom. and its Appl.* (Brno, 1989); J. Janyška, D. Krupka eds.; World Scientific (Singapore, 1990) 236–254.
- [10] L. MANGIAROTTI, G. SARDANASHVILY: *Connections in Classical and Quantum Field Theory*, (World Scientific, Singapore, 2000).
- [11] M. PALESE, E. WINTERROTH: Global Generalized Bianchi Identities for Invariant Variational Problems on Gauge-natural Bundles, to appear in *Arch. Math. (Brno)*.